# 数学分析竞赛资料*

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**美丽有两种**

**一种是深刻而又动人的方程**

**一种是你泛着倦意淡淡的笑容**


*曾经有一份真诚的爱情放在我面前,我没有珍惜,等我失去的时候我才后悔莫及,人世间最痛苦的事莫过于此.如果上天能给我一个再来一次的机会,我会对那个女孩子说三个字,"我爱你!",如果非要在这份爱上加个期限,我希望是一万年.


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# 1 Limits

**问题**.设 $a_1=3, a_n=2a_{n-1}^2-1$,求 $\lim\limits_{n\to\infty}\dfrac{a_n}{2^n a_1 a_2\cdots a_{n-1}}$
解1:令

$$a_1=\frac{1}{2}\left(\alpha+\frac{1}{\alpha}\right)$$

由数学归纳法可得

$$a_n=\frac{1}{2}\left(\alpha^{2^{n-1}}+\frac{1}{\alpha^{2^{n-1}}}\right)$$

则

$$\begin{aligned}
\frac{a_n}{2^n a_1 a_2\cdots a_{n-1}} &= \frac{\frac{1}{2}\left(\alpha^{2^{n-1}}+\frac{1}{\alpha^{2^{n-1}}}\right)}{2^n\cdot\frac{1}{2}\left(\alpha+\frac{1}{\alpha}\right)\cdots\frac{1}{2}\left(\alpha^{2^{n-2}}+\frac{1}{\alpha^{2^{n-2}}}\right)} \\
&= \frac{\left(\alpha-\frac{1}{\alpha}\right)\left(\alpha^{2^{n-1}}+\frac{1}{\alpha^{2^{n-1}}}\right)}{4\left(\alpha-\frac{1}{\alpha}\right)\left(\alpha+\frac{1}{\alpha}\right)\cdots\left(\alpha^{2^{n-2}}+\frac{1}{\alpha^{2^{n-2}}}\right)} \\
&= \frac{\alpha-\frac{1}{\alpha}}{4}
\end{aligned}$$

由 $a_1=3$ 可知 $\alpha=3\pm2\sqrt{2}$.显然 $a_n>0$,从而

$$\alpha=3+2\sqrt{2},\alpha-\frac{1}{\alpha}=4\sqrt{2}$$

故

$$\lim_{n\to\infty}\frac{a_n}{2^n a_1 a_2\cdots a_{n-1}}=\frac{\alpha-\frac{1}{\alpha}}{4}=\sqrt{2}$$

解2:由数学归纳法得 $a_n>1$

$$\begin{aligned}
a_n^2-1 &= (a_n-1)(a_n+1) \\
&= 2^2a_{n-1}^2\left(a_{n-1}^2-1\right) \\
&= 2^2a_{n-1}^2\cdot2^2a_{n-2}^2\left(a_{n-2}^2-1\right) \\
&= 2^{2(n-1)}a_{n-1}^2a_{n-2}^2\cdots a_1^2\left(a_1^2-1\right) \\
&= 2\left(2^n a_1 a_2\cdots a_{n-1}\right)^2
\end{aligned}$$

即

$$\frac{a_n^2-1}{\left(2^n a_1 a_2\cdots a_{n-1}\right)^2}=2$$

则

$$\left(\lim_{n\to\infty}\frac{a_n}{2^n a_1 a_2\cdots a_{n-1}}\right)^2=\lim_{n\to\infty}\frac{a_n^2-1}{\left(2^n a_1 a_2\cdots a_{n-1}\right)^2}+\lim_{n\to\infty}\frac{1}{\left(2^n a_1 a_2\cdots a_{n-1}\right)^2}=2$$

故

$$\lim_{n\to\infty}\frac{a_n}{2^n a_1 a_2\cdots a_{n-1}}=\sqrt{2}$$

**问题**.设 $a_0=1,a_1=3,a_{n+1}=\left(\frac{a_n^2}{a_{n-1}^2}-2\right)a_n$,求 $\lim_{n\to\infty}\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)$

解1:令

$$b_{n+1}=\frac{a_{n+1}}{a_n},b_1=\alpha+\frac{1}{\alpha}$$

则

$$b_{n+1}=b_n^2-2,b_n=\alpha^{2^{n-1}}+\frac{1}{\alpha^{2^{n-1}}}$$

故当 $n\geq 1$ 时

$$\begin{aligned}\frac{1}{a_n}&=\frac{1}{\frac{a_n}{a_{n-1}}\cdots\frac{a_1}{a_0}a_0}=\frac{1}{b_n b_{n-1}\cdots b_1}\\&=\frac{1}{\left(\alpha+\frac{1}{\alpha}\right)\cdots\left(\alpha^{2^{n-1}}+\frac{1}{\alpha^{2^{n-1}}}\right)}\\&=\frac{\alpha^2-1}{\alpha}\left(\frac{1}{\alpha^{2^n}-1}-\frac{1}{\alpha^{2^{n+1}}-1}\right)\end{aligned}$$

从而

$$\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}=1+\frac{\alpha^2-1}{\alpha}\left(\frac{1}{\alpha^2-1}-\frac{1}{\alpha^{2^{n+1}}-1}\right)$$

由 $\alpha+\frac{1}{\alpha}=3$ 得 $\alpha=\frac{3\pm\sqrt{5}}{2}$.不妨取 $\alpha=\frac{3-\sqrt{5}}{2}$,则

$$\lim_{n\to\infty}\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)=1+\alpha=\frac{5-\sqrt{5}}{2}$$

注:取 $\alpha=\frac{3+\sqrt{5}}{2}$ 可以得到相同的结论

解2:令

$$b_{n+1}=\frac{a_{n+1}}{a_n}$$

则

$$b_{n+1}=b_n^2-2$$

$$\frac{1}{b_1 b_2\cdots b_n}=\frac{1}{2}\cdot\frac{b_n^2-b_{n+1}}{b_1 b_2\cdots b_n}=\frac{1}{2}\left(\frac{b_n}{b_1 b_2\cdots b_{n-1}}-\frac{b_{n+1}}{b_1 b_2\cdots b_n}\right)$$

故

$$
\begin{aligned}
1+\frac{1}{b_1}+\cdots+\frac{1}{b_1\cdots b_n}&=1+\frac{1}{3}+\frac{1}{2}\sum_{i=2}^{n}\left(\frac{b_i}{b_1\cdots b_{i-1}}-\frac{b_{i+1}}{b_1\cdots b_i}\right)\\
&=\frac{5}{2}-\frac{b_{n+1}}{2b_1\cdots b_n}
\end{aligned}
$$

由 $b_{n+1}=b_n^2-2$ 可得 $b_{n+1}^2=b_n^4-4b_n^2+4$,从而

$$
b_{n+1}^2-4=b_n^2\left(b_n^2-4\right)=b_n^2b_{n-1}^2\left(b_{n-1}^2-4\right)=\cdots=b_n^2\cdots b_1^2\left(b_1^2-4\right)=5b_n^2\cdots b_1^2
$$

即

$$
b_1b_2\cdots b_n=\sqrt{\frac{b_{n+1}^2-4}{5}}
$$

又易知 $b_n\to+\infty\,(n\to+\infty)$,所以

$$
\begin{aligned}
\lim_{n\to\infty}\left(\frac{1}{a_0}+\frac{1}{a_1}+\cdots+\frac{1}{a_n}\right)&=\lim_{n\to\infty}\left(\frac{5}{2}-\frac{b_{n+1}}{2b_1\cdots b_n}\right)\\
&=\frac{5}{2}-\lim_{n\to\infty}\frac{b_{n+1}}{2\sqrt{\dfrac{b_{n+1}^2-4}{5}}}\\
&=\frac{5-\sqrt{5}}{2}
\end{aligned}
$$

**问题**.求 $\lim\limits_{x\to\infty}x^4\left(\arctan\dfrac{2x^2+5}{x^2+1}-\arctan\dfrac{2x^2+7}{x^2+2}\right)$

解:利用恒等变形

$$
\arctan a-\arctan b=\arctan\frac{a-b}{1+ab},ab>-1
$$

得

$$
\begin{aligned}
&\lim_{x\to\infty}x^4\left(\arctan\frac{2x^2+5}{x^2+1}-\arctan\frac{2x^2+7}{x^2+2}\right)\\
=&\lim_{x\to\infty}x^4\arctan\frac{\dfrac{2x^2+5}{x^2+1}-\dfrac{2x^2+7}{x^2+2}}{1+\dfrac{\left(2x^2+5\right)\left(2x^2+7\right)}{\left(x^2+1\right)\left(x^2+2\right)}}\\
=&\lim_{x\to\infty}x^4\arctan\frac{3}{5x^4+27x^2+37}=\frac{3}{5}
\end{aligned}
$$

**问题**.设 $p(x)$ 是实解析函数,且 $0<\prod\limits_{n=0}^{\infty}p^{(n)}(0)<\infty$,求 $\lim\limits_{x\to\infty}\dfrac{p'(x)}{p(x)}$

解:由无穷乘积满足 $0<\prod\limits_{n=0}^{\infty}p^{(n)}(0)<\infty$ 可得 $p^{(n)}(0)\to1\,(n\to\infty)$,从而

$$
\lim_{n\to\infty}\frac{p^{(n+1)}(0)}{p^{(n)}(0)}=1
$$

因此对 $\forall \varepsilon > 0, \exists n_0\, s.t. n > n_0$ 时成立不等式

$$1-\varepsilon < \frac{p^{(n+1)}(0)}{p^{(n)}(0)} < 1+\varepsilon$$

由于 $p^{(n)}(0)$ 有界对 $\forall n$ 成立,故 $p(x), p'(x)$ 可以展开成幂级数,即

$$p(x) = \sum_{n=0}^{\infty} \frac{p^{(n)}(0)\, x^n}{n!}, p'(x) = \sum_{n=0}^{\infty} \frac{p^{(n+1)}(0)\, x^n}{n!}$$

又当 $x \to \infty$ 时有

$$\sum_{n=0}^{n_0} \frac{p^{(n)}(0)}{n!} x^n = o\big(x^{n_0+1}\big) = o\left(\frac{p^{(n_0+1)}(0)\, x^{n_0+1}}{(n_0+1)!}\right) = o\left(\sum_{n=n_0+1}^{\infty} \frac{p^{(n)}(0)}{n!} x^n\right)$$

从而

$$\lim_{x\to\infty} \frac{p(x)}{\displaystyle\sum_{n=n_0+1}^{\infty} \frac{p^{(n)}(0)}{n!} x^n} = 1$$

同理可得

$$\lim_{x\to\infty} \frac{p'(x)}{\displaystyle\sum_{n=n_0+1}^{\infty} \frac{p^{(n+1)}(0)}{n!} x^n} = 1$$

故

$$\limsup_{x\to\infty} \frac{p'(x)}{p(x)} = \limsup_{x\to\infty} \frac{\displaystyle\sum_{n=n_0+1}^{\infty} \frac{p^{(n+1)}(0)}{n!} x^n}{\displaystyle\sum_{n=n_0+1}^{\infty} \frac{p^{(n)}(0)}{n!} x^n} \leq \limsup_{x\to\infty} \frac{\displaystyle\sum_{n=n_0+1}^{\infty} \frac{(1+\varepsilon) p^{(n)}(0)}{n!} x^n}{\displaystyle\sum_{n=n_0+1}^{\infty} \frac{p^{(n)}(0)}{n!} x^n} = 1+\varepsilon$$

类似的有

$$\liminf_{x\to\infty} \frac{p'(x)}{p(x)} \geq 1-\varepsilon$$

所以

$$1-\varepsilon \leq \liminf_{x\to\infty} \frac{p'(x)}{p(x)} \leq \limsup_{x\to\infty} \frac{p'(x)}{p(x)} \leq 1+\varepsilon$$

令 $\varepsilon \to 0^+$ 即可得到

$$\lim_{x\to\infty} \frac{p'(x)}{p(x)} = 1$$

注: $\displaystyle\sum_{n=0}^{n_0} \frac{p^{(n)}(0)}{n!} x^n = o\left(x^{n_0+1}\right) = o\left(\frac{p^{(n_0+1)}(0)\, x^{n_0+1}}{(n_0+1)!}\right) = o\left(\sum_{n=n_0+1}^{\infty} \frac{p^{(n)}(0)}{n!} x^n\right)$
中最后一步成立的原因是 $\displaystyle\sum_{n=n_0+1}^{\infty} \frac{p^{(n)}(0)}{n!} x^n > \frac{p^{(n_0+1)}(0)}{(n_0+1)!} x^{n_0+1}$

**问题**.求 $\lim\limits_{n\to\infty}\left(\dfrac{\sum\limits_{k=1}^{n}\sin\dfrac{2k}{2n}}{\sum\limits_{k=1}^{n}\sin\dfrac{2k-1}{2n}}\right)^{n}$

解:由积化和差公式得

$$\begin{aligned}\sum_{k=1}^{n}\sin\frac{2k}{2n}&=\frac{\sum\limits_{k=1}^{n}\sin\dfrac{2k}{2n}\sin\dfrac{1}{2n}}{\sin\dfrac{1}{2n}}\\&=\frac{\sum\limits_{k=1}^{n}\left(-\dfrac{1}{2}\right)\left(\cos\dfrac{2k+1}{2n}-\cos\dfrac{2k-1}{2n}\right)}{\sin\dfrac{1}{2n}}\\&=\frac{\left(-\dfrac{1}{2}\right)\left[\cos\left(1+\dfrac{1}{2n}\right)-\cos\dfrac{1}{2n}\right]}{\sin\dfrac{1}{2n}}\end{aligned}$$

同理可得

$$\sum_{k=1}^{n}\sin\frac{2k-1}{2n}=\frac{\left(-\dfrac{1}{2}\right)(\cos 1-1)}{\sin\dfrac{1}{2n}}$$

从而

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{\sum\limits_{k=1}^{n}\sin\dfrac{2k}{2n}}{\sum\limits_{k=1}^{n}\sin\dfrac{2k-1}{2n}}\right)^{n}&=\lim_{n\to\infty}\left(\frac{\cos\left(1+\dfrac{1}{2n}\right)-\cos\dfrac{1}{2n}}{\cos 1-1}\right)^{n}\\&=\lim_{n\to\infty}\left(\frac{\sin\left(\dfrac{1}{2}+\dfrac{1}{2n}\right)}{\sin\dfrac{1}{2}}\right)^{n}\\&=\lim_{n\to\infty}\left[1+\left(\cos\frac{1}{2n}+\sin\frac{1}{2n}\cot\frac{1}{2}-1\right)\right]^{n}\\&=\exp\left[\lim_{n\to\infty}n\cdot\left(\cos\frac{1}{2n}+\sin\frac{1}{2n}\cot\frac{1}{2}-1)\right)\right]\\&=\exp\left[\lim_{n\to\infty}\left(\frac{\sin\dfrac{1}{2n}\cot\dfrac{1}{2}}{\dfrac{1}{n}}-\frac{1-\cos\dfrac{1}{2n}}{\dfrac{1}{n}}\right)\right]\\&=e^{\frac{\cot\frac{1}{2}}{2}}\end{aligned}$$

注:此题也可利用 $Eular$ 公式求解

**问题**.设 $D_n=\begin{vmatrix} 1 & 2 & \cdots & n \\ 1 & 2^2 & \cdots & n^2 \\ \vdots & \vdots & & \vdots \\ 1 & 2^n & \cdots & n^n \end{vmatrix}$,求 $\lim\limits_{n\to\infty} D_n^{\frac{1}{n^2\ln n}}$

解:

$$D_n=n!\begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & n \\ \vdots & \vdots & & \vdots \\ 1 & 2^{n-1} & \cdots & n^{n-1} \end{vmatrix}=n!\prod_{i>j}(i-j)=n!(n-1)!\cdots 1!$$

从而

$$\lim_{n\to\infty} D_n^{\frac{1}{n^2\ln n}}=\exp\left(\lim_{n\to\infty}\frac{\ln(n!(n-1)!\cdots 1!)}{n^2\ln n}\right)$$

另外

$$\begin{aligned}\lim_{n\to\infty}\frac{\ln(n!(n-1)!\cdots 1!)}{n^2\ln n}&=\lim_{n\to\infty}\frac{\ln((n+1)!\cdots 1!)-\ln(n!\cdots 1!)}{(n+1)^2\ln(n+1)-n^2\ln n}\\&=\lim_{n\to\infty}\frac{\ln(n+1)!}{(n+1)^2\ln(n+1)-n^2\ln n}\\&=\lim_{n\to\infty}\frac{\ln(n+1)+\ln(n!)}{n^2(\ln(n+1)-\ln n)+(2n+1)\ln(n+1)}\\&=\lim_{n\to\infty}\frac{\dfrac{1}{n}+\dfrac{\ln(n!)}{n\ln(n+1)}}{\dfrac{2n+1}{n}+\dfrac{\ln\left(1+\dfrac{1}{n}\right)^n}{\ln(n+1)}}=\frac{1}{2}\end{aligned}$$

其中

$$\begin{aligned}\lim_{n\to\infty}\frac{\ln(n!)}{n\ln(n+1)}&=\lim_{n\to\infty}\frac{\ln((n+1)!)-\ln(n!)}{(n+1)\ln(n+2)-n\ln(n+1)}\\&=\lim_{n\to\infty}\frac{\ln(n+1)}{\ln(n+2)+\ln\left(1+\dfrac{1}{n+1}\right)^n}=1\end{aligned}$$

故

$$\lim_{n\to\infty} D_n^{\frac{1}{n^2\ln n}}=\sqrt{e}$$

注:本题也可利用 $Stirling$ 公式 $n!\sim\sqrt{2n\pi}\left(\dfrac{n}{e}\right)^n\,(n\to\infty)$ 进行估计

$$\lim_{n\to\infty}\frac{\ln(n+1)!}{(n+1)^2\ln(n+1)-n^2\ln n}$$
$$=\lim_{n\to\infty}\frac{\frac{1}{2}\ln(2\pi)+\frac{1}{2}\ln(n+1)+(n+1)\ln(n+1)-(n+1)}{(2n+1)\ln(n+1)+n^2\ln\left(1+\frac{1}{n}\right)}$$
$$=\lim_{n\to\infty}\frac{\frac{1}{2n}\left(1+\frac{\ln(2\pi)}{\ln(n+1)}\right)+\left(1+\frac{1}{n}\right)\left(1-\frac{1}{\ln(n+1)}\right)}{2+\frac{1}{n}+\frac{\ln\left(1+\frac{1}{n}\right)^n}{\ln(n+1)}}=\frac{1}{2}$$

**问题**. $\{a_n\}$ 满足 $a_1>1, a_{n+1}\left(a_n^2-1\right)=a_n^3$

(1) 求 $A=\lim\limits_{n\to\infty}\frac{a_n}{\sqrt{n}}$  (2) 求 $\lim\limits_{n\to\infty}\frac{n}{\ln n}\left(\frac{a_n}{\sqrt{n}}-A\right)$

解:(1) $a_2-a_1=\frac{a_1}{a_1^2-1}>0, a_2>1$

由数学归纳法易知 $\{a_n\}$ 严格单调递增且趋于 $+\infty$

利用 $Stolz$ 定理可得

$$\lim_{n\to\infty}\frac{a_n^2}{n}=\lim_{n\to\infty}\left(a_{n+1}^2-a_n^2\right)=\lim_{n\to\infty}\left(\left(\frac{a_n^3}{a_n^2-1}\right)^2-a_n^2\right)$$
$$=\lim_{n\to\infty}\frac{a_n^2}{a_n^2-1}\cdot\frac{2a_n^2-1}{a_n^2-1}=2$$

故

$$A=\lim_{n\to\infty}\frac{a_n}{\sqrt{n}}=\sqrt{2}$$

(2)由 $Stolz$ 定理可得

$$\lim_{n\to\infty}\frac{n}{\ln n}\left(\frac{a_n}{\sqrt{n}}-A\right)=\lim_{n\to\infty}\frac{n}{\ln n}\frac{\frac{a_n^2}{n}-2}{\frac{a_n}{\sqrt{n}}+\sqrt{2}}$$
$$=\frac{1}{2\sqrt{2}}\lim_{n\to\infty}\frac{a_n^2-2n}{\ln n}=\frac{1}{2\sqrt{2}}\lim_{n\to\infty}\frac{a_{n+1}^2-a_n^2-2}{\ln\left(1+\frac{1}{n}\right)}$$
$$=\frac{1}{2\sqrt{2}}\lim_{n\to\infty}n\left[\left(\frac{a_n^3}{a_n^2-1}\right)^2-a_n^2-2\right]$$
$$=\frac{1}{2\sqrt{2}}\lim_{n\to\infty}\frac{a_n^2}{2}\cdot\frac{3a_n^2-2}{a_n^4-2a_n^2+1}=\frac{3}{4\sqrt{2}}$$

**问题**.设 $a_1>0, a_{n+1}=\ln(1+a_n)\,(n\geq 1)$,求 $\lim\limits_{n\to\infty}\frac{n(na_n-2)}{\ln n}$

解:显然 $a_n$ 单调递减趋于0

利用 $Stolz$ 定理得

$$\begin{aligned}\lim_{n\to\infty} na_n &= \lim_{n\to\infty} \frac{1}{\dfrac{1}{a_{n+1}}-\dfrac{1}{a_n}} = \lim_{n\to\infty} \frac{a_n\ln(1+a_n)}{a_n-\ln(1+a_n)} \\ &= \lim_{n\to\infty} \frac{x^2}{x-\ln(1+x)} = 2\end{aligned}$$

从而

$$\begin{aligned}\lim_{n\to\infty} \frac{n(na_n-2)}{\ln n} &= \lim_{n\to\infty} \frac{(n+1)^2a_{n+1}-2(n+1)-n^2a_n+2n}{\ln(1+n)-\ln n} \\ &= \lim_{n\to\infty} \left[\left(n^3+2n^2+n\right)\ln(1+a_n)-n^3a_n-2n\right] \\ &= \lim_{n\to\infty} \left[\left(n^3+2n^2+n\right)\left(a_n-\frac{a_n^2}{2}+\frac{a_n^3}{3}+o\left(a_n^3\right)\right)-n^3a_n-2n\right] \\ &= \lim_{n\to\infty} \left(\frac{n^3a_n^3}{3}+na_n-n^2a_n^2\right)=\frac{2}{3}\end{aligned}$$

**问题**.设 $\{x_n\}$ 对任意具有非负整系数的二次多项式 $p(x)$ 均有 $\lim\limits_{n\to\infty}\left(x_n+x_{p(n)}\right)=0$,判断 $\lim\limits_{n\to\infty} x_n=0$ 是否成立,并给出证明

解: $\lim\limits_{n\to\infty} x_n=0$ 不一定成立

考虑数列

$$x_n=\cos(\pi\log_2(\log_2 n)), n\geq 1$$

则当 $n=2^{2^k}$ 时,$|x_n|=1$,此时 $\lim\limits_{n\to\infty} x_n\neq 0$

下面只需证明 $\lim\limits_{n\to\infty}\left(x_n+x_{p(n)}\right)=0$

事实上,由三角不等式

$$|\cos x-\cos y|\leq |x-y|$$

可得

$$\begin{aligned}&\left|\cos\left[\pi\log_2\left(\log_2\left(an^2+b_n+c\right)\right)\right]+\cos(\pi\log_2(\log_2 n))\right| \\ =&\left|\cos\left[\pi\log_2\left(\log_2\left(an^2+b_n+c\right)\right)\right]-\cos\left(\pi\log_2\left(\log_2 n^2\right)\right)\right| \\ =&\,\pi\log_2\left(\log_2\left(an^2+bn+c\right)\right)-\pi\log_2\left(\log_2 n^2\right) \\ \leq&\,\pi\log_2\left(1+\frac{\log_2(a+b+c)}{2\log_2 n}\right)\to 0\,(n\to\infty)\end{aligned}$$

**问题**. $a_1,b_1,c_1>0, a_1+b_1+c_1=1$,定义 $a_{n+1}=a_n^2+2b_nc_n, b_{n+1}=b_n^2+2c_na_n, c_{n+1}=c_n^2+2a_nb_n$,证明:当 $n\to\infty$ 时 $a_n,b_n,c_n$ 极限都存在,并求出这些极限值

解:易由数学归纳法得

$$a_n+b_n+c_n=1, \forall\, n=1,2,3\cdots$$

定义

$$E_n=\max\{a_n,b_n,c_n\},F_n=\min\{a_n,b_n,c_n\}$$

下面证明

$$F_n\le F_{n+1}\le E_{n+1}\le E_n,\lim_{n\to\infty}(E_n-F_n)=0$$

假设对某个 $n$ 成立

$$a_n\ge b_n\ge c_n$$

则

$$c_n=a_nc_n+b_nc_n+c_n^2\ge a_{n+1},b_{n+1},c_{n+1}\ge a_n^2+a_nb_n+a_nc_n=a_n$$

即

$$F_n\le F_{n+1}\le E_{n+1}\le E_n$$

令

$$a_n-b_n=\alpha\ge 0,b_n-c_n=\beta\ge 0,a_n-c_n=\delta=\alpha+\beta\ge 0$$

则

$$|a_{n+1}-b_{n+1}|=|a_n-b_n||a_n+b_n-2c_n|=\alpha(\delta+\beta)=\delta^2-\beta^2\le\delta^2$$

同理可得

$$|a_{n+1}-c_{n+1}|,|c_{n+1}-b_{n+1}|\le\delta^2$$

从而对任意 $n$ 成立

$$E_{n+1}-F_{n+1}\le(E_n-F_n)^2$$

又因为

$$E_1<1,F_1>0$$

故

$$E_{n+1}-F_{n+1}\le(E_n-F_n)^2\le\cdots\le(E_1-F_1)^{2^n}\to 0\,(n\to\infty)$$

即 $a_n,b_n,c_n$ 均收敛于 $\dfrac{1}{3}$

**问题**.设 $\{a_n\}$ 满足 $a_1=1,a_{n+1}=a_n+\dfrac{1}{\sum\limits_{i=1}^n a_i}$,求 $\lim\limits_{n\to\infty}\dfrac{a_n}{\sqrt{2\ln n}}$

解:记

$$S_n=\sum_{i=1}^n a_i$$

若 $a_n$ 有界,则 $\exists M>0\, s.t. a_n \le M$ 且 $S_n \le nM$,故

$$a_k - a_{k-1} = \frac{1}{S_{k-1}} \ge \frac{1}{(k-1)M}, k=2,3,\cdots$$

将以上 $n-1$ 个式子累加可得

$$a_n - a_1 \ge \frac{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}{M} \to \infty\,(n\to\infty)$$

从而 $a_n$ 无界

由数学归纳法可知 $a_n>0$,从而 $a_n$ 严格单调递增且趋于 $+\infty$,所以

$$1 < \frac{a_{n+1}}{a_n} = 1+\frac{1}{a_nS_n} < 1+\frac{1}{na_n}$$

从而

$$1 < n+1-\frac{na_n}{a_{n+1}} < \frac{1+\frac{1}{a_n}+\frac{1}{na_n}}{1+\frac{1}{na_n}}$$

即

$$\lim_{n\to\infty}\left(n+1-\frac{na_n}{a_{n+1}}\right) = \lim_{n\to\infty}\frac{(n+1)a_{n+1}-na_n}{a_{n+1}} = 1$$

由 $Stolz$ 定理可得

$$\lim_{n\to\infty}\frac{na_n}{S_n} = \lim_{n\to\infty}\frac{(n+1)a_{n+1}-na_n}{a_{n+1}} = 1$$

所以

$$\lim_{n\to\infty}\frac{n}{S_n^2} = \lim_{n\to\infty}\left(\frac{na_n}{S_n}\cdot\frac{1}{a_nS_n}\right) = 0$$

再由 $Stolz$ 定理即得

$$\lim_{n\to\infty}\frac{a_n^2}{2\ln n} = \lim_{n\to\infty}\frac{a_{n+1}^2-a_n^2}{2\ln\left(1+\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{n}{2}\left(\frac{2a_n}{S_n}+\frac{1}{S_n^2}\right) = 1$$

故

$$\lim_{n\to\infty}\frac{a_n}{\sqrt{2\ln n}} = 1$$

**问题**. $a_n>0, a_{n+1}-\dfrac{1}{a_{n+1}} = a_n+\dfrac{1}{a_n}\,(n\ge 1)$,求 $\lim\limits_{n\to\infty}\dfrac{1}{\sqrt{n}}\sum\limits_{i=1}^{n}\dfrac{1}{a_i}$

解:假设 $0<a_n<M$,则

$$a_{n+1}-a_n = \frac{1}{a_n}+\frac{1}{a_{n+1}} \Rightarrow a_n - a_1 = \sum_{i=1}^{n-1}\frac{1}{a_i}+\sum_{i=2}^{n}\frac{1}{a_i} \ge 2\,\frac{n-1}{M}$$

令 $n\to+\infty$, $a_n$ 无界,与假设矛盾!

显然 $a_n$ 严格单调递增,故 $a_n\to+\infty\,(n\to+\infty)$

将 $a_{n+1}-\dfrac{1}{a_{n+1}}=a_n+\dfrac{1}{a_n}\,(n\geq 1)$ 两边平方得

$$a_{n+1}^2+\frac{1}{a_{n+1}^2}=a_n^2+\frac{1}{a_n^2}+4$$

从而

$$a_n+\frac{1}{a_n}=\sqrt{4n+a_1^2+\frac{1}{a_1^2}-2}$$

由 $Stolz$ 公式即得

$$\lim_{n\to\infty}\frac{\sum\limits_{i=1}^{n}\dfrac{1}{a_i}}{\sqrt{n}}=\lim_{n\to\infty}\frac{\sqrt{n}+\sqrt{n+1}}{a_{n+1}}=\lim_{n\to\infty}\frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{4n+a_1^2+\dfrac{1}{a_1^2}-2}+\dfrac{1}{a_{n+1}}}=1$$

**问题**.求 $\lim\limits_{n\to\infty}\left[\dfrac{\ln(n+1)}{n+1}+\dfrac{\ln(n+2)}{n+\dfrac{1}{2}}+\cdots+\dfrac{\ln(n+n)}{n+\dfrac{1}{n}}-\ln n\right]$

解:

$$\frac{\ln(n+1)}{n+1}+\frac{\ln(n+2)}{n+\dfrac{1}{2}}+\cdots+\frac{\ln(n+n)}{n+\dfrac{1}{n}}=\sum_{k=1}^{n}\frac{\ln(n+k)}{n+\dfrac{1}{k}}=\sum_{k=1}^{n}\frac{\ln n+\ln\left(1+\dfrac{k}{n}\right)}{n+\dfrac{1}{k}}$$

注意到

$$\sum_{k=1}^{n}\frac{\ln n+\ln\left(1+\dfrac{k}{n}\right)}{n+1}\leq\sum_{k=1}^{n}\frac{\ln(n+k)}{n+\dfrac{1}{k}}\leq\sum_{k=1}^{n}\frac{\ln n+\ln\left(1+\dfrac{k}{n}\right)}{n}$$

则

$$-\frac{\ln n}{n+1}+\sum_{k=1}^{n}\frac{\ln\left(1+\dfrac{k}{n}\right)}{n}\frac{n}{n+1}\leq\sum_{k=1}^{n}\frac{\ln(n+k)}{n+\dfrac{1}{k}}-\ln n\leq\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\frac{1}{n}$$

$$\lim_{n\to\infty}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\frac{1}{n}=\int_0^1\ln(1+x)\,\mathrm{d}x=2\sqrt{2}-1$$

$$\lim_{n\to\infty}\frac{n}{n+1}\sum_{k=1}^{n}\ln\left(1+\frac{k}{n}\right)\frac{1}{n}=\int_0^1\ln(1+x)\,\mathrm{d}x=2\sqrt{2}-1$$

故

$$\lim_{n\to\infty}\left[\frac{\ln(n+1)}{n+1}+\frac{\ln(n+2)}{n+\dfrac{1}{2}}+\cdots+\frac{\ln(n+n)}{n+\dfrac{1}{n}}-\ln n\right]=2\sqrt{2}-1$$

**问题**.求 $\lim\limits_{n\to\infty}\left[\left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\cdots+\left(\frac{n}{n}\right)^n\right]$

解:利用不等式

$$\left(1-\frac{i}{n}\right)^n<e^{-i},i\in\mathbb{N}^+$$

得

$$\sum_{i=1}^{n}\left(\frac{i}{n}\right)^n=\sum_{i=0}^{n-1}\left(1-\frac{i}{n}\right)^n\le\sum_{i=0}^{n-1}e^{-i}\le\sum_{i=0}^{\infty}e^{-i}=\frac{e}{e-1}$$

对任意固定的 $k\in\mathbb{N}^+$,截取 $\sum\limits_{i=1}^{n}\left(\frac{i}{n}\right)^n$ 的后 $k+1$ 项,则

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}\left(1-\frac{i}{n}\right)^n\ge\lim_{n\to\infty}\sum_{i=0}^{k}\left(1-\frac{i}{n}\right)^n=\sum_{i=0}^{k}\lim_{n\to\infty}\left(1-\frac{i}{n}\right)^n=\sum_{i=0}^{k}e^{-i}$$

再令 $k\to\infty$ 利用夹逼法即得

$$\lim_{n\to\infty}\left[\left(\frac{1}{n}\right)^n+\left(\frac{2}{n}\right)^n+\cdots+\left(\frac{n}{n}\right)^n\right]=\frac{e}{e-1}$$

注:(1) $\left(1-\frac{i}{n}\right)^n$ 的严格递增性可由均值不等式得到

$$\left[1\cdot\left(1-\frac{i}{n}\right)^n\right]^{\frac{1}{n+1}}<\frac{1+n\left(1-\frac{i}{n}\right)}{n+1}=1-\frac{i}{n+1}$$

即

$$\left(1-\frac{i}{n}\right)^n<\left(1-\frac{i}{n+1}\right)^{n+1}$$

(2) 此题可以进行如下推广 (证明方法基本一样)

$$\lim_{n\to\infty}\sum_{k=0}^{n-1}\left(1-\frac{k-a}{n}\right)^n=\frac{e^{a+1}}{e-1}\,(a>0)$$

**问题**.求证: $\lim\limits_{n\to\infty}\dfrac{1+n+\dfrac{n^2}{2!}+\cdots+\dfrac{n^n}{n!}}{e^n}=\dfrac{1}{2}$

证:利用 $Taylor$ 展开式的 $Laplace$ 积分型余项知

$$e^n=\sum_{k=0}^{n}\frac{n^k}{k!}+\frac{1}{n!}\int_0^n e^x(n-x)^n\,\mathrm{d}x=\sum_{k=0}^{n}\frac{n^k}{k!}+\frac{n^{n+1}}{n!}\int_0^1\left[e^t(1-t)\right]^n\mathrm{d}t$$

由 $Stirling$ 公式

$$n!=\sqrt{2n\pi}\left(\frac{n}{e}\right)^n e^{\frac{\theta}{12n}},\theta\in[0,1]$$

可得原命题等价于证明

$$\lim_{n\to\infty}\sqrt{n}\int_0^1\left[e^x(1-x)\right]^n\mathrm{d}x=\sqrt{\frac{\pi}{2}}$$

利用不等式

$$e^{-\frac{x^2}{2}} \geq e^x(1-x)\,(x \geq 0)$$

得

$$\limsup_{n\to\infty} \sqrt{n}\int_0^1 \left[e^x(1-x)\right]^n \mathrm{d}x \leq \limsup_{n\to\infty} \sqrt{n}\int_0^1 e^{-\frac{nx^2}{2}}\,\mathrm{d}x = \sqrt{\frac{\pi}{2}}$$

构造函数

$$f(x) = (1-x)e^x - e^{-\frac{ax^2}{2}}(x \geq 0, a > 1)$$

则

$$f'(x) = xe^x\left(ae^{-\frac{ax^2}{2}-x} - 1\right)$$

因为

$$\lim_{x\to 0^+}\left(ae^{-\frac{ax^2}{2}-x} - 1\right) = a - 1 > 0$$

从而由极限的保号性得

$$\exists\, x_a \in (0,1)\; s.t. x \in [0, x_a] \Rightarrow \left(ae^{-\frac{ax^2}{2}-x} - 1\right) > 0$$

即

$$(1-x)e^x \geq e^{-\frac{x^2}{2}}(x \in [0, x_a]\,)$$

故

$$\liminf_{n\to\infty} \sqrt{n}\int_0^1 \left[e^x(1-x)\right]^2 \mathrm{d}x \geq \liminf_{n\to\infty} \sqrt{n}\int_0^{x_a} e^{-\frac{nax^2}{2}}\,\mathrm{d}x = \sqrt{\frac{\pi}{2a}}$$

再令 $a \to 1^+$ 即得

$$\liminf_{n\to\infty} \sqrt{n}\int_0^1 \left[e^x(1-x)\right]^2 \mathrm{d}x \geq \sqrt{\frac{\pi}{2}}$$

从而

$$\sqrt{\frac{\pi}{2}} \leq \liminf_{n\to\infty} \sqrt{n}\int_0^1 \left[e^x(1-x)\right]^2 \mathrm{d}x \leq \limsup_{n\to\infty} \sqrt{n}\int_0^1 \left[e^x(1-x)\right]^n \mathrm{d}x \leq \sqrt{\frac{\pi}{2}}$$

即

$$\lim_{n\to\infty} \sqrt{n}\int_0^1 \left(e^x(1-x)\right)^n \mathrm{d}x = \sqrt{\frac{\pi}{2}}$$

注:(1) 不等式 $e^{-\frac{x^2}{2}} \geq e^x(1-x)\,(x \geq 0)$ 的证明可采用构造函数法,令

$$h(x) = -\frac{x^2}{2} - x - \ln(1-x), 0 \leq x < 1$$

对上式求导后即得 $h(x)$ 单调递增

(2) 对 $\forall n\geq 0$, 成立 $1+\dfrac{n}{1!}+\cdots+\dfrac{n^n}{n!}>\dfrac{e^n}{2}$

解：利用 $Taylor$ 展开式的 $Laplace$ 积分型余项知

$$e^n=\sum_{k=1}^{n}\frac{k^n}{k!}+\frac{1}{n!}\int_0^n e^t(n-t)^n\,\mathrm{d}t$$

由 $n!=\Gamma(n+1)=\displaystyle\int_0^\infty u^ne^{-u}\,\mathrm{d}u$ 可得

$$\begin{aligned}\sum_{k=0}^{n}\frac{k^n}{k!}>\frac{e^n}{2}&\Leftrightarrow n!>2e^{-n}\int_0^n(n-t)^ne^t\,\mathrm{d}t\\&\Leftrightarrow n!>2\int_0^n u^ne^{-u}\,\mathrm{d}u\\&\Leftrightarrow \int_n^\infty u^ne^{-u}\,\mathrm{d}u>\int_0^n u^ne^{-u}\,\mathrm{d}u\end{aligned}$$

故只需要证明 $\displaystyle\int_n^{2n}u^ne^{-u}\,\mathrm{d}u>\int_0^n u^ne^{-u}\,\mathrm{d}u$，令

$$f(u)=u^ne^{-u}$$

则只需要证明

$$f(n+h)\geq f(n-h)\Leftrightarrow(n+h)^n\geq(n-h)^ne^{2h}\,(0\leq h\leq n)$$

考虑函数

$$g(h)=n\ln(n+h)-n\ln(n-h)-2h$$

则

$$g'(h)=\left(\frac{1}{n-h}-\frac{1}{n+h}\right)h\geq 0,g(0)=0$$

从而原命题得证

**问题.** $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{1-\cos 2^nx}{2^n}\,(x>0)$,证明:极限 $\displaystyle\lim_{x\to0^+}\frac{f(x)}{x}$ 不存在

证:先证明如下引理

$$0<a<1,\frac{11a}{24}<\frac{1-\cos a}{a}<\frac{a}{2}$$

由 $\cos a$ 的 $Taylor$ 级数展开式得

$$1-\cos a=\frac{a^2}{2}-\left(\frac{a^4}{4!}-\frac{a^6}{6!}\right)-\cdots<\frac{a^2}{2}$$

$$1-\cos a=\left(\frac{a^2}{2}-\frac{a^2}{4!}\right)-\left(\frac{a^2}{4!}-\frac{a^4}{4!}\right)-\cdots>\frac{11a^2}{24}$$

从而引理得证!

对任意正整数$k\geq 5$,令$s_k=\dfrac{\pi}{2^k},t_k=\dfrac{3\pi}{2^k}$,则

$$\begin{aligned}
\frac{f(s_k)}{s_k}&=\sum_{n=1}^{\infty}\frac{1-\cos\dfrac{\pi}{2^{k-n}}}{\dfrac{\pi}{2^{k-n}}}=\sum_{n=1}^{k}\frac{1-\cos\dfrac{\pi}{2^{k-n}}}{\dfrac{\pi}{2^{k-n}}}+\sum_{n=k+1}^{\infty}\frac{1-\cos(2^{n-k}\pi)}{2^{n-k}\pi}\\
&=\frac{1-\cos\pi}{\pi}+\frac{1-\cos\dfrac{\pi}{2}}{\dfrac{\pi}{2}}+\frac{1-\cos\dfrac{\pi}{2^2}}{\dfrac{\pi}{2^2}}+\sum_{n=1}^{k-3}\frac{1-\cos\dfrac{\pi}{2^n}}{\dfrac{\pi}{2^n}}\\
&\leq\frac{1-\cos\pi}{\pi}+\frac{1-\cos\dfrac{\pi}{2}}{\dfrac{\pi}{2}}+\frac{1-\cos\dfrac{\pi}{2^2}}{\dfrac{\pi}{2^2}}+\frac{\pi}{2}\sum_{n=3}^{k-1}\frac{1}{2^n}\\
&=\frac{8-2\sqrt{2}}{\pi}+\frac{\pi}{8}-\frac{\pi}{2^k}
\end{aligned}$$

$$\begin{aligned}
\frac{f(t_k)}{t_k}&=\sum_{n=1}^{\infty}\frac{1-\cos\dfrac{3\pi}{2^{k-n}}}{\dfrac{3\pi}{2^{k-n}}}=\sum_{n=1}^{k}\frac{1-\cos\dfrac{3\pi}{2^{k-n}}}{\dfrac{3\pi}{2^{k-n}}}+\sum_{n=k+1}^{\infty}\frac{1-\cos(2^{n-k}3\pi)}{2^{n-k}3\pi}\\
&=\frac{1-\cos 3\pi}{3\pi}+\frac{1-\cos\dfrac{3\pi}{2}}{\dfrac{3\pi}{2}}+\frac{1-\cos\dfrac{3\pi}{2^2}}{\dfrac{3\pi}{2^2}}+\frac{1-\cos\dfrac{3\pi}{2^3}}{\dfrac{3\pi}{2^3}}+\sum_{n=4}^{k-1}\frac{1-\cos\dfrac{3\pi}{2^n}}{\dfrac{3\pi}{2^n}}\\
&\geq\frac{2}{3\pi}+\frac{2}{3\pi}+\frac{4+2\sqrt{2}}{3\pi}+\frac{8-4\sqrt{2-2\sqrt{2}}}{3\pi}+\frac{11}{24}\sum_{n=4}^{k-1}\frac{3\pi}{2^n}\\
&=\frac{16+2\sqrt{2}-4\sqrt{2-2\sqrt{2}}}{3\pi}+\frac{11\pi}{64}\left(1-\frac{1}{2^{k-4}}\right)
\end{aligned}$$

注意到$k\to\infty$时$s_k$与$t_k$均单调递减趋于0,从而

$$\limsup_{k\to\infty}\frac{f(s_k)}{s_k}\leq\frac{8-2\sqrt{2}}{\pi}+\frac{\pi}{8}<2.1$$

$$\liminf_{k\to\infty}\frac{f(t_k)}{t_k}\geq\frac{16+2\sqrt{2}-4\sqrt{2-2\sqrt{2}}}{3\pi}+\frac{11\pi}{64}>2.2$$

故$\lim\limits_{x\to 0^+}\dfrac{f(x)}{x}$不存在

**问题**.求$\lim\limits_{l\to\infty}\lim\limits_{m\to\infty}\lim\limits_{n\to\infty}\sum\limits_{k=1}^{l}\sum\limits_{j=1}^{m}\sum\limits_{i=1}^{n}\dfrac{(-1)^{i+j+k}}{i+j+k}$

解1:先证明如下引理

$$\lim_{t\to\infty}\left|\int_{-1}^{0}\frac{x^t}{(1-x)^p}\,\mathrm{d}x\right|=0\,(p\in\mathbb{N}^+)$$

事实上

$$\left|\int_{-1}^{0}\frac{x^t}{(1-x)^p}\,\mathrm{d}x\right|\leq\int_{-1}^{0}|x|^t\,\mathrm{d}x\to 0\,(t\to\infty)$$

故

$$\begin{aligned}&\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{(-1)^{i+j+k}}{i+j+k}=-\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{-1}^{0}x^{i+j+k-1}\,\mathrm{d}x\\
=&-\int_{-1}^{0}\sum_{i=1}^{n}x^{j-1}\sum_{j=1}^{m}x^{j}\sum_{k=1}^{l}x^{k}\,\mathrm{d}x\\
=&-\int_{-1}^{0}\frac{x^2}{(1-x)^3}(1-x^l)(1-x^m)(1-x^n)\,\mathrm{d}x\\
=&-\int_{-1}^{0}\frac{x^2}{(1-x)^3}\,\mathrm{d}x+I\end{aligned}$$

其中$I$与引理中的积分类似,从而

$$\lim_{l\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{(-1)^{i+j+k}}{i+j+k}=-\int_{-1}^{0}\frac{x^2}{(1-x)^3}\,\mathrm{d}x=\frac{5}{8}-\ln 2$$

解2:

$$\begin{aligned}\text{原式}&=\lim_{l\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\sum_{k=1}^{l}\sum_{j=1}^{m}\sum_{i=1}^{n}\int_{0}^{\infty}(-1)^{i+j+k}e^{-(i+j+k)x}\,\mathrm{d}x\\
&=\lim_{l\to\infty}\lim_{m\to\infty}\lim_{n\to\infty}\int_{0}^{\infty}\left(\sum_{i=1}^{n}(-1)^{i}e^{-ix}\right)\left(\sum_{j=1}^{n}(-1)^{j}e^{-jx}\right)\left(\sum_{k=1}^{n}(-1)^{k}e^{-kx}\right)\mathrm{d}x\\
&=\int_{0}^{\infty}\left(\lim_{n\to\infty}\sum_{i=1}^{n}\left(-e^{-x}\right)^{i}\right)\left(\lim_{m\to\infty}\sum_{j=1}^{m}\left(-e^{-x}\right)^{j}\right)\left(\lim_{l\to\infty}\sum_{k=1}^{l}\left(-e^{-x}\right)^{k}\right)\mathrm{d}x\\
&=\int_{0}^{\infty}\left(\frac{-e^{-x}}{1+e^{-x}}\right)^{3}\mathrm{d}x=-\int_{0}^{1}\frac{t^2}{(1+t)^3}\,\mathrm{d}t=\frac{5}{8}-\ln 2\end{aligned}$$

**问题**.计算$\displaystyle\lim_{t\to0}\lim_{x\to\infty}\frac{\int_{0}^{\sqrt{t}}\int_{x^2}^{t}\sin y^2\,\mathrm{d}y\,\mathrm{d}x}{\left[\left(\frac{2}{\pi}\arctan\frac{x}{t^2}\right)^{x}-1\right]\arctan t^{\frac{3}{2}}}$

解:分别考虑上下两个式子

$$\int_{0}^{\sqrt{t}}\int_{x^2}^{t}\sin y^2\,\mathrm{d}y\,\mathrm{d}x=\frac{1}{2}\int_{0}^{t^2}\frac{\sin u}{u^{1/4}}\,\mathrm{d}u$$

$$\begin{aligned}\left(\frac{2}{\pi}\arctan\frac{x}{t^2}\right)^{x}-1&\sim e^{x\left(\frac{2}{\pi}\arctan\frac{x}{t^2}-1\right)}-1\\
&\sim x\left(\frac{2}{\pi}\arctan\frac{x}{t^2}-1\right)\\
&\to-\frac{2t^2}{\pi}\,(x\to\infty)\end{aligned}$$

$$\arctan t^{2/3} \sim t^{2/3}\,(t \to 0)$$

故原极限等于

$$\lim_{t\to 0}\frac{\dfrac{1}{2}\displaystyle\int_0^{t^2}\frac{\sin u}{u^{1/4}}\,\mathrm{d}u}{-\dfrac{2t^{7/2}}{\pi}}=-\frac{\pi}{4}\lim_{t\to 0}\frac{2t\dfrac{\sin t^2}{t^{1/2}}}{\dfrac{7}{2}t^{5/2}}=-\frac{\pi}{7}$$

**问题**.设 $l, s \in \mathbb{N}$,且 $\lim\limits_{n\to\infty}\dfrac{x_n}{n^l}=x, \lim\limits_{n\to\infty}\dfrac{y_n}{n^s}=y$,求证: $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}x_k y_{n+1-k}}{n^{l+s+1}}=xy\dfrac{l!s!}{(l+s+1)!}$

注:事实上,此题可以进行如下推广

设 $l, s \geq 0, \lim\limits_{n\to\infty}\dfrac{x_n}{n^l}=x, \lim\limits_{n\to\infty}\dfrac{y_n}{n^s}=y$,求证: $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=1}^{n}x_k y_{n+1-k}}{n^{l+s+1}}=xy\mathrm{B}(l+1, s+1)$ (下面给出推广后的证明)

证:根据题设条件,存在收敛于0的数列 $\{a_n\}, \{b_n\}$ 使得

$$x_n=(x+a_n)n^l, y_n=(y+b_n)n^s$$

令

$$M=\sup_{n\geq 1}\frac{|y_n|}{n^s}$$

则

$$\sum_{k=1}^{n}x_k y_{n+1-k}=xy\sum_{k=1}^{n}k^l(n+1-k)^s+x\sum_{k=1}^{n}k^l b_{n+1-k}(n+1-k)^s+\sum_{k=1}^{n}a_k k^l y_{n+1-k}$$

其中

$$\begin{aligned}\sum_{k=1}^{n}x_k y_{n+1-k}&=xy\sum_{k=1}^{n}\left(\frac{k}{n}\right)^l\left(1-\frac{k-1}{n}\right)^s\frac{1}{n}\\&\to xy\int_0^1 t^l(1-t)^s\,\mathrm{d}t\\&=xy\mathrm{B}(l+1, s+1)\,(n\to\infty)\end{aligned}$$

$$\left|\frac{x\sum\limits_{k=1}^{n}k^l b_{n+1-k}(n+1-k)^s}{n^{l+s+1}}\right|\leq|x|\frac{\sum\limits_{k=1}^{n}|b_k|k^s}{n^{s+1}}\xrightarrow{Stolz}0\,(n\to\infty)$$

$$\left|\frac{\sum\limits_{k=1}^{n}a_kk^ly_{n+1-k}}{n^{l+s+1}}\right|\leq M\frac{\sum\limits_{k=1}^{n}a_kk^l}{n^{l+1}}\xrightarrow{Stolz}0\,(n\to\infty)$$

从而

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}x_ky_{n+1-k}}{n^{l+s+1}}=xy\mathrm{B}(l+1,s+1)$$

注:上述证明过程中用到以下引理

设 $f,g\in R[a,b],\xi,\xi'$ 是从属于分划 $P$ 的两个不同的介点集,则

$$\lim_{||P||\to 0}\sum_{k=1}^{n}f(\xi_k)g(\xi_k')\triangle x_k=\int_a^b f(x)g(x)\,\mathrm{d}x$$

**问题**.求 $\lim\limits_{n\to\infty}\dfrac{1}{(2n-1)^{2011}}\sum\limits_{k=0}^{n-1}\displaystyle\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3x\cos^2x\,\mathrm{d}x$

解:由推广的积分第一中值定理,对 $\forall n,\exists\,\theta_n\in(0,1)\;s.t.$

$$\int_{2n\pi}^{(2n+1)\pi}x^{2010}\sin^3x\cos^2x\,\mathrm{d}x=((2n+\theta_n)\pi)^{2010}\int_{2n\pi}^{(2n+1)\pi}\sin^3x\cos^2x\,\mathrm{d}x$$

从而

$$\begin{aligned}&\int_{2n\pi}^{(2n+1)\pi}x^{2010}\sin^3x\cos^2x\,\mathrm{d}x\\=&\left((2n\pi)^{2010}+o(n^{2010})\right)\int_{2n\pi}^{(2n+1)\pi}\sin^3x\cos^2x\,\mathrm{d}x\\=&\left((2n\pi)^{2010}+o(n^{2010})\right)\int_0^{\pi}(\cos^2x-1)\cos^2x\,\mathrm{d}\cos x\\=&\frac{4}{15}\left((2n\pi)^{2010}+o(n^{2010})\right)(n\to\infty)\end{aligned}$$

另外

$$(2n+1)^{2011}-(2n-1)^{2011}=4022(2n)^{2010}+o\left(n^{2010}\right)(n\to\infty)$$

由 $Stolz$ 定理可得

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{(2n-1)^{2011}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^{2010}\sin^3x\cos^2x\,\mathrm{d}x\\=&\lim_{n\to\infty}\frac{\displaystyle\int_{2n\pi}^{(2n+1)\pi}x^{2010}\sin^3x\cos^2x\,\mathrm{d}x}{(2n+1)^{2011}-(2n-1)^{2011}}\\=&\frac{2}{30165}\lim_{n\to\infty}\frac{(2n\pi)^{2010}+o(n^{2010})}{(2n)^{2010}+o(n^{2010})}\\=&\frac{2\pi^{2010}}{30165}\end{aligned}$$

注:本题可以进行如下推广

$$\lim_{n\to\infty}\frac{1}{(2n-1)^{p+1}}\sum_{k=0}^{n-1}\int_{2k\pi}^{(2k+1)\pi}x^p\sin^3 x\cos^2 x\,\mathrm{d}x=\frac{2\pi^p}{15(p+1)}(p>0)$$

**问题**.设 $\lim\limits_{n\to\infty}x_n=+\infty$ ,正项级数 $\sum\limits_{n=1}^{\infty}y_n$ 收敛,设 $n_0\in\mathbb{N}$,若 $n>n_0$ 时

$$x_n<x_{n+1},x_n\le\frac{1}{2}(x_{n-1}+x_{n+1}),y_{n+1}\le y_n$$

求证:

$$\lim_{n\to\infty}\frac{x_ny_n}{x_{n+1}-x_n}=0$$

证1:不妨假设 $n_0=0$.先证明如下引理

正项级数 $\sum\limits_{n=1}^{\infty}y_n$ 收敛, $y_{n+1}\le y_n$,则 $\lim\limits_{n\to\infty}ny_n=0$

由 $Cauchy$ 收敛准则知

$$对\,\forall\varepsilon>0,\exists N\,s.t.当\,k\ge N\,时\left|\sum_{k=n}^{2n}y_k\right|<\varepsilon$$

则

$$\frac{1}{2}\cdot 2ny_{2n}\le\sum_{k=n}^{2n}y_k=\left|\sum_{k=n}^{2n}y_k\right|<\varepsilon$$

又

$$(2n+1)y_{2n+1}=y_{2n+1}+2ny_{2n}\cdot\frac{y_{2n+1}}{y_{2n}}\le y_{2n+1}+2ny_{2n}$$

即

$$(2n+1)y_{2n+1}<y_{2n+1}+2\varepsilon$$

从而由 $\varepsilon$ 的任意性即知 $\lim\limits_{n\to\infty}ny_n=0$

由 $x_{n+1}>x_n$ 可得对 $\forall N,\exists c\in(0,1)\,s.t.$

$$N\left(\frac{x_{N+1}}{x_N}-1\right)>c$$

假设 $n=k$ 时成立

$$\frac{x_{k+1}}{x_k}>1+\frac{c}{k}$$

则当 $n=k+1$ 时有

$$\frac{x_{k+2}}{x_{k+1}}>2-\frac{x_k}{x_{k+1}}>2-\frac{k}{k+c}=1+\frac{c}{k+c}>1+\frac{c}{k+1}$$

从而由数学归纳法即知对 $n \geq N, \exists c \in (0,1)\ s.t. \dfrac{x_{n+1}}{x_n} > 1 + \dfrac{c}{n}$,故

$$\lim_{n\to\infty} \frac{x_n y_n}{x_{n+1} - x_n} = \lim_{n\to\infty} \frac{n y_n}{n\left(\dfrac{x_{n+1}}{x_n} - 1\right)} = 0$$

证2:不妨假设 $n_0 = 0$

由 $x_n \leq \dfrac{1}{2}(x_{n-1} + x_{n+1})$ 可得

$$\frac{x_{n-1}}{x_{n+1} - x_n} \leq \frac{x_{n-1}}{x_n - x_{n-1}}, n \geq 2$$

则

$$\begin{aligned}
&\sum_{k=2}^{n} \left| \frac{x_n y_n}{x_{n+1} - x_n} - \frac{x_{n-1} y_{n-1}}{x_n - x_{n-1}} \right| \\
=& \sum_{k=2}^{n} \left| \frac{(x_n - x_{n-1} + x_{n-1}) y_n}{x_{n+1} - x_{n-1}} - \frac{x_{n-1}(y_{n-1} - y_n + y_n)}{x_n - x_{n-1}} \right| \\
=& \sum_{k=2}^{n} \left| \frac{x_n - x_{n-1}}{x_{n+1} - x_n} y_n + \left( \frac{x_{n-1}}{x_{n+1} - x_n} - \frac{x_{n-1}}{x_n - x_{n-1}} \right) y_n + \frac{x_{n-1}(y_n - y_{n-1})}{x_n - x_{n-1}} \right| \\
\leq& \sum_{k=2}^{n} \left[ \frac{x_n - x_{n-1}}{x_{n+1} - x_n} y_n + \left( \frac{x_{n-1}}{x_n - x_{n-1}} - \frac{x_{n-1}}{x_{n+1} - x_n} \right) y_n + \frac{x_{n-1}(y_{n-1} - y_n)}{x_n - x_{n-1}} \right] \\
\leq& \sum_{k=2}^{n} \left[ 2\,\frac{x_n - x_{n-1}}{x_{n+1} - x_n} y_n + \left( \frac{x_{n-1} y_{n-1}}{x_n - x_{n-1}} - \frac{x_n y_n}{x_{n+1} - x_n} \right) \right] \\
\leq& \, 2\sum_{k=2}^{n} y_n + \frac{x_1 y_1}{x_2 - x_1} < +\infty
\end{aligned}$$

从而 $\left\{ \dfrac{x_n y_n}{x_{n+1} - x_n} \right\}$ 是有界变差数列

根据 $Cauchy$ 收敛准则即知 $\lim\limits_{n\to\infty} \dfrac{x_n y_n}{x_{n+1} - x_n}$ 存在

再由 $Stolz$ 定理可得

$$\lim_{n\to\infty} \frac{x_n y_n}{x_{n+1} - x_n} = \lim_{n\to\infty} \frac{\sum\limits_{k=1}^{n} x_k y_k}{x_{n+1}}$$

记 $S_n = \sum\limits_{k=1}^{n} y_k$,则由 $Abel$ 变换可得

$$0 < \frac{\sum\limits_{k=1}^{n} x_k y_k}{x_{n+1}} < \frac{\sum\limits_{k=1}^{n} x_k y_k}{x_n} = S_n - \frac{\sum\limits_{k=1}^{n-1} S_k (x_{k+1} - x_k)}{x_n}$$

两边同时令 $n\to\infty$ 由 $Stolz$ 定理即得

$$\lim_{n\to\infty}\frac{x_ny_n}{x_{n+1}-x_n}=\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}x_ky_k}{x_{n+1}}=0$$

**问题**.设正项级数 $\sum\limits_{n=1}^{\infty}a_n$ 收敛,并且

$$a_{n+1}<\frac{1}{2}(a_n+a_{n+2}),\frac{1}{a_{n+2}}-\frac{1}{a_{n+1}}\le\frac{1}{3}\left(\frac{1}{a_{n+3}}-\frac{1}{a_n}\right)$$

证明:

$$\lim_{n\to\infty}\frac{a_na_{n+2}(a_n-a_{n+1})}{a_na_{n+1}-2a_na_{n+2}+a_{n+1}a_{n+2}}=0$$

证:由

$$a_{n+1}<\frac{1}{2}(a_n+a_{n+2})$$

得

$$a_n-a_{n+1}>a_{n+1}-a_{n+2}$$

若 $\exists n_0\ s.t.a_{n_0+1}\ge a_{n_0}$,则

$$0\le a_{n_0+1}-a_{n_0}<a_{n_0+2}-a_{n_0+1}$$

从而当 $n\ge n_0$ 时,$\{a_{n+1}-a_n\}$单调递增,这与 $\sum\limits_{n=1}^{\infty}(a_{n+1}-a_n)$ 收敛矛盾!

故 $\{a_n\}$ 严格单调递减.由

$$\frac{1}{a_{n+2}}-\frac{1}{a_{n+1}}\le\frac{1}{3}\left(\frac{1}{a_{n+3}}-\frac{1}{a_n}\right)$$

得

$$\frac{1}{a_{n+2}}-\frac{1}{a_{n+1}}\le\frac{1}{2}\left[\left(\frac{1}{a_{n+3}}-\frac{1}{a_{n+2}}\right)+\left(\frac{1}{a_{n+1}}-\frac{1}{a_n}\right)\right]$$

又

$$\begin{aligned}a_n^2&=\sum_{k=n}^{\infty}(a_k^2-a_{k+1}^2)\\&=\sum_{k=n}^{\infty}(a_k+a_{k+1})(a_k-a_{k+1})\\&<(a_n-a_{n+1})\sum_{k=n}^{\infty}(a_k+a_{k+1})\end{aligned}$$

则

$$\frac{a_n-a_{n+1}}{a_na_{n+1}}>\frac{a_n-a_{n+1}}{a_n^2}>\frac{1}{\sum\limits_{k=n}^{\infty}(a_k+a_{k+1})}$$

从而

$$\lim_{n\to\infty}\left(\frac{1}{a_{n+1}}-\frac{1}{a_n}\right)=\lim_{n\to\infty}\frac{a_n-a_{n+1}}{a_na_{n+1}}=+\infty$$

若对任意大的$N,\exists n_1>N\,s.t.\frac{1}{a_{n_1+2}}-\frac{1}{a_{n_1+1}}\geq\frac{1}{a_{n_1+3}}-\frac{1}{a_{n_1+2}}$,则易知

$$\frac{1}{a_{n_1+3}}-\frac{1}{a_{n_1+2}}\leq\frac{1}{a_{n_1+2}}-\frac{1}{a_{n_1+1}}\leq\frac{1}{a_{n_1+1}}-\frac{1}{a_{n_1}}$$

故

$$\frac{1}{a_{n_1+1}}-\frac{1}{a_{n_1}}\leq\frac{1}{a_{n_1}}-\frac{1}{a_{n_1-1}}\leq\cdots\leq\frac{1}{a_2}-\frac{1}{a_1}$$

令$n_1\to\infty$即推出矛盾!

从而$\left\{\frac{1}{a_{n+1}}-\frac{1}{a_n}\right\}$严格单调递增

再令$x_n=\frac{1}{a_{n+1}}-\frac{1}{a_n},y_n=a_n$,则由上一题结论即知

$$\begin{aligned}&\lim_{n\to\infty}\frac{a_na_{n+2}(a_n-a_{n+1})}{a_na_{n+1}-2a_na_{n+2}+a_{n+1}a_{n+2}}\\=&\lim_{n\to\infty}\frac{a_n\left(\frac{1}{a_{n+1}}-\frac{1}{a_n}\right)}{\left(\frac{1}{a_{n+2}}-\frac{1}{a_{n+1}}\right)-\left(\frac{1}{a_{n+1}}-\frac{1}{a_n}\right)}\\=&0\end{aligned}$$

**问题**.证明: $\lim\limits_{n\to\infty}\left(\max\limits_{0\leq x<+\infty}\left|e^{-x}-\frac{1}{\sum\limits_{k=0}^{n}\frac{x^k}{k!}}\right|\right)^{\frac{1}{n}}=\frac{1}{2}$

证:令

$$f(x)=\frac{1}{\sum\limits_{k=0}^{n}\frac{x^k}{k!}}-e^{-x}$$

注意到

$$f(0)=\lim_{x\to+\infty}f(x)=0$$

从而

$$\exists\, x_n\,s.t.|f(x_n)|=\max_{0\leq x<+\infty}|f(x)|>0,f'(x_n)=0$$

则

$$e^{-x_n}=\frac{\sum\limits_{k=0}^{n-1}\frac{x_n^k}{k!}}{\left(\sum\limits_{k=0}^{n}\frac{x_n^k}{k!}\right)^2}$$

$$f(x_n)=\frac{x_n^n}{n!\left(\sum\limits_{k=0}^{n}\frac{x_n^k}{k!}\right)^2}<\frac{x_n^n}{n!\left(\frac{1}{0!n!}+\frac{1}{1!(n-1)!}+\cdots+\frac{1}{n!0!}\right)x_n^n}=\frac{1}{2^n}$$

故

$$\limsup_{n\to\infty}\left(\max_{0\le x<+\infty}\left|e^{-x}-\frac{1}{\sum\limits_{k=0}^{n}\frac{x^k}{k!}}\right|\right)^{\frac{1}{n}}\le\frac{1}{2}$$

另一方面,取 $x_n=\frac{n}{2}$,则

$$f(y_n)=\frac{e^{y_n}-\sum\limits_{k=0}^{n}\frac{y_n^k}{k!}}{\sum\limits_{k=0}^{n}\frac{y_n^k}{k!}e^{y_n}}=\frac{\sum\limits_{k=n+1}^{\infty}\frac{y_n^k}{k!}}{\sum\limits_{k=0}^{n}\frac{y_n^k}{k!}e^{y_n}}>\frac{\frac{y_n^{n+1}}{(n+1)!}}{e^{2y_n}}=\frac{n^{n+1}}{2^{n+1}e^n(n+1)!}$$

再由 $Stirling$ 公式即知

$$\liminf_{n\to\infty}\left(\max_{0\le x<+\infty}\left|e^{-x}-\frac{1}{\sum\limits_{k=0}^{n}\frac{x^k}{k!}}\right|\right)^{\frac{1}{n}}\ge\frac{1}{2}$$

从而

$$\lim_{n\to\infty}\left(\max_{0\le x<+\infty}\left|e^{-x}-\frac{1}{\sum\limits_{k=0}^{n}\frac{x^k}{k!}}\right|\right)^{\frac{1}{n}}=\frac{1}{2}$$

**问题.证明:** $\lim\limits_{n\to\infty}\left(n\max\limits_{0\le x<+\infty}\left|e^{-x}-\frac{1}{\left(1+\frac{x}{n}\right)^n}\right|\right)=\frac{2}{e^2}$

证:令

$$f(x)=\frac{1}{\left(1+\frac{x}{n}\right)^n}-e^{-x}$$

注意到

$$f(0)=f(+\infty)=0$$

从而

$$\exists\, x_n\, s.t. |f(x_n)|=\max_{0\le x<+\infty}|f(x)|>0, f'(x_n)=0$$

即

$$1+\frac{x_n}{n}=\exp\frac{x_n}{n+1}$$

故

$$f(x_n)=\frac{x_n}{ne^{x_n}}$$

又

$$\exp\frac{x_n}{n+1}=1+\frac{x_n}{n+1}+\frac{1}{2!}\left(\frac{x_n}{n+1}\right)^2+\frac{1}{3!}\left(\frac{x_n}{n+1}\right)^3+\cdots$$

从而

$$1+\frac{1}{n}=\frac{x_n}{2}+\frac{1}{3!}\frac{x_n^2}{n+1}+\frac{1}{4!}\frac{x_n^3}{(n+1)^2}+\cdots$$

显然 $a_n$ 有界,两端令 $n\to\infty$ 即得

$$\lim_{n\to\infty}x_n=2$$

故

$$\lim_{n\to\infty}\left(n\max_{0\le x<+\infty}\left|e^{-x}-\frac{1}{\left(1+\frac{x}{n}\right)^n}\right|\right)=\frac{2}{e^2}$$

**问题**.设数列 $\{y_n\}$ 满足:

$$(n+1)(n-2)y_{n+1}=n(n^2-n-1)y_n-(n-1)^3y_{n-1}, y_2=y_3=1$$

求极限

$$\lim_{n\to\infty}\left(y_n-\frac{1}{n}\right)^{\frac{1}{n^2}}$$

解:令

$$x_n=ny_n$$

由

$$(n-2)x_{n+1}=n(n^2-n+1)x_n-(n-1)^2x_{n-1}$$

易得

$$\frac{x_{n+1}-x_n}{n-1}=(n-1)\frac{x_n-x_{n-1}}{n-2}$$

迭代后即知

$$x_{n+1}-x_n=n!-(n-1)!$$

从而

$$y_n=\frac{x_n}{n}=\frac{(n-1)!-1}{n}$$

故

$$\lim_{n\to\infty}\left(y_n-\frac{1}{n}\right)^{\frac{1}{n^2}}=\lim_{n\to\infty}\left(\frac{(n-1)!}{n}\right)^{\frac{1}{n^2}}=1$$

**问题**.设数列 $\{a_n\}$ 满足 $\lim\limits_{n\to\infty}\dfrac{x_{n+1}-x_n}{n}=a$,求证:

$$\lim_{n\to\infty}\left(\frac{x_{n+1}}{\sqrt[n+1]{(n+1)!}}-\frac{x_n}{\sqrt[n]{n!}}\right)=\frac{ae}{2}$$

证:先证明如下引理

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)=\frac{1}{e}$$

由

$$\left(1+\frac{1}{n}\right)^n<e<\left(1+\frac{1}{n}\right)^{n+1}$$

易得

$$\left(\frac{n+1}{e}\right)^n<n!<e\left(\frac{n+1}{e}\right)^{n+1}$$

令

$$\begin{aligned}a_n&=\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\\&=\sqrt[n]{n!}\left(\left(\frac{\sqrt[n]{(n+1)!}}{\sqrt[n+1]{n!}}\right)^{\frac{1}{n(n+1)}}-1\right)\end{aligned}$$

则

$$\frac{n+1}{e}\left(\left(\frac{e^n}{n+1}\right)^{\frac{1}{n(n+1)}}-1\right)<a_n<\frac{n+1}{e}\sqrt[n]{n+1}\left((e^n)^{\frac{1}{n(n+1)}}-1\right)$$

一方面

$$\begin{aligned}&\lim_{n\to\infty}\frac{n+1}{e}\sqrt[n]{n+1}\left((e^n)^{\frac{1}{n(n+1)}}-1\right)\\=&\lim_{n\to\infty}\frac{\sqrt[n]{n+1}}{e}(n+1)\left(e^{\frac{1}{n+1}}-1\right)=\frac{1}{e}\end{aligned}$$

另一方面

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{n+1}{e}\sqrt[n]{n+1}\left(\left(e^n\right)^{\frac{1}{n(n+1)}}-1\right)\\
=&\lim_{n\to\infty}\frac{1}{e}(n+1)\left(e^{\frac{1}{n+1}}-e^{\frac{\ln(n+1)}{n(n+1)}}\right)\\
=&\lim_{n\to\infty}\frac{1}{e}(n+1)\left[\left(e^{\frac{1}{n+1}}-1\right)-\left(e^{\frac{\ln(n+1)}{n(n+1)}}-1\right)\right]\\
=&\lim_{n\to\infty}\frac{1}{e}\left(\frac{e^{\frac{1}{n+1}}-1}{\dfrac{1}{n+1}}-\frac{e^{\frac{\ln(n+1)}{n(n+1)}}-1}{\dfrac{\ln(n+1)}{n(n+1)}}\cdot\frac{\ln(n+1)}{n}\right)=\frac{1}{e}
\end{aligned}
$$

从而引理得证!

由 $Stolz$ 定理可得

$$\lim_{n\to\infty}\frac{x_{n+1}}{\sum\limits_{k=1}^{n+1}k}=\lim_{n\to\infty}\frac{x_{n+1}-x_n}{n+1}=a$$

即

$$\lim_{n\to\infty}\frac{x_n}{n(n+1)}=\frac{a}{2}$$

又依题意有

$$x_{n+1}=x_n+an+o(n)$$

故

$$
\begin{aligned}
&\frac{x_{n+1}}{\sqrt[n+1]{(n+1)!}}-\frac{x_n}{\sqrt[n]{n!}}\\
=&\frac{-\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)x_n+an\sqrt[n]{n!}+\sqrt[n]{n!}\,o(n)}{\dfrac{n}{e}\cdot\dfrac{n+1}{e}}\\
\to&-\frac{ae}{2}+ae=\frac{ae}{2}\,(n\to\infty)
\end{aligned}
$$

**问题**.设正数列 $a_n$ 满足 $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{n^2\cdot a_n}=b$,求证:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\right)=\frac{b}{e^2}$$

证:不妨设 $b=1$,否则用 $c_n=\dfrac{a_n}{b^n}$ 代替 $a_n$

令 $z_n=\dfrac{a_n}{n^{2n}}$,则

$$\frac{z_{n+1}}{z_n}=\frac{\dfrac{a_{n+1}}{(n+1)^{2(n+1)}}}{\dfrac{a_n}{n^{2n}}}=\frac{a_{n+1}}{(n+1)^2\,a_n}\cdot\frac{1}{\left(1+\dfrac{1}{n}\right)^{2n}}\to\frac{1}{e^2}\,(n\to\infty)$$

$$\lim_{n\to\infty}\sqrt[n]{z_n}=\lim_{n\to\infty}\sqrt[n]{z_1\cdot\frac{z_2}{z_1}\cdots\frac{z_n}{z_{n-1}}}=\lim_{n\to\infty}\frac{z_{n+1}}{z_n}=\frac{1}{e^2}$$

$$\lim_{n\to\infty}\left(\frac{(n+1)\sqrt[n+1]{z_{n+1}}}{n\sqrt[n]{z_n}}\right)^n=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\frac{z_{n+1}}{z_n}\frac{1}{\sqrt[n+1]{z_n}}=e$$

从而

$$\begin{aligned}
&\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\\
=&(n+1)\sqrt[n+1]{z_{n+1}}-n\sqrt[n]{z_n}\\
=&\sqrt[n]{z_n}\left(\frac{(n+1)\sqrt[n+1]{z_{n+1}}}{n\sqrt[n]{z_n}}-1\right)n\\
=&\sqrt[n]{z_n}\frac{\frac{(n+1)\sqrt[n+1]{z_{n+1}}}{n\sqrt[n]{z_n}}-1}{\ln\frac{(n+1)\sqrt[n+1]{z_{n+1}}}{n\sqrt[n]{z_n}}}\ln\left(\frac{(n+1)\sqrt[n+1]{z_{n+1}}}{n\sqrt[n]{z_n}}\right)^n\\
\to&\frac{1}{e^2}\,(n\to\infty)
\end{aligned}$$

**问题**.设 $f_1(x)=x,f_2(x)=x^x,\cdots,f_n(x)=x^{f_{n-1}(x)}$,求极限

$$\lim_{x\to1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}$$

解:先考虑 $n=2$ 的情形,则

$$\lim_{x\to1}\frac{f_2(x)-f_1(x)}{(1-x)^2}=\lim_{x\to1}\frac{x\left(x^{x-1}-1\right)}{(x-1)^2}=1$$

从而当 $n\geq3$ 时,有

$$\begin{aligned}
&\lim_{x\to1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}\\
=&\lim_{x\to1}\frac{x^{f_{n-1}(x)}-x^{f_{n-2}(x)}}{(1-x)^n}\\
=&\lim_{x\to1}\frac{x^{f_{n-2}(x)}\left(x^{f_{n-1}(x)-f_{n-2}(x)}-1\right)}{(1-x)^n}\\
=&\lim_{x\to1}\frac{(f_{n-1}(x)-f_{n-2}(x))\ln x}{(1-x)^n}\\
=&-\lim_{x\to1}\frac{f_{n-1}(x)-f_{n-2}(x)}{(1-x)^{n-1}}\\
=&\cdots=\lim_{x\to1}(-1)^{n-2}\frac{f_2(x)-f_1(x)}{(1-x)^2}\\
=&(-1)^{n-2}=(-1)^n
\end{aligned}$$

**问题**.设二重序列 $\{a_{n,k}\}$ 满足 $a_{n,1}=\frac{1}{2},a_{n,k+1}=a_{n,k}+\frac{1}{n}a_{n,k}^2,k=1,2,\cdots,n,n=1,2,\cdots$,则有如下结论:

(1) 若级数 $\sum_{n=1}^{\infty} x_n$ 收敛于 $A$,则 $\lim_{n\to\infty}(a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n) = \frac{1}{2}A$

(2) 若序列 $\{b_n\}$ 收敛于 $b$,则 $\lim_{n\to\infty}\frac{a_{n,1}b_1 + a_{n,2}b_2 + \cdots + a_{n,n}b_n}{n} = b\ln 2$

证:首先证明如下引理

$$\frac{n+1}{2n-k+3} \leq a_{n,k} \leq \frac{n}{2n-k+1}$$

由题中条件知

$$a_{n,k+1} = a_{n,k} + \frac{1}{n}a_{n,k}^2 \leq a_{n,k} + \frac{1}{n}a_{n,k}\cdot a_{n,k+1}$$

从而

$$\frac{1}{a_{n,k}} - \frac{1}{a_{n,k+1}} \leq \frac{1}{n}, \cdots, \frac{1}{a_{n,1}} - \frac{1}{a_{n,2}} \leq \frac{1}{n}$$

叠加后得

$$a_{n,k} \leq \frac{n}{2n-k+1}$$

又

$$\frac{1}{a_{n,k}} - \frac{1}{a_{n,k+1}} = \frac{1}{n+a_{n,k}} \geq \frac{2n-k+1}{n(2n-k+2)}, \cdots, \frac{1}{a_{n,1}} - \frac{1}{a_{n,2}} \geq \frac{2n}{n(2n+1)}$$

叠加后得

$$\begin{aligned}
\frac{1}{a_{n,1}} - \frac{1}{a_{n,k+1}} &\geq \frac{2n}{n(2n+1)} + \frac{2n-1}{n\cdot 2n} + \cdots + \frac{2n-k+1}{n(2n-k+2)} \\
&\geq \frac{k}{n}\cdot\frac{2n-k+1}{2n-k+2} \\
&\geq \frac{k}{n+1}
\end{aligned}$$

则

$$a_{n,k} \geq \frac{n+1}{2n-k+3}$$

从而引理证毕!

(1) 由引理可知对任意固定的 $k$ 有

$$\lim_{n\to\infty} a_{n,k} = \frac{1}{2}, \lim_{n\to\infty} a_{n,n} = 1$$

令

$$b_{n,k} = a_{n,k} - \frac{1}{2}$$

则令原命题等价于证明

$$\lim_{n\to\infty}\sum_{k=1}^{n} b_{n,k}x_k = 0$$

依题意得

$$\text{对}\,\forall\varepsilon>0,\exists\, k_0\in\mathbb{N}^+\, s.t. n\geq k_0\text{ 时}\Rightarrow\left|\sum_{k=k_0}^{n}x_k\right|<\varepsilon$$

由 $Abel$ 变换得

$$\begin{aligned}\left|\sum_{k=1}^{n}b_{n,k}x_k\right|&\leq\left|\sum_{k=1}^{k_0-1}b_{n,k}x_k\right|+\left|\sum_{k=k_0}^{n}b_{n,k}x_k\right|\\&=\left|\sum_{k=1}^{k_0-1}b_{n,k}x_k\right|+\left|b_{n,n}\sum_{k=k_0}^{n}x_k+\sum_{k=k_0}^{n-1}\left((b_{n,k}-b_{n,k+1})\sum_{i=k_0}^{k}x_i\right)\right|\\&\leq\left|\sum_{k=1}^{k_0-1}b_{n,k}x_k\right|+\varepsilon(2b_{n,n}-b_{n,k_0})\end{aligned}$$

令 $n\to\infty$ 取上极限并由 $\varepsilon$ 的任意性即知

$$\lim_{n\to\infty}\sum_{k=1}^{n}b_{n,k}x_k=0$$

(2) 由引理可知

$$\left(1+\frac{1}{n}\right)\sum_{k=1}^{n}\frac{1}{2n-k+3}\leq\sum_{k=1}^{n}\frac{a_{n,k}}{n}\leq\sum_{k=1}^{n}\frac{1}{2n-k+1}$$

由 $\displaystyle\sum_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+\varepsilon_n\,(\varepsilon_n\to0,n\to\infty)$ 及夹逼原理得

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{a_{n,k}}{n}=\ln 2$$

又由序列 $\{b_n\}$ 收敛于 $b$ 得

$$\text{对}\,\forall\varepsilon>0,\exists\, k'\in\mathbb{N}^+\, s.t. k\geq k'\text{ 时}\Rightarrow|b_k-b|<\varepsilon$$

从而

$$\begin{aligned}\left|\frac{\sum\limits_{k=1}^{n}a_{n,k}b_k}{n}-b\ln 2\right|&\leq\left|\frac{\sum\limits_{k=1}^{n}a_{n,k}(b_k-b)}{n}\right|+\left|b\frac{\sum\limits_{k=1}^{n}a_{n,k}}{n}-b\ln 2\right|\\&\leq\left|\frac{\sum\limits_{k=1}^{k'}a_{n,k}(b_k-b)}{n}\right|+\left|\frac{\sum\limits_{k=k'+1}^{n}a_{n,k}(b_k-b)}{n}\right|+\left|b\frac{\sum\limits_{k=1}^{n}a_{n,k}}{n}-b\ln 2\right|\end{aligned}$$

令$n\to\infty$取上极限即得

$$\lim_{n\to\infty}\frac{\sum\limits_{k=1}^{n}a_{n,k}b_k}{n}=b\ln 2$$

**问题**.设$\{a_k\},\{b_k\},\{\xi_k\}$为非负数列,并且对$\forall k\geq 0$,有

$$a_{k+1}^2\leq(a_k+b_k)^2-\xi_k^2$$

(1) 证明:$\sum\limits_{i=1}^{k}\xi_i^2\leq\left(a_1+\sum\limits_{i=0}^{k}b_i\right)^2$

(2) 若数列$\{b_k\}$还满足$\sum\limits_{k=0}^{\infty}b_k^2<+\infty$，证明:$\lim\limits_{k\to\infty}\frac{1}{k}\sum\limits_{i=1}^{k}\xi_i^2=0$

证:(1) 由

$$a_{k+1}^2\leq(a_k+b_k)^2-\xi_k^2$$

得

$$a_{k+1}\leq a_k+b_k$$

从而

$$a_k\leq a_1+\sum_{i=1}^{k-1}b_i\,(k\geq 2)$$

故

$$\begin{aligned}\sum_{i=1}^{k}\xi_i^2&\leq\sum_{i=1}^{k}\left((a_i+b_i)^2-a_{i+1}^2\right)\\&=a_1^2-a_{k+1}^2+\sum_{i=1}^{k}b_i^2+2\sum_{i=1}^{k}a_ib_i\\&\leq a_1^2-a_{k+1}^2+\sum_{i=1}^{k}b_i^2+2\sum_{i=1}^{k}a_1b_i+2\sum_{i=2}^{k}\sum_{j=1}^{i-1}b_ib_j\\&\leq a_1^2+\sum_{i=0}^{k}b_i^2+2\sum_{i=0}^{k}a_1b_i+2\sum_{i=1}^{k}\sum_{j=0}^{i-1}b_ib_j\\&=\left(a_1+\sum_{i=0}^{k}b_i\right)^2\end{aligned}$$

(2) 由(1)结论知

$$0\leq\frac{\sum\limits_{i=1}^{k}\xi_i^2}{k}\leq\frac{a_1^2+2a_1\sum\limits_{i=0}^{k}b_i+\left(\sum\limits_{i=0}^{k}b_i\right)^2}{k}$$

显然

$$\lim_{k\to\infty}\frac{a_1^2+2a_1\sum\limits_{i=0}^{k}b_i}{k}=0$$

从而本题只需要证明下式成立即可

$$\lim_{k\to\infty}\frac{\left(\sum\limits_{i=0}^{k}b_i\right)^2}{k}=0$$

依题意知对$\forall\varepsilon>0,\exists k_0\in\mathbb{N}^+\ s.t. k\geq k_0\Rightarrow\sum\limits_{i=k_0}^{k}b_i^2<\varepsilon$

又由$Cauchy$不等式得

$$\left(\sum_{i=k_0}^{k}b_i\right)^2\leq(k-k_0+1)\sum_{i=k_0}^{k}b_i^2$$

即

$$\frac{\left(\sum\limits_{i=k_0}^{k}b_i\right)^2}{k}\leq\left(1-\frac{k_0-1}{k}\right)\sum_{i=k_0}^{k}b_i^2$$

从而

$$\begin{aligned}\frac{\left(\sum\limits_{i=0}^{k}b_i\right)^2}{k}&=\frac{\left(\sum\limits_{i=0}^{k_0-1}b_i\right)^2}{k}+\frac{\left(\sum\limits_{i=k_0}^{k}b_i\right)^2}{k}\\&\leq\frac{\left(\sum\limits_{i=0}^{k_0-1}b_i\right)^2}{k}+\left(1-\frac{k_0-1}{k}\right)\sum_{i=k_0}^{k}b_i^2\\&<\frac{\left(\sum\limits_{i=0}^{k_0-1}b_i\right)^2}{k}+\left(1-\frac{k_0-1}{k}\right)\varepsilon\end{aligned}$$

令$k\to\infty$取上极限并由$\varepsilon$的任意性即得

$$\lim_{k\to\infty}\frac{\left(\sum\limits_{i=0}^{k}b_i\right)^2}{k}=0$$

注:第二问还可以得到结论$\lim\limits_{k\to\infty}\dfrac{a_k}{\sqrt{k}}=0$

由$Cauchy$不等式知

$$a_{k+p}-a_k\leq b_k+\cdots+b_{k+p}\leq\sqrt{p+1}\sqrt{b_k^2+\cdots+b_{k+p}^2},k,p=1,2,\cdots$$

则

$$\frac{a_{k+p}-a_k}{\sqrt{k+p}} \le \frac{a_{k+p}-a_k}{\sqrt{1+p}} \le \sqrt{b_k^2+\cdots+b_{k+p}^2}$$

从而不难得到结论!

# 2 Differentiation

**问题**.给定实数$\lambda_1,\lambda_2,\cdots,\lambda_n$满足$\lambda_1^2+\lambda_2^2+\cdots+\lambda_n^2>0\,(k=1,2,\cdots)$,设$f(t)=\dfrac{1}{(1-\lambda_1 t)\cdots(1-\lambda_n t)}$,证明:$f^{(k)}(0)>0,k=1,2,\cdots$

证:对于任意给定的实数$\lambda_1,\lambda_2,\cdots,\lambda_n$,存在足够小的$t$,使得

$$|\lambda_i t|<1,i=1,2,\cdots,n$$

由初等函数的$Taylor$展开式得

$$\begin{aligned}\ln f(t)&=\sum_{m=1}^{n}\ln\frac{1}{1-\lambda_m t}=\sum_{m=1}^{n}(-\ln(1-\lambda_m t))\\&=\sum_{m=1}^{n}\sum_{k=1}^{\infty}\frac{(\lambda_m t)^k}{k}=\sum_{k=1}^{\infty}\frac{\left(\lambda_1^k+\lambda_2^k+\cdots+\lambda_n^k\right)t^k}{k}\\&=\sum_{k=1}^{\infty}c_k t^k\,(c_k>0)\end{aligned}$$

从而

$$\begin{aligned}f(t)&=\exp\left(\sum_{k=1}^{\infty}c_k t^k\right)=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\sum_{k=1}^{\infty}c_k t^k\right)^n\\&=1+c_1 t+\left(\frac{c_1^2}{2}+c_2\right)t^2+\cdots\\&=\sum_{k=1}^{\infty}a_k t^k\,(a_k>0)\end{aligned}$$

故

$$f^{(k)}(0)=a_k\cdot k!>0,k=1,2,\cdots$$

**问题**.$f(t)=1+a_1x+a_2x^2+\cdots$,$\dfrac{f'(x)}{f(x)}$按$x$幂的展开式中所有系数的模不大于2,求证:$|a_n|\le n+1$

证:设

$$\frac{f'(x)}{f(x)}=\sum_{n=0}^{\infty}b_n x^n\,(\,|b_n|\le 2)$$

即

$$a_1+2a_2x+3a_3x^2+\cdots=\left(1+a_1x+a_2x^2+\cdots\right)\left(b_0+b_1x+b_2x^2+\cdots\right)$$

假设$|a_n|\le n+1$不是对所有$n$成立,不妨取$k$为使得$|a_k|>k+1$成立的最小正整数,从而

$$|ka_k|>k(k+1)$$

但是

$$
\begin{aligned}
|ka_k| &= |b_0a_{k-1}+\cdots+b_{k-2}a_1+b_{k-1}| \\
&\le 2(1+|a_1|+\cdots+|a_{k-1}|) \\
&\le 2(1+2+\cdots+k) \\
&= k(k+1)
\end{aligned}
$$

矛盾!

故原命题成立

**问题**.设 $f(x)$ 是 $n$ 次多项式,$f(x)\ge 0$,证明:$F(x)=f(x)+f'(x)+\cdots+f^{(n)}\ge 0$ 恒成立

证1:令

$$G(x)=F(x)e^{-x}$$

则

$$G'(x)=\left(F'(x)-F(x)\right)e^{-x}=-f(x)e^{-x}\le 0$$

又

$$\lim_{x\to\infty}G(x)=\lim_{x\to\infty}F(x)e^{-x}=0$$

故

$$G(x)=F(x)e^{-x}\ge 0$$

即

$$F(x)\ge 0$$

证2:由 $f(x)\ge 0$ 恒成立易知 $f(x)$ 是最高次项为偶数且系数大于0的多项式,显然 $F(x)$ 也是最高次项为偶数且系数大于0的多项式,又

$$\lim_{x\to\infty}F(x)=\infty$$

从而 $F(x)$ 存在最小值点 $x_0$,并且 $F'(x_0)=0$,故

$$F(x)\ge F(x_0)=f(x_0)+F'(x_0)=f(x_0)\ge 0$$

注:证法1,2十分典型,下面给出几道利用这些方法的例子

(1)设 $p(x)$ 为多项式,$p'''(x)-p''(x)-p'(x)+p(x)\ge 0$,证明:$p(x)\ge 0$

证1:引理 $q\pm q'\ge 0\Rightarrow q\ge 0$.此法从略

证2:令

$$h(x)=p'''(x)-p''(x)-p'(x)+p(x)$$

则

$$e^{-x}h(x)=[(p''(x)-p(x))e^{-x}]'=e^{-x}(p'''(x)-p''(x)-p'(x)+p(x))\geq 0$$

即 $(p''(x)-p(x))e^{-x}$ 单调递增且 $\lim\limits_{x\to\infty}(p''(x)-p(x))e^{-x}=0$
注意到 $h(x),p(x)$ 均是最高次项为偶数且系数大于0的多项式
假设 $p(x)$ 存在小于0的点,记

$$p(x_0)=\min_{x\in\mathrm{R}}p(x)<0$$

则

$$p(x_0)<0,p'(x_0)=0,p''(x_0)>0$$

故

$$(p''(x_0)-p(x_0))e^{-x_0}>0$$

矛盾!
从而 $p(x)\geq 0$ 成立
(2) $p(x)$ 是二次可微函数, $p(0)=1,p'(0)=0,p''(x)-5p'(x)+6p(x)\geq 0$,证明: $p(x)\geq 3e^{2x}-2e^{3x},\forall x$
证:依题意,有

$$p''(x)-5p'(x)+6p(x)=[p''(x)-3p'(x)]-2[p'(x)-3p(x)]\geq 0$$

此式等价于

$$[(p'(x)-3p(x))\,e^{-2x}]'\geq 0$$

所以

$$[p'(x)-3p(x)]e^{-2x}\geq p'(0)-3p(0)=-3$$

从而

$$\left(p(x)e^{-3x}\right)'\geq -3e^{2x}\cdot e^{-3x}=-3e^{-x}$$

即

$$\left(p(x)e^{-3x}-3e^{-x}\right)'\geq 0$$

故

$$p(x)e^{-3x}-3e^{-x}\geq -2$$

化简即得

$$p(x)\geq 3e^{2x}-2e^{3x}$$

注:事实上,$p''(x)-5p'(x)+6p(x)=[p''(x)-2p'(x)]-3[p'(x)-2p(x)]$ 同理可证

(3) $f(x)$ 在 $[0,+\infty]$ 二阶可导,$f(0),f'(0)\geq 0,f''(x)\geq f(x)$,证明:

$$f(x)\geq f(0)+f'(0)x$$

证:由

$$f''(x)-f(x)\geq 0$$

可得

$$[(f'(x)+f(x))\,e^{-x}]'\geq 0$$

从而

$$(f'(x)+f(x))\,e^{-x}\geq f'(0)+f(0)$$

又由

$$e^{x}\left(f(x)+f'(x)\right)=\left(e^{x}f(x)\right)'$$

可得

$$\left[e^{x}f(x)-\frac{1}{2}e^{2x}\left(f'(0)+f(0)\right)\right]'=e^{x}[f(x)+f'(x)-e^{x}\left(f(0)+f'(0)\right)]\geq 0$$

故

$$e^{x}f(x)-\frac{1}{2}e^{2x}\left(f'(0)+f(0)\right)\geq\frac{1}{2}\left(f(0)-f'(0)\right)$$

即

$$f(x)\geq\left(\frac{e^{x}}{2}-\frac{e^{-x}}{2}\right)f'(0)+\left(\frac{e^{x}}{2}+\frac{e^{-x}}{2}\right)f(0)$$

显然

$$\frac{e^{x}-e^{-x}}{2}\geq x,\frac{e^{x}+e^{-x}}{2}\geq 1,\forall x\geq 0$$

所以

$$f(x)\geq f(0)+f'(0)x$$

**问题**. $f(x)$ 是实函数,具有三阶连续导数,对 $\forall x,f^{(k)}(x)>0\,(k=0,1,2,3)$, $f'''(x)\leq f(x)$,求证:$f'(x)<2f(x),\forall x$

证1:对任意固定的常数 $a$,定义

$$h(x)=f(x)-f'(x)(x-a)+\frac{f''(x)}{2}(x-a)^{2}$$

则

$$h'(x)=\frac{f'''(x)(x-a)^{2}}{2}>0$$

从而 $h(x)$ 严格单调递增,则

$$f(a+1)-f'(a+1)+\frac{f''(a+1)}{2}>f(a-1)+f'(a-1)+\frac{f''(a-1)}{2}$$

从而

$$f(a+1)-f'(a+1)>\frac{f''(a-1)}{2}-\frac{f''(a+1)}{2}=-f'''(\xi)\geq -f(\xi)\,(a-1<\xi<a+1)$$

故

$$f'(a+1)<f(a+1)+f(\xi)<2f(a+1)$$

由 $a$ 的任意性即知

$$f'(x)<2f(x)$$

证2:令 $c\geq 0$ 是 $f(x)$ 的下确界,则 $f(x)$ 在 $x\to-\infty$ 时单调递减趋于 $c$,并且不难得到 $f'(x),f''(x)$ 在 $x\to-\infty$ 时均单调递减趋于0,又

$$f''(x)f'''(x)\leq f(x)f''(x)<f(x)f''(x)+\left(f'(x)\right)^2$$

将其从 $a$ 到 $x$ 积分可得

$$\frac{1}{2}\left(\left(f''(x)\right)^2-\left(f''(a)\right)^2\right)<f(x)f'(x)-f(a)f'(a)$$

令 $a\to-\infty$,有

$$\left(f''(x)\right)^2\leq 2f(x)f'(x)$$

从而

$$\left(f''(x)\right)^2 f'''(x)\leq 2f^2(x)f'(x)$$

将其从 $a$ 到 $x$ 积分并令 $a\to-\infty$ 得

$$f''(x)\leq\sqrt[3]{2}f(x)$$

从而

$$f'(x)f''(x)\leq\sqrt[3]{2}f(x)f'(x)$$

故再次使用之前的技巧即得

$$f'(x)\leq\sqrt[6]{2}f(x)<2f(x)$$

**问题**.设 $f\in C[0,1]$,在 $(0,1)$ 上可导,且 $f(1)-f(0)=1$,证明:对于 $k=0,1,2,\cdots$, $\exists\xi_k\in(0,1)\,s.t.f'(\xi_k)=\dfrac{n!}{k!(n-k-1)!}\xi_k^k(1-\xi_k)^{n-k-1}$

证:考虑一组 $Bernstein$ 基底

$$B_i^n(x)=C_n^i x^i(1-x)^{n-i},i=0,1,\cdots,n$$

对$k \le n-1$,定义

$$\varphi(x)=\sum_{i=0}^{k} B_i^n(x)$$

显然$\varphi(0)=1,\varphi(1)=0$,设

$$F(x)=f(x)-[\varphi(x)f(0)+(1-\varphi(x))f(1)]$$

则$F(0)=F(1)=0$,由$Rolle$定理得

$$\exists \xi_k \in (0,1)\, s.t. F'(\xi_k)=0$$

即

$$f'(\xi_k)=-\varphi'(\xi_k)$$

又

$$\begin{aligned}\varphi'(x)=&-n(1-x)^{n-1}+[n(1-x)^{n-1}-n(n-1)x(1-x)^{n-2}]+\cdots+\\&\left[\frac{n!}{(k-1)!(n-k)!}x^{k-1}(1-x)^{n-k}-\frac{n!}{k!(n-k-1)!}x^k(1-x)^{n-k-1}\right]\\=&-\frac{n!}{k!(n-k-1)!}x^k(1-x)^{n-k-1}=-nB_k^{n-1}(x)\end{aligned}$$

由此即得

$$f'(\xi_k)=\frac{n!}{k!(n-k-1)!}\xi_k^k(1-\xi_k)^{n-k-1}$$

**问题**.设$x>0$,求证:$\left(1+\dfrac{1}{x}\right)^x\cdot(1+x)^{\frac{1}{x}}\le 4$

证1:两边取对数得

$$2x\ln 2+x^2\ln x-x^2\ln(1+x)-\ln(1+x)\ge 0$$

令

$$f(x)=2x\ln 2+x^2\ln x-x^2\ln(1+x)-\ln(1+x)$$

显然有$f(1)=0$

依题意利用对称性易知只需要考虑$x\in[1,\infty)$的情形,从而只需要证明

$$f'(x)=\frac{x-1}{x+1}+2\ln 2+2x\ln x-2x\ln(1+x)$$

注意到$\dfrac{x-1}{x+1}\ge 0$以及$g(1)=0$,则只需要证明

$$g'(x)=2\left[\ln\left(1-\frac{1}{1+x}\right)+\frac{1}{1+x}\right]\ge 0, x\in[1,\infty)$$

此式显然成立.故原不等式成立

证2:设

$$f(x)=\frac{\ln(1+x)}{x}$$

则

$$f'(x)=\frac{\dfrac{x}{1+x}-\ln(1+x)}{x^2}<0$$

$$f''(x)=\frac{-3(1+x)^2+4(1+x)-3+2(1+x)^2\ln(1+x)}{x^3(1+x)^2}$$

令

$$h(y)=-3y^2+4y-3+2y^2\ln y, y\geq 1$$

则

$$h'(y)=4\left[\,y(\ln y-1)+1\right]>0$$

又 $\lim\limits_{y\to\infty}h(y)<0$，故 $f''(x)<0$，$f(x)$ 为上凸函数，从而

$$f(x)+f\left(\frac{1}{x}\right)\leq 2f\left(\frac{x+\dfrac{1}{x}}{2}\right)\leq 2f(1)=2\ln 2$$

即

$$\left(1+\frac{1}{x}\right)^x\cdot(1+x)^{\frac{1}{x}}\leq 4$$

**问题**.设 $y=y(x)$ 是由方程 $x^3+y^3+xy-1=0$ 确定的隐函数,若存在极限 $\lim\limits_{x\to 0}\dfrac{y(x)-1+\dfrac{1}{3}x+ax^2}{x^3}=A$,求 $a$ 与 $A$ 的值

解:对 $x^3+y^3+xy-1=0$ 两端依次求导可得

$$3x^2+3y^2y'+y+xy'=0$$

$$6x+6yy'^2+3y^2y''+2y'+xy''=0$$

$$6+6y'^3+18yy'y''+(x+3y^2)y'''+3y''=0$$

将 $x=0$ 依次代入以上4式可得

$$y(0)=1, y'(0)=-\frac{1}{3}, y''(0)=0, y'''(0)=-\frac{52}{27}$$

由 $\lim\limits_{x\to 0}\dfrac{y(x)-1+\dfrac{1}{3}x+ax^2}{x^3}=A$ 可得

$$y(x)=1-\frac{1}{3}x-ax^2+Ax^3+o(x^3), x\to 0$$

故

$$a=-\frac{f''(0)}{2!}=0,A=\frac{f'''(0)}{3!}=-\frac{26}{81}$$

**问题**.设 $f(x)$ 在 $[0,1]$ 上三阶可导,$f(0)=-1,f(1)=0,f'(0)=0$,证明:对 $\forall x\in(0,1),\exists\xi\in(0,1)\,s.t.f(x)=-1+x^2+\dfrac{x^2(x-1)}{3!}f'''(\xi)$

解:令

$$\lambda=\frac{3!(f(x)+1-x^2)}{x^2(x-1)},x\in(0,1)$$

$$g(t)=f(t)+1-t^2-\frac{1}{3!}\lambda\, t^2(t-1)$$

则

$$g(0)=g'(0)=g(x)=g(1)=0$$

反复使用 $Rolle$ 定理即知

$$\exists\xi\in(0,1)\,s.t.f(x)=-1+x^2+\frac{x^2(x-1)}{3!}f'''(\xi)$$

**问题**.($Flett$ 均值定理) $f$在[a,b]上可微,$f'(a)=f'(b)$,证明:

$$\exists c\in(a,b)\,s.t.f'(c)=\frac{f(c)-f(a)}{c-a}$$

证:不妨假设 $f'(a)=f'(b)=0$,否则用下式代替

$$g(x)=f(x)-f'(a)x$$

令

$$h(x)=\begin{cases}\dfrac{f(x)-f(a)}{x-a} & x\neq a\\ 0 & x=a\end{cases}$$

则 $h(x)$ 在 [a,b] 上连续,在 (a,b) 上可微

若 $h(b)=0$,则由 $Rolle$ 定理即知

$$\exists c\in(a,b)\,s.t.h'(c)=\frac{f'(c)(c-a)-f(c)+f(a)}{(c-a)^2}=0$$

若 $h(b)>0$,则

$$h'(b)=\frac{f(a)-f(b)}{(b-a)^2}<0$$

从而 $h(x)$ 的最大值点必然在内部取到,故

$$\exists c\in(a,b)\,s.t.h'(c)=0$$

若 $h(b)<0$,同理可证

综上可得

$$\exists c\in(a,b)\, s.t. f'(c)=\frac{f(c)-f(a)}{c-a}$$

**问题**.设 $f(x)$ 在 $(-2,2)$ 内有两阶导数,且 $|f(x)|\le 1, f^2(0)+\left(f'(0)\right)^2=4$,求证:$\exists\xi\in(-2,2)\, s.t. f''(\xi)+f(\xi)=0$

证:由 $Lagrange$ 中值定理得

$$\left|\frac{f(-2)-f(0)}{-2-0}\right|=|f'(\xi_1)|\le 1\,(-2<\xi_1<0)$$

$$\left|\frac{f(2)-f(0)}{2-0}\right|=|f'(\xi_2)|\le 1\,(0<\xi_2<2)$$

令

$$F(x)=f^2(x)+\left(f'(x)\right)^2$$

则

$$F(\xi_1),F(\xi_2)\le 2$$

即 $F(x)$ 在 $[\xi_1,\xi_2]$ 的最大值在内部取到,故

$$\exists\xi\, s.t. F'(\xi)=2f'(\xi)\left(f(\xi)+f''(\xi)\right)=0$$

依题意显然有 $f'(\xi)\neq 0$,从而

$$\exists\xi\in(-2,2)\, s.t. f''(\xi)+f(\xi)=0$$

注:对于本题,我们做一个更为深入的探讨,找寻 $a\,(a>1)$的取值范围使得题中条件改为 $f^2(0)+\left(f'(0)\right)^2=a^2$ 时结论仍成立

解:显然 $f'(0)\neq 0$,不妨假设 $f'(0)>0$

若结论不成立,由 $Darboux$ 定理即知 $f(x)+f''(x)$ 恒为正或者恒为负,故不妨假设 $f(x)+f''(x)$ 恒为正,令

$$\beta=\inf\{x\in(0,2)\,|f'(x)\le 0\}$$

若 $0<\beta<2$,则 $f'(\beta)=0, f'(x)>0\,(0\le x<\beta)$,从而

$$\left[f^2(x)+\left(f'(x)\right)^2\right]'=f'(x)(f(x)+f''(x))>0\,(0\le x<\beta)$$

于是

$$f^2(x)+\left(f'(x)\right)^2>f^2(0)+\left(f'(0)\right)^2=a^2\,(0<x<\beta)$$

所以

$$\left(f'(x)\right)^2 > a^2 - f^2(x)\,(0 < x < \beta)$$

则

$$\liminf_{x\to\beta^-}\left(f'(x)\right)^2 > a^2 - 1 > 0$$

故必有 $\beta = 2$,从而

$$\frac{f'(x)}{\sqrt{a^2 - f^2(x)}} > 1\,(0 < x < 2)$$

即

$$\arcsin\frac{f(x)}{a} > \arcsin\frac{f(0)}{a} + x\,(0 < x < 2)$$

令

$$g(a) = \arcsin\frac{f(x)}{a} - \arcsin\frac{f(0)}{a} - x$$

则

$$g'(a) = \frac{f(0)}{\sqrt{a^2 - f^2(0)}} - \frac{f(x)}{\sqrt{a^2 - f^2(x)}}$$

由

$$\left(\frac{f(x)}{\sqrt{a^2 - f^2(x)}}\right)' = \frac{a^2 f'(x)}{(a^2 - f^2(x))^{\frac{3}{2}}} > 0$$

可得

$$g'(a) < 0$$

令 $a = \dfrac{1}{\sin 1}$,则

$$h(x) = g\left(\frac{1}{\sin 1}\right) = \arcsin(\sin 1 f(x)) - \arcsin(\sin 1 f(0)) - x$$

显然 $h(x)$ 严格单调递增且 $h(0) = 0, h(2) \leq 0$,矛盾!

从而当 $a \geq \dfrac{1}{\sin 1}$ 时要证的结论成立,事实上不难看出 $a \geq \dfrac{1}{\sin 1}$ 是最佳范围

**问题**.$f \in C^3[0,2]$,且 $\lim\limits_{x\to 0^+}\dfrac{f(x)}{x} = 2, f(1) = 1, f(2) = 6$,证明:$\exists \xi \in (0,2)\; s.t. f'''(\xi) = 9$

证:不妨假设 $f'''(x) > 9$ 恒成立,令

$$g(x) = f'''(x) - 9$$

对 $\forall x \in [0,2]$,由积分型余项的 $Taylor$ 展开式得

$$\begin{aligned} f(x) &= 2x + \frac{f''(0)}{2}x^2 + \frac{1}{2}\int_0^x f'''(t)(x-t)^2\,\mathrm{d}t \\ &= 2x + \frac{f''(0)}{2}x^2 + \frac{3}{2}x^3 + G(x) \end{aligned}$$

其中

$$G(x)=\frac{1}{2}\int_0^x (x-t)^2 g(t)\,\mathrm{d}t$$

由 $f(1)=1, f(2)=6$ 可得 $G(2)=4G(1)$,但

$$G(2)>\frac{1}{2}\int_0^1 (2-t)^2 g(t)\,\mathrm{d}t>2\int_0^1 (1-t)^2 g(t)\,\mathrm{d}t=4G(1)$$

从而矛盾!

即 $\exists\xi\in(0,2)\,s.t. f'''(\xi)=9$

注:若题中条件减弱为 $f\in C[0,2]$,在 (0,2) 内三阶可导则结论仍成立

证:由 $Darboux$ 定理知 $f'''(x)$ 恒大于9或者恒小于9

不妨假设 $f'''(x)>9$ 恒成立,令

$$H(x)=f(x)-2x-\frac{3}{2}x^3$$

则

$$H(0)=H'(0)=0, H'''(x)>0\,(0\le x\le 2)$$

由 $f(1)=1, f(2)=6$ 可得 $H(2)=4H(1)$,令

$$F(x)=H(2x)-4H(x)\,(0\le x\le 1)$$

则

$$F(0)=F(1)=F'(0)=0, F''(x)=4\left(H''(2x)-H''(x)\right)>0\,(0\le x\le 1)$$

从而

$$F'(x)>F'(0)=0, F(x)>F(0)=0, x\in(0,1]$$

矛盾!

即 $\exists\xi\in(0,2)\,s.t. f'''(\xi)=9$

**问题**.($Gronwall-Bellman$ 不等式的微分形式) 设 $f$ 在 $[0,+\infty)$ 上可微,$f(0)=0$,存在 $c>0$ 使得 $|f'(x)|\le c\,|f(x)|, \forall\, x\in[0,+\infty)$,求证:$f(x)\equiv 0$

证:记

$$|f(x_0)|=\max_{x\in\left[0,\frac{1}{c+1}\right]}|f(x)|$$

由 $Lagrange$ 中值定理得

$$|f(x_0)-f(0)|=|f'(\xi)x_0|\le c\,|f(\xi)|x_0\le\frac{c}{c+1}|f(x_0)|$$

从而 $|f(x_0)|=0$,则 $f(x)\equiv 0, x\in\left[0,\dfrac{1}{c+1}\right]$
同理可得

$$f(x)\equiv 0, x\in\left[\frac{k}{c+1},\frac{k+1}{c+1}\right], \forall k\in\mathbb{N}$$

令 $k\to\infty$ 即知 $f(x)\equiv 0\,(x\geq 0)$

注:下面来看两道相似的例子

(1) 设 $f$ 在区间 $(-1,1)$ 上二阶可微,$f(0)=f'(0)=0$,且在该区间上满足不等式 $|f''(x)|\leq|f(x)|+|f'(x)|$,证明: $f(x)\equiv 0$

分析:当无法整体解决问题的时候,可以考虑从局部入手

证:考虑闭区间 $[-\delta,\delta]$,令

$$|f'(x_0)|=\max_{x\in[-\delta,\delta]}|f'(x)|$$

由 $Lagrange$ 中值定理得

$$\begin{aligned}
|f'(x_0)-f'(0)|&=|f''(\xi)x_0|\\
&\leq(|f(\xi)|+|f'(\xi)|)\delta\\
&\leq|f'(x_0)|\delta+|f(\xi)-f(0)|\delta\\
&=|f'(x_0)|\delta+|f'(\varsigma)\xi|\delta\\
&\leq|f'(x_0)|\left(\delta+\delta^2\right)
\end{aligned}$$

取 $\delta=\dfrac{1}{2}$ 即得 $|f'(x_0)|=0$，从而 $f(x)\equiv 0, x\in[-\delta,\delta]$

再用相同的方法进行平移即可得到 $f(x)\equiv 0, x\in(-1,1)$

(2) 若将(1)中条件改为 $|f''(x)|^2\leq|f(x)f'(x)|$,可以仿照上述方法直接证明,或者利用平均值不等式转化为(1)中形式

# 3 Integration

**问题**.设 $a_1,b_1$ 是实数,记 $a_n=\int_0^1\max\{b_{n-1},x\}\,\mathrm{d}x,b_n=\int_0^1\min\{a_{n-1},x\}\,\mathrm{d}x$,求极限 $\lim\limits_{n\to\infty}(2011a_n+2012b_n)$

解1:易知对于足够大的 $n$ 有

$$a_n,b_n\in(0,1)$$

不妨假设对所有的 $n$ 上式均成立,从而

$$a_n=\int_0^1\max\{b_{n-1},x\}\,\mathrm{d}x=\int_0^1\frac{|x-b_{n-1}|+(x+b_{n-1})}{2}\,\mathrm{d}x=\frac{1+b_{n-1}^2}{2}$$

同理可得

$$b_n=-\frac{a_{n-1}^2}{2}+a_{n-1}$$

消掉 $a_{n-1}$ 可得

$$b_{n+1}=\frac{(b_{n-1}^2+1)(3-b_{n-1}^2)}{8}$$

从而

$$\begin{aligned}b_{n+1}-(\sqrt{2}-1)=&\frac{(b_{n-1}^2+5-2\sqrt{2})(b_{n-1}+\sqrt{2}-1)(b_{n-1}-(\sqrt{2}-1))}{8}\\<&\frac{(3\sqrt{2}-2)(b_{n-1}-(\sqrt{2}-1))}{4}\end{aligned}$$

故

$$|b_n-(\sqrt{2}-1)|<\frac{3\sqrt{2}-2}{4}|b_{n-2}-(\sqrt{2}-1)|<\frac{3}{4}\left|b_{n-2}-(\sqrt{2}-1)\right|$$

将 $n$ 分为奇数与偶数依次迭代即可得到 $b_n\to\sqrt{2}-1,n\to\infty$
同理可得 $a_n\to2-\sqrt{2},n\to\infty$,故

$$\lim_{n\to\infty}(2011a_n+2012b_n)=2010+\sqrt{2}$$

解2:同解1可得

$$\begin{cases}a_n=\dfrac{1}{2}+\dfrac{b_{n-1}^2}{2}\\ b_n=-\dfrac{a_{n-1}^2}{2}+a_{n-1}\end{cases}$$

则

$$\begin{cases}b_{n+1}-b_n=(a_n-a_{n-1})\dfrac{1-(a_n+a_{n+1})}{2}\\ a_{n+1}-a_n=\dfrac{b_n+b_{n+1}}{2}\dfrac{2-(a_{n-1}+a_{n-2})}{2}(a_{n-1}-a_{n-2})\end{cases}$$

注意到

$$b_n=\frac{a_{n-1}(2-a_{n-1})}{2}\leq\frac{1}{2}$$

故

$$|a_{n+1}-a_n|<\left|\frac{b_n+b_{n-1}}{2}(a_{n-1}-a_{n-2})\right|\leq\frac{1}{2}|a_{n-1}-a_{n-2}|$$

从而易得 $a_n$ 收敛,同理可知 $b_n$ 收敛,令

$$\lim_{n\to\infty}a_n=x,\lim_{n\to\infty}b_n=y$$

则

$$\begin{cases}2y=-x^2+2x\\2x=1+y^2\end{cases}$$

解之即可,后略

解3:由

$$\begin{cases}a_n=\dfrac{1}{2}+\dfrac{b_{n-1}^2}{2}\\b_n=-\dfrac{a_{n-1}^2}{2}+a_{n-1}\end{cases}$$

可得

$$a_{n+1}=\frac{1}{2}+\frac{1}{2}\left(a_{n-1}-\frac{1}{2}a_{n-1}^2\right)^2,a_n\in(0,1)$$

令

$$a_{n+1}=f(a_{n-1}),f(x)=\frac{1}{2}+\frac{1}{2}\left(x-\frac{1}{2}x^2\right)^2$$

则

$$f'(x)=x\left(1-\frac{x}{2}\right)(1-x)>0$$

从而 $a_{2n},a_{2n+1}$ 极限存在

解方程

$$f(x)=\frac{1}{2}+\frac{1}{2}\left(x-\frac{x^2}{2}\right)^2=x$$

得

$$x=2-\sqrt{2}\text{或}2+\sqrt{2}$$

故

$$\lim_{n\to\infty}a_n=2-\sqrt{2}$$

**问题**.求 $\displaystyle\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x}{\left((2x+1)\sqrt{x^2-x+1}+(2x-1)\sqrt{x^2+x+1}\right)\sqrt{x^4+x^2+1}}\mathrm{d}x$

解:令

$$I=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x}{\left((2x+1)\sqrt{x^2-x+1}+(2x-1)\sqrt{x^2+x+1}\right)\sqrt{x^4+x^2+1}}\mathrm{d}x$$

$$a(x)=\sqrt{x^2+x+1},a(-x)=\sqrt{x^2-x+1}$$

则

$$I=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x}{\left(\left(x-\frac{1}{2}\right)a(x)+\left(x+\frac{1}{2}\right)a(-x)\right)a(x)a(-x)}\,\mathrm{d}x$$

又

$$a(x)a'(x)=x+\frac{1}{2},a(-x)a'(-x)=-x+\frac{1}{2}$$

故

$$\begin{aligned}I&=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x}{a^2(x)a^2(-x)\left(a'(x)-a'(-x)\right)}\,\mathrm{d}x\\&=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\left(a'(x)+a'(-x)\right)}{a^2(x)a^2(-x)\left(a'^2(x)-a'^2(-x)\right)}\,\mathrm{d}x\\&=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\left(a'(x)+a'(-x)\right)}{\left(x+\frac{1}{2}\right)a^2(-x)-\left(x-\frac{1}{2}\right)a^2(x)}\,\mathrm{d}x\\&=\frac{1}{2}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{x\left(a'(x)+a'(-x)\right)}{\left(x+\frac{1}{2}\right)(x^2-x+1)-\left(x-\frac{1}{2}\right)(x^2+x+1)}\,\mathrm{d}x\\&=\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{a'(x)+a'(-x)}{3}\,\mathrm{d}x=\frac{a\left(\frac{1}{2}\right)-a\left(-\frac{1}{2}\right)}{3}\\&=\frac{\sqrt{7}-\sqrt{3}}{6}\end{aligned}$$

**问题**.计算 $I=\int_0^1\frac{1}{(x-a)\sqrt[5]{x^2(1-x)^3}}\,\mathrm{d}x\,(a>1)$

解:令

$$x=a\,\frac{1-y}{a-y}$$

则

$$\begin{aligned}I&=-a^{-\frac{2}{5}}(a-1)^{-\frac{3}{5}}\int_0^1(1-y)^{-\frac{2}{5}}y^{-\frac{3}{5}}\,\mathrm{d}y\\&=\frac{-a^{-\frac{2}{5}}(a-1)^{-\frac{3}{5}}\pi}{\sin\frac{2\pi}{5}}\end{aligned}$$

注:解答中的变换是由 $x=\frac{\alpha y+\beta}{cy+d}$ 按照 $x=0,1$分别变成1,0待定出来的

**问题**.计算 $I=\int_0^{\pi/2}\ln\sin x\cdot\ln\cos x\cdot\sin 2x\,\mathrm{d}x$

解:

$$\begin{aligned}
I &= \frac{1}{4}\int_0^{\pi/2} \ln\sin^2 x \cdot \ln\cos^2 x \cdot \sin 2x \,\mathrm{d}x \\
&= \frac{1}{4}\int_0^{\pi/2} \ln\left(\frac{1-y}{2}\right)\ln\left(\frac{1+y}{2}\right)\mathrm{d}y \\
&= \frac{1}{4}\int_0^{1} \ln x\ln(1-x)\,\mathrm{d}x \\
&= -\frac{1}{4}\int_0^{1} \sum_{n=1}^{\infty}\frac{x^n\ln x}{n}\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0^+}\int_0^{1-\varepsilon}\sum_{n=1}^{\infty}\frac{x^n\ln x}{4n}\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0^+}\sum_{n=1}^{\infty}\frac{1}{4n}\int_0^{1-\varepsilon} x^n\ln x\,\mathrm{d}x \\
&= -\lim_{\varepsilon\to 0^+}\sum_{n=1}^{\infty}\left(\frac{(1-\varepsilon)^{n+1}\ln(1-\varepsilon)}{n(1+n)}-\frac{(1-\varepsilon)^2}{n(1+n)^2}\right) \\
&= -\sum_{n=1}^{\infty}\lim_{\varepsilon\to 0^+}\left(\frac{(1-\varepsilon)^{n+1}\ln(1-\varepsilon)}{4n(1+n)}-\frac{(1-\varepsilon)^2}{4n(1+n)^2}\right) \\
&= \sum_{n=1}^{\infty}\frac{1}{4n(1+n)^2} = \frac{1}{4}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+1}-\frac{1}{(1+n)^2}\right) \\
&= \frac{1}{2}-\frac{\pi^2}{24}
\end{aligned}$$

**问题**.设 $0<a<b$,求积分

$$\int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,\mathrm{d}x$$

解:令

$$A = \int_a^b \frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,\mathrm{d}x$$

$$B = \int_a^b \frac{\sqrt[n]{b-x}(1+\sqrt[n]{x-a})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}\,\mathrm{d}x$$

则

$$A+B=a+b$$

利用

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b f(a+b-x)\,\mathrm{d}x$$

可知

$$A=B$$

从而原积分

$$A = \frac{b-a}{2}$$

**问题**.求积分

$$\int_0^{\infty} \frac{1}{\left(x^4 + (1+2\sqrt{2})x^2 + 1\right)\left(x^{100} - x^{98} + \cdots + 1\right)} \mathrm{d}x$$

解:设

$$I = \int_0^{\infty} \frac{1}{\left(x^4 + (1+2\sqrt{2})x^2 + 1\right)\left(x^{100} - x^{98} + \cdots + 1\right)} \mathrm{d}x$$

令 $x = \dfrac{1}{y}$,则

$$I = \int_0^{\infty} \frac{x^{102}}{\left(x^4 + (1+2\sqrt{2})x^2 + 1\right)\left(x^{100} - x^{98} + \cdots + 1\right)} \mathrm{d}x$$

注意到

$$x^{100} - x^{98} + \cdots + 1 = \frac{1 + x^{102}}{1 + x^2}$$

从而

$$2I = \int_0^{\infty} \frac{1 + x^2}{x^4 + (1+2\sqrt{2})x^2 + 1} \mathrm{d}x$$

则

$$I = \frac{1}{2} \int_0^{\infty} \frac{1 + \dfrac{1}{x^2}}{x^2 + \dfrac{1}{x^2} + 1 + 2\sqrt{2}} \mathrm{d}x = \frac{\pi}{2(1+\sqrt{2})}$$

**问题**.证明广义积分 $\displaystyle\int_1^{+\infty} \frac{2\{x\} - 1}{2x - 1} \mathrm{d}x$ 收敛,并求出这个积分值

解:对 $\forall\, k \in \mathbb{N}^+, k \le x < k+1, \{x\} = x - [x] = x - k$,从而

$$\begin{aligned}
\int_1^N \frac{2\{x\} - 1}{2x - 1} \mathrm{d}x &= \sum_{k=1}^{N} \int_k^{k+1} \frac{2(x-k) - 1}{2x - 1} \mathrm{d}x \\
&= N - \sum_{k=1}^{N} k \ln(2k+1) + \sum_{k=1}^{N} k \ln(2k-1) \\
&= N - N \ln(2N+1) - \sum_{k=1}^{N} \ln(2k-1) \\
&= N - N \ln(2N+1) + \ln\left(\frac{\Gamma\left(N + \dfrac{1}{2}\right) 2^N}{\sqrt{\pi}}\right)
\end{aligned}$$

由 $Stirling$ 公式

$$\Gamma(x) = \frac{1}{x}\sqrt{2\pi x}\left(\frac{x}{e}\right)^x e^{\frac{\theta(x)}{12x}}, \theta(x) \in (0,1)$$

可知

$$\begin{aligned}
\int_1^N \frac{2\{x\}-1}{2x-1}\,\mathrm{d}x &= N - N\ln(2N+1) + \ln\left(\frac{\sqrt{2\pi}\left(N+\frac{1}{2}\right)^N e^{-N-\frac{1}{2}+O\left(\frac{1}{N}\right)}\,2^N}{\sqrt{\pi}}\right) \\
&= N - N\ln(2N+1) + \ln\left(\sqrt{2}(2N+1)^N e^{-N-\frac{1}{2}+O\left(\frac{1}{N}\right)}\right) \\
&= \frac{\ln 2 - 1}{2} + O\left(\frac{1}{N}\right)
\end{aligned}$$

故原广义积分收敛,且

$$\int_1^{+\infty} \frac{2\{x\}-1}{2x-1}\,\mathrm{d}x = \lim_{N\to+\infty}\int_1^N \frac{2\{x\}-1}{2x-1}\,\mathrm{d}x = \frac{\ln 2 - 1}{2}$$

**问题**.计算 $I = \lim\limits_{n\to\infty} \frac{1}{n^2}\int_0^n \frac{\sqrt{n^2-x^2}}{2+x^{-x}}\,\mathrm{d}x$

解:令 $x = nt$ 得

$$\begin{aligned}
\frac{1}{n^2}\int_0^n \frac{\sqrt{n^2-x^2}}{2+x^{-x}}\,\mathrm{d}x &= \int_0^1 \frac{\sqrt{1-t^2}}{2+(nt)^{-nt}}\,\mathrm{d}t \\
&= \int_0^{\frac{1}{\sqrt{n}}} \frac{\sqrt{1-t^2}}{2+(nt)^{-nt}}\,\mathrm{d}t + \int_{\frac{1}{\sqrt{n}}}^1 \frac{\sqrt{1-t^2}}{2+(nt)^{-nt}}\,\mathrm{d}t \\
&= I_1 + I_2
\end{aligned}$$

由

$$0 < \frac{\sqrt{1-t^2}}{2+(nt)^{-nt}} < \frac{1}{2}\,(0 \le t \le 1)$$

$$\frac{1}{2+(\sqrt{n})^{-\sqrt{n}}} < \frac{1}{2+(nt)^{-nt}} < \frac{1}{2+n^{-n}}\left(\frac{1}{\sqrt{n}} \le t \le 1\right)$$

可得

$$0 < I_1 < \frac{1}{2\sqrt{n}}$$

$$\frac{1}{2+(\sqrt{n})^{-\sqrt{n}}}\int_{\frac{1}{\sqrt{n}}}^1 \sqrt{1-t^2}\,\mathrm{d}t < I_2 < \frac{1}{2+n^{-n}}\int_{\frac{1}{\sqrt{n}}}^1 \sqrt{1-t^2}\,\mathrm{d}t$$

由夹逼定理即知

$$I = \frac{\pi^2}{8}$$

**问题**.计算 $I = \lim\limits_{n\to\infty} \sqrt{n}\int_0^{+\infty} \frac{\cos x}{(1+x^2)^n}\,\mathrm{d}x$

解:注意到

$$
\begin{aligned}
&\left|\sqrt{n}\int_0^{+\infty}\frac{\cos x}{(1+x^2)^n}\,\mathrm{d}x-\sqrt{n}\int_0^{+\infty}\frac{1}{(1+x^2)^n}\,\mathrm{d}x\right| \\
\le&\sqrt{n}\left|\int_0^{\frac{1}{\sqrt[4]{n}}}\frac{\cos x-1}{(1+x^2)^n}\,\mathrm{d}x\right|+\sqrt{n}\left|\int_{\frac{1}{\sqrt[4]{n}}}^{+\infty}\frac{\cos x-1}{(1+x^2)^n}\,\mathrm{d}x\right| \\
\le&\sqrt{n}\cdot\frac{1}{2\sqrt{n}}\cdot\frac{1}{\sqrt[4]{n}}+2\sqrt{n}\int_{\frac{1}{\sqrt[4]{n}}}^{+\infty}\frac{1}{(1+x^2)^n}\,\mathrm{d}x \\
\le&\frac{1}{2\sqrt[4]{n}}+\frac{2\sqrt{n}}{\left(1+\frac{1}{\sqrt{n}}\right)^{n-1}}\int_0^{+\infty}\frac{1}{1+x^2}\,\mathrm{d}x \\
\le&\frac{1}{2\sqrt[4]{n}}+\frac{\pi\sqrt{n}}{e^{\sqrt{n}-1}}\to 0\,(n\to\infty)
\end{aligned}
$$

其中用到

$$1-\cos x\le\frac{x^2}{2},\left(1+\frac{1}{\sqrt{n}}\right)^{\sqrt{n}+1}>e$$

从而

$$I=\lim_{n\to\infty}\sqrt{n}\int_0^{+\infty}\frac{1}{(1+x^2)^n}\,\mathrm{d}x$$

又由三角换元知

$$\int_0^{+\infty}\frac{1}{(1+x^2)^n}\,\mathrm{d}x=\int_0^{\frac{\pi}{2}}(\cos t)^{2n-2}\,\mathrm{d}t=\frac{\pi}{2}\cdot\frac{(2n-3)!!}{(2n-2)!!}$$

所以

$$I=\lim_{n\to\infty}\sqrt{n}\cdot\frac{\pi}{2}\cdot\frac{(2n-3)!!}{(2n-2)!!}=\frac{\pi}{2}$$

**问题**.当 $x\to 1^-$ 时,求与 $\displaystyle\int_0^{\infty}x^{t^2}\,\mathrm{d}t$ 等价的无穷大量

解:先对广义积分 $\displaystyle\int_0^{\infty}x^{t^2}\,\mathrm{d}t$ 做一个估计

$$
\begin{aligned}
\int_0^{\infty}x^{t^2}\,\mathrm{d}t&=\int_0^{\infty}e^{t^2\ln x}\,\mathrm{d}t=\int_0^{\infty}e^{-t^2\ln\frac{1}{x}}\,\mathrm{d}t \\
&=\left(\ln\frac{1}{x}\right)^{-\frac{1}{2}}\cdot\int_0^{\infty}e^{-y^2}\,\mathrm{d}y \\
&=\frac{\sqrt{\pi}}{2}\left[\ln\left(1+\frac{1}{x}-1\right)\right]^{-\frac{1}{2}} \\
&\sim\frac{\sqrt{\pi}}{2}\left(\frac{1}{x}-1\right)^{-\frac{1}{2}}\,(x\to 1^-)
\end{aligned}
$$

故当 $x\to 1^-$ 时,与 $\displaystyle\int_0^{\infty}x^{t^2}\,\mathrm{d}t$ 等价的无穷大量为 $\dfrac{\sqrt{\pi}}{2}(1-x)^{-\frac{1}{2}}$

注:此题可以进行如下推广

当 $x\to 1^-$ 时,与 $\displaystyle\int_0^{\infty}x^{t^n}\,\mathrm{d}t$ 等价的无穷大量为 $\dfrac{1}{n}\Gamma\left(\dfrac{1}{n}\right)(1-x)^{-\frac{1}{n}}$ (证法类似)

**问题**.设 $f(x)=\dfrac{\ln(1+x)}{1+x}(x>0)$,定义$A(x)=\displaystyle\int_0^x f(t)\,\mathrm{d}t$,令 $A=\displaystyle\sum_{n=1}^{\infty}A\left(\frac{1}{n}\right)$,证明:$\dfrac{1}{3}<A<1$

证:依题意得

$$A(x)=\int_0^x f(t)\,\mathrm{d}t=\int_0^x \frac{\ln(1+x)}{1+t}\,\mathrm{d}t=\frac{\ln^2(1+x)}{2}<\frac{x^2}{2}$$

从而

$$A<\sum_{n=1}^{\infty}\frac{\left(\frac{1}{n}\right)^2}{2}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{12}<1$$

另一方面,由

$$\ln(1+t)=t-\frac{t^2}{2}+\frac{t^3}{3}-\cdots>t-\frac{t^2}{2}$$

可得

$$\begin{aligned}A(x)&>\int_0^x\frac{t-\frac{t^2}{2}}{1+t}\,\mathrm{d}t\\&>\frac{1}{1+x}\int_0^x\left(t-\frac{t^2}{2}\right)\mathrm{d}t\\&=\frac{x^2}{2(1+x)}\left(1-\frac{x}{3}\right)\\&\geq\frac{x^2}{3(1+x)}\end{aligned}$$

从而

$$A>\frac{1}{3}\sum_{n=1}^{\infty}\frac{\left(\frac{1}{n}\right)^2}{1+\frac{1}{n}}=\frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n(1+n)}=\frac{1}{3}$$

综上即得 $\dfrac{1}{3}<A<1$

**问题**.设 $f$ 在 $[a,b]$ 上可微,$f(a)=f(b)=0,|f'(x)|\leq M,\displaystyle\int_a^b f(x)\,\mathrm{d}x=0$,对于函数 $F(x)=\displaystyle\int_a^x f(t)\,\mathrm{d}t$,证明:$|F(x)|\leq\dfrac{M(b-a)^2}{16}$

证:令

$$F(c)=\max_{x\in[a,b]}|F(x)|$$

则

$$F'(c)=f(c)=0$$

故

$$|F(c)|=\left|\int_a^{\frac{a+c}{2}}(f(t)-f(a))\,\mathrm{d}t+\int_{\frac{a+c}{2}}^{c}(f(t)-f(c))\,\mathrm{d}t\right|\leq\frac{M(c-a)^2}{4}$$

同理可得

$$|F(c)| \leq \frac{M(c-b)^2}{4}$$

则

$$|F(x)| \leq \min\left\{\max\left(\frac{M(c-a)^2}{4}, \frac{M(c-b)^2}{4}\right)\right\} = \frac{M(b-a)^2}{16}$$

注:在微分学中也有类似问题,例子如下

设 $f(x)$ 在 $[a,b]$ 上二阶可导,$f(a)=f(b)=f'(a)=f'(b)=0$,$\exists M\ s.t.|f''(x)| \leq M$,求证:$|f(x)| \leq \dfrac{M(b-a)^2}{16}$

证:设

$$f(c) = \max_{x\in[a,b]} |f(x)|$$

则

$$f'(c) = 0$$

对 $\forall x \in [a,c]$,由 $Taylor$ 展开式得

$$f(x) = f(c) + \frac{f''(\xi_1)}{2}(x-c)^2, x \leq \xi_1 \leq c$$

$$f(x) = \frac{f''(\xi_2)}{2}(x-a)^2, a \leq \xi_2 \leq x$$

从而

$$|f(c)| = \left|\frac{f''(\xi_2)}{2}(x-a)^2 - \frac{f''(\xi_1)}{2}(x-c)^2\right| \leq \frac{M(x-a)^2}{2} + \frac{M(x-c)^2}{2}$$

故

$$|f(c)| \leq \min_{x\in[a,c]}\left\{\frac{M(x-a)^2}{2} + \frac{M(x-c)^2}{2}\right\} = \frac{M(a-c)^2}{4}$$

同理可得

$$|f(c)| \leq \frac{M(b-c)^2}{4}$$

从而

$$|f(x)| \leq \min\left\{\max\left(\frac{M(c-a)^2}{4}, \frac{M(c-b)^2}{4}\right)\right\} = \frac{M(b-a)^2}{16}$$

**问题**.($Favard$ 不等式) $f(x):[0,1] \to \mathbb{R}$ 是一个非负上凸函数,证明:

$$\int_0^1 f^p(x)\,\mathrm{d}x \leq \frac{2^p}{p+1}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^p$$

证:不妨考虑 $f(0)=f(1)=0$,$f(x)$ 有连续的二阶导数,则

$$f''(x) \leq 0$$

从而

$$f(x)=-\int_0^1 K(x,t)f''(t)\,\mathrm{d}t$$

其中 $Green$ 函数

$$K(x,t)=\begin{cases} t(1-x) & 0\le t\le x\le 1\\ x(1-t) & 0\le x\le t\le 1\end{cases}$$

由 $Minkowski$ 不等式可得

$$\begin{aligned}\left(\int_0^1 f^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}&=\left(\int_0^1\left(\int_0^1 K(x,t)\left(-f''(t)\right)\,\mathrm{d}t\right)^p\mathrm{d}x\right)^{\frac{1}{p}}\\ &\le\int_0^1\left(\int_0^1 K^p(x,t)\left(-f''(t)\right)^p\mathrm{d}x\right)^{\frac{1}{p}}\mathrm{d}t\\ &=\frac{1}{(p+1)^{\frac{1}{p}}}\int_0^1 t(1-t)\left|f''(t)\right|\,\mathrm{d}t\end{aligned}$$

又

$$\begin{aligned}\int_0^1 f(x)\,\mathrm{d}x&=-\int_0^1\int_0^1 K(x,t)f''(t)\,\mathrm{d}t\,\mathrm{d}x\\ &=-\int_0^1\int_0^1 K(x,t)f''(t)\,\mathrm{d}x\,\mathrm{d}t\\ &=-\frac{1}{2}\int_0^1 t(1-t)f''(t)\,\mathrm{d}t\end{aligned}$$

从而

$$\int_0^1 f^p(x)\,\mathrm{d}x\le\frac{2^p}{p+1}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^p$$

对于一般情形,不难由光滑凹函数的逼近得到

注:令 $p\to+\infty$ 可得不等式

$$\max_{x\in[0,1]}f(x)\le 2\int_0^1 f(x)\,\mathrm{d}x$$

**问题**. $f(x):[0,1]\to\mathbb{R}$ 是一个上凸函数,$f(0)=1$,证明:

$$\int_0^1 xf(x)\,\mathrm{d}x\le\frac{2}{3}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

证1:令

$$F(x)=\int_0^x f(t)\,\mathrm{d}t$$

利用上凸函数的性质可得

$$F(x)=x\int_0^1 f[ux+(1-u)\cdot 0]\,\mathrm{d}u\ge x\int_0^1[uf(x)+(1-u)]\,\mathrm{d}u=\frac{xf(x)}{2}+\frac{x}{2}$$

令

$$I=\int_0^1 xf(x)\,\mathrm{d}x, U=\int_0^1 f(x)\,\mathrm{d}x$$

则原命题等价于证明

$$2U^2-3I\geq 0$$

又

$$I=\int_0^1 x\mathrm{d}F(x)=F(1)-\int_0^1 F(x)\,\mathrm{d}x\leq U-\int_0^1\left(\frac{xf(x)}{2}+\frac{x}{2}\right)\mathrm{d}x=U-\frac{I}{2}-\frac{1}{4}$$

即

$$3I\leq 2U-\frac{1}{2}$$

故

$$2U^2-3I\geq 2U^2-\left(2U-\frac{1}{2}\right)=2\left(U-\frac{1}{2}\right)^2\geq 0$$

则原命题得证

证2:令

$$F(x)=\int_0^x f(t)\,\mathrm{d}t$$

由 $f(x)$ 的上凸性质得

$$\frac{f(t)-f(0)}{t}\geq\frac{f(x)-f(0)}{x}$$

从而

$$\begin{aligned}\int_0^1 F(x)\,\mathrm{d}x&=\int_0^1\int_0^x f(t)\,\mathrm{d}t\,\mathrm{d}x\\&\geq\int_0^1\int_0^x\left(\frac{f(x)-1}{x}t+1\right)\mathrm{d}t\,\mathrm{d}x\\&=\frac{1}{2}\int_0^1(xf(x)+x)\,\mathrm{d}x\end{aligned}$$

故

$$\int_0^1 xf(x)\,\mathrm{d}x=F(1)-\int_0^1 F(x)\,\mathrm{d}x\leq\int_0^1 f(x)\,\mathrm{d}x-\frac{1}{2}\int_0^1(xf(x)+x)\,\mathrm{d}x$$

即

$$\int_0^1 xf(x)\,\mathrm{d}x\leq\frac{2}{3}\left(\int_0^1 f(x)\,\mathrm{d}x-\frac{1}{4}\right)\leq\frac{2}{3}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

注:证1与证2本质上是一样的,但是证2写的更为清晰,下面给出1道相似的例子

$f(x):[0,1]\to\mathbb{R}$ 是一个非负上凸函数,$f(0)=1$,证明:

$$2\int_0^1 x^2f(x)\,\mathrm{d}x+\frac{1}{12}\leq\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

证:令

$$F(x)=\int_0^x f(t)\,\mathrm{d}t$$

由 $f(x)$ 的上凸性可得

$$\frac{f(t)-f(0)}{t}\geq\frac{f(x)-f(0)}{x}$$

从而

$$\begin{aligned}\int_0^1 xF(x)\,\mathrm{d}x&=\int_0^1\int_0^x xf(t)\,\mathrm{d}t\,\mathrm{d}x\\&\geq\int_0^1\int_0^x x\left(\frac{f(x)-1}{x}t+1\right)\mathrm{d}t\,\mathrm{d}x\\&=\frac{1}{2}\int_0^1(x^2f(x)+x^2)\,\mathrm{d}x\end{aligned}$$

则

$$\int_0^1 x^2f(x)\,\mathrm{d}x=F(1)-2\int_0^1 xF(x)\,\mathrm{d}x\leq\int_0^1 f(x)\,\mathrm{d}x-\int_0^1(x^2f(x)+x^2)\,\mathrm{d}x$$

即

$$2\int_0^1 x^2f(x)\,\mathrm{d}x+\frac{1}{12}\leq\int_0^1 f(x)\,\mathrm{d}x-\frac{1}{4}\leq\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

注:此题可推广为

$$\frac{p+2}{2}\int_0^1 x^pf(x)\,\mathrm{d}x+\frac{2pf(0)-(p+1)}{4(p+1)}\leq\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2(p>0)$$

**问题**.设 $f\in C^2[a,b]$, $f(a)=f(b)=0$, $f'(a)=1$, $f'(b)=0$,证明:

$$\int_a^b(f''(x))^2\,\mathrm{d}x\geq\frac{4}{b-a}$$

证:对 $\forall c\in[a,b]$,有

$$\int_a^b(x-c)f''(x)\,\mathrm{d}x=(c-a)-\int_a^b f'(x)\,\mathrm{d}x=c-a$$

由 $Cauchy$ 不等式得

$$(c-a)^2=\left(\int_a^b(x-c)f''(x)\,\mathrm{d}x\right)^2\leq\int_a^b(x-c)^2\,\mathrm{d}x\int_a^b(f''(x))^2\,\mathrm{d}x$$

从而

$$\int_a^b(f''(x))^2\,\mathrm{d}x\geq\frac{(c-a)^2}{\int_a^b(x-c)^2\,\mathrm{d}x}=\frac{3}{(b-a)\left[\left(\frac{b-c}{c-a}\right)^2-\frac{b-c}{c-a}+1\right]}$$

令 $\dfrac{b-c}{c-a}=\dfrac{1}{2}$ 可得 $c=\dfrac{a+2b}{3}$,即

$$\int_a^b \left(f''(x)\right)^2 \mathrm{d}x \geq \frac{4}{b-a}$$

注:下面给出2道相似的例子,从中即可看出应该如何处理这类问题

(1) 设 $f\in C^2[-l,l], f(0)=0$,证明:$\left(\int_{-l}^{l} f(x)\,\mathrm{d}x\right)^2 \leq \dfrac{l^5}{10}\int_{-l}^{l}\left(f''(x)\right)^2 \mathrm{d}x$

证:

$$\begin{aligned}\int_{-l}^{l} f(x)\,\mathrm{d}x &= \int_{-l}^{0} f(x)\mathrm{d}(x+l)+\int_0^l f(x)\mathrm{d}(x-l)\\ &= -\int_{-l}^{0}(x+l)f'(x)\,\mathrm{d}x-\int_0^l (x-l)f'(x)\,\mathrm{d}x\\ &= \frac{1}{2}\int_{-l}^{0}(x+l)^2 f''(x)\,\mathrm{d}x+\frac{1}{2}\int_0^l (x-l)^2 f''(x)\,\mathrm{d}x\end{aligned}$$

令

$$h(x)=\begin{cases}\dfrac{1}{2}(x-l)^2 & 0\leq x\leq l\\ \dfrac{1}{2}(x+l)^2 & -l\leq x\leq 0\end{cases}$$

则由 $Cauchy$ 不等式可得

$$\left(\int_{-l}^{l} f(x)\,\mathrm{d}x\right)^2 \leq \int_{-l}^{l}\left(f''(x)\right)^2 \mathrm{d}x\int_{-l}^{l} h^2(x)\,\mathrm{d}x=\frac{l^5}{10}\int_{-l}^{l}\left(f''(x)\right)^2 \mathrm{d}x$$

(2) $f(x):[0,1]\to\mathbb{R}$ 是可微函数,$f(0)=f(1)=-\dfrac{1}{6}$,求证:

$$\int_0^1 \left(f'(x)\right)^2 \mathrm{d}x \geq 2\int_0^1 f(x)\,\mathrm{d}x+\frac{1}{4}$$

证:对 $\forall t\in\mathbb{R}$,由 $Cauchy$ 不等式得

$$\left(\int_0^1 (x+t)f'(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 (x+t)^2\,\mathrm{d}x\int_0^1 \left(f'(x)\right)^2 \mathrm{d}x$$

即

$$\frac{3}{3t^2+3t+1}\left(\frac{1}{6}+\int_0^1 f(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 \left(f'(x)\right)^2 \mathrm{d}x$$

记

$$s=\frac{3}{3t^2+3t+1}$$

则原不等式成立只需要下式恒成立

$$s\left(\frac{1}{6}+\int_0^1 f(x)\,\mathrm{d}x\right)^2 \geq 2\int_0^1 f(x)\,\mathrm{d}x+\frac{1}{4}$$

即

$$\left(\int_0^1 f(x)\,\mathrm{d}x+\frac{1}{6}-\frac{1}{s}\right)^2+\left(\frac{1}{36}-\frac{1}{4s}\right)-\left(\frac{1}{6}-\frac{1}{s}\right)^2\geq 0$$

令

$$\frac{1}{36}-\frac{1}{4s}=\left(\frac{1}{6}-\frac{1}{s}\right)^2$$

解得 $s=12$,从而易知 $t=-\dfrac{1}{2}$,故

$$\int_0^1\left(f'(x)\right)^2\,\mathrm{d}x\geq 2\int_0^1 f(x)\,\mathrm{d}x+\frac{1}{4}$$

**问题**.设 $f\in C^1[0,+\infty)$,令 $F(x)=\max\limits_{x\leq y\leq 2x}|f(y)|$,证明:

$$\int_0^{\infty}F(x)\,\mathrm{d}x\leq\int_0^{\infty}|f(x)|\,\mathrm{d}x+\int_0^{\infty}x|f'(x)|\,\mathrm{d}x$$

证:由

$$\begin{aligned}f(y)&=\frac{1}{x}\int_x^{2x}(f(y)-f(z))\,\mathrm{d}z+\frac{1}{x}\int_x^{2x}f(z)\,\mathrm{d}z\\&=\frac{1}{x}\int_x^{2x}\int_z^{y}f'(t)\,\mathrm{d}t\,\mathrm{d}z+\frac{1}{x}\int_x^{2x}f(z)\,\mathrm{d}z\end{aligned}$$

可得

$$\begin{aligned}F(x)&=\max_{x\leq y\leq 2x}\frac{1}{x}\int_x^{2x}\int_z^{y}f'(t)\,\mathrm{d}t\,\mathrm{d}z+\frac{1}{x}\int_x^{2x}f(z)\,\mathrm{d}z\\&\leq\frac{1}{x}\int_x^{2x}\int_x^{2x}|f'(t)|\,\mathrm{d}t\,\mathrm{d}z+\frac{1}{x}\int_x^{2x}|f(z)|\,\mathrm{d}z\end{aligned}$$

从而

$$\begin{aligned}\int_0^{\infty}F(x)\,\mathrm{d}x&\leq\int_0^{\infty}\int_x^{2x}|f'(t)|\,\mathrm{d}t\,\mathrm{d}x+\int_0^{\infty}\int_x^{2x}\frac{1}{x}|f(z)|\,\mathrm{d}z\,\mathrm{d}x\\&=\int_0^{\infty}t|f'(t)|\,\mathrm{d}t\int_{\frac{t}{2}}^{t}\mathrm{d}x+\int_0^{\infty}|f(z)|\,\mathrm{d}z\int_{\frac{t}{2}}^{t}\frac{1}{x}\,\mathrm{d}x\\&=\frac{1}{2}\int_0^{\infty}t|f'(t)|\,\mathrm{d}t+\ln 2\int_0^{\infty}|f(z)|\,\mathrm{d}x\\&\leq\int_0^{\infty}|f(x)|\,\mathrm{d}x+\int_0^{\infty}x|f'(x)|\,\mathrm{d}x\end{aligned}$$

**问题**.设 $f(x)$ 在 $[0,1]$ 上二阶连续可微,证明:

$$\int_0^1|f'(x)|\,\mathrm{d}x\leq 9\int_0^1|f(x)|\,\mathrm{d}x+\int_0^1|f''(x)|\,\mathrm{d}x$$

证:对 $\forall x_1\in\left[0,\dfrac{1}{3}\right],x_2\in\left[\dfrac{2}{3},1\right]$,由 *Lagrange* 中值定理得

$$|f'(\xi)|=\left|\frac{f(x_1)-f(x_2)}{x_1-x_2}\right|\leq|f(x_1)-f(x_2)|\leq 3|f(x_1)|+3|f(x_2)|$$

从而对 $\forall x\in[0,1]$ 有

$$\begin{aligned}|f'(x)|=&\left|f'(\xi)+\int_\xi^x f''(t)\,\mathrm{d}t\right|\\ \le&|f'(\xi)|+\int_\xi^x|f''(t)|\,\mathrm{d}t\\ \le&\,3|f(x_1)|+3|f(x_2)|+\int_0^1|f''(x)|\,\mathrm{d}x\end{aligned}$$

在上述不等式两端分别对 $x_1,x_2$ 在 $\left[0,\frac{1}{3}\right],\left[\frac{2}{3},1\right]$ 上进行积分得

$$\begin{aligned}|f'(x)|\le&\,9\int_0^{\frac{1}{3}}|f(x)|\,\mathrm{d}x+9\int_{\frac{2}{3}}^1|f(x)|\,\mathrm{d}x+\int_0^1|f''(x)|\,\mathrm{d}x\\ \le&\,9\int_0^1|f(x)|\,\mathrm{d}x+\int_0^1|f''(x)|\,\mathrm{d}x\end{aligned}$$

最后对 $x$ 在 $[0,1]$ 上积分即得

$$\int_0^1|f'(x)|\,\mathrm{d}x\le 9\int_0^1|f(x)|\,\mathrm{d}x+\int_0^1|f''(x)|\,\mathrm{d}x$$

注:(1) 利用类似的方法可以将结论中 $\int_0^1|f(x)|\,\mathrm{d}x$ 的系数改进为8,即

$$\int_0^1|f'(x)|\,\mathrm{d}x\le 8\int_0^1|f(x)|\,\mathrm{d}x+\int_0^1|f''(x)|\,\mathrm{d}x$$

(2) 下面证明 $\int_0^1|f(x)|\,\mathrm{d}x$ 的最佳系数为4

证:由 $f(x)\in C^2[0,1]$ 可得 $\exists x_0,x_1\in[0,1]\ s.t.$

$$|f(x_0)|=\min_{x\in[0,1]}|f(x)|,|f'(x_1)|=\min_{x\in[0,1]}|f'(x)|$$

由 $Lagrange$ 中值定理知,对 $\forall x\in[0,1],\exists\xi_x\in(0,1)\ s.t.$

$$f(x)=f(x_0)+f'(\xi_x)(x-x_0)$$

显然无论 $f(x_0)\ge$ 还是 $<0$ 均成立

$$|f(x)|=|f(x_0)|+|f'(\xi_x)(x-x_0)|$$

所以

$$|f(x)|\ge|f'(\xi_x)(x-x_0)|\ge|f'(x_1)(x-x_0)|$$

从而

$$\int_0^1|f(x)|\,\mathrm{d}x\ge|f'(x_1)|\left(\frac{x_0^2}{2}+\frac{(1-x_0)^2}{2}\right)\ge\frac{1}{4}|f'(x_1)|$$

故

$$
\begin{aligned}
|f'(x)| &= \left|f'(x_1) + \int_{x_1}^{x} f''(t)\,\mathrm{d}t\right| \\
&\le |f'(x_1)| + \int_{x_1}^{x} |f''(t)|\,\mathrm{d}t \\
&\le 4\int_0^1 |f(x)|\,\mathrm{d}x + \int_0^1 |f''(x)|\,\mathrm{d}x
\end{aligned}
$$

即

$$
\int_0^1 |f'(x)|\,\mathrm{d}x \le 4\int_0^1 |f(x)|\,\mathrm{d}x + \int_0^1 |f''(x)|\,\mathrm{d}x
$$

易知 $f(x) = x - \dfrac{1}{2}$ 时不等式等号成立,从而4是最佳系数

**问题**. $\forall n \in \mathbb{N}^+, I(n) = \displaystyle\int_0^{\frac{\pi}{2}} \left|\frac{\sin(2n+1)x}{\sin x}\right| \mathrm{d}x$,证明:存在 $\displaystyle\lim_{n\to\infty}\left(I(n) - \frac{2\ln n}{\pi}\right)$

证:

$$
\begin{aligned}
I(n) - \frac{2\ln n}{\pi} &= \int_0^{\frac{\pi}{2}} \left|\frac{\sin(2n+1)x}{x}\right| \mathrm{d}x - \frac{2\ln n}{\pi} + \int_0^{\frac{\pi}{2}} |\sin(2n+1)x| \left(\frac{1}{\sin x} - \frac{1}{x}\right) \mathrm{d}x \\
&= \int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,\mathrm{d}x + J(n) + K(n) + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{x - \sin x}{x\sin x}\,\mathrm{d}x
\end{aligned}
$$

其中

$$
J(n) = \sum_{k=1}^{n} \int_{(k-\frac{1}{2})\pi}^{(k+\frac{1}{2})\pi} \frac{|\sin x|}{x}\,\mathrm{d}x - \frac{2\ln n}{\pi}
$$

$$
K(n) = \int_0^{\frac{\pi}{2}} \left(|\sin(2n+1)\pi| - \frac{2}{\pi}\right)\left(\frac{1}{\sin x} - \frac{1}{x}\right) \mathrm{d}x
$$

显然

$$
\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,\mathrm{d}x + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{x - \sin x}{x\sin x}\,\mathrm{d}x
$$

是有限值

由 $Riemann - Lebesgue$ 引理知

$$
\lim_{n\to\infty} K(n) = \frac{1}{\pi}\int_0^{\pi}\left(\sin x - \frac{2}{\pi}\right)\mathrm{d}x \cdot \int_0^{\frac{\pi}{2}}\left(\frac{1}{\sin x} - \frac{1}{x}\right)\mathrm{d}x = 0
$$

又

$$
J(n) = \sum_{k=1}^{n}\left(\int_{(k-\frac{1}{2})\pi}^{k\pi} \frac{|\sin x|}{x}\,\mathrm{d}x + \int_{k\pi}^{(k+\frac{1}{2})\pi} \frac{|\sin x|}{x}\,\mathrm{d}x\right) - \frac{2\ln n}{\pi}
$$

从而

$$
\begin{aligned}
J(n) &\le \sum_{k=1}^{n}\left(\frac{1}{k\pi - \dfrac{\pi}{2}}\int_{-\frac{\pi}{2}}^{0} (-\sin t)\,\mathrm{d}t + \frac{1}{k\pi}\int_0^{\frac{\pi}{2}} \sin t\,\mathrm{d}t\right) - \frac{2\ln n}{\pi} \\
&= \frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{1}{2k-1} + \frac{1}{2k}\right) - \frac{2\ln n}{\pi} = \frac{2}{\pi}\left(\sum_{k=1}^{2n}\frac{1}{k} - \ln n\right)
\end{aligned}
$$

$$
\begin{aligned}
J(n) &\geq \sum_{k=1}^{n}\left(\frac{1}{k\pi}\int_{-\frac{\pi}{2}}^{0}(-\sin t)\,\mathrm{d}t+\frac{1}{k\pi+\frac{\pi}{2}}\int_{0}^{\frac{\pi}{2}}\sin t\,\mathrm{d}t\right)-\frac{2\ln n}{\pi}\\
&=\frac{2}{\pi}\sum_{k=1}^{n}\left(\frac{1}{2k}+\frac{1}{2k+1}\right)-\frac{2\ln n}{\pi}=\frac{2}{\pi}\left(\sum_{k=1}^{2n+1}\frac{1}{k}-1-\ln n\right)
\end{aligned}
$$

由

$$
\sum_{k=1}^{n}\frac{1}{k}=\ln n+\gamma+\varepsilon_n,\varepsilon_n\to 0\,(n\to\infty)
$$

易得

$$
J(n)\leq\frac{2}{\pi}(\ln 2+\gamma+\varepsilon_n)
$$

$$
J(n)\geq\frac{2}{\pi}\left[\ln\left(2+\frac{1}{n}\right)+\gamma-1+\varepsilon_{2n+1}\right]>0
$$

由比较判别法即知 $J(n)$ 收敛,故 $\lim\limits_{n\to\infty}\left(I(n)-\dfrac{2\ln n}{\pi}\right)$存在

注:($Riemann-Lebesgue$ 引理) 设 $f\in R[a,b],g\in R[0,T],g(x+T)=g(x)$,则

$$
\lim_{n\to\infty}\int_a^b f(x)g(nx)\,\mathrm{d}x=\frac{1}{T}\int_0^T g(x)\,\mathrm{d}x\cdot\int_a^b f(x)\,\mathrm{d}x
$$

证:不妨假设 $\displaystyle\int_0^T g(x)\,\mathrm{d}x=0$,否则令

$$
h(x)=g(x)-\frac{1}{T}\int_0^T g(x)\,\mathrm{d}x
$$

从而原命题等价于证明

$$
\lim_{n\to\infty}\int_a^b f(x)g(nx)\,\mathrm{d}x=0
$$

不妨假设 $g(x)\geq 0$,否则用下式代替

$$
g(x)=\frac{|g(x)|+g(x)}{2}-\frac{|g(x)|-g(x)}{2}
$$

对闭区间 $[a,b]$ 做分划 $P,a=x_0<x_1<\cdots<x_m=b$,令

$$
|f(x)|,|g(x)|\leq M
$$

$$
bn=an+k_1T+k_2\,(k_1,k_2\in\mathbb{Z},0\leq k_2<T)
$$

则

$$
\begin{aligned}
\int_a^b g(nx)\,\mathrm{d}x&=\frac{1}{n}\int_{an}^{an+k_1T+k_2}g(x)\,\mathrm{d}x\\
&=\frac{1}{n}\left(\int_{an}^{an+k_1T}g(x)\,\mathrm{d}x+\int_{an+k_1T}^{an+k_1T+k_2}g(x)\,\mathrm{d}x\right)\\
&=\frac{1}{n}\int_{an+k_1T}^{an+k_1T+k_2}g(x)\,\mathrm{d}x\leq\frac{MT}{n}
\end{aligned}
$$

从而

$$\begin{aligned}\left|\int_a^b f(x)g(nx)\,\mathrm{d}x\right| &\le \sum_{k=1}^m \int_{x_{k-1}}^{x_k} |(f(x)-f(x_k))g(nx)|\,\mathrm{d}x + \left|\int_a^b f(x_k)g(nx)\,\mathrm{d}x\right| \\ &\le M\sum_{k=1}^m \omega_k \Delta x_k + \frac{M^2T}{n}\end{aligned}$$

再分别令 $||P|| = \max\limits_{1\le k\le m} \Delta x_k \to 0, n\to\infty$ 即得

$$\lim_{n\to\infty}\int_a^b f(x)g(nx)\,\mathrm{d}x = 0$$

**问题**.讨论 $I=\displaystyle\int_0^{+\infty}\frac{x}{\cos^2 x + x^\alpha \sin^2 x}\,\mathrm{d}x$ 的敛散性,其中 $\alpha\in\mathbb{R}$

解:(1) 当 $\alpha\le 0$ 时

$$I \ge \int_1^{+\infty}\frac{x}{\cos^2 x + x^\alpha\sin^2 x}\,\mathrm{d}x \ge \int_1^{+\infty}\frac{x}{\cos^2 x+\sin^2 x}\,\mathrm{d}x = \int_1^{+\infty} x\,\mathrm{d}x \to +\infty$$

此时积分发散

(2) 当 $0<\alpha\le 2$ 时

$$I \ge \int_1^{+\infty}\frac{x}{x^\alpha\left(\sin^2 x+\cos^2 x\right)}\,\mathrm{d}x = \int_1^{+\infty} x^{1-\alpha}\,\mathrm{d}x \to +\infty$$

此时积分发散

(3) 当 $2<\alpha\le 4$ 时

$$I = \sum_{k=0}^\infty \int_{k\pi}^{(k+1)\pi}\frac{x}{\cos^2 x + x^\alpha\sin^2 x}\,\mathrm{d}x = \sum_{k=0}^\infty a_n$$

则

$$\begin{aligned} a_n &\ge k\pi\int_{k\pi}^{(k+1)\pi}\frac{1}{\cos^2 x + (k+1)^\alpha\pi^\alpha\sin^2 x}\,\mathrm{d}x \\ &= 2k\pi\int_0^{\frac{\pi}{2}}\frac{1}{\cos^2 x+(k+1)^\alpha\pi^\alpha\sin^2 x}\,\mathrm{d}x \\ &= 2k\pi\int_0^{\frac{\pi}{2}}\frac{1}{1+(k+1)^\alpha\pi^\alpha\tan^2 x}\,\mathrm{d}\tan x \\ &= \frac{k\pi^2}{((k+1)\pi)^{\frac{\alpha}{2}}}\end{aligned}$$

此时积分发散

(4) 当 $\alpha > 4$ 时,类似可得

$$\begin{aligned}
a_n &\leq (k+1)\pi \int_{k\pi}^{(k+1)\pi} \frac{1}{\cos^2 x + (k\pi)^\alpha \sin^2 x} \, \mathrm{d}x \\
&= 2(k+1)\pi \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2 x + (k\pi)^\alpha \sin^2 x} \, \mathrm{d}x \\
&= \frac{(k+1)\pi^2}{(k\pi)^{\frac{\alpha}{2}}}
\end{aligned}$$

此时积分收敛

综上所述,当且仅当 $\alpha > 4$ 时原积分收敛

**问题**. $f(x) : [0,1] \to \mathbb{R}$ 是连续可微函数,且 $\displaystyle\int_0^{\frac{1}{2}} f(x) \, \mathrm{d}x = 0$,求证:

$$\int_0^1 \left(f'(x)\right)^2 \, \mathrm{d}x \geq 12 \left( \int_0^1 f(x) \, \mathrm{d}x \right)^2$$

证:由 $Cauchy$ 不等式可得

$$\int_0^{\frac{1}{2}} \left(f'(x)\right)^2 \, \mathrm{d}x \int_0^{\frac{1}{2}} x^2 \, \mathrm{d}x \geq \left( \int_0^{\frac{1}{2}} x f'(x) \, \mathrm{d}x \right)^2 = \left( \frac{1}{2} f\left(\frac{1}{2}\right) \right)^2$$

即

$$\int_0^{\frac{1}{2}} \left(f'(x)\right)^2 \, \mathrm{d}x \geq 24 \left( \frac{1}{2} f\left(\frac{1}{2}\right) \right)^2$$

再由 $Cauchy$ 不等式得

$$\int_{\frac{1}{2}}^1 (1-x)^2 \, \mathrm{d}x \int_{\frac{1}{2}}^1 \left(f'(x)\right)^2 \, \mathrm{d}x \geq \left( \int_{\frac{1}{2}}^1 (1-x) f'(x) \right)^2 = \left( \int_{\frac{1}{2}}^1 f(x) \, \mathrm{d}x - \frac{1}{2} f\left(\frac{1}{2}\right) \right)^2$$

即

$$\int_{\frac{1}{2}}^1 \left(f'(x)\right)^2 \, \mathrm{d}x \geq 24 \left( \int_{\frac{1}{2}}^1 f(x) \, \mathrm{d}x - \frac{1}{2} f\left(\frac{1}{2}\right) \right)^2$$

由 $2(a^2+b^2) \geq (a+b)^2$ 可得

$$\begin{aligned}
\int_0^1 \left(f'(x)\right)^2 \, \mathrm{d}x &\geq 24 \left( \frac{1}{2} f\left(\frac{1}{2}\right) \right)^2 + 24 \left( \int_{\frac{1}{2}}^1 f(x) \, \mathrm{d}x - \frac{1}{2} f\left(\frac{1}{2}\right) \right)^2 \\
&\geq 12 \left( \frac{1}{2} f\left(\frac{1}{2}\right) + \int_{\frac{1}{2}}^1 f(x) \, \mathrm{d}x - \frac{1}{2} f\left(\frac{1}{2}\right) \right)^2 \\
&= 12 \left( \int_0^1 f(x) \, \mathrm{d}x \right)^2
\end{aligned}$$

注:(1) $f(x) : [a,b] \to \mathbb{R}$ 是连续可微函数,且 $\displaystyle\int_a^b f(x) \, \mathrm{d}x = 0$,求证:

$$\int_a^{2b-a} \left(f'(x)\right)^2 \, \mathrm{d}x \geq \frac{3}{2(b-a)^3} \left( \int_a^{2b-a} f(x) \, \mathrm{d}x \right)^2$$

(2) $f(x):[0,1]\to\mathbb{R}$ 是连续可微函数,且 $\int_{\frac{1}{2n+1}}^{\frac{2}{2n+1}} f(x)\,\mathrm{d}x=0\,(n\in\mathbb{N}^{+})$,求证:

$$\int_0^1 \left(f'(x)\right)^2\,\mathrm{d}x \geq \frac{3(2n+1)^2}{4n^2-6n+3}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

证:令

$$g(x)=\begin{cases} x & x\in\left[0,\frac{1}{2n+1}\right) \\ 1-2nx & x\in\left[\frac{1}{2n+1},\frac{2}{2n+1}\right) \\ x-1 & x\in\left[\frac{2}{2n+1},1\right) \end{cases}$$

则

$$\begin{aligned}
&\int_0^1 g(x)f'(x)\,\mathrm{d}x \\
=&\int_0^{\frac{1}{2n+1}} xf'(x)\,\mathrm{d}x+\int_{\frac{1}{2n+1}}^{\frac{2}{2n+1}}(1-2nx)f'(x)\,\mathrm{d}x+\int_{\frac{2}{2n+1}}^1 (x-1)f'(x)\,\mathrm{d}x \\
=&2n\int_{\frac{1}{2n+1}}^{\frac{2}{2n+1}} f(x)\,\mathrm{d}x-\int_0^{\frac{1}{2n+1}} f(x)\,\mathrm{d}x-\int_{\frac{2}{2n+1}}^1 f(x)\,\mathrm{d}x \\
=&-\int_0^1 f(x)\,\mathrm{d}x
\end{aligned}$$

再由 $Cauchy$ 不等式得

$$\left(-\int_0^1 f(x)\,\mathrm{d}x\right)^2=\left(\int_0^1 g(x)f'(x)\,\mathrm{d}x\right)^2\leq\int_0^1 g^2(x)\,\mathrm{d}x\int_0^1\left(f'(x)\right)^2\,\mathrm{d}x$$

又

$$\begin{aligned}
\int_0^1 g^2(x)\,\mathrm{d}x&=\int_0^{\frac{1}{2n+1}} x^2\,\mathrm{d}x+\int_{\frac{1}{2n+1}}^{\frac{2}{2n+1}}(1-2nx)^2\,\mathrm{d}x+\int_{\frac{2}{2n+1}}^1 (x-1)^2\,\mathrm{d}x \\
&=\frac{4n^2-6n+3}{3(2n+1)^2}
\end{aligned}$$

则

$$\int_0^1 \left(f'(x)\right)^2\,\mathrm{d}x \geq \frac{3(2n+1)^2}{4n^2-6n+3}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

(3) $f(x):[a,b]\to\mathbb{R}$ 是连续可微函数,且 $\int_{\frac{1}{2n}}^{\frac{1}{n}} f(x)\,\mathrm{d}x=0\,(n\in\mathbb{N}^{+})$,求证:

$$\int_0^1 \left(f'(x)\right)^2\,\mathrm{d}x \geq \frac{12n^2}{4n^2-10n+7}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

证:令

$$g(x)=\begin{cases} x & x\in\left[0,\frac{1}{2n}\right) \\ 1-(2n-1)x & x\in\left[\frac{1}{2n},\frac{1}{n}\right) \\ x-1 & x\in\left[\frac{1}{n},1\right) \end{cases}$$

则

$$\begin{aligned}
&\int_0^1 g(x)f'(x)\,\mathrm{d}x\\
=&\int_0^{\frac{1}{2n}} xf'(x)\,\mathrm{d}x+\int_{\frac{1}{2n}}^{\frac{1}{n}}[1-(2n-1)x]f'(x)\,\mathrm{d}x+\int_{\frac{1}{n}}^1 (x-1)f'(x)\,\mathrm{d}x\\
=&(2n-1)\int_{\frac{1}{2n}}^{\frac{1}{n}} f(x)\,\mathrm{d}x-\int_0^{\frac{1}{2n}} f(x)\,\mathrm{d}x-\int_{\frac{1}{n}}^1 f(x)\,\mathrm{d}x\\
=&-\int_0^1 f(x)\,\mathrm{d}x
\end{aligned}$$

再由 $Cauchy$ 不等式得

$$\left(-\int_0^1 f(x)\,\mathrm{d}x\right)^2=\left(\int_0^1 g(x)f'(x)\,\mathrm{d}x\right)^2\leq\int_0^1 g^2(x)\,\mathrm{d}x\int_0^1\left(f'(x)\right)^2\,\mathrm{d}x$$

又

$$\begin{aligned}
\int_0^1 g^2(x)\,\mathrm{d}x&=\int_0^{\frac{1}{2n}} x^2\,\mathrm{d}x+\int_{\frac{1}{2n}}^{\frac{1}{n}}[1-(2n-1)x]^2\,\mathrm{d}x+\int_{\frac{1}{n}}^1 (x-1)^2\,\mathrm{d}x\\
&=\frac{4n^2-10n+7}{12n^2}
\end{aligned}$$

则

$$\int_0^1\left(f'(x)\right)^2\,\mathrm{d}x\geq\frac{12n^2}{4n^2-10n+7}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

**问题**. $f(x):[0,1]\to\mathbb{R}$ 是连续函数,且 $\int_0^1 f^3(x)\,\mathrm{d}x=0$，求证:

$$\int_0^1 f^4(x)\,\mathrm{d}x\geq\frac{27}{4}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^4$$

证:令

$$I_n=\int_0^1 f^n(x)\,\mathrm{d}x$$

显然

$$I_2\geq I_1^2$$

对 $\forall r\in\mathbb{R}$,由 $Cauchy$ 不等式可得

$$\left(\int_0^1\left(r+f^2(x)\right)\cdot f(x)\,\mathrm{d}x\right)^2\leq\int_0^1\left(r+f^2(x)\right)^2\,\mathrm{d}x\cdot\int_0^1 f^2(x)\,\mathrm{d}x$$

化简后得

$$\left(I_2-I_1^2\right)r^2+2I_2^2r+I_2I_4\geq0$$

由于上式对$\forall r \in \mathbb{R}$恒成立,故判别式$\triangle \leq 0$,即

$$I_4 \geq \frac{I_2^3}{I_2 - I_1^2}$$

故本题只需要证明

$$\frac{I_2^3}{I_2 - I_1^2} \geq \frac{27}{4} I_1^4$$

事实上,由均值不等式即得

$$\left(I_2 - I_1^2\right) I_1^4 = \frac{1}{2}\left(2I_2 - 2I_1^2\right) \cdot I_1^2 \cdot I_1^2 \leq \frac{4}{27} I_2^3$$

故

$$\int_0^1 f^4(x)\,\mathrm{d}x \geq \frac{27}{4}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^4$$

**问题**. $f(x):[0,1] \to \mathbb{R}$是可积函数,$\int_0^1 xf(x)\,\mathrm{d}x = 0$,令$F(x) = \int_0^x f(y)\,\mathrm{d}y \geq 0$,求证:

$$\int_0^1 f^2(x)\,\mathrm{d}x + 5\int_0^1 F^2(x)\,\mathrm{d}x \geq 10\int_0^1 f(x)F(x)\,\mathrm{d}x$$

证:由分部积分公式可得

$$\int_0^1 F(x)\,\mathrm{d}x = \int_0^1 (1-x)f(x)\,\mathrm{d}x = F(1)$$

$$\int_0^1 f(x)F(x)\,\mathrm{d}x = \int_0^1 F(x)\,\mathrm{d}F(x) = \frac{F^2(1)}{2}$$

由均值不等式得

$$\int_0^1 f^2(x)\,\mathrm{d}x + \int_0^1 F^2(x)\,\mathrm{d}x \geq 2\int_0^1 f(x)F(x)\,\mathrm{d}x = F^2(1)$$

由$Cauchy$不等式得

$$\int_0^1 F^2(x)\,\mathrm{d}x \geq \left(\int_0^1 F(x)\,\mathrm{d}x\right)^2 = F^2(1)$$

从而

$$\int_0^1 f^2(x)\,\mathrm{d}x + 5\int_0^1 F^2(x)\,\mathrm{d}x \geq 5F^2(1) = 10\int_0^1 f(x)F(x)\,\mathrm{d}x$$

注:(1) 此题不等式右端的最佳系数为12,即

$$\int_0^1 f^2(x)\,\mathrm{d}x + 5\int_0^1 F^2(x)\,\mathrm{d}x \geq 12\int_0^1 f(x)F(x)\,\mathrm{d}x$$

(2) 此题可推广为

$$A\int_0^1 f^2(x)\,\mathrm{d}x + B\int_0^1 F^2(x)\,\mathrm{d}x \geq 2(A+B)\int_0^1 f(x)F(x)\,\mathrm{d}x$$

等号成立当且仅当 $A=B=0$ 或 $F(x)=f(x)=0$

**问题**.设 $f\in C^4[0,1]$,并且

$$\int_0^1 f(x)\,\mathrm{d}x+3f\left(\frac{1}{2}\right)=8\int_{\frac{1}{4}}^{\frac{3}{4}} f(x)\,\mathrm{d}x$$

证明:$\exists\, c\in(0,1)\, s.t. f^{(4)}(c)=0$

证1:不妨假设 $f\left(\frac{1}{2}\right)=0$,否则用 $g(x)=f(x)-f\left(\frac{1}{2}\right)$代替 $f(x)$,令

$$F(x)=f(x)-f'\left(\frac{1}{2}\right)\left(x-\frac{1}{2}\right)-\frac{f''\left(\frac{1}{2}\right)}{2}\left(x-\frac{1}{2}\right)^2-\frac{f'''\left(\frac{1}{2}\right)}{6}\left(x-\frac{1}{2}\right)^3$$

则

$$F^{(k)}\left(\frac{1}{2}\right)=0\,(k=0,1,2,3)$$

$$\int_0^1 F(x)\,\mathrm{d}x=8\int_{\frac{1}{4}}^{\frac{3}{4}} F(x)\,\mathrm{d}x$$

令

$$h(x)=\int_{\frac{1}{2}-2x}^{\frac{1}{2}+2x} F(t)\,\mathrm{d}t-8\int_{\frac{1}{2}-x}^{\frac{1}{2}+x} F(t)\,\mathrm{d}t$$

则

$$h\left(\frac{1}{4}\right)=0, h^{(k)}(0)=0\,(k=0,1,2)$$

$$h'''(x)=8\left[F''\left(2x+\frac{1}{2}\right)-F''\left(x+\frac{1}{2}\right)\right]+8\left[F''\left(\frac{1}{2}-2x\right)-F''\left(\frac{1}{2}-x\right)\right]$$

反复由 $Rolle$ 定理有

$$\exists\, b\in\left(0,\frac{1}{4}\right) s.t. h'''(b)=0$$

由 $Lagrange$ 中值定理得 $\exists\, b^+\in\left(b+\frac{1}{2},2b+\frac{1}{2}\right), b^-\in\left(\frac{1}{2}-2b,\frac{1}{2}-b\right) s.t.$

$$F'''(b^+)-F'''(b^-)=0$$

再由 $Rolle$ 定理即知

$$\exists\, c\in(b^-,b^+)\, s.t. F^{(4)}(c)=f^{(4)}(c)=0$$

证2:令

$$g(x)=f\left(x+\frac{1}{2}\right)-f\left(\frac{1}{2}\right)$$

$$G(t)=\int_{-t}^{t} g(x)\,\mathrm{d}x-8\int_{-\frac{t}{2}}^{\frac{t}{2}} g(x)\,\mathrm{d}x$$

从而

$$G(0)=G\left(\frac{1}{2}\right)=0$$

由 $Rolle$ 定理即得 $\exists\, t_0\in\left(0,\frac{1}{2}\right)\, s.t. G'(t_0)=0$,并且

$$G'(t)=g(t)-4\,g\left(\frac{t}{2}\right)-4\,g\left(-\frac{t}{2}\right)+g(-t)$$

显然 $G'(0)=0$,由 $Rolle$ 定理即得 $\exists\, t_1\in(0,t_0)\, s.t. G''(t_1)=0$,并且

$$G''(t)=g'(t)-2\,g'\left(\frac{t}{2}\right)+2\,g'\left(-\frac{t}{2}\right)-g'(-t)$$

显然 $G''(0)=0$,由 $Rolle$ 定理即得 $\exists\, t_2\in(0,t_1)\, s.t. G'''(t_2)=0$,并且

$$G'''(t)=g''(t)-g''\left(\frac{t}{2}\right)-g''\left(-\frac{t}{2}\right)+g''(-t)$$

由 $Lagrange$ 中值定理可得 $\exists\,\theta^+\in\left(\frac{t_2}{2},t_2\right),\theta^-\in\left(-t_2,-\frac{t_2}{2}\right)\, s.t.$

$$\left[g''(t_2)-g''\left(\frac{t_2}{2}\right)\right]-\left[g''(-t_2)-g''\left(\frac{-t_2}{2}\right)\right]=g'''(\theta^+)\,\frac{t_2}{2}-g'''(\theta^-)\,\frac{t_2}{2}=0$$

由于 $t_2\neq 0$,故

$$g'''(\theta^+)=g'''(\theta^-)$$

从而 $\exists\,\theta\in(\theta^-,\theta^+)\, s.t. g^{(4)}(\theta)=0$,即 $f^{(4)}\left(\theta+\frac{1}{2}\right)=0$

令 $\theta+\frac{1}{2}=c$ 即得

$$f^{(4)}(c)=0$$

注:下面给出1道相近的例子,解答较之前类似

设 $f\in C^2[0,1],\int_0^1 f(x)\,\mathrm{d}x=2\int_{\frac{1}{4}}^{\frac{3}{4}} f(x)\,\mathrm{d}x$,证明:$\exists\, c\in(0,1)\, s.t. f''(c)=0$

证:由

$$\int_0^1 f(x)\,\mathrm{d}x=2\int_{\frac{1}{4}}^{\frac{3}{4}} f(x)\,\mathrm{d}x$$

得

$$\int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,\mathrm{d}x-\int_0^{\frac{1}{4}} f(x)\,\mathrm{d}x=\int_{\frac{3}{4}}^{1} f(x)\,\mathrm{d}x-\int_{\frac{1}{2}}^{\frac{3}{4}} f(x)\,\mathrm{d}x$$

令

$$g(x)=\int_x^{x+\frac{1}{4}} f(t)\,\mathrm{d}t$$

则

$$g\left(\frac{1}{4}\right)-g(0)=g\left(\frac{3}{4}\right)-g\left(\frac{1}{2}\right)$$

反复运用 $Rolle$ 定理得

$$\exists\, b \in (0,1)\, s.t. g''(b) = 0$$

即

$$f'(b) = f'\left(b + \frac{1}{4}\right)$$

再由 $Rolle$ 定理即知

$$\exists\, c \in (0,1)\, s.t. f''(c) = 0$$

**问题**.求所有连续可导函数 $f:[0,1]\to(0,+\infty)$ 满足 $f(1)=ef(0)$ 且

$$\int_0^1 \frac{1}{f^2(x)}\,\mathrm{d}x + \int_0^1 \left(f'(x)\right)^2\,\mathrm{d}x \le 2$$

解:由均值不等式得

$$\int_0^1 \left(\frac{1}{f^2(x)} + \left(f'(x)\right)^2\right)\mathrm{d}x \ge 2\int_0^1 \frac{f'(x)}{f(x)}\,\mathrm{d}x = 2\ln\frac{f(1)}{f(0)} = 2$$

从而

$$f'(x) = \frac{1}{f(x)}$$

解之即得

$$f(x) = \sqrt{2x + \frac{2}{e^2-1}}$$

**问题**.设连续函数 $f,g:[0,1]\to(0,+\infty)$,且 $f(x),\dfrac{g(x)}{f(x)}$ 单调递增,求证:

$$\int_0^1 \left(\frac{\displaystyle\int_0^x f(t)\,\mathrm{d}t}{\displaystyle\int_0^x g(t)\,\mathrm{d}t}\right)\mathrm{d}x \le 2\int_0^1 \frac{f(x)}{g(x)}\,\mathrm{d}x$$

证:由 $Chebyshev$ 不等式得

$$\int_0^x f(t)\,\mathrm{d}t \cdot \int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t \le x\int_0^x g(t)\,\mathrm{d}t$$

即

$$\frac{\displaystyle\int_0^x f(t)\,\mathrm{d}t}{\displaystyle\int_0^x g(t)\,\mathrm{d}t} \le \frac{x}{\displaystyle\int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t}$$

由 $Cauchy$ 不等式得

$$\frac{x^4}{4} = \left(\int_0^x \sqrt{\frac{g(t)}{f(t)}}\sqrt{\frac{t^2 f(t)}{g(t)}}\,\mathrm{d}t\right)^2 \le \int_0^x \frac{g(t)}{f(t)}\,\mathrm{d}t\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t$$

从而

$$\frac{\displaystyle\int_0^x f(t)\,\mathrm{d}t}{\displaystyle\int_0^x g(t)\,\mathrm{d}t} \le \frac{4}{x^3}\int_0^x \frac{t^2 f(t)}{g(t)}\,\mathrm{d}t$$

故

$$\begin{aligned}
\int_0^1 \left(\frac{\displaystyle\int_0^x f(t)\,\mathrm{d}t}{\displaystyle\int_0^x g(t)\,\mathrm{d}t}\right)\mathrm{d}x &\le \int_0^1\!\!\int_0^x \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^1\!\!\int_t^1 \frac{4t^2 f(t)}{x^3 g(t)}\,\mathrm{d}t \\
&= 2\int_0^1 \frac{f(t)}{g(t)}\left(1-t^2\right)\mathrm{d}t \\
&\le 2\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t
\end{aligned}$$

注:令 $f(t)=1, g(t)=t+\varepsilon, \varepsilon>0$,则

$$\int_0^1 \left(\frac{\displaystyle\int_0^x f(t)\,\mathrm{d}t}{\displaystyle\int_0^x g(t)\,\mathrm{d}t}\right)\mathrm{d}x = \int_0^1 \frac{x}{\frac{1}{2}x^2+\varepsilon x}\,\mathrm{d}x = 2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon$$

$$\int_0^1 \frac{f(t)}{g(t)}\,\mathrm{d}t = \int_0^1 \frac{1}{t+\varepsilon}\,\mathrm{d}t = \ln(1+\varepsilon) - \ln\varepsilon$$

从而

$$\lim_{\varepsilon\to 0}\frac{2\ln(1+2\varepsilon)-2\ln 2-2\ln\varepsilon}{\ln(1+\varepsilon)-\ln\varepsilon} = 2\lim_{\varepsilon\to 0}\frac{-\dfrac{\ln(1+2\varepsilon)}{\ln\varepsilon}+\dfrac{\ln 2}{\ln\varepsilon}+1}{-\dfrac{\ln(1+\varepsilon)}{\ln\varepsilon}+1} = 2$$

故2是此不等式的最佳系数

**问题**. $f(x):[a,b]\to\mathbb{R}$ 是连续函数,且 $\displaystyle\int_0^1 xf(x)\,\mathrm{d}x=0$,证明:

$$\left|\int_0^1 x^2 f(x)\,\mathrm{d}x\right| \le \frac{1}{6}\max_{x\in[0,1]}|f(x)|$$

解:由

$$\int_0^1 xf(x)\,\mathrm{d}x = 0$$

可得

$$\begin{aligned}
\left|\int_0^1 x^2 f(x)\,\mathrm{d}x\right| &= \left|\int_0^1 \left(x^2-ax\right)f(x)\,\mathrm{d}x\right| \\
&\le \max_{x\in[0,1]}|f(x)|\int_0^1 \left|x^2-ax\right|\mathrm{d}x
\end{aligned}$$

令

$$a=\frac{\sqrt{3}-1}{2}$$

则

$$\int_0^1 \left|x^2-ax\right| \mathrm{d}x=\frac{1}{6}$$

即

$$\left|\int_0^1 x^2 f(x)\,\mathrm{d}x\right| \leq \frac{1}{6}\max_{x\in[0,1]}|f(x)|$$

注:下面给出一道相似的例子

$f(x):[a,b]\to\mathbb{R}$ 是连续函数,且 $\int_0^1 xf(x)\,\mathrm{d}x=0$,证明:

$$\int_0^1 f^2(x)\,\mathrm{d}x \geq 4\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2$$

证:由 $Cauchy$ 不等式知

$$\begin{aligned}\left(\int_0^1 f(x)\,\mathrm{d}x\right)^2 &= \left(\int_0^1\left(1-\frac{3}{2}x\right)f(x)\,\mathrm{d}x\right)^2\\ &\leq \int_0^1\left(1-\frac{3}{2}x\right)^2\mathrm{d}x\cdot\int_0^1 f^2(x)\,\mathrm{d}x\\ &= \frac{1}{4}\int_0^1 f^2(x)\,\mathrm{d}x\end{aligned}$$

**问题**. $f(x):[0,1]\to\mathbb{R}$ 是连续函数, $\int_0^1 f(x)\,\mathrm{d}x=\int_0^1 xf(x)\,\mathrm{d}x$,求证:

(1) $\exists\, c\in(0,1)\; s.t. \int_0^c f(x)\,\mathrm{d}x=0$

(2) $\exists\, d\in(0,1)\; s.t. \int_0^d xf(x)\,\mathrm{d}x=0$

证:(1) 令

$$F(x)=\int_0^x f(x)\,\mathrm{d}x$$

则

$$\int_0^1 F(x)\,\mathrm{d}x=\int_0^1\int_0^x f(t)\,\mathrm{d}t\,\mathrm{d}x=\int_0^1(1-x)f(x)\,\mathrm{d}x=0$$

由推广的积分第一中值定理得

$$\exists\, c\in(0,1)\; s.t. F(c)=\int_0^c f(x)\,\mathrm{d}x=0$$

(2) 令

$$F(x)=\int_0^x f(t)\,\mathrm{d}t$$

$$G(x)=\frac{1}{x}\int_0^x F(t)\,\mathrm{d}t$$

则

$$G(0)=G(1)=0$$

$$G'(x)=\frac{xF(x)-\displaystyle\int_0^x F(t)\,\mathrm{d}t}{x^2}$$

从而由 $Rolle$ 定理得

$$\exists\, d\in(0,1)\; s.t. dF(d)-\int_0^d F(t)\,\mathrm{d}t=\int_0^d xF'(x)\,\mathrm{d}x=0$$

即

$$\int_0^d xf(x)\,\mathrm{d}x=0$$

注:利用本题可以推出许多有趣的结论

(1) $\exists\, c\in(0,1)\; s.t.(c-1)f(c)=f'(c)\displaystyle\int_0^c(x-1)f(x)\,\mathrm{d}x$ ($f$ 可微)

提示:令

$$g(x)=e^{-f(x)}\int_0^x(t-1)f(t)\,\mathrm{d}t$$

(2) $\exists\, c\in(0,1)\; s.t. f(c)=f'(c)\displaystyle\int_0^c f(x)\,\mathrm{d}x$ ($f$ 可微)

提示:令

$$g(x)=e^{-f(x)}\int_0^x f(t)\,\mathrm{d}t$$

(3) $\exists\, c\in(0,1)\; s.t. cf(c)=\displaystyle\int_0^c xf(x)\,\mathrm{d}x$

提示:令

$$g(x)=e^{-x}\int_0^x tf(t)\,\mathrm{d}t$$

(4) $\exists\, c\in(0,1)\; s.t. cf(c)=2\displaystyle\int_c^0 xf(x)\,\mathrm{d}x$

提示:令

$$g(x)=e^{2x}\int_0^x tf(t)\,\mathrm{d}t$$

(5) $\exists\, c\in(0,1)\; s.t. c^2f(c)=\displaystyle\int_0^c xf(x)\,\mathrm{d}x$

提示:令

$$g(x)=\frac{\displaystyle\int_0^x tf(t)\,\mathrm{d}t}{x}$$

**问题.** $f(x):[0,1]\to\mathbb{R}$ 是连续函数,$\displaystyle\int_0^1 f(x)\,\mathrm{d}x=0$,求证:

$$\exists\, c\in(0,1)\; s.t.\int_0^c xf(x)\,\mathrm{d}x=0$$

证:令

$$F(x)=\int_0^x F(t)\,\mathrm{d}t$$

$$H(x)=\begin{cases}\dfrac{\displaystyle\int_0^x F(t)\,\mathrm{d}t}{x} & x\neq 0\\ 0 & x=0\end{cases}$$

由连续函数的介质性不妨假设对 $\forall\, x\in(0,1)$, 成立 $G(x)=\displaystyle\int_0^x tf(t)\,\mathrm{d}t>0$

此时令 $x\to 1^-$ 即得

$$\int_0^1 F(t)\,\mathrm{d}t\leq 0$$

又

$$H'(x)=\frac{xF(x)-\displaystyle\int_0^x F(t)\,\mathrm{d}t}{x^2}=\frac{G(x)}{x^2}>0$$

故

$$\int_0^1 F(t)\,\mathrm{d}t=H(1)>H(0)=0$$

矛盾!

即

$$\exists\, c\in(0,1)\; s.t. \int_0^c xf(x)\,\mathrm{d}x=0$$

注:(1) $\exists\, c\in(0,1)\; s.t. c^2f(c)+\displaystyle\int_0^c f(x)\,\mathrm{d}x=0$

提示:令

$$g(x)=\begin{cases}e^{-\frac{1}{x}}\displaystyle\int_0^x f(t)\,\mathrm{d}t & x\neq 0\\ 0 & x=0\end{cases}$$

(2) $\exists\, c\in(0,1)\; s.t. cf(c)=f'(c)\displaystyle\int_0^c xf(x)\,\mathrm{d}x$ ($f$ 可微)

提示:令

$$g(x)=e^{-f(x)}\int_0^x tf(t)\,\mathrm{d}t$$

(3) $\exists\, c\in(0,1)\; s.t. cf(c)=\displaystyle\int_0^b xf(x)\,\mathrm{d}x$

提示:令

$$g(x)=e^{-x}\int_0^x tf(t)\,\mathrm{d}t$$

(4) $\exists\, c\in(0,1)\; s.t. c^2f(c)=\displaystyle\int_0^c \left(x+x^2\right)f(x)\,\mathrm{d}x$

提示:令

$$g(x)=\int_0^x \left((x-1)t+t^2\right)f(t)\,\mathrm{d}t$$

再结合 $Flett$ 均值定理即可

**问题**.设 $f(x)$ 在 [a,b] 上大于0,且满足 $Lipschitz$ 条件 $|f(x_1)-f(x_2)|\leq L|x_1-x_2|\,(L>0)$,又对于 $a\leq c\leq d\leq b$ 有

$$\int_c^d \frac{\mathrm{d}x}{f(x)}=\alpha,\int_a^b \frac{\mathrm{d}x}{f(x)}=\beta$$

求证:

$$\int_a^b f(x)\,\mathrm{d}x\leq \frac{e^{2L\beta}-1}{2\alpha L}\int_c^d f(x)\,\mathrm{d}x$$

证1:由积分第一中值定理得

$$\exists\, x_0\in[c,d\,]\; s.t.\; \frac{d-c}{f(x_0)}=\int_c^d \frac{\mathrm{d}x}{f(x)}=\alpha$$

则

$$f(x)\leq f(x_0)+L|x-x_0|$$

从而

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x&\leq \int_a^{x_0}(f(x_0)+L(x_0-x))\,\mathrm{d}x+\int_{x_0}^b(f(x_0)+L(x-x_0))\,\mathrm{d}x\\&=f(x_0)(b-a)+\frac{L(x_0-a)^2}{2}+\frac{L(b-x_0)^2}{2}\\ \beta&\geq \int_a^{x_0}\frac{\mathrm{d}x}{L(x_0-x)+f(x_0)}+\int_{x_0}^b\frac{\mathrm{d}x}{L(x-x_0)+f(x_0)}\\&=\frac{1}{L}\ln\left[\left(1+\frac{L(x_0-a)}{f(x_0)}\right)\left(1+\frac{L(b-x_0)}{f(x_0)}\right)\right]\end{aligned}$$

即

$$\left(1+\frac{\alpha L(x_0-a)}{d-c}\right)\left(1+\frac{\alpha L(b-x_0)}{d-c}\right)\leq e^{L\beta}$$

记

$$m=1+\frac{\alpha L(x_0-a)}{d-c},n=1+\frac{\alpha L(b-x_0)}{d-c}$$

则

$$mn\leq e^{L\beta},m\geq 1,n\geq 1$$

由 $Cauchy$ 不等式得

$$1=\frac{1}{(d-c)^2}\left(\int_c^d \frac{1}{\sqrt{f(x)}}\cdot\sqrt{f(x)}\,\mathrm{d}x\right)^2\leq \frac{\alpha}{(d-c)^2}\int_c^d f(x)\,\mathrm{d}x$$

再由 $m^2n^2 \geq m^2+n^2-1\,(m,n\geq 1)$ 即知

$$\begin{aligned}\int_a^b f(x)\,\mathrm{d}x &\leq \frac{(d-c)^2}{\alpha}\left(\frac{b-a}{d-c}+\frac{\alpha L\left[(x_0-a)^2+(b-x_0)^2\right]}{2(d-c)^2}\right)\\ &= \frac{(d-c)^2}{\alpha}\cdot\frac{m^2+n^2-2}{2\alpha L}\\ &\leq \frac{(d-c)^2}{\alpha}\cdot\frac{m^2n^2-1}{2\alpha L}\\ &\leq \frac{e^{2L\beta}-1}{2\alpha L}\int_c^d f(x)\,\mathrm{d}x\end{aligned}$$

证2:记

$$f(x_0)=\min_{x\in[a,b]}f(x)=m$$

则

$$m\leq f(x)\leq m+L|x-x_0|$$

从而

$$\begin{aligned}&2L\int_c^d\frac{\mathrm{d}x}{f(x)}\cdot\int_a^b f(x)\,\mathrm{d}x\\ \leq\,&2L\,\frac{d-c}{m}\left[\int_a^{x_0}(m+L(x_0-x))\,\mathrm{d}x+\int_{x_0}^b(m+L(x-x_0))\,\mathrm{d}x\right]\\ \leq\,&\int_c^d f(x)\,\mathrm{d}x\left[\frac{2(b-a)L}{m}+\frac{L^2(x_0-a)^2+L^2(b-x_0)^2}{m^2}\right]\\ =\,&\int_c^d f(x)\,\mathrm{d}x\left[\left(1+\frac{L(x_0-a)}{m}\right)^2+\left(1+\frac{L(b-x_0)}{m}\right)^2-2\right]\end{aligned}$$

再由 $s^2t^2\geq s^2+t^2-1\,(s,t\geq 1)$ 即知

$$\begin{aligned}&2L\int_c^d\frac{\mathrm{d}x}{f(x)}\cdot\int_a^b f(x)\,\mathrm{d}x\\ \leq\,&\int_c^d f(x)\,\mathrm{d}x\left[\left(1+\frac{L(x_0-a)}{m}\right)^2\left(1+\frac{L(b-x_0)}{m}\right)^2-1\right]\\ =\,&\int_c^d f(x)\,\mathrm{d}x\left(e^{2\ln\left(1+\frac{L(x_0-a)}{m}\right)\left(1+\frac{L(b-x_0)}{m}\right)}-1\right)\\ =\,&\int_c^d f(x)\,\mathrm{d}x\left[\exp\left(2L\int_a^{x_0}\frac{\mathrm{d}x}{m+L(x_0-x)}+2L\int_{x_0}^b\frac{\mathrm{d}x}{m+L(x-x_0)}\right)-1\right]\\ \leq\,&\int_c^d f(x)\,\mathrm{d}x\left[\exp\left(2L\int_a^b\frac{\mathrm{d}x}{f(x)}\right)-1\right]\end{aligned}$$

即

$$\int_a^b f(x)\,\mathrm{d}x\leq\frac{e^{2L\beta}-1}{2\alpha L}\int_c^d f(x)\,\mathrm{d}x$$

注:事实上,两种证法在本质上类似

**问题**.($Hardy$ 不等式) 设 $p>1$,$f$ 是 $(0,+\infty)$ 上非负连续函数,且 $\int_0^{+\infty} f^p(x)\,\mathrm{d}x$ 收敛,令 $g(x)=\int_0^x f(t)\,\mathrm{d}t\,(x\geq 0)$,证明:

$$\int_0^{+\infty}\frac{g^p(x)}{x^p}\,\mathrm{d}x\leq\frac{p^p}{(p-1)^p}\int_0^{+\infty}f^p(x)\,\mathrm{d}x$$

证1:由 $H\ddot{o}lder$ 不等式可得

$$\frac{g^p(x)}{x^{p-1}}=\frac{\left(\int_0^x f(t)\,\mathrm{d}t\right)^p}{x^{p-1}}\leq\frac{\int_0^x f^p(t)\,\mathrm{d}t\cdot x^{p-1}}{x^{p-1}}=\int_0^x f^p(t)\,\mathrm{d}t\to 0\,(x\to 0)$$

对 $\forall A>0, 0<\mu<A$ 有

$$\begin{aligned}\int_\mu^A\frac{g^p(x)}{x^p}\,\mathrm{d}x=&\frac{x^{1-p}g^p(x)}{1-p}\bigg|_\mu^A+\frac{p}{p-1}\int_\mu^A x^{1-p}g^{p-1}(x)f(x)\,\mathrm{d}x\\ \leq&\frac{\mu^{1-p}g^p(x)}{p-1}+\frac{p}{p-1}\left(\int_\mu^A f^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\mu^A\frac{g^p(x)}{x^p}\,\mathrm{d}x\right)^{1-\frac{1}{p}}\\ \leq&\frac{\mu^{1-p}g^p(x)}{p-1}+\frac{p}{p-1}\left(\int_0^{+\infty} f^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_\mu^A\frac{g^p(x)}{x^p}\,\mathrm{d}x\right)^{1-\frac{1}{p}}\end{aligned}$$

令 $\mu\to 0^+$ 可得

$$\int_0^A\frac{g^p(x)}{x^p}\,\mathrm{d}x\leq\frac{p}{p-1}\left(\int_0^{+\infty}f^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_0^A\frac{g^p(x)}{x^p}\,\mathrm{d}x\right)^{1-\frac{1}{p}}$$

即

$$\left(\int_0^A\frac{g^p(x)}{x^p}\,\mathrm{d}x\right)^{\frac{1}{p}}\leq\frac{p}{p-1}\left(\int_0^{+\infty}f^p(x)\,\mathrm{d}x\right)^{\frac{1}{p}}$$

再令 $A\to+\infty$ 即得

$$\int_0^{+\infty}\frac{g^p(x)}{x^p}\,\mathrm{d}x\leq\frac{p^p}{(p-1)^p}\int_0^{+\infty}f^p(x)\,\mathrm{d}x$$

证2:由 $H\ddot{o}lder$ 不等式可得

$$\frac{g^p(x)}{x^{p-1}}=\frac{\left(\int_0^x f(t)\,\mathrm{d}t\right)^p}{x^{p-1}}\leq\frac{\int_0^x f^p(t)\,\mathrm{d}t\cdot x^{p-1}}{x^{p-1}}=\int_0^x f^p(t)\,\mathrm{d}t\to 0\,(x\to 0)$$

从而对 $\forall A>0$ 有

$$\int_0^A\frac{g^p(x)}{x^p}\,\mathrm{d}x=\frac{x^{1-p}g^p(x)}{1-p}\bigg|_0^A+\frac{p}{p-1}\int_0^A\frac{f(x)g^{p-1}(x)}{x^{p-1}}\,\mathrm{d}x\leq\frac{p}{p-1}\int_0^A\frac{f(x)g^{p-1}(x)}{x^{p-1}}\,\mathrm{d}x$$

由带权的$Young$不等式

$$ab \leq \frac{\varepsilon^{1-p}}{p}a^p + \frac{\varepsilon}{q}b^q, \frac{1}{p} + \frac{1}{q} = 1, \forall\, a, b \geq 0, \varepsilon \in (0,1)$$

可得到

$$\int_0^A \frac{g^p(x)}{x^p}\,\mathrm{d}x \leq \frac{1}{(p-1)\varepsilon^{p-1}}\int_0^A f^p(x)\,\mathrm{d}x + \varepsilon\int_0^A \frac{g^p(x)}{x^p}\,\mathrm{d}x$$

即

$$\int_0^A \frac{g^p(x)}{x^p}\,\mathrm{d}x \leq \frac{1}{(p-1)(1-\varepsilon)\varepsilon^{p-1}}\int_0^A f^p(x)\,\mathrm{d}x$$

又由均值不等式可得

$$(p-1)(1-\varepsilon)\varepsilon^{p-1} \leq \left(\frac{p-1}{p}\right)^p$$

上式当且仅当$\varepsilon = \dfrac{p-1}{p}$时取等号,故

$$\int_0^A \frac{g^p(x)}{x^p}\,\mathrm{d}x \leq \frac{p^p}{(p-1)^p}\int_0^A f^p(x)\,\mathrm{d}x$$

令$A \to +\infty$即得

$$\int_0^{+\infty} \frac{g^p(x)}{x^p}\,\mathrm{d}x \leq \frac{p^p}{(p-1)^p}\int_0^{+\infty} f^p(x)\,\mathrm{d}x$$

**问题**. $f(x):[0,1] \to \mathbb{R}$是可积函数,且$\displaystyle\int_0^1 f(x)\,\mathrm{d}x = 0, m \leq f(x) \leq M (m \neq M)$,令$F(x) = \displaystyle\int_0^x f(t)\,\mathrm{d}t$,求证:

$$\int_0^1 F^2(x)\,\mathrm{d}x \leq \frac{m^2M^2}{3(M-m)^2}$$

证:令

$$E = \{x\,|F(x) = 0, x \in [0,1]\}$$

则$E$中除去恒为0的区间后只含有可数个小区间$I_n = [a_n, b_n]$,并且$F(a_n) = F(b_n) = 0$,$F$在$I_n$上不变号

若$F(x) > 0$,当$x \in I_n$时有

$$F(x) = F(x) - F(a_n) = \int_{a_n}^x f(t)\,\mathrm{d}t \leq M(x - a_n)$$

$$F(x) = F(x) - F(b_n) = -\int_x^{b_n} f(t)\,\mathrm{d}t \leq m(x - b_n)$$

从而

$$0 < F(x) \le \min\{M(x-a_n), m(x-b_n)\}$$

故

$$\begin{aligned}\int_{I_n} F^2(x)\,\mathrm{d}x &\le \int_{a_n}^{b_n} \left(\min\{M(x-a_n), m(x-b_n)\}\right)^2 \mathrm{d}x \\ &= \int_{a_n}^{\frac{Ma_n - mb_n}{M-m}} M^2(x-a_n)^2\,\mathrm{d}x + \int_{\frac{Ma_n-mb_n}{M-m}}^{b_n} m^2(x-b_n)^2\,\mathrm{d}x \\ &= \frac{m^2M^2}{3(M-m)^2}(b_n-a_n)^3 \\ &= \frac{m^2M^2}{3(M-m)^2}|I_n|^3\end{aligned}$$

同理知 $F(x) < 0$ 时有

$$\int_{I_n} F^2(x)\,\mathrm{d}x \le \frac{m^2M^2}{3(M-m)^2}|I_n|^3$$

从而

$$\int_0^1 F^2(x)\,\mathrm{d}x \le \frac{m^2M^2}{3(M-m)^2}\sum_n |I_n|^3 \le \frac{m^2M^2}{3(M-m)^2}\left(\sum_n |I_n|\right)^3 \le \frac{m^2M^2}{3(M-m)^2}$$

**问题**.设函数 $f(x) > 0, f''(x) > 0, 0 < x < 1, \int_0^1 f(x)\,\mathrm{d}x = 1$,求证:

$$\int_0^1 |f(x) - t|\,\mathrm{d}x \le \frac{(1-t)^2+1}{2}, \forall\, t \in \mathbb{R}$$

证:

**问题**.设 $p \ge 1, f \in C^1(-\infty, +\infty)$,并且满足

$$\int_{-\infty}^{+\infty} |f(x)|^p\,\mathrm{d}x < +\infty, \int_{-\infty}^{+\infty} |f'(x)|^p\,\mathrm{d}x < +\infty$$

证明:

(1) $\lim\limits_{x\to\infty} f(x) = 0$

(2) $|f(x)|^p \le \dfrac{p-1}{2}\displaystyle\int_{-\infty}^{+\infty} |f(t)|^p\,\mathrm{d}t + \dfrac{1}{2}\displaystyle\int_{-\infty}^{+\infty} |f'(t)|^p\,\mathrm{d}t, \forall\, x \in (-\infty, +\infty)$

证:仅考虑 $p > 1$ 的情形,$p = 1$ 的情形更简单

(1) 对 $\forall\, x_1, x_2 \in \mathbb{R}$,由 $Young$ 不等式

$$ab \le \frac{1}{p}a^p + \frac{1}{q}b^q, \frac{1}{p} + \frac{1}{q} = 1, \forall\, a, b \ge 0$$

可得

$$
\begin{aligned}
|f(x_2)|^p - |f(x_1)|^p &= \int_{x_1}^{x_2} p|f(t)|^{p-2} f(t) f'(t)\,\mathrm{d}t \\
&\le \int_{x_1}^{x_2} p|f(t)|^{p-1}|f'(t)|\,\mathrm{d}t \\
&\le \frac{1}{p}\int_{x_1}^{x_2} |f'(t)|^p\,\mathrm{d}t + (\,p-1)\int_{x_1}^{x_2} |f(t)|^p\,\mathrm{d}t \\
&< \int_{x_1}^{x_2} |f'(t)|^p\,\mathrm{d}t + (\,p-1)\int_{x_1}^{x_2} |f(t)|^p\,\mathrm{d}t
\end{aligned}
$$

从而由$Cauchy$收敛准则易知极限$\lim\limits_{x\to\infty}|f(x)|^p$存在

再根据积分$\displaystyle\int_{-\infty}^{+\infty}|f(x)|^p\,\mathrm{d}x$的收敛性有$\lim\limits_{x\to\infty}|f(x)|^p = 0$,从而$\lim\limits_{x\to\infty}f(x) = 0$

(2) 由(1)结论及$Young$不等式得

$$|f(x)|^p = \int_{-\infty}^{x} p|f(t)|^{p-2}f(t)f'(t)\,\mathrm{d}t \le (\,p-1)\int_{-\infty}^{x}|f(t)|^p\,\mathrm{d}t + \int_{-\infty}^{x}|f'(t)|^p\,\mathrm{d}t$$

$$|f(x)|^p = \int_{x}^{+\infty} -p|f(t)|^{p-2}f(t)f'(t)\,\mathrm{d}t \le (\,p-1)\int_{x}^{+\infty}|f(t)|^p\,\mathrm{d}t + \int_{x}^{+\infty}|f'(t)|^p\,\mathrm{d}t$$

两式相加即得

$$|f(x)|^p \le \frac{p-1}{2}\int_{-\infty}^{+\infty}|f(t)|^p\,\mathrm{d}t + \frac{1}{2}\int_{-\infty}^{+\infty}|f'(t)|^p\,\mathrm{d}t, \forall\, x\in(-\infty,+\infty)$$

# 4 Series

**问题**.试求保证不等式$e^x+e^{-x}\le 2e^{cx^2},x\in\mathbb{R}$成立实数$c$的条件

解1:由对称性只需考虑$x\ge 0$即可.令

$$f(x)=\ln\left(\frac{e^x+e^{-x}}{2}\right)$$

则

$$c\ge\frac{1}{x^2}\ln\left(\frac{e^x+e^{-x}}{2}\right)=\frac{f(x)}{x^2}$$

由幂级数展开式易得

$$\sinh x<x\cosh x\,(x>0)$$

从而

$$\frac{f(x)}{x^2}=\frac{1}{x^2}\int_0^x f'(t)\,\mathrm{d}t=\frac{1}{x^2}\int_0^x\frac{\sinh t}{\cosh t}\,\mathrm{d}t<\frac{1}{2}$$

考虑$x\to 0$时的$Taylor$展开式得

$$\begin{aligned}\frac{1}{x^2}\ln\left(\frac{e^x+e^{-x}}{2}\right)&=\frac{1}{x^2}\ln\left(1+\frac{x^2}{2}+o(x^2)\right)\\&=\frac{1}{x^2}\left(\frac{x^2}{2}-\frac{x^4}{8}+o(x^2)\right)\\&\to\frac{1}{2}\end{aligned}$$

故

$$c\ge\frac{1}{2}$$

解2:

$$\begin{aligned}0\le e^{cx^2}-\frac{e^x+e^{-x}}{2}&=\sum_{n=1}^{\infty}\left(\frac{c^n}{n!}-\frac{1}{(2n)!}\right)x^{2n}\\&=c-\frac{1}{2}+\sum_{n=2}^{\infty}\left(\frac{c^n}{n!}-\frac{1}{(2n)!}\right)x^{2n}\end{aligned}$$

若对任意给定的$c<\dfrac{1}{2}$,令$x\to 0$即得$c\ge\dfrac{1}{2}$,矛盾!

事实上,当$c\ge\dfrac{1}{2}$时

$$\frac{e^x+e^{-x}}{2}=\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\le\sum_{n=0}^{\infty}\frac{x^{2n}}{2^n n!}=e^{x^2/2}\le e^{cx^2}$$

从而实数$c$的取值范围是$c\ge\dfrac{1}{2}$

**问题.**求$\displaystyle\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n+2kn+n^2k}$

解1:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n+2kn+n^2k}=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{kn(k+n+2)}\\
&=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\int_0^{\infty}\frac{1}{kn}e^{-(k+n+2)x}\,\mathrm{d}x=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\int_0^{\infty}\frac{e^{-kx}}{k}\frac{e^{-nx}}{n}e^{-2x}\,\mathrm{d}x\\
&=\int_0^{\infty}\sum_{n=1}^{\infty}\frac{e^{-nx}}{n}\sum_{k=1}^{\infty}\frac{e^{-kx}}{k}e^{-2x}\,\mathrm{d}x=\int_0^{\infty}e^{-2x}\ln^2(1-e^{-x})\,\mathrm{d}x\\
&=\int_0^{\infty}y\ln^2(1-y)\,\mathrm{d}y=\int_0^{\infty}(1-x)\ln^2x\,\mathrm{d}x=\frac{7}{4}
\end{aligned}$$

解2:

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{kn(k+n+2)}&=\frac{1}{n(n+2)}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+n+2}\right)\\
&=\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+2}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{n+2}\right)
\end{aligned}$$

从而

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k^2n+2kn+n^2k}\\
&=\frac{1}{2}\left[\left(1-\frac{1}{3}\right)\left(1+\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\cdots\right]\\
&=\frac{1}{2}\left[\left(1+\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}+\frac{1}{5}\right)+\frac{1}{4}\left(\frac{1}{5}+\frac{1}{6}\right)+\cdots\right]\\
&=\frac{1}{2}\left[\frac{11}{6}+\frac{25}{24}+\left(\frac{1}{3\cdot4}+\frac{1}{4\cdot5}+\cdots\right)+\left(\frac{1}{3\cdot5}+\frac{1}{4\cdot6}+\frac{1}{5\cdot7}+\cdots\right)\right]=\frac{7}{4}
\end{aligned}$$

**问题.**求$\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n\cdot3^m+m\cdot3^n)}$

解:

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n\cdot3^m+m\cdot3^n)}\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n^2}{n\cdot3^m(n\cdot3^m+m\cdot3^n)}\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n^2}{m\cdot3^n}\left(\frac{1}{n\cdot3^m}-\frac{1}{n\cdot3^m+m\cdot3^n}\right)\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{mn}{3^m3^n}-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{n^2m}{3^n(n\cdot3^m+m\cdot3^n)}
\end{aligned}$$

由原级数中 $m$ 与 $n$ 的对称性可知

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n\cdot 3^m+m\cdot 3^n}=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{n}{3^n}\right)^2=\frac{9}{32}$$

注:利用变量代换可以更直观地发现此级数的特点.令

$$a_n=\frac{3^n}{n},a_m=\frac{3^m}{m}$$

则

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n\cdot 3^m+m\cdot 3^n)}\\
=&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_m(a_m+a_n)}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_n(a_m+a_n)}\\
=&\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{1}{a_m(a_m+a_n)}+\frac{1}{a_n(a_m+a_n)}\right]\\
=&\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_ma_n}=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{n}{3^n}\right)^2=\frac{9}{32}
\end{aligned}$$

**问题**. $\{a_n\}$ 满足 $a_0=0,a_1=1,a_{n+2}+2na_{n+1}+\left(n^2-n+\frac{2}{9}\right)a_n=0$, 求 $\sum\limits_{n=1}^{\infty}a_n\frac{x^n}{n!}$

解:令

$$y(x)=\sum_{n=1}^{\infty}a_n\frac{x^n}{n!}$$

则

$$y''(x)=\sum_{n=0}^{\infty}a_{n+2}\frac{x^n}{n!},xy''(x)=\sum_{n=0}^{\infty}na_{n+1}\frac{x^n}{n!}$$

$$xy'(x)=\sum_{n=0}^{\infty}na_n\frac{x^n}{n!},x(xy'(x))'=\sum_{n=0}^{\infty}n^2a_n\frac{x^n}{n!}$$

故

$$y''(x)+2xy''(x)+x(xy'(x))'+\frac{2}{9}y(x)-xy'(x)=0$$

化简得

$$(x+1)^2y''(x)+\frac{2}{9}y(x)=0$$

解此二阶齐次常微分方程得

$$y=c_1(x+1)^{\frac{1}{3}}+c_2(x+1)^{\frac{2}{3}}$$

再利用初始条件 $y(0)=0, y'(0)=1$ 即得

$$y=3(x+1)^{\frac{2}{3}}-3(x+1)^{\frac{1}{3}}$$

**问题**.设 $T_1=T_2=1, T_3=2, T_n=T_{n-1}+T_{n-2}+T_{n-3}$,求 $\sum\limits_{n=1}^{\infty}\frac{T_n}{\pi^n}$

解1:易知

$$\frac{T_{n+1}}{T_n}=1+\frac{T_{n-1}}{T_n}+\frac{T_{n-2}}{T_n}<3$$

故

$$T_n=\frac{T_n}{T_{n-1}}\cdots\frac{T_2}{T_1}T_1<3^{n-1}$$

从而 $S=\sum\limits_{n=1}^{\infty}\frac{T_n}{\pi^n}$ 收敛,并且

$$\begin{aligned}
S&=\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{2}{\pi^3}+\sum_{n=4}^{\infty}\frac{T_{n-1}+T_{n-2}+T_{n-3}}{\pi^n}\\
&=\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{2}{\pi^3}+\frac{1}{\pi}\sum_{n=4}^{\infty}\frac{T_{n-1}}{\pi^{n-1}}+\frac{1}{\pi^2}\sum_{n=4}^{\infty}\frac{T_{n-2}}{\pi^{n-2}}+\frac{1}{\pi^3}\sum_{n=4}^{\infty}\frac{T_{n-3}}{\pi^{n-3}}\\
&=\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{2}{\pi^3}+\frac{1}{\pi}\left(S-\frac{1}{\pi}-\frac{1}{\pi^2}\right)+\frac{1}{\pi^2}\left(S-\frac{1}{\pi}\right)+\frac{S}{\pi^3}\\
&=\frac{1}{\pi}+S\left(\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{1}{\pi^3}\right)
\end{aligned}$$

即

$$S=\frac{\pi^2}{\pi^3-\pi^2-\pi-1}$$

解2:令

$$S=\sum_{n=1}^{\infty}T_nx^n$$

则

$$xS=\sum_{n=2}^{\infty}T_{n-1}x^n, x^2S=\sum_{n=3}^{\infty}T_{n-2}x^n, x^3S=\sum_{n=4}^{\infty}T_{n-3}x^n$$

由

$$T_n=T_{n-1}+T_{n-2}+T_{n-3}$$

得

$$S-(T_1x+T_2x^2+T_3x^3)=(x+x^2+x^3)S-(T_1x^2-T_2x^3)-T_1x^3$$

即

$$S=\frac{x}{1-x-x^2-x^3}$$

令 $x=\dfrac{1}{\pi}$ 即得

$$S=\frac{\pi^2}{\pi^3-\pi^2-\pi-1}$$

**问题**.若序列 $\{na_n\}$ 单调,正项级数 $\sum\limits_{n=1}^{\infty}a_n$ 收敛,证明: $\lim\limits_{n\to\infty}n\ln n\cdot a_n=0$

证:若 $\{na_n\}$ 单调递增,则

$$na_n\geq a_1\Rightarrow a_n\geq\frac{a_1}{n}\Rightarrow\sum_{n=1}^{\infty}a_n\geq a_1\sum_{n=1}^{\infty}\frac{1}{n}$$

这与 $\sum\limits_{n=1}^{\infty}a_n$ 收敛矛盾,故 $\{na_n\}$ 单调递减

由 $Cauchy$ 收敛准则知

$$\text{对}\,\forall\varepsilon>0,\exists N\,s.t.\text{当}\,n>m>N\,\text{时},\sum_{k=m}^{n}a_n<\varepsilon$$

取

$$m=[\sqrt{n}\,]-1\,(n\,\text{足够大})$$

则

$$\varepsilon>\sum_{k=m}^{n-1}ka_k\cdot\frac{1}{k}\geq na_n\sum_{k=m}^{n-1}\frac{1}{k}\geq na_n\sum_{k=m}^{n-1}\int_k^{k+1}\frac{1}{x}\,\mathrm{d}x=na_n\int_m^n\frac{1}{x}\,\mathrm{d}x>na_n\ln\frac{n}{\sqrt{n}}$$

由 $\varepsilon$ 的任意性即得

$$\lim_{n\to\infty}n\ln n\cdot a_n=0$$

证2:先证明如下引理

$$\sum_{n=1}^{\infty}a_nb_n\,\text{收敛},b_n\,\text{递减趋于0,则}\lim_{n\to\infty}\sum_{k=1}^{n}a_kb_n=0$$

显然

$$\begin{aligned}\left|\sum_{k=1}^{n}a_kb_n\right|&\leq\left|\sum_{k=1}^{N}a_kb_n\right|+\left|\sum_{k=N+1}^{n}a_kb_n\right|\\&\leq\left|\sum_{k=1}^{N}a_kb_n\right|+\left|\sum_{k=N+1}^{n}(a_kb_k)\frac{b_n}{b_k}\right|\end{aligned}$$

再利用 $Abel$ 变换即可

依题意易知 $na_n$ 递减趋于0,从而由引理可得

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}\cdot na_n=0$$

再由

$$\sum_{k=1}^{n}\frac{1}{k}=\ln n+\varepsilon_n+\gamma,\varepsilon_n\to 0\,(n\to\infty)$$

即得

$$\lim_{n\to\infty} n\ln n\cdot a_n=0$$

问题.求$\displaystyle\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n(n-1)}\right)$

解:

$$\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n(n-1)}\right)=\ln\left(\lim_{n\to\infty}\prod_{k=2}^{n}\frac{k^2-k-1}{k(k-1)}\right)=\ln\left(\lim_{n\to\infty}\frac{\displaystyle\prod_{k=2}^{n}(k^2-k-1)}{n!(n-1)!}\right)$$

令

$$k^2-k-1=(k-\alpha)(k-\beta)$$

则

$$\alpha+\beta=1,\alpha\beta=-1$$

不妨取$\alpha=\dfrac{1-\sqrt{5}}{2}$,则

$$\begin{aligned}
&\lim_{n\to\infty}\frac{\displaystyle\prod_{k=2}^{n}(k^2-k-1)}{n!(n-1)!}=\lim_{n\to\infty}\frac{\displaystyle\prod_{k=2}^{n}(k-\alpha)(k-\beta)}{n!(n-1)!}\\
&=\lim_{n\to\infty}\frac{(2-\alpha)\cdots(n-\alpha)}{n!}\cdot\frac{(2-\beta)\cdots(n-\beta)}{(n-1)!}\\
&=\frac{n^{-\alpha}(-\alpha)(1-\alpha)}{\Gamma(-\alpha)}\cdot\frac{n^{-\beta}n(-\beta)(1-\beta)}{\Gamma(-\beta)}\\
&=\frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}=\frac{1}{\dfrac{\Gamma(1-\alpha)}{-\alpha}\dfrac{\Gamma(1-\beta)}{-\beta}}\\
&=\frac{-1}{\Gamma(1-\alpha)\Gamma(1-\beta)}=-\frac{\cos\dfrac{\sqrt{5}\,\pi}{2}}{\pi}
\end{aligned}$$

故

$$\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n(n-1)}\right)=\ln\left(-\frac{\cos\dfrac{\sqrt{5}\,\pi}{2}}{\pi}\right)$$

注:解答中利用了$Eular-Gauss$公式

$$\Gamma(x)=\lim_{n\to\infty}\frac{n!\,n^x}{x(x+1)\cdots(x+n)},n\neq 0,-1,-2,\cdots$$

**问题**.证明正项级数 $\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2n(2n)!!}$ 收敛,并求出此级数的和

解:对 $\forall n\in\mathbb{N}$,由

$$\int_0^{\frac{\pi}{2}}\sin^{2n}x\,\mathrm{d}x<\int_0^{\frac{\pi}{2}}\sin^{2n-1}x\,\mathrm{d}x$$

可得

$$\frac{(2n-1)!!}{(2n)!!}<\frac{1}{n\pi}$$

从而

$$0<\frac{(2n-1)!!}{2n(2n)!!}<\frac{1}{2\sqrt{\pi}n^{3/2}}$$

由比较判别法即知原级数收敛,令

$$s(x)=\sum_{n=1}^{\infty}\frac{(2n-1)!!}{2n(2n)!!}x^{2n}$$

易知 $s(x)$ 在 $[-1,1]$ 上收敛,从而原级数的和 $S=\lim\limits_{x\to1^-}s(x)$,又由

$$(1-x)^{-\frac{1}{2}}=1+\sum_{n=1}^{\infty}\frac{(2n-1)!!}{(2n)!!}x^n$$

可得

$$s'(x)=\frac{1}{x\sqrt{1-x^2}}-\frac{1}{x}(|x|<1)$$

则

$$\begin{aligned}S&=\lim_{x\to1^-}\left(s(0)+\int_0^x s'(t)\,\mathrm{d}t\right)\\&=\int_0^1\left(\frac{1}{x\sqrt{1-x^2}}-\frac{1}{x}\right)\mathrm{d}t\\&\xrightarrow{\text{三角换元}}2\int_0^{\frac{\pi}{4}}\tan t\,\mathrm{d}t\\&=\ln 2\end{aligned}$$

注:(1) $\frac{(2n-1)!!}{(2n)!!}$ 的估计可以利用更为初等的方法,利用

$$e^x\geq 1+x,x\in\mathbb{R},x\geq\ln(1+x),x\geq-1$$

可得

$$\frac{(2n-1)!!}{(2n)!!}=\prod_{i=1}^{n}\frac{2i-1}{2i}\leq\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\frac{1}{i}\right)\leq\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\ln\frac{i+1}{i}\right)=\frac{1}{\sqrt{n+1}}$$

(2) 证明不等式:$\ln 2-\frac{1}{\sqrt{n\pi}}<\sum_{k=1}^{n}\frac{(2k-1)!!}{2k(2k)!!}<\ln 2$

证:由

$$\frac{(2n-1)!!}{(2n)!!}<\frac{1}{n\pi}$$

可得

$$\begin{aligned}\sum_{k=n+1}^{\infty}\frac{(2k-1)!!}{2k(2k)!!}&<\sum_{k=n+1}^{\infty}\frac{1}{2k\sqrt{k\pi}}\\&<\sum_{k=n+1}^{\infty}\frac{1}{\sqrt{\pi}}\left(\frac{1}{\sqrt{k-1}}-\frac{1}{\sqrt{k}}\right)\\&=\frac{1}{\sqrt{n\pi}}\end{aligned}$$

从而原不等式成立

**问题**.设 $e^{e^x}=\sum_{n=0}^{\infty}a_nx^n$,试确定系数 $a_0,a_1,a_2,a_3$,并证明:当 $n\geq 2$ 时成立 $a_n>\frac{e}{(\gamma\ln n)^n}$,其中 $\gamma$ 是一个大于 $e$ 的常数

解:易知前四项的系数分别为

$$a_0=a_1=a_2=e,a_3=\frac{5e}{6}$$

利用幂级数展开式可知

$$e^{e^x}=\sum_{k=0}^{\infty}\frac{e^{kx}}{k!}=\sum_{k=0}^{\infty}\frac{1}{k!}\left(\sum_{n=0}^{\infty}\frac{(kx)^n}{n!}\right)=\sum_{n=0}^{\infty}\frac{x^n}{n!}\left(\sum_{k=0}^{\infty}\frac{k^n}{k!}\right)$$

从而

$$a_n=\frac{1}{n!}\sum_{k=0}^{\infty}\frac{k^n}{k!}=\frac{1}{n!}\sum_{k=1}^{\infty}\frac{k^n}{k!}>\frac{k^n}{n!\,k!},\forall k\geq 1$$

对于任意固定的 $n$,只需要找出适当的 $k$ 满足题中的不等式即可.又因为对于前有限个 $n$,我们可以适当放大 $\gamma$ 满足题意,因此本题只需在等价意义下完成证明.若对任意固定足够大的 $n$,取 $k=\left[\frac{n}{\ln n}\right]$,则

$$\begin{aligned}\frac{k^n}{n!\,k!}&\sim\frac{\left[\frac{n}{\ln n}\right]^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot\sqrt{2\pi\left[\frac{n}{\ln n}\right]}\left(\frac{\left[\frac{n}{\ln n}\right]}{e}\right)^{\left[\frac{n}{\ln n}\right]}}\\&>\frac{\left(\frac{n}{\ln n}-1\right)^n}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\cdot\sqrt{2\pi\frac{n}{\ln n}}\left(\frac{n}{e\ln n}\right)^{\frac{n}{\ln n}}}\\&>\frac{e}{(\ln n)^n}\cdot\frac{1}{2\pi n}\left[\left(1-\frac{\ln n}{n}\right)(\ln n)^{\frac{1}{\ln n}}\right]^n\end{aligned}$$

故只需要成立不等式

$$\gamma\cdot\left(1-\frac{\ln n}{n}\right)(\ln n)^{\frac{1}{\ln n}}>(2\pi n)^{\frac{1}{n}}$$

显然对于足够大的$N$,当$n\geq N$时存在$\gamma>e$使得上式成立,然后放大$\gamma$即可使得不等式对一切$n\geq 2$均成立

**问题**.正项级数$\sum_{n=1}^{\infty}\frac{1}{a_n}$收敛,证明:级数$\sum_{n=1}^{\infty}\frac{n^2a_n}{(a_1+a_2+\cdots+a_n)^2}$收敛

证:先证明如下引理

$$\sum_{n=1}^{\infty}\frac{n}{a_1+a_2+\cdots+a_n}\leq 2\sum_{n=1}^{\infty}\frac{1}{a_n}$$

由$Cauchy$不等式得

$$(1+2+\cdots+n)^2\leq(a_1+a_2+\cdots+a_n)\left(\frac{1^2}{a_1}+\frac{2^2}{a_2}+\cdots+\frac{n^2}{a_n}\right)$$

即

$$\frac{2n+1}{a_1+a_2+\cdots+a_n}\leq\frac{4(2n+1)}{n^2(n+1)^2}\sum_{j=1}^{n}\frac{j^2}{a_j}$$

则

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{2n+1}{a_1+a_2+\cdots+a_n}&\leq 4\sum_{n=1}^{\infty}\sum_{j=1}^{n}\frac{2n+1}{n^2(n+1)^2}\frac{j^2}{a_j}\\&=4\sum_{j=1}^{\infty}\sum_{n=j}^{\infty}\frac{2n+1}{n^2(n+1)^2}\frac{j^2}{a_j}\\&=4\sum_{n=1}^{\infty}\frac{1}{a_n}\end{aligned}$$

从而引理得证!

记

$$A_n=a_1+a_2+\cdots+a_n$$

则

$$\begin{aligned}\frac{n^2a_n}{(a_1+a_2+\cdots+a_n)^2}&=\frac{n^2(A_n-A_{n-1})}{A_n^2}\\&\leq\frac{n^2(A_n-A_{n-1})}{A_nA_{n-1}}\\&=\left(\frac{(n-1)^2}{A_{n-1}}-\frac{n^2}{A_n}\right)+\frac{2n-1}{A_{n-1}}\end{aligned}$$

由平均值不等式可知

$$\frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}}\leq\frac{a_1+a_2+\cdots+a_n}{n}=\frac{A_n}{n}$$

即

$$\frac{A_n}{n^2} \le \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

再结合引理所证结论易知级数 $\sum_{n=1}^{\infty} \frac{n^2 a_n}{(a_1 + a_2 + \cdots + a_n)^2}$ 收敛

**问题**.讨论 $\sum_{n=1}^{\infty} \frac{\cos\left(\frac{\pi}{2}\ln n\right)}{n}$ 的敛散性

解:取

$$m = \left[e^{4N}\right] + 1, n = \left[e^{4N+1}\right], N \in \mathbb{N}^+$$

则

$$2N\pi < \frac{\pi}{2}\ln m < 2N\pi + \frac{\pi}{2e^4}$$

$$2N\pi + \frac{\pi}{2} + \frac{\pi}{2}\ln(1 - e^{-5}) < \frac{\pi}{2}\ln n < 2N\pi + \frac{\pi}{2}$$

令

$$f(x) = \frac{\cos\left(\frac{\pi}{2}\ln x\right)}{x}$$

则

$$\begin{aligned}
\sum_{k=m}^{n-1} f(k) > \int_m^n f(x)\,\mathrm{d}x = \int_m^n \frac{\cos\left(\frac{\pi}{2}\ln x\right)}{x}\,\mathrm{d}x \\
> \frac{2}{\pi}\sin\left[\frac{\pi}{2}\left(1 + \ln(1 - e^{-5})\right)\right] - \frac{2}{\pi}\sin\frac{\pi}{2e^4} \\
> \frac{1}{\pi} - \frac{1}{e^4} > 0
\end{aligned}$$

从而由 $Cauchy$ 收敛准则知原级数发散

注:本题可以推广为级数 $\sum_{n=1}^{\infty} \frac{\cos(\ln n)}{n^p}$ $(p \le 1)$ 发散

# 5 Miscellany

**问题**.已知 $0<x,y<1$,证明:$x^y+y^x>1$

证:由 $Bernoulli$ 不等式得

$$\frac{1}{\left(\frac{1}{x}\right)^y}=\frac{1}{\left(1+\frac{1}{x}-1\right)^y}>\frac{1}{1+\left(\frac{1}{x}-1\right)y}=\frac{x}{x+y-xy}>\frac{x}{x+y}$$

同理可得

$$\frac{1}{\left(\frac{1}{y}\right)^x}>\frac{y}{x+y}$$

从而

$$x^y+y^x=\frac{1}{\left(\frac{1}{x}\right)^y}+\frac{1}{\left(\frac{1}{y}\right)^x}>\frac{x}{x+y}+\frac{y}{x+y}=1$$

**问题**.设 $a\in\mathbb{Z},b\in\mathbb{Z}^+$,由辗转相除法可得

$$\begin{cases} a=bq_1+r_1,0<r_1<b \\ b=r_1q_2+r_2,0<r_2<r_1 \\ \cdots\cdots \\ r_{n-1}=r_nq_{n+1}+r_{n+1},r_{n+1}=0 \end{cases}$$

求证:(1) $n\le\frac{2\ln b}{\ln 2}$ (2) $n\le\frac{2\ln(b+1)}{\ln 2}-1$

证:(1) 记 $b=r_0$ .当 $k\ge 1$ 时,有

$$r_{2k-2}\ge r_{2k-1}+r_{2k}\ge 2r_{2k}+1$$

则

$$r_{2k}\le\frac{r_{2k-2}-1}{2}<\frac{r_{2k-2}}{2}$$

从而

$$r_{2k}\le\frac{r_{2k-2}}{2}-\frac{1}{2}<\frac{r_{2k-4}}{2}-\frac{1}{2}<\cdots<\frac{b}{2^k}-\frac{1}{2}$$

记 $m$ 是满足 $\frac{b}{2^k}-\frac{1}{2}<1$ 的最小正整数,则

$$\frac{b}{2^m}<\frac{3}{2}\le\frac{b}{2^{m-1}}$$

即

$$m\le 2+\frac{\ln b}{\ln 2}-\frac{\ln 3}{\ln 2}$$

又由 $r_{2m}=0$ 知 $n+1\le 2m$,故

$$n\le 2m-1<\frac{2\ln b}{\ln 2}$$

(2) 由

$$r_{2k}\le\frac{r_{2k-2}}{2}-\frac{1}{2}$$

可得

$$r_{2k}+1\le\frac{1}{2}(r_{2k-2}+1)$$

从而

$$r_{2k}\le\frac{b+1}{2^k}-1$$

记 $m'$ 是满足 $\dfrac{b+1}{2^k}-1<1$ 的最小正整数,则

$$\frac{b+1}{2^{m'}}<2\le\frac{b+1}{2^{m'-1}}$$

即

$$m'\le\frac{\ln(b+1)}{\ln 2}$$

又由 $r_{2m'}=0$ 知 $n+1\le 2m'$,故

$$n\le\frac{2\ln(b+1)}{2\ln 2}-1$$

**问题**.证明:对任意 $n,x$ 成立不等式 $\left|\sum\limits_{k=1}^{n}\dfrac{\sin kx}{k}\right|<2\sqrt{\pi}$

证:显然本题只需要考虑 $x\in(0,\pi)$ 即可.令

$$S_n=\sum_{k=1}^{n}\sin kx$$

由积化和差公式得

$$|S_n|=\frac{\left|\sum\limits_{k=1}^{n}\left[\cos\left(k+\frac{1}{2}\right)x-\cos\left(k-\frac{1}{2}\right)x\right]\right|}{2\sin\frac{x}{2}}\le\frac{1}{\sin\frac{x}{2}}<\frac{\pi}{x}$$

再由 $Abel$ 变换得

$$\begin{aligned}\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right|&=\left|\frac{S_n}{n}+\sum_{k=1}^{n-1}S_k\left(\frac{1}{k}-\frac{1}{k+1}\right)\right|\\&\le\frac{|S_n|}{n}+\frac{\pi}{x}\cdot\sum_{k=1}^{n-1}\left(\frac{1}{k}-\frac{1}{k+1}\right)\\&=\frac{|S_n|}{n}+\frac{\pi}{x}\left(1-\frac{1}{n}\right)\\&<\frac{\pi}{x}\end{aligned}$$

对 $\forall a>0$ 有

1) 当 $nx \le a$ 时,$\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \le nx \le a$

2) 当 $nx > a$ 时,则 $\exists m\in\mathbb{N}^+\ s.t. mx \le a, (m+1)x > a$

从而

$$\begin{aligned}\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| &= \left|\sum_{k=1}^{m}\frac{\sin kx}{k}\right| + \left|\sum_{k=m+1}^{n}\frac{\sin kx}{k}\right| \\ &\le a+\frac{\pi}{(m+1)x} \\ &< a+\frac{\pi}{a}\end{aligned}$$

令 $a=\sqrt{\pi}$ 即知原不等式成立

注:若原题弱化为证明对 $\forall n,x,\exists M>0\ s.t. \left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \le M$ 成立,则还可以利用 $Dirichlet$ 积分的方法给出证明

证:同样只需考虑 $x\in(0,\pi)$.令

$$S_n=\sum_{k=1}^{n}\frac{\sin kx}{k}$$

由积化和差公式可得

$$\begin{aligned}|S_n| &= \left|\sum_{k=1}^{n}\int_0^x \cos kt\,\mathrm{d}t\right| = \left|\int_0^x\sum_{k=1}^{n}\cos kt\,\mathrm{d}t\right| \\ &= \left|\int_0^x \frac{\left[\sin\left(n+\frac{1}{2}\right)t-\sin\frac{t}{2}\right]}{2\sin\frac{t}{2}}\,\mathrm{d}t\right| \\ &\le \left|\int_0^x\frac{\sin\left(n+\frac{1}{2}\right)}{t}\,\mathrm{d}t\right| + \left|\int_0^x\left(\frac{1}{2\sin\frac{t}{2}}-\frac{1}{t}\right)\sin\left(n+\frac{1}{2}\right)t\,\mathrm{d}t\right| + \left|\frac{x}{2}\right| \\ &\le \int_0^\pi\frac{\sin x}{x}\,\mathrm{d}x+\int_0^\pi\left(\frac{1}{2\sin\frac{t}{2}}-\frac{1}{t}\right)\mathrm{d}t+\frac{\pi}{2}\end{aligned}$$

显然

$$\int_0^\pi\frac{\sin x}{x}\,\mathrm{d}x+\int_0^\pi\left(\frac{1}{2\sin\frac{t}{2}}-\frac{1}{t}\right)\mathrm{d}t$$

是有限值

从而 $\exists M>0\ s.t.$ 不等式 $\left|\sum_{k=1}^{n}\frac{\sin kx}{k}\right| \le M$ 对 $\forall n,x$ 成立

**问题**.($Carleman$ 不等式) 设 $\sum\limits_{n=1}^{\infty} a_n$ 为收敛的正项级数,证明:

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \le e \sum_{n=1}^{\infty} a_n$$

证:由不等式

$$\left(\frac{n+1}{e}\right)^n < n!$$

可得

$$\begin{aligned}
\sum_{n=1}^{N} \sqrt[n]{a_1 a_2 \cdots a_n} &= \sum_{n=1}^{N} \frac{\sqrt[n]{(a_1)(2a_2)\cdots(na_n)}}{\sqrt[n]{n!}} \\
&\le \sum_{n=1}^{N} \frac{a_1 + 2a_2 + \cdots + na_n}{n\sqrt[n]{n!}} \\
&\le e \sum_{n=1}^{N} \frac{a_1 + 2a_2 + \cdots + na_n}{n(n+1)} \\
&= e \sum_{n=1}^{N} \sum_{k=1}^{n} \frac{ka_k}{n(n+1)} \\
&= e \sum_{k=1}^{N} \sum_{n=k}^{N} \frac{ka_k}{n(n+1)} \\
&= e \sum_{l=1}^{N} ka_k \left(\frac{1}{k} - \frac{1}{N+1}\right) \\
&\le e \sum_{k=1}^{N} a_k
\end{aligned}$$

令 $N \to \infty$ 即得

$$\sum_{n=1}^{\infty} \sqrt[n]{a_1 a_2 \cdots a_n} \le e \sum_{n=1}^{\infty} a_n$$

注:对每个 $N$ 构造数列

$$a_n = \begin{cases} \dfrac{1}{n} & 1 \le n \le N \\ 0 & n > N \end{cases}$$

由 $Stolz$ 定理得

$$\lim_{N\to\infty} \frac{\sum\limits_{n=1}^{N} \sqrt[n]{a_1 a_2 \cdots a_n}}{\sum\limits_{n=1}^{N} a_n} = \lim_{N\to\infty} \frac{N}{\sqrt[N]{N!}} = e$$

这表明不等式右边的系数 $e$ 不能再改进

**问题**.若 $\alpha$ 是无理数,证明: $\{n\alpha\}\,(n=1,2,\cdots)$ 在 [0,1] 上稠密

证:对 $\forall\,\varepsilon>0, x_0\in[0,1]$,将 [0,1] $N$ 等分,且 $\dfrac{1}{N}<\varepsilon$.则 $\{\alpha\},\cdots,\{(N+1)\alpha\}$ 中至少有两数落在同一等分中.从而存在 $k_1>k_2$ 使得

$$0<\{k_1\alpha\}-\{k_2\alpha\}=(k_1-k_2)\alpha-([k_1\alpha]-[k_2\alpha])<\varepsilon$$

或者

$$-\varepsilon<\{k_1\alpha\}-\{k_2\alpha\}=(k_1-k_2)\alpha-([k_1\alpha]-[k_2\alpha])<0$$

由于两种情形类似,这里我们只对第一种情形进行处理.令

$$k_1-k_2=p\in\mathbb{N}^+, [k_1\alpha]-[k_2\alpha]=q\in\mathbb{N}^+$$

则

$$0<p\alpha-q=\beta<\varepsilon, p\alpha=q+\beta$$

对 $\forall\,k\in\mathbb{N}^+, 1\le k<\dfrac{1}{\beta}$ 满足

$$\{kp\alpha\}=k\beta$$

故 $\{kp\alpha\}$ 在 [0,1] 上均匀分布并且相邻两项之差为 $\beta$,即 $\exists\,k_0\in\mathbb{N}^+\ s.t.\{k_0p\alpha\}$ 与 $x_0$ 的距离小于 $\varepsilon$.再令 $n_0=k_0p$ 即可得到

$$|\{n_0\alpha\}-x_0|<\varepsilon$$

所以当 $\alpha$ 是无理数时,$\{n\alpha\}\,(n=1,2,\cdots)$ 在 [0,1] 上稠密

注:设 $x_n=\sin n, n\in\mathbb{N}^+$,则数列 $\{x_n\}$ 的极限点集合为 $[-1,1]$

证:由连续周期函数的一致连续性得

$$\text{对}\,\forall\,\varepsilon>0, \exists\,\delta>0\ s.t.|x'-x''|<\delta\Rightarrow|\sin x'-\sin x''|<\varepsilon$$

依题意只需要证明

$$\text{对}\,\forall\,\varepsilon>0, \forall\,A\in[-1,1], \exists\,n_0\in\mathbb{N}^+\ s.t.|\sin n_0-A|<\varepsilon$$

取 $\varepsilon=\dfrac{\delta}{2\pi}, x_0=\dfrac{\arcsin A}{2\pi}, \alpha=\dfrac{1}{2\pi}$,由上题可知

$$\exists\,n_0\in\mathbb{N}^+\ s.t.\left|\left\{n_0\cdot\frac{1}{2\pi}\right\}-\frac{\arcsin A}{2\pi}\right|<\varepsilon=\frac{\delta}{2\pi}$$

即

$$\left|n_0-\left[\frac{n_0}{2\pi}\right]\cdot 2\pi-\arcsin A\right|<\delta$$

从而

$$\left|\sin\left(n_0-2\pi\left[\frac{n_0}{2\pi}\right]\right)-A\right|=|\sin n_0-A|<\varepsilon$$

所以数列 $\{x_n\}$ 的极限点集合为 $[-1,1]$

**问题**.设 $P$ 表示素数的全体,证明:级数 $\sum\limits_{p\in P}\frac{1}{p}$ 发散

证:假设级数 $\sum\limits_{n=1}^{\infty}\frac{1}{p_n}\,(p_n\in P)$ 收敛,则

$$\exists k\in\mathbb{N}^+\ s.t.\ \sum_{n=k+1}^{\infty}\frac{1}{p_n}<\frac{1}{2}$$

令 $Q=p_1p_2\cdots p_k$,显然 $1+nQ\,(n\in\mathbb{N}^+)$ 的素因子全包含于 $p_{k+1},p_{k+2},\cdots$,故

$$\sum_{n=1}^{\infty}\frac{1}{1+nQ}\le\sum_{t=1}^{\infty}\left(\sum_{n=k+1}^{\infty}\frac{1}{p_n}\right)^t<\sum_{t=1}^{\infty}\frac{1}{2^t}$$

从而级数 $\sum\limits_{n=1}^{\infty}\frac{1}{1+nQ}$ 收敛,矛盾!

**问题**.设 $f(x)=\frac{1}{5}(x^3+4x^2+6x-6)$,定义数列 $\{x_n\}$ 满足 $x_{n+1}=f(x_n),n=0,1,\cdots$,求 $x_0$ 的取值范围使得 $\{x_n\}$ 收敛并求其值

解:依题意知

$$f'(x)=\frac{3}{5}\left[\left(x+\frac{3}{4}\right)^2+\frac{2}{9}\right]$$

由

$$f(x)=\frac{1}{5}(x^3+4x^2+6x-6)=x$$

可解得

$$x=1,-2,-3$$

1) $x_0>1$ 或 $x_0<-3$,易知 $\{x_n\}$ 均不收敛

2) $x_0=1$ 或 $-3$,则 $x_n\equiv 1$或 $-3$

3) $-2\le x_0<1$,显然 $x_0\ge x_1$,从而 $\{x_n\}$ 单调递减.又

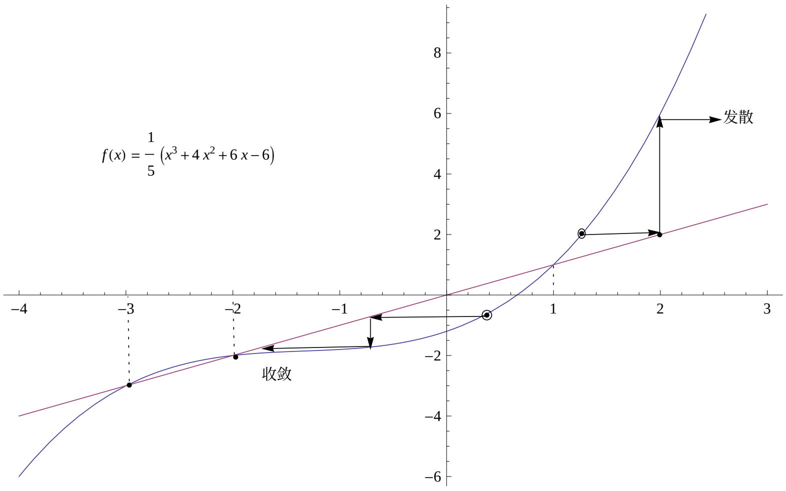
$$f(x)\ge f(-2)=-2$$

从而 $x_n$ 有下界,故 $\{x_n\}$ 收敛,易知 $x_n\to -2\,(n\to\infty)$

4) $-3<x_0\le -2$,显然 $x_1\ge x_0$,从而 $x_n$ 单调递增,易知 $x_n\to -2\,(n\to\infty)$

综上可得 $x_0$ 的取值范围是 $[-3,1]$,且

$$\lim_{n\to\infty}x_n=\begin{cases}1 & x_0=1\\ -3 & x_0=-3\\ -2 & 3<x_0<1\end{cases}$$




**问题**.设 $a_n > 0, \sum_{n\geq 1} a_n = 1$,证明:$F = \left\{ \sum_{n\in A} a_n : A \subset \mathbb{N} \right\}$ 为闭集 ($A$可以取$\varnothing$)

证:$F$ 与数集 $A$ 有关,不妨记作 $F(A)$

由 $\sum_{n\geq 1} a_n = 1$ 可得

$$\forall \varepsilon > 0, \exists m \in \mathbb{N}\, s.t. \sum_{n\geq m} a_n < \varepsilon$$

任取 $a \in \overline{F}(A)$,则 $\exists A_1, A_2, A_3, \ldots s.t. F(A_n) \to a (n \to +\infty)$

下面来构造 $A\, s.t. F(A) = a$ 即可完成证明

若 $A_n$ 只有有限项包含1,则取 $A_n$ 子列使得任意项都不包含1,否则选取 $A_n$ 子列使得任意项都包含1.前者表示$1 \notin A$,后者表示$1 \in A$

若 $A_n$ 只有有限项包含2,则取之前所得子列的子列使得任意项都不包含2,否则选取子列使得任意项都包含2.前者表示$2 \notin A$,后者表示$2 \in A$

这样不断进行下去即可得到数集 $A_n$.任取 $m \in A$,则 $\exists k \geq m, k \in \mathbb{N}\, s.t. A_k$ 含有 $A_n$ 所包含小于等于 $m$ 的所有项,故

$$\begin{aligned} F(A) &= \sum_{n\in A, n\leq m} a_n + \sum_{n\in A_k, n>m} a_n = \sum_{n\in A, n\leq m} a_n + \sum_{n\in A, n>m} a_n \\ &= F(A_k) - \sum_{n\in A_k, n>m} a_n + \sum_{n\in A, n>m} a_n \end{aligned}$$

从而

$$F(A_k) - \varepsilon < F(A) < F(A_k) + 2\varepsilon$$

令 $m \to +\infty$ 即得

$$a - \varepsilon \leq F(A) \leq a + 2\varepsilon$$

再由 $\varepsilon$ 的任意性可知 $F(A) = a$

# 后记

特别感谢博士数学论坛版主$tian27546$提供了大量精彩的题目与解答, 以及这一年多以来在数学分析方面对自己的帮助. 另外, 有些题目也参考了论坛网友$maorenfeng88$, $Hansschwarzkopf$, $mathjgs$以及百度吧友$ytdwdw$, 饼饼饼饼酱等人的解答, 在此一并表示感谢!

最后祝福所有喜欢数学和不喜欢数学的朋友!