全国大学生数学竞赛非数学类模拟七

清疏竞赛考研数学

2023 年 10 月 9 日

摘要

$$\mathbb{N}=\{1,2,\cdots\},\mathbb{N}_0=\{0,1,2,\cdots\}.$$

模拟试题应当规定时间独立完成并给予反馈.

1 填空题

填空题 1.1 设 $f(x)$ 连续且 $f(0)\neq 0$, 则 $\lim\limits_{x\to 0}\frac{\int_0^x(x-t)f(t)dt}{x\int_0^x f(x-t)dt}=$ $\underline{\frac{1}{2}}$

填空题 1.2 设函数 $y=y(x)$ 由 $\int_0^x x\sin t^2dt=\int_0^{x+y}e^{-u^2}du$ 确定, 则 $y'(0)=$ $\underline{-1}$

$x\int_0^x \sin t^2 dt-\int_0^{x+y}e^{-u^2}du=0$

$y'(0)=-\frac{\int_0^x \sin t^2dt+x\sin x^2-e^{-(x+y)^2}}{-e^{-(x+y)^2}}\Big|_{x=0}=-1$

填空题 1.3 曲面 $r(u,v)=(\sin u,\cos u\sin v,\sin v)$ 在点 $\left(\frac{1}{2},\frac{\sqrt{3}}{4},\frac{1}{2}\right)$ 的切平面方程为 $\underline{-\frac{\sqrt{3}}{8}(x-\frac{1}{2})-\frac{3}{4}(y-\frac{\sqrt{3}}{4})+\frac{3\sqrt{3}}{8}(z-\frac{1}{2})=0}$. $r(u,v)=(f(u,v),g(u,v),h(u,v))$, 则切平面方向为 $\left(\frac{\partial(g,h)}{\partial(u,v)},-\frac{\partial(f,h)}{\partial(u,v)},\frac{\partial(f,g)}{\partial(u,v)}\right)$

填空题 1.4 假如 z,w 是 x,y 的连续可微函数且满足 $xw^3+yz^2+z^3=-1,zw^3-xz^3+y^2w=1$, 在 $(x,y,z,w)=(1,-1,-1,1)$. 计算 $\frac{\partial z}{\partial x}=$ $\underline{-\frac{5}{4}}$

填空题 1.5 设 $f(x)=\int_1^x\frac{\ln(1+t)}{t}dt$, 则 $\int_0^1\frac{f(x)}{\sqrt{x}}dx=$ $\underline{-2\pi-4\ln 2+8}$

1.1 $\lim\limits_{x\to 0}\frac{\int_0^x(x-t)f(t)dt}{x\int_0^x f(t)dt}=\lim\limits_{x\to 0}\frac{[x\int_0^x f(t)dt-\int_0^x tf(t)dt]'}{\int_0^x f(t)dt+xf(x)}$

$=\lim\limits_{x\to 0}\frac{\int_0^x f(t)dt}{\int_0^x f(t)dt+xf(x)}=\lim\limits_{x\to 0}\frac{1}{1+\frac{xf(x)}{\int_0^x f(t)dt}}=A$

又 $\lim\limits_{x\to 0}\frac{x}{\int_0^x f(t)dt}=\lim\limits_{x\to 0}\frac{1}{f(x)}=\frac{1}{f(0)}$, 故 $A=\frac{1}{2}$

1.3. $\frac{\partial(g,h)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v}\end{vmatrix}=\begin{vmatrix}-\sin u\sin v & \cos u\cos v\\ 0 & \cos v\end{vmatrix}=-\sin u\sin v\cos v$

$-\frac{1}{4}x$

2 选择题答案区

$\frac{\partial(f,h)}{\partial(u,v)}=\begin{vmatrix}\cos u & 0\\ 0 & \cos v\end{vmatrix}=\cos u\cdot\cos v$

$\frac{\partial(f,g)}{\partial(u,v)}=\begin{vmatrix}\cos u & 0\\ -\sin u\sin v & \cos u\cos v\end{vmatrix}=\cos^2 u\cos v$

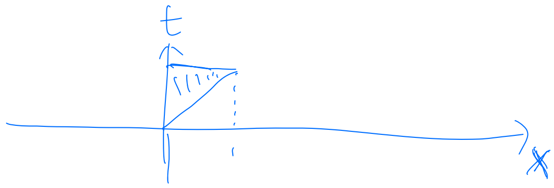
代入 $(u,v)=\left(\frac{\pi}{6},\frac{\pi}{6}\right)$，故 $\left(-\frac{\sqrt{3}}{8},-\frac{3}{4},\frac{3\sqrt{3}}{8}\right)$

1.4 $\begin{cases}xw^3+yz^2+z^3=-1\\ zw^3-xz^3+y^2w=1\end{cases}\Rightarrow\begin{cases}w^3+3xw^2\frac{\partial w}{\partial x}+2yz\frac{\partial z}{\partial x}+3z^2\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial x}w^3+3zw^2\frac{\partial w}{\partial x}-z^3-3xz^2\frac{\partial z}{\partial x}+y^2\frac{\partial w}{\partial x}=0\end{cases}$

即 $\begin{cases}1+3\frac{\partial w}{\partial x}+2\frac{\partial z}{\partial x}+3\frac{\partial z}{\partial x}=0\\ \frac{\partial z}{\partial x}-3\frac{\partial w}{\partial x}+1-3\frac{\partial z}{\partial x}+\frac{\partial w}{\partial x}=0\end{cases}\Rightarrow\begin{cases}5a+3b=-1\\ -2a-2b=-1\end{cases}$

$\begin{cases}10a+6b=-2\\ -6a-6b=-3\end{cases}\Rightarrow 4a=-5\Rightarrow a=-\frac{5}{4}$

1.5 $\int_0^1\frac{f(x)}{\sqrt{x}}dx=\int_0^1\int_1^x\frac{\ln(1+t)}{t\sqrt{x}}dt\,dx=-\int_0^1\int_x^1\frac{\ln(1+t)}{t\sqrt{x}}dt\,dx$



$=-\int_0^1dt\int_0^t\frac{\ln(1+t)}{t\sqrt{x}}dx$

$=-\int_0^1 2\frac{\ln(1+t)}{t}\sqrt{t}\,dt$

$=-2\int_0^1\frac{\ln(1+t)}{\sqrt{t}}dt$

$t=u^2$

$=-4\int_0^1\frac{\ln(1+u^2)}{u}u\,du$

$=-4\int_0^1\ln(1+u^2)du$

$=-4[u\ln(1+u^2)|_0^1-\int_0^1 u\,d(\ln(1+u^2))]$

$=$ 可求~~原~~函数

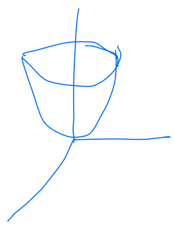
$=-2\pi-4\ln2+8$

3 解答题

解答题 3.1 计算 $\iiint_D 2xdV$ 这里 D 由 $5(z^2+y^2)=x, x=5$ 围成的有界区域.

解: 不妨考虑 D: $z=5(x^2+y^2)$ 和 $z=5$ 围成 计算 $\iiint_D 2z\,dV$

$z=5r^2=5 \Rightarrow r=1$



$$\iiint_D 2x\,dv = \int_0^{2\pi} d\theta \int_0^1 r\,dr \int_{5r^2}^{5} 2z\,dz$$

$$= 2\pi \int_0^1 \left[5^2-(5r^2)^2\right] \cdot r\,dr$$

$$= \pi \int_0^1 \left[25-25x^2\right] dx$$

$$= \frac{50}{3}\pi$$

解答题 3.2 设 $\varphi,\psi\in C^1(\mathbb{R}),\varphi(0)=-2,\psi(0)=1$ 且对任意一光滑闭曲线 L, 都有

$$\int_L 2\left[x\varphi(y)+\psi(y)\right]dx+\left[x^2\varphi(y)+2xy^2-2x\varphi(y)\right]dy=0.$$

(1): 求 φ,ψ.

(2): 对简单曲线 $L:(0,0)\to\left(\pi,\frac{\pi}{2}\right)$, 计算上述积分.

解 (1): $2x\varphi(y)+2y^2-2\varphi(y)=2x\varphi'(y)+2\psi'(y)$

故 $2\varphi(y)=2\varphi'(y)$, $y^2-\varphi(y)=\psi'(y)$

故 $\varphi(y)=-2e^y$, $\psi'(y)=y^2+2e^y$

$\psi(y)=\frac{y^3}{3}+2e^y-1$

(2): $\int_L \left(-4xe^y+\frac{2}{3}y^3+4e^y-2\right)dx+\left[-2x^2e^y+2xy^2+4xe^y\right]dy$

$$\frac{d}{dy}\left[-2x^2e^y+\left(\frac{2}{3}y^3+4e^y-2\right)x+C(y)\right]$$

$=-2x^2e^y+(2y^2+4e^y)x+c'(y)$

$=-2x^2e^y+2xy^2+4xe^x$

故 $C(y)=0$, 故所求 $=\left[-2x^2e^y+\left(\frac{2}{3}y^3+4e^y-2\right)x\right]_{(0,0)}^{(\pi,\frac{\pi}{2})}$

$=-2\pi^2e^{\frac{\pi}{2}}+\pi\left(\frac{\pi^3}{12}+4e^{\frac{\pi}{2}}-2\right)$

解答题 3.3 设 $f, g \in C^1[a,b] \bigcap D^2(a,b)$ 且满足 $f''(x), g''(x) \neq 0, \forall x \in (a,b)$, 证明存在 $c \in (a,b)$, 使得

$$\frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}.$$

证明：$F(x) = f(x) - f(a) - (x-a)f'(a)$, $F'(x) = f'(x) - f'(a)$

$G(x) = g(x) - g(a) - (x-a)g'(a)$, $G'(x) = g'(x) - g'(a)$

由Cauchy中值定理, $\dfrac{F(b)-F(a)}{G(b)-G(a)} = \dfrac{F'(c_1)}{G'(c_1)} = \dfrac{f'(c_1)-f'(a)}{g'(c_1)-g'(a)} = \dfrac{f''(c)}{g''(c)}$, $c \in (a,b)$.

因此我们完成了证明.

解答题 3.4 设 $(a_n)_{n=1}^{\infty}$, $(b_n)_{n=1}^{\infty}$ 是两个正数列, 证明下述结果等价.

(a): 存在正实数列 $(c_n)_{n=1}^{\infty}$, 使得 $\sum\limits_{n=1}^{\infty} \frac{a_n}{c_n}$, $\sum\limits_{n=1}^{\infty} \frac{c_n}{b_n}$ 都收敛.

(b): $\sum\limits_{n=1}^{\infty} \sqrt{\frac{a_n}{b_n}}$ 收敛.

证明: (a) $\Rightarrow$ (b): 设正实数列 (c_n) 使 $\sum\limits_{n=1}^{+\infty} \frac{a_n}{c_n}$, $\sum\limits_{n=1}^{+\infty} \frac{c_n}{b_n}$ 收敛.

则 $\sum\limits_{n=1}^{+\infty} \sqrt{\frac{a_n}{b_n}} \le \frac{1}{2} \sum\limits_{n=1}^{+\infty} \left(\frac{a_n}{c_n} + \frac{c_n}{b_n}\right) < +\infty$, 故 $\sum\limits_{n=1}^{+\infty} \sqrt{\frac{a_n}{b_n}}$ 收敛.

(b) $\Rightarrow$ (a): 若 $\sum\limits_{n=1}^{+\infty} \sqrt{\frac{a_n}{b_n}}$ 收敛, 取 $c_n = \sqrt{a_n b_n}$,

则 $\sum\limits_{n=1}^{+\infty} \frac{a_n}{c_n} = \sum\limits_{n=1}^{+\infty} \sqrt{\frac{a_n}{b_n}} = \sum\limits_{n=1}^{+\infty} \frac{c_n}{b_n}$. 证毕!

解答题 3.5 让 $a_0=1, a_1=\frac{1}{2}$, $a_{n+1}=\frac{na_n^2}{1+(n+1)a_n}, n\geqslant 1$, 证明 $\sum_{k=0}^{\infty}\frac{a_{k+1}}{a_k}$ 收敛并求值.

证明: $\frac{[1+(n+1)a_n]a_{n+1}}{a_n}=na_n=\frac{a_{n+1}+(n+1)a_{n+1}a_n}{a_n}$

故 $\frac{a_{n+1}}{a_n}=na_n-(n+1)a_{n+1}$, $n\geq 1$ 且 na_n 递减有下界.

$$\sum_{k=0}^{\infty}\frac{a_{k+1}}{a_k}=\frac{a_1}{a_0}+\sum_{k=1}^{+\infty}(ka_k-(k+1)a_{k+1})$$
$$=\frac{1}{2}+1\cdot a_1-\lim_{n\to+\infty}na_n$$
$$=1-\lim_{n\to+\infty}na_n \text{ 收敛.}$$

故 $\lim_{n\to+\infty}\frac{a_{n+1}}{a_n}=0$, 故 $\exists N\geq 1, \forall n\geq N$, 有 $a_{n+1}\leq\frac{1}{2}a_n$. 则

$a_{n+1}\leq\frac{1}{2}a_n\leq\left(\frac{1}{2}\right)^2a_{n-1}\leq\cdots\leq\left(\frac{1}{2}\right)^{n-N+1}a_N$.

故 $\lim_{n\to+\infty}na_n=0$. 故 $\sum_{k=0}^{\infty}\frac{a_{k+1}}{a_k}=1$。

解答题 3.6 设 $f(x) = e^{\frac{x^2}{2}} \int_x^{\infty} e^{-\frac{y^2}{2}} dy$, 证明

(1): f 是 $[0,+\infty)$ 的递减函数且 $\lim_{x\to+\infty} f(x) = 0$.

(2): 试计算 $\lim_{x\to+\infty} xf(x)$.

证明:

(1): $f'(x) = x\cdot e^{\frac{x^2}{2}}\int_x^{\infty} e^{-\frac{y^2}{2}}dy - e^{\frac{x^2}{2}}\cdot e^{-\frac{x^2}{2}}$

$= xe^{\frac{x^2}{2}}\int_x^{\infty} e^{-\frac{y^2}{2}}dy - 1$

下证 $\int_x^{\infty} e^{-\frac{y^2}{2}}dy \leq \frac{1}{x}e^{-\frac{x^2}{2}}$,

令 $F(x) = \int_x^{\infty} e^{-\frac{y^2}{2}}dy - \frac{1}{x}e^{-\frac{x^2}{2}}$ $\quad F'(x) = -e^{-\frac{x^2}{2}} + \frac{e^{-\frac{x^2}{2}}}{x^2} - \frac{1}{x}\cdot e^{-\frac{x^2}{2}}\cdot(-x)$

$= \frac{e^{-\frac{x^2}{2}}}{x^2} > 0.$

故 $F(x) \leq F(+\infty) = 0$, 故 $f'(x) \leq 0$, 故 $f(x)$ 递减.

(2): 我们直接求 $\lim_{x\to+\infty} xf(x)$, 从而 $\lim_{x\to+\infty} f(x) = 0$.

$\lim_{x\to+\infty} xf(x) = \lim_{x\to+\infty} \frac{\int_x^{+\infty} e^{-\frac{y^2}{2}}dy}{\frac{1}{x}e^{-\frac{x^2}{2}}} = \lim_{x\to+\infty} \frac{-e^{-\frac{x^2}{2}}}{-\frac{1}{x^2}\cdot e^{-\frac{x^2}{2}} - x\cdot\frac{1}{x}e^{-\frac{x^2}{2}}}$

$= \lim_{x\to+\infty} \frac{-1}{-\frac{1}{x^2} - 1} = 1.$