第十届清疏竞赛班非数学类 20

积分换元法

重积分换元法：

$$\omega:\begin{cases} x = x\left(u,v\right) \\ y = y\left(u,v\right) \end{cases}$$

$$\iint_D f\left(x,y\right)dxdy = \iint_{\omega^{-1}\left(D\right)} f\left(x\left(u,v\right),y\left(u,v\right)\right)Jdudv$$

$$J = \left|\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}\right| = \left|\det\begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix}\right|$$

区域变换时视为边界对应边界

$$\left|\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}\right|\cdot\left|\frac{\partial\left(u,v\right)}{\partial\left(x,y\right)}\right| = 1$$

对于多元积分,也是类似的

二重例：

D:$2 \le \dfrac{x}{x^2+y^2},\dfrac{y}{x^2+y^2} \le 4$, 计算$\displaystyle\iint_D \frac{1}{xy}dxdy$

计算：

$$\begin{cases} u = \dfrac{x}{x^2+y^2} \\ v = \dfrac{y}{x^2+y^2} \end{cases},\left|\frac{\partial\left(u,v\right)}{\partial\left(x,y\right)}\right| = \left\|\begin{matrix} \dfrac{y^2-x^2}{\left(x^2+y^2\right)^2} & -\dfrac{2xy}{\left(x^2+y^2\right)^2} \\ -\dfrac{2xy}{\left(x^2+y^2\right)^2} & \dfrac{x^2-y^2}{\left(x^2+y^2\right)^2} \end{matrix}\right\|$$

$$= \left|-\frac{\left(y^2-x^2\right)^2}{\left(x^2+y^2\right)^4} - \frac{4x^2y^2}{\left(x^2+y^2\right)^4}\right| = \frac{1}{\left(x^2+y^2\right)^2},\left|\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}\right| = \left(x^2+y^2\right)^2$$

$$\iint_D \frac{1}{xy}dxdy = \int_2^4\int_2^4 \frac{\left(x^2+y^2\right)^2}{xy}dudv = \int_2^4\int_2^4 \frac{1}{uv}dudv = \ln^2 2.$$

三重例：$z=y^2, z=4y^2\left(y>0\right), z=x, z=2x, z=2$围成$\Omega$

计算$\iiint\limits_{\Omega}\frac{z\sqrt{z}}{y^3}\cos\frac{z}{y^2}dxdydz$

$$\begin{cases} u=\frac{z}{y^2} \\ v=\frac{z}{x} \\ w=z \end{cases} \Rightarrow \begin{cases} x=\frac{z}{v} \\ y=\sqrt{\frac{z}{u}} \\ z=w \end{cases}$$

$z=y^2, z=4y^2$对应着$u=1, u=4$,

$z=x, z=2x$对应着$v=1, v=2$,

$z=2$对应着$w=2, z=0$对应着$w=0$

$$\left\|\begin{matrix} 0 & -\frac{w}{v^2} & \frac{1}{v} \\ -\frac{1}{2}\sqrt{\frac{w}{u^3}} & 0 & \frac{1}{2}\sqrt{\frac{1}{uw}} \\ 0 & 0 & 1 \end{matrix}\right\| = \frac{w}{2v^2}\left(\frac{w}{u}\right)^{\frac{3}{2}}$$

所以

$$\iiint\limits_{\Omega}\frac{z\sqrt{z}}{y^3}\cos\frac{z}{y^2}dxdydz = \iiint\limits_{\Omega}u^{\frac{3}{2}}\cos u\cdot\frac{w}{2v^2}\left(\frac{w}{u}\right)^{\frac{3}{2}}dudvdw$$

$$=\frac{1}{2}\int_1^4\cos udu\int_1^2\frac{1}{v^2}dv\int_0^2 w^{\frac{5}{2}}dw=\frac{4\sqrt{2}}{7}\left(\sin 4-\sin 1\right)$$

凸集：

设$A \subset \mathbb{R}^n$, 如果A是凸集, 那么对$x, y \in A$, 且$\lambda \in [0,1]$都有
$\lambda x + (1-\lambda) y \in A$.

命题：设$A \subset \mathbb{R}^n$，那么包含A的最小凸集(凸包)为

$$coA = \left\{ \sum_{j=1}^{s} t_j x_j : s \in \mathbb{N}, \sum_{j=1}^{n} t_j = 1, t_j \geq 0, x_j \in A，j = 1, 2, \cdots, s \right\}$$

证明：显然设M是包含A的最小凸集, 那么$M \supset coA$，
可以自行归纳证明coA的确是凸集, 那么$coA \supset M$,
因此$CoA = M$.

抽象二重例子：

计算$\iint\limits_D xdxdy, D$是以$(x_i, y_i), i = 1, 2, 3$为顶点三角形.

证明：

因为三角形是三个顶点的凸包, 因此对$(x, y) \in D$, 可以有

$$\begin{cases} x = t_1 x_1 + t_2 x_2 + (1 - t_1 - t_2) x_3 \\ y = t_1 y_1 + t_2 y_2 + (1 - t_1 - t_2) y_3 \end{cases}, t_1, t_2 \geq 0, t_1 + t_2 \leq 1,$$

$$J = \left\| \begin{matrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{matrix} \right\|$$

因此$\iint\limits_D xdxdy = J \iint\limits_D \left(t_1 x_1 + t_2 x_2 + (1 - t_1 - t_2) x_3 \right) dt_1 dt_2$

$$= J \int_0^1 \int_0^{1-t_1} \left(t_1 x_1 + t_2 x_2 + (1 - t_1 - t_2) x_3 \right) dt_2 = \frac{x_1 + x_2 + x_3}{6} J.$$

抽象三重例子：

设$H\left(x\right)=\sum_{1\le i,j\le 3}a_{ij}x_ix_j, A=\left(a_{ij}\right)$是3阶正定矩阵, 计算

$$\iiint_{H(x)\le 1}e^{\sqrt{H(x)}}dx_1dx_2dx_3$$

证明：

设$T=\left(t_{ij}\right)$是正交矩阵,使得:

$$T^TAT=\begin{pmatrix}\lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3\end{pmatrix},\lambda_i>0,i=1,2,3.$$

则令$x=Ty$, 于是$x_i=\sum_{k=1}^{n}t_{ik}y_k, \left(J\right)_{(i,j)}=\dfrac{\partial x_i}{\partial y_j}=t_{ij}$

所以雅可比矩阵就是T，因此$\left|J\right|=\left\|T\right\|=1$

$$\text{因此}\iiint_{H(x)\le 1}e^{\sqrt{H(x)}}dx_1dx_2dx_3=\iiint_{\sum_{i=1}^{3}\lambda_iy_i^2\le 1}e^{\sqrt{\sum_{i=1}^{3}\lambda_iy_i^2}}dy_1dy_2dy_3$$

$$=\frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}}\iiint_{\sum_{i=1}^{3}z_i^2\le 1}e^{\sqrt{\sum_{i=1}^{3}z_i^2}}dz_1dz_2dz_3$$

$$=\frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^1 r^2e^rdr$$

$$=\frac{4\pi\left(e-2\right)}{\sqrt{\lambda_1\lambda_2\lambda_3}}.$$

对称性的本质是换元法：

$$\iiint\limits_{x^2+y^2+2z^2\le x+y+2z}\left(x^2+y^2+z^2\right)dV.$$

正常做题：

$$\iiint\limits_{x^2+y^2+2z^2\le x+y+2z}\left(x^2+y^2+z^2\right)dV$$

$$=\iiint\limits_{\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2+2\left(z-\frac{1}{2}\right)^2\le 1}\left(x^2+y^2+z^2\right)dV$$

$$=\iiint\limits_{x^2+y^2+2z^2\le 1}\left(x+\frac{1}{2}\right)^2+\left(y+\frac{1}{2}\right)^2+\left(z+\frac{1}{2}\right)^2dV$$

$$=\iiint\limits_{x^2+y^2+2z^2\le 1}x^2+y^2+z^2+x+y+z+\frac{3}{4}dV$$

$$=\iiint\limits_{x^2+y^2+2z^2\le 1}x^2+y^2+z^2+\frac{3}{4}dV$$

$$=\frac{1}{\sqrt{2}}\iiint\limits_{x^2+y^2+z^2\le 1}\frac{3}{4}dV+\frac{1}{\sqrt{2}}\iiint\limits_{x^2+y^2+z^2\le 1}x^2+y^2+\frac{z^2}{2}dV$$

$$=\frac{3}{4\sqrt{2}}\frac{4}{3}\pi+\frac{5}{6\sqrt{2}}\iiint\limits_{x^2+y^2+z^2\le 1}x^2+y^2+z^2dV$$

$$=\frac{\pi}{\sqrt{2}}+\frac{5}{6\sqrt{2}}\int_0^{2\pi}d\theta\int_0^{\pi}\sin\varphi d\varphi\int_0^1 r^4dr$$

$$=\frac{\pi}{\sqrt{2}}+\frac{4\pi}{6\sqrt{2}}=\frac{5\pi}{3\sqrt{2}}.$$

其中 $\iiint\limits_{x^2+y^2+2z^2\le 1}xdV=0$来自对称性, 证明来自换元法：

$$\iiint\limits_{x^2+y^2+2z^2\le 1}xdV=\iiint\limits_{(-x)^2+y^2+2z^2\le 1}-xdV=-\iiint\limits_{x^2+y^2+2z^2\le 1}xdV.$$

如下公式记忆即可：

第一类曲线(曲面)积分换元法：

第一类曲线积分换元法.

$$D_F=\begin{pmatrix}F_x\\F_y\end{pmatrix},\left\|D_F\right\|=\sqrt{F_x^2+F_y^2}$$

$$\int_{F(x,y)=0}f(x,y)ds\overset{\begin{cases}x=x(u,v)\\y=y(u,v)\end{cases}}{=}\int_{F(x(u,v),y(u,v))=0}f\left(x(u,v),y(u,v)\right)\frac{|J|\left\|D_F\right\|}{\left\|J\cdot D_F\right\|}ds$$

第一类曲面积分换元法.

$$D_F=\begin{pmatrix}F_x\\F_y\\F_z\end{pmatrix},\left\|D_F\right\|=\sqrt{F_x^2+F_y^2+F_z^2}$$

$$\int_{F(x,y,z)=0}f(x,y,z)dS$$

$$\overset{\begin{cases}x=x(u,v,w)\\y=y(u,v,w)\\z=z(u,v,w)\end{cases}}{=}\int_{F(x(u,v,w),y(u,v,w))=0}f\left(x(u,v,w),y(u,v,w)\right)\frac{|J|\left\|D_F\right\|}{\left\|J\cdot D_F\right\|}dS$$

使用情景,以曲线积分为例:

$$(1):\begin{cases} x = a + ru \\ y = b + rv \end{cases} \Rightarrow J = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = r^2, D_F = \begin{pmatrix} F_x \\ F_y \end{pmatrix}$$

$$\|J \cdot D_F\| = \left\|\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} F_x \\ F_y \end{pmatrix}\right\| = \left\|\begin{pmatrix} rF_x \\ rF_y \end{pmatrix}\right\| = r\left\|\begin{pmatrix} F_x \\ F_y \end{pmatrix}\right\|$$

$$\Rightarrow \frac{|J|\|D_F\|}{\|J \cdot D_F\|} = r$$

如果是$\begin{cases} x = a + r_1 u \\ y = b + r_2 v \end{cases} \Rightarrow \dfrac{|J|\|D_F\|}{\|J \cdot D_F\|} = \dfrac{r_1 r_2 \sqrt{F_x^2 + F_y^2}}{\sqrt{r_1^2 F_x^2 + r_2^2 F_y^2}}$,这一般用不上

可以看到对重积分,(1)这种换元,要乘的是r^2,但是对曲线积分却是r

(2):重积分和曲线积分正交变换不变

正交变换即J是正交矩阵,$|\det J| = 1$,

对曲线积分,$\|J \cdot D_F\| = \sqrt{D_F^T J^T J \cdot D_F} = \|D_F\| \Rightarrow \dfrac{|J|\|D_F\|}{\|J \cdot D_F\|} = 1$

故正交变换不变.(旋转,平移,翻转).

例子：

计算$\iint\limits_{\Omega}(x+y+z)dS$,其中$\Omega$：$x^2+y^2+z^2=1, x+y+z\geq 0.$

证明：

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = T \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

,T是正交矩阵，那么

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = T^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} & & \\ & & \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\iint\limits_{\Omega}(x+y+z)dS = \sqrt{3}\iint\limits_{\Omega'} wdS, \Omega': w\geq 0, u^2+v^2+w^2=1$$

这是因为$u^2+v^2+w^2 = (u,v,w)I_n \begin{pmatrix} u \\ v \\ w \end{pmatrix}$

$$= (x,y,z)TI_nT^T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = (x,y,z) \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2+y^2+z^2 = 1.$$

因此

$$\iint\limits_{\Omega}(x+y+z)dS = \sqrt{3}\iint\limits_{\text{上半球面}} zdS = \sqrt{3}\iint\limits_{x^2+y^2+z^2\leq 1} z\sqrt{1+\frac{x^2+y^2}{z^2}}dxdy$$

$$= \sqrt{3}\iint\limits_{x^2+y^2\leq 1} 1dxdy = \sqrt{3}\pi.$$

第一类曲面积分的参数方程计算方法

$$\iint\limits_{F(x,y,z)=0} f(x,y,z)\,dS \overset{\begin{cases}x=x(u,v)\\y=y(u,v)\\z=z(u,v)\end{cases}}{=} \iint\limits_{u,v\text{范围}} f\left(x(u,v),y(u,v),z(u,v)\right)\sqrt{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2+\left|\frac{\partial(}{\partial(}\right.}$$

$$\sqrt{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2+\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2}=\sqrt{EG-F^2}$$

$$E=\sum_{cyc}x_u^2,F=\sum_{cyc}x_u x_v,G=\sum_{cyc}x_v^2$$

对于球坐标$\begin{cases}x=x_0+r\sin\psi\cos\theta\\y=y_0+r\sin\psi\sin\theta\\z=z_0+r\cos\psi\end{cases},\psi\in[0,\pi],\theta\in[0,2\pi],$

此时$\sqrt{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|^2+\left|\frac{\partial(y,z)}{\partial(u,v)}\right|^2+\left|\frac{\partial(x,z)}{\partial(u,v)}\right|^2}=r^2\sin\psi$

例子：设f连续,化简$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}f(1-\sin\theta\cos\varphi)\sin\varphi\cos\varphi d\theta d\varphi$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}f(1-\sin\theta\cos\varphi)\sin\varphi\cos\varphi d\theta d\varphi$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}f(1-\cos\theta\sin\varphi)\cos\varphi\sin\varphi d\theta d\varphi$$

$$=\iint\limits_{\text{第一象限单位球面}}f(1-x)z\,dS$$

$$=\iint\limits_{x^2+y^2\le 1,\text{第一象限}}f(1-x)\,dxdy$$

$$=\int_0^1 dx\int_0^{\sqrt{1-x^2}}f(1-x)\,dy$$

$$=\int_0^1\sqrt{1-x^2}\cdot f(1-x)\,dx$$

$$=\int_0^1\sqrt{2x-x^2}\cdot f(x)\,dx$$

第二类曲线积分换元法:

例子: $\lim\limits_{r\to+\infty}\oint_{x^2+xy+y^2=r^2}\frac{ydx-xdy}{\left(x^2+y^2\right)^a}$(逆时针)

重点掌握：第二类曲线积分换元法, 对应的可以推广到空间和第二类曲面积分

做换元$x=x\left(u,v\right),y=y\left(u,v\right)$, 那么$\begin{cases}\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}>0,\text{方向不变}\\\frac{\partial\left(x,y\right)}{\partial\left(u,v\right)}<0,\text{方向改变}\end{cases}$.

$$\lim_{r\to+\infty}\oint_{x^2+xy+y^2=r^2}\frac{ydx-xdy}{\left(x^2+y^2\right)^a}$$

$$x^2+xy+y^2=\left(x,y\right)\begin{pmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix},$$

则$x^2+xy+y^2=\left(u,v\right)\begin{pmatrix}\frac{3}{2} & 0\\ 0 & \frac{1}{2}\end{pmatrix}\begin{pmatrix}u\\ v\end{pmatrix}=\frac{3}{2}u^2+\frac{v^2}{2}$

$$J=\frac{1}{2}\begin{vmatrix}1 & -1\\ 1 & 1\end{vmatrix}=1>0.$$

$$\oint_{x^2+xy+y^2=r^2}\frac{ydx-xdy}{\left(x^2+y^2\right)^a}=\oint_{\frac{3}{2}u^2+\frac{v^2}{2}=r^2}\frac{\frac{u+v}{\sqrt{2}}d\frac{u-v}{\sqrt{2}}-\frac{u-v}{\sqrt{2}}d\frac{u+v}{\sqrt{2}}}{\left(u^2+v^2\right)^a}$$

$$=\frac{1}{2}\oint_{\frac{3}{2}u^2+\frac{v^2}{2}=r^2}\frac{\left(u+v\right)d\left(u-v\right)-\left(u-v\right)d\left(u+v\right)}{\left(u^2+v^2\right)^a}$$

$$=\oint_{\frac{3}{2}u^2+\frac{v^2}{2}=r^2}\frac{vdu-udv}{\left(u^2+v^2\right)^a}$$

$$\overset{\begin{cases}u=\sqrt{\frac{2}{3}}r\cos\theta\\ v=\sqrt{2}r\sin\theta\end{cases}}{=}\int_0^{2\pi}\frac{\sqrt{2}r\sin\theta d\sqrt{\frac{2}{3}}r\cos\theta-\sqrt{\frac{2}{3}}r\cos\theta d\sqrt{2}r\sin\theta}{\left(2r^2\sin^2\theta+\frac{2}{3}r^2\cos^2\theta\right)^a}$$

$$=\frac{1}{r^{2a-2}}\int_0^{2\pi}\frac{\sqrt{2}\sin\theta d\sqrt{\frac{2}{3}}\cos\theta-\sqrt{\frac{2}{3}}\cos\theta d\sqrt{2}\sin\theta}{\left(2\sin^2\theta+\frac{2}{3}\cos^2\theta\right)^a}$$

故原极限在$a=1$存在, 此时极限值为

$$\int_0^{2\pi}\frac{\sqrt{2}\sin\theta d\sqrt{\frac{2}{3}}\cos\theta-\sqrt{\frac{2}{3}}\cos\theta d\sqrt{2}\sin\theta}{2\sin^2\theta+\frac{2}{3}\cos^2\theta}$$

$$=\frac{2}{\sqrt{3}}\int_0^{2\pi}\frac{\sin\theta d\cos\theta-\cos\theta d\sin\theta}{2\sin^2\theta+\frac{2}{3}\cos^2\theta}$$

$$=-\frac{2}{\sqrt{3}}\int_0^{2\pi}\frac{d\theta}{2\sin^2\theta+\frac{2}{3}\cos^2\theta}$$

$$=-\frac{2}{\sqrt{3}}\int_0^{2\pi}\frac{\frac{1}{\cos^2\theta}d\theta}{2\tan^2\theta+\frac{2}{3}}$$

$$=-\frac{4}{\sqrt{3}}\int_0^{\pi}\frac{\frac{1}{\cos^2\theta}d\theta}{2\tan^2\theta+\frac{2}{3}}$$

$$=-\frac{4}{\sqrt{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\frac{1}{\cos^2\theta}d\theta}{2\tan^2\theta+\frac{2}{3}}$$

$$=-\frac{4}{\sqrt{3}}\int_0^{\frac{\pi}{2}}\frac{\frac{1}{\cos^2\theta}d\theta}{\tan^2\theta+\frac{1}{3}}$$

$$=-\frac{4}{\sqrt{3}}\int_0^{\frac{\pi}{2}}\frac{d\tan\theta}{\tan^2\theta+\frac{1}{3}}$$

$$=-\frac{4}{\sqrt{3}}\int_0^{\infty}\frac{dx}{x^2+\frac{1}{3}}=-2\pi.$$

延续之前的方法,可以导出$Possion$公式

$\left(\text{类似的对重积分和第一类曲面积分都对}\right)$：

$$\oint_{x^2+y^2=r^2} f\left(ax+by\right)ds = \oint_{u^2+v^2=r^2} f\left(\sqrt{a^2+b^2}v\right)ds.$$

注意$possion$公式无需记忆,而且可能有变形,

本质上就是换元法.

记忆$\left(\text{无需证明}\right)$余面积公式$\left(\text{类似的,对二重积分也正确}\right)$:

$$\int_{\left(x-x_0\right)^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2\le r^2} f\left(x,y,z\right)dV$$

$$= \int_0^r dt \int_{\left(x-x_0\right)^2+\left(y-y_0\right)^2+\left(z-z_0\right)^2=t^2} f\left(x,y,z\right)dS.$$

例子：

$$\iiint\limits_{x^2+y^2+z^2\le 1} x^2+y^2+z^2 dV$$

$$= \int_0^1 dt \int_{x^2+y^2+z^2=t^2} x^2+y^2+z^2 dS$$

$$= \int_0^1 dt \int_{x^2+y^2+z^2=t^2} t^2 dS$$

$$= 4\pi \int_0^1 t^4 dt$$

$$= \frac{4\pi}{5}.$$