2014 年第五届全国大学生数学竞赛决赛
(非数学类）参考答案

一、解答下列各题

1.【参考解答】:【解法一】:

$$\text{原式} = \int_0^{2\pi} \frac{\sin^2 t}{t^2} \mathrm{d}\, t \int_0^t x \, \mathrm{d}\, x = \frac{1}{2}\int_0^{2\pi} \sin^2 t \, \mathrm{d}\, t = 2\int_0^{\frac{\pi}{2}} \sin^2 t \, \mathrm{d}\, t = \frac{\pi}{2}.$$

【解法二】: 令 $f(x) = \int_x^{2\pi} \frac{\sin^2 t}{t^2} \mathrm{d}\, t$，则 $f'(x) = -\frac{\sin^2 x}{x^2}$ 且 $f\left(2\pi\right) = 0$.

$$\text{原式} = \int_0^{2\pi} x f(x) \, \mathrm{d}\, x = \left[\frac{1}{2} x^2 f(x)\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} x^2 f'(x) \, \mathrm{d}\, x$$

$$= \frac{1}{2}\int_0^{2\pi} x^2 \frac{\sin^2 x}{x^2} \mathrm{d}\, x = \frac{1}{2}\int_0^{2\pi} \sin^2 x \, \mathrm{d}\, x = \frac{\pi}{2}.$$

2.【参考解答】: $I = \int_0^1 f(x) \, \mathrm{d}\, x = \int_0^1 f(x) \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} \mathrm{d}\, x$

$$\le \left(\int_0^1 \left(1+x^2\right) f^2(x) \, \mathrm{d}\, x\right)^{1/2} \left(\int_0^1 \frac{1}{1+x^2} \mathrm{d}\, x\right)^{1/2} = \left(\int_0^1 \left(1+x^2\right) f^2(x) \, \mathrm{d}\, x\right)^{1/2} \left(\frac{\pi}{4}\right)^{1/2}$$

$\int_0^1 \left(1+x^2\right) f^2(x) \, \mathrm{d}\, x \ge \frac{4}{\pi}$，取 $f(x) = \frac{4}{\pi\left(1+x^2\right)}$ 即可.

3.【参考解答】: 由两方程定义的曲面在 $P_0\left(x_0, y_0, z_0\right)$ 的切面分别为

$$F_x\left(P_0\right)(x-x_0) + F_y\left(P_0\right)(y-y_0) + F_z\left(P_0\right)(z-z_0) = 0,$$
$$G_x\left(P_0\right)(x-x_0) + G_y\left(P_0\right)(y-y_0) + G_z\left(P_0\right)(z-z_0) = 0.$$

上述两切面的交线就是 Γ 在 P_0 点的切线，该切线在 xOy 面上的投影就是 S 过 $\left(x_0, y_0\right)$ 的切线. 消去 $z - z_0$，有

$$\left(F_x G_z - G_x F_z\right)_{P_0}(x-x_0) + \left(F_y G_z - G_y F_z\right)_{P_0}(y-y_0) = 0.$$

这里 $x - x_0$ 的系数 $\frac{\partial\left(F, G\right)}{\partial\left(x, z\right)} \ne 0$，故上式是一条直线的方程，就是所求的切线.

4.【参考解答】: 由关系式

$$AB = A - B + E \Rightarrow \left(A+E\right)\left(B-E\right) = 0.$$

$$\Rightarrow rank(A+B) \le rank(A+E) + rank\left(B-E\right) \le 3$$

因为 $rank\left(A+B\right) = 3$，所以

$$rank(A+E) + rank\left(B-E\right) = 3.$$

又$rank(A+E)\geq 2,$考虑到B非单位，所以$rank\left(B-E\right)\geq 1$，只有$rank(A+E)=2.$

$$A+E=\begin{pmatrix}2&2&1\\3&5&a\\1&2&3\end{pmatrix}\to\begin{pmatrix}0&-2&-5\\0&-1&a-9\\1&2&3\end{pmatrix}\to\begin{pmatrix}0&0&13-2a\\0&-1&a-9\\1&2&3\end{pmatrix}$$

从而$a=\dfrac{13}{2}.$

二、【参考证明】: 由泰勒公式

$$f\left(x+h\right)=f\left(x\right)+f'(x)h+\frac{f''\left(x\right)}{2}h^2+\frac{f'''\left(x\right)}{6}h^3+\frac{f^{(4)}\left(\xi\right)}{24}h^4 \qquad (1)$$

$$f''\left(x+\theta h\right)=f''\left(x\right)+f'''\left(x\right)\theta h+\frac{f^{(4)}\left(\eta\right)}{2}\theta^2h^2 \qquad (2)$$

其中ξ介于x与$x+h$之间，η介于x与$x+\theta h$之间，由(1)(2)式和已知条件

$$f\left(x+h\right)=f\left(x\right)+f'(x)h+\frac{f''\left(x+\theta h\right)}{2}h^2$$

可得$4\left(1-3\theta\right)f'''\left(x\right)=\left[6f^{(4)}\left(\eta\right)\theta^2-f^{(4)}\left(\xi\right)\right]h$.

当$\theta\neq\dfrac{1}{3}$时，令$h\to 0$得$f'''(x)=0$，此时f是不超过二次的多项式;

当$\theta=\dfrac{1}{3}$时，有$\dfrac{2}{3}f^{(4)}\left(\eta\right)=f^{(4)}\left(\xi\right)$. 令$h\to 0$，注意到$\xi\to x,\eta\to x,$ 有$f^{(4)}(x)=0$，此时f是不超过三次的多项式.

三、【参考证明】: 由题设可知 $f'(0)=-1$，则所给方程可变形为$\left(1+x\right)f'(x)+\left(1+x\right)f(x)-\int_0^x f(t)\,\mathrm{d}\,t=0$，两端关于$x$求导并整理得

$$\left(1+x\right)f''(x)+\left(2+x\right)f'(x)=0$$

这是一个可降阶的二阶微分方程，可用分离变量法求得$f'(x)=\dfrac{Ce^{-x}}{1+x}.$

由$f'(0)=-1$得$C=-1$，即$f'(x)=-\dfrac{e^{-x}}{1+x}<0.$函数$f(x)$单调递减. 而$f(0)=1$，所以当$x\geq 0$，$f(x)\leq 1$.

对$f'(t)=-\dfrac{e^{-t}}{1+t}<0$在$\left[0,x\right]$上进行积分，得

$$f\left(x\right)=f\left(0\right)-\int_0^x\frac{e^{-t}}{1+t}\mathrm{d}\,t\geq 1-\int_0^x e^{-t}\,\mathrm{d}\,t=e^{-x}.$$

四、【参考证明】: $I=\int_0^1\mathrm{d}\,y\int_0^1 f(x,y)\,\mathrm{d}\,x=-\int_0^1\mathrm{d}\,y\int_0^1 f(x,y)\,\mathrm{d}\left(1-x\right)$

对于固定y，$\left(1-x\right)f\left(x,y\right)\Big|_{x=0}^{x=1}=0,$ 由分部积分法可得

$$\int_0^1 f(x,y)\,\mathrm{d}\left(1-x\right)=-\int_0^1\left(1-x\right)\frac{\partial f(x,y)}{\partial x}\,\mathrm{d}\,x$$

交换积分次序后可得 $I=\int_0^1\left(1-x\right)\mathrm{d}\,x\int_0^1\frac{\partial f(x,y)}{\partial x}\,\mathrm{d}\,y$. 因为 $f\left(x,0\right)=0$，所以 $\frac{\partial f(x,y)}{\partial x}=0$；从而 $\left(1-y\right)\frac{\partial f(x,y)}{\partial x}\bigg|_{y=0}^{y=1}=0.$ 再由分部积分法得

$$\int_0^1\frac{\partial f(x,y)}{\partial x}\,\mathrm{d}\,y=-\int_0^1\frac{\partial f(x,y)}{\partial y}\,\mathrm{d}\left(1-y\right)=\int_0^1\left(1-y\right)\frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}\,y.$$

$$I=\int_0^1\left(1-x\right)\mathrm{d}\,x\int_0^1\left(1-y\right)\frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}\,y=\iint_D\left(1-x\right)\left(1-y\right)\frac{\partial^2 f}{\partial x\partial y}\,\mathrm{d}\,x\,\mathrm{d}\,y$$

因为 $\frac{\partial^2 f}{\partial x\partial y}\le A$，且 $\left(1-x\right)\left(1-y\right)$ 在 D 上非负，故

$$I\le A\iint_D\left(1-x\right)\left(1-y\right)\mathrm{d}\,x\,\mathrm{d}\,y=\frac{A}{4}.$$

五、【参考解答】：由高斯公式，有

$$I_t=\iiint_V\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)\mathrm{d}\,V=\iiint_V\left(2xz+2yz+x^2+y^2\right)f'\left(\left(x^2+y^2\right)z\right)\mathrm{d}\,V$$

由对称性，有 $\iiint_V\left(2xz+2yz\right)f'\left(\left(x^2+y^2\right)z\right)\mathrm{d}\,V=0.$ 从而

$$I_t=\iiint_V\left(x^2+y^2\right)f'\left(\left(x^2+y^2\right)z\right)\mathrm{d}\,V=\int_0^1\left[\int_0^{2\pi}\mathrm{d}\,\theta\int_0^t f'\left(r^2z\right)r^3\,\mathrm{d}\,r\right]\mathrm{d}\,z$$

$$=2\pi\int_0^1\left[\int_0^t f'\left(r^2z\right)r^3\,\mathrm{d}\,r\right]\mathrm{d}\,z$$

$$\lim_{t\to0+}\frac{I_t}{t^4}=\lim_{t\to0+}\frac{2\pi\int_0^1\left[\int_0^t f'\left(r^2z\right)r^3\,\mathrm{d}\,r\right]\mathrm{d}\,z}{t^4}=\lim_{t\to0+}\frac{2\pi\int_0^1 f'\left(t^2z\right)t^3\,\mathrm{d}\,z}{4t^3}$$

$$=\lim_{t\to0+}\frac{\pi}{2}\int_0^1 f'\left(t^2z\right)\mathrm{d}\,z=\frac{\pi}{2}f'\left(0\right).$$

六、【参考证明】：(必要性) 设 A,B 是两个 n 阶正定矩阵，从而为对称矩阵，即

$$\left(AB\right)^T=AB.$$

又 $A^T=A,B^T=B,$ 所以 $\left(AB\right)^T=B^TA^T=BA,$ 所以 $AB=BA.$

(充分性) 因为 $AB=BA.$ 则 $\left(AB\right)^T=B^TA^T=BA=AB,$ 所以 AB 为实对称矩阵. 因为 A,B 是正定矩阵，存在可逆矩阵 P,Q，使得

$$A=P^TP,B=Q^TQ\Rightarrow AB=P^TPQ^TQ.$$

所以$\left(P^T\right)^{-1}ABP^T=PQ^TQP^T=\left(QP^T\right)^T\left(QP^T\right)$，即$\left(P^T\right)^{-1}ABP^T$是正定矩阵．所以矩阵$\left(P^T\right)^{-1}ABP^T$的特征值 全为正实数,而$AB$相似于$\left(P^T\right)^{-1}ABP^T$,所以AB的特征值全为正实数,所以AB为正定矩阵.

七、【参考证明】: 由 $\lim\limits_{n\to\infty} na_n=0$，知 $\lim\limits_{n\to\infty}\dfrac{\sum\limits_{k=0}^{n}k\mid a_k\mid}{n}=0,$ 故对任意的$\varepsilon>0$，存在N_1，使得当$n>N_1$时，有

$$0\leq\frac{\sum\limits_{k=0}^{n}k\mid a_k\mid}{n}<\frac{\varepsilon}{3},n\mid a_n\mid<\frac{\varepsilon}{3}.$$

又因为 $\lim\limits_{x\to1^-}\sum\limits_{n=0}^{\infty}a_nx^n=A.$ 所以存在$\delta>0$，当$1-\delta<x<1$时,

$$\left|\sum_{n=0}^{\infty}a_nx^n-A\right|<\frac{\varepsilon}{3}.$$

取N_2，使得当$n>N_2$时，$\dfrac{1}{n}<\delta$，从而$1-\delta<1-\dfrac{1}{n}$，取$x=1-\dfrac{1}{n}$，则

$$\left|\sum_{n=0}^{\infty}a_n\left(1-\frac{1}{n}\right)^n-A\right|<\frac{\varepsilon}{3}.$$

取$N=\max\left\{N_1,N_2\right\}$，当$n>N$时

$$\begin{aligned}\left|\sum_{k=0}^{n}a_k-A\right|&=\left|\sum_{k=0}^{n}a_k-\sum_{k=0}^{n}a_kx^k-\sum_{k=n+1}^{\infty}a_kx^k+\sum_{k=0}^{\infty}a_kx^k-A\right|\\&\leq\left|\sum_{k=0}^{n}a_k\left(1-x^k\right)\right|+\left|\sum_{k=n+1}^{\infty}a_kx^k\right|+\left|\sum_{k=0}^{\infty}a_kx^k-A\right|\end{aligned}$$

取$x=1-\dfrac{1}{n}$，则有

$$\begin{aligned}\left|\sum_{k=0}^{n}a_k\left(1-x^k\right)\right|&=\left|\sum_{k=0}^{n}a_k\left(1-x\right)\left(1+x+x^2+\cdots+x^{k-1}\right)\right|\\&\leq\sum_{k=0}^{n}\left|a_k\right|\left(1-x\right)k=\frac{\sum\limits_{k=0}^{n}k\left|a_k\right|}{n}<\frac{\varepsilon}{3}\end{aligned}$$

$$\left|\sum_{k=n+1}^{\infty}a_kx^k\right|\leq\frac{1}{n}\sum_{k=n+1}^{\infty}k\mid a_k\mid x^k<\frac{\varepsilon}{3n}\sum_{k=n+1}^{\infty}x^k\leq\frac{\varepsilon}{3n}\frac{1}{1-x}=\frac{\varepsilon}{3n\cdot\frac{1}{n}}=\frac{\varepsilon}{3}$$

又因为$\left|\sum_{k=0}^{\infty} a_k x^k - A\right| < \frac{\varepsilon}{3}$，则$\left|\sum_{k=0}^{n} a_k - A\right| < 3\frac{\varepsilon}{3} = \varepsilon.$