2019 年第十届全国大学生数学竞赛决赛
(非数学专业）参考答案

一、填空题

(1) $a+b=-3$ (2) $I=\dfrac{\pi \ln a}{2a}$ (3) $I=\dfrac{9}{2}$ (4) $\dfrac{\partial^2 z}{\partial x \partial y}=\dfrac{F_2^2F_{11}-2F_1F_2F_{12}+F_1^2F_{22}}{F_2^3}$

(5) $y_1^2+y_2^2+\cdots+y_{n-1}^2$

二、**【参考解析】**: 由于 $f(x)$ 在区间 $(-1,1)$ 内三阶可导，$f(x)$ 在 $x=0$ 处有 Taylor 公式

$$f(x)=f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+o\left(x^3\right)$$

又 $f(0)=0, f'(0)=1, f''(0)=0, f'''(0)=-1$，所以

$$f(x)=x-\frac{1}{6}x^3+o\left(x^3\right)$$

由于 $a_1\in(0,1)$，数列 $\left\{a_n\right\}$ 严格单调且 $\lim\limits_{n\to\infty}a_n=0$，则 $a_n>0$，且 $\left\{\dfrac{1}{a_n^2}\right\}$ 为严格单调增加趋于正无穷的数列，注意到 $a_{n+1}=f\left(a_n\right)$，故由 Stolz 定理及式，有

$$\lim_{n\to\infty}na_n^2=\lim_{n\to\infty}\frac{n}{\dfrac{1}{a_n^2}}=\lim_{n\to\infty}\frac{1}{\dfrac{1}{a_{n+1}^2}-\dfrac{1}{a_n^2}}=\lim_{n\to\infty}\frac{a_n^2a_{n+1}^2}{a_n^2-a_{n+1}^2}=\lim_{n\to\infty}\frac{a_n^2f^2\left(a_n\right)}{a_n^2-f^2\left(a_n\right)}$$

$$=\lim_{n\to\infty}\frac{a_n^2\left(a_n-\dfrac{1}{6}a_n^3+o\left(a_n^3\right)\right)^2}{a_n^2-\left(a_n-\dfrac{1}{6}a_n^3+o\left(a_n^3\right)\right)^2}=\lim_{n\to\infty}\frac{a_n^4-\dfrac{1}{3}a_n^6+\dfrac{1}{36}a_n^8+o\left(a_n^4\right)}{\dfrac{1}{3}a_n^4-\dfrac{1}{36}a_n^6+o\left(a_n^4\right)}=3$$

三、**【参考解析】**: 令 $y=x-\dfrac{1}{nx}$，则 $y'=1+\dfrac{1}{nx^2}>0$ 故函数 $y(x)$ 在 $[\alpha,\beta]$ 上严格单调增加.记 $y(x)$ 的反函数为 $x(y)$，则定义在 $\left[\alpha-\dfrac{1}{n\alpha},\beta-\dfrac{1}{n\beta}\right]$ 上，且 $x'(y)=\dfrac{1}{y'(x)}=\dfrac{1}{1+\dfrac{1}{nx^2}}>0$. 于是

$$\int_\alpha^\beta f'\left(nx-\frac{1}{x}\right)\mathrm{d}x=\int_{\alpha-\frac{1}{n\alpha}}^{\beta-\frac{1}{n\beta}}f'(ny)x'(y)\,\mathrm{d}y.$$

根据积分中值定理，存在 $\xi_n\in\left[\alpha-\dfrac{1}{n\alpha},\beta-\dfrac{1}{n\beta}\right]$，使得

$$\int_{\alpha-\frac{1}{n\alpha}}^{\beta-\frac{1}{n\beta}} f'(ny)x'(y)\,\mathrm{d}\,y = x'\left(\xi_n\right)\int_{\alpha-\frac{1}{n\alpha}}^{\beta-\frac{1}{n\beta}} f'(ny)\,\mathrm{d}\,y$$

$$= \frac{x'\left(\xi_n\right)}{n}\left[f\left(n\beta-\frac{1}{\beta}\right)-f\left(n\alpha-\frac{1}{\alpha}\right)\right]$$

因此$\left|\int_a^\beta f'\left(nx-\frac{1}{x}\right)\mathrm{d}\,x\right| \le \frac{\left|x'\left(\xi_n\right)\right|}{n}\left\|f\left(n\beta-\frac{1}{\beta}\right)\right|+\left|f\left(n\alpha-\frac{1}{\alpha}\right)\right\| \le \frac{2\left|x'\left(\xi_n\right)\right|}{n}$.

注意到$0 < x'\left(\xi_n\right) = \dfrac{1}{1+\dfrac{1}{n\xi_n^2}} < 1$，则

$$\left|\int_\alpha^\beta f'\left(nx-\frac{1}{x}\right)\mathrm{d}\,x\right| \le \frac{2}{n}，即 \lim_{n\to\infty}\int_\alpha^\beta f'\left(nx-\frac{1}{x}\right)\mathrm{d}\,x = 0$$

四、【参考解析】：采用“先二后一”法，并利用对称性，得

$$I = 2\int_0^1 \mathrm{d}\,z \iint_D \frac{\mathrm{d}\,x\,\mathrm{d}\,y}{\left(1+x^2+y^2+z^2\right)^2}，其中 D: 0\le x\le 1, 0\le y\le x$$

交换积分次序，得

$$I = \int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^1\left(\frac{1}{1+z^2}-\frac{1}{1+\sec^2\theta+z^2}\right)\mathrm{d}\,z$$

$$= \frac{\pi^2}{16}-\int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^1\frac{1}{1+\sec^2\theta+z^2}\mathrm{d}\,z$$

作变量代换：$z=\tan t$，并利用对称性，得

$$\int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^1\frac{1}{1+\sec^2\theta+z^2}\mathrm{d}\,z = \int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2 t}{\sec^2\theta+\sec^2 t}\mathrm{d}t$$

$$= \int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta}{\sec^2\theta+\sec^2 t}\mathrm{d}\,t = \frac{1}{2}\int_0^{\frac{\pi}{4}}\mathrm{d}\,\theta\int_0^{\frac{\pi}{4}}\frac{\sec^2\theta+\sec^2 t}{\sec^2\theta+\sec^2 t}\mathrm{d}\,t$$

$$= \frac{1}{2}\times\frac{\pi^2}{16} = \frac{\pi^2}{32}$$

所以，$I = \dfrac{\pi^2}{16}-\dfrac{1}{2}\dfrac{\pi^2}{16} = \dfrac{\pi^2}{32}$

五、【参考解析】：级数通项$a_n = \dfrac{1}{3}\cdot\dfrac{2}{5}\cdot\dfrac{3}{7}\cdots\cdot\dfrac{n}{2n+1}\cdot\dfrac{1}{n+1} = \dfrac{2(2n)!!}{(2n+1)!(n+1)}\left(\dfrac{1}{\sqrt{2}}\right)^{2n+2}$ 令

$$f(x) = \sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!!(n+1)}x^{2n+2},$$

则收敛区间为$(-1,1)$，则$\sum_{n=1}^{\infty} a_n = 2\left[f\left(\frac{1}{\sqrt{2}}\right)-\frac{1}{2}\right]$，由逐项可导性质，得

$$f'(x)=2\sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!!}x^{2n+1}=2g(x)\,,$$

其中$g(x)=\sum_{n=0}^{\infty}\frac{(2n)!!}{(2n+1)!!}x^{2n+1}$，因为

$$g'(x)=1+\sum_{n=1}^{\infty}\frac{(2n)!!}{(2n-1)!!}x^{2n}=1+x\sum_{n=1}^{\infty}\frac{(2n-2)!!}{(2n-1)!!}2nx^{2n-1}$$

$$=1+x\frac{\mathrm{d}}{\mathrm{d}x}\left(\sum_{n=1}^{\infty}\frac{(2n-2)!!}{(2n-1)!!}x^{2n}\right)=1+x\frac{\mathrm{d}}{\mathrm{d}x}[xg(x)]$$

所以$g(x)$满足$g(0)=0,\quad g'(x)-\frac{x}{1-x^2}g(x)=\frac{1}{1-x^2}$. 解这个一阶线性方程，得

$$g(x)=e^{\int\frac{x}{1-x^2}\mathrm{d}x}\left(\int\frac{1}{1-x^2}e^{-\int\frac{x}{1-x^2}\mathrm{d}x}\mathrm{d}x+C\right)=\frac{\arcsin x}{\sqrt{1-x^2}}+\frac{C}{\sqrt{1-x^2}}$$

由$g(0)=0$得$C=0$，故$g(x)=\frac{\arcsin x}{\sqrt{1-x^2}}$，所以

$$f(x)=(\arcsin x)^2,\quad f\left(\frac{1}{\sqrt{2}}\right)=\frac{\pi^2}{16}$$

且$\sum_{n=1}^{\infty}a_n=2\left(\frac{\pi^2}{16}-\frac{1}{2}\right)=\frac{\pi^2-8}{8}$

六、【参考解析】: 存在n阶可逆矩阵H,Q，使得$A=H\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}Q$，因为$A^2=O$，所以

$$A^2=H\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}QH\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}Q=O$$

对QH作相应分块为$QH=\begin{pmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{pmatrix}$，则有

$$\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}QH\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}=\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}\begin{pmatrix}R_{11} & R_{12}\\ R_{21} & R_{22}\end{pmatrix}\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}=\begin{pmatrix}R_{11} & O\\ O & O\end{pmatrix}=O$$

因此，$R_{11}=O$. 而$Q=\begin{pmatrix}O & R_{12}\\ R_{21} & R_{22}\end{pmatrix}H^{-1}$，所以

$$A=H\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}Q=H\begin{pmatrix}I_r & O\\ O & O\end{pmatrix}\begin{pmatrix}O & R_{12}\\ R_{21} & R_{22}\end{pmatrix}H^{-1}=H\begin{pmatrix}O & R_{12}\\ O & O\end{pmatrix}H^{-1}$$

显然，$r(A)=r\left(R_{12}\right)=r$，所以$R_{12}$为行满秩矩阵.

因为$r<\frac{n}{2}$，所以存在可逆矩阵S_1,S_2，使得$S_1R_{12}S_2=\left(I_r,O\right)$，令$P=H\begin{pmatrix} S_1^{-1} & O \\ O & S_2 \end{pmatrix}$，则有

$$P^{-1}AP=\begin{pmatrix} S_1 & O \\ O & S_2^{-1} \end{pmatrix}H^{-1}AH\begin{pmatrix} S_1^{-1} & O \\ O & S_2 \end{pmatrix}=\begin{pmatrix} O & I_r & O \\ O & O & O \end{pmatrix}$$

七、【参考解析】：$\sum_{n=1}^{\infty}a_nu_n$收敛，所以对任意给定$\varepsilon>0$，存在自然数$N_1$，使得当$n>N_1$时，有

$$-\frac{\varepsilon}{2}<\sum_{k=N_1}^{n}a_ku_k<\frac{\varepsilon}{2}.$$

因为$\left\{u_n\right\}_{n=1}^{\infty}$单调递减的正数列，所以$0<\frac{1}{u_{N_1}}\le\frac{1}{u_{N_1+1}}\le\cdots\le\frac{1}{u_n}$. 注意到当$m<n$时，有

$$\sum_{k=m}^{n}\left(A_k-A_{k-1}\right)b_k=A_nb_n-A_{m-1}b_m+\sum_{k=m}^{n-1}\left(b_k-b_{k+1}\right)A_k$$

令$A_0=0,\quad A_k=\sum_{i=1}^{k}a_i\quad(k=1,2,\cdots,n)$，得到

$$\sum_{k=1}^{n}a_kb_k=A_nb_n+\sum_{k=1}^{n-1}\left(b_k-b_{k+1}\right)A_k$$

下面证明：对于任意自然数n，如果$\left\{a_n\right\},\left\{b_n\right\}$满足

$$b_1\ge b_2\ge\cdots\ge b_n\ge0,m\le a_1+a_2+\cdots+a_n\le M$$

则有$b_1m\le\sum_{k=1}^{n}a_kb_k=b_1M$.

事实上，$m\le A_k\le M,\quad b_k-b_{k+1}\ge0$，即得到

$$mb_1=mb_n+\sum_{k=1}^{n-1}\left(b_k-b_{k+1}\right)m\le\sum_{k=1}^{n}a_kb_k\le Mb_n+\sum_{k=1}^{n-1}\left(b_k-b_{k+1}\right)M=Mb_1$$

令$b_1=\frac{1}{u_n},b_2=\frac{1}{u_{n-1}},\cdots$可以得到

$$-\frac{\varepsilon}{2}u_n^{-1}<\sum_{k=N_1}^{n}a_k<\frac{\varepsilon}{2}u_n^{-1},$$

即$\left|\sum_{k=N_1}^{n}a_ku_n\right|<\frac{\varepsilon}{2}$. 又由$\lim\limits_{n\to\infty}u_n=0$知，存在自然数$N_2$，使得$n>N_2$，

$$\left|\left(a_1+a_2+\cdots+a_{N_1-1}\right)u_n\right|<\frac{\varepsilon}{2}$$

取$N=\max\left\{N_1,N_2\right\}$，则当$n>N$时，有

$$\left|\left(a_1+a_2+\cdots+a_n\right)u_n\right|<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

因此 $\lim\limits_{n\to\infty}\left(a_1+a_2+\ldots+a_n\right)u_n=0$.