2013年西西寒假辅导班数学分析第三课竞赛讲义

讲课时间：2月5日—15：00-17:00

例题1：已知

$$\lim_{x\to\infty} x^n[(x+4)\arctan(x+4)-4(x+3)\arctan(x+3)+6(x+2)\arctan(x+2) \tag{1}$$

$$-4(x+1)\arctan(x+1)+x\arctan x] \tag{2}$$

$$=A \tag{3}$$

其中A是有限的非零常数,求$n+A$的值

解：当$|x+k|>1$时,$(x+k)\arctan(x+k)$可以展开成下列级数形式

$$(x+k)\arctan(x+k)=(x+k)\left[\frac{\pi}{2}-\frac{1}{x+k}+\frac{1}{3(x+k)^3}-\frac{1}{5(x+k)^5}+\cdots\right] \tag{4}$$

$$=\frac{\pi}{2}(x+k)-1+\frac{1}{3(x+k)^2}-\frac{1}{5(x+k)^4}+\cdots,k=0,1,2,3,4 \tag{5}$$

将展开式形为$\frac{\pi}{2}(x+k)$的项,乘以系数$1,-4,6,-4,1$后相加,得到

$$\frac{\pi}{2}[(x+4)-4(x+3)+6(x+2)-4(x+1)+x]=0$$

将展开式中形为-1的项,乘以系数$1,-4,6,-4,1$后相加,得到

$$(-1)-4(-1)+6(-1)-4(-1)+(-1)=0$$

将展开式形为$\frac{1}{3(x+k)^2}$的项,乘以系数$1,-4,6,-4,1$后相加.得到

$$\frac{1}{3(x+4)^2}-\frac{4}{3(x+3)^2}+\frac{6}{3(x+2)^2}-\frac{4}{3(x+1)^2}+\frac{1}{3x^2} \tag{6}$$

$$=\frac{40x^4+320x^3+840x^2+800x+192}{x^2(x+1)^2(x+2)^2(x+3)^2(x+4)^2} \tag{7}$$

这式子乘以x^6再取$x\longrightarrow\infty$,有

$$\lim_{x\to\infty} x^6\frac{40x^4+320x^3+840x^2+800x+192}{x^2(x+1)^2(x+2)^2(x+3)^2(x+4)^2}=40$$

展开式中其余分母幂次比$\frac{1}{3(x+k)^2}$更高的各项,乘以系数$1,-4,6,-4,1$后相加,和式的幂次都低于x^{-6},乘以x^6,再取$x\longrightarrow\infty$的极限，极限值都等于0,
所以必有

$$n=6,A=40$$

,故

$$n+A=46$$

例题2：设f是在R上有四阶连续可导的函数,$x \in [0,1]$,满足

$$\int_0^1 f(x)dx + 3f\left(\frac{1}{2}\right) = 8\int_{\frac{1}{4}}^{\frac{3}{4}} f(x)dx$$

证明：存在$c \in (0,1)$,使得$f^{(4)}(c) = 0$

证明：令

$$G(t) = \int_{-t}^{t} g(x)dx - 8\int_{-\frac{t}{2}}^{\frac{t}{2}} g(x)dx$$

其中

$$g(x) = f\left(x + \frac{1}{2}\right) - f\left(\frac{1}{2}\right)$$

易得

$$G(0) = 0, G\left(\frac{1}{2}\right) = 0$$

由洛尔定理有存在$t_0 \in (0, 1/2)$使得$G'(t_0) = 0$.由于

$$G'(t) = g(t) - 4g(t/2) - 4g(-t/2) + g(-t)$$

则$G'(0) = 0, G'(t_0) = 0$,则由洛尔定理有：$G''(t_1) = 0$,又

$$G''(t) = g'(t) - 2g'(t/2) + 2g'(-t/2) - g'(-t)$$

显然$G''(0) = 0$,故由中值定理有$G'''(t_2) = 0$

又

$$G'''(t) = (g''(t) - g''(t/2)) - (g''(-t/2) - g''(-t))$$

即

$$G'''(t_2) = (g''(t_2) - g''(t_2/2)) - (g''(-t_2/2) - g''(-t_2))$$

又由拉格朗日中值定理有存在$\theta_+ \in (t_2/2, t_2), \theta_- \in (-t_2, -t_2/2)$,

$$(g''(t_2) - g''(t_2/2)) - (g''(-t_2/2) - g''(-t_2)) = g'''(\theta_+)\frac{t_2}{2} - g'''(\theta_-)\frac{t_2}{2}$$

注意$t_2 \neq 0$即

$$g'''(\theta_+) - g'''(\theta_-) = 0$$

再利用拉格朗日中值定理

$$g''''(\theta) = 0$$

即

$$f^{(4)}\left(\theta + \frac{1}{2}\right) = 0$$

将$\theta + \dfrac{1}{2} \longrightarrow \theta$,即有

$$f^{(4)}(\theta) = 0$$

例题3：求证：$\sum\limits_{n=1}^{\infty}\frac{1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}}{(n+1)(n+2)}$ 收敛,并求值

解：由于 $\lim\limits_{n\to\infty}\frac{u_n}{n^{\frac{3}{2}}}=0$,由比较判别法易得.

其次我们注意有

$$
\begin{align}
&\sum_{n=1}^{\infty}\frac{H_n}{(n+1)(n+2)} \tag{8}\\
&=\sum_{n=1}^{\infty}\left(\frac{H_n}{n+1}-\frac{H_n}{n+2}\right) \tag{9}\\
&=\sum_{n=1}^{\infty}\left(\frac{H_n}{n+1}-\frac{H_{n+1}-\frac{1}{n+1}}{n+2}\right) \tag{10}\\
&=\sum_{n=1}^{\infty}\left(\frac{H_n}{n+1}-\frac{H_{n+1}}{n+2}\right)+\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)} \tag{11}\\
&=\frac{1}{2}-\lim_{n\to\infty}\frac{H_{n+1}}{n+2}+\frac{1}{2}-\lim_{n\to\infty}\frac{1}{n+2} \tag{12}\\
&=1 \tag{13}
\end{align}
$$

变式1：求

$$\sum_{n=1}^{\infty}\frac{H_n}{n(n+1)}$$

变式2：求

$$\sum_{n=1}^{\infty}\frac{H_n}{(2n+1)(2n+2)}$$

变式3：求求

$$\sum_{n=1}^{\infty}\frac{H_n^2}{n^2}$$

变式4：求

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3}$$

变式5：求

$$\sum_{n=2}^{\infty}\frac{(2n+1)H_n^2}{(n-1)n(n+1)(n+2)}$$

答案：1：$\frac{\pi^2}{6},2:\frac{\pi^2}{12}-\ln^2 2,3:\frac{17\pi^4}{360},4:\frac{\pi^4}{72},5:\frac{13}{12}$

证明下变式4：显然有

$$-n\int_0^1(1-x)^{n-1}\ln x dx=-\sum_{k=1}^{n}C_n^k\frac{(-1)^k}{k}=H_n$$

证明：考虑积分

$$\int_0^1 \frac{1-(1-x)^n}{x}dx = \int_0^1 \sum_{k=1}^{n} C_n^k (-1)^{k+1} x^{k-1} dx = \sum_{k=1}^{n} \frac{C_n^k(-1)^{k+1}}{k}$$

另外一方面

$$\int_0^1 \frac{1-(1-x)^n}{x}dx = \int_0^1 \frac{1-u^n}{1-u}du = H_n$$

所以

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = -\sum_{n=1}^{\infty} \frac{1}{n^2} \int_0^1 (1-x)^{n-1} \ln x dx = -\int_0^1 \sum_{n=1}^{\infty} \frac{(1-x)^{n-1}}{n^2} \ln x dx$$

由于

$$\sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2} = \frac{Li_2(1-x)}{1-x}$$

故

$$\sum_{n=1}^{\infty} \frac{H_n}{n^3} = -\int_0^1 \frac{Li_2(1-x)\ln x}{1-x}dx = \frac{1}{2}(Li_2(1-x))^2|_0^1 = \frac{1}{2}\left(\frac{\pi^2}{6}\right)^2$$

例题4：求2×2整数矩阵B,C使之满足

$$B^3+C^3 = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

解：令

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$$

则

$$A^2 = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$$

故

$$A^2+3A+2I=0$$

即$A^3+3A^2+2A=0$
因此有

$$(A+I)^3 = A^3+3A^2+3A+I = A+I \Longrightarrow A = (A+I)^3 - I$$

故

$$B = A+I = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$C = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

例题5：设$T_1=T_2=1,T_3=2$,且$T_n=T_{n-1}+T_{n-2}+T_{n-3}$.求$\sum\limits_{n=1}^{\infty}\dfrac{T_n}{\pi^n}$的值

解：显然有

$$\frac{T_{n+1}}{T_n}=1+\frac{T_{n-1}}{T_n}+\frac{T_{n-2}}{T_n}<1+1+1=3$$

故$T_n<3^{n-1}$,故

$$S=\sum_{n=1}^{\infty}\frac{T_n}{\pi^n}$$

是收敛的. 故

$$\begin{aligned}
S=\sum_{n=1}^{\infty}\frac{T_n}{\pi^n}&=\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{2}{\pi^3}+\sum_{n=4}^{\infty}\frac{T_{n-1}+T_{n-2}+T_{n-3}}{\pi^n} &(14)\\
&=\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{2}{\pi^3}+\frac{1}{\pi}\sum_{n=4}^{\infty}\frac{T_{n-1}}{\pi^{n-1}}+\frac{1}{\pi^2}\sum_{n=4}^{\infty}\frac{T_{n-2}}{\pi^{n-2}}+\frac{1}{\pi^3}\sum_{n=4}^{\infty}\frac{T_{n-3}}{\pi^{n-3}} &(15)\\
&=\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{2}{\pi^3}+\frac{1}{\pi}\left(S-\frac{1}{\pi}-\frac{1}{\pi^2}\right)+\frac{1}{\pi^2}\left(S-\frac{1}{\pi}\right)+\frac{1}{\pi^3}S &(16)\\
&=\frac{1}{\pi}+S\left(\frac{1}{\pi}+\frac{1}{\pi^2}+\frac{1}{\pi^3}\right) &(17)
\end{aligned}$$

即

$$S=\frac{\pi^2}{\pi^3-\pi^2-\pi-1}$$

解法2：注意

$$\begin{aligned}
&(1-x-x^2-x^3)\sum_{n=1}^{\infty}T_nx^{n-1} &(18)\\
&=T_1+x(T_2-T_1)+x^2(T_3-T_2-T_1)+\sum_{n=1}^{\infty}x^{n+2}(T_{n+3}-T_{n+2}-T_{n+1}-T_n) &(19)\\
&=T_1+x(1-1)+x^2(2-1-1)+\sum_{n=1}^{\infty}x^{n+2}\cdot 0 &(20)\\
&=1 &(21)
\end{aligned}$$

即

$$\sum_{n=1}^{\infty}T_nx^n=\frac{x}{1-x-x^2-x^3}$$

所以

$$\sum_{n=1}^{\infty}\frac{T_n}{\pi^n}=\frac{\pi^2}{\pi^3-\pi^2-\pi-1}$$

例题6：设正数列$\{a_n\}$满足$\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{n^2\cdot a_n}=b$,求证：

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1}-\frac{\sqrt[n]{a_n}}{n}\right)=\frac{b}{e^2}$$

证明:令

$$z_n = \frac{a_n}{n^{2n}}$$

则

$$\frac{z_{n+1}}{z_n} = \frac{a_{n+1}}{n^2 a_n}\left[\left(1+\frac{1}{n}\right)^n\right]^{-2}\left(1+\frac{1}{n}\right)^{-2} \longrightarrow be^{-2}$$

则有熟悉结论有

$$\lim_{n\to\infty}\sqrt[n]{z_n} = be^{-2}$$

$$\Longrightarrow \left(\frac{(n+1)(z_{n+1})^{\frac{1}{n+1}}}{n(z_n)^{\frac{1}{n}}}\right)^n = \left(1+\frac{1}{n}\right)^n \frac{z_{n+1}}{z_n}(z_n)^{-\frac{1}{n+1}} \longrightarrow e$$

注意恒等式

$$\frac{\sqrt[n+1]{a_{n+1}}}{n+1} - \frac{\sqrt[n]{a_n}}{n} = (z_n)^{\frac{1}{n}}\left(\frac{\frac{(n+1)(z_{n+1})^{\frac{1}{n+1}}}{n(z_n)^{\frac{1}{n}}}-1}{\ln\left(\frac{(n+1)(z_{n+1})^{\frac{1}{n+1}}}{n(z_n)^{\frac{1}{n}}}\right)}\ln\left(\frac{(n+1)(z_{n+1})^{\frac{1}{n+1}}}{n(z_n)^{\frac{1}{n}}}\right)^n\right) \longrightarrow be^{-2}$$

习题：若正数列$\{a_n\}$满足$\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{n^2\cdot a_n}=b$,求

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{n+1} - 2\frac{\sqrt[n]{a_n}}{n} + \frac{\sqrt[n-1]{a_{n-1}}}{n-1}\right)$$

例题7:设数列$\{x_n\}$满足$\lim\limits_{n\to\infty}\dfrac{x_{n+1}-x_n}{n}=2013$,求证：

$$\lim_{n\to\infty}\left(\frac{x_{n+1}}{\sqrt[n+1]{(n+1)!}} - \frac{x_n}{\sqrt[n]{n!}}\right) = \frac{2013e}{2}$$

设数列 $\{a_n\}$ 满足 $\lim\limits_{n\to\infty}\dfrac{x_{n+1}-x_n}{n}=a$，求证：

$$\lim_{n\to\infty}\left(\frac{x_{n+1}}{\sqrt[n+1]{(n+1)!}} - \frac{x_n}{\sqrt[n]{n!}}\right) = \frac{ae}{2}$$

证：先证明如下引理

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) = \frac{1}{e}$$

由

$$\left(1+\frac{1}{n}\right)^n < e < \left(1+\frac{1}{n}\right)^{n+1}$$

易得

$$\left(\frac{n+1}{e}\right)^n < n! < e\left(\frac{n+1}{e}\right)^{n+1}$$

令

$$
\begin{aligned}
a_n &= \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \\
&= \sqrt[n]{n!}\left(\left(\frac{\sqrt[n]{(n+1)!}}{\sqrt[n+1]{n!}}\right)^{\frac{1}{n(n+1)}} - 1\right)
\end{aligned}
$$

则

$$
\frac{n+1}{e}\left(\left(\frac{e^n}{n+1}\right)^{\frac{1}{n(n+1)}} - 1\right) < a_n < \frac{n+1}{e}\sqrt[n]{n+1}\left((e^n)^{\frac{1}{n(n+1)}} - 1\right)
$$

一方面

$$
\begin{aligned}
\frac{n+1}{e}\sqrt[n]{n+1}\left((e^n)^{\frac{1}{n(n+1)}} - 1\right) &= \frac{\sqrt[n]{n+1}}{e}(n+1)\left(e^{\frac{1}{n+1}} - 1\right) \\
&\sim \frac{1}{e}\,(n \to \infty)
\end{aligned}
$$

另一方面

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{n+1}{e}\sqrt[n]{n+1}\left((e^n)^{\frac{1}{n(n+1)}} - 1\right) \\
=&\lim_{n\to\infty}\frac{1}{e}(n+1)\left(e^{\frac{1}{n+1}} - e^{\frac{\ln(n+1)}{n(n+1)}}\right) \\
=&\lim_{n\to\infty}\frac{1}{e}(n+1)\left(\left(e^{\frac{1}{n+1}} - 1\right) - \left(e^{\frac{\ln(n+1)}{n(n+1)}} - 1\right)\right) \\
=&\frac{1}{e}
\end{aligned}
$$

从而引理得证

由 $Stolz$ 定理可得

$$
\lim_{n\to\infty}\frac{x_{n+1}}{\sum\limits_{k=1}^{n+1}k} = \lim_{n\to\infty}\frac{x_{n+1}-x_n}{n+1} = a
$$

即

$$
\lim_{n\to\infty}\frac{x_n}{n(n+1)} = \frac{a}{2}
$$

又依题意有

$$
x_{n+1} = x_n + an + o(n)
$$

故

$$
\begin{aligned}
&\frac{x_{n+1}}{\sqrt[n+1]{(n+1)!}} - \frac{x_n}{\sqrt[n]{n!}} \\
=&\frac{-\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right)x_n + an\sqrt[n]{n!} + \sqrt[n]{n!}\,o(n)}{\frac{n}{e}\cdot\frac{n+1}{e}} \\
\to&-\frac{ae}{2} + ae = \frac{ae}{2}\,(n \to \infty)
\end{aligned}
$$

例题8：设$f_1(x)=x,f_2(x)=x^x,f_3(x)=x^{x^x},\cdots,f_n(x)=x^{f_{n-1}(x)}$,求极限

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}$$

解：注意到

$$f_n(x)=e^{f_{n-1}(x)\ln x}$$

故

$$f_n(x)-f_{n-1}(x)=e^{f_{n-1}(x)\ln x}-e^{f_{n-2}(x)\ln x}=\ln x(f_{n-1}(x)-f_{n-2}(x))e^c,c\in(f_{n-1}(x)\ln x,f_{n-2}(x)\ln x)$$

注意到

$$\lim_{x\to 1}f_{n-1}(x)\ln x=\lim_{x\to 1}f_{n-2}(x)\ln x=0,\lim_{x\to 1}\frac{\ln x}{1-x}=-1$$

所以

$$I_n=\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=-\lim_{x\to 1}\frac{f_{n-1}(x)-f_{n-2}(x)}{(1-x)^{n-1}}=-I_{n-1}=\cdots=(-1)^{n-1}$$

又

$$\lim_{x\to 1}\frac{f_n(x)-f_{n-1}(x)}{(1-x)^n}=(-1)^n$$

注:

$$\lim_{x\to 1}f_n(x)=1$$

例9:设$0<a<b$,求积分

$$\int_a^b\frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}dx$$

解：令

$$A=\int_a^b\frac{\sqrt[n]{x-a}(1+\sqrt[n]{b-x})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}dx$$

$$B=\int_a^b\frac{\sqrt[n]{b-x}(1+\sqrt[n]{x-a})}{\sqrt[n]{x-a}+2\sqrt[n]{-x^2+(a+b)x-ab}+\sqrt[n]{b-x}}dx$$

显然

$$A+B=b-a$$

其次，利用我们常用结论

$$\int_a^b f(x)dx=\int_a^b f(a+b-x)dx$$

得到

$$A=B$$

故原积分

$$A=\frac{b-a}{2}$$

例题10：求积分：

$$\int_0^\infty \frac{1}{(x^4+(1+\sqrt{2})x^2+1)(x^{100}-x^{98}+\cdots+1)}dx$$

解：设

$$I=\int_0^\infty \frac{1}{(x^4+(1+\sqrt{2})x^2+1)(x^{100}-x^{98}+\cdots+1)}dx$$

令$x=\frac{1}{y}$ 则

$$I=\int_0^\infty \frac{x^{102}}{(x^4+(1+\sqrt{2})x^2+1)(x^{100}-x^{98}+\cdots+1)}dx$$

注意

$$x^{100}-x^{98}+\cdots+1=\frac{1+x^{102}}{1+x^2}$$

所以

$$\frac{1}{2}I=\int_0^\infty \frac{1+x^2}{x^4+(1+\sqrt{2})x^2+1}dx$$

即

$$I=2\int_0^\infty \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+3+\sqrt{2}}dx=\frac{\pi}{2(1+\sqrt{2})}$$

例题11：求所有的连续可导函数$f:[0,1]\longrightarrow(0,\infty)$,满足$f(1)=ef(0)$,且

$$\int_0^1 \frac{dx}{(f(x))^2}+\int_0^1 (f'(x))^2dx\leq 2$$

解：注意

$$\begin{aligned}
0 &\leq \int_0^1\left(f'(x)-\frac{1}{f(x)}\right)^2dx=\int_0^1(f'(x))^2dx-2\int_0^1\frac{f'(x)}{f(x)}dx+\int_0^1\frac{dx}{(f(x))^2} && (22)\\
&=\left(\int_0^1(f'(x))^2dx+\int_0^1\frac{dx}{(f(x))^2}dx\right)-2\int_0^1(\ln f(x))'dx && (23)\\
&=\left(\int_0^1(f'(x))^2dx+\int_0^1\frac{dx}{(f(x))^2}dx\right)-2\ln\frac{f(1)}{f(0)} && (24)\\
&\leq 0 && (25)
\end{aligned}$$

所以

$$f(x)f'(x)=1$$

$$\Longrightarrow f(x)=\sqrt{2x+C},C>0$$

由于

$$\frac{f(1)}{f(0)}=e\Longrightarrow C=\frac{2}{e^2-1}$$

故

$$f(x)=\sqrt{2x+\frac{2}{e^2-1}}$$

例题12：设$f,g:[0,1]\longrightarrow(0,+\infty)$是连续的,且$f,\dfrac{g}{f}$递增的,求证：

$$\int_0^1\left(\frac{\int_0^x f(t)dt}{\int_0^x g(t)dt}\right)dx\le 2\int_0^1\frac{f(x)}{g(x)}dx$$

并说明右边系数2是最佳的.

证明：由切比雪夫不等式有

$$\left(\frac{1}{x}\int_0^x f(t)dt\right)\left(\frac{1}{x}\int_0^x\frac{g(t)}{f(t)}dt\right)\le\frac{1}{x}\int_0^x g(t)dt$$

即

$$\frac{\int_0^x f(t)dt}{\int_0^x g(t)dt}\le\frac{x}{\int_0^x\frac{g(t)}{f(t)}dt}$$

另外由柯西不等式有

$$\left(\int_0^x\frac{g(t)}{f(t)}dt\right)\left(\int_0^x\frac{t^2f(t)}{g(t)}dt\right)\ge\left(\int_0^x tdt\right)^2=\frac{x^4}{4}$$

即

$$\frac{1}{\int_0^x\frac{g(t)}{f(t)}dt}\le\frac{4}{x^4}\int_0^x\frac{t^2f(t)}{g(t)}dt$$

所以有

$$\frac{\int_0^x f(t)dt}{\int_0^x g(t)dt}\le\frac{4}{x^3}\int_0^x\frac{t^2f(t)}{g(t)}dt$$

故有

$$\begin{aligned}\int_0^1\frac{\int_0^x f(t)dt}{\int_0^x g(t)dt}dx&\le\int_0^1\left(\int_0^x\frac{4t^2f(t)}{x^3g(t)}dt\right)dx=\int_0^1\left(\int_t^1\frac{4t^2f(t)}{x^3g(t)}dx\right)dt &(26)\\&=\int_0^1\frac{4t^2f(t)}{g(t)}\left(\int_t^1\frac{dx}{x^3}\right)dt &(27)\\&=2\int_0^1\frac{f(t)}{g(t)}(1-t^2)dt &(28)\\&\le 2\int_0^1\frac{f(t)}{g(t)}dt &(29)\end{aligned}$$

另外一方面:我们令

$$f(t)=1,g(t)=t+\varepsilon,\varepsilon>0$$

则

$$\int_0^1 \frac{\displaystyle\int_0^x f(t)dt}{\displaystyle\int_0^x g(t)dt}dx = \int_0^1 \frac{x}{\dfrac{1}{2}x^2 + \varepsilon x}dx = 2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon$$

$$\int_0^1 \frac{f(t)}{g(t)}dt = \int_0^1 \frac{dt}{t+\varepsilon} = \ln(1+\varepsilon) - \ln\varepsilon$$

所以

$$\lim_{\varepsilon\to 0} \frac{2\ln(1+2\varepsilon) - 2\ln 2 - 2\ln\varepsilon}{\ln(1+\varepsilon) - \ln\varepsilon} = 2\lim_{\varepsilon\to 0} \frac{-\dfrac{\ln(1+2\varepsilon)}{\ln\varepsilon} + \dfrac{\ln 2}{\ln\varepsilon} + 1}{-\dfrac{\ln(1+\varepsilon)}{\ln\varepsilon} + 1} = 2$$