第十届清疏竞赛班非数学类 25

$B_r=\left\{x\in\mathbb{R}^3:\left\|x\right\|\le r\right\}$, 设$f\in C^2\left(\mathbb{R}^3-B_1\right)$满足$\Delta f=0$,

若$F\left(x\right)=\dfrac{1}{4\pi\left\|x\right\|^2}\displaystyle\iint_{\partial B_{\|x\|}}f\left(y\right)dS\left(y\right)$, 证明$\Delta F=0$.

知识点:

证明(多元微分学训练题)在球坐标下拉普拉斯算子有表达式:

$$\Delta u=\frac{1}{r^2}\partial_r\left(r^2\partial_r\right)+\frac{1}{r^2\sin\varphi}\partial_\varphi\left(\sin\varphi\partial_\varphi\right)+\frac{1}{r^2\sin^2\varphi}\partial_\theta^2,$$

这里$\partial_r=\dfrac{\partial}{\partial r},\partial_\varphi=\dfrac{\partial}{\partial_\varphi},\partial_\theta^2=\dfrac{\partial^2}{\partial_\theta\partial_\theta}$

$x=r\sin\varphi\cos\theta,y=r\sin\varphi\sin\theta,z=r\cos\varphi,\varphi\in\left[0,\pi\right],\theta\in\left[0,2\pi\right],r\ge0$

并推导极坐标下Δ的表达式.

证明:

$$\Delta F\left(x\right)=\Delta\left(\frac{1}{4\pi\left\|x\right\|^2}\iint_{\partial B_{\|x\|}}f\left(y\right)dS\left(y\right)\right)\overset{y=\|x\|y}{=}\Delta\left(\frac{1}{4\pi}\iint_{\partial B_1}f\left(\left\|x\right\|\cdot y\right)dS\left(y\right)\right)$$

$$=\frac{1}{4\pi}\iint_{\partial B_1}\Delta_x f\left(\left\|x\right\|\cdot y\right)dS\left(y\right)=\frac{1}{4\pi}\iint_{\partial B_1}\frac{1}{r^2}\partial_r\left(r^2\partial_r f\left(ry\right)\right)dS\left(y\right)$$

$$=\frac{1}{4\pi}\int_0^{2\pi}d\theta\int_0^\pi\frac{\sin\varphi}{r^2}\partial_r\left(r^2\partial_r f\left(r\sin\varphi\cos\theta,r\sin\varphi\sin\theta,r\cos\varphi\right)\right)d\varphi$$

$$=-\int_0^{2\pi}d\theta\int_0^\pi\sin\varphi\left[\frac{1}{r^2\sin\varphi}\partial_\varphi\left(\sin\varphi\partial_\varphi f\right)+\frac{1}{r^2\sin^2\varphi}\partial_\theta^2 f\right]d\varphi$$

$$=-\int_0^{2\pi}d\theta\int_0^\pi\left[\frac{1}{r^2}\partial_\varphi\left(\sin\varphi\partial_\varphi f\right)\right]d\varphi-\int_0^{2\pi}d\theta\int_0^\pi\frac{1}{r^2\sin\varphi}\partial_\theta^2 fd\varphi$$

$$=-\frac{1}{r^2}\int_0^{2\pi}\left(\sin\varphi\partial_\varphi f\mid_0^\pi\right)d\theta-\int_0^\pi d\varphi\int_0^{2\pi}\frac{1}{r^2\sin\varphi}\partial_\theta^2 fd\theta$$

$$=-\int_0^\pi\frac{\partial_\theta f\left(r\sin\varphi\cos\theta,r\sin\varphi\sin\theta,r\cos\varphi\right)\mid_0^{2\pi}}{r^2\sin\varphi}d\varphi=0,$$

从而我们证明了$\Delta F=0$.

$0<\rho<1$, 计算 $\dfrac{1}{\sqrt{2\pi\left(1-\rho^{2}\right)}}\iint_{\mathbb{R}^2}\max\left\{x, y\right\}e^{-\frac{x^2-2\rho xy+y^2}{2\left(1-\rho^2\right)}}dxdy$

积累：

$\max\left\{x,y\right\}=\dfrac{x+y+\left|x-y\right|}{2}, \min\left\{x,y\right\}=\dfrac{x+y-\left|x-y\right|}{2}$

做 $x+y=u, x-y=v$, 即 $\begin{cases} x=\dfrac{u+v}{2} \\ y=\dfrac{u-v}{2} \end{cases}$, 那么:

$x^2-2\rho xy+y^2=\left(\rho+1\right)v^2+\left(1-\rho\right)u^2, \left|J\right|=\left\|\begin{matrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{matrix}\right\|=\dfrac{1}{2}$

$$\frac{1}{\sqrt{2\pi\left(1-\rho^{2}\right)}}\iint_{\mathbb{R}^2}\frac{x+y+\left|x-y\right|}{2}e^{-\frac{x^2-2\rho xy+y^2}{2\left(1-\rho^2\right)}}dxdy$$

$$=\frac{1}{4\sqrt{2\pi\left(1-\rho^{2}\right)}}\iint_{\mathbb{R}^2}\left(u+\left|v\right|\right)e^{-\frac{\left(\rho+1\right)v^2+\left(1-\rho\right)u^2}{2\left(1-\rho^2\right)}}dudv$$

$$=\frac{1}{4\sqrt{2\pi\left(1-\rho^{2}\right)}}\iint_{\mathbb{R}^2}\left|v\right|e^{-\frac{\left(\rho+1\right)v^2+\left(1-\rho\right)u^2}{2\left(1-\rho^2\right)}}dudv$$

$$=\frac{1}{4\sqrt{2\pi\left(1-\rho^{2}\right)}}\int_{-\infty}^{\infty}e^{-\frac{u^2}{2\left(1+\rho\right)}}du\int_{-\infty}^{\infty}\left|v\right|e^{-\frac{v^2}{2\left(1-\rho\right)}}dv$$

$$=\frac{\sqrt{1-\rho}}{2}.$$

$f \in C\left(\mathbb{R}\right)$满足$f\left(x\right) > 0$且对所有$t \in \mathbb{R}$, 都有

$\int_{-\infty}^{\infty} e^{-\left|t-x\right|} f\left(x\right) dx \le 1$, 证明：对任何$b > a$, 都有:

$\int_a^b f\left(x\right) dx \le \frac{b-a}{2} + 1.$

证明：

注意到$\int_a^b \int_{-\infty}^{\infty} e^{-\left|t-x\right|} f\left(x\right) dxdt \le b-a$

另外一方面, 注意到$\int_a^b \int_{-\infty}^{\infty} e^{-\left|t-x\right|} f\left(x\right) dxdt$

$= \int_{-\infty}^{\infty} f\left(x\right) dx \int_a^b e^{-\left|t-x\right|} dt \ge \int_a^b f\left(x\right) dx \int_a^b e^{-\left|t-x\right|} dt$

$= \int_a^b f\left(x\right) dx \left[\int_a^x e^{t-x} dt + \int_x^b e^{x-t} dt\right]$

$= \int_a^b f\left(x\right)\left[2 - e^{a-x} - e^{x-b}\right] dx$

因此$\int_a^b f\left(x\right) dx \le \frac{b-a}{2} + \frac{1}{2}\int_a^b f\left(x\right)\left[e^{a-x} + e^{x-b}\right] dx$

$\int_a^b e^{a-x} f\left(x\right) dx \le \int_{-\infty}^{\infty} e^{-\left|a-x\right|} f\left(x\right) dx \le 1$, 同理$\int_a^b e^{x-b} f\left(x\right) dx \le 1$,

因此我们证明了$\int_a^b f\left(x\right) dx \le \frac{b-a}{2} + 1.$

计算曲线积分

$$\oint_\Gamma \left(y^2+z^2\right)dx+\left(z^2+x^2\right)dy+\left(x^2+y^2\right)dz$$

其中$\Gamma:\begin{cases}x^2+y^2+z^2=2Rx\\ x^2+y^2=2rx\end{cases},0<r<R,z\ge 0.$

方向与z轴正向符合右手螺旋法则.

证明:

由$Stokes$定理,我们知道

$$\oint_\Gamma \left(y^2+z^2\right)dx+\left(z^2+x^2\right)dy+\left(x^2+y^2\right)dz$$

$$=\iint_S \begin{vmatrix} dydz & dzdx & dxdy \\ \partial_x & \partial_y & \partial_z \\ y^2+z^2 & z^2+x^2 & x^2+y^2 \end{vmatrix}$$

$$=\iint_S \left(2y-2z\right)dydz-\left(2x-2z\right)dzdx+\left(2x-2y\right)dx\,dy$$

S只要是任何Γ围成的光滑曲面都是对的.

可取S：$x^2+y^2+z^2=2Rx$，此时$\left(x-R\right)^2+y^2+z^2=R^2$

球面外法向量为:$\left(\dfrac{x-R}{R},\dfrac{y}{R},\dfrac{z}{R}\right)$,因此

$$\iint_S \left(2y-2z\right)dydz-\left(2x-2z\right)dzdx+\left(2x-2y\right)dxdy$$

$$=\iint_S \left(2y-2z\right)\frac{x-R}{R}dS-\left(2x-2z\right)\frac{y}{R}dS+\left(2x-2y\right)\frac{z}{R}dS$$

$$=\frac{2}{R}\iint_S \left(y-z\right)\left(x-R\right)-\left(x-z\right)y+\left(x-y\right)zdS$$

$$=\frac{2}{R}\iint_S -z\left(x-R\right)+xzdS$$

$$=2\iint_S zdS$$

$$=2\iint_{x^2+y^2\le 2rx} z\frac{R}{z}dxdy$$

$$=2R\iint_{x^2+y^2\le 2rx} 1dxdy$$

$$=2\pi r^2R.$$

$z=f\left(x,y\right)$是区域$D=\left\{\left(x,y\right)\in\mathbb{R}^2:0\le x,y\le 1\right\}$上的可微函数.

若$f\left(0,0\right)=0$, 且$dz\,|_{\left(0,0\right)}=3dx+2dy$, 求极限

$$\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}}f\left(t,u\right)dudt}{1-\sqrt[4]{1-x^4}}$$

注：

本题可以改为$f\in C\left(D\right)$且在D处一点可微, 那么二元函数的中值定理就不在适用.

证明：$\frac{\partial f}{\partial x}\left(0,0\right)=3,\frac{\partial f}{\partial y}\left(0,0\right)=2$

$$\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}}f\left(t,u\right)dudt}{1-\sqrt[4]{1-x^4}}=\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}}f\left(t,u\right)dudt}{1-\left(1-\frac{1}{4}x^4+o\left(x^4\right)\right)}=4\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}}f\left(t,u\right)dudt}{x^4}$$

注意到是原点局部的积分，因此对$t,u>0$, 有渐进写法：

$$f\left(t,u\right)=f\left(0,0\right)+3t+2u+o\left(\sqrt{t^2+u^2}\right),$$

$$\frac{t+u}{\sqrt{2}}\le\sqrt{t^2+u^2}\le t+u,o\left(\sqrt{t^2+u^2}\right)=o\left(t+u\right),\text{则}$$

$$4\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}}\left(3t+2u\right)dudt+o\left(\int_0^{x^2}\int_x^{\sqrt{t}}\left(t+u\right)dudt\right)}{x^4}=4\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}}\left(3t+2u\right)dudt}{x^4}$$

严格写法1：

$\forall\varepsilon>0$, 存在$\delta>0$, 当$t,u\in\left(0,\delta^2\right)$, 有

$3t+2u-\varepsilon\left(t+u\right)\le f\left(t,u\right)\le 3t+2u+\varepsilon\left(t+u\right)$.

当$x\in\left(0,\delta\right)$

$$4\lim_{x\to 0^+}\frac{-\int_0^{x^2}\int_{\sqrt{t}}^{x}f\left(t,u\right)dudt}{x^4}\le 4\lim_{x\to 0^+}\frac{-\int_0^{x^2}\int_{\sqrt{t}}^{x}3t+2u-\varepsilon\left(t+u\right)dudt}{x^4}$$

$$=\frac{1}{5}\lim_{x\to 0^+}\frac{x^4\left(2\varepsilon x+5\varepsilon-6x-10\right)}{x^4}=\frac{1}{5}\left(5\varepsilon-10\right)$$

$$4\lim_{x\to 0^+}\frac{-\int_0^{x^2}\int_{\sqrt{t}}^{x}f\left(t,u\right)dudt}{x^4}\ge-\frac{1}{5}\left(5\varepsilon+10\right),$$

因此由ε任意性我们知$\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}}f\left(t,u\right)dudt}{1-\sqrt[4]{1-x^4}}=-2$.

方法2：不洞察到本质, 直接怎么说怎么列.

含参积分求导公式当$a\left(x\right), b\left(x\right), f\left(x,s\right)$连续可微, 我们有

$$F\left(x\right)=\int_{a\left(x\right)}^{b\left(x\right)} f\left(x,s\right)ds,$$

$$F'\left(x\right)=b'\left(x\right)f\left(x,b\left(x\right)\right)-a'\left(x\right)f\left(x,a\left(x\right)\right)+\int_{a\left(x\right)}^{b\left(x\right)}\partial_x f\left(x,s\right)ds$$

$$\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_x^{\sqrt{t}} f\left(t,u\right)dudt}{1-\sqrt[4]{1-x^4}}$$

$$=-4\lim_{x\to 0^+}\frac{\int_0^{x^2}\int_{\sqrt{t}}^{x} f\left(t,u\right)dudt}{x^4}$$

$$\overset{\text{含参积分求导公式}}{=}-\lim_{x\to 0^+}\frac{\int_0^{x^2} f\left(t,x\right)dt}{x^3}$$

此时我们可以对$f\left(t,x\right)$运用可微定义的$peano$余项形式类似方法1证明.

或者$\lim_{x\to 0^+}\frac{\int_0^{x^2} f\left(t,x\right)dt}{x^3}=\lim_{x\to 0^+}\frac{x^2 f\left(\xi\left(x\right),x\right)}{x^3}=\lim_{x\to 0^+}\frac{f\left(\xi\left(x\right),x\right)}{x}$

$$f\left(\xi\left(x\right),x\right)=3\xi\left(x\right)+2x+o\left(\sqrt{x^2+\xi^2\left(x\right)}\right), 0<\xi\left(x\right)<x^2$$

因此可以知道$-\lim_{x\to 0^+}\frac{\int_0^{x^2} f\left(t,x\right)dt}{x^3}=-2.$

设$f\left(x,y\right)$在$D=\left\{\left(x,y\right)\in\mathbb{R}^2:x^2+y^2\le r^2\right\}$上具有二阶连续偏导数, 且

$\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=x^2+y^2$,求$\lim\limits_{r\to0^+}\dfrac{\iint_D\left(x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}\right)dxdy}{\left(\tan r-\sin r\right)^2}$.

证明:

适用下述公式的二重积分版本.

$$\iiint\limits_{\Omega}\nabla\left(x^2+y^2\right)\cdot\nabla f dV=-\iiint\limits_{\Omega}\left(x^2+y^2\right)\Delta f dV+\iint\limits_{S}\left(x^2+y^2\right)\frac{\partial f}{\partial n}dS$$

因此

$$\lim_{r\to0^+}\frac{\iint_D\left(x\dfrac{\partial f}{\partial x}+y\dfrac{\partial f}{\partial y}\right)dxdy}{\left(\tan r-\sin r\right)^2}$$

$$=2\lim_{r\to0^+}\frac{-\iint\left(x^2+y^2\right)^2dxdy+\oint\left(x^2+y^2\right)\left(f_x\dfrac{x}{r}+f_y\dfrac{y}{r}\right)ds}{r^6}$$

$$=2\lim_{r\to0^+}\frac{-2\pi\int_0^r\rho^5d\rho}{r^6}+2\lim_{r\to0^+}\frac{\oint\left(f_x\dfrac{x}{r}+f_y\dfrac{y}{r}\right)ds}{r^4}$$

$$=-\frac{2\pi}{3}+2\lim_{r\to0^+}\frac{\iint\Delta f dxdy}{r^4}$$

$$=-\frac{2\pi}{3}+2\lim_{r\to0^+}\frac{\iint\left(x^2+y^2\right)dxdy}{r^4}$$

$$=-\frac{2\pi}{3}+4\pi\lim_{r\to0^+}\frac{\int_0^r\rho^3d\rho}{r^4}$$

$$=\frac{\pi}{3}$$