全国大学生数学竞赛非数学类模拟二

清疏竞赛考研数学

2023 年 9 月 3 日

摘要

$$\mathbb{N} = \{1, 2, \cdots\}, \mathbb{N}_0 = \{0, 1, 2, \cdots\}.$$

模拟试题应当规定时间独立完成并给予反馈.

1 填空题

填空题 1.1 给定 $[0,1]$ 上的正值连续函数 f, 计算 $\lim_{n\to\infty} \sqrt[n]{f\left(\frac{1}{n}\right) f\left(\frac{2}{n}\right) \cdots f\left(\frac{n-1}{n}\right)} =$ $e^{\int_0^1 \ln f(x)dx}$

填空题 1.2 过点 $(2,0,0)$ 做曲面 $x^2 + \frac{y^2}{4} + \frac{z^2}{9} = 1$ 的全部切线, 则全部切线组成的曲面方程为 $\frac{y^2}{4} + \frac{z^2}{9} = \frac{1}{3}(x-2)^2$

$xx_0 + \frac{yy_0}{4} + \frac{zz_0}{9} = 1$

$xy \to \frac{x+x_0}{2}\ \frac{y+y_0}{2}$

$x \to \frac{x+x_0}{2}$

填空题 1.3 设 $n > 0$ 且 $\boxed{\int_1^\infty \frac{x^m \arctan x}{2+x^n} dx}$ 收敛, 则 m, n 的范围是 = $n-m>1$

$\arctan x \to \frac{\pi}{2}$, 故 $\sim \frac{x^m}{2+x^n} \sim \frac{1}{x^{n-m}}$, 故 $n-m>1$

填空题 1.4 计算积分 $\int_0^{\frac{\pi}{2}} (1 + \sin x) \ln \sin x dx =$ $(1-\frac{\pi}{2})\ln 2 - 1$

填空题 1.5 微分方程 $4y'' + 4xy' + x^2 y = 2e^{x-\frac{x^2}{4}}$ 的通解为 $C_1 e^{-\frac{x^2}{4} - \frac{\sqrt{2}}{2}x} + C_2 e^{-\frac{x^2}{4} + \frac{\sqrt{2}}{2}x} + e^{x-\frac{x^2}{4}}$

变系数必先猜后证. 令 $y(x) = e^{-\frac{x^2}{4}} \cdot g(x)$, 则代入方程得

$$-2e^{-\frac{x^2}{4}}(g(x) - 2g''(x)) = 2e^{x-\frac{x^2}{4}}$$

故 $4g''(x) - 2g(x) = 2e^x$

可解 g 得 y

$\ln x^a = a\ln x$, $e^{\ln x} = x$ $\ln(xy)=\ln x+\ln y$

2 选择题答案区

1. $\sqrt[n]{f(\frac1n)\cdots f(\frac{n-1}{n})} = e^{\frac{\ln[f(\frac1n)f(\frac2n)\cdots f(\frac{n-1}{n})]}{n}} = e^{\frac{\sum_{k=1}^{n-1}\ln f(\frac kn)}{n}}$

故 $\lim\limits_{n\to+\infty}\sqrt[n]{f(\frac1n)\cdots f(\frac{n-1}{n})} = e^{\int_0^1 \ln f(x)dx}$.

2. 设切点 (x_0,y_0,z_0)，则 $x_0^2+\frac{y_0^2}{4}+\frac{z_0^2}{9}=1$，切面 $xx_0+\frac{yy_0}{4}+\frac{zz_0}{9}=1$

由 $2x_0+\frac{0y_0}{4}+\frac{0z_0}{9}=1 \Rightarrow x_0=\frac12$，$\frac{y_0^2}{4}+\frac{z_0^2}{9}=\frac34$

切线：$\frac{x-2}{x_0-2}=\frac{y-0}{y_0-0}=\frac{z-0}{z_0-0}=t$

目标消去 x_0,y_0,z_0，故 $\frac{x-2}{-\frac32}=t=\frac{y}{y_0}=\frac{z}{z_0}$

$t=-\frac23(x-2)$，$y_0=\frac yt$，$z_0=\frac zt$，$\frac{y^2}{4t^2}+\frac{z^2}{9t^2}=\frac34$

故 $\frac{y^2}{4}+\frac{z^2}{9}=\frac34\cdot\frac49(x-2)^2=\frac13(x-2)^2$

4. $\int_0^{\frac\pi2}(1+\sin x)\ln\sin x\,dx$

① $\int_0^{\frac\pi2}\ln\sin x\,dx \overset{x=\frac\pi2-y}{=} \int_0^{\frac\pi2}\ln\cos x\,dx = \frac12\int_0^{\frac\pi2}\ln(\sin x\cdot\cos x)dx$

$=\frac12\int_0^{\frac\pi2}\ln\frac12+\ln\sin(2x)dx$

$=\frac\pi4\ln\frac12+\frac12\int_0^{\frac\pi2}\ln\sin(2x)dx$

$=\frac\pi4\ln\frac12+\int_0^{\frac\pi4}\ln\sin(2x)dx$

$\overset{2x=y}{=}\frac\pi4\ln\frac12+\frac12\int_0^{\frac\pi2}\ln\sin x\,dx$

故 $\int_0^{\frac\pi2}\ln\sin x\,dx = -\frac\pi2\ln2$

记下: $\int_a^b f(x)dx=\int_a^b f(b+a-x)dx=\frac12\int_a^b[f(x)+f(b+a-x)]dx$

② $\int_0^{\frac\pi2}\sin x\ln\sin x\,dx = \int_0^{\frac\pi2}\ln\sin x\, d(1-\cos x) = (1-\cos x)\ln\sin x\Big|_0^{\frac\pi2} - \int_0^{\frac\pi2}(1-\cos x)\,d\ln\sin x$

$= -\int_0^{\frac\pi2}\frac{1-\cos x}{\sin x}\cdot\cos x\,dx = -\int_0^{\frac\pi2}\frac{\cos x-\cos^2x}{\sin x}dx$

$=-\int_0^{\frac\pi2}\left(\frac{\cos x-1}{\sin x}+\sin x\right)dx = -\int_0^{\frac\pi2}\left(\frac{-2\sin^2\frac x2}{2\sin\frac x2\cos\frac x2}+\sin x\right)dx = -\int_0^{\frac\pi2}(-\tan\frac x2+\sin x)dx=\ln2-1$

3 解答题

解答题 3.1 设 $z=f(u,v)$, u,v 由方程组 $\begin{cases} u+v=g(xy) \\ u-v=h\left(\frac{x}{y}\right) \end{cases}$ 确定的 x,y 的函数, 这里 f,g,h 连续可微, 求 $\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$.

证. $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial y}$

$\left.\begin{cases} \frac{\partial u}{\partial x}+\frac{\partial v}{\partial x}=yg'(xy) \\ \frac{\partial u}{\partial x}-\frac{\partial v}{\partial x}=\frac{1}{y}h'\left(\frac{x}{y}\right) \end{cases}\right\}$ 解出 $\frac{\partial u}{\partial x},\frac{\partial v}{\partial x}$

$\left.\begin{cases} \frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}=xg'(xy) \\ \frac{\partial u}{\partial y}-\frac{\partial v}{\partial y}=-\frac{x}{y^2}h'\left(\frac{x}{y}\right) \end{cases}\right\}$ 解出 $\frac{\partial u}{\partial y},\frac{\partial v}{\partial y}$

故 $\frac{\partial z}{\partial x}=\frac{\partial f}{\partial u}\left[\frac{1}{2}yg'(xy)+\frac{1}{2y}h'\left(\frac{x}{y}\right)\right]+\frac{\partial f}{\partial v}\left[\frac{1}{2}yg'(xy)-\frac{1}{2y}h'\left(\frac{x}{y}\right)\right]$

$\frac{\partial z}{\partial y}=\frac{\partial f}{\partial u}\left[\frac{1}{2}xg'(xy)-\frac{x}{2y^2}h'\left(\frac{x}{y}\right)\right]+\frac{\partial f}{\partial v}\left[\frac{1}{2}xg'(xy)+\frac{x}{2y^2}h'\left(\frac{x}{y}\right)\right]$

解答题 3.2 设 $f \in C[0,1]$ 且满足 $\int_0^1 x^2 f(x)\,dx = 1$, 证明

(1): 存在 $\theta \in [0,1]$, 使得 $|f(\theta)| \geqslant 3$.

(2): 若还有 $\int_0^1 xf(x)\,dx = 0$, 证明存在 $\theta \in [0,1]$, 使得 $|f(\theta)| \geqslant 10.2$.

证: (1): 由积分中值定理: $\int_0^1 x^2 f(x)dx = f(\theta)\int_0^1 x^2 dx = \frac{f(\theta)}{3} = 1$, 故 $|f(\theta)| = 3$.

(2): $\int_0^1 (x^2 - ax) f(x) dx = 1$, 设 $|f(x_0)| = \max\limits_{x\in[0,1]} |f(x)|$,

$$1 = \left|\int_0^1 x(x-a) f(x) dx\right| \leq M\int_0^1 |x(x-a)| dx = \begin{cases} \frac{1}{6}(2-3a), & a \leq 0 \\ \frac{1}{6}(3a-2), & a \geq 1 \\ \frac{1}{6}(2a^3-3a+2), & 0<a<1 \end{cases}.$$

而 $\left[\frac{1}{6}(2a^3-3a+2)\right]' = \frac{1}{6}[6a^2-3] = 0$, $a = \frac{1}{\sqrt{2}}$, 代入上述等式.

故 $1 \leq M\frac{2-\sqrt{2}}{6} \implies M \geq \frac{6}{2-\sqrt{2}} = 3(2+\sqrt{2}) \geq 10.2$.

证毕!

解答题 3.3 对 $n \in \mathbb{N}, a > 0$, $L_n = \{(x, y) \in \mathbb{R}^2 : x^{2n+1} + y^{2n+1} = ax^n y^n\}$ 围成区域记为 D_n, 计算

$$\lim_{n\to\infty} nS_{D_n},$$

这里 S_{D_n} 表示 D_n 的面积.

证明: $y = tx$, $\begin{cases} x = \dfrac{at^n}{1+t^{2n+1}} \\ y = \dfrac{at^{n+1}}{1+t^{2n+1}} \end{cases}$, $t \in [0, +\infty)$

$$\iint_{D_n} dxdy = \frac{1}{2}\int_{L_n} -ydx + x\,dy$$，代入参数方程

$$S_{D_n} = \frac{a^2}{2}\int_0^{+\infty} \frac{t^{2n}}{(t^{2n+1}+1)^2}\,dt = \frac{a^2}{4n+2}$$

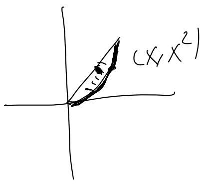
故 $\lim\limits_{n\to+\infty} S_{D_n} = \dfrac{a^2}{4}$.

解答题 3.4 设区域 D 由 $y=x^2, y=x$ 所围成,

(1): 求区域 D 绕 $y=x$ 旋转一周形成的旋转体体积.

(2): 求区域 D 绕 $y=x$ 旋转一周形成的旋转体表面积.

证:(1):



$$V=\iint_D 2\pi \frac{|x-y|}{\sqrt{2}}\,dxdy$$

$$=\sqrt{2}\pi\iint_D (x-y)\,dxdy$$

$$=\sqrt{2}\pi\int_0^1 dx\int_{x^2}^{x} x-y\,dy=\frac{\sqrt{2}\pi}{60}$$

(2): $$S=\int_L 2\pi\frac{x-x^2}{\sqrt{2}}\,ds \quad , L: y=x^2, 0\le x\le 1.$$

$$=\sqrt{2}\pi\int_0^1 (x-x^2)\sqrt{1+4x^2}\,dx$$

$$=\sqrt{2}\pi\int_0^1\sqrt{1+4x^2}\,d\left(\tfrac{1}{2}x^2\right)-\sqrt{2}\pi\int_0^1 x^2\sqrt{1+4x^2}\,dx \qquad x=\frac{\tan t}{2}$$

$$=\sqrt{2}\pi\left[-\frac{1}{12}+\frac{-3\ln(\sqrt{5}-2)+26\sqrt{5}}{192}\right]$$

解答题 3.5 对每个 $n \in \mathbb{N}$, 设

$$a_n > 0, n = 1, 2, \cdots, s = \sum_{n=1}^{\infty} \frac{1}{a_n} < \infty.$$

记

$$t_n = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_n=1}^{\infty} \left(\frac{1}{a_{k_1}} \frac{1}{a_{k_1} + a_{k_2}} \cdot \cdots \cdot \frac{1}{a_{k_1} + a_{k_2} + \cdots + a_{k_n}} \right).$$

计算 $\sum\limits_{n=1}^{\infty} t_n$.

证明: $t_n = \sum_{k_1=1}^{+\infty} \cdots \sum_{k_n=1}^{+\infty} \frac{a_{k_2}}{a_{k_1} a_{k_2}} \frac{1}{a_{k_1}+a_{k_2}} \cdots \frac{1}{a_{k_1}+a_{k_2}+\cdots+a_{k_n}}$

$= \frac{1}{2} \sum_{k_1=1}^{+\infty} \cdots \sum_{k_n=1}^{+\infty} \frac{a_{k_2}+a_{k_1}}{a_{k_1} a_{k_2}} \frac{1}{a_{k_1}+a_{k_2}} \cdots \frac{1}{a_{k_1}+a_{k_2}+\cdots+a_{k_n}}$

$= \cdots\cdots$

$= \frac{1}{n!} \sum_{k_1=1}^{+\infty} \cdots \sum_{k_n=1}^{+\infty} \frac{1}{a_{k_1} a_{k_2} \cdots a_{k_n}}$

$= \frac{1}{n!} \cdot s^n$

故 $\sum_{n=1}^{+\infty} t_n = e^s - 1$

解答题 3.6 设 a_n 递减到 0 且满足

$$\frac{1}{2}(a_n + a_{n+2}) \geqslant a_{n+1}, n = 1, 2, \cdots.$$

则对 $\theta \in (0, 2\pi)$，证明

(1): $\sum_{n=1}^{\infty} a_n \cos n\theta$ 收敛.

(2): $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta \geqslant 0$.

证：(1)：学习阿贝尔，狄利克雷判别法.

$$\sum_{n=1}^{m} a_n \cos n\theta = \sum_{n=1}^{m-1}(a_n - a_{n+1}) S_n + a_m S_m \quad , \quad S_n = \sum_{k=1}^{n} \cos k\theta$$

$$= \frac{\sin\frac{\theta}{2} \sum_{k=1}^{n} \cos k\theta}{\sin\frac{\theta}{2}}$$

$$= \frac{\sum_{k=1}^{n}\left[\sin(\frac{1}{2}+k)\theta + \sin(\frac{1}{2}-k)\theta\right]}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sum_{k=1}^{n}\left[\sin\frac{2k+1}{2}\theta - \sin\frac{2k-1}{2}\theta\right]}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\frac{2n+1}{2}\theta - \sin\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

故 $|S_n| \leq M_\theta$，$\forall n \in \mathbb{N}$，令 $m \to \infty$，$a_m S_m \to 0$。

$$\sum_{n=1}^{m-1}\left|(a_n - a_{n+1}) S_n\right| \leq M_\theta \sum_{n=1}^{m-1}(a_n - a_{n+1}) = M_\theta(a_1 - a_m) \leq M_\theta a_1.$$

故 $\lim\limits_{m\to\infty} \sum_{n=1}^{m-1}(a_n - a_{n+1}) S_n$ 存在。

因此 $\sum_{n=1}^{\infty} a_n \cos n\theta$ 收敛.

(2)：$D_n(\theta) = \frac{1}{2} + \sum_{k=1}^{n} \cos k\theta = \frac{\sin(n+\frac{1}{2})\theta}{2\sin\frac{\theta}{2}}$

$K_n(\theta) = \sum_{k=0}^{n} D_k(\theta) = \frac{\left[\sin\frac{(n+1)}{2}\theta\right]^2}{2\sin^2\frac{\theta}{2}} \geq 0$

则 $\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos k\theta = \sum_{k=0}^{n-1}(a_{k-1} - a_k) D_k(\theta) + a_n D_n(\theta)$

$$= \sum_{k=0}^{n-2}(a_k - 2a_{k+1} + a_{k+2}) K_k(\theta) + (a_{n-1} - a_n) K_{n-1}(\theta) + a_n D_n(\theta)$$

对固定的 $\theta \in (0, 2\pi)$，$K_n(\theta)$，$D_n(\theta)$ 有界，因此

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta = \sum_{k=0}^{\infty}(a_k - 2a_{k+1} + a_{k+2}) K_k(\theta) \geq 0.$$ 证毕！