第四章不定积分

习题四

4. 1

1．求下列不定积分．

(1) $\int x^2\sqrt[3]{x}\,\mathrm{d}x$；

解 $\int x^2\sqrt[3]{x}\,\mathrm{d}x=\dfrac{1}{\dfrac{7}{3}+1}x^{\frac{7}{3}+1}+C=\dfrac{3}{10}x^{\frac{10}{3}}+C$

(2) $\int \dfrac{(1-x)^2}{\sqrt{x}}\mathrm{d}x$；

解

$$\int \frac{(1-x)^2}{\sqrt{x}}\mathrm{d}x=\int\left(x^{\frac{3}{2}}-2x^{\frac{1}{2}}+x^{-\frac{1}{2}}\right)\mathrm{d}x=\int x^{\frac{3}{2}}\mathrm{d}x-2\int x^{\frac{1}{2}}\mathrm{d}x+\int x^{-\frac{1}{2}}\mathrm{d}x=\frac{2}{5}x^{\frac{5}{2}}-\frac{4}{3}x^{\frac{3}{2}}+2x^{\frac{1}{2}}+C$$

(3) $\int\left(2\mathrm{e}^x+\dfrac{3}{x}\right)\mathrm{d}x$；

解 $\int\left(2\mathrm{e}^x+\dfrac{3}{x}\right)\mathrm{d}x=2\int \mathrm{e}^x dx+3\int\dfrac{\mathrm{d}x}{x}=2\mathrm{e}^x+3\ln|x|+C$

(4) $\int\dfrac{2\cdot 3^x-5\cdot 2^x}{3^x}\mathrm{d}x$；

解 $\int\dfrac{2\cdot 3^x-5\cdot 2^x}{3^x}\mathrm{d}x=\int\left[2-5\left(\dfrac{2}{3}\right)^x\right]\mathrm{d}x=2\int\mathrm{d}x-5\int\left(\dfrac{2}{3}\right)^x\mathrm{d}x=2x-\dfrac{5}{\ln\dfrac{2}{3}}\left(\dfrac{2}{3}\right)^x+C$

(5) $\int\cot^2 x\,\mathrm{d}x$；

解 $\int\cot^2 x\,\mathrm{d}x=\int(\csc^2 x-1)\mathrm{d}x=\int\csc^2 x\,\mathrm{d}x-\int dx=-\cot x-x+C$

(6) $\int\dfrac{1+\cos^2 x}{1+\cos 2x}\mathrm{d}x$；

解 $\int\dfrac{1+\cos^2 x}{1+\cos 2x}\mathrm{d}x=\int\dfrac{1+\cos^2 x}{2\cos^2 x}\mathrm{d}x=\dfrac{1}{2}\int\dfrac{1}{\cos^2 x}\mathrm{d}x+\dfrac{1}{2}\int\mathrm{d}x=\dfrac{1}{2}\tan x+\dfrac{x}{2}+C$

(7) $\int \frac{\sqrt{1+x^2}}{\sqrt{1-x^4}}\mathrm{d}x$；

解 $\int \frac{\sqrt{1+x^2}}{\sqrt{1-x^4}}\mathrm{d}x = \int \frac{\sqrt{1+x^2}}{\sqrt{(1-x^2)(1+x^2)}}\mathrm{d}x = \int \frac{\mathrm{d}x}{\sqrt{(1-x^2)}} = \arcsin x + C$

(8) $\int \frac{3x^4+2x^2}{x^2+1}\mathrm{d}x$.

解

$$\int \frac{3x^4+2x^2}{x^2+1}\mathrm{d}x = \int\left(3x^2-1+\frac{1}{x^2+1}\right)\mathrm{d}x = 3\int x^2\mathrm{d}x - \int \mathrm{d}x + \int \frac{\mathrm{d}x}{x^2+1} = x^3 - x + \arctan x + C$$

2. 一曲线通过点 $(\mathrm{e}^2,3)$，且在任一点处的切线的斜率等于该点横坐标的倒数，求该曲线的方程.

解 由题设得

$$y' = \frac{1}{x}$$

所以

$$y = \int \frac{\mathrm{d}x}{x} = \ln|x| + C$$

又因为曲线过点 $(\mathrm{e}^2,3)$，所以曲线定义在 $x>0$ 上，且

$$3 = \ln \mathrm{e}^2 + C$$

解得 $C=1$，故所求曲线方程为

$$y = \ln x + 1$$

3. 一物体由静止开始运动，经过 t s 后的速度是 $3t^2$m/s，问：

(1) 在 3s 后物体离开出发点的距离是多少？

(2) 物体走完 360m 需要多长时间？

解 设路程函数为 $s(t)$，则

$$s'(t) = 3t^2$$

所以

$$s(t) = \int 3t^2 \mathrm{d}t = t^3 + C$$

又 $s(0)=0$，所以 $C=0$，故

$$s(t)=t^3$$

(1) $s(3)=3^3=27\text{m}$

(2) 由 $s(t)=t^3=360$ 得

$$t=\sqrt[3]{360}\approx 7.1\text{s}$$

4. 证明：函数 $\arcsin(2x-1),\arccos(1-2x)$ 和 $\arctan\sqrt{\dfrac{x}{1-x}}$ 都是 $\dfrac{1}{\sqrt{x-x^2}}$ 的原函数.

证 因为

$$[\arcsin(2x-1)]'=\frac{1}{\sqrt{1-(2x-1)^2}}\cdot 2=\frac{1}{\sqrt{x-x^2}}$$

$$[\arccos(1-2x)]'=-\frac{1}{\sqrt{1-(1-2x)^2}}\cdot(-2)=\frac{1}{\sqrt{x-x^2}}$$

$$\left[\arctan\sqrt{\frac{x}{1-x}}\right]'=2\frac{1}{1+\left(\sqrt{\dfrac{x}{1-x}}\right)^2}\cdot\frac{1}{2}\frac{1}{\sqrt{\dfrac{x}{1-x}}}\cdot\frac{1-x-x(-1)}{(1-x)^2}=\frac{1}{\sqrt{x-x^2}}$$

所以结论成立.

4.2

1. 用第一类换元积分法计算下列不定积分.

(1) $\displaystyle\int\frac{\mathrm{d}x}{1-2x}$；

解 $\displaystyle\int\frac{\mathrm{d}x}{1-2x}=-\frac{1}{2}\int\frac{1}{1-2x}(1-2x)'\,\mathrm{d}x=-\frac{1}{2}\int\frac{1}{1-2x}\mathrm{d}(1-2x)=-\frac{1}{2}\ln|1-2x|+C$

(2) $\displaystyle\int(3x+2)^{100}\mathrm{d}x$；

解

$$\int(3x+2)^{100}\mathrm{d}x=\frac{1}{3}\int(3x+2)^{100}\cdot(3x+2)'\,\mathrm{d}x$$

$$=\frac{1}{3}\int(3x+2)^{100}\mathrm{d}(3x+2)=\frac{1}{3}\cdot\frac{1}{101}(3x+2)^{101}+C=\frac{1}{303}(3x+2)^{101}+C$$

(3) $\displaystyle\int x\mathrm{e}^{-x^2}\mathrm{d}x$；

解 $\displaystyle\int x\mathrm{e}^{-x^2}\mathrm{d}x=-\frac{1}{2}\int\mathrm{e}^{-x^2}\left(-x^2\right)'\mathrm{d}x=-\frac{1}{2}\int\mathrm{e}^{-x^2}\mathrm{d}\left(-x^2\right)=-\frac{1}{2}\mathrm{e}^{-x^2}+C$

(4) $\int \frac{\sin \lg x}{x} \mathrm{d}x$；

解 $\int \frac{\sin \lg x}{x} \mathrm{d}x = \ln 10 \int \sin \lg x \cdot (\lg x)' \mathrm{d}x = \ln 10 \int \sin \lg x \mathrm{d}(\lg x) = -\ln 10 \cos \lg x + C$

(5) $\int \sqrt{\frac{\arcsin x}{1-x^2}} \mathrm{d}x$；

解

$$\int \sqrt{\frac{\arcsin x}{1-x^2}} \mathrm{d}x = \int \sqrt{\arcsin x} \cdot (\arcsin x)' \mathrm{d}x = \int \sqrt{\arcsin x}\, \mathrm{d}(\arcsin x)$$

$$= \frac{1}{\frac{1}{2}+1} (\arcsin x)^{\frac{1}{2}+1} + C = \frac{2}{3} (\arcsin x)^{\frac{3}{2}} + C$$

(6) $\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} \mathrm{d}x \ (a > 0)$；

解

$$\int \frac{\sqrt{x}}{\sqrt{a^3 - x^3}} \mathrm{d}x = \frac{2}{3} \int \frac{1}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - \left(x^{\frac{3}{2}}\right)^2}} \left(x^{\frac{3}{2}}\right)' \mathrm{d}x$$

$$= \frac{2}{3} \int \frac{1}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - \left(x^{\frac{3}{2}}\right)^2}} \mathrm{d}\left(x^{\frac{3}{2}}\right) = \frac{2}{3} \arcsin \frac{x^{\frac{3}{2}}}{a^{\frac{3}{2}}} + C = \frac{2}{3} \arcsin \left(\frac{x}{a}\right)^{\frac{3}{2}} + C$$

(7) $\int \frac{3^x}{1+9^x} \mathrm{d}x$；

解 $\int \frac{3^x}{1+9^x} \mathrm{d}x = \frac{1}{\ln 3} \int \frac{1}{1+\left(3^x\right)^2} \cdot \left(3^x\right)' \mathrm{d}x = \frac{1}{\ln 3} \int \frac{1}{1+\left(3^x\right)^2} \mathrm{d}\left(3^x\right) = \frac{1}{\ln 3} \arctan\left(3x\right) + C$

(8) $\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} \mathrm{d}x$；

解

$$\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}} \mathrm{d}x = \int \frac{1}{\sqrt[3]{\sin x - \cos x}} (\sin x - \cos x)' dx$$

$$= \int (\sin x - \cos x)^{-\frac{1}{3}} \mathrm{d}(\sin x - \cos x) = \frac{1}{-\frac{1}{3}+1} (\sin x - \cos x)^{-\frac{1}{3}+1} + C = \frac{3}{2} (\sin x - \cos x)^{\frac{2}{3}} + C$$

(9) $\int \frac{1+\sin 3x}{\cos^2 3x}\mathrm{d}x$；

解

$$\int \frac{1+\sin 3x}{\cos^2 3x}\mathrm{d}x = \int \frac{1}{\cos^2 3x}\mathrm{d}x + \int \frac{\sin 3x}{\cos^2 3x}\mathrm{d}x$$

$$= \frac{1}{3}\int \frac{1}{\cos^2 3x}\mathrm{d}(3x) - \frac{1}{3}\int \frac{1}{\cos^2 3x}\mathrm{d}(\cos 3x) = \frac{1}{3}\tan 3x + \frac{1}{3}\frac{1}{\cos 3x} + C$$

(10) $\int \frac{\mathrm{d}x}{x\ln x \ln\ln x}$；

解 $\int \frac{1}{x\ln x \ln\ln x}\mathrm{d}x = \int \frac{1}{\ln x \ln\ln x}\mathrm{d}(\ln x) = \int \frac{1}{\ln\ln x}\mathrm{d}(\ln\ln x) = \ln\left|\ln\ln x\right| + C$

(11) $\int \frac{\ln\tan x}{\cos x \sin x}\mathrm{d}x$；

解

$$\int \frac{\ln\tan x}{\cos x\sin x}\mathrm{d}x = \int \frac{\ln\tan x}{\tan x}\cdot\frac{1}{\cos^2 x}\mathrm{d}x$$

$$= \int \frac{\ln\tan x}{\tan x}\mathrm{d}(\tan x) = \int \ln\tan x\,\mathrm{d}(\ln\tan x) = \frac{1}{2}(\ln\tan x)^2 + C$$

(12) $\int \tan^3\frac{x}{3}\sec^2\frac{x}{3}\mathrm{d}x$；

解 $\int \tan^3\frac{x}{3}\sec^2\frac{x}{3}\mathrm{d}x = 3\int \tan^3\frac{x}{3}\sec^2\frac{x}{3}\mathrm{d}\left(\frac{x}{3}\right) = 3\int \tan^3\frac{x}{3}\mathrm{d}\left(\tan\frac{x}{3}\right) = \frac{3}{4}\tan^4\frac{x}{3} + C$

(13) $\int \cos x\cos\frac{x}{2}\mathrm{d}x$；

解

$$\int \cos x\cos\frac{x}{2}\mathrm{d}x = \int \frac{1}{2}\left(\cos\frac{x}{2} + \cos\frac{3x}{2}\right)\mathrm{d}x = \frac{1}{2}\int \cos\frac{x}{2}\mathrm{d}x + \frac{1}{2}\int \cos\frac{3x}{2}\mathrm{d}x$$

$$= \int \cos\frac{x}{2}\mathrm{d}\left(\frac{x}{2}\right) + \frac{1}{3}\int \cos\frac{3x}{2}\mathrm{d}\left(\frac{3x}{2}\right) = \sin\frac{x}{2} + \frac{1}{3}\sin\frac{3x}{2} + C$$

(14) $\int \sec^4 x\mathrm{d}x$；

解 $\int \sec^4 x\mathrm{d}x = \int \sec^2 x\sec^2 x\mathrm{d}x = \int \left(1+\tan^2 x\right)\mathrm{d}\tan x = \tan x + \frac{1}{3}\tan^3 x + C$

(15) $\int \tan^3 x\sec x\mathrm{d}x$；

解 $\int \tan^3 x\sec x\mathrm{d}x = \int \tan^2 x(\sec x\tan x)\mathrm{d}x = \int \left(\sec^2 x - 1\right)\mathrm{d}\sec x = \frac{1}{3}\sec^3 x - \sec x + C$

(16) $\int \frac{\sin 2x}{\sqrt{1-\cos^4 x}}\mathrm{d}x$；

解

$$\int \frac{\sin 2x}{\sqrt{1-\cos^4 x}}\mathrm{d}x = \int \frac{2\sin x\cos x}{\sqrt{1-\cos^4 x}}\mathrm{d}x = -\int \frac{2\cos x}{\sqrt{1-\cos^4 x}}\mathrm{d}\cos x$$
$$= -\int \frac{1}{\sqrt{1-\cos^4 x}}\mathrm{d}\left(\cos^2 x\right) = -\arcsin\left(\cos^2 x\right)+C$$

(17) $\int \frac{1}{1+\sin x}\mathrm{d}x$；

解

$$\int \frac{1}{1+\sin x}\mathrm{d}x = \int \frac{1-\sin x}{1-\sin^2 x}\mathrm{d}x = \int \frac{1-\sin x}{\cos^2 x}\mathrm{d}x$$
$$= \int \frac{\mathrm{d}x}{\cos^2 x} - \int \frac{\sin x}{\cos^2 x}\mathrm{d}x = \tan x + \int \frac{1}{\cos^2 x}\mathrm{d}\cos x = \tan x - \frac{1}{\cos x} + C$$

(18) $\int \frac{x-1}{x^2+2x+3}\mathrm{d}x$.

解

$$\int \frac{x-1}{x^2+2x+3}\mathrm{d}x = \int \frac{x+1-2}{x^2+2x+3}\mathrm{d}x = \int \frac{x+1}{x^2+2x+3}\mathrm{d}x - 2\int \frac{\mathrm{d}x}{x^2+2x+3}$$
$$= \frac{1}{2}\int \frac{1}{x^2+2x+3}\mathrm{d}\left(x^2+2x+3\right) - 2\int \frac{1}{(x+1)^2+\left(\sqrt{2}\right)^2}\mathrm{d}(x+1)$$
$$= \frac{1}{2}\ln\left(x^2+2x+3\right) - 2\cdot\frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}} + C = \frac{1}{2}\ln\left(x^2+2x+3\right) - \sqrt{2}\arctan\frac{x+1}{\sqrt{2}} + C$$

2. 用第二类换元积分法计算下列不定积分.

(1) $\int \frac{\mathrm{d}x}{1+\sqrt{2x}}$；

解　令 $t=\sqrt{2x}$，则 $x=\frac{t^2}{2}, \mathrm{d}x = t\mathrm{d}t$，所以

$$\int \frac{\mathrm{d}x}{1+\sqrt{2x}} = \int \frac{t\mathrm{d}t}{1+t} = \int\left(1-\frac{1}{1+t}\right)\mathrm{d}t = \int \mathrm{d}t - \int \frac{1}{1+t}\mathrm{d}(1+t)$$
$$= t - \ln(1+t) + C = \sqrt{2x} - \ln\left(1+\sqrt{2x}\right) + C$$

(2) $\int \frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}}$；

解　令 $t=\sqrt{1+\mathrm{e}^x}$，则 $x=\ln\left(t^2-1\right), \mathrm{d}x = \frac{2t}{t^2-1}\mathrm{d}t$，所以

$$\int \frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}} = \int \frac{\frac{2t}{t^2-1}\mathrm{d}t}{t} = 2\int \frac{\mathrm{d}t}{t^2-1} = 2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \ln\frac{\sqrt{1+\mathrm{e}^x}-1}{\sqrt{1+\mathrm{e}^x}+1} + C$$

(3) $\int \frac{x^2}{\sqrt{a^2-x^2}}\mathrm{d}x\ (a>0)$；

解 令 $x = a\sin t\left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$，则 $\mathrm{d}x = a\cos t\mathrm{d}t$，所以

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\mathrm{d}x = \int \frac{(a\sin t)^2}{\sqrt{a^2-(a\sin t)^2}}\cdot a\cos t\mathrm{d}t = \int \frac{a^2\sin^2 t}{a\cos t}\cdot a\cos t\mathrm{d}t$$

$$= a^2\int \sin^2 t\mathrm{d}t = a^2\int \frac{1-\cos 2t}{2}\mathrm{d}t = \frac{a^2}{2}\left(t - \frac{1}{2}\sin 2t\right) + C = \frac{a^2}{2}\arcsin\frac{x}{a} - \frac{x}{2}\sqrt{a^2-x^2} + C$$

(4) $\int \frac{\sqrt{x^2+a^2}}{x^2}\mathrm{d}x\,(a>0)$；

解 令 $x = a\tan t\left(-\frac{\pi}{2} < t < \frac{\pi}{2}\right)$，则 $\mathrm{d}x = a\sec^2 t\mathrm{d}t$，所以

$$\int \frac{\sqrt{x^2+a^2}}{x^2}\mathrm{d}x = \int \frac{\sqrt{(a\tan t)^2+a^2}}{(a\tan t)^2}\cdot a\sec^2 t\mathrm{d}t = \int \frac{a\sec t}{a^2\tan^2 t}\cdot a\sec^2 t\mathrm{d}t = \int \frac{1}{\sin^2 t\cos t}\mathrm{d}t$$

$$= \int \frac{\sin^2 t + \cos^2 t}{\sin^2 t\cos t}\mathrm{d}t = \int \left(\frac{1}{\cos t} + \frac{\cos t}{\sin^2 t}\right)\mathrm{d}t = \int \sec t\mathrm{d}t + \int \frac{1}{\sin^2 t}\mathrm{d}\sin t$$

$$= \ln\left|\sec t + \tan t\right| - \frac{1}{\sin t} + C = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{\sqrt{x^2+a^2}}{x} + C_1$$

$$= \ln\left|x + \sqrt{x^2+a^2}\right| - \frac{\sqrt{x^2+a^2}}{x} + C\ (C = C_1 - \ln a)$$

(5) $\int \frac{\sqrt{x^2-9}}{x}\mathrm{d}x$；

解 令 $x = 3\sec t\left(0 \le t < \frac{\pi}{2}\text{或}\pi \le t < \frac{3\pi}{2}\right)$，则 $\mathrm{d}x = 3\sec t\tan t\mathrm{d}t$，所以

$$\int \frac{\sqrt{x^2-9}}{x}\mathrm{d}x = \int \frac{\sqrt{(3\sec t)^2-9}}{3\sec t}\cdot 3\sec t\tan t\mathrm{d}t = \int \frac{3\tan t}{3\sec t}\cdot 3\sec t\tan t\mathrm{d}t = 3\int \tan^2 t\mathrm{d}t$$

$$= 3\int\left(\sec^2 t - 1\right)\mathrm{d}t = 3\tan t - 3t + C = \sqrt{x^2-9} - 3\arccos\frac{3}{x} + C$$

(6) $\int \frac{dx}{x+\sqrt{1-x^2}}$；

解 令 $x=a\sin t\left(-\frac{\pi}{2}\le t\le\frac{\pi}{2}\right)$，则 $dx=a\cos t dt$，所以

$$\begin{aligned}
&\int \frac{dx}{x+\sqrt{1-x^2}}=\int\frac{a\cos t dt}{\sin t+\sqrt{1-\sin^2 t}}\\
&=\int\frac{\cos t}{\sin t+\cos t}dt=\frac{1}{2}\int\frac{\cos t+\sin t+\cos t-\sin t}{\sin t+\cos t}dt\\
&=\frac{1}{2}\int\left(1+\frac{\cos t-\sin t}{\sin t+\cos t}\right)dt=\frac{1}{2}\int dt+\frac{1}{2}\int\frac{1}{\sin t+\cos t}d(\sin t+\cos t)\\
&=\frac{t}{2}+\frac{1}{2}\left|\ln\sin t+\cos t\right|+C=\frac{1}{2}\arcsin x+\frac{1}{2}\ln\left|x+\sqrt{1-x^2}\right|+C
\end{aligned}$$

(7) $\int \frac{1}{1+\sqrt{x^2+2x+2}}dx$.

解 $\int\frac{1}{1+\sqrt{x^2+2x+2}}dx=\int\frac{1}{1+\sqrt{(x+1)^2+1}}dx$

令 $x+1=\tan t\left(-\frac{\pi}{2}<t<\frac{\pi}{2}\right)$，则 $dx=\sec^2 t dt$，所以

$$\begin{aligned}
&\int\frac{1}{1+\sqrt{x^2+2x+2}}dx=\int\frac{\sec^2 t dt}{1+\sqrt{\tan^2 t+1}}=\int\frac{1}{\cos t(1+\cos t)}dt=\int\left(\frac{1}{\cos t}-\frac{1}{1+\cos t}\right)dt\\
&=\int\sec t dt-\int\frac{1}{\cos^2\frac{t}{2}}d\left(\frac{t}{2}\right)=\ln\left|\sec t+\tan t\right|-\tan\frac{t}{2}+C\\
&=\ln\left|x+1+\sqrt{x^2+2x+2}\right|-\frac{\sqrt{x^2+2x+2}-1}{x+1}+C
\end{aligned}$$

4.3

1．用分部积分法计算下列不定积分.

(1) $\int x\sin x dx$；

解 $\int x\sin x dx=\int x d\cos x=-x\cos x+\int\cos x dx=-x\cos x+\sin x+C$

(2) $\int xe^{-x}dx$；

解 $\int xe^{-x}dx=-\int x de^{-x}=-xe^{-x}+\int e^{-x}dx=-xe^{-x}-e^{-x}+C$

(3) $\int \arctan x dx$；

解

$$\int \arctan x\mathrm{d}x = x\arctan x - \int x\mathrm{d}(\arctan x)$$

$$= x\arctan x - \int x\cdot\frac{1}{1+x^2}\mathrm{d}x = x\arctan x - \frac{1}{2}\int\frac{1}{1+x^2}\mathrm{d}\left(1+x^2\right) = x\arctan x - \frac{1}{2}\ln\left(1+x^2\right)+C$$

(4) $\int x\ln(x-1)\mathrm{d}x$；

解

$$\int x\ln(x-1)\mathrm{d}x = \frac{1}{2}\int\ln(x-1)\mathrm{d}\left(x^2-1\right) = \frac{1}{2}\left[\left(x^2-1\right)\ln(x-1) - \int\left(x^2-1\right)\mathrm{d}(\ln(x-1))\right]$$

$$= \frac{1}{2}\left(x^2-1\right)\ln(x-1) - \frac{1}{2}\int\left(x^2-1\right)\cdot\frac{1}{x-1}\mathrm{d}x = \frac{1}{2}\left(x^2-1\right)\ln(x-1) - \frac{1}{2}\int(x+1)\mathrm{d}x$$

$$= \frac{1}{2}\left(x^2-1\right)\ln(x-1) - \frac{x^2}{4} - \frac{x}{2} + C$$

(5) $\int \ln^2 x\mathrm{d}x$；

解

$$\int\ln^2 x\mathrm{d}x = x\ln^2 x - \int x\mathrm{d}\left(\ln^2 x\right) = x\ln^2 x - \int x\cdot 2\ln x\cdot\frac{1}{x}\mathrm{d}x$$

$$= x\ln^2 x - 2\int\ln x\mathrm{d}x = x\ln^2 x - 2\left[x\ln x - \int x\mathrm{d}\ln x\right]$$

$$= x\ln^2 x - 2x\ln x + 2\int x\cdot\frac{1}{x}\mathrm{d}x = x\ln^2 x - 2x\ln x + 2x + C$$

(6) $\int\frac{\arcsin x}{\sqrt{1+x}}\mathrm{d}x$；

解

$$\int\frac{\arcsin x}{\sqrt{1+x}} = 2\int\arcsin x\,\mathrm{d}\left(\sqrt{1+x}\right) = 2\left[\sqrt{1+x}\arcsin x - \int\sqrt{1+x}\mathrm{d}(\arcsin x)\right]$$

$$= 2\sqrt{1+x}\arcsin x - 2\int\sqrt{1+x}\frac{1}{\sqrt{1-x^2}}\mathrm{d}x = 2\sqrt{1+x}\arcsin x - 2\int\frac{1}{\sqrt{1-x}}\mathrm{d}x$$

$$= 2\sqrt{1+x}\arcsin x + 2\int\frac{1}{\sqrt{1-x}}\mathrm{d}(1-x) = 2\sqrt{1+x}\arcsin x + 4\sqrt{1-x} + C$$

(7) $\int\frac{x}{\sin^2 x}\mathrm{d}x$；

解 $\int\frac{x}{\sin^2 x}\mathrm{d}x = -\int x\mathrm{d}\cot x = -x\cot x + \int\cot x\mathrm{d}x = -x\cot x + \ln|\sin x| + C$

(8) $\int\mathrm{e}^{-2x}\sin\frac{x}{2}\mathrm{d}x$；

解

$$\int e^{-2x}\sin\frac{x}{2}dx=-\frac{1}{2}\int\sin\frac{x}{2}d\left(e^{-2x}\right)=-\frac{1}{2}\left[e^{-2x}\sin\frac{x}{2}-\int e^{-2x}\cdot\cos\frac{x}{2}\cdot\frac{1}{2}dx\right]$$

$$=-\frac{1}{2}e^{-2x}\sin\frac{x}{2}-\frac{1}{8}\int\cos\frac{x}{2}d\left(e^{-2x}\right)=-\frac{1}{2}e^{-2x}\sin\frac{x}{2}-\frac{1}{8}\left[e^{-2x}\cos\frac{x}{2}-\int e^{-2x}\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2}dx\right]$$

$$=-\frac{1}{8}\left(4\sin\frac{x}{2}+\cos\frac{x}{2}\right)e^{-2x}-\frac{1}{16}\int e^{-2x}\sin\frac{x}{2}dx$$

解得

$$\int e^{-2x}\sin\frac{x}{2}dx=-\frac{2}{17}\left(4\sin\frac{x}{2}+\cos\frac{x}{2}\right)e^{-2x}+C$$

(9) $\int x^2\cos^2\frac{x}{2}dx$；

解

$$\int x^2\cos^2\frac{x}{2}dx=\int x^2\cdot\frac{1+\cos x}{2}dx=\frac{1}{2}\int x^2dx+\frac{1}{2}\int x^2\cos xdx$$

$$=\frac{x^3}{6}+\frac{1}{2}\int x^2d\sin x=\frac{x^3}{6}+\frac{1}{2}\left[x^2\sin x-\int\sin x\cdot 2xdx\right]$$

$$=\frac{x^3}{6}+\frac{x^2}{2}\sin x+\int xd\cos x=\frac{x^3}{6}+\frac{x^2}{2}\sin x+x\cos x-\int x\cos xdx$$

$$=\frac{x^3}{6}+\frac{x^2}{2}\sin x+x\cos x-\sin x+C$$

(10) $\int e^{\sqrt[3]{x}}dx$；

解 令 $t=\sqrt[3]{x}$， 则 $x=t^3,dx=3t^2dt$， 所以

$$\int e^{\sqrt[3]{x}}dx=\int e^t\cdot 3t^2dt=\int 3t^2de^t=3t^2e^t-\int e^t\cdot 6tdt=3t^2e^t-\int 6tde^t$$

$$=3t^2e^t-\left[6te^t-\int e^t\cdot 6dt\right]=3t^2e^t-6te^t+6e^t+C=3e^{\sqrt[3]{x}}\left(x^{\frac{2}{3}}-2x^{\frac{1}{3}}+2\right)+C$$

(11) $\int\cos\ln xdx$.

解

$$\int\cos\ln xdx=x\cos\ln x-\int x\cdot(-\sin\ln x)\cdot\frac{1}{x}dx=x\cos\ln x+\int\sin\ln xdx$$

$$=x\cos\ln x+x\sin\ln x-\int x\cos\ln x\cdot\frac{1}{x}dx=x(\cos\ln x+\sin\ln x)-\int\cos\ln xdx$$

解得

$$\int\cos\ln xdx=\frac{x}{2}(\cos\ln x+\sin\ln x)+C$$

2. 已知 $(1+\sin x)\ln x$ 是 $f(x)$ 的一个原函数，求 $\int xf'(x)\mathrm{d}x$.

解　由题设知

$$f'(x)=\left[(1+\sin x)\ln x\right]'=\frac{1}{x}(1+\sin x)+\cos x\ln x$$

所以

$$\int xf'(x)\mathrm{d}x=\int x\mathrm{d}f(x)=xf(x)-\int f(x)\mathrm{d}x$$

$$=x\left[\frac{1}{x}(1+\sin x)+\cos x\ln x\right]-(1+\sin x)\ln x+C=x\cos x\ln x+(1+\sin x)(1-\ln x)+C$$

3. 当 $x\ge 0$ 时，$F(x)$ 是 $f(x)$ 的一个原函数，已知 $f(x)F(x)=\sin^2 2x$，且，

$F(0)=1$，$F(x)\ge 0$，求 $f(x)$.

解　由题设知 $F'(x)=f(x)$，所以

$$f(x)F(x)=F'(x)F(x)=\sin^2 2x$$

于是

$$\int F'(x)F(x)\mathrm{d}x=\int \sin^2 2x\mathrm{d}x$$

由此得

$$\frac{1}{2}F^2(x)=\int\frac{1-\cos 4x}{2}\mathrm{d}x=\frac{1}{2}\int\mathrm{d}x-\frac{1}{2}\cos 4x\mathrm{d}x=\frac{x}{2}-\frac{1}{8}\sin 4x+C$$

由 $F(0)=1$ 得　$C=\frac{1}{2}$，所以

$$F^2(x)=x-\frac{1}{4}\sin 4x+1$$

解得

$$F(x)=\frac{\sqrt{4x-\sin 4x+4}}{2}$$

从而

$$f(x)=\frac{\sin^2 2x}{F(x)}=\frac{2\sin^2 2x}{\sqrt{4x-\sin 4x+4}}$$

4.4

1. 计算下列有理函数的不定积分.

(1) $\int\frac{x^3}{x+3}\mathrm{d}x$；

解 $\int \frac{x^3}{x+3}\mathrm{d}x = \int\left(x^2-3x+9-\frac{27}{x+3}\right)\mathrm{d}x = \frac{x^3}{3}-\frac{3}{2}x^2+9x-27\ln|x+3|+C$

(2) $\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x$；

解

$$\int \frac{x^2+1}{(x+1)^2(x-1)}\mathrm{d}x = \int\left[\frac{1}{2}\frac{1}{x-1}+\frac{1}{2}\frac{1}{x+1}-\frac{1}{(x+1)^2}\right]\mathrm{d}x$$
$$=\frac{1}{2}\ln|x-1|+\frac{1}{2}\ln|x+1|+\frac{1}{x+1}+C=\frac{1}{2}\ln|x^2-1|+\frac{1}{x+1}+C$$

(3) $\int \frac{\mathrm{d}x}{x^4-1}$；

解

$$\int \frac{\mathrm{d}x}{x^4-1} = \int \frac{\mathrm{d}x}{(x-1)(x+1)(x^2+1)} = \int\left(\frac{1}{4}\frac{1}{x-1}-\frac{1}{4}\frac{1}{x+1}-\frac{1}{2}\frac{1}{x^2+1}\right)\mathrm{d}x$$
$$=\frac{1}{4}\ln|x-1|-\frac{1}{4}\ln|x+1|-\frac{1}{2}\arctan x+C=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+C$$

(4) $\int \frac{-x^2-2}{(x^2+x+1)^2}\mathrm{d}x$；

解

$$\int \frac{-x^2-2}{(x^2+x+1)^2}\mathrm{d}x = \int\left(-\frac{1}{x^2+x+1}+\frac{x-1}{(x^2+x+1)^2}\right)\mathrm{d}x$$
$$=-\int \frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\mathrm{d}\left(x+\frac{1}{2}\right)+\frac{1}{2}\int\frac{1}{(x^2+x+1)^2}\mathrm{d}(x^2+x+1)-\frac{3}{2}\int\frac{1}{\left[\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]^2}\mathrm{d}\left(x+\frac{1}{2}\right)$$
$$=-\frac{2}{\sqrt{3}}\arctan\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}-\frac{1}{2}\frac{1}{x^2+x+1}-\frac{3}{2}\left[\frac{2}{3}\frac{x+\frac{1}{2}}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}+\frac{2}{3}\int\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\mathrm{d}\left(x+\frac{1}{2}\right)\right]$$
$$=-\frac{x+1}{x^2+x+1}-\frac{4}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C$$

2．计算下列三角函数有理式的不定积分.

(1) $\int \frac{\mathrm{d}x}{3+\cos x}$；

解　令 $u=\tan\frac{x}{2}(-\pi<x<\pi)$，则 $x=2\arctan u$，$\mathrm{d}x=\frac{2}{1+u^2}\mathrm{d}u,\cos x=\frac{1-u^2}{1+u^2}$，所以

$$\int \frac{\mathrm{d}x}{3+\cos x}=\int \frac{\frac{2}{1+u^2}\mathrm{d}u}{3+\frac{1-u^2}{1+u^2}}=\int \frac{\mathrm{d}u}{u^2+2}=\frac{1}{\sqrt{2}}\arctan\frac{u}{\sqrt{2}}+C=\frac{1}{\sqrt{2}}\arctan\frac{\tan\frac{x}{2}}{\sqrt{2}}+C$$

(2) $\int \frac{\mathrm{d}x}{2\sin x-\cos x+5}$.

解　令 $u=\tan\frac{x}{2}(-\pi<x<\pi)$，则 $x=2\arctan u$，$\mathrm{d}x=\frac{2}{1+u^2}du,\sin x=\frac{2u}{1+u^2}$，$\cos x=\frac{1-u^2}{1+u^2}$，所以

$$\int \frac{\mathrm{d}x}{2\sin x-\cos x+5}=\int \frac{\frac{2}{1+u^2}\mathrm{d}u}{2\frac{2u}{1+u^2}-\frac{1-u^2}{1+u^2}+5}=\int \frac{\mathrm{d}u}{3u^2+2u+2}=\frac{1}{3}\int \frac{1}{\left(u+\frac{1}{3}\right)^2+\left(\frac{\sqrt{5}}{3}\right)^2}\mathrm{d}\left(u+\frac{1}{3}\right)$$

$$=\frac{1}{\sqrt{5}}\arctan\frac{u+\frac{1}{3}}{\frac{\sqrt{5}}{3}}+C=\frac{1}{\sqrt{5}}\arctan\left(\frac{3\tan\frac{x}{2}+1}{\sqrt{5}}\right)+C$$

3. 计算下列无理函数的不定积分.

(1) $\int \frac{x^{\frac{1}{3}}}{x^{\frac{3}{2}}+x^{\frac{4}{3}}}\mathrm{d}x$；

解　令 $t=\sqrt[6]{x}$，则 $x=t^6,dx=6t^5\mathrm{d}t$，所以

$$\int \frac{x^{\frac{1}{3}}}{x^{\frac{3}{2}}+x^{\frac{4}{3}}}\mathrm{d}x=\int \frac{t^2}{t^7+t^8}\cdot 6t^5\mathrm{d}t=6\int \frac{\mathrm{d}t}{t(t+1)}$$

$$=6\int\left(\frac{1}{t}-\frac{1}{t+1}\right)\mathrm{d}t=6\ln t-6\ln(1+t)+C=6\ln\frac{t}{t+1}+C=6\ln\frac{\sqrt[6]{x}}{\sqrt[6]{x}+1}+C$$

(2) $\int\sqrt{\frac{1-x}{1+x}}\frac{\mathrm{d}x}{x}$.

解 令 $t=\sqrt{\frac{1-x}{1+x}}$，则 $x=\frac{1-t^2}{1+t^2},\mathrm{d}x=-\frac{4t}{\left(1+t^2\right)^2}\mathrm{d}t$，所以

$$\int\sqrt{\frac{1-x}{1+x}}\frac{\mathrm{d}x}{x}=\int t\cdot\frac{-\frac{4t}{\left(1+t^2\right)^2}\mathrm{d}t}{\frac{1-t^2}{1+t^2}}=-4\int\frac{t^2}{\left(1-t^2\right)\left(1+t^2\right)}\mathrm{d}t=2\int\left(\frac{1}{1+t^2}-\frac{1}{1-t^2}\right)\mathrm{d}t$$

$$=2\arctan t+\ln\left|\frac{1-t}{1+t}\right|+C=2\arctan\sqrt{\frac{1-x}{1+x}}+\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right|+C$$

总习题四

1. 在下列等式中，正确的结果是（ ）

（A）$\int f'(x)\mathrm{d}x=f(x)$

（B）$\int \mathrm{d}f(x)=f(x)$

（C）$\frac{\mathrm{d}}{\mathrm{d}x}\int f(x)\mathrm{d}x=f(x)$

（D）$\mathrm{d}\int f(x)\mathrm{d}x=f(x)$

解 选 C.

2. 计算下列不定积分.

(1) $\int\frac{\cos\sqrt{x}-1}{\sqrt{x}\sin^2\sqrt{x}}\mathrm{d}x$；

解

$$\int\frac{\cos\sqrt{x}-1}{\sqrt{x}\sin^2\sqrt{x}}\mathrm{d}x=2\int\frac{\cos\sqrt{x}-1}{\sin^2\sqrt{x}}\mathrm{d}\sqrt{x}=2\int\frac{\cos\sqrt{x}}{\sin^2\sqrt{x}}\mathrm{d}\sqrt{x}-2\int\frac{1}{\sin^2\sqrt{x}}\mathrm{d}\sqrt{x}$$

$$=2\int\frac{1}{\sin^2\sqrt{x}}\mathrm{d}\left(\sin\sqrt{x}\right)-2\int\frac{1}{\sin^2\sqrt{x}}\mathrm{d}\sqrt{x}=-\frac{2}{\sin\sqrt{x}}+2\cot\sqrt{x}+C$$

(2) $\int\frac{x\ln x}{\left(1+x^2\right)^2}\mathrm{d}x$；

解

$$\int \frac{x\ln x}{\left(1+x^2\right)^2}\mathrm{d}x=\frac{1}{2}\int \frac{\ln x}{\left(1+x^2\right)^2}\mathrm{d}\left(1+x^2\right)=-\frac{1}{2}\int \ln x\mathrm{d}\left(\frac{1}{1+x^2}\right)$$
$$=-\frac{1}{2}\left[\frac{\ln x}{1+x^2}-\int \frac{1}{1+x^2}\cdot\frac{1}{x}\mathrm{d}x\right]=-\frac{1}{2}\frac{\ln x}{1+x^2}+\frac{1}{2}\int\left(\frac{1}{x}-\frac{x}{1+x^2}\right)\mathrm{d}x$$
$$=-\frac{1}{2}\frac{\ln x}{1+x^2}+\frac{1}{2}\ln|x|-\frac{1}{4}\ln\left(1+x^2\right)+C$$

(3) $\int \sqrt{x}\sin\sqrt{x}\mathrm{d}x$；

解　令 $t=\sqrt{x}$，则 $x=t^2,\mathrm{d}x=2t\mathrm{d}t$，所以

$$\int \sqrt{x}\sin\sqrt{x}\mathrm{d}x=\int t\sin t\cdot 2t\mathrm{d}t=-2\int t^2\mathrm{d}\cos t=-2\left[t^2\cos t-\int \cos t\cdot 2t\mathrm{d}t\right]$$
$$=-2t^2\cos t+4\int t\mathrm{d}\sin t=-2t^2\cos t+4\left[t\sin t-\int \sin t\mathrm{d}t\right]$$
$$=-2t^2\cos t+4t\sin t+4\cos t+C=-2x\cos\sqrt{x}+4\sqrt{x}\sin\sqrt{x}+4\cos\sqrt{x}+C$$

(4) $\int \frac{\ln x-1}{\ln^2 x}\mathrm{d}x$；

解

$$\int \frac{\ln x-1}{\ln^2 x}\mathrm{d}x=\int \frac{1}{\ln x}\mathrm{d}x-\int \frac{1}{\ln^2 x}\mathrm{d}x=\frac{1}{\ln x}\cdot x-\int x\mathrm{d}\left(\frac{1}{\ln x}\right)-\int \frac{1}{\ln^2 x}dx$$
$$=\frac{x}{\ln x}-\int x\left(-\frac{1}{\ln^2 x}\right)\cdot\frac{1}{x}\mathrm{d}x-\int \frac{dx}{\ln^2 x}=\frac{x}{\ln x}+\int \frac{\mathrm{d}x}{\ln^2 x}-\int \frac{\mathrm{d}x}{\ln^2 x}=\frac{x}{\ln x}+C$$

(5) $\int \frac{x^3\arccos x}{\sqrt{1-x^2}}\mathrm{d}x$.

解　令 $x=\cos t\left(0<t<\pi\right)$，则 $t=\arccos x,\mathrm{d}x=-\sin t\mathrm{d}t$，所以

$$\int \frac{x^3\arccos x}{\sqrt{1-x^2}}\mathrm{d}x=\int \frac{\cos^3 t\cdot t}{\sin t}\cdot(-\sin t)\mathrm{d}t=-\int t\cos^3 t\mathrm{d}t=-\int t\left(1-\sin^2 t\right)\mathrm{d}\sin t$$
$$=-\int t\mathrm{d}\left(\sin t-\frac{\sin^3 t}{3}\right)=-t\left(\sin t-\frac{\sin^3 t}{3}\right)+\int\left(\sin t-\frac{\sin^3 t}{3}\right)\mathrm{d}t$$
$$=-t\left(\sin t-\frac{\sin^3 t}{3}\right)-\frac{2}{3}\cos t-\frac{1}{9}\cos^3 t+C=-\frac{1}{3}\sqrt{1-x^2}\left(2+x^2\right)\arccos x-\frac{1}{9}x\left(x^2+6\right)+C$$

3. 设 n 为正整数，证明：递推公式

$$\int \sec^n xdx=\frac{1}{n-1}\cdot\tan x\sec^{n-2}x+\frac{n-2}{n-1}\int \sec^{n-2}x\mathrm{d}x\ \left(n\geq 2\right).$$

证

$$\int \sec^n x\mathrm{d}x = \int \sec^{n-2} x\mathrm{d}\tan x = \sec^{n-2} x\tan x - \int \tan x\mathrm{d}\left(\sec^{n-2} x\right)$$

$$= \sec^{n-2} x\tan x - \int \tan x(n-2)\sec^{n-3} x\cdot(\sec x\tan x)\mathrm{d}x = \sec^{n-2} x\tan x - (n-2)\int \tan^2 x\sec^{n-2} x\mathrm{d}x$$

$$= \sec^{n-2} x\tan x - (n-2)\int\left(\sec^2 x - 1\right)\sec^{n-2} x\mathrm{d}x$$

$$= \sec^{n-2} x\tan x - (n-2)\int \sec^n x\mathrm{d}x + (n-2)\int \sec^{n-2} x\mathrm{d}x$$

解得

$$\int \sec^n x\mathrm{d}x = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\mathrm{d}x$$