

## 微分算子法

$f(x)$ 形式	$y^*(x)$ 表达式	附注
① $f(x) = e^{kx}$	$y^*(x) = \frac{1}{F(D)} e^{kx} = \frac{1}{F(k)} e^{kx}$ , 其中 $F(k) \neq 0, F(k)$ 为 $F(D)$ 中的 D 用 $k$ 代替所得值.	若 $F(k) = 0$ , 不妨设 $k$ 为 $F(k)$ 的 $m$ 重根, 则 $\frac{1}{F(D)} e^{kx} = x^m \frac{1}{F^{(m)}(D)} e^{kx} = x^m \frac{1}{F^{(m)}(k)} e^{kx}$ , 其中 $F^{(m)}(D)$ 表示 $F(D)$ 对 D 的 $m$ 阶导数.
② $f(x) = \sin ax$ 或 $\cos ax$	$y^*(x) = \frac{1}{F(D^2)} \sin ax$ $= \frac{\sin ax}{F(-a^2)}$ , 或 $y^*(x) = \frac{1}{F(D^2)} \cos ax$ $= \frac{\cos ax}{F(-a^2)}$ , 其中 $F(-a^2) \neq 0$ .	若 $F(-a^2) = 0$ , 不妨设 $(-a^2)$ 为 $F(-a^2)$ 的 $m$ 重根, 则 $\frac{1}{F(D^2)} \sin ax = x^m \cdot \frac{1}{F^{(m)}(D^2)} \sin ax$ , $\frac{1}{F(D^2)} \cos ax = x^m \cdot \frac{1}{F^{(m)}(D^2)} \cos ax$ .
③ $f(x) = e^{kx} v(x)$	$y^*(x) = \frac{1}{F(D)} e^{kx} v(x)$ $= e^{kx} \frac{1}{F(D+k)} v(x)$ .	

④ $f(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m$	(i) 若 $p_n \neq 0$ , 则 $y^*(x) = \frac{1}{F(D)} (a_0 x^m + \dots + a_m)$ $= Q(D) (a_0 x^m + \dots + a_m)$ , 其中 $Q(D)$ 为 1 除以按升幂排列的 $F(D)$ 的商式, 其最高次数取到 $f(x)$ 的次数 $m$ .	(i) $\frac{1}{p_n} - \frac{p_{n-1}}{p_n^2} D + \dots$ $\frac{1}{p_n + p_{n-1} D + \dots + D^n} \sqrt{\frac{1}{1 + \frac{p_{n-1} D}{p_n} + \frac{p_{n-2} D^2}{p_n} + \dots} - \frac{p_{n-1} D}{p_n} - \frac{p_{n-2} D^2}{p_n} - \dots}$ 当商式中出现 D 的最高次数为 $m$ 时除法停止; $Q(D) = \frac{1}{p_n} - \frac{p_{n-1} D}{p_n} + \dots$ D 的 $m$ 次多项式
	(ii) 若 $p_n = 0$ , 则 $y^*(x) = \frac{1}{F(D)} (a_0 x^m + \dots + a_m) = \frac{1}{DF_1(D)} (a_0 x^m + \dots + a_m) = \frac{1}{D} Q_1(D) (a_0 x^m + \dots + a_m)$ , 其中 $Q_1(D)$ 为 $\frac{1}{F_1(D)}$ 的商式, 次数为 $m$ 次.	ii) $\frac{1}{p_{n-1}} - \frac{p_{n-2}}{p_{n-1}^2} D + \dots$ $\frac{1}{p_{n-1} + p_{n-2} D + \dots} \sqrt{\frac{1}{1 + \frac{p_{n-2} D}{p_{n-1}} + \dots} - \frac{p_{n-2} D}{p_{n-1}} - \dots}$ $Q_1(D) = \frac{1}{p_{n-1}} - \frac{p_{n-2} D}{p_{n-1}^2} + \dots$ D 的 $m$ 次多项式.

注 ① D 表示微分, 则  $\frac{1}{D}$  表示积分, 例  $\frac{1}{D} \cos x = \int \cos x dx = \sin x + C$ , 积分常数  $C$  不写!

$$\begin{aligned} ② \frac{1}{D^{2n+1}} \sin ax &= \frac{1}{D^{2n} D} \sin ax = \frac{1}{(-a^2)^n} \frac{1}{D} \sin ax = (-1)^{n+1} \frac{\cos ax}{a^{2n+1}}, \\ \frac{1}{D^{2n+1}} \cos ax &= \frac{1}{D^{2n} D} \cos ax = \frac{1}{(-a^2)^n} \frac{1}{D} \cos ax = (-1)^n \frac{\sin ax}{a^{2n+1}}, \\ \frac{1}{kD+b} \sin ax &= \frac{(kD-b)}{k^2 D^2 - b^2} \sin ax = \frac{(kD-b)}{k^2 (-a^2) - b^2} \sin ax = -\frac{1}{k^2 a^2 + b^2} (ka \cos ax - b \sin ax), \\ \frac{1}{kD+b} \cos ax &= \frac{(kD-b)}{(kD+b)(kD-b)} \cos ax = \frac{1}{k^2 a^2 + b^2} (ka \sin ax + b \cos ax). \end{aligned}$$

【例 6.15】解下列微分方程：

- (1)  $y'' + 4y' + 4y = e^{\alpha x}$ , 其中  $\alpha$  为实数;
- (2)  $y'' + \alpha^2 y = \sin x$ , 其中  $\alpha > 0$  的常数;
- (3)  $y'' - 4y' + 4y = (1 + x + \dots + x^{23})e^{2x}$ .

【解】(1) ① 对应的特征方程  $\lambda^2 + 4\lambda + 4 = 0$ , 特征值  $\lambda_1 = \lambda_2 = -2$ , 对应的齐次方程的通解为  
 $y = (C_1 + C_2 x)e^{-2x}$ .

② 非齐次方程的一个特解  $y^*$  为

$$y^* = \frac{1}{D^2 + 4D + 4} e^{\alpha x} = \begin{cases} \frac{1}{(\alpha + 2)^2} e^{\alpha x}, & \alpha \neq -2 \text{ 时}, \\ \frac{1}{2} x^2 e^{\alpha x}, & \alpha = -2 \text{ 时}. \end{cases}$$

故非齐次方程的通解为

$$y = \begin{cases} (C_1 + C_2 x)e^{-2x} + \frac{1}{(\alpha + 2)^2} e^{\alpha x}, & \alpha \neq -2 \text{ 时}, \\ (C_1 + C_2 x)e^{-2x} + \frac{1}{2} x^2 e^{-2x}, & \alpha = -2 \text{ 时}. \end{cases}$$

(2) ① 特征方程  $\lambda^2 + \alpha^2 = 0$ , 特征值  $\lambda = \pm \alpha i$ , 对应的齐次方程的通解为

$$y = C_1 \cos \alpha x + C_2 \sin \alpha x.$$

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③ 非齐次方程的一个特解  $y^*$  为

$$y^* = \frac{1}{D^2 + \alpha^2} \sin x = \begin{cases} \frac{1}{\alpha^2 - 1} \sin x, & \alpha \neq 1, \\ -\frac{1}{2} x \cos x, & \alpha = 1. \end{cases}$$

故非齐次方程的通解为

$$y = \begin{cases} C_1 \cos \alpha x + C_2 \sin \alpha x + \frac{1}{\alpha^2 - 1} \sin x, & \alpha \neq 1, \\ C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x, & \alpha = 1. \end{cases}$$

(3) 特征方程为  $\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda - 2)^2 = 0 \Rightarrow \lambda = 2$  (重根),

对应齐次方程通解为  $y(x) = e^{2x}(C_1 + C_2 x)$ .

$$\text{非齐次方程特解 } y^*(x) = \frac{1}{D^2 - 4D + 4} e^{2x} (1 + x + \dots + x^{23})$$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} (1 + x + \dots + x^{23})$$

$$= e^{2x} \frac{1}{D^2} (1 + x + \dots + x^{23})$$

$$= e^{2x} \left( \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \dots + \frac{x^{25}}{24 \cdot 25} \right),$$

$$\text{故原方程的通解为 } y = e^{2x}(C_1 + C_2 x) + e^{2x} \left( \frac{x^2}{1 \cdot 2} + \frac{x^3}{2 \cdot 3} + \dots + \frac{x^{25}}{24 \cdot 25} \right).$$

$$\begin{array}{ll}
 (1) y'' - y = \sin x & (2) y'' + y' + y = \cos 3x \\
 (3) y'' - 2y' - 3y = x^2 + 2x - 1 & (4) y'' - 6y' + 9y = (x+1)e^{3x} \\
 (5) y'' - 2y' + 2y = xe^x \cos x
 \end{array}$$

【解】(1) 特征方程  $\lambda^3 - 1 = 0$ , 即  $(\lambda - 1)(\lambda^2 + \lambda + 1) = 0$

$$\Rightarrow \lambda_1 = 1, \lambda_{2,3} = \frac{-1 \pm \sqrt{3}i}{2},$$

对应齐次方程的通解为  $y(x) = C_1 e^x + e^{-\frac{x}{2}}(C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x)$ .

$$\begin{aligned}
 \text{非齐次方程特解为 } y^*(x) &= \frac{1}{D^3 - 1} \sin x = \frac{1}{D^2 \cdot D - 1} \sin x = -\frac{1}{D+1} \sin x \\
 &= -\frac{D-1}{D^2 - 1} \sin x = \frac{1}{2} (\cos x - \sin x),
 \end{aligned}$$

$$\text{故原方程通解为 } y(x) = C_1 e^x + e^{-\frac{x}{2}}(C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x) + \frac{1}{2} (\cos x - \sin x).$$

(2) 特征方程为  $\lambda^3 + \lambda^2 + \lambda + 1 = 0 \Rightarrow (\lambda + 1)(\lambda^2 + 1) = 0$

$$\Rightarrow \lambda_1 = -1, \lambda_{2,3} = \pm i,$$

对应齐次方程的通解为  $y(x) = C_1 e^{-x} + C_2 \cos x + C_3 \sin x$ .

$$\begin{aligned}
 \text{齐次方程的特解为 } y^*(x) &= \frac{1}{D^3 + D^2 + D + 1} \cos 3x = \frac{1}{D^2 D + D^2 + D + 1} \cos 3x \\
 &= \frac{1}{-9D - 9 + D + 1} \cos 3x = -\frac{1}{8} \frac{1}{D+1} \cos 3x \\
 &= -\frac{1}{8} \frac{D-1}{D^2 - 1} \cos 3x = \frac{1}{80} (-3 \sin 3x - \cos 3x),
 \end{aligned}$$

$$\text{故原方程的通解为 } y(x) = C_1 e^{-x} + C_2 \cos x + C_3 \sin x - \frac{1}{80} (3 \sin 3x + \cos 3x).$$

(3) 特征方程为  $\lambda^3 - 2\lambda^2 - 3\lambda = 0 \Rightarrow \lambda(\lambda^2 - 2\lambda - 3) = 0$

$$\Rightarrow \lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 3,$$

对应齐次方程通解为  $y(x) = C_1 + C_2 e^{-x} + C_3 e^{3x}$

$$\begin{aligned}
 \text{非齐次方程特解为 } y^*(x) &= \frac{1}{D^3 - 2D^2 - 3D} (x^2 + 2x - 1) \\
 &= \frac{1}{D} \frac{1}{D^2 - 2D - 3} (x^2 + 2x - 1) \\
 &= \frac{1}{D} \left(-\frac{1}{3} + \frac{2}{9}D - \frac{7}{27}D^2\right) (x^2 + 2x - 1) \\
 &= \left(-\frac{1}{3D} + \frac{2}{9} - \frac{7}{27}D\right) (x^2 + 2x - 1) \\
 &= -\frac{1}{9}x^3 - \frac{1}{9}x^2 + \frac{7}{27}x - \frac{20}{27},
 \end{aligned}$$

$$\text{故原方程的通解为 } y(x) = C_1 + C_2 e^{-x} + C_3 e^{3x} - \frac{1}{9}x^3 - \frac{1}{9}x^2 + \frac{7}{27}x - \frac{20}{27}.$$

(4) 特征方程为  $\lambda^2 - 6\lambda + 9 = 0 \Rightarrow (\lambda - 3)^2 = 0 \Rightarrow \lambda = 3$  (重根),

对应齐次方程的通解为  $y(x) = e^{3x}(C_1 + C_2 x)$

$$\begin{aligned}
\text{非齐次方程特解为 } y^*(x) &= \frac{1}{D^2 - 6D + 9}(x+1)e^{3x} \\
&= e^{3x} \frac{1}{(D+3)^2 - 6(D+3) + 9}(x+1) \\
&= e^{3x} \frac{1}{D^2}(x+1) = (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{3x},
\end{aligned}$$

$$\text{故原方程通解为 } y(x) = e^{3x}(C_1 + C_2x) + (\frac{1}{6}x^3 + \frac{1}{2}x^2)e^{3x}.$$

$$(5) \text{ 特征方程为 } \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = 1 \pm i,$$

$$\text{对应齐次方程的通解为 } y(x) = e^x(C_1 \cos x + C_2 \sin x).$$

$$\begin{aligned}
\text{非齐次方程的特解为 } y^*(x) &= \frac{1}{D^2 - 2D + 2}e^x x \cos x = e^x \frac{1}{(D+1)^2 - 2(D+1) + 2} x \cos x \\
&= e^x \frac{1}{D^2 + 1} x \cos x,
\end{aligned}$$

因为  $\cos x$  是  $e^{ix}$  的实部，所以 先求  $\frac{1}{D^2 + 1} x e^{ix}$  再取实部，即得  $\frac{1}{D^2 + 1} x \cos x$ .

$$\begin{aligned}
\frac{1}{D^2 + 1} e^{ix} x &= e^{ix} \frac{1}{(D+i)^2 + 1} x = e^{ix} \frac{1}{D^2 + 2Di} x \\
&= e^{ix} \frac{1}{D(D+2i)} x = e^{ix} \frac{1}{D} (\frac{1}{2i} + \frac{D}{4}) x \\
&= e^{ix} (\frac{1}{2i} \frac{1}{D} + \frac{1}{4}) x = e^{ix} (\frac{1}{2i} \frac{1}{2} x^2 + \frac{1}{4} x) = e^{ix} (-\frac{i}{4} x^2 + \frac{1}{4} x),
\end{aligned}$$

$$Re(\frac{1}{D^2 + 1} e^{ix} x) = Re[e^{ix} (-\frac{i}{4} x^2 + \frac{1}{4} x)] = \frac{x}{4} (x \sin x + \cos x),$$

$$y^*(x) = \frac{1}{4} x (x \sin x + \cos x) e^x,$$

$$\text{故 原方程的通解为 } y(x) = e^x(C_1 \cos x + C_2 \sin x) + \frac{1}{4} x e^x (x \sin x + \cos x).$$