第十届清疏竞赛班非数学类 19

多元函数微分学

多元函数极值判定和多元函数$Taylor$公式：

设$f\in C^k\left(\mathbb{R}^n\right)$, 那么考虑$g\left(\lambda\right)=f\left(a+\lambda b\right),\forall a,b\in\mathbb{R}^n$

则$g\in C^k\left(\mathbb{R}\right)$, 对$g$使用一元函数$taylor$公式即可得到多元函数$taylor$ 公式

为了方便以$k=2$为例, 顺便证明极值的判定.

$$g\left(\lambda\right)=f\left(a_1+\lambda b_1,a_2+\lambda b_2,\cdots,a_n+\lambda b_n\right)$$

$$g'\left(\lambda\right)=\sum_{i=1}^{n}f_{x_i}\left(a+\lambda b\right)b_i,\ g''\left(\lambda\right)=\sum_{i=1}^{n}\sum_{j=1}^{n}f_{x_ix_j}\left(a+\lambda b\right)b_ib_j$$

$$g\left(\lambda\right)=g\left(0\right)+g'\left(0\right)\lambda+\frac{g''\left(\theta\right)}{2}\lambda^2=f\left(a\right)+\sum_{i=1}^{n}f_{x_i}\left(a\right)b_i\lambda+\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}f_{x_ix_j}\left(a+\theta b\right)b_ib_j}{2}$$

取$\lambda=1$, 则

$$f\left(a+h\right)=f\left(a\right)+\sum_{i=1}^{n}f_{x_i}\left(a\right)h_i+\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}f_{x_ix_j}\left(a+\theta h\right)h_ih_j}{2}$$

$\dfrac{\sum_{i=1}^{n}\sum_{j=1}^{n}f_{x_ix_j}\left(a+\theta b\right)h_ih_j}{2}$还能写成$\dfrac{1}{2}\left(h_1\dfrac{\partial}{\partial x_1}+h_2\dfrac{\partial}{\partial x_2}+\cdots+h_n\dfrac{\partial}{\partial x_n}\right)^2$

这是因为

$$\left(h_1\frac{\partial}{\partial x_1}+h_2\frac{\partial}{\partial x_2}+\cdots+h_n\frac{\partial}{\partial x_n}\right)^2=h^T\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\cdots,\frac{\partial}{\partial x_n}\right)\begin{pmatrix}\frac{\partial}{\partial x_1}\\\frac{\partial}{\partial x_2}\\\vdots\\\frac{\partial}{\partial x_n}\end{pmatrix}h$$

$$f\left(a+h\right)=f\left(a\right)+Df\left(a\right)\cdot h+\frac{h^TD^2f\left(a+\theta h\right)h}{2}$$

$$D^2f=\begin{pmatrix}f_{x_1x_1}&f_{x_1x_2}&\cdots&f_{x_1x_n}\\\vdots&\vdots&\vdots&\vdots\\\vdots&\vdots&\vdots&\vdots\\f_{x_1x_n}&f_{x_2x_n}&\cdots&f_{x_nx_n}\end{pmatrix}$$

多元函数极值判定:

$$f\left(a+h\right)=f\left(a\right)+Df\left(a\right)\cdot h+\frac{h^{T}D^{2}f\left(a+\theta h\right)h}{2},\theta\in\left[0,1\right]$$

若$f\in C^{1}\left(D\right)$在内点$a\in D$取得极值, 所以在a处, $Df\left(a\right)=0$.

若$\mathrm{f}\in C^{2}\left(D\right)$且$Df\left(a\right)=0$, 则有二阶极值判断充分条件：

$\left(1\right)$: $D^{2}f\left(a\right)$正$\left(负\right)$定, 则f在a处取得极小$\left(大\right)$值.

证明：

因为$D^{2}f\left(a\right)$正定，所以$D^{2}f\left(a\right)$顺序主子式都为正,

因为$D^{2}f$连续性, 所以存在a的邻域U, 使得$D^{2}f\left(x\right)$顺序主子式都为正, $\forall x\in U$.

此时就得到$D^{2}f\left(x\right)$正定, $\forall x\in U$.

现在当h充分小, 使得$a+h\in U$, 则$f\left(a+h\right)=f\left(a\right)+\dfrac{h^{T}D^{2}f\left(a+\theta h\right)h}{2}\geq f\left(a\right)$.

证毕！

$\left(2\right)$： $D^{2}f\left(a\right)$半正$\left(负\right)$定, 则f在a处是否取极值无法判断.

原因分析：从$\left(1\right)$的证明可以看到, $D^{2}f\left(a\right)$实际上顺序主子式可能为0

则扰动一下， 正负性未知, 因此无法判断.

$\left(3\right)$: 如果$D^{2}f\left(a\right)$不定, 即既不半正定也不半负定, 不是极值.

证明：因为$D^{2}f\left(a\right)$不定, 所以存在$x,y\in\mathbb{R}^{n}$, 使得

$x^{T}D^{2}f\left(a\right)x<0,y^{T}D^{2}f\left(a\right)y>0$, 结合连续性, 我们知道存在$a$的邻域$U$, 使得

$x^{T}D^{2}f\left(b\right)x<0,y^{T}D^{2}f\left(b\right)y>0,\forall b\in U$.

此时当h充分小, 使得$a+h\in U$, 就有

$$f\left(a+x\right)=f\left(a\right)+\frac{x^{T}D^{2}f\left(a+\theta x\right)x}{2}<f\left(a\right)$$

$$f\left(a+y\right)=f\left(a\right)+\frac{y^{T}D^{2}f\left(a+\theta y\right)y}{2}>f\left(a\right),$$

因此a不是极值.

$\left(4\right)$: 若a是f的极小$\left(大\right)$值点, 则$D^{2}f\left(a\right)$半正$\left(负\right)$定.

证明：

因为a是f的极小值点, 所以存在a的邻域U, 使得$f\left(a+h\right)\geq f\left(a\right)$， $\forall h$ 充分小.

此时$f\left(a+h\right)=f\left(a\right)+\dfrac{h^{T}D^{2}f\left(a+\theta h\right)h}{2}\geq f\left(a\right)\Rightarrow h^{T}D^{2}f\left(a+\theta h\right)h\geq 0$

$\Rightarrow\lambda^{2}h^{T}D^{2}f\left(a+\lambda\theta h\right)h\geq 0,\forall\lambda\in\left(0,1\right)\Rightarrow h^{T}D^{2}f\left(a+\lambda\theta h\right)h\geq 0,\forall\lambda\in\left(0,1\right)$,

则$\lim\limits_{\lambda\to 0^{+}}h^{T}D^{2}f\left(a+\lambda\theta h\right)h\geq 0\Rightarrow h^{T}D^{2}f\left(a\right)h\geq 0\Rightarrow D^{2}f\left(a\right)$半正定

$f \in C^2\left(\mathbb{R}^2\right)$,定义$g_\alpha\left(t\right) = f\left(t\cos\alpha, t\sin\alpha\right)$满足$\frac{dg_\alpha\left(0\right)}{dt} = 0, \frac{d^2 g_\alpha\left(0\right)}{dt^2} > 0, \forall \alpha \in \left[0, 2\pi\right]$

证明$f\left(0,0\right)$是f的极小值.

证明：

$g_\alpha{}'\left(t\right) = \cos\alpha f_x\left(t\cos\alpha, t\sin\alpha\right) + \sin\alpha f_y\left(t\cos\alpha, t\sin\alpha\right)$

则$\cos\alpha f_x\left(0,0\right) + \sin\alpha f_y\left(0,0\right) = 0, \forall\alpha \in \left[0, 2\pi\right]$

$\left(\cos\alpha, \sin\alpha\right) \cdot \left(f_x\left(0,0\right), f_y\left(0,0\right)\right) = 0, \forall\alpha \in \left[0, 2\pi\right]$

即$\left(f_x\left(0,0\right), f_y\left(0,0\right)\right)$和任何向量都正交, 所以$\left(f_x\left(0,0\right), f_y\left(0,0\right)\right) = 0$.

$g_\alpha{}''\left(t\right) =$

$\cos^2\alpha f_{xx}\left(t\cos\alpha, t\sin\alpha\right) + 2\cos\alpha\sin\alpha f_{xy}\left(t\cos\alpha, t\sin\alpha\right) + \sin^2\alpha f_{yy}\left(t\cos\alpha, t\sin\alpha\right)$

则$\cos^2\alpha f_{xx}\left(0,0\right) + 2\cos\alpha\sin\alpha f_{xy}\left(0,0\right) + \sin^2\alpha f_{yy}\left(0,0\right) > 0, \forall\alpha \in \left[0, 2\pi\right]$

$$\left(\cos\alpha, \sin\alpha\right)\begin{pmatrix} f_{xx}\left(0,0\right) & f_{xy}\left(0,0\right) \\ f_{xy}\left(0,0\right) & f_{yy}\left(0,0\right) \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} > 0, \forall\alpha \in \left[0, 2\pi\right]$$

从而$D^2 f\left(0\right)$正定,因此我们证明了$f\left(0,0\right)$是f的极小值.

结论和证明都需要记忆：

设区域$V\subset\mathbb{R}^2$，定义在V上的f关于每个分量连续,且关于其中一个分量单调,证明$f\in C\left(V\right)$.

证明：

不妨假设$\left(0,0\right)\in V$,去证$f\left(x,y\right)$在$\left(0,0\right)$连续,假设f关于x递增

$\forall\varepsilon>0$,取$\delta>0$,使得$\left|f\left(x,0\right)-f\left(0,0\right)\right|\le\dfrac{\varepsilon}{2},\forall x\in\left[-\delta,\delta\right]$

此时就有$\left|f\left(\delta,0\right)-f\left(0,0\right)\right|\le\dfrac{\varepsilon}{2},\left|f\left(-\delta,0\right)-f\left(0,0\right)\right|\le\dfrac{\varepsilon}{2}$

取$\eta>0$,使得$\forall y\in\left(-\eta,\eta\right)$都有

$\left|f\left(\delta,y\right)-f\left(\delta,0\right)\right|\le\dfrac{\varepsilon}{2},\left|f\left(-\delta,y\right)-f\left(-\delta,0\right)\right|\le\dfrac{\varepsilon}{2}.$

故对$\left(x,y\right)\in\left(-\delta,\delta\right)\times\left(-\eta,\eta\right)$,由于$f$关于$x$递增,就有

$\left|f\left(x,y\right)-f\left(0,0\right)\right|\le\max\left\{\left|f\left(\delta,y\right)-f\left(0,0\right)\right|,\left|f\left(-\delta,y\right)-f\left(0,0\right)\right|\right\}$

而$\left|f\left(\delta,y\right)-f\left(0,0\right)\right|\le\left|f\left(\delta,y\right)-f\left(\delta,0\right)\right|+\left|f\left(\delta,0\right)-f\left(0,0\right)\right|\le\varepsilon.$

类似的$\left|f\left(-\delta,y\right)-f\left(0,0\right)\right|\le\varepsilon$,因此$\left|f\left(x,y\right)-f\left(0,0\right)\right|\le\varepsilon.$

这样我们就完成了证明.

思想：通过单调性,对x一致的控制住.

设区域$V\subset\mathbb{R}^2$，定义在V上的f关于每个分量偏导数存在,

且其中一个分量偏导数有下界,证明$f\in C\left(V\right)$.

证明：

设$f_x\ge-c,c>0$,那么考虑$f+cx$,则$f+cx$关于每个分量连续，

且其中一个分量单调,因此$f+cx\in C\left(V\right)$,所以$f\in C\left(V\right)$.

设$B\subset\mathbb{R}^n\left(n\ge 2\right)$的单位开球, 设$u,v\in C^2\left(\bar{B}\right)$, 满足

$$\begin{cases}-\Delta u-\left(1-u^2-v^2\right)u=0,x\in B\\-\Delta v-\left(1-u^2-v^2\right)v=0,x\in B\\u\left(x\right)=v\left(x\right)=0,x\in\partial B\end{cases}$$

证明：$u^2+v^2\le 1$

证明：

$w=u^2+v^2$, 设w在$a\in B$达到最大值(否则只能在边界达到最大值, 此时平凡)

则$w_{x_i}=2uu_{x_i}+2vv_{x_i}$

$w_{x_ix_i}=2\left(u_{x_i}\right)^2+2uu_{x_ix_i}+2\left(v_{x_i}\right)^2+2vv_{x_ix_i},i=1,2,\cdots,n.$

而由极值必要条件, 我们就有$w_{x_i}\left(a\right)=0,i=1,2,\cdots,n.$

即$w_{x_ix_i}\left(a\right)=2u\left(a\right)u_{x_ix_i}\left(a\right)+2v\left(a\right)v_{x_ix_i}\left(a\right)$

$\Delta w\left(a\right)=2u\left(a\right)\Delta u\left(a\right)+2v\left(a\right)\Delta v\left(a\right)$

结合$\begin{cases}-\Delta u-\left(1-w\right)u=0,x\in B\\-\Delta v-\left(1-w\right)v=0,x\in B\end{cases}$

故$\Delta w\left(a\right)=2\left(w-1\right)u^2+2\left(w-1\right)v^2=2\left(w-1\right)w$

所以结合一开始极值判定的第四点, 我们有$D^2w\left(a\right)$半负定

$-D^2w\left(a\right)$半正定, 故$tr\left(-D^2w\left(a\right)\right)=-\Delta w\left(a\right)=2\left(1-w\right)w\ge 0$,

因此$w\left(a\right)\le 1$, 这样我们就完成了证明.

在$D=\left\{(x,y)\in\mathbb{R}^2:|x|+|y|<1\right\}$, 给定$f\in C\left(\overline{D}\right)\bigcap C^1(D)$, $|f|\le 1$.

证明：存在一个点$(x_0,y_0)\in D$, 使得$\left|\frac{\partial f}{\partial x}(x_0,y_0)\right|^2+\left|\frac{\partial f}{\partial y}(x_0,y_0)\right|^2\le 8.$

证明：

$g(x,y)=f(x,y)-2\left(|x|+|y|\right)\in C\left(\overline{D}\right)$, 在$\partial D$上，$g=f-2\le -1$

$g(0,0)=f(0,0)\ge -1$, 因此存在$(x_0,y_0)\in D$，使得g在(x_0,y_0)达到最大值.

断言(x_0,y_0)为所求, 不妨设$x_0\ge 0, y_0\ge 0.$

(1): 当$x_0>0, y_0>0$, 此时在(x_0,y_0)邻域内, $g\in C^1$, 因此

$g_x(x_0,y_0)=f_x(x_0,y_0)-2=0, g_y(x_0,y_0)=f_y(x_0,y_0)-2=0$

故$\left|\frac{\partial f}{\partial x}(x_0,y_0)\right|^2+\left|\frac{\partial f}{\partial y}(x_0,y_0)\right|^2=8.$

(2): 当$x_0>0, y_0=0$, 对任意$\alpha\in[0,2\pi)$, 因为$g(x_0,0)$是最大值, 所以

$$0\ge\lim_{t\to 0^+}\frac{g(x_0+t\cos\alpha,t\sin\alpha)-g(x_0,0)}{t}$$

$$=f_x(x_0,0)\cos\alpha+f_y(x_0,0)\sin\alpha-\lim_{t\to 0^+}\frac{2\left(|x_0+t\cos\alpha|+|t\sin\alpha|-x_0\right)}{t}$$

$$=f_x(x_0,0)\cos\alpha+f_y(x_0,0)\sin\alpha-\lim_{t\to 0^+}\frac{2t\cos\alpha+2t|\sin\alpha|}{t}$$

$$=f_x(x_0,0)\cos\alpha+f_y(x_0,0)\sin\alpha-2\left(\cos\alpha+|\sin\alpha|\right).$$

取$\alpha=0,\alpha=\pi$, 则

$f_x(x_0,0)\le 2, -f_x(x_0,0)\le -2$

取$\alpha=\frac{\pi}{2},\alpha=\frac{3\pi}{2}$, 则

$f_y(x_0,0)\le 2, -f_y(x_0,0)\le 2$

因此$\left|\frac{\partial f}{\partial x}(x_0,y_0)\right|^2+\left|\frac{\partial f}{\partial y}(x_0,y_0)\right|^2\le 8.$

(3): 当$x_0=0, y_0>0$, 证明是完全一致的.

(4): 当$x_0=0, y_0=0$, 此时类似(2)就有

$0\ge f_x(x_0,0)\cos\alpha+f_y(x_0,0)\sin\alpha-2\left(|\cos\alpha|+|\sin\alpha|\right)$,

取合适的α就能导出$\left|\frac{\partial f}{\partial x}(x_0,y_0)\right|^2+\left|\frac{\partial f}{\partial y}(x_0,y_0)\right|^2\le 8.$

$D=\left\{\left(x,y\right)\in\mathbb{R}^2:x^2+y^2<1\right\}$, 正值有界函数$f\in C^2\left(D\right)$满足$\Delta\ln f\geq f^2$,

证明：$f\left(x,y\right)\leq\dfrac{2}{1-x^2-y^2},\forall\left(x,y\right)\in D.$

证明：

$F=\ln f\in C^2\left(D\right),h\triangleq\ln\dfrac{2}{1-x^2-y^2},$

$\Delta\left(h-F\right)\leq\Delta h-e^{2F}=e^{2h}-e^{2F}.$

注意$\lim\limits_{\left(x,y\right)\to\partial D}\left(h-F\right)=+\infty$, 因此存在$\left(x_0,y_0\right)\in D$，使得$h-F$取得最小值.

反证：假设最小值为负, 那么$\Delta\left(h-F\right)\left(x_0,y_0\right)=e^{2h}-e^{2F}<0.$

由刚开始极值判定的第四点, $D^2\left(h-F\right)$半正定,因此

$\Delta\left(h-F\right)\left(x_0,y_0\right)=tr\left(D^2\left(h-F\right)\left(x_0,y_0\right)\right)\geq 0$, 因此这是一个矛盾！

我们证明了$h\geq F$，这恰好就是$f\left(x,y\right)\leq\dfrac{2}{1-x^2-y^2},\forall\left(x,y\right)\in D.$