# 全国大学生数学竞赛非数学类模拟四

清疏竞赛考研数学

2023 年 9 月 11 日


**摘要**

$$\mathbb{N}=\{1,2,\cdots\},\mathbb{N}_0=\{0,1,2,\cdots\}.$$

模拟试题应当规定时间独立完成并给予反馈.


## 1 填空题

**填空题 1.1** 计算 $\lim\limits_{x\to+\infty}\left((2x)^{1+\frac{1}{2x}}-x^{1+\frac{1}{x}}-x\right)=$ ln2

**填空题 1.2** $f(x)=\frac{1}{1+|x|}+\frac{1}{1+|x-1|}$ 在 $\mathbb{R}$ 上的最大值为 $=$ $\frac{3}{2}$

**填空题 1.3** 两曲面 $3x^2+2y^2=2z+1,x^2+y^2+z^2-4y-2z+2=0$ 在交点 $(1,1,2)$ 处两法线的夹角 *(取锐角)* 为 $\frac{\pi}{2}$

**填空题 1.4** 计算级数 $\sum\limits_{n=0}^{\infty}\frac{2^{n+1}(n!)^2}{(2n+2)!}=$ $\frac{\pi^2}{8}$

**填空题 1.5** 计算 $\int_0^1\left[(e-1)\sqrt{\ln(1+(e-1)x)}+e^{x^2}\right]dx=$ $e$

1.1 : $(2x)^{1+\frac{1}{2x}}=2x\cdot e^{\frac{\ln(2x)}{2x}}$

$=2x[1+\frac{\ln(2x)}{2x}+\frac{1}{2}\frac{\ln^2(2x)}{4x^2}+o(\frac{\ln^2 x}{x^2})]$

$$\frac{o\left(\frac{\ln^2(2x)}{4x^2}\right)}{\frac{\ln^2 x}{x^2}}=\underbrace{\frac{o\left(\frac{\ln^2(2x)}{4x^2}\right)}{\frac{\ln^2(2x)}{4x^2}}}_{\to 0}\cdot\underbrace{\frac{\frac{\ln^2(2x)}{4x^2}}{\frac{\ln^2 x}{x^2}}}_{\to 1}\Rightarrow o\left(\frac{\ln^2(2x)}{4x^2}\right)=o\left(\frac{\ln^2 x}{x^2}\right)$$

$$x^{1+\frac{1}{x}} = x\cdot e^{\frac{\ln x}{x}} = x\left[1+\frac{\ln x}{x}+\frac{\ln^2 x}{2x^2}+o\left(\frac{\ln^2 x}{x^2}\right)\right]$$

故原极限 $=\lim\limits_{x\to+\infty}\left[\ln 2+\frac{\ln^2(2x)}{4x}-\frac{\ln^2 x}{2x}+o\left(\frac{\ln^2 x}{x}\right)\right]=\ln 2.$

## 2 选择题答案区

1.2：$f(x)=\frac{1}{1+|x|}+\frac{1}{1+|x-1|}=\begin{cases}\frac{2x+1}{x(1+x)} & x\geq 1\\ \frac{3}{(1+x)(2-x)}, & 0\leq x<1\\ \frac{3-2x}{(x-1)(x-2)}, & x<0\end{cases}$

求导并令 $f'(x)=0$，得 $x_0$，再比较 $x\to\infty$ 极限值和不可导点 0,1 处函数值就得在 $x=0,1$ 取最大.

1.3 ：对 $3x^2+2y^2-2z-1=0$ ，对 $x^2+y^2+z^2-4y-2z+2=0$

$(6x,4y,-2)$ ， $(2x,2y-4,2z-2)$

故法线方向分别是 $(3,2,-1)$， $(1,-1,1)$

故 $|\cos\text{夹角}|=\frac{a\cdot b}{|a||b|}=0$，故 $\frac{\pi}{2}$.

1.4 记忆 $\arcsin^2 x$ 模型，或者 $\frac{\arcsin x}{\sqrt{1-x^2}}$ 模型. $2^n n!=(2n)!!$

$\frac{\arcsin x}{\sqrt{1-x^2}}=\sum\limits_{n=0}^{+\infty}\frac{(2n)!!}{(2n+1)!!}x^{2n+1}$ ，$\arcsin^2 x=\sum\limits_{n=0}^{+\infty}\frac{(2n)!!\cdot 2}{(2n+1)!!\,(2n+2)}x^{2n+2}$

$\sum\limits_{n=0}^{+\infty}\frac{2^{n+1}(n!)^2}{(2n+2)!}=\sum\limits_{n=0}^{+\infty}\frac{2^{n+1}\cdot(n!)^2}{(2n+2)!!\,(2n+1)!!}=\sum\limits_{n=0}^{+\infty}\frac{2^{n+1}(n!)^2}{2^{n+1}(n+1)!\,(2n+1)!!}$

$=\sum\limits_{n=0}^{\infty}\frac{n!}{(n+1)(2n+1)!!}$

$=\sum\limits_{n=0}^{+\infty}\frac{(2n)!!}{2^{n-1}(2n+2)(2n+1)!!}$

故 $\frac{\arcsin^2 x}{2x^4}=\sum\limits_{n=0}^{+\infty}\frac{(2n)!!}{(2n+1)!!(2n+2)}(x^2)^{n-1}$ ，令 $x=\frac{1}{\sqrt{2}}$，故所求为 $\frac{\left(\frac{\pi}{4}\right)^2}{\frac{1}{4}\cdot 2}=\frac{\pi^2}{8}$.

1.5 ： $\ln[1+(e-1)x]=t^2$ ，则 $x=\frac{e^{t^2}-1}{e-1}$，



$dx=\frac{2te^{t^2}}{e-1}dt$，故 $\int_0^1[(e-1)\sqrt{\ln(1+(e-1)x)}]dx=\int_0^1 2t^2e^{t^2}dt$

$$I=\int_0^1\left[2x^2e^{x^2}+e^{x^2}\right]dx=\int_0^1(2x^2+1)e^{x^2}dx$$
$$=xe^{x^2}\Big|_0^1=e$$

# 3 解答题

**解答题 3.1** 证明: 方程 $\frac{\partial^2 z}{\partial x^2}+2xy^2\frac{\partial z}{\partial x}+2\left(y-y^3\right)\frac{\partial z}{\partial y}+x^2y^2z=0$ 在 $x=uv, y=\frac{1}{v}$ 替换下形式不变.

$$\begin{cases} u=xy \\ v=\frac{1}{y} \end{cases}$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=yz_u$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=xz_u-\frac{1}{y^2}z_v$$

$$\frac{\partial^2 z}{\partial x^2}=y\left[z_{uu}\frac{\partial u}{\partial x}+z_{uv}\frac{\partial v}{\partial x}\right]=y^2z_{uu}$$

$$y^2z_{uu}+2xy^3z_u+2(y-y^3)\left(xz_u-\frac{1}{y^2}z_v\right)+x^2y^2z=0$$

$$\frac{1}{v^2}z_{uu}+2uv\cdot\frac{1}{v^3}z_u+\left(\frac{2}{v}-\frac{2}{v^3}\right)(uvz_u-v^2z_v)+u^2z=0$$

$$z_{uu}+2uz_u+\left(2v-\frac{2}{v}\right)(uvz_u-v^2z_v)+v^2u^2z=0$$

故 $z_{uu}+2uv^2z_u+2(v-v^3)z_v+v^2u^2z=0$.

**解答题 3.2** 计算

$$\iiint_{\Omega}\left(y^2+z^2\right)dxdydz,$$

这里

$$\Omega=\left\{(x,y,z): x^2+y^2\leqslant a^2\,(a>0)\,,|z|\leqslant\frac{h}{2}\,(h>0)\right\}.$$

解: $\begin{cases} x=r\cos\theta \\ y=r\sin\theta \\ z=z \end{cases}$ : $0\le r\le a$, $\theta\in[0,2\pi]$, $|z|\le\frac{h}{2}$

故 $\iiint_{\Omega}(y^2+z^2)dxdydz=\frac{1}{2}\iiint_{\Omega}(x^2+y^2)dxdydz+\iiint_{\Omega}z^2dxdydz$

$=\frac{1}{2}\int_0^{2\pi}d\theta\int_0^a r^3dr\int_{-\frac{h}{2}}^{\frac{h}{2}}dz+\int_0^{2\pi}d\theta\int_0^a rdr\int_{-\frac{h}{2}}^{\frac{h}{2}}z^2dz$

$=\frac{h}{2}\cdot2\pi\cdot\frac{a^4}{4}+2\pi\cdot\frac{1}{2}a^2\cdot\frac{2}{3}\left(\frac{h}{2}\right)^3$

$=\frac{\pi h}{4}a^4+\frac{\pi a^2h^3}{12}$

**解答题 3.3** 设 $f$ 在 $\mathbb{R}$ 上 $n+1$ 阶可微. 对 $a<b$, 若

$$\ln\left(\frac{f(b)+f'(b)+\cdots+f^{(n)}(b)}{f(a)+f'(a)+\cdots+f^{(n)}(a)}\right)=b-a,$$

证明存在 $c\in(a,b)$, 使得 $f^{(n+1)}(c)=f(c)$.

证: $F(x)=f(x)+f'(x)+\dots+f^{(n)}(x)$,

于是 $\ln\frac{F(b)}{F(a)}=b-a$, 于是 $\frac{F(b)}{F(a)}=e^{b-a}$

故: $\frac{F(b)}{e^b}=\frac{F(a)}{e^a}$, 因此令 $g(x)=\frac{F(x)}{e^x}$, 由罗尔, $\exists c\in(a,b)$, 使得

$$g'(c)=\frac{F'(c)-F(c)}{e^c}=0=\frac{f'(c)+f''(c)+\dots+f^{(n+1)}(c)-f(c)-f'(c)-\dots-f^{(n)}(c)}{e^c}$$

故 $f^{(n+1)}(c)=f(c)$, 故我们完成了证明.

**解答题 3.4** 设连续函数 $f:[1,8]\to\mathbb{R}$ 满足

$$\int_1^2 f\left(t^3\right)^2 dt+2\int_1^2 f\left(t^3\right) dt=\frac{2}{3}\int_1^8 f\left(t\right) dt-\int_1^2 \left(t^2-1\right)^2 dt.$$

求 $f$ 表达式.

解: $\frac{2}{3}\int_1^8 f(t)dt \overset{t=u^3}{=} \frac{2}{3}\int_1^2 f(u^3)\cdot 3u^2 du - \int_1^2 (t^2-1)^2 dt$

$= \int_1^2 2t^2 f(t^3)\, dt - \int_1^2 (t^2-1)^2 dt.$

注意到 $\int_1^2 \left[f^2(t^3)+2f(t^3)-2t^2 f(t^3)+(t^2-1)^2\right] dt$

$= \int_1^2 \left[f(t^3)-(t^2-1)\right]^2 dt \geq 0$

故 $f(t^3)=t^2-1$, 故 $f(t)=t^{\frac{2}{3}}-1$, $t\in[1,8]$

**解答题 3.5** 给定递增连续函数 $f:[0,1]\to(0,+\infty)$, 对每个 $a\geqslant 0$, 证明

$$\int_0^1 \frac{x^{a+1}}{f(x)}dx \leqslant \frac{a+1}{a+2}\int_0^1 \frac{x^a}{f(x)}dx.$$

证: $\frac{1}{a+1}\int_0^1 \frac{x^{a+1}}{f(x)}dx \leq \frac{1}{a+2}\int_0^1 \frac{x^a}{f(x)}dx$

$$\Leftarrow\Rightarrow \int_0^1 x^a dx \int_0^1 \frac{x^{a+1}}{f(x)}dx \leq \int_0^1 x^{a+1}dx \int_0^1 \frac{x^a}{f(x)}dx$$

$$\Leftarrow\Rightarrow \iint_{[0,1]^2} \frac{x^a y^{a+1}}{f(y)}dxdy \leq \iint_{[0,1]^2} \frac{x^{a+1}y^a}{f(y)}dxdy$$

$$\Leftarrow\Rightarrow \iint_{[0,1]^2} \frac{x^a y^a (y-x)}{f(y)}dxdy \leq 0$$

$$\Leftarrow\Rightarrow \iint_{[0,1]^2} \frac{y^a x^a (x-y)}{f(x)}dxdy \leq 0$$

$$\Leftarrow\Rightarrow \frac{1}{2}\iint_{[0,1]^2} x^a y^a (y-x)\left[\frac{1}{f(y)}-\frac{1}{f(x)}\right]dxdy \leq 0.$$

最后一个等价利用 $A=B=\frac{A+B}{2}$ 和 $f$ 递增.

**解答题 3.6** 设 $f:[0,1]\to\mathbb{R}$ 是连续可微的递增函数，满足

$$\lim_{x\to 0^+}\frac{f(x)}{x}=1.$$

证明

(1): 序列

$$x_n=f\left(\frac{1}{1}\right)+f\left(\frac{1}{2}\right)+\cdots+f\left(\frac{1}{n}\right)-\int_1^n f\left(\frac{1}{x}\right)\mathrm{d}x$$

收敛.

(2): 计算

$$\lim_{n\to\infty}\left(f\left(\frac{1}{n+1}\right)+f\left(\frac{1}{n+2}\right)+\cdots+f\left(\frac{1}{2021n}\right)\right).$$

证明:

(1): $x_{n+1}-x_n=\sum_{k=1}^{n+1}f(\frac{1}{k})-\int_1^{n+1}f(\frac{1}{x})dx-\sum_{k=1}^{n}f(\frac{1}{k})+\int_1^n f(\frac{1}{x})dx$

$=f(\frac{1}{n+1})-\int_n^{n+1}f(\frac{1}{x})dx\le f(\frac{1}{n+1})-f(\frac{1}{n+1})\int_n^{n+1}1dx=0.$

故 $x_n$ 递减.

$x_n=\sum_{k=1}^{n}f(\frac{1}{k})-\int_1^n f(\frac{1}{x})dx=\sum_{k=1}^{n}\int_k^{k+1}f(\frac{1}{k})dx-\int_1^n f(\frac{1}{x})dx$

$\ge\sum_{k=1}^{n}\int_k^{k+1}f(\frac{1}{x})dx-\int_1^n f(\frac{1}{x})dx$

$=\int_1^{n+1}f(\frac{1}{x})dx-\int_1^n f(\frac{1}{x})dx$

$=\int_n^{n+1}f(\frac{1}{x})dx=f(\frac{1}{\theta_n}),\ n\le\theta_n\le n+1$

又 $\frac{1}{\theta_n}\in[0,1]$，故 $x_n\ge\min_{x\in[0,1]}f(x)$.

由单调有界必收敛知 $\lim_{n\to+\infty}x_n$ 存在.

(2): $\lim_{n\to+\infty}\sum_{k=1}^{2020n}f(\frac{1}{n+k})$.

由于 $\lim_{x\to0^+}\frac{f(x)}{x}=1$，故 $\forall\varepsilon\in(0,1)$，$\exists\delta>0$，当 $0<x<\delta$ 时有 $|\frac{f(x)}{x}-1|\le\varepsilon$.

即 $(1-\varepsilon)x\le f(x)\le(1+\varepsilon)x$. 而 $\sum_{k=1}^{2020n}\frac{1}{n+k}=\frac{1}{n}\sum_{k=1}^{2020n}\frac{1}{1+\frac{k}{n}}$.

又 $\frac{1}{n}\sum_{k=1}^{2020n}\frac{1}{1+\frac{k}{n}}\le\sum_{k=1}^{2020n}\int_{\frac{k-1}{n}}^{\frac{k}{n}}\frac{1}{1+x}dx=\int_0^{2020}\frac{1}{1+x}dx=\ln 2021$

$$\frac{1}{n}\sum_{k=1}^{2020n}\frac{1}{1+\frac{k}{n}} \geq \sum_{k=1}^{2020n}\int_{\frac{k}{n}}^{\frac{k+1}{n}}\frac{1}{1+x}dx=\int_{\frac{1}{n}}^{\frac{2020n+1}{n}}\frac{1}{1+x}dx$$

$$= \ln\frac{1+\frac{2020n+1}{n}}{1+\frac{1}{n}} \to \ln 2021.$$

即 $\lim\limits_{n\to+\infty}\frac{1}{n}\sum\limits_{k=1}^{2020n}\frac{1}{1+\frac{k}{n}}=\ln 2021.$

当 $0<\frac{1}{n+k}\leq\frac{1}{n+1}<\delta$，即 $n>\frac{1}{\delta}-1$，$k=1,2,\cdots,2020n$，

有 $(1-\varepsilon)\sum\limits_{k=1}^{2020n}\frac{1}{n+k}\leq\sum\limits_{k=1}^{2020n}f\left(\frac{1}{n+k}\right)\leq(1+\varepsilon)\sum\limits_{k=1}^{2020n}\frac{1}{n+k}$，

令 $n\to+\infty$，有 $\ln 2021\cdot(1-\varepsilon)\leq\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{2020n}f\left(\frac{1}{n+k}\right)\leq\ln 2021\cdot(1+\varepsilon)$，

由 $\varepsilon$ 任意性知 $\lim\limits_{n\to+\infty}\sum\limits_{k=1}^{2020n}f\left(\frac{1}{n+k}\right)=\ln 2021.$