第十届清疏竞赛班非数学类 2:

1:本次课之后需要详细阅读第九届非数学类(3)

2:本次课需要熟练记忆 Stolz 定理

$Stolz$定理:

设下述两种之一情况发生,

(a):$y_{n+1} > y_n, \lim\limits_{n\to\infty} y_n = \infty$,

(b):$y_{n+1} < y_n, \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} x_n = 0$:

如果$\lim\limits_{n\to\infty} \dfrac{x_{n+1} - x_n}{y_{n+1} - y_n} = a \in [-\infty, +\infty]$则我们一定有:

$$\lim_{n\to\infty} \frac{x_n}{y_n} = a.$$

$\left(\text{经典结论}\right)$应用1:

$a_n > 0$,如果$\lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = a \in \left[0, +\infty\right]$,那么有$\lim\limits_{n\to\infty} \sqrt[n]{a_n} = \lim\limits_{n\to\infty} \dfrac{a_{n+1}}{a_n} = a$.

证明:

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} e^{\frac{\ln a_n}{n}} = e^{\frac{\ln a_{n+1} - \ln a_n}{1}} = e^{\ln a} = a$$

(记忆)应用2:$\lim\limits_{n\to\infty}\dfrac{\sqrt[n]{n!}}{n}$.

证明：

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\frac{e^{\frac{\ln n!}{n}}}{n}=\lim_{n\to\infty}\frac{e^{\frac{\sum_{k=1}^{n}\ln k}{n}}}{n}=\lim_{n\to\infty}e^{\frac{\sum_{k=1}^{n}\ln k}{n}-\ln n}$$

$$=\lim_{n\to\infty}e^{\frac{\sum_{k=1}^{n}\ln k-n\ln n}{n}}=\lim_{n\to\infty}e^{\frac{\ln(n+1)-(n+1)\ln(n+1)+n\ln n}{1}}$$

$$=\lim_{n\to\infty}e^{-n\ln(n+1)+n\ln n}=\lim_{n\to\infty}e^{n\ln\frac{n}{n+1}}=\lim_{n\to\infty}e^{n\left(\frac{n}{n+1}-1\right)}=\lim_{n\to\infty}e^{-\frac{n}{n+1}}=e^{-1}$$

即我们有$\sqrt[n]{n!}\sim\dfrac{n}{e}$.

记忆：

$wallis$公式:$\dfrac{(2n)!!}{(2n-1)!!}\sim\sqrt{\pi n}$

$string$公式：$n!\sim\sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$

(美国数学月刊,第九届非数学决赛推广)

应用4:设正值函数$f(x)$满足$\lim\limits_{n\to\infty}\frac{f(n)}{n}=a>0$, 计算

$$\lim_{n\to\infty}\left[\sqrt[n+1]{\prod_{k=1}^{n+1}f(k)}-\sqrt[n]{\prod_{k=1}^{n}f(k)}\right]$$

证明:

不妨设$a=1$,

$$\lim_{n\to\infty}\left[\sqrt[n+1]{\prod_{k=1}^{n+1}f(k)}-\sqrt[n]{\prod_{k=1}^{n}f(k)}\right]=\lim_{n\to\infty}\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}}-e^{\frac{\ln\prod_{k=1}^{n}f(k)}{n}}\right]$$

$$=\lim_{n\to\infty}e^{\frac{\ln\prod_{k=1}^{n}f(k)}{n}}\cdot\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}}-1\right]$$

$$\frac{f(n)}{n}=1+o(1),f(n)=n+o(n)$$

$$\ln f(n)=\ln(n+o(n))=\ln(n)+\ln(1+o(1))=\ln n+o(1)$$

$$\sum_{k=1}^{n}\ln f(k)=\sum_{k=1}^{n}\ln k+no(1)$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}o(1)}{n}\overset{stolz}{=}\lim_{n\to\infty}o(1)=0\Rightarrow\sum_{k=1}^{n}o(1)=no(1).$$

$$\frac{\ln\prod_{k=1}^{n}f(k)}{n}=\frac{\sum_{k=1}^{n}\ln k}{n}+o(1)$$

$$e^{\frac{\ln\prod_{k=1}^{n}f(k)}{n}}\cdot\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}}-1\right]=e^{\frac{\sum_{k=1}^{n}\ln k}{n}+o(1)}\cdot\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}}-1\right]$$

$$=e^{\frac{\sum_{k=1}^{n}\ln k}{n}}\cdot\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}}-1\right]$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\ln k}{n\ln n}=\lim_{n\to\infty}\frac{\ln(n+1)}{(n+1)\ln(n+1)-n\ln n}=\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n+1}=1$$

故$\sum_{k=1}^{n}\ln k\sim n\ln n,$

$$\lim_{n\to\infty}\frac{1}{n}\left(\sum_{k=1}^{n}\ln k-n\ln n\right)\overset{\text{见前面的应用}}{=}-1$$

$$\lim_{n\to\infty}e^{\frac{\sum_{k=1}^{n}\ln k}{n}}\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}}-1\right]=\lim_{n\to\infty}ne^{\frac{\sum_{k=1}^{n}\ln k}{n}-\ln n}\cdot\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}}-1\right]$$

$$=e^{-1}\lim_{n\to\infty}n\cdot\left[e^{\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}}-1\right]$$

$$\frac{\ln\prod_{k=1}^{n+1}f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n}f(k)}{n}=\frac{n\sum_{k=1}^{n+1}\ln f(k)-(n+1)\sum_{k=1}^{n}\ln f(k)}{n(n+1)}$$

$$=\frac{n\sum_{k=1}^{n+1}\ln f(k)-n\sum_{k=1}^{n}\ln f(k)-\sum_{k=1}^{n}\ln f(k)}{n(n+1)}$$

$$=\frac{n\ln f(n+1)-\sum_{k=1}^{n}\ln f(k)}{n(n+1)}=\frac{\ln f(n+1)}{n+1}-\frac{\sum_{k=1}^{n}\ln f(k)}{n(n+1)}$$

$$\frac{\sum_{k=1}^{n}\ln f(k)}{n(n+1)}=\frac{\sum_{k=1}^{n}\ln k}{n(n+1)}+o\left(\frac{1}{(n+1)}\right)$$

$$\frac{\ln f(n+1)}{n+1}=\frac{\ln(n+1)+o(1)}{n+1}$$

$$e^{-1}\lim_{n\to\infty} n\cdot\left[e^{\frac{\ln\prod_{k=1}^{n+1} f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n} f(k)}{n}}-1\right]$$

$$=e^{-1}\lim_{n\to\infty} n\cdot\left(\frac{\ln\prod_{k=1}^{n+1} f(k)}{n+1}-\frac{\ln\prod_{k=1}^{n} f(k)}{n}\right)$$

$$=e^{-1}\lim_{n\to\infty} n\frac{n\ln f(n+1)-\sum_{k=1}^{n}\ln f(k)}{n(n+1)}$$

$$=e^{-1}\lim_{n\to\infty}\frac{n\ln f(n+1)-\sum_{k=1}^{n}\ln f(k)}{n}$$

$$\ln f(n)=\ln n+o(1)$$

$$e^{-1}\lim_{n\to\infty}\frac{n\ln f(n+1)-\sum_{k=1}^{n}\ln f(k)}{n}$$

$$=e^{-1}\lim_{n\to\infty}\left[\ln(n+1)+o(1)-\left(\frac{\sum_{k=1}^{n}\ln k}{n}+o(1)\right)\right]$$

$$=e^{-1}\lim_{n\to\infty}\left[\ln n+\ln\left(1+\frac{1}{n}\right)-\frac{\sum_{k=1}^{n}\ln k}{n}+o(1)\right]$$

$$=e^{-1}\lim_{n\to\infty}\left[\ln n-\frac{\sum_{k=1}^{n}\ln k}{n}+o(1)\right]=e^{-1}$$

对于$\lim_{n\to\infty}\frac{f(n)}{n}=a,\lim_{n\to\infty}\frac{\frac{f(n)}{a}}{n}=1,$

$$\lim_{n\to\infty}\left[\sqrt[n+1]{\prod_{k=1}^{n+1}\frac{f(k)}{a}}-\sqrt[n]{\prod_{k=1}^{n}\frac{f(k)}{a}}\right]=e^{-1}$$

$$\Rightarrow\lim_{n\to\infty}\left[\sqrt[n+1]{\prod_{k=1}^{n+1}f(k)}-\sqrt[n]{\prod_{k=1}^{n}f(k)}\right]=\frac{a}{e}$$

第九届真题是$f\left(k\right)=k$,

本题仔细品味如果把$f\left(k\right)$换成k这个逻辑没有任何道理

但是做余项计算$\left(\text{渐进方法}\right)$,我们成功的把$f\left(k\right)$替换为了k

$\left(\text{第八届非数学类决赛压轴题}\right)$应用5：

计算$\lim\limits_{n\to\infty} n\left(H_n-\ln n-\gamma\right)$

证明：$H_n=\sum\limits_{k=1}^{n}\dfrac{1}{k}$,$\left(\text{记忆}\right)$$\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}\dfrac{1}{k}}{\ln n}=\lim\limits_{n\to\infty}\dfrac{\dfrac{1}{n+1}}{\ln\dfrac{n+1}{n}}=\lim\limits_{n\to\infty}\dfrac{\dfrac{1}{n+1}}{\dfrac{1}{n}}=1$

$$H_n\sim\ln n, H_n-\ln n=\sum_{k=1}^{n}\frac{1}{k}-\sum_{k=1}^{n-1}\ln\frac{k+1}{k}=\frac{1}{n}+\sum_{k=1}^{n-1}\left(\frac{1}{k}-\ln\frac{k+1}{k}\right)$$

$$=\frac{1}{n}+\sum_{k=1}^{n-1}\left(\frac{1}{k}-\ln\left(1+\frac{1}{k}\right)\right)$$

$$\ln\left(1+x\right)=x+O\left(x^2\right)$$

$$\frac{1}{n}+\sum_{k=1}^{n-1}\left(\frac{1}{k}-\ln\left(1+\frac{1}{k}\right)\right)=\frac{1}{n}+\sum_{k=1}^{n-1}O\left(\frac{1}{k^2}\right),\text{而}\sum_{k=1}^{n-1}\left|O\left(\frac{1}{k^2}\right)\right|\le C\sum_{k=1}^{n-1}\frac{1}{k^2}$$

因此$\lim\limits_{n\to\infty}H_n-\ln n\triangleq\gamma\approx0.577$

$$\lim_{n\to\infty}n\left(H_n-\ln n-\gamma\right)=\lim_{n\to\infty}\frac{H_n-\ln n-\gamma}{\frac{1}{n}}=\lim_{n\to\infty}\frac{H_{n+1}-H_n-\ln\left(n+1\right)+\ln n}{\frac{1}{n+1}-\frac{1}{n}}$$

$$=\lim_{n\to\infty}\frac{\frac{1}{n+1}-\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n+1}-\frac{1}{n}}=-\lim_{n\to\infty}n^2\left[\frac{1}{n+1}-\frac{1}{n}+\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right]$$

$$=-\lim_{n\to\infty}n^2\left[-\frac{1}{n\left(n+1\right)}+\frac{1}{2n^2}+o\left(\frac{1}{n^2}\right)\right]=\frac{1}{2}.$$

(经典习题)应用6: $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}a_k}{n}=A$, 证明: $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}\frac{a_k}{k}}{\ln n}=A$

证明:

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\frac{a_k}{k}}{\ln n}\overset{stolz}{=}\lim_{n\to\infty}n\frac{a_{n+1}}{n+1}=\lim_{n\to\infty}a_{n+1}=\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}a_k}{n}=A,$$ 这是错的！因为$stolz$不能逆用.

$abel$变换(见第九届数学类2pdf.)

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}\frac{a_k}{k}}{\ln n}=\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\sum\limits_{j=1}^{k}a_j+\frac{1}{n}\sum\limits_{j=1}^{n}a_j}{\ln n}$$

$$=\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\sum\limits_{j=1}^{k}a_j}{\ln n}=\lim_{n\to\infty}n\left(\frac{1}{n}-\frac{1}{n+1}\right)\sum_{j=1}^{n}a_j$$

$$=\lim_{n\to\infty}\frac{1}{n+1}\sum_{j=1}^{n}a_j=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}a_j=A.$$

本想法非数学类决赛考过:

$f(x)$连续, $\lim\limits_{x\to+\infty}\int_0^x f(y)dy$存在, 去证明 $\lim\limits_{x\to+\infty}\dfrac{\int_0^x yf(y)dy}{x}=0$

$$事实上\lim_{x\to+\infty}\frac{\int_0^x yd\int_0^y f(z)dz}{x}=\lim_{x\to+\infty}\frac{x\int_0^x f(y)dy-\int_0^x\left(\int_0^y f(z)dz\right)dy}{x}$$

$$=\lim_{x\to+\infty}\left[\int_0^x f(y)dy-\frac{\int_0^x\left(\int_0^y f(z)dz\right)dy}{x}\right]$$

$$=\lim_{x\to+\infty}\int_0^x f(y)dy-\lim_{x\to+\infty}\int_0^x f(z)dz=0$$

应用7：计算$\lim\limits_{k\to\infty}\dfrac{1}{k^{m+2}}\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{k}\left|i-j\right|^{m}$

证明：

$$\text{计算}\lim_{k\to\infty}\frac{1}{k^{m+2}}\sum_{i=1}^{k}\sum_{j=1}^{k}\left|i-j\right|^{m}$$

$$=\lim_{k\to\infty}\frac{\sum\limits_{i=1}^{k+1}\sum\limits_{j=1}^{k+1}\left|i-j\right|^{m}-\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{k}\left|i-j\right|^{m}}{(m+2)k^{m+1}}$$

$$=\lim_{k\to\infty}\frac{\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{k+1}\left|i-j\right|^{m}+\sum\limits_{j=1}^{k+1}\left|k+1-j\right|^{m}-\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{k}\left|i-j\right|^{m}}{(m+2)k^{m+1}}$$

$$=\lim_{k\to\infty}\frac{\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{k}\left|i-j\right|^{m}+2\sum\limits_{j=1}^{k+1}\left|k+1-j\right|^{m}-\sum\limits_{i=1}^{k}\sum\limits_{j=1}^{k}\left|i-j\right|^{m}}{(m+2)k^{m+1}}$$

$$=\lim_{k\to\infty}\frac{2\sum\limits_{j=1}^{k+1}\left|k+1-j\right|^{m}}{(m+2)k^{m+1}}=\lim_{k\to\infty}\frac{2\sum\limits_{j=1}^{k}j^{m}}{(m+2)k^{m+1}}=\lim_{k\to\infty}\frac{2(k+1)^{m}}{(m+2)(m+1)k^{m}}$$

$$=\frac{2}{(m+2)(m+1)}$$

$\left(by\text{逆神}\right)$应用9：经典$stolz$习题一般总结.

设$x\to 0^+$,有$f\left(x\right)=x-Ax^{1+a}+Bx^{1+2a}+O\left(x^{1+3a}\right),A,a>0$, 如果

a_n递减到0,且$a_{n+1}=f\left(a_n\right)$,那么我们有

$$\lim_{n\to\infty}\frac{n^{1+\frac{1}{a}}}{\ln n}\left[a_n-\frac{1}{n^{\frac{1}{a}}\left(aA\right)^{\frac{1}{a}}}\right]=\frac{1}{\left(aA\right)^{2+\frac{1}{a}}}\left(B-\frac{\left(a+1\right)A^2}{2}\right)$$

证明：

$$\lim_{n\to\infty}n^{\frac{1}{a}}a_n=\lim_{n\to\infty}\sqrt[a]{na_n^a}$$

$$\lim_{n\to\infty}\frac{1}{na_n^a}=\lim_{n\to\infty}\frac{\frac{1}{a_n^a}}{n}=\lim_{n\to\infty}\frac{\frac{1}{a_{n+1}^a}-\frac{1}{a_n^a}}{1}=\lim_{n\to\infty}\frac{1}{f^a\left(a_n\right)}-\frac{1}{a_n^a}$$

$$=\lim_{x\to 0^+}\left(\frac{1}{f^a\left(x\right)}-\frac{1}{x^a}\right)=\lim_{x\to 0^+}\frac{x^a-f^a\left(x\right)}{x^af^a\left(x\right)}=\lim_{x\to 0^+}\frac{x^a-f^a\left(x\right)}{x^{2a}}$$

$$=a\lim_{x\to 0^+}\frac{x-f\left(x\right)}{x^{2a}}x^{a-1}=a\lim_{x\to 0^+}\frac{Ax^{1+a}}{x^{2a}}x^{a-1}=aA$$

因此$\lim\limits_{n\to\infty}n^{\frac{1}{a}}a_n=\dfrac{1}{\left(aA\right)^{\frac{1}{a}}},n\sim\dfrac{a_n^{-a}}{aA}$

$$\lim_{n\to\infty}\frac{n^{1+\frac{1}{a}}}{\ln n}\left[a_n-\frac{1}{n^{\frac{1}{a}}\left(aA\right)^{\frac{1}{a}}}\right]=\lim_{n\to\infty}\frac{n^{1+\frac{1}{a}}}{\ln n}\left[\left(a_n^{-a}\right)^{-\frac{1}{a}}-\left(naA\right)^{-\frac{1}{a}}\right]$$

$$=-\frac{1}{a}\lim_{n\to\infty}\frac{n^{1+\frac{1}{a}}}{\ln n}\left[a_n^{-a}-naA\right]\left(naA\right)^{-\frac{1}{a}-1}$$

$$=-\frac{\left(aA\right)^{-\frac{1}{a}-1}}{a}\lim_{n\to\infty}\frac{a_n^{-a}-naA}{\ln n}$$

$$=-\frac{\left(aA\right)^{-\frac{1}{a}-1}}{a}\lim_{n\to\infty}n\left[a_{n+1}^{-a}-a_n^{-a}-aA\right]$$

$$=-\frac{\left(aA\right)^{-\frac{1}{a}-1}}{a}\lim_{n\to\infty}\frac{a_n^{-a}}{aA}\left[a_{n+1}^{-a}-a_n^{-a}-aA\right]$$

$$=-\frac{\left(aA\right)^{-\frac{1}{a}-1}}{a}\lim_{x\to 0^+}\frac{x^{-a}}{aA}\left[f^{-a}\left(x\right)-x^{-a}-aA\right]$$

$$\overset{\text{函数极限自己算了}}{=}\frac{1}{\left(aA\right)^{2+\frac{1}{a}}}\left(B-\frac{\left(a+1\right)A^2}{2}\right)$$

$\left(\text{考试大概率不考}\right)$

注意：我们这里只讲了递推函数的$taylor$展开第一项为x的处理方法,

一般的情况该如何解决？

能不能不靠题目给出阶, 自己计算出各个阶, 等待下回分解.

$\left(\text{要考}\right)$习题：写出本结果在$a_{n+1}=\sin a_n, a_1=1, a_{n+1}=\ln\left(1+a_n\right), a_1=1$的情况并单独证明

$\left(\text{如果你实在不想算上面那个带字母的情况，但这个具体的形式必须算}\right)$.

习题：完成第十届非数学类决赛第一大题.

应用11：

估计$\sum_{k=1}^{n}\frac{2^{k-1}}{k}(n\to\infty)$

证明：

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{2^{k-1}}{k}}{\frac{2^n}{n}}=\lim_{n\to\infty}\frac{\frac{2^n}{n+1}}{\frac{2^{n+1}}{n+1}-\frac{2^n}{n}}=\lim_{n\to\infty}\frac{\frac{1}{n+1}}{\frac{2}{n+1}-\frac{1}{n}}=\lim_{n\to\infty}\frac{1}{1-\frac{1}{n}}=1$$

$$\sum_{k=1}^{n}\frac{2^{k-1}}{k}\sim\frac{2^n}{n}$$

$$\lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{2^{k-1}}{k}-\frac{2^n}{n}}{\frac{2^n}{n^2}}=\lim_{n\to\infty}\frac{\frac{2^n}{n+1}-\frac{2^{n+1}}{n+1}+\frac{2^n}{n}}{\frac{2^{n+1}}{(n+1)^2}-\frac{2^n}{n^2}}=\lim_{n\to\infty}\frac{\frac{1}{n+1}-\frac{2}{n+1}+\frac{1}{n}}{\frac{2}{(n+1)^2}-\frac{1}{n^2}}$$

$$=\lim_{n\to\infty}\frac{n(n+1)}{\left(n-\sqrt{2}-1\right)\left(n+\sqrt{2}-1\right)}=1,$$

$$\sum_{k=1}^{n}\frac{2^{k-1}}{k}-\frac{2^n}{n}\sim\frac{2^n}{n^2}$$

习题：猜测一下$\sum_{k=1}^{n}\frac{2^{k-1}}{k^2}$的等价无穷大.

$\left(by\text{逆神, 考虑到计算较大, 想偷懒的同学可以考虑}a=A=1\right)$

习题$\left(\text{选做}\right)$:

设$x\to0^+$,有$f(x)=x-Ax^{1+a}+Bx^{1+2a}\ln x+O\left(x^{1+3a}\right)$,$A,a>0$, 如果

a_n递减到0,且$a_{n+1}=f(a_n)$,那么我们有

$$\lim_{n\to\infty}\frac{n}{\ln n}\left[n^{\frac{1}{a}}(aA)^{\frac{1}{a}}a_n-1+\frac{B}{2a^3A}\frac{\ln^2 n}{n}\right]=-\frac{2B\ln(aA)+a(1+a)A^2}{2a^3A^2}$$