Cointegration in Finance:
An Application to Index Tracking

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Abstract

The purpose of this paper is to construct and test two different index tracking strategies – one based upon cointegration analysis of the price processes of assets (CIT strategy), and the other based on a market equilibrium and continuous time portfolio optimisation approach (MIT strategy). Within a broad empirical analysis it is found that both tracking strategies are able to track an index (FTSE100, DJ Industrial, DJ Composite Average) accurately, even if only a relatively small subset of constituent stocks is used. Thereby, it is also suggested that (particularly in the British stock market) the CIT strategy is preferred since there is some (out of sample) evidence indicating that log-price spreads between index and CIT tracking portfolio follow a stationary process. Moreover, regarding the attempt to perform simple enhanced indexation, no empirical evidence was found that would suggest that either of the two tracking strategies was suitable for such an approach.

1 Introduction

By the 1950s with the development of the simple one period CAPM, and later on with Merton's (1971, 1973) ICAPM in a continuous time finance framework, it was suggested that the composition of the 'market portfolio' (often approximated by an index) plays a key role in setting up efficient investment strategies. Moreover, some empirical studies suggest that on average actively managed mutual funds are not able to outperform market indices after accounting for costs (Frino, Gallagher and Oetomo, 2005). Frino, Gallagher and Oetomo (2005) document that total assets benchmarked to the S&P 500 index exceed US$1 trillion, and similar numbers are recorded across other Western countries.

In view of the importance of indices as approximations of a 'market portfolio', this paper tackles the problem of tracking an index. In particular, it is of special interest to find an index tracking strategy that does not include all of the index's constituent stocks but only a small subset. This is most important if a very large index has to be tracked, meaning that it is unpractical or even impossible to manage a fund including 100, 1000 or even more assets which is a not uncommon size of indices (especially if they are meant to represent a country's overall stock market).

In the following, two approaches are stated, empirically implemented and tested. The first strategy is based on cointegration analysis and aims to find a tracking portfolio whose price process is in the long run closely tied together with the index's price process. The second strategy rests on market equilibrium and portfolio optimisation theory in continuous time finance. Hence, the first methodology tries to find long run equilibrium relations in price processes, while the latter approach considers only assets' returns and does not analyse common stochastic trends in a system of asset price processes. In addition to the index tracking problem, a short analysis of enhanced indexation as proposed by Alexander, Giblin and Weddington (2001) is researched.

For my analysis, I made use of three datasets - assets of the FTSE100, the Dow Jones Industrial and the Dow Jones Composite Average indices - all reaching over a period of more than 20 years. Such broad datasets were chosen to be able to compare the results of indices of different sizes and in diverse markets and get some idea about the robustness of the investigated strategies.

Alexander (1999), Alexander, Giblin and Weddington (2001), Alexander and Dimitriu (2002, 2004) have already attempted to analyse certain properties of named cointegration index tracking approach for the Dow Jones industrial index. However, their analysis based on a rather small dataset and unfortunately they all made use of inaccurate critical values to perform an Engle-Granger cointegration test, which actually is the fundament of whole analysis.

The paper is organised as follows. Section 2 discusses principles of cointegration analysis, test methods and applications in finance. In section 3, I introduce two different tracking strategies and list some indicators of how to measure the performance of an index tracking fund. Section 4 shows the practical implementation and my empirical results. Finally, section 5 concludes and discusses potential future research closely related to this paper. In addition, some further selected empirical results may be found in the appendix.

2 On Cointegration

2.1 Principles of Cointegration Analysis and Estimation Methods

Cointegration analysis was first introduced by Granger (1981), more extensively discussed in Engle and Granger (1987) and is nowadays covered in every standard econometrics textbook. It is an econometrics tool that allows the discussion of certain relationships between several unit root non-stationary time series processes. It gives the opportunity to detect and characterise the comovement and hence, a long run equilibrium relationship of a system of diverse processes integrated of order one or higher that share one or more common (stochastic) trend. However, not only long term interrelations may be described, but also disequilibrium situations that may appear in the short run and subsequent mean-reversion tendencies towards the system's "steady state".

There are several ways a cointegration system may be estimated and tested, among which the Engle-Granger, introduced by Engle and Granger (1987), and the Johansen methods, presented by Johansen (1988, 1991) and Johansen and Juselius (1990), are the most famous and widespread. Nevertheless, there are also suggestions on how these two major approaches may be improved and/or the introduction of alternative estimation and testing methodologies. Since this paper applies the approach of Engle and Granger (1987), the main focus lies on this method, though a short sketch of other methodologies is provided.

2.1.1 Integrated Processes and Cointegration

A time series $x = (x_t)$ is said to be integrated of order d and denoted as $x \sim I(d)$ if, after differencing d times, it turns out to be stationary, or mathematically written as $x_t(1-B)^d = z_t$, where B is the backshift or backward operator ($x_t B^k = x_{t-k}$) and z is a stationary series ($z \sim I(0)$). So, z may be characterized by an ARMA process (satisfying stationarity conditions) or using Wold's representation theorem described as $z_t = a(B)e_t$, where $a(B)$ is some linear filter and e_t is some noise series, while x, in this case, was represented as an ARIMA process. Further, Granger (1981) shows that in general a combination of integrated processes $x \sim I(d_x)$ and $y \sim I(d_y)$ results in some series $z \sim I(max[d_x, d_y])$. However, he further states that in the special case where x and y are cointegrated time series, denoted as

$x, y \sim CI(max[d_x, d_y], b)$, a certain combination of these two series results in a new series $z \sim I(d_z)$, where $d_z < max[d_x, d_y]$ or $d_z = max[d_x, d_y] - b$ and $b > 0$.

Engle and Granger (1987) listed some nice implications and properties that may result in a certain setting of cointegration. Assuming for a moment a simplified setting where x and y are both integrated of order $d = 1$ and $x, y \sim CI(d, b)$ with a cointegrating vector $\alpha = [1, -\alpha_1]^T$ and $b = d$, then the series $z_t = [x_t, y_t]\alpha$ is stationary, $z \sim I(0)$ with a mean equal to zero. Thus, for both series x and y there exists no finite variance, every innovation has a permanent effect on the series' paths, the series may travel virtually anywhere, and the theoretical autocorrelation coefficient between any lags goes to one as t goes to infinity. On the other hand, for $z_t = [x, y]\alpha$, which is stationary, there exists a finite variance, the effect of an innovations disappears as time goes by (even though there may be some persistence, z is always mean-reverting), the series is expected to cross its origin after some finite interval of time, and autocorrelation between z_t and z_{t-k} is decreasing in k (Engle and Granger, 1987).

Since the paper by Granger and Newbold (1974) and the discussion about spurious correlation, it is also well known that correlation and regression analysis using $I(1)$ processes (in general) produce misleading results and thus are not valid. As a possible solution to the problem of spurious correlation, it is suggested to use first differences of data (which are now stationary) rather than the raw data in levels. However, in the presence of cointegration in a system of unit root non-stationary variables, i.e. the number of variables exceeds the number of unit roots and thus, the time series share some common stochastic trends, differencing all variables individually to achieve stationary results leads to a loss of information. This phenomenon is known as 'overdifferencing' and leads to difficulties in the estimation of a VARMA model, which may be used to explain the system's motion (Tsay, 2005). Hence, working with level data and doing a cointegration analysis is important since it may not be possible to detect all drivers in a system when working with differenced data only (cp. Bos-saerts, 1988).

2.1.2 Estimation of Cointegration Systems using the Engle-Granger Approach

The methodology of Engle and Granger (1987) is a two step estimation approach. Firstly, a (potential) cointegrating vector is estimated using an optimization set-up as simple as ordinary least squares estimation (LSE) (and the errors may be tested further for non-stationarity using some unit root test). Secondly, an error correction model (ECM) is constructed again applying simple LSE, while the previous estimations may be used in place of the unknown, true cointegrating vector[1]. Let p be an n column vector describing a system of n variables following $I(1)$ processes. Moreover, let these n processes be cointegrated with one cointegrating vector α, i.e. there exists α such that $\alpha^T p = z$ and $z \sim I(0)$. Engle and Granger (1987) showed that α may be approximated very well by running a regression of p_i on a constant and all remaining variables, hence minimizing the variance of e of

$$p_{i,t} = c - \sum_{j \neq i} \alpha_j p_{j,t} + e_t \qquad \text{(cointegrating regression)}$$

and setting $\alpha_i = 1$. If cointegration is present in the system, then α obtained from named regression is a good approximation since it minimizes the error variance (the variance of the estimated z) and asymptotically the variance of any linear combination of p is infinite except in the case of a combination given by a cointegrating vector. At this point, one also notices that this method only allows to find one cointegrating vector even though many more may exist, and that results may differ depending on what p_i was chosen as the dependent variable in the cointegrating regression. While the estimated cointegrating vector describes long run equilibrium relationships in the system, one needs to construct an ECM in a second step to characterise short run disequilibrium situations and subsequent mean-reversion motions. The ECM then takes the form

$$\triangle p_t = \delta + \beta(B) \triangle p_t + \gamma^T z_{t-1} + \varepsilon_t \qquad \text{(error correction model, ECM)}$$

where $\triangle$ is the differencing operator ($\triangle = (1 - B)$), B is the backshift operator ($B^k x_t = x_{t-k}$), δ is a vector of constants, β is a linear filter, γ is a column vector that determines the speed of mean-reversion towards the equilibrium, ε_t is some noise, and z_t is the disequilibrium term obtained by $\alpha^T p_t$.

2.1.3 Test of Cointegration using the Engle-Granger Approach

The Engle-Granger method is pretty simple to perform. It is though only valid conditional on the existence of some cointegration relation. To test a system for cointegration is somewhat trickier than estimating a cointegrating vector or an ECM. If the true cointegrating vector was known, one could simply apply some unit root test to the residuals z ($z_t = \alpha^T p_t$). However, the cointegrating vector α must usually be estimated (via the cointegrating regression). But this estimation is only identified/ valid under the hypothesis of stationary

[1]Note that the second step regression has the same limiting distribution either if the true cointegrating vector is known or the consistent estimates obtained from the first step regression are used instead.

residuals (which means a rejection of the above stated null in a unit root test)[2]. Hence, there is a logical problem in the setup of a test for cointegration, which empirically leads to overrejections of the null in unit root tests performed on residuals of a cointegrating regression (cp. Bossaerts, 1988). Nevertheless, Engle and Granger (1987) discuss seven testing approaches using some limited simulations and conclude that an augmented Dickey-Fuller (ADF) test applied on z is most reasonable and preferred with respect to the power of the tests (let's call it the EG-ADF test). The provided simulation results in Engle and Granger (1987) and the more extended ones in Engle and Yoo (1987) suggest that the critical values have to be higher than the values obtained when simulating the limiting distribution of a DF test stated by Dickey and Fuller (1979). But in both papers, neither a limiting distribution is derived nor are any accurate critical values provided. Finally, MacKinnon (1990, 1994, 1996) was first to present accurate simulation results based on response surface regression estimation to obtain critical values for the EG-ADF test solving the (above) addressed problem of overrejection[3].

A further, more extensive discussion in the literature is devoted to the issue of the applicability of unit root tests in the presence of conditional heteroskedasticity, and recently, jump processes, which is particularly important with regard to cointegration analysis with financial data. There is a lot of research in the field of modelling first and especially second order moments of asset returns. This starts with the empirical studies of Mandelbrot (1963) and Fama (1965) who both observed 'fat-tailed' distributions of asset returns and temporal dependencies of variances of asset returns, and the ARCH resp. GARCH models first introduced by Engle (1982) and Bollerslev (1986). Moreover, there is (initiated by latter two papers) a vast literature especially on various (G)ARCH models. But before one has to worry about heteroskedasticity, it may be worth to discuss whether the disequilibrium term z is heteroskedastic at all. Alexander (2001) discusses the property of common features of time series data and mentions that there are sometimes common volatility patterns found in financial markets (in particular equity markets). And yet, in general, it still appears pretty unlikely that a cointegrating vector would coincide with a linear combination that 'neutralises' the common feature of time varying and temporally persistent volatilities since in general baskets of assets show properties of heteroskedasticity (e.g. any stock index). Nevertheless, this seems rather likely in a setup of index tracking since one actually tries to replicate an index using some combination of its constituents. Indeed, some standard tests

[2] Cf. discussion about spurious regression (Granger and Newbold, 1974).

[3] For my own response surface regression estimates, see the appendix.

of empirical results suggest that if a cointegrating vector for an index and its constituents is estimated, the error of the cointegrating relation reveals no volatility clustering[4].

However, faced with problems of volatility clustering and serial correlation, some alternatives to the conventional Dickey-Fuller τ test (DF or τ test) have been introduced and discussed as well as many studies based on simulation performed to test the accuracy of these tests (mostly) compared to the DF test. An immediate improvement on the DF test is the augmented DF (ADF) test introduced by Said and Dickey (1984) and mentioned within a simpler example in Dickey and Fuller (1979, 1981). It is shown that by adding already some few autoregression terms/ lags to the LS regression setup (used to construct the τ test statistic), the new ADF test is more accurate in the presence of serial correlation and is able to approximate sophisticated ARIMA models pretty well (Said and Dickey, 1984). Moreover, the limiting distribution remains the same as for the earlier DF test. Dickey and Fuller (1981) further present a likelihood ratio test that appears to be more flexible than the earlier τ test since it not only may be used to test for unit roots, but different nested model specifications may be compared as well. Phillips (1987) and Phillips and Perron (1988) introduce another modification of the DF test. It is shown that even for very general and sophisticated time series models, it is not necessary to specify the model exactly, but a simple LSE is sufficient to consistently estimate or test for a unit root. Moreover, a test statistic for a general case (e.g. serial correlated and heterogeneously distributed innovations) is proposed and its limiting distribution derived, as well as the simple relation to the DF statistic being shown. Perron and Ng (1996) further suggest some modifications of the test statistic, which leads to a much more accurate size in the case where the coefficient of the MA component in the error term process is going towards -1 (strong negative serial correlation). Breitung (2002) presents a nonparametric test (a generalization of a variance ratio test) that is closely related to Bierens' (1997) test and is apparently an improvement regarding power. It needs no specifications of any short run dynamics and is robust against misspecifications and structural breaks. Simulations show that it is preferred in the setting of a nonlinear data generating process where an ADF experiences some size distortions, but it is not able to outperform the ADF test otherwise. Finally, Luger (2003) introduces a nonparametric test that allows (amongst other things) for an unknown drift and heteroskedastic innovations and only assumes that the error term is symmetrically distributed. It is based on an analysis of the signs of adjusted changes, i.e. changes relative to some reference change

[4] Engle's ARCH LM test and/or Ljung-Box test; see empirical part of this paper.

estimate from the sample. A more extensive survey of unit root tests is also provided by Phillips and Xiao (1998)[5].

Quite a few empirical papers were published with special regard to the performance of unit root tests applied on data series generated by an AR(I)MA-GARCH model. In this literature, there is general agreement that DF tests are asymptotically robust to heteroskedasticity and there is only a finite sample problem. In general, this also applies to GARCH(p,q) models as long as they are not integrated and not degenerated (Kim and Schmidt, 1993). Kim and Schmidt (1993) further analysed in more detail the accuracy of the DF test if the error variance follows a GARCH(1,1) process. They show that it is not near-integration of a GARCH model, but a large 'alpha' (volatility factor) and degenerateness (constant equal zero) and near-degenerateness that are responsible for size distortions. In addition, it is illustrated that size distortions caused by a huge 'alpha' may be much reduced if White's (1980) heteroskedasticity consistent covariance matrix is applied to estimate the τ statistic. Boswijk (2000) analysing the DF test suggests that a likelihood ratio test is preferred to a τ test if 'alpha' gets more weight and the GARCH model is near integrated. Rodrigues and Rubia (2004) suggest further that in the case of a big 'alpha', Breitung's nonparametric test must be preferred to any DF test. However, given some (in empirical finance) reasonable parameter choice, there is no dominance over the DF τ tests (Boswijk, 2000). Moreover, Li, Ling and McAleer (2002, 2003) provide similar conclusions as Boswijk (2000). However, the simulation results in Li, Ling and McAleer (2003) also suggest that for some common parameters in financial econometrics (low 'alpha', big 'beta') in a GARCH(1,1) model, the DF τ test has more power than the equivalent likelihood ratio test. Finally, Chua and Suardi (2006) find that Breitung's nonparametric test performs best in presence of a jump process and GARCH errors, but a DF test does not seem to suffer much from the inclusion of a jump process either. Finally, one may conclude that for an application in finance Breitung's test is the most preferred, but for example an ADF τ test may also serve as long as a large enough sample is provided.

Some further discussion about the presence of heteroskedasticity and the accuracy of the estimation of a cointegrating vector (first step regression in Engle-Granger approach) is done by Hansen (1992). He concludes that even if the volatility process is integrated a cointegrating vector (but not the regression's intercept which is of no economic interest) may be estimated consistently by a simple LSE.

[5] However, it does not contain some of the newer tests as the one by Breitung (2002) or Luger (2003).

Finally, Gregory, Nason and Watt (1996) and Gregory and Hansen (1996) analyse Engle-Granger cointegration tests in the presence of structural breaks in the parameters of cointegrating vectors and find that if the true model is indeed cointegrated with a one-time regime shift in the cointegrating vector, the ADF test may falsely not reject the null of no cointegration. Hence, the power of the test falls sharply in the presence of a structural break. Thus, it is suggested to include a level shift dummy variable indicating a regime switch. However, it is also shown that if it is not known whether there is a structural break but a dummy is included anyway, test statistics no longer have asymptotic standard distributions.

Hence, as long as no structural breaks are assumed, the application of an EG-ADF test is very reasonable with financial datasets (which are usually pretty large) and especially in the case of cointegration index tracking since even the feature of heteroskedasticity disappears.

2.1.4 Johansen Approach and other Methods for Estimation of Cointegration Systems and Test of Cointegration

In contrast to the two step Engle-Granger method, the approach presented by Johansen (1988, 1991) directly ties in with an ECM respectively a VAR(q) model. Let pt again be the system of processes. Then following Johansen and Juselius (1990) one has to start with a general VAR(q) model written as

$$p_t = \sum_{i=1}^{k} \Pi_i p_{t-i} + \Pi p_{t-k} + \mu + \Phi D_t + \varepsilon_t$$

where $[\varepsilon_1, ..., \varepsilon_T] \sim IIN\,(0, \Lambda)$, μ is a vector of constants and D_t are some seasonal dummies.

Then, rewriting it in a form that appears to be better for interpretations and tests one gets the model

$$\triangle p_t = \sum_{i=1}^{k-1} \Gamma_i \triangle p_{t-i} + \Pi p_{t-k} + \mu + \Phi D_t + \varepsilon_t$$

with $\Gamma_i = \sum_{j=1}^{i} \Pi_j - I$ and $\Pi = \sum_{i=1}^{k} \Pi_i - I$, (Johansen and Juselius, 1990).

This estimation is proposed in two steps: the VAR(q) part of the model is estimated using an LSE conditional on Π being known, and Π is estimated by MLE given the LSE results. This now allows to perform tests about Π. Johansen and Juselius (1990) state three cases:

1) $rank(\Pi) = n$, all processes in the system are in fact stationary,

2) $rank(\Pi) = 0$, there is no cointegration and in fact $\Pi = 0$, i.e. the model reduces to a conventional VAR(q) model

3) $0 < rank(\Pi) = r < n$, there are r cointegrating vectors in the system[6].

The test proposed in Johansen (1988) tests the null of $rank(\Pi) \leq R$ against the alternative hypothesis of $rank(\Pi) > R$ using a likelihood ratio test (since Π is estimated by MLE). This leads then to a 'trace statistic' of the form

$$Tr = -T \sum_{i=1+R}^{n} \ln\left(1 - \widehat{\lambda}_i\right) \qquad \text{(trace statistic)}$$

where $\widehat{\lambda}_i$, $i = \{1, \ldots, n\}$ are the estimated eigenvalues of Π sorted from the largest to the smallest (Johansen, 1988, 1991). Critical values have to be simulated according to the limiting distribution derived in Johansen (1988, 1991), or alternatively an χ^2 distribution may be used as an approximation. Certainly, the big advantage over the Engle-Granger approach is that it is now possible to estimate all cointegrating vectors in the system at once.

Lucas (1997a) suggests a straightforward enhancement of the Johansen test by applying a PMLE (pseudo maximum likelihood estimation) to estimate Π instead of a simple MLE as originally proposed. He argues that a PMLE is less sensitive and more robust against outliers and non-Gaussian distributions of error terms. Therefore, a modified test statistic and new limiting distributions are derived. Lucas (1997a) shows that if residuals are for example student's t distributed, the test based on PMLE performs better then Johansen's original method. This is particularly interesting if one remembers Nelson's (1990) results that innovations following a specific GARCH process are unconditionally student's t distributed.

A further approach is introduced by Bossaerts (1988), whose test is based upon canonical correlation analysis to detect linear combinations of p_t and p_{t-1} that are maximally correlated and checks later on for cointegration respectively identifies the actual cointegrating vectors. Stock and Watson (1988) suggest two further methodologies. The first method ties in with the assumption that the processes may be represented through a VAR(q) model and parameters are estimated and tested within this model specification. The second technique is based on an analysis of the eigenvalues of a corrected autocovariance matrix. Finally, Stock and Watson (1988) show that both tests are asymptotically similar.

[6] Where n is equal to the number of components in the system p_t and Π may be decomposed as $\Pi = \alpha\beta^T$, where β is a $p \times r$ matrix of r cointegrating vectors and α is some $r \times p$ weighting matrix.

2.2 Cointegration in Finance

Price processes of financial assets are usually characterised through random walk models (or at least assumed to be $I(1)$). Thus, it is not surprising that applications of cointegration analysis may be found in miscellaneous fields in finance. Alexander (1999, 2001) offers a comprehensive overview of recent academic literature.

Bossaerts (1988) states that cointegration does not imply any market inefficiency and later on Dwyer and Wallace (1992) show, based on theoretical arguments, that the presence of cointegration in (some) financial markets is compatible with market efficiency (one neither implies nor excepts the other). Though, Dwyer and Wallace (1992) defined market efficiency as the absence of arbitrage in the sense that "there are no risk-free returns above opportunity cost available to agents given transaction costs and agents' information". Moreover, Alexander and Dimitriu (2005) discuss the use of cointegration methods for enhanced indexation and conclude that even though some abnormal returns may be found, this does not contradict market efficiency. This is justified by the argument that abnormal returns only may be observed in one out of two states in a regime switching world and thus, may be seen as a kind of timing risk premia (Alexander and Dimitriu, 2005).

Some research covers the field of cointegration between exchange rates and it is generally agreed on that it is implausible that two markets are cointegrated (Alexander, 2001). In addition, Alexander (2001) mentions that there is empirical evidence suggesting that three or more exchange rates are cointegrated.

A very strong cointegration relation is expected (cp. arbitrage pricing theory) and empirically confirmed in a system of futures and their underlying asset. However, Alexander (2001) also mentions that arbitrage opportunities seem to be impossible to find in such relations. Furthermore, Duan and Pliska (1999) discuss whether the presence of cointegration in financial markets affects the pricing of spread options and conclude that it does not as long as volatilities are constant, but it does have a considerable impact if volatilities are modelled stochastically.

Often certain commodity prices are expected to be cointegrated following a rational argument of mean-reverting carry costs (Alexander, 1999). Foster, Havenner and Walburger (1995) indeed construct and estimate a trend state space model for seven markets of weekly live cattle prices and find some evidence for cointegration in a system of the observed markets. Furthermore, they develop a trading strategy based on the found cointegration relation which equals a new asset with a stationary price process that may be exploited very easily to achieve almost risk free profits (Foster, Havenner and Walburger, 1995).

Term structures are often considered in cointegration analysis. Interest rates of different maturities are expected to be tied together strongly by arguments of no arbitrage and indeed the theoretical argumentation is supported by quite some empirical evidence (Alexander, 2001).

Finally, stock markets are the subject of much research done in the field of cointegration. Alexander (1999) argues that for example "market indices in different countries should be cointegrated if purchasing power parity holds." Some studies show such cointegrating relations in international markets and it is suggested that the US market is apparently something of an international leader (Alexander, 2001). Further, Cerchi and Havenner (1988) construct and estimate a state space trend and cycle model for a basket of (only) five randomly chosen US (retail) stocks and indeed find that these assets share one common stochastic trend and that there are three cyclical states describing the system. Moreover, they construct (successfully) some simple trading strategies that enable the exploitation of mean-reversion tendencies in the system. Lucas (1997b) presents a very good analysis of continuous time portfolio optimisation given a world with cointegrated asset prices. He provides an analytical derivation by solving a Bellman function and provides a closed-form solution of an optimal investment strategy which appears to differ significantly depending on the degree of cointegration (number of cointegrating vectors). In addition, he provides simulation results indicating that it seems to be better to overestimate the degree of cointegration for short term investments, but the opposite appears to be true for a long run investment horizon.

3 Index Tracking and Enhanced Indexation

3.1 Cointegration Index Tracking

It appears natural to expect that an index, which actually is nothing else than a basket of stocks, may in some way be tied together with a certain portfolio consisting of its constituent stocks. Hence, one may try to find some cointegrating relation of a system p of an index and (some of) its constituents[7]. Thus, if a cointegrating vector α may be found such that $\alpha^T p = z$, where $z \sim I(0)$, then $-\alpha_{\{i \neq 1\}}$ is a tracking portfolio of the index. In this case, the cointegrating relation ensures that the price spread between tracking portfolio and index

[7]The first element of pt represents the log-price process of the index; the n following components are log-price processes of n constituent stocks of the index.

does not diverge, but always reverts to zero, which makes such a tracking strategy attractive since alternative methods usually do not guarantee this stability.

Alexander (1999) argues for the use of the Engle-Granger method to find a cointegrating vector α. The reasons for this choice are that the EG approach identifies the one α that produces a z with minimum variance, which is certainly important when tracking an index, and it is very simple to implement. Besides, one does not have to care about any further cointegrating vectors that may exist (only one is needed), the choice of the index as the dependent variable in the cointegrating regression also seems pretty obvious (no other choice of a de-pendent variable need be considered), and usually large samples are available in financial analysis such that one does not have to worry about small sample problems.

Thus, if there is a cointegrating relation, one may estimate a tracking portfolio strategy using a simple linear regression of the log-price of the index on log-prices of n selected stocks denoted by,

(cointegration index tracking regression)

$$p_{index,t} = c + \sum_{i=1}^{n} w_i p_{i,t} + \varepsilon_t$$
$$s.t. \quad \sum_{i=1}^{n} w_i = 1$$

where c is some constant, $p_{j,t}$ is the log-price of asset j at time t, w_i is the approximate weight of asset i in the tracking portfolio, and ε_t is the stationary spread between the index and the tracking strategy. Note, that this regression is only valid given there is a cointegrating relation since otherwise it is a case of spurious regression (cf. Granger and Newbold, 1974). Moreover, it is obvious that by using logarithmic prices the first difference leads to returns, but it is not mathematically correct to build a portfolio with these returns. Nevertheless, it may be considered a good approximation.

In addition, if the system p was cointegrated of a higher degree than one, say there were now $r+1$ common random walk components (with $r > 0$), then one could reduce the number of assets needed to track the index to $n = N - r$ stocks with at least one 'non-redundant stochastic trend'[8,9].

[8] N equals the number of stocks included in the index.

[9] With 'redundant stochastic trends', I would like to express that all the asset price process' stochastic trends may be replicated through a portfolio of other assets, hence, there may still be some stationary noise that can't be captured. This is in constrast to redundancy in a sense of usual no arbitrage arguments, i.e. there is a self-financing strategy that costs and pays off the same as the considered asset whatsoever/ almost

Alexander and Dimitriu (2002) have done empirical analysis in this direction and suggest that the Dow Jones Industrial index may be tracked using only 20 out of 30 stocks. But it has to be noted that this conclusions base on Engle-Granger cointegration tests using inaccurate critical values, i.e. some conclusions are not exactly true. Alexander and Dimitriu (2002) argue that there may be different approaches to choosing the universe of $N - r$ assets reaching from "proprietary selection models, technical analysis [to] just stock picking skills of a portfolio manager", and this choice does not matter for the later cointegration analysis at all. This is however not quite right. The choice of the $N - r$ stocks is only free in so far as it has to be satisfied that there is still a cointegrating relation (all the stochastic trends of the index must be replicable by the basket of assets). In other words, there are exactly m different possible asset selections ($m > 0$). Which of the m possible universes of assets is finally picked may be decided by using technical analysis, the stock picking skills of a manager or asking a monkey (in other words randomly).

Another practical approach discussed by Alexander (1999), Alexander, Giblin and Weddington (2001) and Alexander and Dimitriu (2002, 2004, 2005) deals with enhanced indexation. This approach attempts to track an enhanced index, i.e. an existing index plus some risk free return (usually equally distributed over time). In contrast to the above described index tracking, this approach has unfortunately no reasonable model and no strong underlying theory and appears somewhat artificial. Unfortunately, all named authors come to some questionable conculsions because of their use of inaccurate critical values for their EG cointegration tests.

A further application, but building on weak theoretical foundations as well, was a cointegration index tracking strategy with respect to the restriction that some fixed portfolio weight had to be held in cash. The motivation of such a constraint was to face a typical problem in portfolio management of minimising fixed and flexible transaction costs in a setting where there are stochastic inflows and outflows of wealth available for investment in the tracking fund (cf. problem discussed by Buckley and Korn, 1998). To date, there has not been any attempt in the academic literature to approach Buckley and Korn's (1998) problem by applying cointegration analysis.

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surely.

3.2 Index Tracking using a Continuous Time Portfolio Optimisation Approach

Here, I would like to present an alternative method that does not consider cointegration analysis, but is based on an asset-liability model used in continuous time portfolio optimisation (cf. Müller, 2000 and Denzler, Müller and Scherer, 2001). It uses some simplifying assumptions similar to the ones found in a Black-Scholes world (for details cf. e.g. Merton, 1990). The model builds on the result that the portfolio choice problem in continuous time can be reduced to a static optimization problem under the assumption that assets are driven by a geometric Brownian motion and investors have a constant relative risk aversion profile. I will proceed by maximizing expected power utility over terminal wealth[10].

Let there be a probability space $(\Omega, \mathcal{F}, P)$[11]. There are N risky assets ($i = 1, 2, \ldots, N$) and one riskless security ($i = 0$). Their price processes S_i are described as,

$$\begin{aligned} \frac{dS_{0,t}}{S_{0,t}} &= rdt \\ \left[\frac{dS_{i,t}}{S_{i,t}} \right]_{i=1,\ldots,N} &= \mu dt + \sigma dW_t \end{aligned}$$

where r is the riskless return, μ is a column vector of N drift terms, W is a column vector of d independent Wiener processes, and $\sigma\sigma^T$ denotes the risky asset returns' $N \times N$ covariance matrix V. Moreover, there is a liability L with a process,

$$\frac{dL_t}{L_t} = \mu_L dt + \sigma_L \begin{bmatrix} dW_t \\ dW_{L,t} \end{bmatrix}$$

where μ_L is a drift, W_L is a Brownian motion independent of W, and $\sigma_L \sigma_L^T$ denotes the variance of the liability's return process. Then, the investment strategy $\widehat{x}_t = (x_{0,t}, x_t)^T$, with $\sum_{i=0}^{N+1} \widehat{x}_{i,t} = 1$ generates the wealth process Y_t,[12]

$$\frac{dY_t}{Y_t} = \left(r + \pi^T x_t \right) dt + x_t^T \sigma dW_t$$

with $\pi = \mu - r1_{(N\times1)}$.

[10] Note that the same result can be achieved by maximizing a more general class of time additive utility functions defined over a stream of intermediate consumption. However, for the sake of this paper it is sufficient to look at the restricted case of maximizing CRRA utility over terminal wealth.

[11] Some space Ω, with a Brownian filtration $\mathcal{F}_t = \sigma\left(W_s : 0 \leq s \leq t\right)$, and some probability measure P.

[12] Note that the riskless portfolio weight is $x_{0,t}$ and the risky weights is x_t. Moreover, x_t is adapted to the Brownian filtration F generated by the above introduced $d + 1$ dimensional Brownian motion.

Further, a fund of the named portfolio and the earlier mentioned liability is defined as,

$$F_t = \frac{Y_t}{L_t}$$

Applying Itô's Lemma to F and dividing by F,

$$\frac{dF_t}{F_t} = \frac{dY_t}{Y_t} - \frac{dL_t}{L_t} - \frac{dY_t}{Y_t}\frac{dL_t}{L_t} + \left(\frac{dL_t}{L_t}\right)^2$$

leads to the process,

$$\frac{dF_t}{F_t} = \left(r + \pi^T x_t - \mu_L - \gamma^T x_t + \sigma_L \sigma_L^T\right) dt + x_t^T \sigma dW_t - \sigma_L \begin{bmatrix} dW_t \\ dW_{L,t} \end{bmatrix}$$

where $\gamma = \left([\sigma_L]_{i=1,\ldots,d} \cdot \sigma^T\right)^T$ is an N column vector describing the covariances of the liability with the asset universe[13]. Moreover, the solution of the above stochastic differential equation is,

$$F_t = F_0 \exp\left\{\left(\mu_{AL,x} - \frac{\sigma_{AL,x}}{2}\right) T + \left(\begin{pmatrix} x^T\sigma \\ 0 \end{pmatrix}^T - \sigma_L\right)\begin{bmatrix} W_t - W_0 \\ W_{L,t} - W_{L,0} \end{bmatrix}\right\}$$

with

$$\begin{aligned}
\mu_{AL,x} &= r + \pi^T x_t - \mu_L - \gamma^T x_t + \sigma_L \sigma_L^T \\
\sigma_{AL,x} &= \left(\begin{pmatrix} x^T\sigma \\ 0 \end{pmatrix}^T - \sigma_L\right)\left(\begin{pmatrix} x^T\sigma \\ 0 \end{pmatrix}^T - \sigma_L\right)^T \\
&= x^T \sigma \sigma^T x - 2\gamma^T x + \sigma_L \sigma_L^T
\end{aligned}$$

.

Suppose now that the investor's utility function $u(F_t)$ displays constant relative risk aversion, i.e.

$$u(F_t) = \begin{matrix} \frac{F_t^{1-c}}{1-c}, & for \;\; c \neq 1 \\ \ln(F_t), & for \;\; c = 1 \end{matrix}$$

An investor then chooses a portfolio that maximizes his expected utility $u(F_t)$,

$$\max_{x_s \in \mathbb{R}^N,\; s\in[0,t]} \left\{E\left[u(F_t)\right]\right\}$$

[13] Note that $[\sigma_L]_{i=1,\ldots,d}$ is the row vector of the first d elements of σ_L.

It turns out, either way, for any $c \in \mathbb{R}^1_+$, the dynamic maximization problem is equivalent to the following static problem for $s \in [0, t]$,

$$\max_{x_s \in \mathbb{R}^N} \left\{ \pi^T x_s - \frac{c}{2} x_s^T \sigma \sigma^T x_s + (c-1) \gamma^T x_s \right\}$$

$$s.t. \quad Ax_s \geq b$$

$$Dx_s = e$$

The proof is straightforward and can be found in Müller (2000) and Denzler, Müller and Scherer (2001).

A straightforward solution to this maximisation problem in an unconstraint case is given as,

$$x^* = \frac{1}{c} \left(\sigma \sigma^T \right)^{-1} \pi + \left(1 - \frac{1}{c} \right) \left(\sigma \sigma^T \right)^{-1} \gamma$$

It is noteworthy that optimal portfolio weights remain constant over time and are independent of the chosen investment horizon. Moreover, an equilibrium situation implies now risk premia the risky assets should pay in a rational market. This is denoted as,

$$\pi^* = c^M \left(\sigma \sigma^T \right) x^M$$

where c^M is the risk aversion coefficient of the representative investor (market risk aversion), and x^M is the market portfolio. Note that the market portfolio includes no liability, what appears economically reasonabe.

Now, since the aim of an index tracking strategy is to perform as closely as possible to the index, one may simply apply the above derived optimisation problem to find an optimal investment strategy, while selecting the index as a liability and choosing an artificial risk aversion coefficient approaching infinity. This leads to an optimal tracking strategy since on one hand an investor with an infinite risk aversion coefficient tries to reduce respectively hedge risk as much as possible, no matter what it costs. On the other hand, the expected return of the liability/ index matches the expected return of the optimal portfolio perfectly, provided markets are in equilibrium and the liability is traded in the market. The optimal portfolio and its expected excess return are,

$$x^* = \lim_{c \to \infty} \left(\frac{1}{c} \left(\sigma \sigma^T \right)^{-1} \pi + \left(1 - \frac{1}{c} \right) \left(\sigma \sigma^T \right)^{-1} \gamma \right) = \left(\sigma \sigma^T \right)^{-1} \gamma$$

$$\pi_{x^*} = \pi^{*T} x^* = \pi^* \left(\sigma \sigma^T \right)^{-1} \gamma$$

From market equilibrium, it is known that every (traded) asset is compensated according to its risk in the market. Hence, the following equilibrium condition is true,

$$\left(\left(\sigma\sigma^T\right)x^M\right)^{-1}\pi^* = c^M = \left(\gamma^T x^M\right)^{-1}\pi_L^*$$

Rewriting and applying some simple calculus yields,

$$\begin{aligned}
\pi_L^* &= \gamma^T x^M \left(\left(\sigma\sigma^T\right)x^M\right)^{-1}\pi^* \\
&= \gamma^T \left(\sigma\sigma^T\right)^{-1}\left(\sigma\sigma^T\right)x^M\left(\left(\sigma\sigma^T\right)x^M\right)^{-1}\pi^* \\
&= \gamma^T \left(\sigma\sigma^T\right)^{-1}\pi^* \\
&= x^{*T}\pi^* = \pi_{x^*}
\end{aligned}$$

what proves the statement made earlier.

Provided one is unlimited in the use of all assets available in the market to track an (in the market) traded index, the optimal tracking strategy is given by,

$$x^* = \left(\sigma\sigma^T\right)^{-1}\gamma$$

what is approximately equal to Roll's (1992) TEV (tracking error variance) minimisation model,

$$x^* = \left(\sigma\sigma^T\right)^{-1}\gamma \approx \left(X^T X\right)^{-1}X^T y$$

where X is a matrix of (demeaned) returns of the universe of assets (each column one asset's time series) and y is a vector of (demeaned) returns of the index. Hence, the tracking strategy is given by the coefficients of the regression of the index returns on the returns of the asset universe.

However, the TEV model is limited to the application of tracking an index under the two discussed conditions[14], while the approach presented here based on continuous time portfolio optimisation also allows to track an index using only a subset of the stocks' universe or is able to track an enhanced index. It only has to be assured that the tracking strategy's expected return matches the benchmark, what may be done by setting an appropriate restriction in the maximisation problem. The best possible replication of the index's diffusion is afterwards achieved in the (constraint) optimisation process while choosing a risk aversion coefficient approaching infinity.

[14] The index is traded in the market and one may use the complete universe of assets to build up a tracking strategy.

3.3 Performance Measurement in an Index Tracking Problem

Unfortunately, there is not only one criterion that measures a tracking strategy's adequacy. One notes that measuring the suitability by doing an analysis based on utility is not straightforward because it is not obvious with what pay-off to tie in when defining the utility function. Instead, criteria include differential return/ tracking error (difference of returns between tracking portfolio and index), tracking error variance (variance of differential returns), correlation of returns, information ratio (ratio between mean tracking error and its standard deviation), turnover (sum of changes of portfolio weights; relevant to estimate transaction costs), and long term comovement (stationarity test of process of level differences between tracking strategy and index) (cf. e.g. Aleander, 1999, 2001 or Alexander and Dimitriu, 2002, 2004).

It is obvious that not every investor ranks the importance of these criteria in the same way. Compare for example two investors, one with a very long and the other with a very short term investment horizon. For the very long horizon it is especially important that the two price processes are tied together (price spreads are mean-reverting), even though there are big differential returns and a low correlation. For the short run view the opposite is true. Differential returns should be as low as possible and the correlation close to one, while the long term relation is not of interest at all.

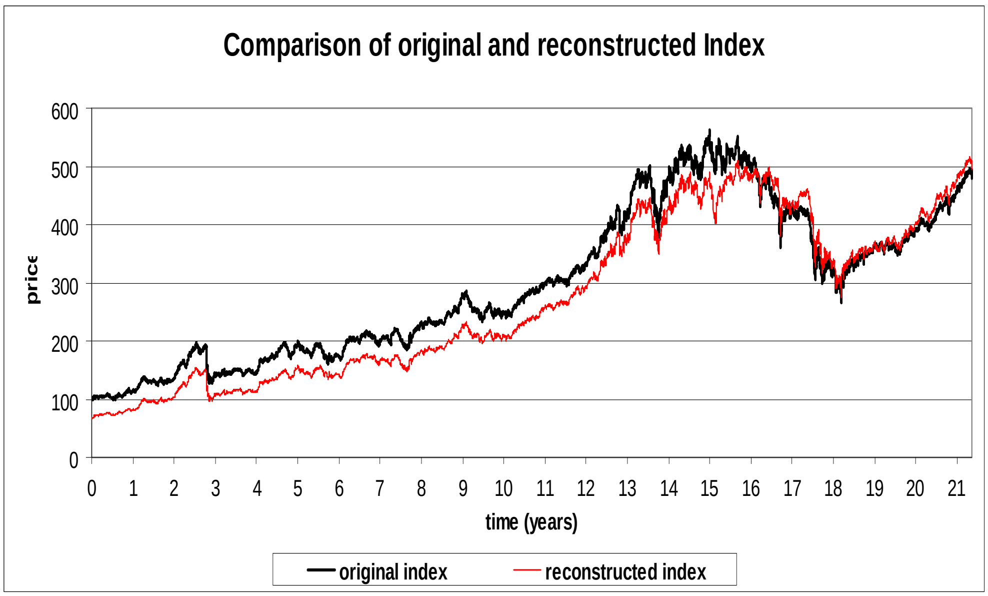
One may guess that the approach based on cointegration rather conforms to the requirements of a long term investor since it is based on long term comovements. In contrast, the latter method is somewhat expected to better meet the demands of a short run investor.

4 Practical Implementation of Index Tracking Strategies

4.1 Data

For the empirical analysis, three datasets were used: the Dow Jones Industrial index (DJI), the Dow Jones Composite Average index (DJCA), and the FTSE100 index. For the DJI, I downloaded the daily price history reaching from 29.10.1986 to 12.03.2008 of the index and its 30 most recent constituents. The DJCA dataset consists of daily prices of the index and 58 of its most recent components over the period from 13.03.1986 to 12.03.2008. The FTSE100 dataset contains daily prices of the index and 51 of its most recent constituents for the period from 01.01.1985 to 12.03.2008[15]. All data was downloaded using DataStream[16]. The choice of different datasets allows checking for the robustness of the tested methods[17].

Next, I reconstructed the indices based on the stocks in my prepared data sets. The reconstructed indices have then been used in the further analysis. Since I use only a selected subset of the indices' constituents, this reconstruction allows the overcoming of problems such as 'survivor bias', special dividend policies or other difficulties caused by the particular design of an index[18].



Comparison of original and reconstructed FTSE100 indices.

4.2 Implementation and Empirical Results

Using all three datasets, both index tracking strategies were constructed and tested. In the following section, I discuss in detail the performance of the two introduced index tracking strategies applied to the dataset of the FTSE100, while I mention some results about the US stocks and some additional figures that may be found in the appendix.

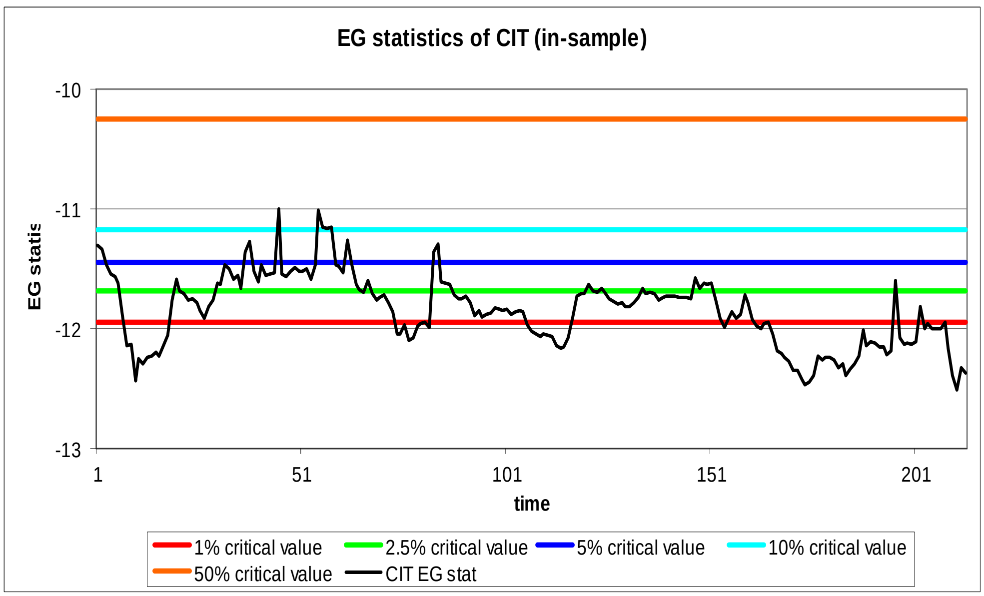
[15]The significant reduction of stocks used (then the number of actual constituents) is caused by the lack of data provided.

[16]http://www.datastream.com.

[17]Firstly, the DJ was chosen twice to check the robustness against the size of an index, and secondly, the FTSE was analysed in addition to check for robustness against different markets.

[18]Cf. Frino, Gallagher, Neubert and Oetomo (2003) and Frino, Gallagher and Oetomo (2005) for a treatment about named issues.

For the cointegration index tracking method, I performed unit root tests for all (log-)price processes to check whether they are $I(1)$ and cointegration analysis may be considered at all. Applying an ADF test, in the case of the log-price data series the null of a unit root can never be rejected on a 1% significance level (no rejection of $I(d)$, $d > 0$), and in the case of the first differences of the same series, the same null is always rejected with a much higher confidence than 99.99% (clear rejection of $I(b)$, $b > 1$). Hence, these two tests suggest that the given data series are all $I(1)$. Then, I constructed rolling windows for calibrations and measured the tracking strategy's performance with subsequent 'out of sample' data. I ran an Engle-Granger cointegration test for every rolling window (calibration period), in the sense of performing an ADF test on the residuals of the EG cointegrating regression.



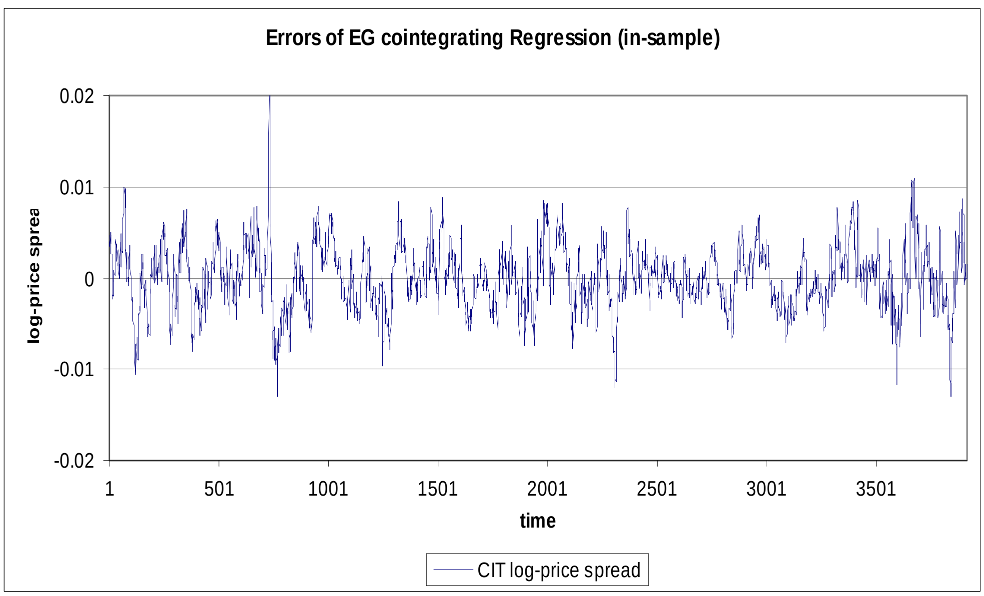
Engle-Granger statistics of (in-sample) cointegration tests (case of FTSE100 index).

For the decision about the number of autoregression terms included in the 'ADF test model', I made use of the Akaike (Akaike and Hirotugu, 1974), Baysian (Schwarz, 1978), and Hannan-Quinn information criteria (Hannan and Quinn, 1979) to test for a model of up to 10 lags[19]. Moreover, since there are no useful critical values tabulated for a test, as in my analysis, I estimated appropriate critical values using response surface regressions (cf. section 3 in the appendix)[20]. An optimal length for the rolling windows was chosen based on

[19] For a treatment about model selection cf. Bierens (2006).

[20] Compare also more detailed Monte Carlo simulation setups presented by MacKinnon, 1990, 1994, 1996. In the literature (MacKinnon), critical values only for systems up to 12 integrated processes are tabulated. Hence, I estimated some critical values for bigger systems as needed within this paper.

a comparison of different window lengths and their corresponding EG cointegration tests[21]. It seems reasonable that the calibration period should not be too short to ensure an accurate estimate and not too long, since the weights of the constituents in the index may change significantly over time. I found that for the DJI around 9 years, for the DJCA 13 years and for the FTSE100 up to 15 years calibration seem optimal. However, in the displayed graph for the FTSE100, rejection of the null of no cointegration appears to be pretty clear (even if not with a very high confidence). In the case of the US markets, it is not always obvious whether there is a cointegrating relation[22]. But it is also known that the power of the test used here may be not as great if the errors of the cointegrating regression are near-integrated, i.e. the mean-reversion of the process is weak. Therefore, I analysed the residuals of the cointegrating regression as well as the mean-reversion coefficient of their process[23].



Residuals of Engle-Granger cointegrating regression (case of FTSE100 index).

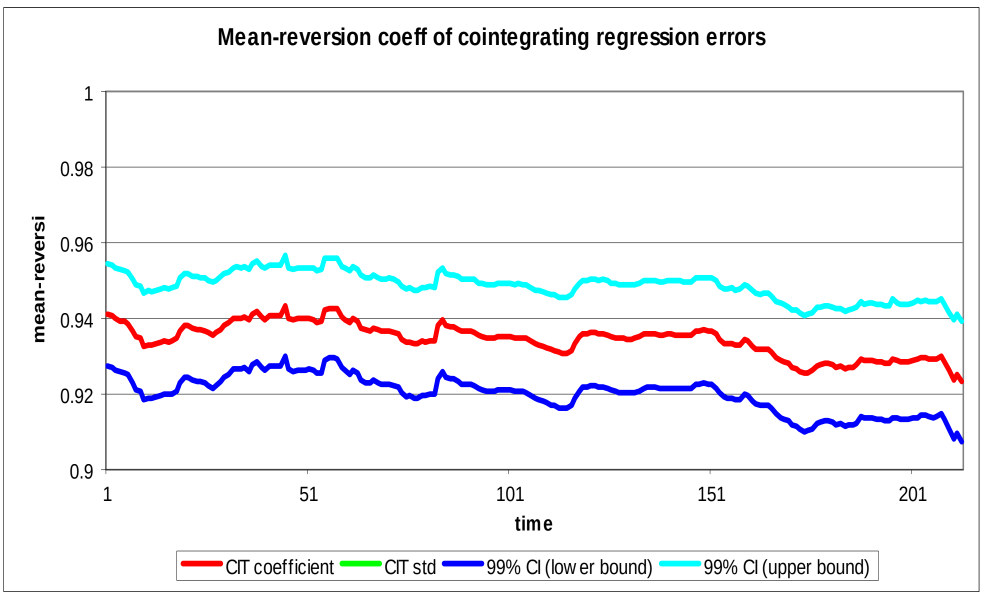
From a visual test, I conclude (in the British and the US market) that the residuals seem to follow a stationary process. Here, it is also noteworthy that the residuals do not feature any volatility clustering, i.e. Engle's ARCH LM test and the Ljung-Box test both show no

[21] EG cointegration tests based on window lengths of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 years were compared.

[22] Only in some periods one rejects the null of no cointegration.

[23] Note that these residuals and the subsequent mean-reversion coefficients are only valid provided there is a cointegrating relation.

rejection of the null hypothesis of homoskedasticity in the residuals time series at all. The plot of the mean-reversion coefficients further shows that the mean-reversion appears to be quite weak. Hence, the not so strong rejection rate of the Engle-Granger tests (particularly in the case of the US market) may be caused by the fact that the cointegration errors are $I(0)$, but the process is not strongly mean-reverting. In fact, this argument seems pretty plausible since the EG statistics and the mean-reversion coefficients are highly correlated ($\rho = 0.91$ in the case of the British stocks, and $\rho = 0.98$ for those in the US). In addition, in the more critical case of the US stocks, the EG statistics are still located only on the left side of the distribution of critical values, indicating that it appears unlikely that there is no cointegration[24].



Mean-reversion coefficients of residuals of Engle-Granger cointegrating regression (case of FTSE100 index).

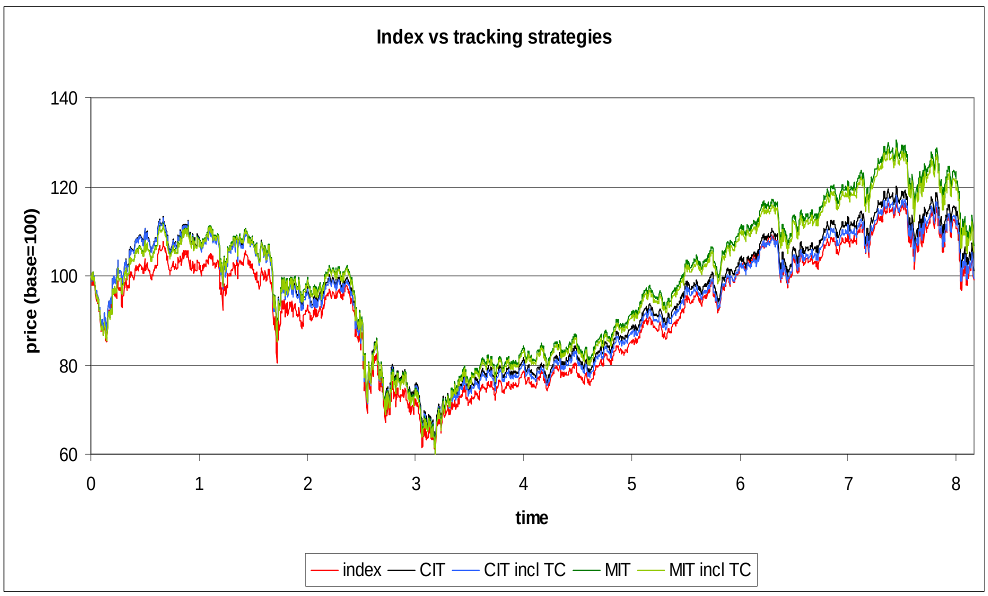
For the approach based upon continuous time portfolio optimization, I did not use a calibration period, but estimated the covariance matrix only once for the complete analysis. For this estimation, I made use of a bootstrapping approach presented by Efron (1979)[25].

[24] A joint test was not performed since such a joint test's limiting distribution and critical values have not yet been analysed in the literature.

[25] From the whole data sample, a 'random sub-sample' was picked and based upon this 'sub-sample' the covariance matrix is (consistently) estimated what ensures that there is no bias when using the same data as the later index tracking analysis is based on; note in the the theoretical model, the covariance matrix is constant over time, and thus I have estimated it only once for the overall index tracking analysis.

To quantify the 'out of sample' performance of the two strategies, I had a look at the final price development of the strategies compared to the index, tested the log-price spread between the index and the strategies of unit roots, reported turnover and transaction costs, studied differential returns and correlations of the strategies with the index, and compared the conditional variances of the index to the ones of the tracking portfolios.

In the following section, I concentrated again on the British market and chose a calibration period of 15 years and a non-rebalancing time of 10 (trading) days. Rebalancing every 10 days appears to be pretty optimal among the six tested lengths of non-rebalancing periods (in the trade off between more accuracy and lower transaction costs)[26]. First of all, I displayed the price path of the index and the tracking strategies for a first visual test.



FTSE100 index and cointegration index tracking strategy (CIT) and tracking strategy based upon CTPO (MIT); TC indicates that transaction costs are inluded.

This graph indicates that the cointegration index tracking strategy (CIT) seems more strongly tied together with the index than the tracking portfolio based upon continuous time portfolio optimisation (MIT). In fact, applying an ADF unit root test to (out of sample) log-price spreads between index and CIT suggests them to be $I(0)$ (rejection of unit root on a significance level of 1%)[27], while a unit root for spreads between index and MIT can not be rejected with any reasonable confidence. However, in the US market the null of

[26] Periods of 1 day, 10 days, 1 month, 3 months, half a year, 1 year were tested.

[27] Here, an ADF unit root test including a constant was performed. This may be disputed, since a test without a constant could be argued to be more reasonable in this setting. Moreover, in contrast to the

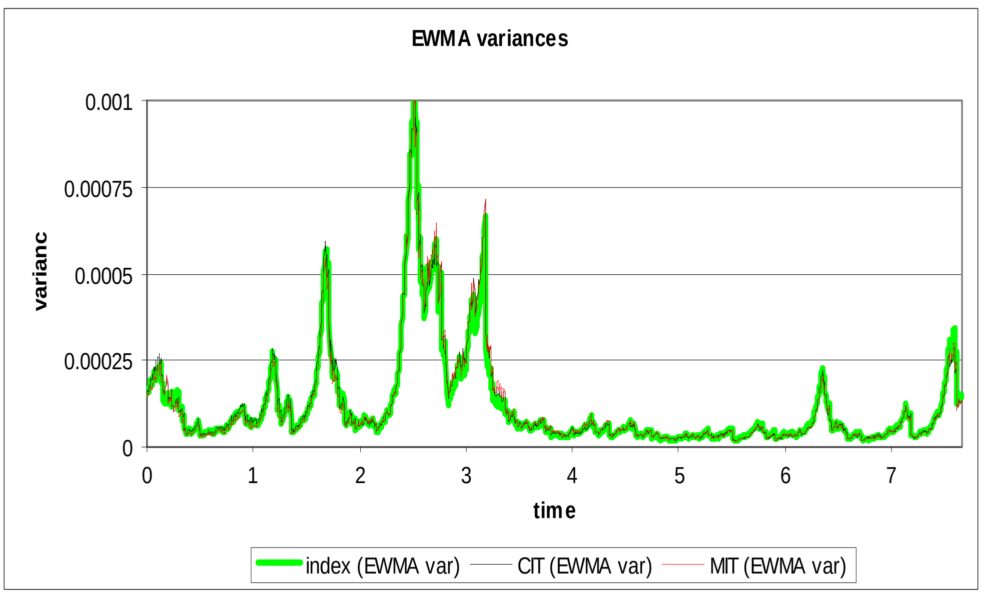
non-stationarity can not be rejected in either case (for spreads between index and CIT resp. MIT).

Next, I analysed turnover and transaction costs, where the latter is a fixed proportion of the turnover. Transaction costs are simply defined as,

$$TC_t = 0.002 \sum_{i=1}^{N} abs\left(w_{i,t} - w_{i,t-1}\right)$$

where TC_t denotes the transaction costs relative to the overall fund value at time t and $w_{i,t}$ is the portfolio weight of asset i at time t. This matches findings in empirical finance literature pretty well (Alexander and Dimitriu, 2002). It turns out that in the case of a rebalancing frequency of 10 days, transaction costs are negligible (annual $TC < 0.25\%$) in both strategies, while TC of MIT are a little lower then TC of CIT.

In absolute terms, daily differential returns are on average around 0.1% and not autocorrelated. Moreover, they appear to be symmetrically distributed in the case of CIT.



Conditional variances of FTSE100 index and tracking strategies.

Further, I estimated conditional variances of the returns of index and tracking strategies

in-sample log-price spreads process, which showed no heteroskedasticity, the out of sample log-price spreads series features some volatility clustering. I then find that the conclusion about the stationarity of the out of sample log-price spreads process may depend on the specific unit root test applied – an ADF test without constant leads for example to the conclusion that a unit root may not be rejected. Hence, the conclusion of out of sample log-price spreads being $I(0)$ has to be treated with caution.

using a simple exponentially weighted moving average (EWMA) model given as,

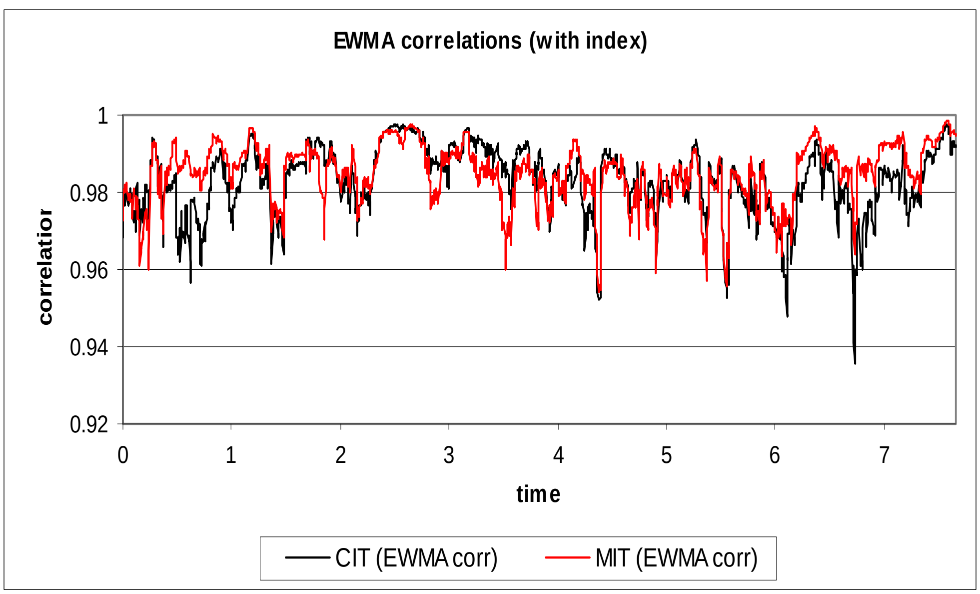
$$\begin{aligned}\sigma_t^2 &= \lambda\sigma_{t-1}^2 + (1-\lambda)(r_{t-1}-\mu_{t-1})^2 \\ &= (1-\lambda)\sum_{i=1}^{n}\lambda^{i-1}(r_{t-1}-\mu_{t-1})^2 + \lambda^n\sigma_{t-n}^2\end{aligned}$$

where r_t is the return in period t, μ_t is the expected return at time t, σ_t^2 is the variance at time t, and λ is some weighting parameter that is usually calibrated applying MLE. For the parameter λ, I chose (by rule of thumb) an estimate provided by Hull (2006) of $\lambda = 0.94$ which is argued to perform quite well across different financial time series. I found that these EWMA estimates of the index and both strategies essentially coincide.

Using the same EWMA model I further estimated conditional correlations between the returns of the index and both tracking strategies. Then the model becomes,

$$\rho_{ij,t} = \frac{\sigma_{ij,t}}{\sigma_{i,t}\sigma_{j,t}}$$

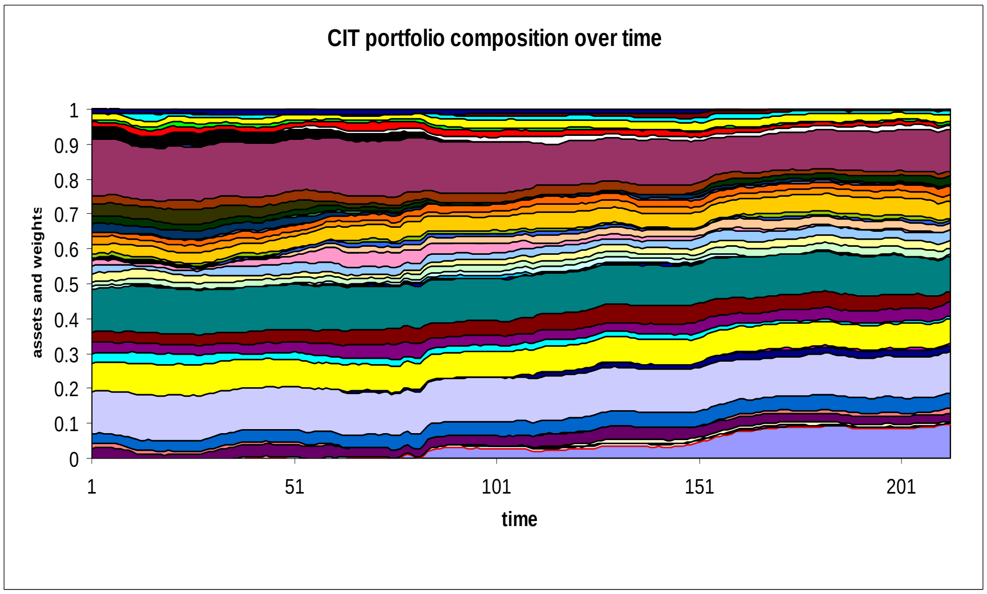
with $\sigma_{ij,t} = (1-\lambda)\sum_{i=1}^{n}\lambda^{i-1}(r_{i,t-1}-\mu_{i,t-1})(r_{j,t-1}-\mu_{j,t-1}) + \lambda^n\sigma_{ij,t-n}^2$ and σ_t as stated earlier.

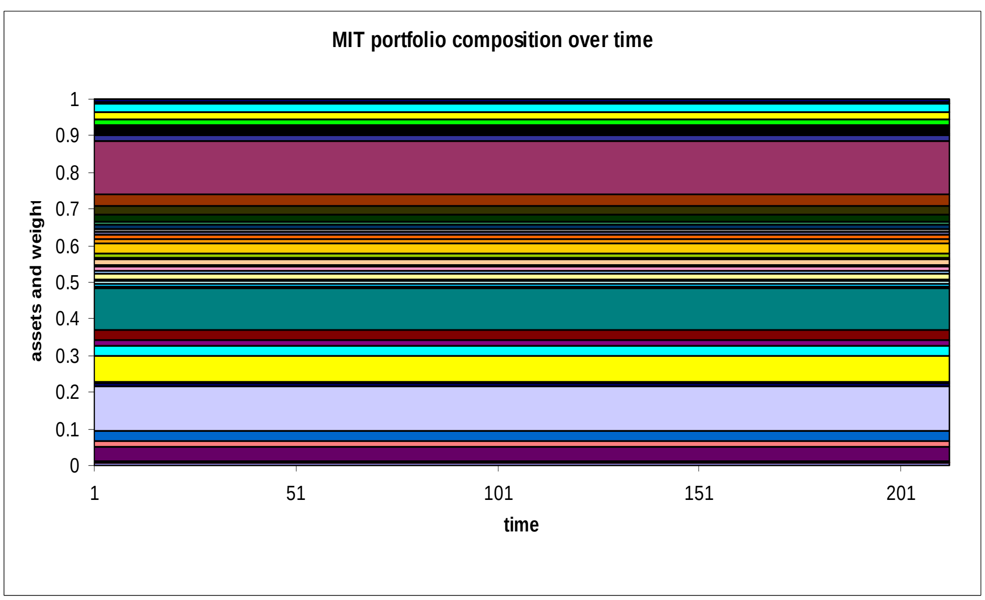


Conditional correlations between FTSE100 index and tracking strategies.

The results of this estimated EWMA model indicate that the returns of the index and either tracking portfolio are highly correlated at all times. Moreover, the unconditional correlation in both cases was over 99%.

Finally, the development of the tracking portfolio composition is analysed. While MIT has a fixed structure by definition (constant portfolio weights over time), in the case of CIT the optimal weights change a little at each rebalancing day. These changes are however not very large, i.e. the portfolio composition is not very volatile. Moreover, when roughly comparing the largest positions, it seems that the two strategies' structures are quite close to each other.





Portfolio composition of CIT and MIT over time.

Lastly, it is important to note that in the case of both Dow Jones indices, the above discussed properties/ index tracking performance indicators (but the first EG cointegration test and the log-price spread unit root test[28]) are similar to the case of the FTSE100 dataset.

It follows that both strategies are able to track an index accurately. Moreover, in the FTSE100 dataset, the CIT appears to be more closely tied together with the index (in the long term), while it is easy for the MIT price path to drift away from the index price path (even though expected return and risk of MIT and index coincide and there is a high correlation). However, in the two US datasets, this advantage of the CIT over the MIT is not statistically significant (in out of sample tests). Hence, an approach based upon cointegration analysis might not be better in any financial market. Nevertheless it is shown to be preferred in the case of the British stock market. However, that this slightly different finding in the two markets indicates that the cointegration approach is not robust would be not the right conclusion at this point. Finally, I would like to remark that the MIT strategy tends to outperform the index regularly, which makes it rather attractive to use it as a strategy when taking a long position but not when selling short.

Next, I analysed (using the same methodology as just presented) the possibility of tracking an index while only making use of a subset of its constituent stocks[29]. This is in particular interesting for very large indices where it is impractical or even impossible to hold all stocks. The calibration period was 9, 13 and 15 years for the DJI, the DJCA and the FTSE100 datasets, respectively. The non-rebalancing time was chosen to be 10 days. For simplicity, the (pre)selection of stocks (subset of constituents) used to construct the two tracking strategies was done by ranking the stocks according to their weight in the index at the beginning of the sample period. Of course, this is not at all an optimal selection, since one should only weed out the stocks with ‘redundant stochastic trends’ as explained earlier. However, if with a ‘bad’ selection an index may still be tracked, it must also be possible to track the same index using a more optimal selection methodology. Indeed, it turns out that for the British stock market, the CIT and MIT strategies still work pretty well, if only around 25 stocks (out of the 51) were used, and in the case of the CIT strategy (out of sample) log-price spreads are still suggested to be $I(0)$. Moreover, the CIT strategy is able to track the DJI when using only half of its constituents and the DJCA with around 25 out

[28] Where it appears not that easy to judge whether the null of no cointegration may be rejected with a satisfactory confidence, even though some arguments are shown that rather suggest the presence of cointegration.

[29] 25 different basket sizes were tested to track the given indices.

of 58 stocks. In contrast, the MIT strategy shows in the US market that only 10 to 15 stocks are necessary to track either the DJI or DJCA. However, it has to be noted that as the size of the 'asset sub-set' decreases, the correlation of MIT and the index decreases as well as the volatility of MIT exceeds the volatility of the index. Moreover, MIT tends to generally outperform the index and there is no force that ties the tracking strategy and the index together (in the long run). Hence, I conclude that in general, it seems possible to track an index accurately while using only a subset of its constituents. Further, the CIT strategy is (particularly in the British market) preferred to the MIT strategy, since it guarantees a long run mean reversion (towards zero) of log-price spreads between index and CIT.

Finally, I tried to perform some enhanced indexation strategies similar to the analysis discussed by Alexander and Dimitriu (2002). Note that these authors made use of inaccurate critical values for the EG cointegration test analysis, and consequently ended up with incorrect conclusions. I have simply enhanced an index while adding to each day's return a small constant positive return and then tried to track the new enhanced index in the exact same manner as before[30]. In contrast to the above mentioned research, I found that the CIT strategy is not able to track any enhanced index. Moreover, I found that the MIT strategy is in principle able to 'track' all enhanced indices, but matching the expected return of an enhanced index results in a sizeable increase in volatility in the MIT strategy, which leads to a huge loss in tracking accuracy. Hence, even though one might use MIT to track an enhanced index, the performance is poor. Thus, I conclude that for an enhanced indexation strategy originally proposed by Alexander, Giblin and Weddington (2001) and Alexander and Dimitriu (2002), no convincing empirical evidence can be found.

5 Conclusions and Prospects for Future Research

I performed and compared two different index tracking strategies, one based upon cointegration analysis (CIT) and the other resting on market equilibrium theory and portfolio optimisation in continuous time (MIT). It was shown that both strategies are able to track indices of different sizes and diverse markets quite accurately, while even using only a subset of constituent stocks. I also suggest that the CIT strategy is somewhat preferable to the MIT strategy, since it ensures log-price spreads between index and CIT to be $I(0)$. However, in out of sample tests, this advantage is not found to be statistically significant in

[30] I have used enhancements of 1%, 2%, 3%, 4%, 5%, 10%. I also used the same length of calibration periods and a 10 day non-rebalancing period as earlier.

every stock market. Moreover, I found that the MIT strategy (as a solution of a restricted optimisation setup) tends to outperform the index a little, what is especially problematic if the tracking portfolio is intended to be sold short. On the other side, one may ask whether this steady outperformance may be exploited in a trading strategy. This question was not discussed within this paper, but might be the subject of further research. Anyway, it is clear that (in contrast to CIT) the MIT strategy does not ensure any mean reversion of price spreads between index and tracking portfolio. Finally, it is shown that in a simple enhanced indexation approach, where one tries to track an artificial, enhanced index, both tracking strategies are either not valid at all or perform poorly.

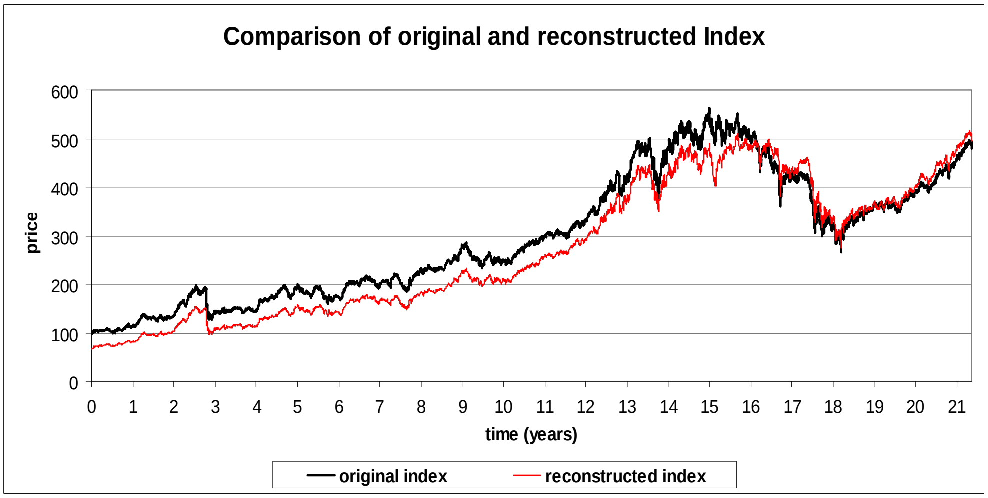
Further research should be done to test the robustness of the CIT approach against different markets since (in out of sample tests) I was only able to reject the null of non-stationarity of the log-price spreads (between index and CIT portfolio) in the British stock market, while in the US market the results are not so unambiguous.

Although not observed within this paper, certainly a very interesting question is whether some more sophisticated trading strategies based upon the cointegrating relation between index and CIT strategy might be attainable. One could possibly think about to exploiting the mean reversion property of the process of log-price spreads between index and CIT strategy. Hence, it could be possible to achieve almost risk free profits if one can trade a financial instrument with a stationary price process. However, it also has to be considered that the mean-reversion is typically quite weak.

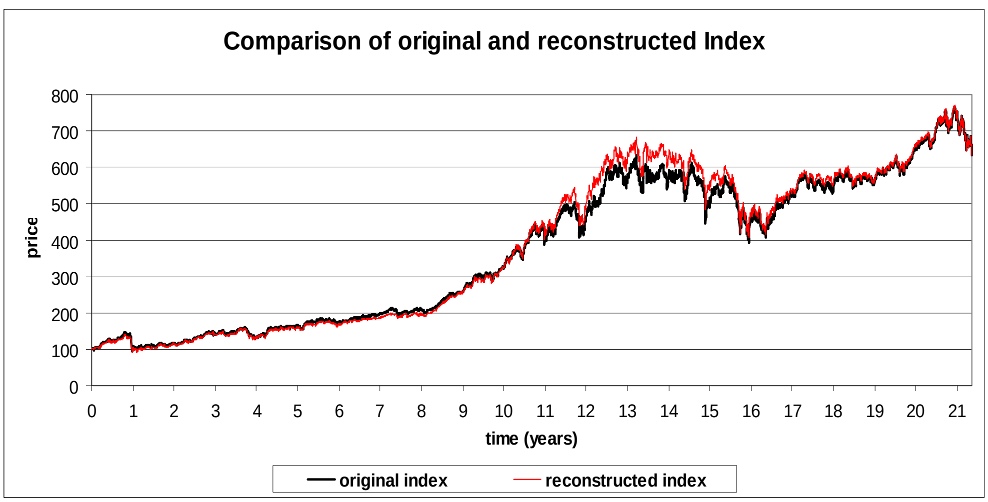
6 Appendix

6.1 Reconstruction of Indices

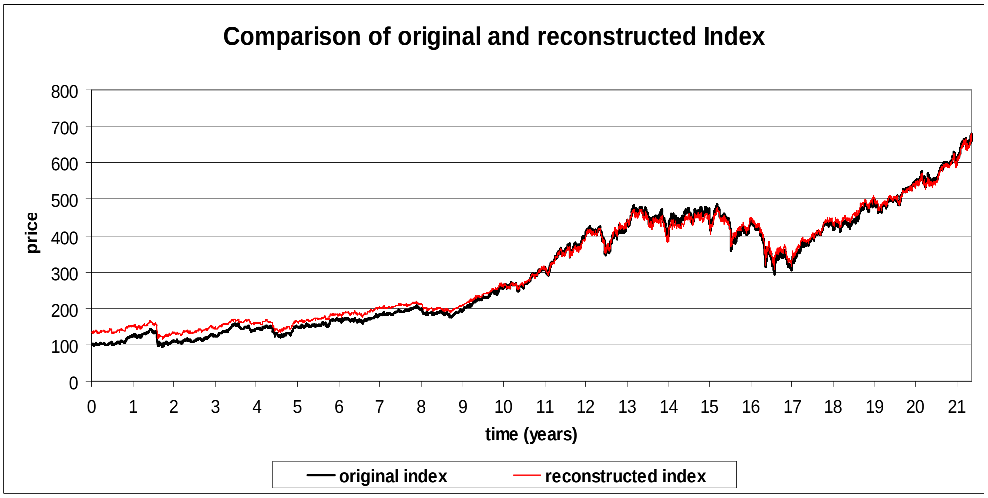
6.1.1 FTSE100



6.1.2 DJI

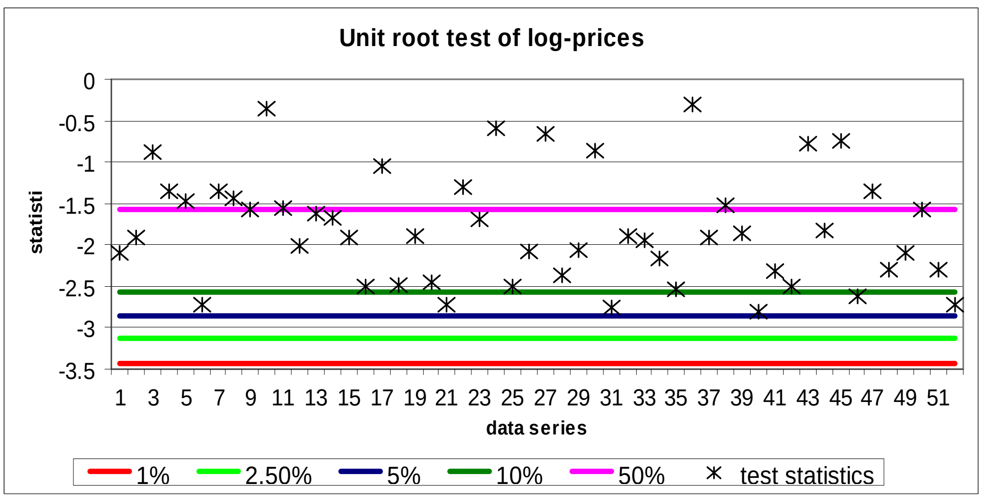


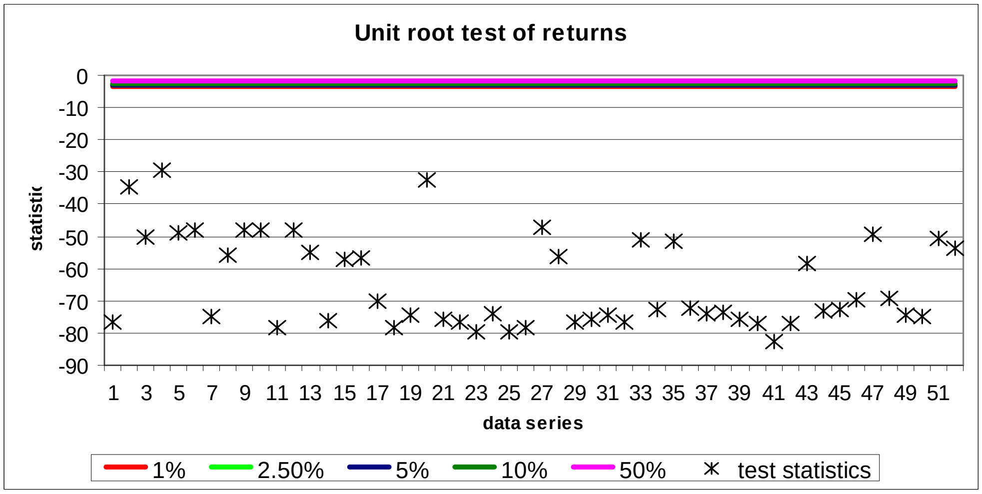
6.1.3 DJCA



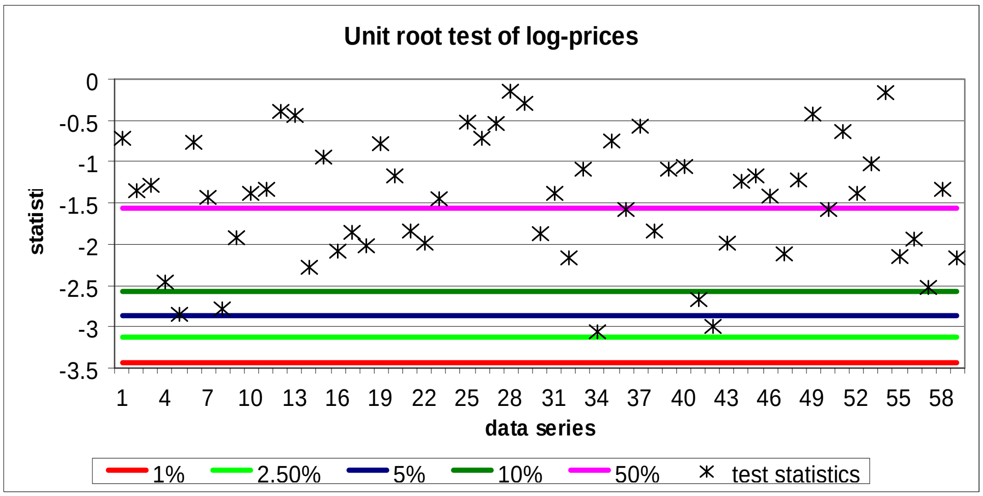
6.2 Unit Root (ADF) Test of Log-Price Time Series

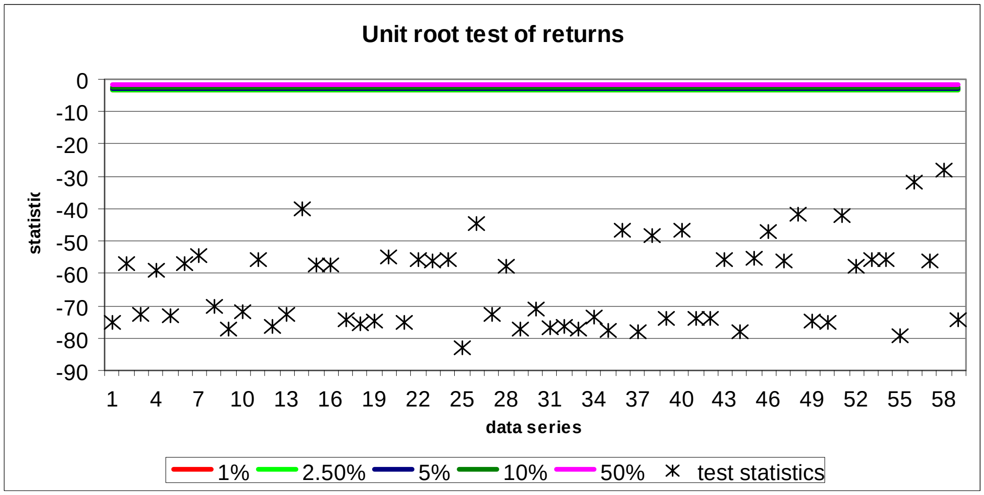
6.2.1 FTSE100




Unit root test of returns
statistic
0
-10
-20
-30
-40
-50
-60
-70
-80
-90
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41 43 45 47 49 51
data series
1%
2.50%
5%
10%
50%
test statistics

6.2.2 DJCA


Unit root test of log-prices
statisti
0
-0.5
-1
-1.5
-2
-2.5
-3
-3.5
1 4 7 10 13 16 19 22 25 28 31 34 37 40 43 46 49 52 55 58
data series
1%
2.50%
5%
10%
50%
test statistics



6.3 Response Surface Regression

There are only very few accurate critical values of Engle-Granger cointegration tests tabulated in the academic literature. MacKinnon (1990, 1994, 1996) is the first useful source to get tables with critical values. However, his estimates can only be used for hypothesis testing in a system with no more than 12 $I(1)$ processes. In this paper, I analysed systems including far more variables (up to 59). Hence, I estimated some critical values for myself using a response surface regression setup as first introduced by MacKinnon (1990). It is simply a Monte Carlo simulation, where first a system of N $I(1)$ processes is simulated (conditional on there being no cointegration in the system) and then an Engle-Granger test statistic is computed from the simulated dataset. I repeated this procedure 200,000 times for 11 different sample sizes[31]. Moreover, I defined 10,000 different significance levels reaching from 0.01% to 99.99% and made use of the earlier estimated test statistics to estimate critical values corresponding to the different sample sizes. Thus, I ended up with 200,000 test statis-tics and extracted therefrom 10,000 critical values for a system of N (not cointegrated) I(1) processes for each one of the 11 different sample sizes. Lastly, I ran (10,000 times) the ac-tual response surface regression first proposed by MacKinnon (1990). This is given as,

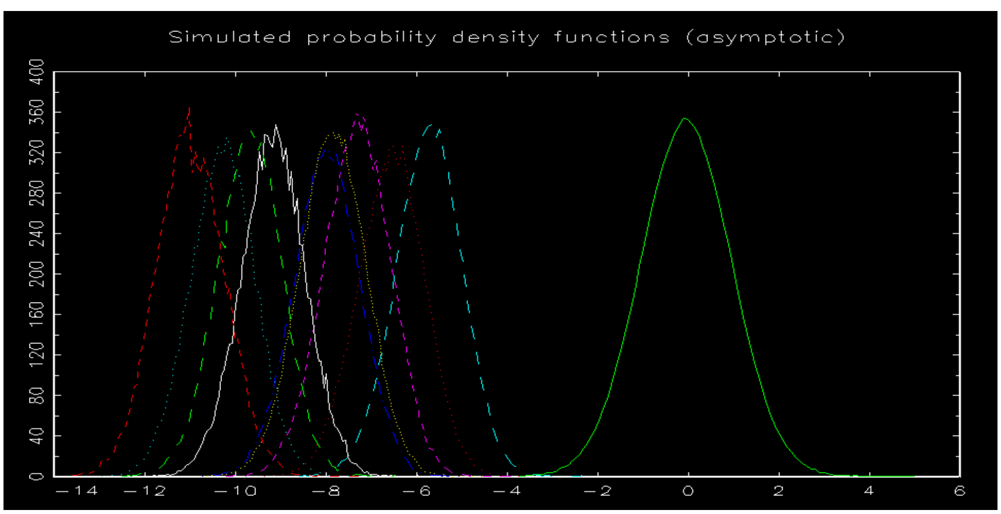
$$CV_p(T_i) = \theta_p^{(\infty)} + \theta_p^{(1)} T_i^{-1} + \theta_p^{(2)} T_i^{-2} + \theta_p^{(3)} T_i^{-3} + \varepsilon_i$$

[31] For system sizes 16, 21, 26, 30, 31 the sample sizes {100,125,150,200,250,300,400,500,750,1000,1500}, for system sizes 41, 46 the sample sizes {125,150,175,200,250,300,500,700,900,1000,1500}, and for the system sizes 52, 59 the sample sizes {150,175,200,250,300,500,700,900,1000,1500,2000} were used.

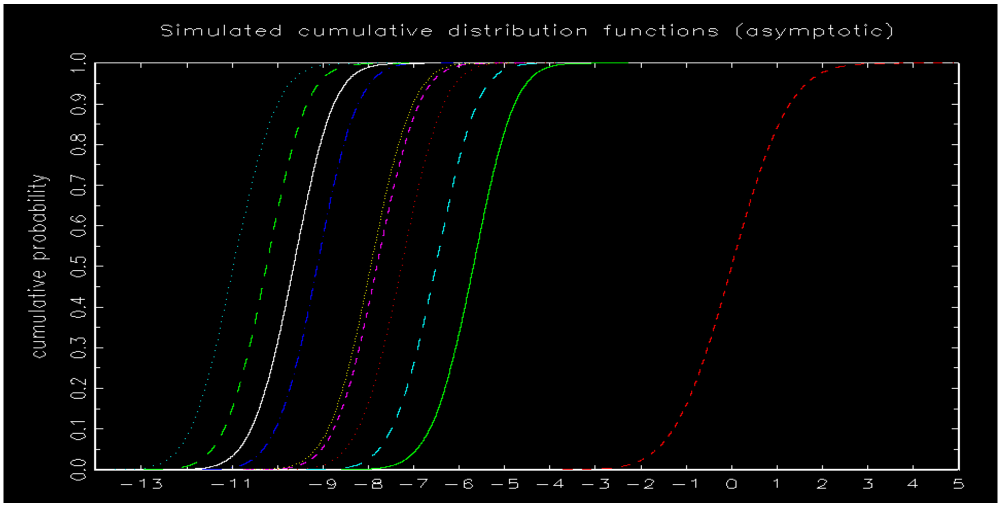
where $CV_p(T_i)$ denotes the critical value on the significance level p corresponding to the sample size T_i, $\theta_p^{(\infty)}$ is the asymptotic critical value on the significance level p, and the further θs allow estimating small sample critical values. Hence, I estimated the described re-sponse surface regression and the according asymptotic critical values as well as the equations to approximate small sample critical values for $N = \{16, 21, 26, 30, 31, 41, 46, 52, 59\}$.

System size	Critical values (asymptotic)					
	1%	2.5%	5%	10%	50%	100%
16	-7.47	-7.18	-6.93	-6.65	-5.69	-2.26
21	-8.28	-8.01	-7.77	-7.49	-6.52	-3.47
26	-9.02	-8.76	-8.51	-8.24	-7.27	-3.72
30	-9.54	-9.27	-9.16	-8.89	-7.82	-4.43
31	-9.66	-9.40	-9.16	-8.89	-7.95	-4.84
41	-10.81	-10.56	-10.32	-10.06	-9.13	-5.90
46	-11.37	-11.09	-10.85	-10.58	-9.65	-6.39
52	-11.94	-11.69	-11.45	-11.17	-10.25	-7-10
59	-12.70	-12.43	-12.19	-11.92	-10.98	-7.20

Table: Selected asymptotic critical values for Engle-Granger cointegration test in a system of N I(1) processes.



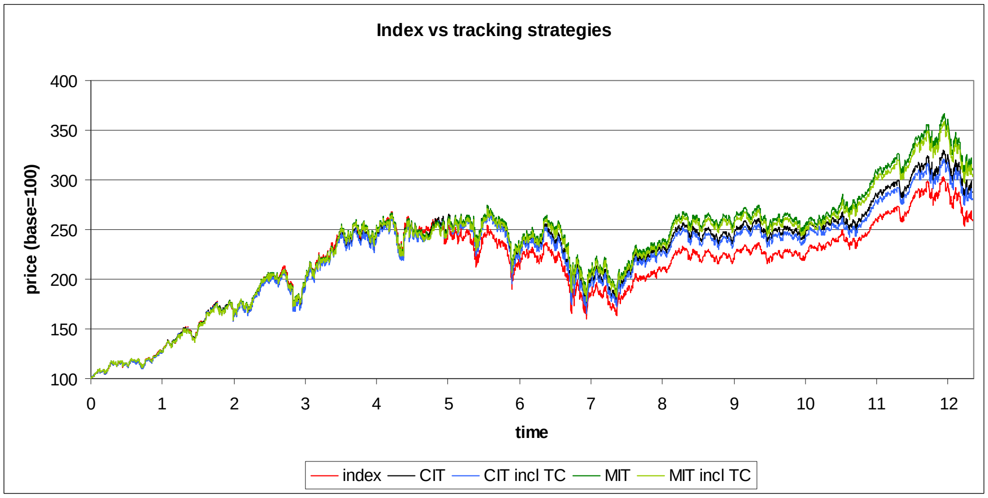
Simulated asymptotic probability density functions for Engle-Granger cointegration test statistics in a system of N $I(1)$ processes.

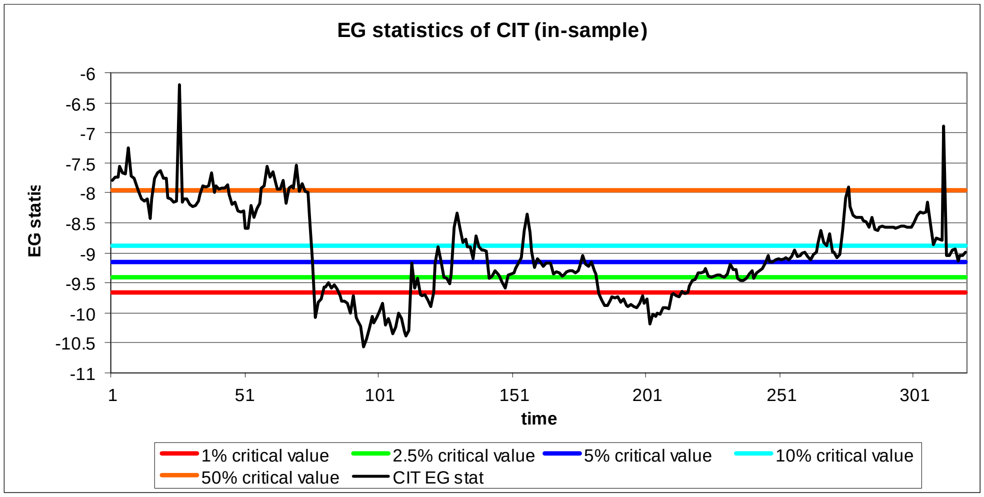


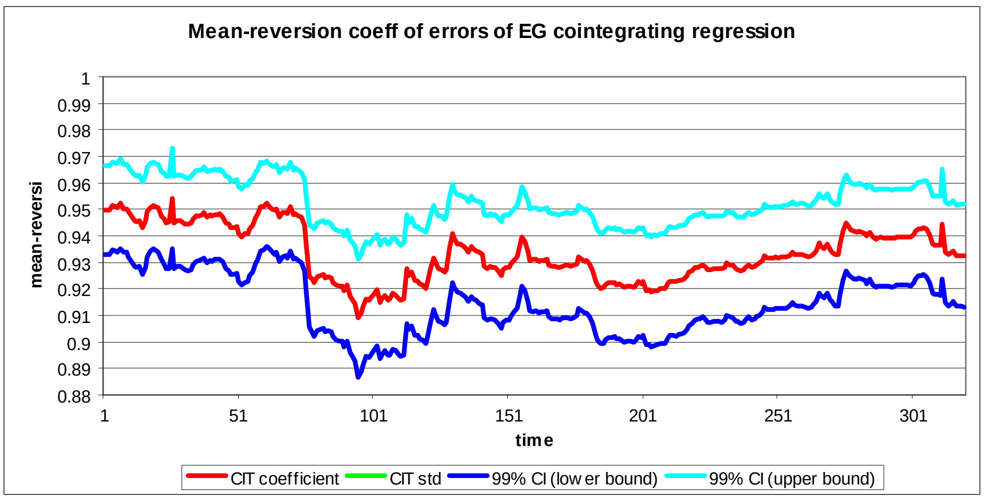
Simulated asymptotic cumulative distribution functions for Engle-Granger cointegration test statistics in a system of N $I(1)$ processes.

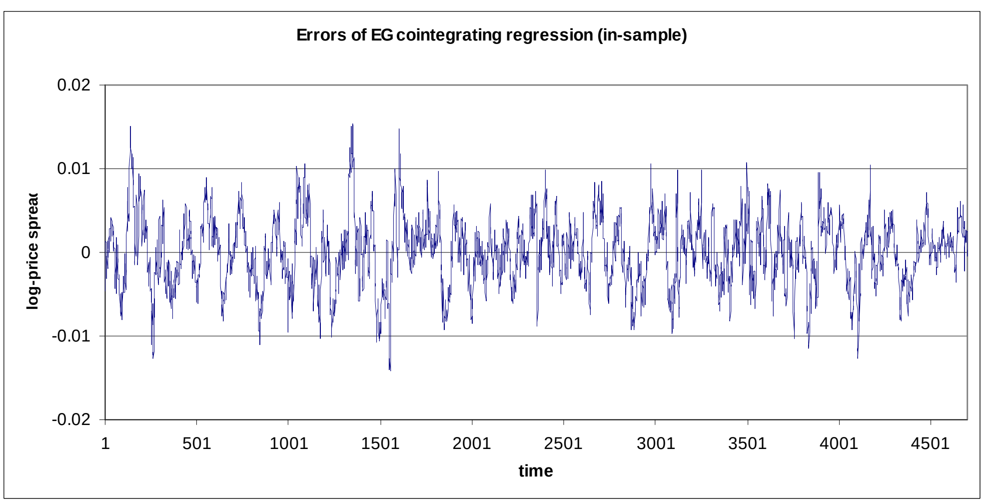
6.4 Index Tracking Strategies

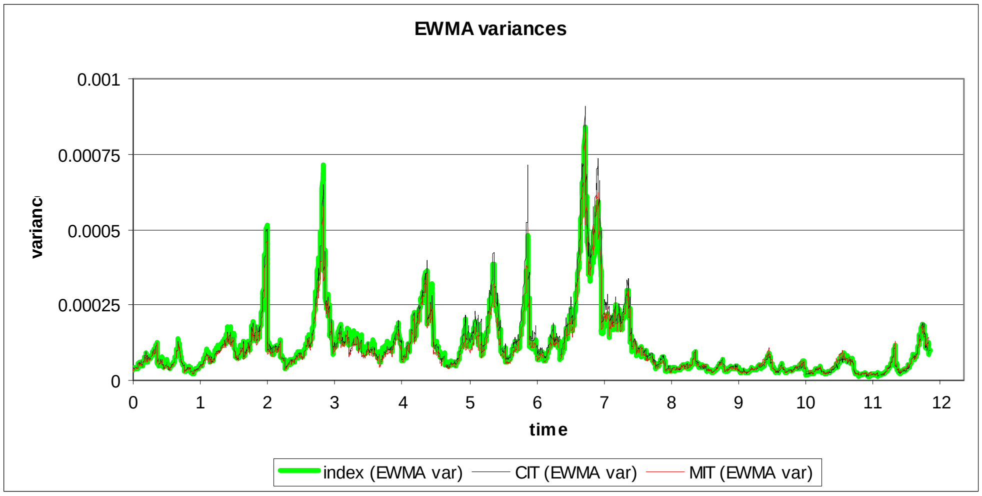
6.4.1 DJI

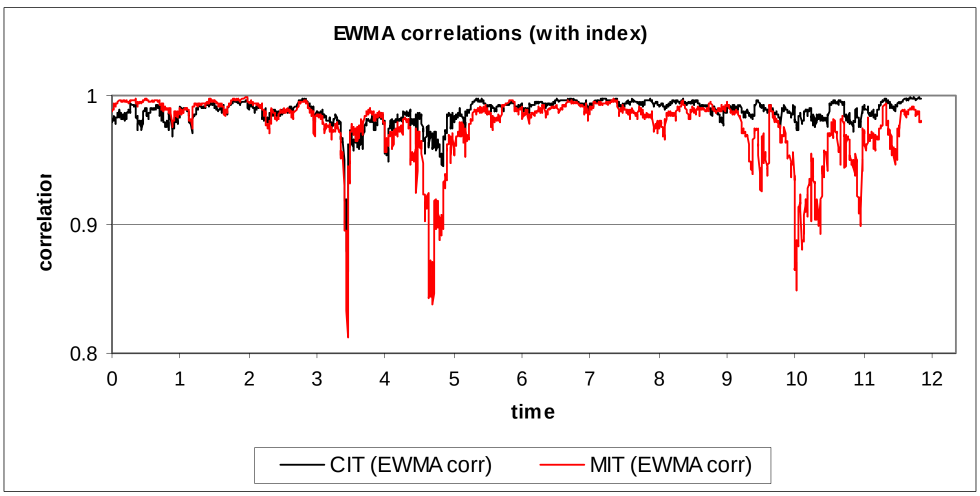



EG statistics of CIT (in-sample)
EG statis
-6
-6.5
-7
-7.5
-8
-8.5
-9
-9.5
-10
-10.5
-11
1
51
101
151
201
251
301
time
1% critical value
2.5% critical value
5% critical value
10% critical value
50% critical value
CIT EG stat

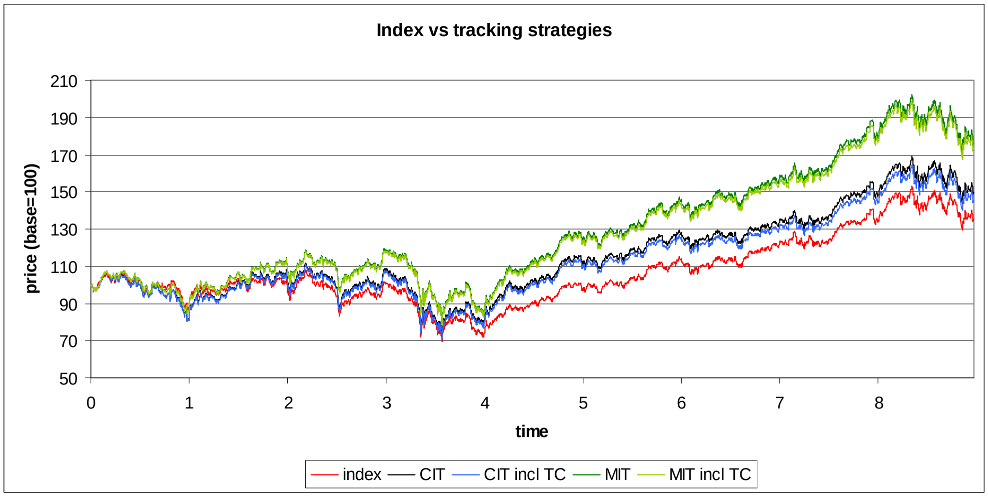

Mean-reversion coeff of errors of EG cointegrating regression
mean-reversi
1
0.99
0.98
0.97
0.96
0.95
0.94
0.93
0.92
0.91
0.9
0.89
0.88
1
51
101
151
201
251
301
time
CIT coefficient
CIT std
99% CI (lower bound)
99% CI (upper bound)


Errors of EG cointegrating regression (in-sample)
log-price spreac
0.02
0.01
0
-0.01
-0.02
1
501
1001
1501
2001
2501
3001
3501
4001
4501
time

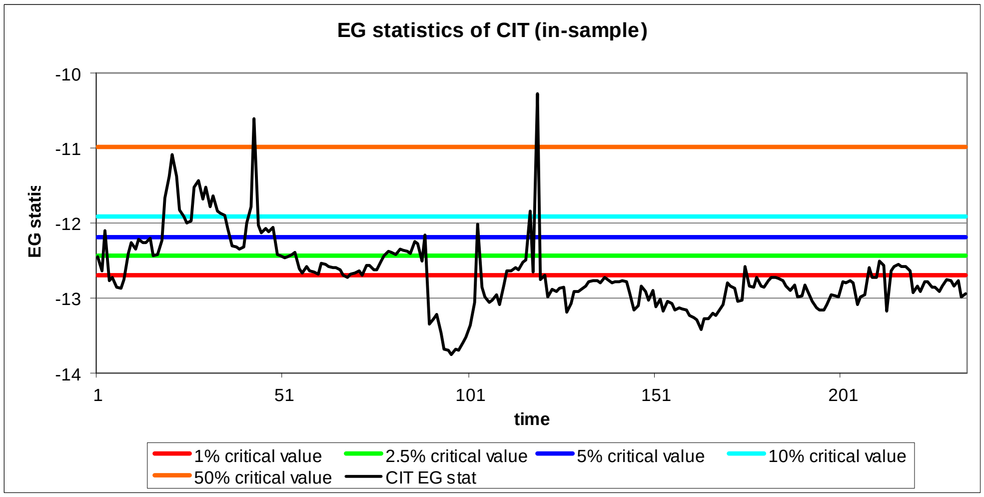

EWMA variances
0.001
0.00075
0.0005
0.00025
0
varianc
0 1 2 3 4 5 6 7 8 9 10 11 12
time
index (EWMA var)
CIT (EWMA var)
MIT (EWMA var)


EWMA correlations (with index)
1
0.9
0.8
correlatio
0 1 2 3 4 5 6 7 8 9 10 11 12
time
CIT (EWMA corr)
MIT (EWMA corr)

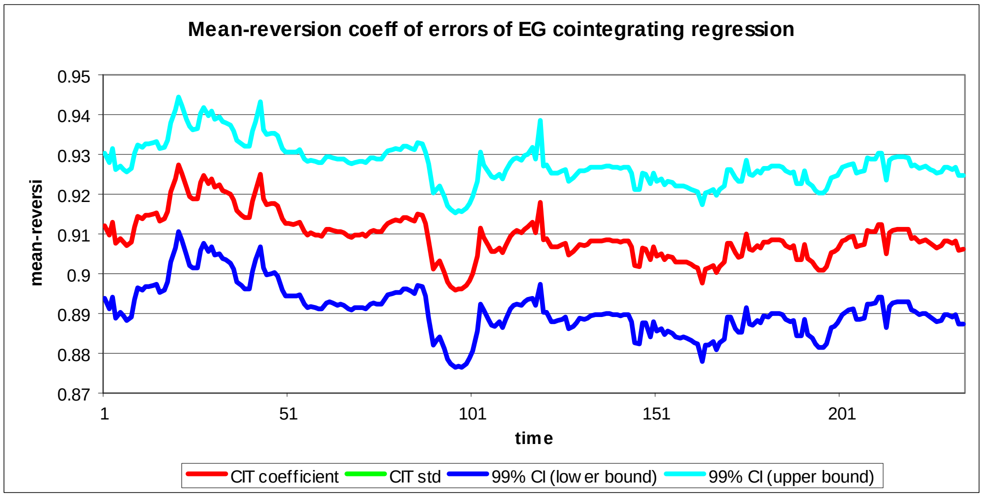
6.4.2 DJCA



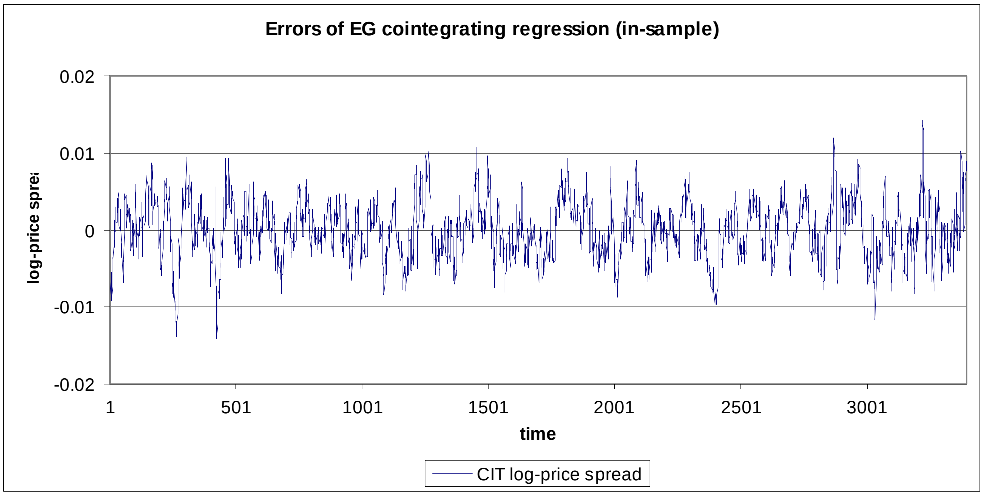
Index vs tracking strategies
price (base=100)
210
190
170
150
130
110
90
70
50
0
1
2
3
4
5
6
7
8
time
index
CIT
CIT incl TC
MIT
MIT incl TC

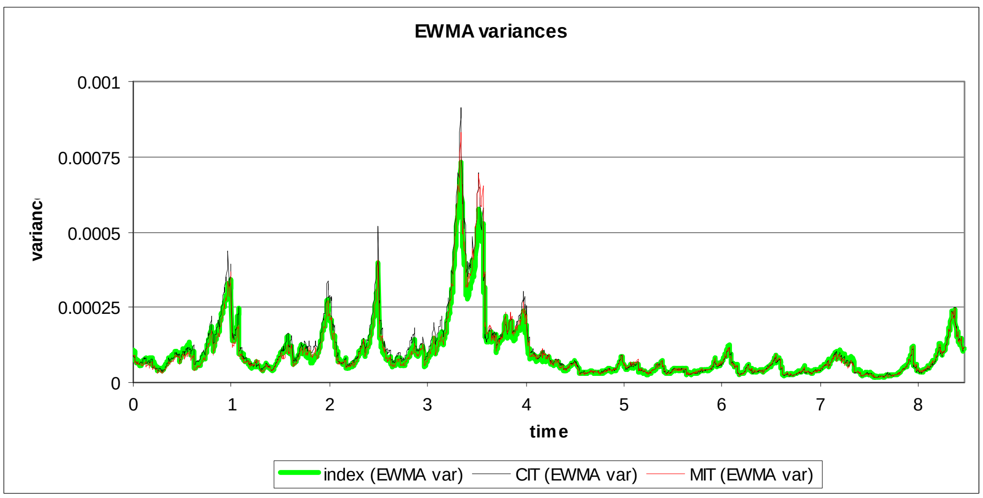


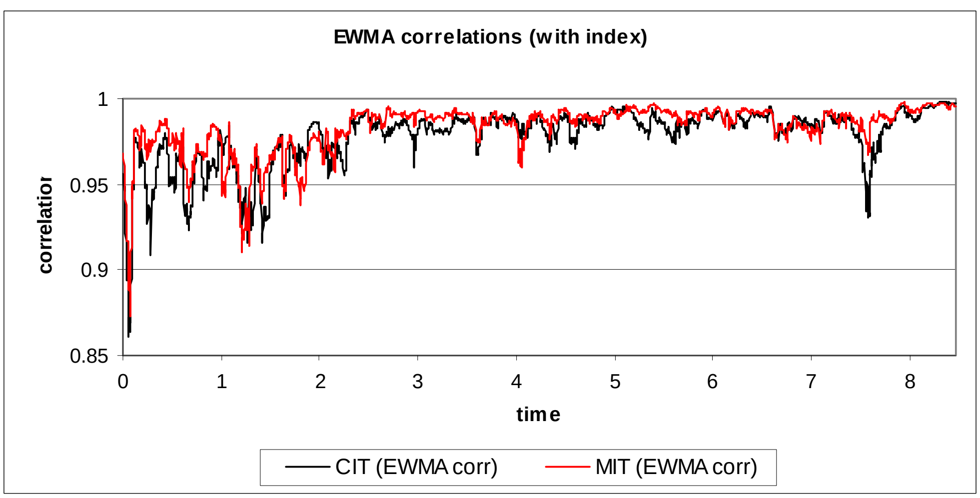
EG statistics of CIT (in-sample)
EG statis
-10
-11
-12
-13
-14
1
51
101
151
201
time
1% critical value
2.5% critical value
5% critical value
10% critical value
50% critical value
CIT EG stat



Mean-reversion coeff of errors of EG cointegrating regression
mean-reversi
0.95
0.94
0.93
0.92
0.91
0.9
0.89
0.88
0.87
1
51
101
151
201
time
CIT coefficient
CIT std
99% CI (low er bound)
99% CI (upper bound)

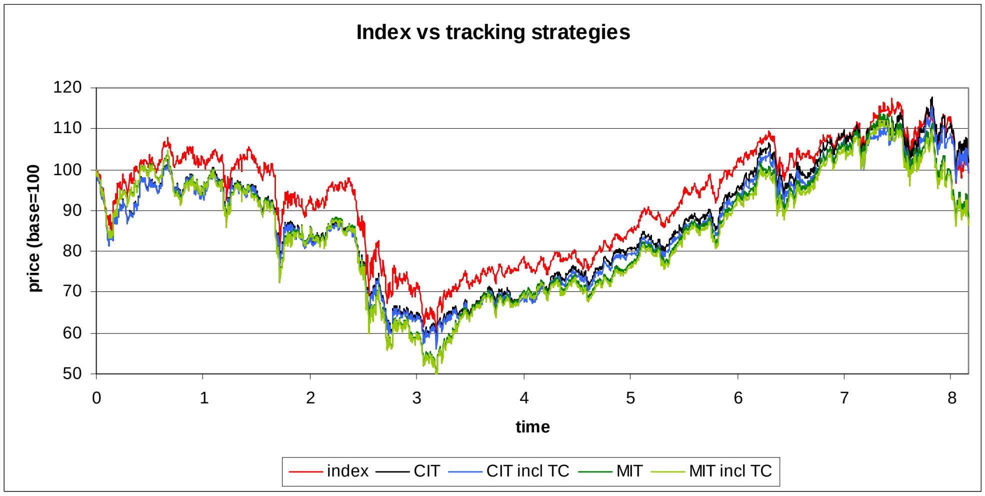

Errors of EG cointegrating regression (in-sample)
0.02
0.01
0
-0.01
-0.02
log-price sprea
1
501
1001
1501
2001
2501
3001
time
CIT log-price spread

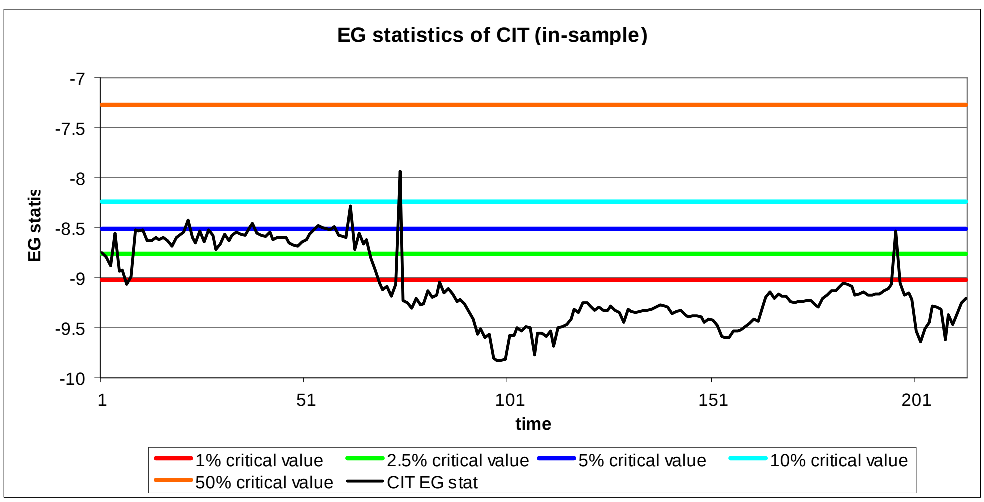

EWMA variances
0.001
0.00075
0.0005
0.00025
0
varianc
0
1
2
3
4
5
6
7
8
time
index (EWMA var)
CIT (EWMA var)
MIT (EWMA var)

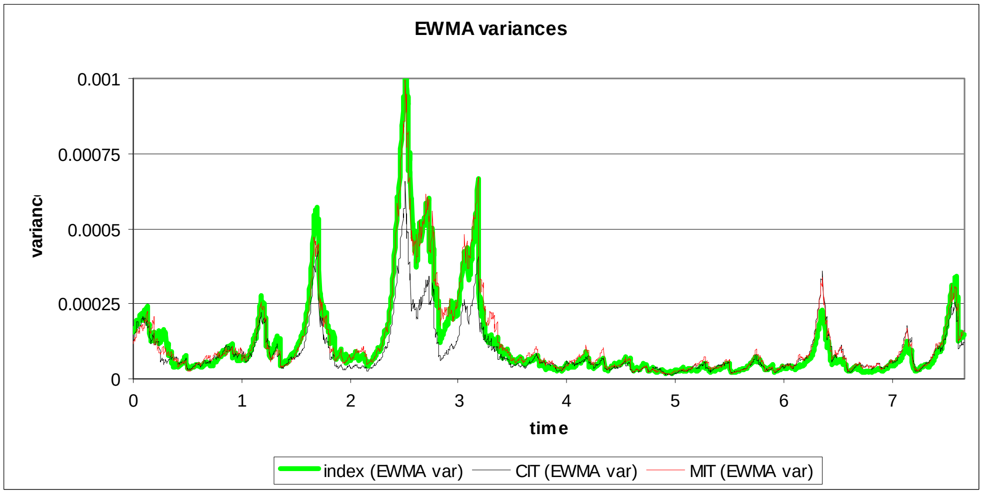

EWMA correlations (with index)
1
0.95
0.9
0.85
correlatio
0
1
2
3
4
5
6
7
8
time
CIT (EWMA corr)
MIT (EWMA corr)

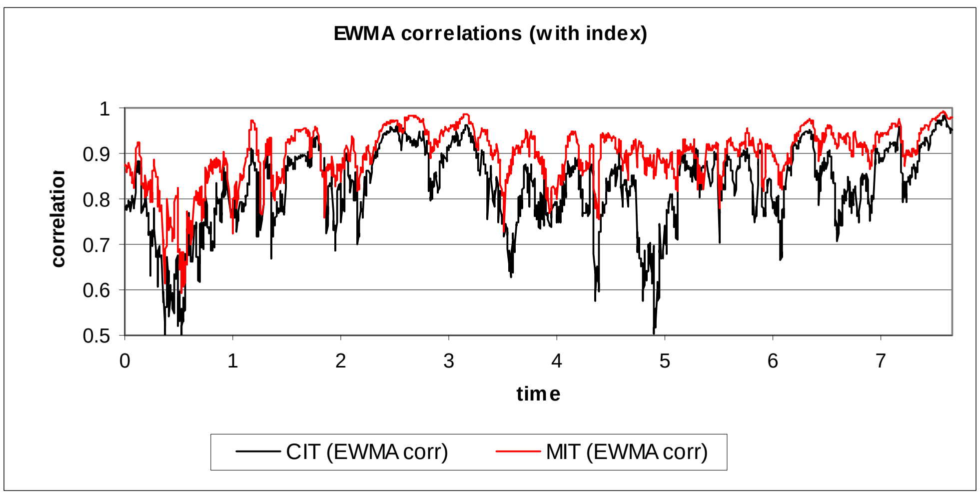
6.5 Index Tracking Strategies using a small basket of stocks

6.5.1 FTSE100 (25 stocks)

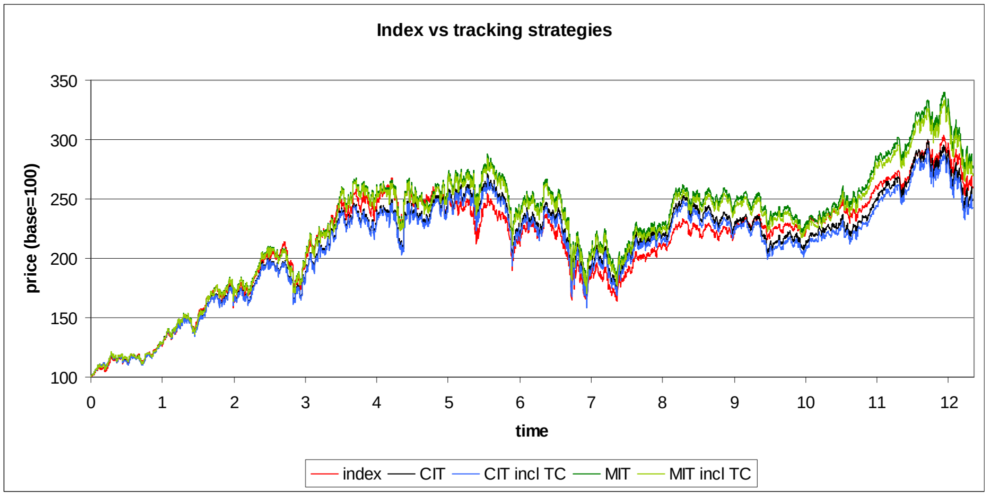


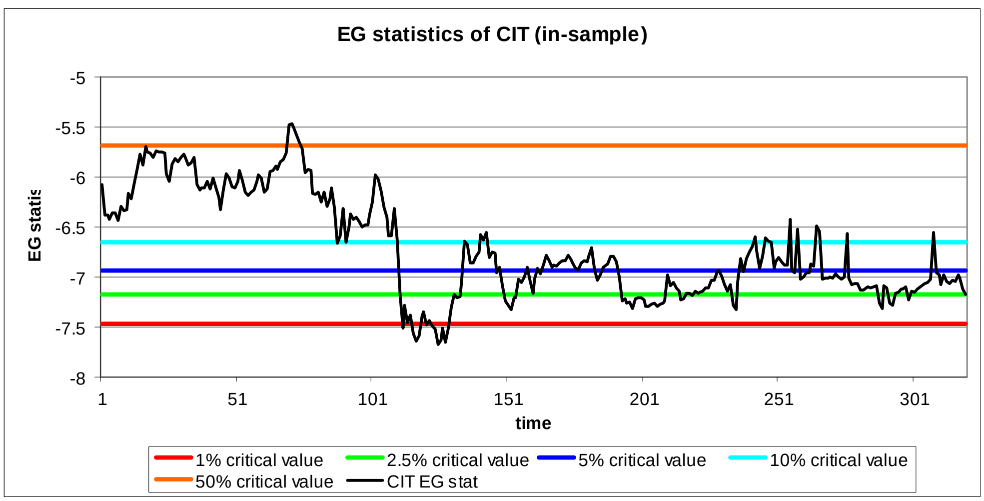


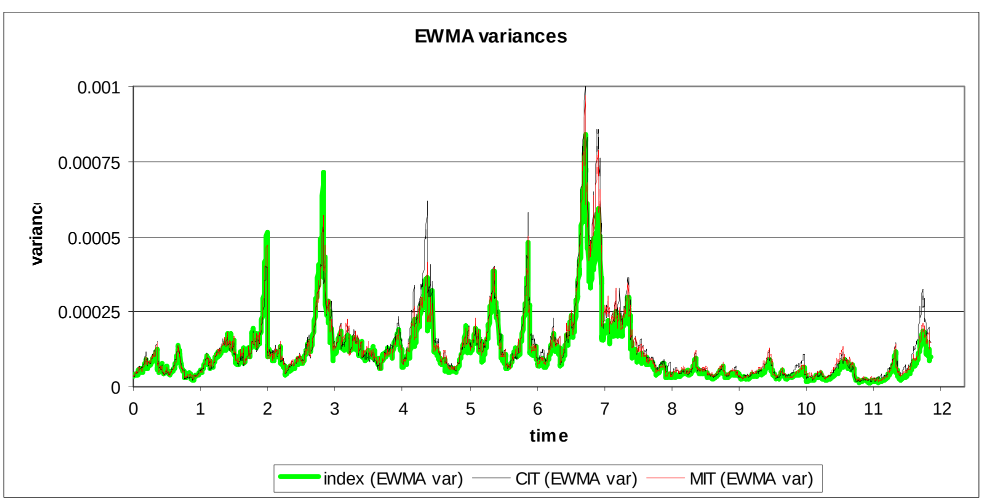

EWMA variances
varianc
0.001
0.00075
0.0005
0.00025
0
0
1
2
3
4
5
6
7
time
index (EWMA var)
CIT (EWMA var)
MIT (EWMA var)

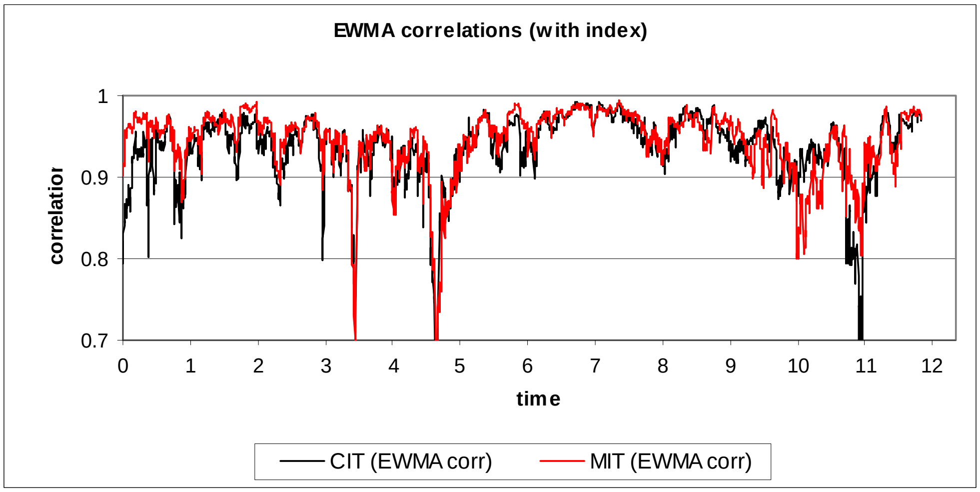

EWMA correlations (with index)
correlatio
1
0.9
0.8
0.7
0.6
0.5
0
1
2
3
4
5
6
7
time
CIT (EWMA corr)
MIT (EWMA corr)

6.5.2 DJI (15 stocks)

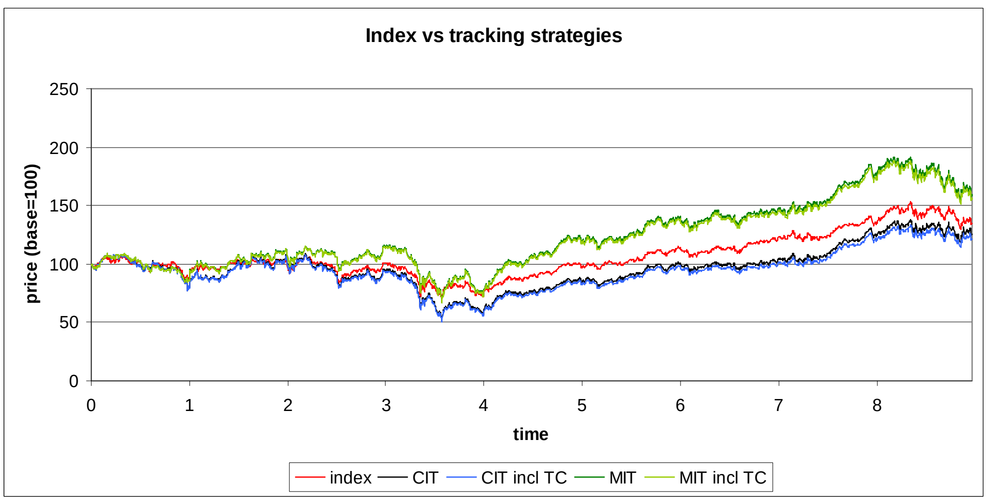


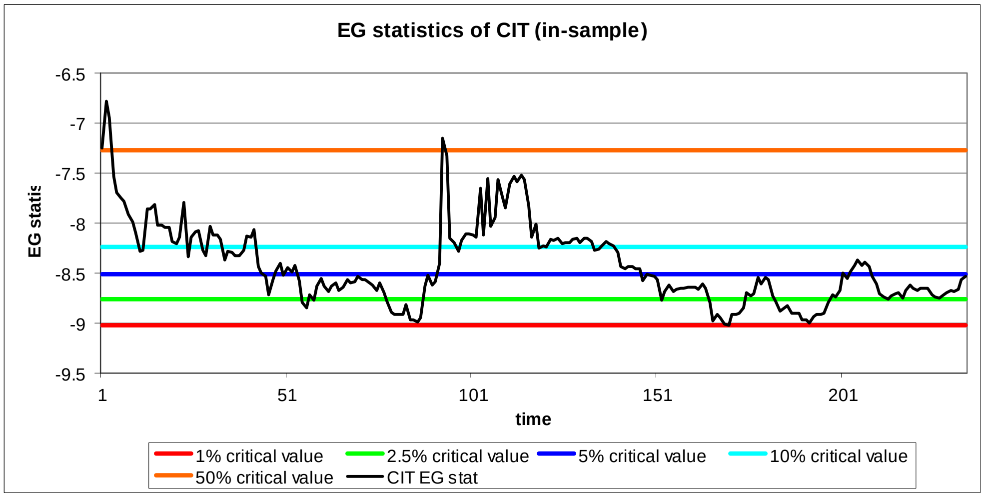


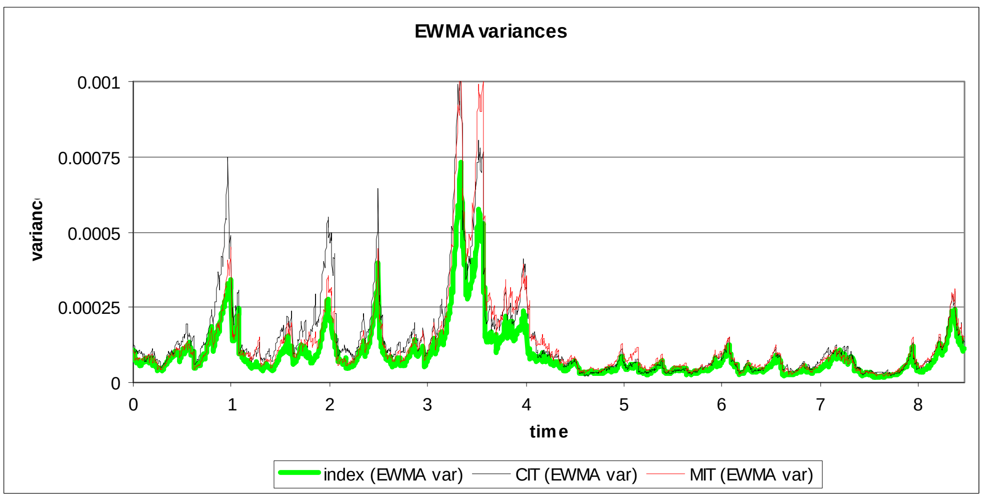


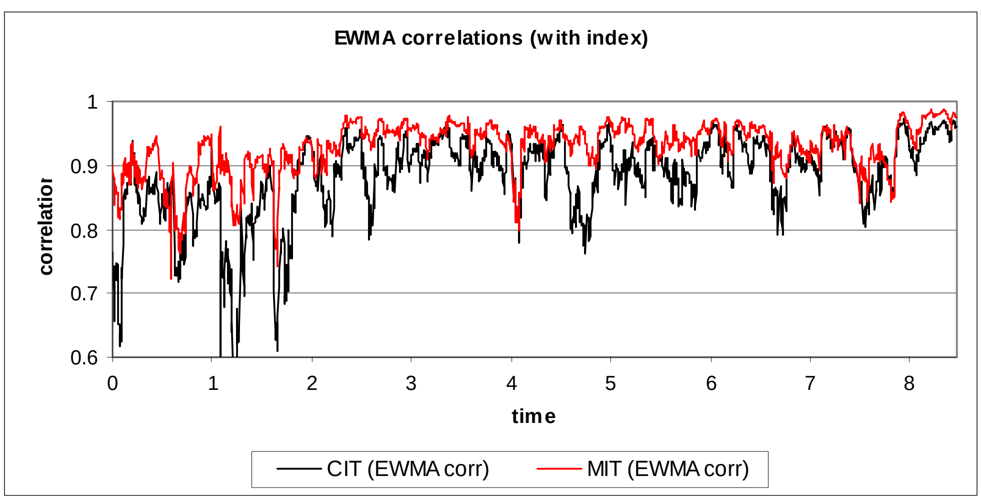


6.5.3 DJCA (25 stocks)



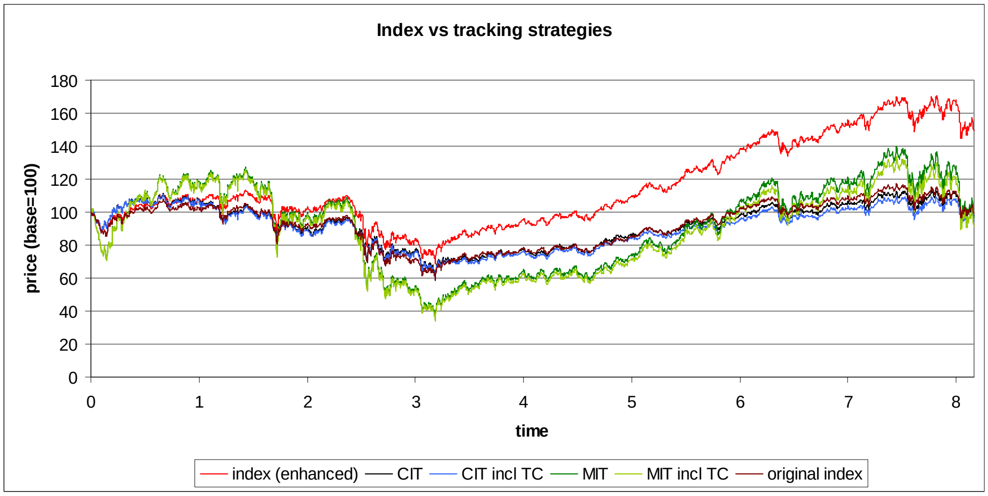

EG statistics of CIT (in-sample)
-6.5
-7
-7.5
-8
-8.5
-9
-9.5
EG statis
1
51
101
151
201
time
1% critical value
2.5% critical value
5% critical value
10% critical value
50% critical value
CIT EG stat

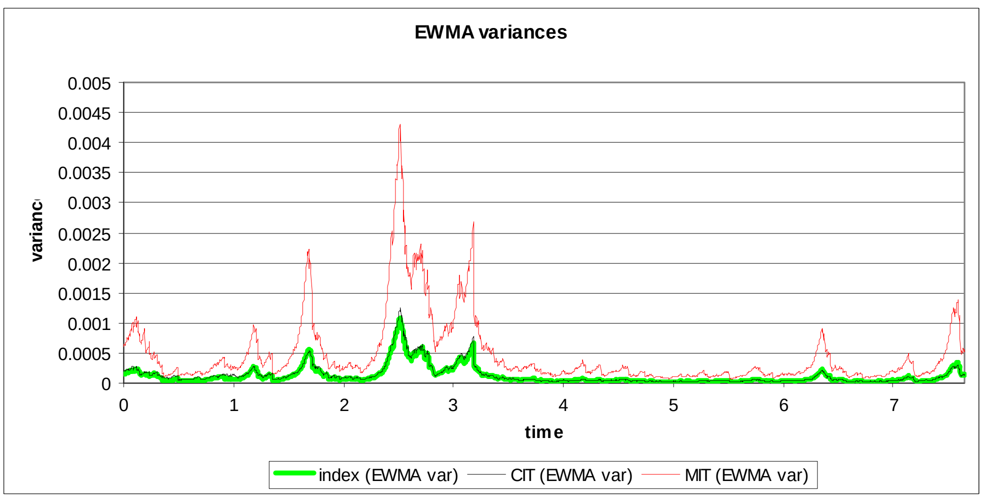

EWMA variances
0.001
0.00075
0.0005
0.00025
0
varianc
0
1
2
3
4
5
6
7
8
time
index (EWMA var)
CIT (EWMA var)
MIT (EWMA var)

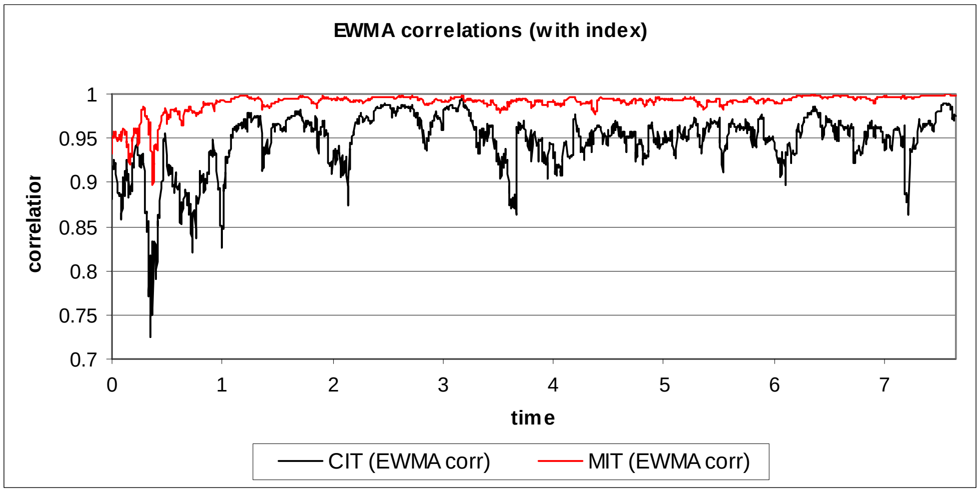

EWMA correlations (with index)
1
0.9
0.8
0.7
0.6
correlatio
0
1
2
3
4
5
6
7
8
time
CIT (EWMA corr)
MIT (EWMA corr)

6.6 Enhanced Indexation

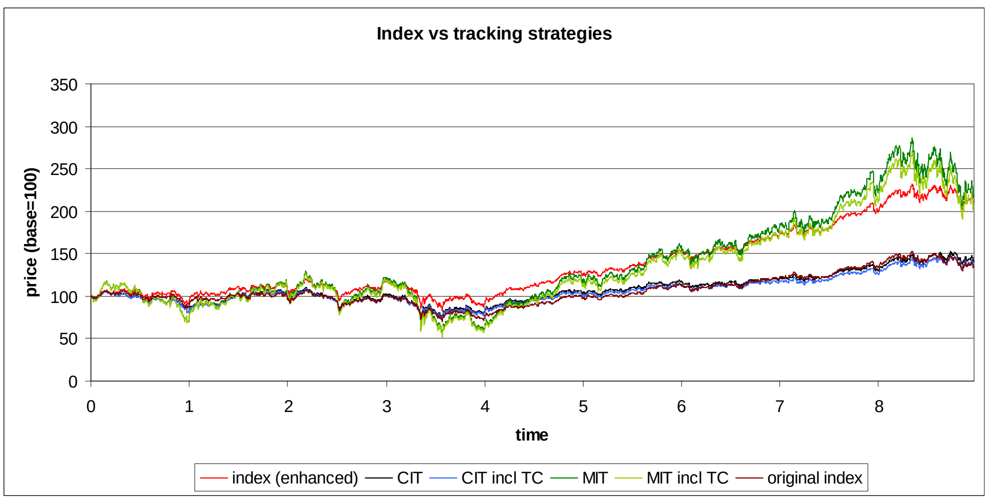
6.6.1 FTSE100 (+5%)

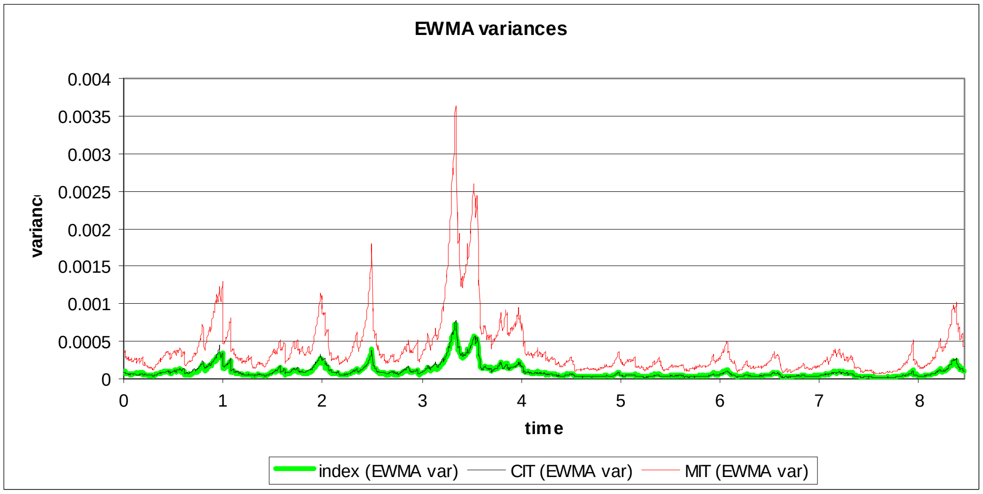


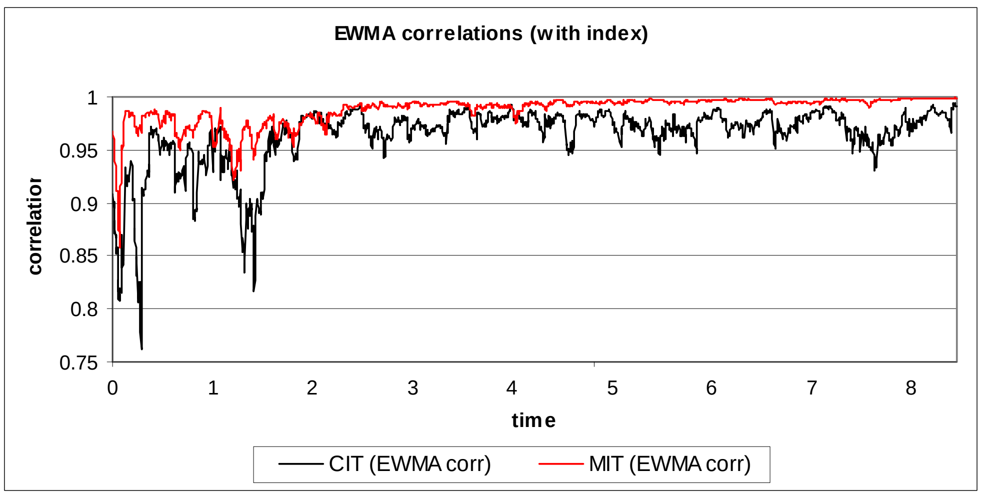



EWMA correlations (with index)
correlation
time
CIT (EWMA corr)
MIT (EWMA corr)

6.6.2 DJCA (+5%)


Index vs tracking strategies
price (base=100)
time
index (enhanced)
CIT
CIT incl TC
MIT
MIT incl TC
original index





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