第十届清疏竞赛班非数学类 14:

积分不等式 3

课后记得看第九届非数学类 12-14 以及番外篇,番外篇最好看一下视频.

给定$[0,+\infty)$上的递减函数$f(x)$，若$\int_0^\infty x^3 f(x)dx$收敛, 证明

$$\left(\int_0^\infty x^2 f(x)dx\right)^2 \le \frac{8}{9}\int_0^\infty xf(x)dx\int_0^\infty x^3 f(x)dx.$$

证明：

经典方法部分：

若$\lim\limits_{x\to+\infty} f(x)=A\in\mathbb{R}\cup\{+\infty,-\infty\}-\{0\}$

那么不妨设$A>0$,那么存在$X>0$,使得$f(x)\ge\frac{A}{2},\forall x\ge X$,

所以$\int_X^\infty x^3 f(x)dx \ge \int_X^\infty x^3\frac{A}{2}dx=+\infty$,因此矛盾！

所以$\lim\limits_{x\to+\infty} f(x)=0$,结合$f$递减，于是我们有$f(x)\ge 0,\forall x\ge 0.$

本题证明部分：

方法1:

如果$f\in C^1$,于是

$$\left(\int_0^\infty x^2 f(x)dx\right)^2 \le \frac{8}{9}\int_0^\infty xf(x)dx\int_0^\infty x^3 f(x)dx.$$

$$\left(\int_0^\infty x^2 f(x)dx\right)^2=\frac{1}{9}\left(\int_0^\infty f(x)dx^3\right)^2=\left(x^3 f(x)\big|_0^\infty-\int_0^\infty x^3 f'(x)dx\right)^2$$

$$\int_0^\infty xf(x)dx=\frac{1}{2}\left(\int_0^\infty f(x)dx^2\right)=\frac{1}{2}\left(x^2 f(x)\big|_0^\infty-\int_0^\infty x^2 f'(x)dx\right)$$

$$\int_0^\infty x^3 f(x)dx=\frac{1}{4}\left(\int_0^\infty f(x)dx^4\right)=\frac{1}{4}\left(x^4 f(x)\big|_0^\infty-\int_0^\infty x^4 f'(x)dx\right)$$

我们证明$\lim\limits_{x\to+\infty} x^n f(x)=0, n=2,3,4$

实际上只需要$\lim\limits_{x\to+\infty} x^4 f(x)=0$,

这里涉及一个重要的技巧我们放到后面.

$\int_0^\infty x^3 f(x)dx$收敛,证明

因此只需证明$\frac{1}{9}\left(\int_0^\infty x^3 f'(x)dx\right)^2 \le \frac{8}{9}\cdot\frac{1}{2}\cdot\frac{1}{4}\left(\int_0^\infty x^4 f'(x)dx\right)\left(\int_0^\infty x^2 f'(x)dx\right)$

这等价于$\left(\int_0^\infty x^3 f'(x)dx\right)^2 \le \left(\int_0^\infty x^4\left[-f'(x)\right]dx\right)\left(\int_0^\infty x^2\left[-f'(x)\right]dx\right)$

由$Cauchy$不等式这是显然的.

证明：$\lim\limits_{x\to+\infty} x^4 f\left(x\right)=0.$

由$Cauchy$积分收敛准则，$\lim\limits_{A\to\infty}\int_{\frac{A}{2}}^{A} x^3 f\left(x\right)dx=0$

那么$\int_{\frac{A}{2}}^{A} x^3 f\left(x\right)dx \geq \int_{\frac{A}{2}}^{A} x^3 f\left(A\right)dx=\frac{7A^4}{64} f\left(A\right)$

因此$\lim\limits_{A\to+\infty}\frac{7A^4}{64} f\left(A\right)=0 \Rightarrow \lim\limits_{x\to+\infty} x^4 f\left(x\right)=0.$

方法2:

当$f \in C^1$, 显然$\lim\limits_{x\to+\infty} f(x)=0$.

运用二重积分换序, 我们有

$$\int_0^\infty x^n f(x)dx=-\int_0^\infty x^n\int_x^\infty f'(y)dydx=-\frac{1}{n+1}\int_0^\infty x^{n+1}f'(x)dx.$$

此时我们就规避掉了$\lim\limits_{x\to+\infty} x^n f(x)=0$的证明.

方法3:

$$\left(\int_0^\infty x^2 f(x)dx\right)^2\le\frac{8}{9}\int_0^\infty xf(x)dx\int_0^\infty x^3 f(x)dx.$$

运用$Cauchy$不等式的证明思想:

$$g(t)=t^2\int_0^\infty x^3 f(x)dx-t\int_0^\infty x^2 f(x)dx+\frac{2}{9}\int_0^\infty xf(x)dx$$

下证明$g(t)\ge 0, \forall t\in\mathbb{R}$.

只需考虑$t>0$,

$$g(t)=\int_0^\infty\left(x^3t^2-tx^2+\frac{2}{9}x\right)f(x)dx$$

$$=\int_0^\infty\frac{x(3tx-2)(3tx-1)}{9}f(x)dx$$

$$\ge\int_0^{\frac{1}{3t}}\frac{x(3tx-2)(3tx-1)}{9}f(x)dx+\int_{\frac{1}{3t}}^{\frac{2}{3t}}\frac{x(3tx-2)(3tx-1)}{9}f(x)dx$$

$$\ge f\left(\frac{1}{3t}\right)\int_0^{\frac{1}{3t}}\frac{x(3tx-2)(3tx-1)}{9}dx+f\left(\frac{1}{3t}\right)\int_{\frac{1}{3t}}^{\frac{2}{3t}}\frac{x(3tx-2)(3tx-1)}{9}dx$$

$$=f\left(\frac{1}{3t}\right)\int_0^{\frac{2}{3t}}\frac{x(3tx-2)(3tx-1)}{9}dx=0.$$

这样我们就完成了证明.

运用牛顿莱布尼兹公式类：

$f\in C^1[0,1]$满足$\int_0^1 f(x)dx=0$, 证明：

$$2\int_0^1 |f(x)|^2 dx \le \left(\int_0^1 |f(x)|dx\right)\left(\int_0^1 |f'(x)|dx\right).$$

证明：

$$f(x)=f(0)+\int_0^x f'(y)dy$$

$$2\int_0^1 f^2(x)dx=2\int_0^1 f(x)\left[f(0)+\int_0^x f'(y)dy\right]dx$$

$$=2\int_0^1 f(x)\int_0^x f'(y)dydx$$

$$\le 2\int_0^1 |f(x)|\int_0^x |f'(y)|dydx$$

$$f(x)=f(1)-\int_x^1 f'(y)dy$$

$$2\int_0^1 f^2(x)dx=2\int_0^1 f(x)\left[f(1)-\int_x^1 f'(y)dy\right]dx$$

$$=-2\int_0^1 f(x)\int_x^1 f'(y)dydx$$

$$\le 2\int_0^1 |f(x)|\int_x^1 |f'(y)|dydx$$

$$4\int_0^1 f^2(x)dx\le 2\int_0^1 |f(x)|\int_0^1 |f'(y)|dydx$$

因此$2\int_0^1 |f(x)|^2 dx \le \left(\int_0^1 |f(x)|dx\right)\left(\int_0^1 |f'(x)|dx\right).$

配凑柯西不等式类

1:

$f \in C^2[a,b], f(a)=f(b)=0, f'(a)=1, f'(b)=0$, 证明:

$\int_a^b f''(x)^2 dx \geq \frac{4}{b-a}.$

证明:

如果我们证明了$a=0, b=1$的时候上述是正确的, 那么

考虑$g(x)=\frac{f(a+(b-a)x)}{b-a}$,因此$g(1)=g(0)=0, g'(1)=0, g'(0)=1$

此时$\int_0^1 g''(x)^2 dx \geq 4$,即$(b-a)^4 \int_0^1 \left|\frac{f''(a+(b-a)x)}{b-a}\right|^2 dx \geq 4$

即$(b-a)\int_a^b |f''(x)|^2 dx \geq 4.$

不妨设$a=0, b=1$,

$\int_0^1 f''(x)^2 dx \int_0^1 g^2(x) dx \geq \left(\int_0^1 f''(x) g(x) dx\right)^2$

待定$g(x)=x-c.$

$\int_0^1 f''(x)^2 dx \int_0^1 (x-c)^2 dx \geq \left(\int_0^1 f''(x)(x-c) dx\right)^2$

$=\left(\int_0^1 f''(x)(x-c) dx\right)^2=\left(\int_0^1 x f''(x) dx+c\right)^2$

$=\left(-\int_0^1 f'(x) dx+c\right)^2=c^2,$

$\int_0^1 f''(x)^2 dx \geq \frac{c^2}{\int_0^1 (x-c)^2 dx}=\frac{3c^2}{3c^2-3c+1}$

$\left(\frac{3c^2}{3c^2-3c+1}\right)'=\frac{3c(2-3c)}{(3c^2-3c+1)^2}$,你期望不等式越精确越好

因此取$c=\frac{2}{3}$,此时$\int_0^1 f''(x)^2 dx \geq 4.$

2:

3: $f \in C^2[0,2]$ 证明：

$$\int_0^1 |f''(x)|^2 dx \ge \frac{3}{2}\left[f(0)+f(2)-2f(1)\right]^2$$

证明：

不妨设$f(0)=f(2)=0, f(1)=1$，只需证$\int_0^2 |f''(x)|^2 dx \ge 6$

$$\int_0^2 |f''(x)|^2 dx \int_0^2 |g(x)|^2 dx \ge \left(\int_0^2 f''(x) g(x) dx\right)^2$$

$$g(x)=\begin{cases} x: 0 \le x \le 1 \\ 2-x: 1 \le x \le 2 \end{cases}$$

$$\int_0^1 x^2 dx = \int_1^2 (2-x)^2 dx = \frac{1}{3}$$

$$\frac{1}{3}\int_0^2 |f''(x)|^2 dx = \frac{1}{3}\int_0^1 |f''(x)|^2 dx + \frac{1}{3}\int_1^2 |f''(x)|^2 dx$$

$$\ge \left[\int_0^1 x f''(x) dx\right]^2 + \left[\int_1^2 (2-x) f''(x) dx\right]^2$$

$$=\left[f'(1)-1\right]^2 + \left[f'(1)+1\right]^2 = 2 + |f'(1)|^2 \ge 2,$$

因此就有$\int_0^2 |f''(x)|^2 dx \ge 6.$

不妨设的原因：

$$\int_0^1 |f''(x)|^2 dx \ge \frac{3}{2}\left[f(0)+f(2)-2f(1)\right]^2$$

当$f(0)+f(2)-2f(1)=0$显然

当$f(0)+f(2)-2f(1) \ne 0$, 事实上注意到$cf-ax-b$

一共三个参数可用,而如上操作不影响积分不等式,

因此我们适当选取c,a,b改变f在三个点的值.

$$\begin{pmatrix} -2 & -1 & f(2) \\ 0 & -1 & f(0) \\ -1 & -1 & f(1) \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{注意到} \begin{vmatrix} -2 & -1 & f(2) \\ 0 & -1 & f(0) \\ -1 & -1 & f(1) \end{vmatrix} \ne 0$$

因此a,b,c存在.

3: $f \in C^1[0,1]$满足$f(0)=f(1)=-\dfrac{1}{6}$, 证明:

$$\int_0^1 \left|f'(x)\right|^2 dx \ge 2\int_0^1 f(x)dx + \frac{1}{4}$$

分析:

注意到左右不是齐次的, 不是自然的不等式,

但是我们一定可以得到一个自然不等式

$$\int_0^1 \left|f'(x)\right|^2 dx \ge c\left(\int_0^1 f(x)dx + \right)^2$$

证明:

$$\int_0^1 \left|f'(x)\right|^2 dx \int_0^1 (x+c)^2 dx \ge \left(\int_0^1 f'(x)(x+c)dx\right)^2$$

$$=\left(\frac{1}{6}+\int_0^1 f(x)dx\right)^2$$

$$\int_0^1 \left|f'(x)\right|^2 dx \ge \frac{\left(\frac{1}{6}+\int_0^1 f(x)dx\right)^2}{\frac{3c^2+3c+1}{3}}$$

$$s\left(\frac{1}{6}+t\right)^2 \ge 2t+\frac{1}{4}.$$

$$\Delta=\left(\frac{s}{3}-2\right)^2-4s\left(\frac{s}{36}-\frac{1}{4}\right)=\frac{12-s}{3}\le 0$$

可取$\dfrac{12-\dfrac{3}{3c^2+3c+1}}{3}=\dfrac{3(2c+1)^2}{3c^2+3c+1}\le 0$,即$c=-\dfrac{1}{2}$.

直接求导类：

$f:[0,+\infty)\to[0,M]$是一个连续函数，且$\int_0^{\infty}(1+x)f(x)dx<\infty$，证明

$$\left(\int_0^{\infty}f(x)dx\right)^2\le 2M\int_0^{\infty}xf(x)dx.$$

证明：

由$\int_0^{\infty}(1+x)f(x)dx<\infty$知上述积分都收敛.

因此

$$F(x)=\left(\int_0^{x}f(y)dy\right)^2-2M\int_0^{x}yf(y)dy.$$

$$F'(x)=2f(x)\int_0^{x}f(y)dy-2Mxf(x)\le 2Mxf(x)-2Mxf(x)=0,$$

因此$F(x)\le F(0)=0\Rightarrow\left(\int_0^{\infty}f(x)dx\right)^2\le 2M\int_0^{\infty}xf(x)dx.$

反向切比雪夫不等式：

$f,g\in R[a,b]$且$m_1\le f(x)\le M_1, m_2\le g(x)\le M_2$, 证明：

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{(b-a)^2}\int_a^b f(x)dx\int_a^b g(x)dx\right|\le\frac{(M_2-m_2)(M_1-m_1)}{4}$$

证明：

不妨设$a=0,b=1,$即证明

$$\left|\int_0^1 f(x)g(x)dx-\int_0^1 f(x)dx\int_0^1 g(x)dx\right|\le\frac{(M_2-m_2)(M_1-m_1)}{4},$$

记$A=\int_0^1 f(x)dx, B=\int_0^1 g(x)dx.$

$$\int_0^1 f(x)g(x)dx-\int_0^1 f(x)dx\int_0^1 g(x)dx=\int_0^1 (f(x)-A)(g(x)-B)dx$$

$$\left|\int_0^1 f(x)g(x)dx-\int_0^1 f(x)dx\int_0^1 g(x)dx\right|^2\le\int_0^1 |f(x)-A|^2dx\int_0^1 |g(x)-B|^2dx$$

$$=\left(\int_0^1 |f(x)|^2dx-\left(\int_0^1 f(x)dx\right)^2\right)\left(\int_0^1 |g(x)|^2dx-\left(\int_0^1 g(x)dx\right)^2\right)$$

$$\int_0^1 (M_1-f)(f-m_1)dx=M_1A+m_1A-M_1m_1-\int_0^1 |f(x)|^2dx$$

$$M_1A+m_1A-M_1m_1-A^2=(M_1-A)(A-m_1)$$

$$\int_0^1 |f(x)|^2dx-\left(\int_0^1 f(x)dx\right)^2=(M_1-A)(A-m_1)-\int_0^1 (M_1-f)(f-m_1)dx$$

$$\le(M_1-A)(A-m_1)\le\frac{(M_1-m_1)^2}{4}$$

类似的$\int_0^1 |g(x)|^2dx-\left(\int_0^1 g(x)dx\right)^2\le(M_2-B)(B-m_2)\le\frac{(M_2-m_2)^2}{4}$

$$\left|\int_0^1 f(x)g(x)dx-\int_0^1 f(x)dx\int_0^1 g(x)dx\right|^2\le\frac{(M_1-m_1)^2}{4}\cdot\frac{(M_2-m_2)^2}{4}$$

因此$\left|\int_0^1 f(x)g(x)dx-\int_0^1 f(x)dx\int_0^1 g(x)dx\right|\le\frac{(M_2-m_2)(M_1-m_1)}{4}$

化重积分法：

设$f\in C[a,b]$满足$0\le f(x)\le M$，证明：

$$0\le\left(\int_a^b f(x)dx\right)^2-\left(\int_a^b f(x)\sin xdx\right)^2-\left(\int_a^b f(x)\cos xdx\right)^2\le\frac{M^2(b-a)^4}{12}$$

证明：

$$\left(\int_a^b f(x)dx\right)^2=\int_a^b f(x)dx\int_a^b f(y)dy$$

$$\left(\int_a^b f(x)\sin xdx\right)^2=\int_a^b f(x)\sin xdx\int_a^b f(y)\sin ydx$$

$$\left(\int_a^b f(x)\cos xdx\right)^2=\int_a^b f(x)\cos xdx\int_a^b f(y)\cos ydy$$

因此

$$\left(\int_a^b f(x)dx\right)^2-\left(\int_a^b f(x)\sin xdx\right)^2-\left(\int_a^b f(x)\cos xdx\right)^2$$

$$=\int_a^b\int_a^b\left(1-\cos(x-y)\right)f(x)f(y)dxdy\le M^2\int_a^b\int_a^b\left(1-\cos(x-y)\right)dxdy$$

$$\le M^2\int_a^b\int_a^b\frac{(x-y)^2}{2}dxdy=\frac{M^2(b-a)^4}{12}.$$

这里用到了$1-\cos t\le\dfrac{t^2}{2},t\ge0.$

$$\int_0^{\frac{\pi}{2}} x\left(\frac{\sin(nx)}{\sin x}\right)^4 dx \le C, n\in\mathbb{N}$$

分析：估阶角度：

$$\int_0^{\frac{\pi}{2}} x\left(\frac{\sin(nx)}{\sin x}\right)^4 dx \sim \int_0^{\frac{\pi}{2}} x\left(\frac{\sin(nx)}{x}\right)^4 dx = \int_0^{\frac{\pi}{2}} \frac{\sin^4(nx)}{x^3}dx$$

$$= n^2\int_0^{\frac{\pi}{2}n} \frac{\sin^4(x)}{x^3}dx \sim n^2\int_0^{\infty} \frac{\sin^4(x)}{x^3}dx = \ln 2\cdot n^2$$

证明：

$$\int_0^{\theta} x\left(\frac{\sin(nx)}{\sin x}\right)^4 dx + \int_\theta^{\frac{\pi}{2}} x\left(\frac{\sin(nx)}{\sin x}\right)^4 dx$$

直接利用$\left|\sin(nx)\right| \le n\left|\sin x\right|$（归纳法证明，结果记忆）

$$\int_0^{\theta} x\left(\frac{\sin(nx)}{\sin x}\right)^4 dx \le \frac{n^4}{2}\theta^2$$

利用$jordan$不等式$\sin x \ge \frac{2}{\pi}x, x\in\left[0,\frac{\pi}{2}\right]$（作图看割线斜率）

$$\int_\theta^{\frac{\pi}{2}} x\left(\frac{\sin(nx)}{\sin x}\right)^4 dx \le \int_\theta^{\frac{\pi}{2}} x\left(\frac{1}{\frac{2}{\pi}x}\right)^4 dx = \frac{\pi^4}{32\theta^2} - \frac{\pi^2}{8}$$

$$\int_0^{\frac{\pi}{2}} x\left(\frac{\sin(nx)}{\sin x}\right)^4 dx \le \frac{n^4}{2}\theta^2 + \frac{\pi^4}{32\theta^2} - \frac{\pi^2}{8}$$

取$\theta = \frac{\pi}{2n}$是经典题目的上界也使得上述不等式右边最小

取$\theta = \frac{\pi}{2n} < \frac{\pi}{2}$，于是就有这个方法最好的界为$\frac{n^2}{4}\pi^2 - \frac{\pi^2}{8}$