第十届清疏竞赛班非数学类 24
多元积分

对光滑曲面S，Ω是S围成区域, $n=\left(\cos\alpha,\cos\beta,\cos\gamma\right)$是单位外法向量.

课内学过,对足够好的P,Q,R,我们有

$$\iint_S P\cos\alpha+Q\cos\beta+R\cos\gamma dS=\iiint_\Omega\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\right)dV$$

理解:分部积分转移导数:

设f,g光滑,且fg在$\partial\Omega$上为0,则有$\iiint_\Omega\frac{\partial f}{\partial x}gdV=-\iiint_\Omega f\frac{\partial g}{\partial x}dV$.

证明：

$$\iiint_\Omega\left(\frac{\partial f}{\partial x}g+f\frac{\partial g}{\partial x}\right)dV=\iiint_\Omega\frac{\partial\left(fg\right)}{\partial x}dV=\iint_{\partial\Omega}fg\cdot\cos\alpha dV=0.$$

理解:$\int_L\left(Pdx+Qdy\right)=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy$也可以写作gauss公式的形式：

$\int_L\left(Pdx+Qdy\right)=\int_L\left(P\cos\alpha+Q\sin\alpha\right)ds$, 这里$\left(\cos\alpha,\sin\alpha\right)$切线方向.

外法向应该是$\left(\cos\left(\alpha-\frac{\pi}{2}\right),\sin\left(\alpha-\frac{\pi}{2}\right)\right)=\left(\sin\alpha,-\cos\alpha\right)$

$$\int_L\left(Q\sin\alpha+\left(-P\right)\left(-\cos\alpha\right)\right)ds=\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dxdy.$$

经典习题：

设$f\left(x,y\right)\in C^4\left(D\right),D=\left\{\left(x,y\right)\in\mathbb{R}^2:0\le x,y\le 1\right\},f$在$D$边界为0,

且在D上,我们有$\left|\dfrac{\partial^4 f}{\partial x^2\partial y^2}\right|\le b$,证明$\left|\iint\limits_D f\left(x,y\right)dxdy\right|\le\dfrac{b}{144}$.

分析：$\dfrac{\partial^4 g}{\partial x^2\partial y^2}$要为常数,则$g$是一个$x$的二次函数$\times y$的二次函数.

那么g能提供零边界条件最多的就是$g\left(x,y\right)=x\left(1-x\right)y\left(1-y\right)$

证明：

$$\left|\iint\limits_D f\left(x,y\right)dxdy\right|=\frac{1}{4}\left|\iint\limits_D f\left(x,y\right)\frac{\partial^4\left[x\left(1-x\right)y\left(1-y\right)\right]}{\partial x^2\partial y^2}dxdy\right|$$

$$\overset{f(x,y)\frac{\partial^3\left[x(1-x)y(1-y)\right]}{\partial x^2\partial y}\text{有0边界条件}}{=}\frac{1}{4}\left|\iint\limits_D\frac{\partial f\left(x,y\right)}{\partial y}\frac{\partial^3\left[x\left(1-x\right)y\left(1-y\right)\right]}{\partial x^2\partial y}dxdy\right|$$

$$\overset{\frac{\partial f(x,y)}{\partial y}\frac{\partial^2\left[x(1-x)y(1-y)\right]}{\partial x^2}\text{有0边界条件}}{=}\frac{1}{4}\left|\iint\limits_D\frac{\partial^2 f\left(x,y\right)}{\partial y^2}\frac{\partial^2\left[x\left(1-x\right)y\left(1-y\right)\right]}{\partial x^2}dxdy\right|$$

$$\overset{\frac{\partial^2 f(x,y)}{\partial y^2}\frac{\partial\left[x(1-x)y(1-y)\right]}{\partial x}\text{有0边界条件}}{=}\frac{1}{4}\left|\iint\limits_D\frac{\partial^3 f\left(x,y\right)}{\partial x\partial y^2}\frac{\partial\left[x\left(1-x\right)y\left(1-y\right)\right]}{\partial x}dxdy\right|$$

$$\overset{\frac{\partial^3 f(x,y)}{\partial x\partial y^2}\left(x(1-x)y(1-y)\right)\text{有0边界条件}}{=}\frac{1}{4}\left|\iint\limits_D\frac{\partial^4 f\left(x,y\right)}{\partial x^2\partial y^2}\left(x\left(1-x\right)y\left(1-y\right)\right)dxdy\right|$$

故

$$\left|\iint\limits_D f\left(x,y\right)dxdy\right|\le\frac{1}{4}\iint\limits_D\left|\frac{\partial^4 f\left(x,y\right)}{\partial x^2\partial y^2}\right|\cdot x\left(1-x\right)y\left(1-y\right)dxdy$$

$$\le\frac{b}{4}\iint\limits_D x\left(1-x\right)y\left(1-y\right)dxdy=\frac{b}{4}\left(\int_0^1 x\left(1-x\right)dx\right)^2=\frac{b}{144}.$$

总结：分部积分转移导数思想和需要零边界思想.

下面等式非常重要, 但也可以不记.

设u,v光滑, 证明:$\iiint\limits_{\Omega} \nabla u \cdot \nabla v dV = -\iiint\limits_{\Omega} u \Delta v dV + \iint\limits_{S} u \dfrac{\partial v}{\partial n} dS$

$\dfrac{\partial v}{\partial n}$: 表示$v$对外法向的方向导数.

证明：

$$\iint\limits_{S} u \frac{\partial v}{\partial n} dS = \iint\limits_{S} u \frac{\partial v}{\partial n} dS = \iint\limits_{S} \left(u \frac{\partial v}{\partial x} dydz + u \frac{\partial v}{\partial y} dzdx + u \frac{\partial v}{\partial z} dxdy \right)$$

$$= \iiint\limits_{\Omega} \left(\sum_{cyc} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \sum_{cyc} \frac{\partial^2 v}{\partial x^2} \right) dV$$

$$= \iiint\limits_{\Omega} \nabla u \cdot \nabla v dV + \iiint\limits_{\Omega} u \Delta v dV \text{，证毕！}$$

真题：

如果f在$\partial\Omega$上为0，二阶连续可微，证明：$\exists C>0$，使得

$$C\iiint_{\Omega} f^2 dV+\frac{1}{C}\iiint_{\Omega}(\Delta f)^2 dV \ge 2\iiint_{\Omega}|\nabla f|^2 dV.$$

证明：

方法1：

在$\iiint_{\Omega}\nabla u\cdot\nabla v dV=-\iiint_{\Omega}u\Delta v dV+\iint_{S}u\frac{\partial v}{\partial n}dS$取$u=v=f$，则

$$\iiint_{\Omega}|\nabla f|^2 dV=-\iiint_{\Omega}f\Delta f dV+\iint_{S}f\frac{\partial f}{\partial n}dS=-\iiint_{\Omega}f\Delta f dV$$

$$\iiint_{\Omega}|\nabla f|^2 dV\le\iiint_{\Omega}|f\Delta f|dV=\iiint_{\Omega}\left|\sqrt{C}f\frac{1}{\sqrt{C}}\Delta f\right|dV$$

$$\le\frac{1}{2}\iiint_{\Omega}C|f|^2 dV+\frac{1}{2}\frac{1}{C}\iiint_{\Omega}(\Delta f)^2 dV$$

因此就证明了$C\iiint_{\Omega} f^2 dV+\frac{1}{C}\iiint_{\Omega}(\Delta f)^2 dV \ge 2\iiint_{\Omega}|\nabla f|^2 dV$.

方法2：

$$\iiint_{\Omega}|\nabla f|^2 dV=\sum_{cyc}\iiint_{\Omega}\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial x}dV\overset{\text{分部积分转移导数}}{=}-\sum_{cyc}\iiint_{\Omega}f\frac{\partial^2 f}{\partial x^2}dV$$

$=-\iiint_{\Omega}f\Delta f dV$，然后运用方法1就可以证明.

真题：$\Delta f=\sqrt{x^2+y^2+z^2}$,区域$\Omega$是单位球,

计算$\iiint\limits_{\Omega}\left(xf_x+yf_y+zf_z\right)dV$

证明：

$$\iiint\limits_{\Omega}\left(xf_x+yf_y+zf_z\right)dV$$

$$=\sum_{cyc}\iiint\limits_{\Omega}xf_x dV$$

$$=\frac{1}{2}\sum_{cyc}\iiint\limits_{\Omega}\frac{\partial\left(x^2+y^2+z^2\right)}{\partial x}\frac{\partial f}{\partial x}dV$$

$$=-\frac{1}{2}\sum_{cyc}\iiint\limits_{\Omega}\left(x^2+y^2+z^2\right)\frac{\partial^2 f}{\partial x^2}dV+\frac{1}{2}\sum_{cyc}\iint\limits_{\partial\Omega}\left(x^2+y^2+z^2\right)\frac{\partial f}{\partial x}dydz$$

$$=-\frac{1}{2}\iiint\limits_{\Omega}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}dV+\frac{1}{2}\sum_{cyc}\iint\limits_{\partial\Omega}\frac{\partial f}{\partial x}dydz$$

$$=-\frac{1}{2}\iiint\limits_{\Omega}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}dV+\frac{1}{2}\sum_{cyc}\iiint\limits_{\Omega}\frac{\partial^2 f}{\partial x^2}dV$$

$$=-\frac{1}{2}\iiint\limits_{\Omega}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}dV+\frac{1}{2}\iiint\limits_{\Omega}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}dV$$

$$=-\frac{1}{2}\int_0^1 dr\iint\limits_{x^2+y^2+z^2=r^2}\left(x^2+y^2+z^2\right)^{\frac{3}{2}}dS+\frac{1}{2}\int_0^1 dr\iint\limits_{x^2+y^2+z^2=r^2}\left(x^2+y^2+z^2\right)^{\frac{1}{2}}dS$$

$$=-2\pi\int_0^1 r^5 dr+2\pi\int_0^1 r^3 dr$$

$$=-\frac{\pi}{3}+\frac{\pi}{2}=\frac{\pi}{6}.$$

真题：

$f\left(x,y\right)$在$x^2+y^2\le a^2$上连续可微，且在边界上$f\left(x,y\right)=a^2$

且在圆$x^2+y^2\le a^2$上，$\left|f_x\right|^2+\left|f_y\right|^2\le a^2, a>0$，证明：

$$\left|\iint_{x^2+y^2\le a^2} f\left(x,y\right)dxdy\right|\le\frac{4}{3}\pi a^4$$

证明：

本题当然也可以使用分部积分转移导数的观点，

我们换一个观点：直接从曲线$\left(面\right)$积分出发配凑.

$$\int_{x^2+y^2=a^2}-yfdx+xfdy$$

$$=\iint_{x^2+y^2\le a^2}2f\left(x,y\right)+xf_x+yf_y dxdy$$

$$=a^2\int_{x^2+y^2=a^2}-ydx+xdy$$

$$=a^2\iint_{x^2+y^2\le a^2}2dxdy$$

故$\displaystyle\iint_{x^2+y^2\le a^2}2f\left(x,y\right)+xf_x+yf_y dxdy=a^2\iint_{x^2+y^2\le a^2}2dxdy$

即$\displaystyle\iint_{x^2+y^2\le a^2}f\left(x,y\right)dxdy=\iint_{x^2+y^2\le a^2}a^2dxdy-\frac{1}{2}\iint_{x^2+y^2\le a^2}xf_x+yf_y dxdy$

$$=\pi a^4-\frac{1}{2}\iint_{x^2+y^2\le a^2}xf_x+yf_y dxdy$$

$$\left|\iint_{x^2+y^2\le a^2}xf_x+yf_y dxdy\right|\le\iint_{x^2+y^2\le a^2}\left|xf_x+yf_y\right|dxdy$$

$$\le a\iint_{x^2+y^2\le a^2}\sqrt{x^2+y^2}dxdy=a\int_0^a dr\int_{x^2+y^2=r^2}\sqrt{x^2+y^2}ds$$

$$=2\pi a\int_0^a r^2dr=\frac{2\pi}{3}a^4$$

因此$\displaystyle\left|\iint_{x^2+y^2\le a^2}f\left(x,y\right)dxdy\right|\le\pi a^4+\frac{\pi}{3}a^4=\frac{4}{3}\pi a^4.$

设$D=\left\{\left(x,y\right)\in\mathbb{R}^2:x^2+y^2\le 1\right\}$,$U$是$\mathbb{R}^2$包含$D$的开区域,$f\in C^1\left(U\right)$,且$f\mid_{\partial D}=0$,证明$\left|\iint_D f\left(x,y\right)dxdy\right|\le\frac{\pi}{3}\max_{\left(x,y\right)\in D}\sqrt{\left|f_x\right|^2+\left|f_y\right|^2}$.

证明:

$$0=\int_{x^2+y^2=1}-yfdx+xfdy$$

$$=\iint_D 2f+xf_x+yf_ydxdy$$

故$\iint_D fdxdy=-\frac{1}{2}\iint_D xf_x+yf_ydxdy$

因此

$$\left|\iint_D f\left(x,y\right)dxdy\right|\le\frac{1}{2}\frac{2}{3}\pi\cdot\max_{\left(x,y\right)\in D}\sqrt{\left|f_x\right|^2+\left|f_y\right|^2}=\frac{\pi}{3}\max_{\left(x,y\right)\in D}\sqrt{\left|f_x\right|^2+\left|f_y\right|^2}.$$

设$\lambda, a \in \mathbb{R}, a < 0, G$是包含单位开圆$B$和其边界的连通开集,$u \in C^{2}\left(G\right)$

若$\Delta u = \lambda u, \left(x, y\right) \in B, \dfrac{\partial u}{\partial n} = au \left(x, y\right) \in \partial B$, 若$u$不恒为0，证明$\lambda < 0$

证明：

强行配凑和分部积分转移导数思想仍然适用，但本题不是那么直接.

在$\iiint\limits_{\Omega} \nabla u \cdot \nabla v dV = -\iiint\limits_{\Omega} u \Delta v dV + \iint\limits_{S} u \dfrac{\partial v}{\partial n} dS$令$v = u$就有

$$\iiint\limits_{\Omega} \left|\nabla u\right|^{2} dV = -\iiint\limits_{\Omega} u \Delta u dV + \iint\limits_{S} u \frac{\partial u}{\partial n} dS = -\lambda \iiint\limits_{\Omega} u^{2} dV + a \iint\limits_{S} u^{2} dS$$

若u不恒为0, 那么$0 \le \iiint\limits_{\Omega} \left|\nabla u\right|^{2} dV < -\lambda \iiint\limits_{\Omega} u^{2} dV, \iiint\limits_{\Omega} u^{2} dV > 0,$

于是$-\lambda > 0$, 因此我们就证明了$\lambda < 0$.

$Poincare$不等式

设D是由简单光滑闭曲线L围成区域, $f\left(x, y\right)$在$\bar{D}$上有连续偏导数,

且在边界上$f\left(x, y\right) = 0$, 证明：

$$\iint\limits_{D} f^{2}\left(x, y\right) dxdy \le \max_{D}\left\{x^{2} + y^{2}\right\} \iint\limits_{D} \left[\left|f_{x}\right|^{2} + \left|f_{y}\right|^{2}\right] dxdy$$

证明：

$$0 = \int_{L} -yf^{2}dx + xf^{2}dy = \iint\limits_{D} 2f^{2} + 2xff_{x} + 2yff_{y} dxdy$$

因此$\iint\limits_{D} f^{2} dxdy = -\iint\limits_{D} f\left(xf_{x} + yf_{y}\right) dxdy \le \iint\limits_{D} \left|f\left(xf_{x} + yf_{y}\right)\right| dxdy$

$$\le \sqrt{\iint\limits_{D} \left|f\right|^{2} dxdy} \cdot \sqrt{\iint\limits_{D} \left|xf_{x} + yf_{y}\right|^{2} dxdy}$$

$$\le \sqrt{\iint\limits_{D} \left|f\right|^{2} dxdy} \cdot \sqrt{\iint\limits_{D} \left(x^{2} + y^{2}\right)\left(\left|f_{x}\right|^{2} + \left|f_{y}\right|^{2}\right) dxdy}$$

故$\sqrt{\iint\limits_{D} \left|f\right|^{2} dxdy} \le \sqrt{\iint\limits_{D} \left(x^{2} + y^{2}\right)\left(\left|f_{x}\right|^{2} + \left|f_{y}\right|^{2}\right) dxdy}$

$$\le \sqrt{\max_{D}\left\{x^{2} + y^{2}\right\} \iint\limits_{D} \left(\left|f_{x}\right|^{2} + \left|f_{y}\right|^{2}\right) dxdy}$$

因此$\iint\limits_{D} f^{2}\left(x, y\right) dxdy \le \max_{D}\left\{x^{2} + y^{2}\right\} \iint\limits_{D} \left[\left|f_{x}\right|^{2} + \left|f_{y}\right|^{2}\right] dxdy.$

$B\left(a,R\right)$表示$\mathbb{R}^3$中a为心，半径为R的开球

设$u\in C^1\left(B\left(a,R\right)\right)$且$\iint_{\partial B\left(a,r\right)}\frac{\partial u}{\partial n}dS=0,\forall r\in\left(0,R\right)$

证明：$\frac{1}{4\pi r^2}\iint_{\partial B\left(a,r\right)}udS=u\left(a\right),\forall r\in\left(0,R\right)$.

证明：

$$\frac{d\left(\frac{1}{4\pi r^2}\iint_{\partial B\left(a,r\right)}u\left(x\right)dS\left(x\right)\right)}{dr}\overset{x=a+ry}{=}\frac{d\left(\frac{1}{4\pi}\iint_{\partial B\left(0,1\right)}u\left(a+ry\right)dS\left(y\right)\right)}{dr}$$

$$=\frac{1}{4\pi}\iint_{\partial B\left(0,1\right)}\frac{d}{dr}u\left(a+ry\right)dS\left(y\right)$$

$$=\frac{1}{4\pi}\iint_{\partial B\left(0,1\right)}\sum_{i=1}^{3}u_i\left(a+ry\right)\cdot y_i dS\left(y\right)$$

$$\overset{y=\frac{x-a}{r}}{=}\frac{1}{4\pi}\iint_{\partial B\left(0,1\right)}\sum_{i=1}^{3}u_i\left(a+ry\right)\cdot y_i dS\left(y\right)$$

$$=\frac{1}{4\pi r^2}\iint_{\partial B\left(a,r\right)}\sum_{i=1}^{3}u_i\left(x\right)\cdot\frac{x_i-a}{r}dS\left(x\right)$$

熟记以$\left(a,b,c\right)$为心半径为r的球面上一点$\left(x,y,z\right)$

的单位外法向量为$\left(\frac{x-a}{r},\frac{y-b}{r},\frac{z-c}{r}\right)$.

因此$\frac{1}{4\pi r^2}\iint_{\partial B\left(a,r\right)}\sum_{i=1}^{3}u_i\left(x\right)\cdot\frac{x_i-a}{r}dS\left(x\right)$

$$=\frac{1}{4\pi r^2}\iint_{\partial B\left(a,r\right)}\frac{\partial u}{\partial n}dS\left(y\right)=0.$$

因此运用积分中值定理，我们有

$$\frac{1}{4\pi r^2}\iint_{\partial B\left(a,r\right)}udS=\lim_{r\to 0}\frac{1}{4\pi r^2}\iint_{\partial B\left(a,r\right)}udS$$

$$=\lim_{r\to 0}\frac{4\pi r^2}{4\pi r^2}u\left(\xi,\eta,\zeta\right)=u\left(a\right),\text{其中}\left(\xi,\eta,\zeta\right)\in\partial B\left(a,r\right).$$

设$h \in C^2\left(\mathbb{R}^3\right), \partial B\left(M, r\right)$是$M = \left(x, y, z\right)$为心, 半径为$r > 0$的球面,

定义$M_h\left(x, y, z, r\right) = \dfrac{1}{4\pi r^2} \oiint\limits_{\partial B\left(M, r\right)} h\left(\xi, \eta, \zeta\right) dS\left(\xi, \eta, \zeta\right)$

证明：$\Delta M_h\left(x, y, z, r\right) = \left(\dfrac{\partial^2}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial}{\partial r}\right) M_h\left(x, y, z, r\right)$,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

并证明$\lim\limits_{r \to 0^+} \dfrac{\partial}{\partial r} M_h\left(x, y, z, r\right) = 0.$

证明：

$$\frac{\partial M_h}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{4\pi r^2} \oiint\limits_{\partial B\left(M, r\right)} h\left(\xi, \eta, \zeta\right) dS\left(\xi, \eta, \zeta\right)\right)$$

$$= \frac{\partial}{\partial r}\left(\frac{1}{4\pi} \oiint\limits_{\partial B\left(0, 1\right)} h\left(\xi + rx, \eta + ry, \zeta + rz\right) dS\left(\xi, \eta, \zeta\right)\right)$$

$$= \frac{1}{4\pi} \oiint\limits_{\partial B\left(0, 1\right)} h_1 x + h_2 y + h_3 z dS\left(\xi, \eta, \zeta\right)$$

$$= \frac{1}{4\pi r^2} \oiint\limits_{\partial B\left(M, r\right)} h_1 \frac{\xi - x}{r} + h_2 \frac{\eta - y}{r} + h_3 \frac{\zeta - z}{r} dS\left(\xi, \eta, \zeta\right)$$

$$\overset{Gauss}{=} \frac{1}{4\pi r^2} \iiint\limits_{B\left(M, r\right)} \Delta h\left(\xi, \eta, \zeta\right) dV\left(\xi, \eta, \zeta\right)$$

$$\overset{\text{积分中值定理}}{=} \frac{1}{4\pi r^2} \frac{4}{3} \pi r^3 \cdot \Delta h\left(\xi_0, \eta_0, \zeta_0\right) \to 0, \text{当} r \to 0.$$

进一步

$$\frac{\partial^2 M_h}{\partial r^2}$$

$$= -\frac{1}{2\pi r^3}\iiint\limits_{B(M,r)} \Delta h(\xi,\eta,\zeta)\,dV(\xi,\eta,\zeta) + \frac{1}{4\pi r^2}\frac{\partial\left(\iiint\limits_{B(M,r)} \Delta h(\xi,\eta,\zeta)\,dV(\xi,\eta,\zeta)\right)}{\partial r}$$

$$= -\frac{1}{2\pi r^3}\iiint\limits_{B(M,r)} \Delta h(\xi,\eta,\zeta)\,dV(\xi,\eta,\zeta)$$

$$+\frac{1}{4\pi r^2}\frac{\partial\left(\int_0^r ds \iint\limits_{\partial B(M,s)} \Delta h(\xi,\eta,\zeta)\,dS(\xi,\eta,\zeta)\right)}{\partial r}$$

$$= -\frac{1}{2\pi r^3}\iiint\limits_{B(M,r)} \Delta h(\xi,\eta,\zeta)\,dV(\xi,\eta,\zeta) + \frac{1}{4\pi r^2}\iint\limits_{\partial B(M,r)} \Delta h(\xi,\eta,\zeta)\,dS(\xi,\eta,\zeta)$$

因此$\left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}\right)M_h(x,y,z,r) = \frac{1}{4\pi r^2}\iint\limits_{\partial B(M,r)} \Delta h(\xi,\eta,\zeta)\,dS(\xi,\eta,\zeta)$

$$= \frac{1}{4\pi}\iint\limits_{\partial B(0,1)} \Delta h(x+r\xi, y+r\eta, z+r\zeta)\,dS(\xi,\eta,\zeta)$$

$$\overset{*}{=} \Delta\left[\frac{1}{4\pi}\iint\limits_{\partial B(0,1)} h(x+r\xi, y+r\eta, z+r\zeta)\,dS(\xi,\eta,\zeta)\right]$$

$$= \Delta \iint\limits_{\partial B(M,r)} h(\xi,\eta,\zeta)\,dS(\xi,\eta,\zeta)$$

$$= \Delta M_h(x,y,z,r).$$

其中*的理解是难点.

我们完成了证明.

$B_r=\left\{x\in\mathbb{R}^3:\left\|x\right\|\le r\right\}$, 设$f\in C^2\left(\mathbb{R}^3-B_1\right)$满足$\Delta f=0$,

若$F\left(x\right)=\iint\limits_{\partial B_{\|x\|}} f\left(y\right)dS\left(y\right)$, 证明$\Delta F=0$.

知识点：

证明(多元微分学训练题)在球坐标下拉普拉斯算子有表达式:

$$\Delta u=\frac{1}{r^2}\partial_r\left(r^2\partial_r\right)+\frac{1}{r^2\sin\varphi}\partial_\varphi\left(\sin\varphi\partial_\varphi\right)+\frac{1}{r^2\sin^2\varphi}\partial_\theta^2,$$

这里$\partial_r=\dfrac{\partial}{\partial r},\partial_\varphi=\dfrac{\partial}{\partial_\varphi},\partial_\theta^2=\dfrac{\partial^2}{\partial_\theta\partial_\theta}$

$x=r\sin\varphi\cos\theta,y=r\sin\varphi\sin\theta,z=r\cos\varphi,\varphi\in\left[0,\pi\right],\theta\in\left[0,2\pi\right],r\ge0$

并推导极坐标下Δ的表达式.