第十届清疏竞赛班非数学类 15:

定积分的计算

课后学习第九届非数学类 15

最重要的方法：

$$\int_a^b f(x)dx=\int_a^b f(b+a-x)dx=\frac{1}{2}\int_a^b\left[f(b+a-x)+f(x)\right]dx$$

$$=\int_a^{\frac{a+b}{2}}\left[f(b+a-x)+f(x)\right]dx=\int_{\frac{a+b}{2}}^b\left[f(b+a-x)+f(x)\right]dx$$

$$\int_0^\infty f(x)dx=\int_0^1 f(x)dx+\int_1^\infty f(x)dx=\int_0^1\left[f(x)+\frac{f\left(\frac{1}{x}\right)}{x^2}\right]dx$$

例子：计算

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}dx, \int_0^{\infty} \frac{\ln x}{1+x+x^2}dx, \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\lambda} x}dx, \int_0^1 \frac{\arcsin\sqrt{x}}{\sqrt{1-x+x^2}}dx, \int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x}dx$$

$$\int_{-\pi}^{\pi} \frac{x\sin x \cdot \arctan e^x}{1+\cos^2 x}dx, \int_0^{2\pi} \sin(\sin x + nx)dx, n \in \mathbb{N}.$$

$$(1): \int_0^1 \frac{\ln(1+x)}{1+x^2}dx = \int_0^{\frac{\pi}{4}} \frac{\ln(1+\tan y)}{1+\tan^2 y}d\tan y = \int_0^{\frac{\pi}{4}} \ln(1+\tan y)dy$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-y\right)\right)dy = \int_0^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan y}{1+\tan y}\right)dy$$

$$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan y}\right)dy = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[\ln\left(\frac{2}{1+\tan y}\right) + \ln(1+\tan y)\right]dy = \frac{\pi \ln 2}{8}.$$

$$(2): \int_0^{\infty} \frac{\ln x}{1+x+x^2}dx = \int_0^{\infty} \frac{\ln\frac{1}{x}}{1+\frac{1}{x}+\frac{1}{x^2}} \frac{1}{x^2}dx = -\int_0^{\infty} \frac{\ln x}{x^2+x+1}dx$$

$$\Rightarrow \int_0^{\infty} \frac{\ln x}{1+x+x^2}dx = 0$$

$$(3): \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\lambda} x}dx = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\lambda} x}dx + \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\lambda}\left(\frac{\pi}{2}-x\right)}dx\right]$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\lambda} x}dx + \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{\tan^{\lambda} x}}dx\right]$$

$$= \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{\lambda} x}dx + \int_0^{\frac{\pi}{2}} \frac{\tan^{\lambda} x}{1+\tan^{\lambda} x}dx\right] = \frac{\pi}{4}.$$

(4):记得熟悉一下$\arcsin x,\arctan x$的各种恒等式

$$\left(\arcsin\sqrt{x}+\arcsin\sqrt{1-x}\right)'=0\Rightarrow\arcsin\sqrt{x}+\arcsin\sqrt{1-x}=\frac{\pi}{2}$$

$$\left(\arcsin hx\right)'=\frac{1}{\sqrt{1+x^2}}$$

$$\int_0^1\frac{\arcsin\sqrt{x}}{\sqrt{1-x+x^2}}dx=\frac{1}{2}\left[\int_0^1\frac{\arcsin\sqrt{x}}{\sqrt{1-x+x^2}}+\frac{\arcsin\sqrt{1-x}}{\sqrt{1-(1-x)+(1-x)^2}}dx\right]$$

$$=\frac{1}{2}\left[\int_0^1\frac{\arcsin\sqrt{x}}{\sqrt{1-x+x^2}}+\frac{\arcsin\sqrt{1-x}}{\sqrt{1-x+x^2}}dx\right]$$

$$=\frac{\pi}{4}\int_0^1\frac{1}{\sqrt{1-x+x^2}}dx=\frac{\pi}{4}\int_0^1\frac{1}{\sqrt{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}}dx$$

$$=\frac{\pi}{4}\sqrt{\frac{4}{3}}\int_0^1\frac{1}{\sqrt{\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)^2+1}}dx$$

$$=\frac{\pi}{4}\sqrt{\frac{4}{3}}\frac{\sqrt{3}}{2}\arcsin h\left(\frac{2}{\sqrt{3}}x-\frac{1}{\sqrt{3}}\right)\Big|_0^1$$

$$=\frac{\pi}{4}\ln 3.$$

(5):

$$\int_{-\pi}^{\pi}\frac{\sin(nx)}{(1+2^x)\sin x}dx=\int_0^{\pi}\left[\frac{\sin(nx)}{(1+2^x)\sin x}+\frac{\sin(nx)}{(1+2^{-x})\sin x}\right]dx$$

$$=\int_0^{\pi}\frac{\sin(nx)}{\sin x}dx=I_n$$

$$I_{n+2}-I_n=\int_0^{\pi}\frac{\sin((n+2)x)-\sin(nx)}{\sin x}dx=\int_0^{\pi}2\cos((n+1)x)\frac{\sin x}{\sin x}dx$$

$$=\int_0^{\pi}2\cos((n+1)x)dx=0.$$

因此当n为偶数,我们有$I_n=I_0=\int_0^{\pi}\frac{\sin(nx)}{\sin x}dx=0$

n为奇数,我们有$I_n=I_1=\int_0^{\pi}\frac{\sin(x)}{\sin x}dx=\pi$

$(6):$

$$\left(\arctan x+\arctan\frac{1}{x}\right)'=0,\text{所以}\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2},x>0$$

$$\int_{-\pi}^{\pi}\frac{x\sin x\cdot\arctan e^{x}}{1+\cos^2 x}dx=\int_0^{\pi}\left[\frac{x\sin x\cdot\arctan e^{x}}{1+\cos^2 x}+\frac{x\sin x\cdot\arctan e^{-x}}{1+\cos^2 x}\right]dx$$

$$=\frac{\pi}{2}\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left[\frac{x\sin x}{1+\cos^2 x}+\frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\right]dx$$

$$=\frac{\pi^2}{2}\int_0^{\frac{\pi}{2}}\frac{\sin x}{1+\cos^2 x}dx=-\frac{\pi^2}{2}\arctan\cos x\,|_0^{\frac{\pi}{2}}=\frac{\pi^3}{8}.$$

$(7):\int_0^{2\pi}\sin(\sin x+nx)dx, n\in\mathbb{N}.$

$$\int_0^{2\pi}\sin(\sin x+nx)dx$$

$$=\int_0^{\pi}\left[\sin(\sin x+nx)+\sin\left(\sin(2\pi-x)+n(2\pi-x)\right)\right]dx$$

$$=\int_0^{\pi}\left[\sin(\sin x+nx)+\sin(-\sin x-nx)\right]dx=0.$$

递推法：

课后需熟$gamma$函数和$beta$函数的定义和性质.

计算$\int_0^1 x^m \ln^n x dx, \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx, m, n \in \mathbb{N}$,

方法1：

$$\int_0^1 x^m \ln^n x dx \overset{x=e^{-y}}{=} \int_0^{\infty} e^{-(m+1)y} (-y)^n dy$$

$$= \frac{(-1)^n}{m+1} \int_0^{\infty} e^{-z} \left(\frac{z}{m+1} \right)^n dz$$

$$= \frac{(-1)^n}{(m+1)^{n+1}} \int_0^{\infty} e^{-z} z^n dz$$

$$= \frac{(-1)^n \Gamma(n+1)}{(m+1)^{n+1}} = \frac{(-1)^n n!}{(m+1)^{n+1}}$$

方法2：可以直接分部积分, 细节自行完成.

$$I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin(nx) dx = \int_0^{\frac{\pi}{2}} \cos^{n-1} x (\cos x \cdot \sin(nx)) dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x (\sin(n+1)x + \sin(n-1)x) dx$$

$$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x (\sin nx \cdot \cos x + \cos nx \cdot \sin x) dx + \frac{I_{n-1}}{2}$$

$$= \frac{I_n + I_{n-1}}{2} - \frac{1}{2} \int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos nx d \cos x$$

$$= \frac{I_n + I_{n-1}}{2} - \frac{1}{2n} \int_0^{\frac{\pi}{2}} \cos nx d \cos^n x$$

$$= \frac{I_n + I_{n-1}}{2} - \frac{-1 + n\int_0^{\frac{\pi}{2}} \cos^n x \cdot \sin nx dx}{2n}$$

$$= \frac{I_n + I_{n-1}}{2} - \frac{-1 + nI_n}{2n} = \frac{I_{n-1}}{2} + \frac{1}{2n}$$

$2^n I_n = 2^{n-1} I_{n-1} + \dfrac{2^{n-1}}{n}, I_0 = 0$, 所以

$$I_n = \frac{1}{2^n} \sum_{k=1}^{n} \frac{2^{k-1}}{k}.$$

级数方法

积累一些经典级数和函数的大概样子, 例如

$$\sum\frac{\sin(nx)}{n},\sum\frac{\sin(nx)}{n!},\sum\frac{q^n\sin(nx)}{n!}$$

计算$\int_0^\pi \ln\left(1-2a\cos x+a^2\right)dx, a\in(0,1)\cup(1,+\infty)$

当$a\in(0,1)$, 利用$\ln\left(1-2a\cos x+a^2\right)=-2\sum_{n=1}^\infty\frac{a^n\cos(nx)}{n}$

$$\int_0^\pi \ln\left(1-2a\cos x+a^2\right)dx=\int_0^\pi -2\sum_{n=1}^\infty\frac{a^n\cos(nx)}{n}dx$$

$$=-2\sum_{n=1}^\infty\frac{a^n}{n}\int_0^\pi\cos(nx)\,dx=0.$$

当$a\in(1,+\infty)$, $\int_0^\pi \ln\left(1-2a\cos x+a^2\right)dx=\pi\ln a^2+\int_0^\pi\ln\left(\frac{1}{a^2}-\frac{2}{a}\cos x+1\right)dx$

$$=2\pi\ln a.$$

计算$\int_0^1\ln x\cdot\ln(1-x)\,dx$

$$\ln(1-x)=-\sum_{n=1}^\infty\frac{x^n}{n}, |x|<1$$

$$\int_0^1\ln x\cdot\ln(1-x)\,dx=-\int_0^1\ln x\cdot\sum_{n=1}^\infty\frac{x^n}{n}dx$$

$$=-\sum_{n=1}^\infty\frac{1}{n}\int_0^1 x^n\ln x\,dx=\sum_{n=1}^\infty\frac{1}{n(n+1)^2}$$

$$=\sum_{n=1}^\infty\frac{1}{(n+1)}\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$=\sum_{n=1}^\infty\frac{1}{n(n+1)}-\sum_{n=1}^\infty\frac{1}{(n+1)^2}$$

$$=1-\left(\frac{\pi^2}{6}-1\right)=2-\frac{\pi^2}{6}$$

注意：考试中需要说明换序的理由,
如果没时间就用控制收敛Levi定理去说.

严格证明:$\int_0^1 \ln x \cdot \sum_{n=1}^{\infty} \frac{x^n}{n} dx = \sum_{n=1}^{\infty} \int_0^1 \ln x \cdot \frac{x^n}{n} dx,$

事实上,$\sum_{n=1}^{\infty} \int_0^1 \ln x \cdot \frac{x^n}{n} dx = \lim_{m \to \infty} \sum_{n=1}^{m} \int_0^1 \ln x \cdot \frac{x^n}{n} dx$

$$= \lim_{m \to \infty} \int_0^1 \ln x \cdot \sum_{n=1}^{m} \frac{x^n}{n} dx = \lim_{m \to \infty} \int_0^1 \ln x \cdot \left(\sum_{n=1}^{\infty} \frac{x^n}{n} - \sum_{n=m+1}^{\infty} \frac{x^n}{n} \right) dx$$

只需证明$\lim_{m \to \infty} \int_0^1 \ln x \cdot \sum_{n=m+1}^{\infty} \frac{x^n}{n} dx = 0.$

$\forall \eta \in (0,1), \left| \int_0^1 \ln x \cdot \sum_{n=m+1}^{\infty} \frac{x^n}{n} dx \right| \le \int_0^1 |\ln x| \cdot \frac{x^{m+1}}{1-x} dx$

$$\le \int_0^{\eta} |\ln x| \cdot \frac{\eta^{m+1}}{1-x} dx + \int_{\eta}^1 |\ln x| \cdot \frac{1}{1-x} dx$$

$$\le \eta^{m+1} \int_0^1 \frac{|\ln x|}{1-x} dx + \int_{\eta}^1 |\ln x| \cdot \frac{1}{1-x} dx$$

所以$\lim_{m \to \infty} \left| \int_0^1 \ln x \cdot \sum_{n=m+1}^{\infty} \frac{x^n}{n} dx \right| \le \int_{\eta}^1 \frac{|\ln x|}{1-x} dx$

由$\eta \to 1^-$和$\int_0^1 \frac{|\ln x|}{1-x} dx < \infty$知$\lim_{m \to \infty} \left| \int_0^1 \ln x \cdot \sum_{n=m+1}^{\infty} \frac{x^n}{n} dx \right| = 0$

计算

$$\int_0^1 \frac{\ln^k x}{1+x}dx = \int_0^1 \sum_{j=0}^{\infty} (-x)^j \ln^k x dx$$

$$= \sum_{j=0}^{\infty} (-1)^j \int_0^1 x^j \ln^k x dx = \sum_{j=0}^{\infty} (-1)^j \frac{(-1)^j k!}{(j+1)^{k+1}}$$

$$= k! \sum_{j=0}^{\infty} \frac{1}{(j+1)^{k+1}} = k! \zeta(k+1)$$

计算$\int_0^1 \left(\left[\frac{2}{x} \right] - 2\left[\frac{1}{x} \right] \right) dx$

$$\int_0^1 \left(\left[\frac{2}{x} \right] - 2\left[\frac{1}{x} \right] \right) dx = \int_1^{\infty} \frac{[2x]-2[x]}{x^2} dx$$

$$= \sum_{j=1}^{\infty} \left(\int_j^{j+\frac{1}{2}} \frac{[2x]-2[x]}{x^2} dx + \int_{j+\frac{1}{2}}^{j+1} \frac{[2x]-2[x]}{x^2} dx \right)$$

$$= \sum_{j=1}^{\infty} \left(\int_j^{j+\frac{1}{2}} \frac{2j-2j}{x^2} dx + \int_{j+\frac{1}{2}}^{j+1} \frac{2j+1-2j}{x^2} dx \right)$$

$$= \sum_{j=1}^{\infty} \left(\int_{j+\frac{1}{2}}^{j+1} \frac{1}{x^2} dx \right) = \sum_{j=1}^{\infty} \left(\frac{1}{j+\frac{1}{2}} - \frac{1}{j+1} \right)$$

对于这个级数,$digamma$函数的引入是更好算的，
有兴趣的同学可以去$wolfram$搜素这个函数学习.

$$\sum_{j=1}^{\infty} \left(\frac{1}{j+\frac{1}{2}} - \frac{1}{j+1} \right) = \lim_{n\to\infty} \sum_{j=1}^{n} \left(\frac{2}{2j+1} - \frac{1}{j+1} \right)$$

而$H_n = \sum_{j=1}^{n} \frac{1}{j} = \ln n + \gamma + o(1)$, 所以

$$\sum_{j=1}^{n} \frac{1}{j+1} = \ln(n+1) + \gamma - 1 + o(1)$$

$$\sum_{j=1}^{n} \frac{2}{2j+1} = 2\left(H_{2n+1} - 1 - \frac{1}{2} H_n \right)$$

代入即知$\int_0^1 \left(\left[\frac{2}{x} \right] - 2\left[\frac{1}{x} \right] \right) dx = 2\ln 2 - 1$

重积分方法

$$\int_0^{\infty}\frac{\arctan(\beta x)-\arctan(\alpha x)}{x}dx,\int_0^{\infty}\frac{\cos(\beta x)}{\alpha^2+x^2}dx\ (\alpha,\beta>0)$$

解:

$$\int_0^{\infty}\frac{\arctan(\beta x)-\arctan(\alpha x)}{x}dx=\int_0^{\infty}\int_{\alpha}^{\beta}\frac{1}{1+y^2x^2}dydx$$

$$=\int_{\alpha}^{\beta}\int_0^{\infty}\frac{1}{1+y^2x^2}dxdy=\int_{\alpha}^{\beta}\frac{\arctan yx}{y}\Big|_0^{\infty}dy$$

$$=\int_{\alpha}^{\beta}\frac{\pi}{2y}dy=\frac{\pi}{2}\ln\frac{\beta}{\alpha}.$$

$$\int_0^{\infty}\frac{\cos(\beta x)}{\alpha^2+x^2}dx=\int_0^{\infty}\cos(\beta x)dx\int_0^{\infty}e^{-(\alpha^2+x^2)y}dy$$

$$=\int_0^{\infty}e^{-\alpha^2y}dy\int_0^{\infty}\cos(\beta x)e^{-x^2y}dx$$

$$=\frac{1}{2}\int_0^{\infty}e^{-\alpha^2y}dy\int_{-\infty}^{\infty}\cos(\beta x)e^{-x^2y}dx$$

$$=\frac{1}{2}\int_0^{\infty}e^{-\alpha^2y}dy\int_{-\infty}^{\infty}\mathrm{Re}\,e^{i\beta x}e^{-x^2y}dx$$

$$=\frac{1}{2}\mathrm{Re}\int_0^{\infty}e^{-\alpha^2y}dy\int_{-\infty}^{\infty}e^{i\beta x-x^2y}dx$$

$$=\frac{1}{2}\mathrm{Re}\int_0^{\infty}e^{-\alpha^2y}dy\int_{-\infty}^{\infty}e^{-y\left(x-\frac{i\beta}{2y}\right)^2-\frac{\beta^2}{4y}}dx$$

$$=\frac{1}{2}\mathrm{Re}\int_0^{\infty}e^{-\alpha^2y-\frac{\beta^2}{4y}}dy\int_{-\infty}^{\infty}e^{-y\left(x-\frac{i\beta}{2y}\right)^2}dx$$

$$=\frac{\sqrt{\pi}}{2}\mathrm{Re}\int_0^{\infty}\frac{1}{\sqrt{y}}e^{-\alpha^2y-\frac{\beta^2}{4y}}dy$$

$$=\sqrt{\pi}\int_0^{\infty}e^{-\alpha^2z^2-\frac{\beta^2}{4z^2}}dz$$

$$=\frac{\pi e^{-\alpha\beta}}{2\alpha}$$

Froullani积分：第九届非数学类15学习.

倒代换类型：

$$\int_0^{\infty} e^{-ay^2-\frac{b}{y^2}}dy, a,b>0$$

$$\int_0^{\infty} e^{-ay^2-\frac{b}{y^2}}dy=\frac{1}{\sqrt{a}}\int_0^{\infty} e^{-y^2-\frac{ab}{y^2}}dy$$

先计算$a=1$的情况，事实上

$$I=\int_0^{\infty} e^{-y^2-\frac{b}{y^2}}dy=e^{-2\sqrt{b}}\int_0^{\infty} e^{-\left(y-\frac{\sqrt{b}}{y}\right)^2}dy$$

$$=e^{-2\sqrt{b}}\int_0^{\infty}\frac{\sqrt{b}}{y^2}e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2}dy$$

$$=\frac{e^{-2\sqrt{b}}}{2}\int_0^{\infty}\left(1+\frac{\sqrt{b}}{y^2}\right)e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2}dy$$

$$=\frac{e^{-2\sqrt{b}}}{2}\int_0^{\infty} e^{-\left(\frac{\sqrt{b}}{y}-y\right)^2}d\left(\frac{\sqrt{b}}{y}-y\right)$$

$$=\frac{e^{-2\sqrt{b}}}{2}\int_0^{\infty} e^{-x^2}dx=\frac{\sqrt{\pi}}{2}e^{-2\sqrt{b}}$$

对于一般情况$\int_0^{\infty} e^{-ay^2-\frac{b}{y^2}}dy=\frac{\sqrt{\pi}}{2\sqrt{a}}e^{-2\sqrt{ab}}$.

含参积分求导(和重积分方法可以互相转化):

计算:$\int_1^{\infty}\frac{\arctan(\alpha x)}{x^2\sqrt{x^2-1}}dx$

$F(\alpha)=\int_1^{\infty}\frac{\arctan(\alpha x)}{x^2\sqrt{x^2-1}}dx,$

$F'(\alpha)=\int_1^{\infty}\frac{1}{(1+\alpha^2x^2)x\sqrt{x^2-1}}dx=\frac{\pi}{2}\left(1-\frac{\alpha}{\sqrt{1+\alpha^2}}\right)$

$F(0)=0,$ 所以 $F(\alpha)=\int_0^{\alpha}\frac{\pi}{2}\left(1-\frac{y}{\sqrt{1+y^2}}\right)dy$

$=\frac{\pi}{2}\left(1+\alpha-\sqrt{1+\alpha^2}\right)$

$\psi(s)=\int_0^{\infty}\frac{\ln(1+sx)}{x(1+x)}dx, s>0,$ 证明 $\psi(s)+\psi\left(\frac{1}{s}\right)=\frac{\pi^2}{3}+\frac{\ln^2 s}{2}$

$\psi'(s)=\int_0^{\infty}\frac{1}{(1+sx)(1+x)}dx=\frac{\ln s}{s-1}$

$-\frac{1}{s^2}\psi'\left(\frac{1}{s}\right)=-\frac{1}{s^2}\frac{\ln\frac{1}{s}}{\frac{1}{s}-1}=\frac{\ln s}{s-s^2}$

显然 $\left[\psi(s)+\psi\left(\frac{1}{s}\right)\right]'=\frac{\ln s}{s},$ 只需证明 $\psi(1)=\frac{\pi^2}{6}$

而 $\psi(1)=\int_0^1\frac{\ln s}{s-1}ds=-\sum_{n=1}^{\infty}\int_0^1\ln s\cdot s^n ds=\frac{\pi^2}{6}.$