M1522.000900 Data Structure

Spring 2025, Kang

Homework 1 Answer Sheet [1/2]

2023-12753 EunSu Yeo

※ Write your answers on the "your answer" columns.

※ Do **NOT** write anything on "score" columns.

※ Before submission, delete all the blue-colored texts, and convert the file into **PDF-format**.

※ Write your proof on the solution sheet and leave the "your answer" column blank.

Question		Your Answer	Points	Score
1	(1)	Satisfy: Antisymmetric, Transitive. Doesn't satisfy: Reflexive, Symmetric	2.5	
	(2)	Satisfy: Reflexive, Symmetric, Transitive. Doesn't satisfy: Antisymmetric	2.5	
	(3)	Satisfy: Antisymmetric, Transitive. Doesn't satisfy: Reflexive, Symmetric	2.5	
	(4)	Satisfy: Reflexive Doesn't satisfy: Symmetric, Antisymmetric, Transitive.	2.5	
2	(1)		2	
	(2)		2	
	(3)		2	
	(4)		2	
	(5)		2	
3	(1)	$T(n) = 1023$	5	
	(2)	$T(n) = 2T(n-1) + 1, (T(1) = 1)$	5	
	(3)	$T(n) = 2^n - 1$	5	
4	(1)	$C(n) = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$	5	
	(2)		5	
	(3)		5	

M1522.000900 Data Structure

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Homework 1 Answer Sheet [2/2]

2023-12753 EunSu Yeo

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※ Write your proof on the solution sheet and leave the “your answer” column blank.

Question		Your Answer	Points	Score
5		$\frac{n!}{5}$ is efficient for $n \in [2.04,3.1]$, 1.1^n when $n \in [3.1,21.4]$, $10\log(\log n)$ when $n \in [2,2.04] \cup [152.89,\infty)$, $n^{\frac{2}{3}}$ when $n \in [21.4,152.89]$	10	
6	(1)	$\theta(n)$	2.5	
	(2)	$\theta(n+m)$	2.5	
	(3)	$\theta(n^2)$	2.5	
	(4)	$\theta(n)$	2.5	
7	(1)	Yes, program B improves in time complexity over program A.	7.5	
	(2)	X is $\frac{logn}{n}$times faster than Y	7.5	
8	(1)	$\theta(n)$	5	
	(2)	$\theta(logn)$	10	
Total			100	

Homework 1
Solution Sheet

2023-12753

EunSu Yeo

1. Question 1 [10 points]
1.1. Q1(1) [2.5 points]

Solution.

Reflexive:

A person can't be a senior to himself.

So $aR_1a\ for\ all\ a \in S$ is impossible.

Due to the definition of reflexive, R_1 doesn't satisfy reflexive.

Symmetric:

Let's assume a person a is a senior to a person b. Then b can't be a senior to a

So $if\ aR_1b, then\ bR_1a$ is impossible.

Due to the definition of symmetric, R_1 doesn't satisfy symmetric.

Antisymmetric:

The condition of definition of antisymmetric is $aR_1b\ and\ bR_1a$. but as I described in Symmetric, it is impossible. As the condition of definition is false the proposition is true.

So R_1 satisfy antisymmetric.

Transitive:
$if\ aR_1b\ and\ bR_1c, then\ aR_1c\ for\ all\ a, b, c \in S$

If a is senior to b and b is senior to c then a is also senior to c. Due to the definition of Transitive, R_1 satisfy Transitive.

1.2. Q1(2) [2.5 points]

Solution.

Reflexive:

$aR_2a\ for\ all\ a \in S$ is possible. Due to the definition of reflexive, R_2 satisfy reflexive.

Symmetric:

$if\ aR_2b, then\ bR_2a$ is possible. Due to the definition of symmetric, R_2 satisfy symmetric.

Antisymmetric:

$if\ aR_2b\ and\ bR_2a\ , then\ a = b\ for\ all\ a, b \in S$ is not true.

When a and b are different people.

It satisfies the given condition. So due to the definition of Antisymmetric, R_2 doesn't satisfy antisymmetric.

Transitive:

$if\ aR_2b\ and\ bR_2c, then\ aR_2c\ for\ all\ a, b, c \in S$

is possible. Due to the definition of Transitive, R_1 satisfy Transitive.

1.3. Q1(3) [2.5 points]

Solution.

Reflexive:

$a(R_1 \cap R_2)a\ for\ all\ a \in S$ is impossible due to R_1. So $(R_1 \cap R_2)$ doesn't satisfy reflexive.

Symmetric:

$if\ a(R_1 \cap R_2)b, then\ b(R_1 \cap R_2)a$ is impossible due to R_1. Due to the definition of symmetric, $(R_1 \cap R_2)$ doen't satisfy symmetric.

Antisymmetric:

$if\ a(R_1 \cap R_2)b\ and\ b(R_1 \cap R_2)a\ , then\ a = b\ for\ all\ a, b \in S$

It's impossible to satisfy the given if condition due to R_1. As the condition of definition is false the proposition is true. So $R_1 \cap R_2$ satisfy antisymmetric.

Transitive:

$if\ a(R_1 \cap R_2)b\ and\ b(R_1 \cap R_2)c, then\ a(R_1 \cap R_2)c\ for\ all\ a,b,c \in S$

is possible. Due to the definition of Transitive, $R_1 \cap R_2$ satisfy Transitive.

1.4. Q1(4) [2.5 points]

Solution.

Reflexive:

$a(R_1 \cup R_2)a\ for\ all\ a \in S$ is true due to R_2. So $(R_1 \cup R_2)$ satisfy reflexive.

Symmetric:

$if\ a(R_1 \cup R_2)b, then\ b(R_1 \cup R_2)a$ is false due to R_1. In case when a and b only satisfy the relation R_1,the next condition doesn't satisfy. So this is the error case. Due to the definition of symmetric, $(R_1 \cup R_2)$ doesn't satisfy symmetric.

Antisymmetric:

$if\ a(R_1 \cup R_2)b\ and\ b(R_1 \cup R_2)a\ , then\ a = b\ for\ all\ a,b \in S$

If a and b is in same apartment and they are different people. Still it satisfies the condition. So we can not assert a=b. So $(R_1 \cup R_2)$ doesn't satisfy antisymmetric.

Transitive:

$if\ a(R_1 \cup R_2)b\ and\ b(R_1 \cup R_2)c, then\ a(R_1 \cup R_2)c\ for\ all\ a,b,c \in S$

In case a is senior to b and b and c is living in the same department it doesn't satisfies the condition. Due to the definition of Transitive, $(R_1 \cup R_2)$ doesn't satisfy Transitive.

2. Question 2 [10 points]

2.1. Q2(1) [2 points]

Solution.

Assume a triangle has two right angles. This means that two of its angles are 90 degrees each. The sum of angles in any triangle must be 180 degrees. However, if two angles are already 90 degrees, their sum is 180 degrees, leaving no degree for a third angle. This contradicts the definition of a triangle, which must have three angles. Therefore, a triangle cannot have more than one right angle

2.2. Q2(2) [2 points]

Solution.

Assume $\sqrt{3}$ is rational. Then it can be written as p/q, in which p and q are integers with no common factor without 1. Then let's square both sides.

$$3q^2 = p^2$$

Then we can know that p is divided by 3. Then lets put p=3k(k is integer that is not 0)

$$3k^2 = q^2$$

Then we can know that q is also divided by 3. This contradicts out initial assumption. So $\sqrt{3}$ is irrational.

2.3. Q2(3) [2 points]

Solution.

Assume there are only finite prime numbers. Which are $p_1, p_2, p_3 \ldots p_n$.

Then the number $p_1p_2p_3..p_n + 1$ is not divided by any of prime numbers. So this number is prime number, which is contradicting the assumption. So, there must be infinitely many prime numbers.

2.4. Q2(4) [2 points]

Solution.

Assume both a and b are odd, square of an odd number is always 1 when modulo 4

Then $a^2 + b^2 \equiv 2\ (mod 4)$, a square of c can only be 0 when modulo 4. So a and b can't be both odd numbers.

2.5. Q2(5) [2 points]

Solution.

Assume there is no hash collision, meaning h is an one-to-one function.

Then each element in X maps unique elements in Y each. Since |X|>|Y|, there are more elements in X than in Y. This contradicts the assumption due to the pigeonhole principle. So hash collision must occur.

3. Question 3 [15 points]

3.1. Q3(1) [5 points]

Solution.

T(n)=2T(n-1)+1

T(1)=1

T(2)=3

T(3)=7

T(4)=15

T(5)=31

…

T(10)=1023

3.2. Q3(2) [5 points]

Solution.

T(n)=2*T(n-1)+1

3.3. Q3(3) [5 points]

Solution.

T(n)+1=2(T(n-1)+1)

Set T(n)+1=S(n)

S(n)=2S(n-1)

$S(n) = 2^{n-1}S(1) = 2^{n-1} * 2 = 2^n$

$T(n) = 2^n - 1$

4. Question 4 [15 points]

4.1. Q4(1) [5 points]

Solution.

$$C(n) = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!\,n!}$$

Catalan number recurrence relation.

4.2. Q4(2) [5 points]

Solution.

For n=1, using the answer in 4.1

$$C(1) = \sum_{i=0}^{0} C(i)C(0) = C(0)^2 = 1$$

$$C(1) = \frac{1}{2}\binom{2}{1} = 1$$

So, the base case holds.

4.3. Q4(3) [5 points]

Solution.

Assume the closed-form solution holds for some n smaller or equal than k.

$$C(k-i) = \frac{1}{k-i+1}\binom{2(k-i)}{k-i}, 0 \le i \le k$$

Then show C(k+1)

$$C(k+1) = \sum_{i=0}^{k} C(i)C(k-i)$$

$$= \sum_{i=0}^{k}\{\frac{1}{i+1} * \frac{1}{(k-i)+1} * \binom{2i}{i}\binom{2(k-i)}{k-i}\}$$

$$= \sum_{i=0}^{k}\{\frac{\big(2(k-i)\big)!}{i+1}\frac{(2i)!}{(k-i+1)}\frac{1}{i!\,i!}\frac{1}{(k-i)!\,(k-1)!}\}$$

$$= \sum_{i=0}^{k}\frac{\big(2(k-i)\big)!}{(i+1)!\,i!}\frac{(2i)!}{\big((k-i)+1\big)!\,(k-i)!}$$

$$= \sum_{i=0}^{k}\frac{\big(2(k-i)\big)!}{(i+1)!\,(k-i)!}\frac{(2i)!}{\big((k-i)+1\big)!\,i!}$$

$$= \sum_{i=0}^{k}\frac{\big(2(k-i)\big)!\,(2i)!}{(2k+2)!}\binom{k+1}{i+1}\binom{k+1}{i}\binom{2k+2}{k+1}$$

$$= \binom{2k+2}{k+1} \sum_{i=0}^{k} \frac{(2(k-i))!\,(2i)!}{(2k+2)!} \binom{k+1}{i+1} \binom{k+1}{i}$$

By calculation,

$$\sum_{i=0}^{k} \frac{\binom{k+1}{i+1}\binom{k+1}{i}}{\binom{2k}{2i}(2k+1)(2k+2)} = \frac{1}{k+2}$$

So $\sum_{i=0}^{k} \frac{(2(k-i))!(2i)!}{(2k+2)!} \binom{k+1}{i+1} \binom{k+1}{i} = \frac{1}{k+2}$

Then we can finally figure out that

$$C(k+1) = \frac{1}{k+2}\binom{2k+2}{k+1}$$

And our answer is correct by mathematical induction.

5. Question 5 [10 points]

Solution.

we know that

$$\theta(n!) > \theta(1.1^n) > \theta\left(n^{\frac{2}{3}}\right) > \theta(\log(logn))$$

So, for N that is sufficiently large $10\log(\log n)$ might be the most efficient.

Based on this information and information in HW1, let's compare the complexity of the four algorithms given.

Case1) $\frac{n!}{5}$ is efficient for

$n \in [2.04,3.1]$

Due to information #1 & #2

Case2) 1.1^n is efficient for

$n \in [3.1,21.4]$

Due to information #1 & #5

Case3) $10\log(\log n)$ is efficient for

$n \in [2,2.04] \cup [152.89,\infty)$

Due to information #2 & #6

And also log(log2)=0. So the smallest value is 10log(logn) when n is very close to 2, which shows 10loglogn is the most efficient graph when n is close to 2.

Case4) $n^{\frac{2}{3}}$ is efficient for

$n \in [21.4,152.89]$

Due to information #5 & #6

6. Question 6 [10 points]

6.1. Q6(1) [2.5 points]

Solution.

While loop is based on n so the time complexity is $\theta(n)$

6.2. Q6(2) [2.5 points]

Solution.

For loop based on n and m so the time complexity is $\theta(n), \theta(m)$ each.

$\theta(n+m)$ is the answer.

6.3. Q6(3) [2.5 points]

Solution.

Double for loop is based on n so the time complexity is $\theta(n^2)$.

$\theta(n^2)$ is the answer.

6.4. Q6(4) [2.5 points]

Solution.

$$\begin{aligned} T(n) &= T(n-1) + \theta(1) \\ &= T(n-2) + \theta(1) + \theta(1) \\ &= \cdots = n * \theta(1) = \theta(n) \end{aligned}$$

7. Question 7 [15 points]

7.1. Q7(1) [7.5 points]

Solution.

Yes, program B improves in time complexity over program A.

Proof)

Let's assume $n = 2^k$

Then $T_A(2^k) = 2T_A(2^{k-1}) + 4^k$

$$= 2\left(2T_A(2^{k-1}) + 4^{k-1}\right) + 4^k$$

$$= 2^2 T_A(2^{k-2}) + \left(1 + \frac{1}{2}\right) 4^k$$

$$= \cdots = 2^k T_A(1) + \left(1 + \frac{1}{2} + \cdots + \frac{1}{2}^{k-1}\right) 4^k$$

$< C * 4^k = Cn^2 \ (for\ some\ C \geq 2)$

$T_A(n) = O(n^2)$

Let's assume $n = 3^k$

Then $T_B(3^k) = 3T_B(3^{k-1}) + 3^k$

$$= 3\left(3T_B(3^{k-2}) + 3^{k-1}\right) + 3^k$$

$$= 3^2 T_B(3^{k-2}) + 2 * 3^k$$

$$= \cdots = 3^k T_B(1) + k * 3^k$$

$= 3^k(c + k) = nlogn + cn$
$< Cnlogn\ (for\ some\ C\ that\ is\ integer)$

$T_B(n) = O(nlogn)$

Q7(2) [7.5 points]

Solution.

Due to Q7.1

Let's assume

$T_A(n) = C_A n^2, T_B(n) = C_B nlogn$

$(C_A, C_B\ are\ constants)$

$ratio\ of\ time\ complexity = \frac{T_A(n)}{T_B(n)}$

$= \frac{C_A n}{C_B logn} \cong \frac{n}{logn}\ (when\ n\ is\ very\ large)$

But the speed is reciprocal of time complexity. So, X is $\frac{logn}{n}$times faster than Y

8. Question 8 [15 points]

8.1. Q8(1) [7.5 points]

Solution.

$\theta(n)$

powerN is called n times to calculate powerN(n).

8.2. Q8(2) [7.5 points]

Solution.

Let's set $n = 2^k$ then

$powerN(2^k)\ calls\ powerN(2^{k-1})\ and\ multiply\ the\ value.$

$T(powerN(2^k) = T(powerN(2^{k-1}) + O(1)$

$= T(powerN(0)) + O(k) = O(logn)$