第十届清疏竞赛班非数学类 18

多元函数

设$f\left(x,y\right)$在区域$D\left(\text{连通的}\right)$偏导数$f_x=0, f_y=0$, 证明f在D为常数.

证明：

第一步：

若D为凸区域,即对任何$x,y\in D$, 都有$\lambda x+\left(1-\lambda\right)y\in D, \lambda\in\left[0,1\right]$.

对任何$a,b\in D$

考虑$g\left(\lambda\right)=f\left(\left(1-\lambda\right)a+\lambda b\right)=f\left(a+\lambda\left(b-a\right)\right),\lambda\in\left[0,1\right]$

因为f_x,f_y连续,所以f是连续可微，因此$g\in C^1\left[0,1\right]$, 由链式法则, 有

$g'\left(\lambda\right)=0$, 故$g\left(1\right)=f\left(b\right)=g\left(0\right)=f\left(a\right)$, 由$a,b$任意性可知$f$为常数.

$\left(\text{非数兴趣}\right)$第二步：

若D不为凸区域,取定$a\in D$，考虑$C=\left\{x\in D:f\left(x\right)=f\left(a\right)\right\}$

只需证明$C=D$,这等价于只需证C是是非空闭集也是开集.

显然C非空以及C是D的闭子集，再证C是开的.

对$b\in C$,取以b为心的小开球$B\subset D$，B是凸的,由第一步的结果我们知道

$f\left(x\right)=f\left(b\right),\forall x\in B$,所以$B\subset C$，因此$C$是$D$的开子集，然后我们就运用

D是连通的说明了$D=C$.

重极限训练

(1)：$\lim\limits_{(x,y)\to(0,0)}(x+y)\cdot\sin\frac{1}{x}\cdot\sin\frac{1}{y}$

第一种定义（必须去心邻域有定义）：

考虑$\left(0,\frac{1}{n}\right)\to(0,0)$，此时$\left(0+\frac{1}{n}\right)\cdot\sin\frac{1}{0}\cdot\sin\frac{1}{\frac{1}{n}}$不存在.

第二种定义（只考虑去心邻域内有定义的点）

$$\lim_{(x,y)\to(0,0)}\left|(x+y)\cdot\sin\frac{1}{x}\cdot\sin\frac{1}{y}\right|\le\lim_{(x,y)\to(0,0)}\left(|x|+|y|\right)=0.$$

因此$\lim\limits_{(x,y)\to(0,0)}(x+y)\cdot\sin\frac{1}{x}\cdot\sin\frac{1}{y}=0.$

(2)：$\lim\limits_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}$

解：

$x^2y^2+(x-y)^2=0\Leftrightarrow xy=0$且$x=y\Leftrightarrow x=0,y=0.$

$$\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}\overset{y=\ln(1+x)}{=}\lim_{(x,y)\to(0,0)}\frac{1}{1+\left(\frac{1}{x}-\frac{1}{\ln(1+x)}\right)^2}=\frac{1}{1+\frac{1}{4}}$$

$$\lim_{(x,y)\to(0,0)}\frac{x^2y^2}{x^2y^2+(x-y)^2}\overset{y=x}{=}\lim_{(x,y)\to(0,0)}\frac{1}{1+\left(\frac{1}{x}-\frac{1}{x}\right)^2}=1$$

因此不存在

(3)：$\lim\limits_{\substack{x\to+\infty\\y\to+\infty}}\left(\frac{xy}{x^2+y^2}\right)^{x^2}$

$$0<\lim_{\substack{x\to+\infty\\y\to+\infty}}\left(\frac{xy}{x^2+y^2}\right)^{x^2}=\lim_{\substack{x\to+\infty\\y\to+\infty}}e^{x^2\ln\frac{xy}{x^2+y^2}}\le\lim_{\substack{x\to+\infty\\y\to+\infty}}e^{x^2\ln\frac{xy}{2xy}}=0.$$

因此$\lim\limits_{\substack{x\to+\infty\\y\to+\infty}}\left(\frac{xy}{x^2+y^2}\right)^{x^2}=0$

$(4):\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y}{x^4+y^2}$

解：$\lim\limits_{(x,y)\to(0,0)}\dfrac{x^2y}{x^4+y^2}\overset{y=kx^2}{=}\lim\limits_{(x,y)\to(0,0)}\dfrac{kx^4}{x^4+k^2x^4}=\dfrac{k}{1+k^2},k\in\mathbb{Z}$

因此极限不存在.

$(5):\lim\limits_{(x,y)\to(\infty,a)}\left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}}$，$a\in\mathbb{R}$.

解：$\lim\limits_{(x,y)\to(\infty,a)}\left(1+\dfrac{1}{x}\right)^{\frac{x^2}{x+y}}=\lim\limits_{(x,y)\to(\infty,a)}e^{\frac{x^2}{x+y}\ln\left(1+\frac{1}{x}\right)}=\lim\limits_{(x,y)\to(\infty,a)}e^{\frac{x}{x+y}}=e.$

$(6):\lim\limits_{(x,y)\to(0,0)}\dfrac{x^3y+xy^4+x^2y}{x+y}$

解：第一种定义不存在, 考虑$(x,-x)$即可.

考虑第二种定义：

考虑$y=u(x)-x$

$$\lim_{(x,y)\to(0,0)}\frac{x^3\left(u(x)-x\right)+x\left(u(x)-x\right)^4+x^2\left(u(x)-x\right)}{u(x)}$$

$$=\lim_{(x,y)\to(0,0)}\frac{x(u-x)\left(u^3-3u^2x+3ux^2-x^3+x^2+x\right)}{u}$$

把u取成x^3,此时

$$\lim_{(x,y)\to(0,0)}\frac{x(u-x)\left(u^3-3u^2x+3ux^2-x^3+x^2+x\right)}{u}$$

$$=\lim_{(x,y)\to(0,0)}(x-1)(x+1)^2\left(x^7-x^6-2x^5+2x^4+x^3-x^2+1\right)$$

$$=-1$$

$\lim\limits_{(x,0)\to(0,0)}\dfrac{x^3y+xy^4+x^2y}{x+y}=\lim\limits_{(x,0)\to(0,0)}0=0$, 故极限不存在

$\left(7\right)$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2+y^4}$

解：

$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$, 此时 $\left|\dfrac{xy^2}{x^2+y^2+y^4}\right| = r^3 \dfrac{\left|\cos\theta\sin^2\theta\right|}{r^2+r^4\sin^4\theta}$

$= r\dfrac{\left|\cos\theta\sin^2\theta\right|}{1+r^2\sin^4\theta} \le r \to 0,$

因此 $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^2+y^4} = 0.$

也就是说极坐标方法需要放缩到与θ无关才能够证明.

$\left(8\right)$: $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^3-3y^4}{3x^2-2xy+y^2}$

解：$\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^3}{3x^2-2xy+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{2x^3}{2x^2+\left(x-y\right)^2}$

因此 $\lim\limits_{(x,y)\to(0,0)} \left|\dfrac{2x^3}{2x^2+\left(x-y\right)^2}\right| \le \lim\limits_{(x,y)\to(0,0)} \left|\dfrac{2x^3}{2x^2}\right| = \lim\limits_{(x,y)\to(0,0)} \left|x\right| = 0,$

$\lim\limits_{(x,y)\to(0,0)} \dfrac{3y^4}{3x^2-2xy+y^2} = \lim\limits_{(x,y)\to(0,0)} \dfrac{3y^4}{3\left(x-\dfrac{y}{3}\right)^2+\dfrac{2}{3}y^2},$

因此 $\lim\limits_{(x,y)\to(0,0)} \left|\dfrac{3y^4}{3x^2-2xy+y^2}\right| \le \lim\limits_{(x,y)\to(0,0)} \dfrac{3y^4}{\dfrac{2}{3}y^2} = \dfrac{9}{2}\lim\limits_{(x,y)\to(0,0)} y^2 = 0.$

因此 $\lim\limits_{(x,y)\to(0,0)} \dfrac{2x^3-3y^4}{3x^2-2xy+y^2} = 0.$

齐次函数：$f(tx,ty)=t^n f(x,y), n\in\mathbb{N}, x,y\in\mathbb{R}, t>0.$

若$f\in D(\mathbb{R}^2)$,则f是齐n次函数的充分必要条件是$x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf(x,y)$.

证明：若f是齐n次函数,则$\dfrac{d}{dt}f(tx,ty)=\dfrac{d}{dt}t^n f(x,y)$

即$xf_1(tx,ty)+yf_2(tx,ty)=nt^{n-1}f(x,y)$

因此$txf_1(tx,ty)+tyf_2(tx,ty)=nt^n f(x,y)=nf(tx,ty)$

因此有$x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf(x,y)$.

反之若$x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}=nf(x,y)$,对固定的$x,y\in\mathbb{R}$

可以令$g(t)=f(tx,ty), g'(t)=xf_1(tx,ty)+yf_2(tx,ty)$

$tg'(t)=txf_1(tx,ty)+tyf_2(tx,ty)=nf(tx,ty)=ng(t)$

解微分非常就得到$g(t)=Ct^n, C=g(1)=f(x,y)$

此时$f(tx,ty)=f(x,y)\cdot t^n$,完成了证明

设f为n次可微齐次函数,$f(x_0,y_0)=a$,求$\left[x^2f_{xx}+2xyf_{xy}+y^2f_{yy}\right]_{(x_0,y_0)}$

$g(t)=f(tx,ty)=t^n f(x,y)$

$g'(t)=xf_1(tx,ty)+yf_2(tx,ty)=nt^{n-1}f(x,y)$

$g''(t)=x^2f_{11}(tx,ty)+2xyf_{12}(tx,ty)+y^2f_{22}(tx,ty)=n(n-1)t^{n-2}f(x,y)$.

因此$x^2f_{11}(x,y)+2xyf_{12}(x,y)+y^2f_{22}(x,y)=n(n-1)f(x,y)$

因此$\left[x^2f_{xx}+2xyf_{xy}+y^2f_{yy}\right]_{(x_0,y_0)}=n(n-1)a.$

设$u=f\left(r\right),r=\sqrt{x^2+y^2+z^2},f\in D^2\left(0,+\infty\right),\Delta u=0,$ 求$u.$

定义：$\Delta u=\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}$

计算：

$$r_x=\frac{x}{r},r_y=\frac{y}{r},r_z=\frac{z}{r}$$

$$u_x=\frac{x}{r}f'\left(r\right),u_{xx}=\frac{f'\left(r\right)}{r}+x\cdot\frac{x}{r}\cdot\frac{rf''\left(r\right)-f'\left(r\right)}{r^2}$$

$$=\frac{f'\left(r\right)}{r}+\frac{x^2}{r}\cdot\frac{rf''\left(r\right)-f'\left(r\right)}{r^2}$$

$$\Delta u=\frac{3f'\left(r\right)}{r}+\frac{r^2}{r^3}\left(rf''\left(r\right)-f'\left(r\right)\right)$$

$$=\frac{2f'\left(r\right)}{r}+\frac{rf''\left(r\right)}{r}=0$$

故$2f'\left(r\right)+rf''\left(r\right)=0,$ 于是解得$f\left(r\right)=\dfrac{c_1}{r}+c_2$

故$u=\dfrac{c_1}{\sqrt{x^2+y^2+z^2}}+c_2.$

三元函数u,v,w由方程组$\begin{cases} u+v+w=x \\ uv+vw+wu=y \\ uvw=z \end{cases}$确定, 求$u,v,w$的偏导数.

解：

$$\begin{cases} du+dv+dw=dx \\ (v+w)du+(u+w)dv+(v+u)dw=dy \\ vwdu+uwdv+uvdw=dz \end{cases}$$

$$du=\frac{\begin{vmatrix} dx & 1 & 1 \\ dy & u+w & v+u \\ dz & uw & uv \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & v+u \\ vw & uw & uv \end{vmatrix}},dv=\ldots,dw=\ldots,$$

$$=p(u,v,w)dx+q(u,v,w)dy+r(u,v,w)dz$$

即有$\dfrac{\partial u}{\partial x}=p(u,v,w),\dfrac{\partial u}{\partial y}=q(u,v,w),\dfrac{\partial u}{\partial z}=r(u,v,w)$

具体计算自行完成.

函数相关性：
记忆结论：
可微函数u是v的函数的充分必要条件是$\left|\dfrac{\partial\left(u,v\right)}{\partial\left(x,y\right)}\right| = 0,$

这里$\dfrac{\partial\left(u,v\right)}{\partial\left(x,y\right)} \triangleq \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix}.$

可微函数$z = f\left(x,y\right)$是形如$ax + by$的函数充要条件是$b\dfrac{\partial z}{\partial x} = a\dfrac{\partial z}{\partial y}, \left(ab \neq 0\right)$

证明：
可以直接套$\left|\begin{pmatrix} z_x & z_y \\ a & b \end{pmatrix}\right| = bz_x - az_y = 0.$

也可以如下证明
$\left(\text{要注意另外一个}v\text{要保证}\left|\dfrac{\partial\left(u,v\right)}{\partial\left(x,y\right)}\right| \neq 0\right)$：

不妨设$a \neq 0$, 考虑$u = ax + by, v = y$, 那么z是u的函数等价于$\dfrac{\partial z}{\partial v} = 0.$

而$y = v, x = \dfrac{u - bv}{a}$

所以$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\cdot\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\cdot\dfrac{\partial y}{\partial v} = -\dfrac{b}{a}\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0,$

因此充分必要条件就是$b\dfrac{\partial z}{\partial x} = a\dfrac{\partial z}{\partial y}.$

设$D \subset \mathbb{R}^2$是包含原点凸区域, $f \in C^1\left(D\right)$, 若$xf_x + yf_y = 0, \left(x,y\right) \in D$
则f为常数.
证明：对$\left(x,y\right) \in D$, 考虑$g\left(t\right) = f\left(tx,ty\right)$, 则$g'\left(t\right) = xf_x + yf_y$
$tg'\left(t\right) = txf_x\left(tx,ty\right) + tyf_y\left(tx,ty\right) = 0 \Rightarrow g'\left(t\right) = 0\ \left(\text{在}t = 0\text{还需要连续}\right)$
因此$f\left(x,y\right) = g\left(1\right) = g\left(0\right) = f\left(0,0\right).$
我们完成了证明.

若$f\in C^2\left(\mathbb{R}^2\right)$满足

$\left(1\right): f\left(x,y\right)=f\left(y,x\right)$

$\left(2\right): f\left(x,y\right)+f\left(y,z\right)+f\left(z,x\right)=3f\left(\dfrac{x+y+z}{3},\dfrac{x+y+z}{3}\right)$

求全体f.

解：

不妨设$f\left(0,0\right)=0$.

$$\frac{\partial^2 f\left(x,y\right)}{\partial x\partial y}=\frac{1}{3}\left[f_{11}+2f_{12}+f_{22}\right]\left(\frac{x+y+z}{3},\frac{x+y+z}{3}\right)$$

由z的任意性我们可以知道$\dfrac{\partial^2 f\left(x,y\right)}{\partial x\partial y}\equiv c$

故$f\left(x,y\right)=cxy+g\left(x\right)+h\left(y\right),g,h\in C^1\left(\mathbb{R}\right)$.

由$\left(1\right)$可知$g=h$,由$f\left(0,0\right)=0$知$g\left(0\right)=0$,

把f代入$\left(2\right)$需要

$$c\left(xy+yz+zx\right)+2\left(g\left(x\right)+g\left(y\right)+g\left(z\right)\right)=\frac{c}{3}\left(x+y+z\right)^2+6g\left(\frac{x+y+z}{3}\right)$$

令$y,z=0$,就有$2g\left(x\right)+4g\left(0\right)=\dfrac{c}{3}x^2+6g\left(\dfrac{x}{3}\right)$

$$故\frac{1}{3}g\left(\frac{x}{3^{n-1}}\right)=\frac{c}{18}\frac{x^2}{3^{2\left(n-1\right)}}+g\left(\frac{x}{3^n}\right)$$

$$\Rightarrow 3^{n-1}g\left(\frac{x}{3^{n-1}}\right)=\frac{c}{18}\frac{x^2}{3^{n-2}}+3^n g\left(\frac{x}{3^n}\right)$$

$$\Rightarrow \sum_{n=1}^{m}\left(3^{n-1}g\left(\frac{x}{3^{n-1}}\right)-3^n g\left(\frac{x}{3^n}\right)\right)=\sum_{n=1}^{m}\frac{c}{18}\frac{x^2}{3^{n-2}}$$

$$\Rightarrow g\left(x\right)-3^m g\left(\frac{x}{3^m}\right)=\sum_{n=1}^{m}\frac{c}{18}\frac{x^2}{3^{n-2}}$$

令$m\to\infty$,利用$g\left(\dfrac{x}{3^m}\right)=g'\left(0\right)\cdot\dfrac{x}{3^m}+o\left(\dfrac{1}{3^m}\right)$

就有$g\left(x\right)-g'\left(0\right)x=\displaystyle\sum_{n=1}^{\infty}\frac{c}{18}\frac{x^2}{3^{n-2}}=\frac{cx^2}{18}\frac{3}{1-\frac{1}{3}}=\frac{cx^2}{4}$

故$f\left(x,y\right)=cxy+g'\left(0\right)\left(x+y\right)+\dfrac{c}{4}\left(x^2+y^2\right)$

因此所求全体$f\left(x,y\right)=cxy+a\left(x+y\right)+\dfrac{c}{4}\left(x^2+y^2\right)+b,a,b,c\in\mathbb{R}$