第十届清疏竞赛班非数学类 27

级数计算(2)

构造幂级数解微分方程：

计算$\sum_{n=0}^{\infty}\frac{1}{\left(4n\right)!}$

计算：

$$f\left(x\right)=\sum_{n=0}^{\infty}\frac{x^{4n}}{\left(4n\right)!},f'\left(x\right)=\sum_{n=1}^{\infty}\frac{x^{4n-1}}{\left(4n-1\right)!},f''\left(x\right)=\sum_{n=1}^{\infty}\frac{x^{4n-2}}{\left(4n-2\right)!},f'''\left(x\right)=\sum_{n=1}^{\infty}\frac{x^{4n-3}}{\left(4n-3\right.}$$

$$f+f'+f''+f'''=\sum_{n=0}^{\infty}\frac{x^{n}}{n!}=e^{x},f\left(0\right)=1,f'\left(0\right)=f''\left(0\right)=f'''\left(0\right)=0.$$

这样计算太复杂了.

回忆$\cos x=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}x^{2n}}{\left(2n\right)!}$, 可以猜想$\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{\left(2n\right)!}=\frac{e^{x}+e^{-x}}{2}$

验证的确成立, 因此只需解：

$$f+f''=\sum_{n=0}^{\infty}\frac{x^{2n}}{\left(2n\right)!}=\frac{e^{x}+e^{-x}}{2}，f\left(0\right)=1,f'\left(0\right)=0.$$

故$f\left(x\right)=\frac{1}{4}\left(e^{-x}+e^{x}+2\cos x\right)$, 所以$f\left(1\right)=\frac{1}{4}\left(e^{-1}+e+2\cos 1\right)$

因此$\sum_{n=0}^{\infty}\frac{1}{\left(4n\right)!}=\frac{1}{4}\left(e^{-1}+e+2\cos 1\right)=\frac{\cosh 1+\cos 1}{2}$.

从答案可以看到, 一开始就可以猜：

$\cosh x=\sum_{n=0}^{\infty}\frac{x^{2n}}{\left(2n\right)!},\cos x=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}x^{2n}}{\left(2n\right)!}$, 所以两式相加除以2直接得到

$$\sum_{n=0}^{\infty}\frac{x^{4n}}{\left(4n\right)!}=\frac{\cosh x+\cos x}{2}.$$

习题：计算$\sum_{n=0}^{\infty}\frac{1}{\left(3n\right)!}$

积累常见级数和(习题)

$$\sum_{n=0}^{\infty} a^n \cos nx, \sum_{n=0}^{\infty} a^n \sin nx, \sum_{n=0}^{\infty} \frac{\sin nx}{n!}, \sum_{n=0}^{\infty} \frac{\cos nx}{n!}, \sum_{n=0}^{\infty} \frac{\sin nx}{n^m}, \sum_{n=0}^{\infty} \frac{\cos nx}{n^m}$$

$|a| < 1, m \in \mathbb{N}$

最后两个只计算m是具体值的时候即可(例如$m=1,2$)

例子：

$$t > 0, \sum_{n=1}^{\infty} e^{-nt} \sin nx = \operatorname{Im} \sum_{n=1}^{\infty} e^{-nt} e^{inx} = \operatorname{Im} \sum_{n=1}^{\infty} e^{-n(t-ix)} = \operatorname{Im} \frac{e^{-(t-ix)}}{1-e^{-(t-ix)}}$$

$$= \operatorname{Im} \frac{1}{e^{(t-ix)}-1} = \operatorname{Im} \frac{1}{e^t \cos x - 1 - e^t \sin x \cdot i}$$

$$= \operatorname{Im} \frac{e^t \cos x - 1 + e^t \sin x \cdot i}{\left(e^t \cos x - 1\right)^2 + \left(e^t \sin x\right)^2} = \frac{e^t \sin x}{e^{2t} + 1 - 2e^t \cos x}$$

$$= \frac{\sin x}{e^t + e^{-t} - 2\cos x} = \frac{1}{2} \frac{\sin x}{\cosh t - \cos x}.$$

二重级数换序法：

计算$\displaystyle\sum_{n=1}^{\infty}\frac{\sinh\pi}{\cosh(2n\pi)-\cosh\pi}$

计算：

$$\sum_{n=1}^{\infty}e^{-nt}\sin nx=\frac{1}{2}\frac{\sin x}{\cosh t-\cos x}$$

$$e^{ix}=\cos x+i\sin x,e^{-ix}=\cos x-i\sin x$$

$$\cosh(ix)=\frac{e^{ix}+e^{-ix}}{2}=\cos x,\sin x=\frac{e^{ix}-e^{-ix}}{2i}=\frac{\sinh(ix)}{i}$$

$$\sum_{n=1}^{\infty}e^{-nt}\frac{\sinh(inx)}{i}=\frac{1}{2}\frac{\frac{\sinh(ix)}{i}}{\cosh t-\cosh(ix)}$$

从而$\displaystyle\sum_{n=1}^{\infty}e^{-nt}\sinh(nx)=\frac{1}{2}\frac{\sinh(x)}{\cosh t-\cosh(x)}$.

当然上述恒等式可以直接用等比级数求和证明.

$$\sum_{n=1}^{\infty}\frac{\sinh\pi}{\cosh(2n\pi)-\cosh\pi}=2\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}e^{-2mn\pi}\sinh(m\pi)$$

$$=2\sum_{m=1}^{\infty}\sinh(m\pi)\sum_{n=1}^{\infty}e^{-2mn\pi}=2\sum_{m=1}^{\infty}\sinh(m\pi)\frac{e^{-2m\pi}}{1-e^{-2m\pi}}$$

$$=2\sum_{m=1}^{\infty}\sinh(m\pi)\frac{1}{e^{2m\pi}-1}=\sum_{m=1}^{\infty}\frac{e^{m\pi}-e^{-m\pi}}{e^{2m\pi}-1}$$

$$=\sum_{m=1}^{\infty}\frac{e^{2m\pi}-1}{e^{2m\pi}-1}e^{-m\pi}=\sum_{m=1}^{\infty}e^{-m\pi}=\frac{e^{-\pi}}{1-e^{-\pi}}=\frac{1}{e^{\pi}-1}$$

转化为部分和极限的方法：

$$\prod_{n=2}^{\infty} e\left(1-\frac{1}{n^2}\right)^{n^2}$$

计算：

$$x_m=\prod_{n=2}^{m} e\left(1-\frac{1}{n^2}\right)^{n^2}=e^{m-1}\prod_{n=2}^{m}\left(\frac{(n-1)(n+1)}{n^2}\right)^{n^2}$$

$$=e^{m-1}\prod_{n=2}^{m}\left(\frac{n-1}{n}\right)^{n^2}\prod_{n=2}^{m}\left(\frac{n+1}{n}\right)^{n^2}$$

$$=e^{m-1}\prod_{n=2}^{m}\left(\frac{n-1}{n}\right)^{n^2}\prod_{n=3}^{m+1}\left(\frac{n}{n-1}\right)^{(n-1)^2}$$

$$=\frac{1}{16}e^{m-1}\left(\frac{m+1}{m}\right)^{m^2}\prod_{n=3}^{m}\left(\frac{n-1}{n}\right)^{n^2}\prod_{n=3}^{m}\left(\frac{n}{n-1}\right)^{(n-1)^2}$$

$$=\frac{1}{16}e^{m-1}\left(\frac{m+1}{m}\right)^{m^2}\prod_{n=3}^{m}\left(\frac{n-1}{n}\right)^{n^2}\prod_{n=3}^{m}\left(\frac{n}{n-1}\right)^{n^2}\prod_{n=3}^{m}\left(\frac{n}{n-1}\right)^{(n-1)^2-n^2}$$

$$=\frac{1}{16}e^{m-1}\left(\frac{m+1}{m}\right)^{m^2}\prod_{n=3}^{m}\left(\frac{n-1}{n}\right)^{2n-1}$$

于是

$$\ln\left(16e\cdot x_m\right)=m+m^2\ln\frac{m+1}{m}+\sum_{n=3}^{m}\left[\left(2n-1\right)\ln\left(n-1\right)-\left(2n-1\right)\ln n\right]$$

$$=m+m^2\ln\frac{m+1}{m}+\sum_{n=3}^{m}\left[\left(2n-1\right)\ln\left(n-1\right)-\left(2n+1\right)\ln n\right]$$

$$+\sum_{n=3}^{m}\left[\left(2n+1\right)\ln n-\left(2n-1\right)\ln n\right]$$

$$=m+m^2\ln\frac{m+1}{m}+5\ln 2-\left(2m+1\right)\ln m+2\sum_{n=3}^{m}\ln n$$

$$=2m-2m\ln m-\ln m+5\ln 2-\frac{1}{2}+2\sum_{n=3}^{m}\ln n+o\left(1\right)$$

$$=2m-2m\ln m-\ln m+5\ln 2-\frac{1}{2}+2\sum_{n=1}^{m}\ln n-2\ln 2+o\left(1\right)$$

$$=2m-2m\ln m-\ln m+3\ln 2-\frac{1}{2}+2\sum_{k=1}^{m}\ln k+o\left(1\right)$$

$$=2\ln\frac{e^m m!}{m^m\sqrt{m}}+3\ln 2-\frac{1}{2}+o\left(1\right)\to 2\ln\frac{e^m\sqrt{2\pi m}\left(\dfrac{m}{e}\right)^m}{m^m\sqrt{m}}+3\ln 2-\frac{1}{2}$$

$$=2\ln\sqrt{2\pi}+3\ln 2-\frac{1}{2}$$

因此$\displaystyle\prod_{n=2}^{\infty}e\left(1-\frac{1}{n^2}\right)^{n^2}=\frac{e^{2\ln\sqrt{2\pi}+3\ln 2-\frac{1}{2}}}{16e}=\frac{\pi}{e^{\frac{3}{2}}}$

计算级数(无初等,只了解方法):$\sum_{k=2}^{\infty}(-1)^k\left[k\ln\frac{k}{k-\frac{1}{2}}-(k-1)\ln\frac{k-\frac{1}{2}}{k-1}\right]$

计算:

$$k\ln\frac{k}{k-\frac{1}{2}}-(k-1)\ln\frac{k-\frac{1}{2}}{k-1}=\frac{1}{4k}+O\left(\frac{1}{k^2}\right),k\to\infty$$

因此$\sum_{k=2}^{\infty}(-1)^k\left[\frac{1}{4k}+O\left(\frac{1}{k^2}\right)\right]=$ 条件+ 绝对 = 条件.

$$\sum_{k=2}^{n}(-1)^k\left[k\ln\frac{k}{k-\frac{1}{2}}-(k-1)\ln\frac{k-\frac{1}{2}}{k-1}\right]$$

$$=\sum_{k=2}^{n}\left[(-1)^k k\ln\frac{k}{k-\frac{1}{2}}-(k-1)(-1)^{k-1}\ln\frac{k-1}{k-\frac{1}{2}}\right]$$

$$=\sum_{k=2}^{n}\left[(-1)^k k\ln k-(k-1)(-1)^{k-1}\ln(k-1)\right]$$

$$+\sum_{k=2}^{n}\left[-(-1)^k k\ln\left(k-\frac{1}{2}\right)+(k-1)(-1)^{k-1}\ln\left(k-\frac{1}{2}\right)\right]$$

$$=-\ln\frac{1}{2}+(-1)^n n\ln n+\sum_{k=1}^{n}\left[(-1)^{k-1}(2k-1)\ln\left(k-\frac{1}{2}\right)\right]$$

只需考虑偶数的情况,因此

$$\ln 2+2n\ln\left(2n\right)+\sum_{k=1}^{2n}\left[\left(-1\right)^{k-1}\left(2k-1\right)\ln\left(k-\frac{1}{2}\right)\right]$$

$$=\ln 2+2n\ln\left(2n\right)+\sum_{k=1}^{n}\left[\left(4k-3\right)\ln\left(2k-\frac{3}{2}\right)-\left(4k-1\right)\ln\left(2k-\frac{1}{2}\right)\right]$$

$$=\ln 2+2n\ln\left(2n\right)+\sum_{k=1}^{n}\left[\left(4k-3\right)\ln\left(\frac{4k-3}{2}\right)-\left(4k-1\right)\ln\left(\frac{4k-1}{2}\right)\right]$$

$$=\ln 2+2n\ln\left(2n\right)+\sum_{k=1}^{n}\left[\left(4k-3\right)\ln\left(4k-3\right)-\left(4k-1\right)\ln\left(4k-1\right)\right]$$

$$+\sum_{k=1}^{n}\left[-\left(4k-3\right)\ln 2+\left(4k-1\right)\ln 2\right]$$

$$=\ln 2+2n\ln\left(2n\right)+2\ln 2\cdot n+\sum_{k=1}^{n}\left[\left(4k-3\right)\ln\left(4k-3\right)-\left(4k-1\right)\ln\left(4k-1\right)\right]$$

回忆$\sum_{k=1}^{n}k\ln k$用欧拉麦克劳林可以得到任意阶的渐进估计, 类似的

可以得到$\sum_{k=1}^{n}\left(4k-3\right)\ln\left(4k-3\right)$, $\sum_{k=1}^{n}\left(4k-1\right)\ln\left(4k-1\right)$的渐进估计

带进去就可以得到一个常数C，这个C显然无初等表示.

可以用软件检查，发现C会依赖于$Glaisher-kinkelin$常数.

恒等式变形：

计算$\displaystyle\sum_{n=1}^{\infty}\frac{\ln 2-\left(\frac{1}{n+1}+\cdots+\frac{1}{2n}\right)}{n}$

分析：

$$\sum_{k=1}^{2n}\frac{\left(-1\right)^{k}}{k}=\sum_{k=1}^{n}\left(\frac{1}{2k}-\frac{1}{2k-1}\right)$$

$$\sum_{k=1}^{n}\left(\frac{1}{2k}+\frac{1}{2k-1}\right)=\sum_{k=1}^{2n}\frac{1}{k}$$

$$\frac{1}{n+1}+\cdots+\frac{1}{2n}=\sum_{k=1}^{2n}\frac{1}{k}-\sum_{k=1}^{n}\frac{1}{k}$$

$$=\sum_{k=1}^{n}\left(\frac{1}{2k}+\frac{1}{2k-1}\right)-\sum_{k=1}^{n}\frac{1}{k}$$

$$=\sum_{k=1}^{n}\left(\frac{1}{2k-1}-\frac{1}{2k}\right)=\sum_{k=1}^{2n}\frac{\left(-1\right)^{k-1}}{k}$$

这个恒等式要积累一下.

证明：

熟知$\displaystyle\sum_{k=1}^{\infty}\frac{\left(-1\right)^{k-1}}{k}=\ln 2,\sum_{k=1}^{\infty}\frac{x^{k}}{k}=-\ln\left(1-x\right)$

$$\sum_{n=1}^{\infty}\frac{\ln 2-\left(\frac{1}{n+1}+\cdots+\frac{1}{2n}\right)}{n}=\sum_{n=1}^{\infty}\frac{\sum_{k=2n+1}^{\infty}\frac{\left(-1\right)^{k-1}}{k}}{n}$$

$$=\sum_{n=1}^{\infty}\frac{\sum_{k=2n+1}^{\infty}\int_0^1\left(-x\right)^{k-1}dx}{n}=\sum_{n=1}^{\infty}\frac{\int_0^1\sum_{k=2n+1}^{\infty}\left(-x\right)^{k-1}dx}{n}$$

$$=\sum_{n=1}^{\infty}\frac{\int_0^1\frac{\left(-x\right)^{2n}}{1+x}dx}{n}=\sum_{n=1}^{\infty}\int_0^1\frac{x^{2n}}{\left(1+x\right)n}dx$$

$$=\int_0^1\sum_{n=1}^{\infty}\frac{x^{2n}}{\left(1+x\right)n}dx=-\int_0^1\frac{\ln\left(1-x^{2}\right)}{1+x}dx$$

$$
\begin{aligned}
&= -\int_0^1 \frac{\ln\left(1-x\right)}{1+x}\,dx - \int_0^1 \frac{\ln\left(1+x\right)}{1+x}\,dx \\
&= -\int_0^1 \frac{\ln y}{2-y}\,dx - \int_0^1 \ln\left(1+x\right) d\ln\left(1+x\right) \\
&= -\int_0^1 \frac{\ln\dfrac{y}{2}}{2-y}\,dy - \int_0^1 \frac{\ln 2}{2-y}\,dy - \frac{\ln^2 2}{2} \\
&= -\int_0^{\frac{1}{2}} \frac{\ln t}{1-t}\,dt - \ln^2 2 - \frac{\ln^2 2}{2} \\
&= -\int_{\frac{1}{2}}^1 \frac{\ln\left(1-t\right)}{t}\,dt - \frac{3\ln^2 2}{2} \\
&= -\left(\int_0^1 \frac{\ln\left(1-t\right)}{t}\,dt - \int_0^{\frac{1}{2}} \frac{\ln\left(1-t\right)}{t}\,dt\right) - \frac{3\ln^2 2}{2} \\
&= \left(\int_0^1 \sum_{t=1}^{\infty} \frac{t^{n-1}}{n}dt + \int_0^{\frac{1}{2}} \frac{\ln\left(1-t\right)}{t}\,dt\right) - \frac{3\ln^2 2}{2} \\
&= \left(\sum_{n=1}^{\infty} \frac{1}{n^2} + \int_0^{\frac{1}{2}} \frac{\ln\left(1-t\right)}{t}\,dt\right) - \frac{3\ln^2 2}{2} \\
&= \frac{1}{2}\left(\frac{\pi^2}{6} - \ln^2 2\right)
\end{aligned}
$$

积累经典积分：

$$\int_0^{\frac{1}{2}}\frac{\ln\left(1-t\right)}{t}dt=\int_0^{\frac{1}{2}}\ln\left(1-t\right)d\ln t=\ln^2 2+\int_0^{\frac{1}{2}}\frac{\ln t}{1-t}dt$$

$$=\ln^2 2+\int_{\frac{1}{2}}^{1}\frac{\ln\left(1-t\right)}{t}dt$$

$$\int_0^{\frac{1}{2}}\frac{\ln\left(1-t\right)}{t}dt-\int_{\frac{1}{2}}^{1}\frac{\ln\left(1-t\right)}{t}dt=\ln^2 2$$

$$\int_0^{\frac{1}{2}}\frac{\ln\left(1-t\right)}{t}dt+\int_{\frac{1}{2}}^{1}\frac{\ln\left(1-t\right)}{t}dt=-\frac{\pi^2}{6}$$

因此$\int_0^{\frac{1}{2}}\frac{\ln\left(1-t\right)}{t}dt=\frac{\ln^2 2}{2}-\frac{\pi^2}{12}$

实际上还可以得到:$\sum_{n=1}^{\infty}\frac{1}{n^2 2^n}=-\frac{\ln^2 2}{2}+\frac{\pi^2}{12}$

习题：证明如下恒等式并计算级数.

恒等式：$H_n=-n\int_0^1\left(1-x\right)^{n-1}\ln x dx$

计算:$\sum_{n=1}^{\infty}C_{2n}^{n}x^n H_n$

轮换对称二重级数计算：

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m\left(m+1\right)\left(n+1\right)\left(n+m\right)}$$

看到二重级数可以尝试一下对称性的方法.

证明：

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m\left(m+1\right)\left(n+1\right)\left(n+m\right)}$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{n}{m\left(m+1\right)n\left(n+1\right)\left(n+m\right)}$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m}{m\left(m+1\right)n\left(n+1\right)\left(n+m\right)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{m+n}{m\left(m+1\right)n\left(n+1\right)\left(n+m\right)}$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m\left(m+1\right)n\left(n+1\right)}$$

$$=\frac{1}{2}\left(\sum_{m=1}^{\infty}\frac{1}{m\left(m+1\right)}\right)^{2}=\frac{1}{2}.$$

特殊例子：

$$\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\frac{\pi}{2^{k+1}}$$

计算：

$$\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}=\frac{\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}}$$

$$=\frac{\cos\frac{x}{2}\cdots\cos\frac{x}{2^{n-1}}\sin\frac{x}{2^{n-1}}}{2\sin\frac{x}{2^n}}$$

$$=\frac{\cos\frac{x}{2}\cdots\cos\frac{x}{2^{n-2}}\sin\frac{x}{2^{n-2}}}{2^2\sin\frac{x}{2^n}}$$

$\cdots$

$$=\frac{\cos\frac{x}{2}\sin\frac{x}{2}}{2^{n-1}\sin\frac{x}{2^n}}=\frac{\sin x}{2^n\sin\frac{x}{2^n}}$$

$$\ln\left(\cos\frac{x}{2}\cdots\cos\frac{x}{2^n}\right)=\ln\frac{\sin x}{2^n\sin\frac{x}{2^n}}=\ln\sin\,x-n\ln 2-\ln\sin\frac{x}{2^n}$$

两边求导有$-\sum_{k=1}^{n}\frac{1}{2^k}\tan\frac{x}{2^k}=\frac{\cos x}{\sin x}-\frac{\cos\frac{x}{2^n}}{2^n\sin\frac{x}{2^n}}$

赋予$x=\frac{\pi}{2}$,就有

$$-\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\frac{\pi}{2^{k+1}}=-\lim_{n\to\infty}\frac{\cos\frac{\pi}{2^{n+1}}}{2^n\sin\frac{\pi}{2^{n+1}}}=-\lim_{n\to\infty}\frac{1}{2^n\frac{\pi}{2^{n+1}}}=-\frac{2}{\pi}$$

$$\sum_{k=1}^{\infty}\frac{1}{2^k}\tan\frac{\pi}{2^{k+1}}=\frac{2}{\pi}.$$

利用傅里叶计算级数：

展开$x\left(1-x\right), x\in\left[0,1\right]$. 计算$\sum_{n=1}^{\infty}\frac{1}{n^2}$

我们让$x\left(1-x\right)$周期1延拓, 然后

$$x\left(1-x\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(2\pi nx\right)+b_n\sin\left(2\pi nx\right)\right)$$

$$\int_0^1 x\left(1-x\right)\cos\left(2\pi mx\right)dx$$

$$=\int_0^1\left[\frac{a_0}{2}+\sum_{n=1}^{\infty}\left(a_n\cos\left(2\pi nx\right)+b_n\sin\left(2\pi nx\right)\right)\right]\cos\left(2\pi mx\right)dx$$

$$=a_n\int_0^1\cos^2\left(2\pi mx\right)dx=\frac{a_n}{2}$$

$$a_n=2\int_0^1 x\left(1-x\right)\cos\left(2\pi nx\right)dx, b_n=2\int_0^1 x\left(1-x\right)\sin\left(2\pi nx\right)dx$$

$$a_0=\frac{1}{3}, a_n=-\frac{1}{\pi^2 n^2}, b_n=0$$

$x\left(1-x\right)=\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos\left(2\pi nx\right)}{n^2}$, 赋予$x=0$,

这样就有$\frac{1}{6}-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}=0\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.

习题：构造一个周期函数的傅里叶级数, 计算$\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{\left(2n+1\right)^3}$

提示：可以把$x\left(1-x\right)$作奇延拓之后在$\left[-1,1\right]$展开傅里叶级数

习题：

$a\notin\mathbb{Z}$, 计算$\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}$

选取$f\left(x\right)=\cos\left(ax\right), x\in\left[-\pi,\pi\right]$, 计算傅里叶级数