全国大学生数学竞赛非数学类模拟六

清疏竞赛考研数学

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摘要

$$\mathbb{N}=\{1,2,\cdots\},\mathbb{N}_0=\{0,1,2,\cdots\}.$$

模拟试题应当规定时间独立完成并给予反馈.

$1.1: g(x)=\lim\limits_{r\to 0^+} e^{\frac{\ln[(x+1)^{r+1}-x^{r+1}]}{r}} = \lim\limits_{r\to 0^+} e^{\frac{(x+1)^{r+1}-x^{r+1}-1}{r}} = e^{\lim\limits_{r\to 0^+}[\ln(x+1)\cdot(x+1)^{r+1}-\ln x\cdot x^{r+1}]}$

$= e^{(x+1)\ln(1+x)-x\ln x} = \frac{(x+1)^{x+1}}{x^x}.\quad \lim\limits_{x\to+\infty}\frac{(x+1)^{x+1}}{x^{x+1}} = \lim\limits_{x\to+\infty} e^{(x+1)\ln(1+\frac{1}{x})} = e$

1 填空题

填空题 1.1 设 $g(x)=\lim\limits_{r\to 0^+}\left((x+1)^{r+1}-x^{r+1}\right)^{\frac{1}{r}}$, 则 $\lim\limits_{x\to+\infty}\frac{g(x)}{x}=$ $\underline{\quad e \quad}$

填空题 1.2 平面 $lx+my+nz=\lambda$ 和椭球面 $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ 相切, 则 $a^2\ell^2+b^2m^2+c^2n^2=$ $\underline{\quad \lambda^2 \quad}$

填空题 1.3 设 f 连续且满足 $x=\int_0^x f(y)\,dy+\int_0^x yf(x-y)\,dy$, 则 $f(x)=$ $\underline{\quad e^{-x} \quad}$

填空题 1.4 给定二阶连续可微函数 $f(x,y,z)$, 做换元 $\begin{cases} x=a_{11}u+a_{12}v+a_{13}w \\ y=a_{21}u+a_{22}v+a_{23}w \\ z=a_{31}u+a_{32}v+a_{33}w \end{cases}$, 这里 $A=(a_{ij})$ 是实正交矩阵. 则方程 $\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}=0$ 变为 $\underline{\frac{\partial^2 f}{\partial u^2}+\frac{\partial^2 f}{\partial v^2}+\frac{\partial^2 f}{\partial w^2}}=0$

填空题 1.5 设 $[x]$ 表示不超过 x 的最大整数, 则 $\int_0^1\left(\frac{1}{x}-\left[\frac{1}{x}\right]\right)dx=$ $\underline{\quad 1-\gamma \quad}$

1.2. 设 $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}+\frac{zz_0}{c^2}=1$, (x_0,y_0,z_0) 切点

$\frac{x_0}{a^2}=kl,\ \frac{y_0}{b^2}=km,\ \frac{z_0}{c^2}=kn.\quad \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}+\frac{z_0^2}{c^2}=a^2k^2l^2+k^2m^2b^2+k^2n^2c^2=1$

$lx_0+my_0+nz_0=a^2l^2k+m^2b^2k+kn^2c^2=\lambda$

$\begin{cases} a^2l^2+m^2b^2+n^2c^2=\frac{1}{k^2} \\ a^2l^2+m^2b^2+n^2c^2=\frac{\lambda}{k} \end{cases} \Rightarrow \frac{1}{k^2}=\frac{\lambda}{k} \Rightarrow k=\frac{1}{\lambda}.$

1.3 $x=\int_0^x f(y)dy+\int_0^x (x-y)f(y)dy=\int_0^x f(y)dy+x\int_0^x f(y)dy-\int_0^x yf(y)dy$

$1=f(x)+\int_0^x f(y)dy \quad 0=f'(x)+f(x)\ ,\ f(0)=1$

2 选择题答案区

1.4. 积累=阶导之和在正交变换下不改变，计算不困难.

1.5 $\int_0^1 \left(\frac{1}{x}-[\frac{1}{x}]\right)dx=\int_1^{+\infty}\frac{1}{y^2}(y-[y])dy=\sum_{k=1}^{+\infty}\int_k^{k+1}\frac{y-[y]}{y^2}dy$

$=\sum_{k=1}^{+\infty}\int_k^{k+1}\frac{y-k}{y^2}dy=\sum_{k=1}^{+\infty}\left[-\frac{1}{k+1}-\ln k+\ln(k+1)\right]$

$=\lim_{n\to+\infty}\left[\sum_{k=1}^{n}\left(-\frac{1}{k+1}\right)+\ln(n+1)\right]$

$=\lim_{n\to+\infty}\left[\ln(n+1)-\sum_{k=1}^{n}\frac{1}{k+1}\right]=1-\gamma$

熟知 $\sum_{k=1}^{n}\frac{1}{k}-\ln n\to\gamma\approx 0.577$. 所以

3 解答题

解答题 3.1 设 $u \in C^{\infty}(\mathbb{R}^2)$ 满足

$$u = x + y\sin u.$$

证明

$$\frac{\partial^n u}{\partial y^n} = \frac{\partial^{n-1}}{\partial x^{n-1}}\left(\sin^n u \cdot \frac{\partial u}{\partial x}\right), \forall n \geqslant 1.$$

证明： $u-x-y\sin u=0$. 即 $\frac{\partial u}{\partial x} = -\frac{-1}{1-y\cos u} = \frac{1}{1-y\cos u}$

$$\frac{\partial u}{\partial y} = \frac{\sin u}{1-y\cos u}$$

故 $\frac{\partial u}{\partial y} = \sin u \cdot \frac{\partial u}{\partial x}$，因此$n=1$时,命题成立.

设 $\frac{\partial^n u}{\partial y^n} = \frac{\partial^{n-1}}{\partial x^{n-1}}\left(\sin^n u \cdot \frac{\partial u}{\partial x}\right)$ $n\geq 1$，则

$$\begin{aligned}
\frac{\partial^{n+1}u}{\partial y^{n+1}} &= \frac{\partial}{\partial y}\left(\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\sin^n u\cdot\frac{\partial u}{\partial x}\right)\right)\\
&= \frac{\partial^{n-1}}{\partial x^{n-1}}\left(\frac{\partial}{\partial y}\left(\sin^n u\cdot\frac{\partial u}{\partial x}\right)\right)\\
&= \frac{\partial^{n-1}}{\partial x^{n-1}}\left(n\sin^{n-1}u\cdot\cos u\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial x}+\sin^n u\cdot\frac{\partial^2 u}{\partial x\partial y}\right)\\
&= \frac{\partial^{n-1}}{\partial x^{n-1}}\left(n\sin^n u\cdot\cos u\cdot\left(\frac{\partial u}{\partial x}\right)^2+\sin^n u\cdot\frac{\partial^2 u}{\partial x\partial y}\right)
\end{aligned}$$

$\frac{\partial^2 u}{\partial x\partial y} = \cos u\cdot\left(\frac{\partial u}{\partial x}\right)^2+\sin u\cdot\frac{\partial^2 u}{\partial x^2}$，所以

$$\frac{\partial^{n+1}u}{\partial y^{n+1}} = \frac{\partial^{n-1}}{\partial x^{n-1}}\left((n+1)\sin^n u\cdot\cos u\cdot\left(\frac{\partial u}{\partial x}\right)^2+\sin^{n+1}u\cdot\frac{\partial^2 u}{\partial x^2}\right)$$

$$\frac{\partial^n}{\partial x^n}\left(\sin^{n+1}u\cdot\frac{\partial u}{\partial x}\right) = \frac{\partial^{n-1}}{\partial x^{n-1}}\left((n+1)\sin^n u\cdot\cos u\cdot\left(\frac{\partial u}{\partial x}\right)^2+\sin^{n+1}u\,\frac{\partial^2 u}{\partial x^2}\right).$$

因此由数归,我们完成了证明.

$$\cos^{(n)} x = \cos\left(x+\frac{n}{2}\pi\right)$$

解答题 3.2 设 $f(x)=\frac{\sin x}{x}, x>0$, 证明

$$\left|f^{(n)}(x)\right|<\frac{1}{n+1}, \forall n \in \mathbb{N}.$$

证明: $f(x)=\frac{\sin x}{x}=\int_0^1 \cos(xt)\,dt$

$$f^{(n)}(x)=\int_0^1 t^n \cos\left(xt+\frac{n}{2}\pi\right)dt$$

故 $\left|f^{(n)}(x)\right| \leqslant \int_0^1 t^n\,dt=\frac{1}{n+1}$.

若 $\left|f^{(n)}(x)\right|=\frac{1}{n+1}$, 则 $\left|\cos\left(xt+\frac{n}{2}\pi\right)\right|=1$, $\forall t\in[0,1]$, 矛盾.

故 $\left|f^{(n)}(x)\right|<\frac{1}{n+1}$.

解答题 3.3 定义 $C(\alpha)$ 为 $(1+x)^{\alpha}$ 在 $x=0$ 的 *Taylor* 级数的 x^{1992} 的系数, 计算

$$\int_0^1 \left(C(-y-1) \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy.$$

解: $(1+x)^a = \sum_{k=0}^{+\infty} C_a^k x^k$, $C(-y-1) = C_{-y-1}^{1992}$

$C_a^0 = 1$, $C_a^k = \frac{a(a-1)\cdots(a-k+1)}{k!}, k>0$

$$= \frac{(-y-1)(-y-2)\cdots(-y-1992)}{1992!}$$

$$= \frac{(y+1)(y+2)\cdots(y+1992)}{1992!}$$

故 $\int_0^1 \left(C(-y-1) \cdot \sum_{k=1}^{1992} \frac{1}{y+k} \right) dy = \frac{1}{1992!} \int_0^1 \sum_{k=1}^{1992} \frac{(y+1)\cdots(y+1992)}{y+k} dy$

$$= \frac{1}{1992!} \int_0^1 \frac{d[(y+1)\cdots(y+1992)]}{dy} dy$$

$$= \frac{1993! - 1992!}{1992!}$$

$$= 1993 - 1$$

$$= 1992.$$

解答题 3.4 设 ρ 为点 (x,y,z) 到 x 轴的距离，Ω 为一棱台，其六个顶点分别为

$$(0,0,1),(0,1,1),(1,1,1),(0,0,2),(0,2,2),(2,2,2).$$

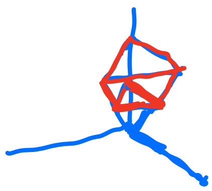
计算

$$\iiint_{\Omega}\frac{1}{\rho^2}dxdydz.$$

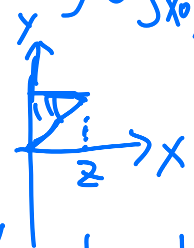
注意 允许查阅棱台定义.

解答：

$\rho^2=y^2+z^2$



$$\iiint_{\Omega}\frac{1}{\rho^2}dxdydz=\int_1^2dz\iint_{S_{xoy}}\frac{1}{y^2+z^2}dxdy$$

$S_{xoy}=$ 

故 $$\iiint_{\Omega}\frac{1}{\rho^2}dxdydz=\int_1^2dz\int_0^z dy\int_0^y\frac{1}{y^2+z^2}dx$$

$$=\int_1^2dz\int_0^z\frac{y}{y^2+z^2}dy$$

$$=\int_1^2\frac{\ln 2}{2}dz$$

$$=\frac{\ln 2}{2}.$$

解答题 3.5 设

$$x_1=\frac{1}{2}, x_n=\frac{\sqrt{x_{n-1}^2+4x_{n-1}}+x_{n-1}}{2}, n=2,3,\cdots.$$

证明 $\sum\limits_{n=1}^{\infty}\frac{1}{x_n^2}$ 收敛并求值.

证明:

$$\frac{1}{x_n}=\frac{2}{\sqrt{x_{n-1}^2+4x_{n-1}}+x_{n-1}}$$

$$=\frac{2\left(\sqrt{x_{n-1}^2+4x_{n-1}}-x_{n-1}\right)}{4x_{n-1}}$$

$$=\frac{2x_n-x_{n-1}-x_{n-1}}{2x_{n-1}}$$

$$=\frac{x_n-x_{n-1}}{x_{n-1}}$$

$$\frac{1}{x_n^2}=\frac{x_n-x_{n-1}}{x_nx_{n-1}}=\frac{1}{x_{n-1}}-\frac{1}{x_n}$$

$$\sum_{j=2}^{n}\frac{1}{x_j^2}=\frac{1}{x_1}-\frac{1}{x_n}=2-\frac{1}{x_n}.$$

$f(x)=\frac{\sqrt{x^2+4x}+x}{2}$, $f\uparrow$, $f(x)=x$ 无解. f 无不动点.

而递增递推, f 一定单调, f 收敛则 f 有不动点.

因此 $\lim\limits_{n\to+\infty}\frac{1}{x_n}=0$. 故 $\sum\limits_{n=1}^{+\infty}\frac{1}{x_n^2}=2$.

解答题 3.6 给定实数 $a_0, a_1, \ldots, a_n$ 和 $x \in (0,1)$, 满足

$$\frac{a_0}{1-x} + \frac{a_1}{1-x^2} + \cdots + \frac{a_n}{1-x^{n+1}} = 0.$$

证明存在 $y \in (0,1)$ 使得

$$a_0 + a_1 y + \cdots + a_n y^n = 0.$$

证明: $\displaystyle\sum_{j=0}^{n} \frac{a_j}{1-x^{j+1}} = \sum_{j=0}^{n} a_j \sum_{k=0}^{+\infty} x^{k(j+1)}$

$\displaystyle= \sum_{k=0}^{+\infty} x^k \left[\sum_{j=0}^{n} a_j x^{kj}\right]$

定义 $P(t) = \sum_{j=0}^{n} a_j t^j$, 则 $0 = \sum_{j=0}^{n} \frac{a_j}{1-x^{j+1}} = \sum_{k=0}^{+\infty} x^k P(x^k) \cdots (*)$.

故若 $P \neq 0$, 则 $P(t) > 0$, 或 $P(t) < 0$, $\forall t \in (0,1)$,

那么 $(*)$ 右边也将 >0 或 <0, 矛盾!

故 $\exists y \in (0,1)$, 使 $P(y) = 0$ 即 $\sum_{j=0}^{n} a_j y^j = 0$