已知六面骰子, 投若干次, 六个面出现次数分别为 $k_1, k_2, k_3, k_4, k_5, k_6$, 用最大似然估计推到每面出现的概率.

多次投骰子符合多项分布, 令 $n = \sum\limits_{i=1}^{6} k_i$, 模型的参数为 $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6)$, 其概率质量函数为:

$$P(X_1 = k_1, X_2 = k_2, \ldots, X_6 = k_6) = \frac{n!}{\prod\limits_{i=1}^{6} k_i!} \prod_{i=1}^{6} \theta_i^{k_i}$$

则最大似然估计, 有:

$$\begin{aligned}
\hat{\theta}_{MLE} &= \operatorname*{argmax}_{\theta} \frac{n!}{\prod\limits_{i=1}^{6} k_i!} \prod_{i=1}^{6} \theta_i^{k_i} \\
&= \operatorname*{argmax}_{\theta} \log \frac{n!}{\prod\limits_{i=1}^{6} k_i!} + \sum_{i=1}^{6} k_i \log \theta_i \\
&= \operatorname*{argmax}_{\theta} \sum_{i=1}^{6} k_i \log \theta_i
\end{aligned}$$

同时存在约束条件:

$$\sum_{i=1}^{6} \theta_i - 1 = 0$$

用拉格朗日乘数法求最值, 构造拉格朗日函数:

$$f(\theta_1, \theta_2, \ldots, \theta_6) = \sum_{i=1}^{6} k_i \log \theta_i + \lambda(\sum_{i=1}^{6} \theta_i - 1)$$

f 对各变量的偏导数为 0, 有:

$$\begin{cases} \frac{k_i}{\theta_i} + \lambda = 0 (1 \leq i \leq 6) \\ \sum_{i=1}^{6} \theta_i - 1 = 0 \end{cases}$$

得:

$$\begin{cases} \theta_i = -\frac{k_i}{\lambda} (1 \leq i \leq 6) \\ \lambda = -n \end{cases}$$

因此, 结果为:

$$\hat{\theta}_{MLE} = (\frac{k_1}{\sum_{i=1}^{6} k_i}, \frac{k_2}{\sum_{i=1}^{6} k_i}, \frac{k_3}{\sum_{i=1}^{6} k_i}, \frac{k_4}{\sum_{i=1}^{6} k_i}, \frac{k_5}{\sum_{i=1}^{6} k_i}, \frac{k_6}{\sum_{i=1}^{6} k_i})$$