第十届清疏竞赛班非数学类 3:

本次课之后需要看第九届非数学类 2

设$x\to 0^{+}$,有$f\left(x\right)=x-Ax^{1+a}+Bx^{1+2a}+O\left(x^{1+3a}\right)$,$A,a>0$

计算:$\lim\limits_{x\to 0^{+}}x^{-a}\left[f^{-a}\left(x\right)-x^{-a}-aA\right]$

计算:

利用$\left(1+x\right)^{-a}=1-ax+\dfrac{\left(-a\right)\left(-a-1\right)}{2}x^{2}+O\left(x^{3}\right)$

$$f^{-a}\left(x\right)=\left(x-Ax^{1+a}+Bx^{1+2a}+O\left(x^{1+3a}\right)\right)^{-a}$$

$$=x^{-a}\left(1-Ax^{a}+Bx^{2a}+O\left(x^{3a}\right)\right)^{-a}$$

$$=x^{-a}\left(1-a\left(-Ax^{a}+Bx^{2a}+O\left(x^{3a}\right)\right)+\frac{\left(-a\right)\left(-a-1\right)}{2}\left(-Ax^{a}+Bx^{2a}+O\left(x^{3a}\right)\right)^{2}+O\left(x^{3a}\right)\right)$$

$$=x^{-a}\left(1-a\left(-Ax^{a}+Bx^{2a}\right)+\frac{\left(-a\right)\left(-a-1\right)}{2}\left(-Ax^{a}+Bx^{2a}+O\left(x^{3a}\right)\right)^{2}+O\left(x^{3a}\right)\right)$$

$$=x^{-a}\left(1-a\left(-Ax^{a}+Bx^{2a}\right)+\frac{\left(-a\right)\left(-a-1\right)}{2}\left(-Ax^{a}\right)^{2}+O\left(x^{3a}\right)\right)$$

$$=x^{-a}\left(1+Aax^{a}-aBx^{2a}+\frac{a\left(a+1\right)}{2}A^{2}x^{2a}+O\left(x^{3a}\right)\right)$$

$$=x^{-a}\left(1+Aax^{a}+\left(\frac{a\left(a+1\right)}{2}A^{2}-aB\right)x^{2a}+O\left(x^{3a}\right)\right)$$

$$=x^{-a}+Aa+\left(\frac{a\left(a+1\right)}{2}A^{2}-aB\right)x^{a}+O\left(x^{2a}\right)$$

$$\lim_{x\to 0^{+}}x^{-a}\left[f^{-a}\left(x\right)-x^{-a}-aA\right]=\lim_{x\to 0^{+}}x^{-a}\left(\left(\frac{a\left(a+1\right)}{2}A^{2}-aB\right)x^{a}+O\left(x^{2a}\right)\right)$$

$$=\frac{a\left(a+1\right)}{2}A^{2}-aB$$

证明：$String$公式

$$n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}\left(1+O\left(\frac{1}{n}\right)\right)$$

证明：

我们去估计$\ln n!=\sum_{k=1}^{n}\ln k$, 记$b_1\left(x\right)=x-\left[x\right]-\frac{1}{2}$

由欧拉麦克劳林公式,我们知道

$$\sum_{k=1}^{n}\ln k=\int_1^n \ln x dx+\frac{\ln n+\ln 1}{2}+\int_1^n\frac{b_1\left(x\right)}{x}dx$$

$$=n\ln n-n+\frac{\ln n}{2}+\int_1^n\frac{1}{x}db_2\left(x\right)$$

$$=n\ln n-n+\frac{\ln n}{2}+\frac{b_2\left(n\right)}{n}-\frac{b_2\left(1\right)}{1}+\int_1^n\frac{b_2\left(x\right)}{x^2}dx$$

$$=n\ln n-n+\frac{\ln n}{2}+\frac{b_2\left(n\right)}{n}-\frac{b_2\left(1\right)}{1}+\int_1^\infty\frac{b_2\left(x\right)}{x^2}dx-\int_n^\infty\frac{b_2\left(x\right)}{x^2}dx$$

$$=n\ln n-n+\frac{\ln n}{2}+C+O\left(\frac{1}{n}\right)-\int_n^\infty\frac{b_2\left(x\right)}{x^2}dx$$

$$\left|\int_n^\infty\frac{b_2\left(x\right)}{x^2}dx\right|\le M\int_n^\infty\frac{1}{x^2}dx=\frac{M}{n},$$

$$\ln n!=\sum_{k=1}^{n}\ln k=n\ln n-n+\frac{\ln n}{2}+C+O\left(\frac{1}{n}\right)$$

$$e^x=1+O\left(x\right)$$

$$n!=e^{n\ln n-n+\frac{\ln n}{2}+C+O\left(\frac{1}{n}\right)}=e^{n\ln n-n+\frac{\ln n}{2}+C}e^{O\left(\frac{1}{n}\right)}=e^{n\ln n-n+\frac{\ln n}{2}+C}\left(1+O\left(\frac{1}{n}\right)\right)$$

$$=\left(\frac{n}{e}\right)^{n}\sqrt{n}e^{C}\left(1+O\left(\frac{1}{n}\right)\right)$$

$$\lim_{n\to\infty}\frac{n!}{\left(\frac{n}{e}\right)^n\sqrt{n}}=C'$$

回忆上次数学类laplace方法导出了$wallis$公式$\frac{(2n)!!}{(2n-1)!!}\sim\sqrt{\pi n}$

积累：$(2n)!!=2^n n!\Rightarrow\frac{(2n)!!}{\sqrt{n}\frac{(2n)!}{(2n)!!}}=\frac{(2n)!!^2}{\sqrt{n}(2n)!}=\frac{\left(2^n n!\right)^2}{\sqrt{n}(2n)!}$

$$=\frac{4^n C'^2\left(\frac{n}{e}\right)^{2n}n}{\sqrt{n}\left(\frac{2n}{e}\right)^{2n}\sqrt{2n}C'}\to\sqrt{\pi}$$

$$\frac{4^n C'^2\left(\frac{n}{e}\right)^{2n}n}{\sqrt{n}\left(\frac{2n}{e}\right)^{2n}\sqrt{2n}C'}=\frac{C'}{\sqrt{2}}=\sqrt{\pi}=C'=\sqrt{2\pi}=e^C$$

因此证明了$String$公式$:n!=\sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+O\left(\frac{1}{n}\right)\right)$

分部积分思想：

估计$\int_0^x \cos\frac{1}{t}dt, x\to 0^+$

例如：如果$f(x)=\int_0^x \cos\frac{1}{t}dt$，求$f'(0)$,

证明：

$$\frac{1}{x}=y$$

$$\frac{1}{x}\int_0^x \cos\frac{1}{t}dt \overset{t=\frac{1}{u}}{=} \frac{1}{x}\int_{\frac{1}{x}}^{\infty}\frac{\cos u}{u^2}du = y\int_y^{\infty}\frac{\cos u}{u^2}du$$

$$=y\int_y^{\infty}\frac{1}{u^2}d\sin u = -\frac{\sin y}{y}-y\int_y^{\infty}\sin u d\frac{1}{u^2}$$

$$=-\frac{\sin y}{y}+y\int_y^{\infty}\frac{\sin u}{u^3}du$$

$$\left|\int_y^{\infty}\frac{\sin u}{u^3}du\right| \le \int_y^{\infty}\frac{1}{u^3}du=\frac{1}{2y^2}$$

因此$-\frac{\sin y}{y}+y\int_y^{\infty}\frac{\sin u}{u^3}du = O\left(\frac{1}{y}\right)$

从而$\lim\limits_{x\to 0^+}\frac{1}{x}\int_0^x \cos\frac{1}{t}dt = 0$（另外一侧极限类似处理）

因此$f'(0)=0.$

$\left(\text{傅里叶型积分}\right) f\left(x\right)\in C^{2}\left[0,\pi\right]$, 估计 $\int_0^{\pi} f\left(x\right)\sin nxdx, n\to\infty$

$$\int_0^{\pi} f\left(x\right)\sin nxdx = -\frac{1}{n}\int_0^{\pi} f\left(x\right)d\cos nx$$

$$= -\frac{1}{n}\left[f\left(\pi\right)\cos\left(n\pi\right) - f\left(0\right) - \int_0^{\pi} f'\left(x\right)\cos nxdx\right]$$

$$= -\frac{1}{n}\left[f\left(\pi\right)\cos\left(n\pi\right) - f\left(0\right) - \frac{1}{n}\int_0^{\pi} f'\left(x\right)d\sin nx\right]$$

$$= -\frac{1}{n}\left[f\left(\pi\right)\cos\left(n\pi\right) - f\left(0\right) - \frac{1}{n}\left[-\int_0^{\pi} f''\left(x\right)\sin nxdx\right]\right]$$

$$= -\frac{f\left(\pi\right)\left(-1\right)^{n} - f\left(0\right)}{n} - \frac{1}{n^{2}}\int_0^{\pi} f''\left(x\right)\sin nxdx$$

由黎曼引理 $\int_0^{\pi} f''\left(x\right)\sin nxdx = o\left(1\right)$

$$\int_0^{\pi} f\left(x\right)\sin nxdx = -\frac{f\left(\pi\right)\left(-1\right)^{n} - f\left(0\right)}{n} + o\left(\frac{1}{n^{2}}\right)$$

$$\limsup_{n\to\infty} n\int_0^{\pi} f\left(x\right)\sin nxdx = \left|f\left(\pi\right)\right| + f\left(0\right)$$

$$\liminf_{n\to\infty} n\int_0^{\pi} f\left(x\right)\sin nxdx = -\left|f\left(\pi\right)\right| + f\left(0\right)$$

非数记住即可：上极限就是子列极限的最大值

下极限就是子列极限的最小值.

特别的，上极限 = 下极限且是有限值, 意味着函数极限存在.

强行替换思想:

计算 $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\frac{\sin^2 xt}{\sin^2 t}dt$

分析：想要把$\frac{1}{\sin^2 t}$替换成$\frac{1}{t^2}$

证明：

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\frac{\sin^2 xt}{t^2}-\arctan\frac{1}{t}\frac{\sin^2 xt}{\sin^2 t}dt$$

$$=\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\sin^2 xt\left(\frac{1}{t^2}-\frac{1}{\sin^2 t}\right)dt$$

$$=\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\sin^2 xt\frac{\sin^2 t-t^2}{t^2\sin^2 t}dt$$

注意到$\lim\limits_{t\to0}\frac{\sin^2 t-t^2}{t^2\sin^2 t}=\lim\limits_{t\to0}\frac{(\sin t-t)(\sin t+t)}{t^4}=-\frac{1}{3}$

$t=0$不是$\frac{\sin^2 t-t^2}{t^2\sin^2 t}$的瑕点, 所以

$$\left|\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\sin^2 xt\frac{\sin^2 t-t^2}{t^2\sin^2 t}dt\right|\le\frac{\pi}{2}\int_0^{\frac{\pi}{2}}\left|\frac{\sin^2 t-t^2}{t^2\sin^2 t}\right|dt$$

因此 $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\frac{\sin^2 xt}{\sin^2 t}-\arctan\frac{1}{t}\frac{\sin^2 xt}{t^2}dt=0$

所以只需计算 $\lim\limits_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\frac{\sin^2 xt}{t^2}dt$

积累恒等式$\arctan\frac{1}{t}+\arctan t=\frac{\pi}{2}, t>0$, 它能实现arctan 在0和∞之间的切换

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\arctan\frac{1}{t}\frac{\sin^2 xt}{t^2}dt=\lim_{x\to+\infty}\frac{\pi}{2x}\int_0^{\frac{\pi}{2}}\frac{\sin^2 xt}{t^2}dt-\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\frac{\arctan t\sin^2 xt}{t^2}dt$$

$$=\lim_{x\to+\infty}\frac{\pi}{2}\int_0^{\frac{\pi}{2}x}\frac{\sin^2 t}{t^2}dt-\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\frac{\arctan t\sin^2 xt}{t^2}dt$$

$$=\frac{\pi}{2}\int_0^{\infty}\frac{\sin^2 t}{t^2}dt-\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\frac{\sin^2 xt}{t}dt$$

$$=\frac{\pi}{2}\int_0^{\infty}\frac{\sin^2 t}{t^2}dt-\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}x}\frac{\sin^2 t}{t}dt$$

$$\overset{洛}{=}\frac{\pi}{2}\int_0^{\infty}\frac{\sin^2 t}{t^2}dt\text{(积分计算课再算)}$$

其中$\lim\limits_{t\to 0}\left|\dfrac{\arctan t}{t^2}-\dfrac{t}{t^2}\right|=0$

$$\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\left|\frac{\arctan t\sin^2 xt}{t^2}-\frac{t\sin^2 xt}{t^2}\right|dt$$

$$\le\lim_{x\to+\infty}\frac{1}{x}\int_0^{\frac{\pi}{2}}\left|\frac{\arctan t}{t^2}-\frac{t}{t^2}\right|dt=0.$$

和式积分的处理：

$$\lim_{t\to+\infty}t\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{k^2+t^2}}$$

证明：

$$f(x)=\frac{1}{\sqrt{x^2+t^2}},f'(x)=-x\left(x^2+t^2\right)^{-\frac{3}{2}}$$

由莱布尼兹判别法知级数$\sum\limits_{k=0}^{\infty}\dfrac{(-1)^k}{\sqrt{k^2+t^2}}$收敛

$$\lim_{t\to+\infty}t\sum_{k=0}^{\infty}\frac{(-1)^k}{\sqrt{k^2+t^2}}=\lim_{t\to+\infty}\lim_{n\to\infty}t\sum_{k=0}^{2n-1}\frac{(-1)^k}{\sqrt{k^2+t^2}}$$

$$=\lim_{t\to+\infty}\lim_{n\to\infty}t\sum_{k=0}^{n-1}\left(\frac{(-1)^{2k}}{\sqrt{(2k)^2+t^2}}+\frac{(-1)^{2k+1}}{\sqrt{(2k+1)^2+t^2}}\right)$$

$$=\lim_{t\to+\infty}t\sum_{k=0}^{\infty}\left(\frac{1}{\sqrt{(2k)^2+t^2}}-\frac{1}{\sqrt{(2k+1)^2+t^2}}\right)$$

$$\overset{\text{拉中}}{=}\lim_{t\to+\infty}t\sum_{k=0}^{\infty}\theta\left(\theta^2+t^2\right)^{-\frac{3}{2}}$$

$$\overset{\text{保持阶不变,你可以自行放大放小}\theta\text{来严格实现}}{=}\lim_{t\to+\infty}t\sum_{k=0}^{\infty}(2k)\left((2k)^2+t^2\right)^{-\frac{3}{2}}$$

$$=\lim_{t\to+\infty}t\sum_{k=0}^{\infty}\frac{2k}{\left((2k)^2+t^2\right)^{\frac{3}{2}}}$$

$\frac{x}{\left(x^2+t^2\right)^{\frac{3}{2}}}$关于$x$在充分大时递减. 而前面有限和不影响极限, 所以

$$\lim_{t\to+\infty}\frac{t}{2}\sum_{k=0}^{\infty}\int_{2k}^{2k+2}\frac{x}{\left(x^2+t^2\right)^{\frac{3}{2}}}dx\le\lim_{t\to+\infty}t\sum_{k=0}^{\infty}\frac{2k}{\left((2k)^2+t^2\right)^{\frac{3}{2}}}\le\lim_{t\to+\infty}\frac{t}{2}\sum_{k=0}^{\infty}\int_{2k-2}^{2k}\frac{x}{\left(x^2+t^2\right)^{\frac{3}{2}}}dx$$

$$\lim_{t\to+\infty}\frac{t}{2}\sum_{k=0}^{\infty}\int_{2k}^{2k+2}\frac{x}{\left(x^2+t^2\right)^{\frac{3}{2}}}dx=\lim_{t\to+\infty}\frac{t}{2}\int_0^{\infty}\frac{x}{\left(x^2+t^2\right)^{\frac{3}{2}}}dx\overset{x=tu}{=}\lim_{t\to+\infty}\frac{1}{2}\int_0^{\infty}\frac{u}{\left(u^2+1\right)^{\frac{3}{2}}}du$$

$$=\frac{1}{2}\int_0^{\infty}\frac{u}{\left(u^2+1\right)^{\frac{3}{2}}}du=\frac{1}{4}\int_0^{\infty}\frac{1}{\left(u^2+1\right)^{\frac{3}{2}}}du^2=\frac{1}{4}\int_0^{\infty}\frac{1}{\left(z+1\right)^{\frac{3}{2}}}dz=\frac{1}{2}$$

注意：初学者务必要写一遍完整的严谨的过程, 足够熟练之后才允许如此观点处理。

真题：

$$\lim_{n\to\infty}\sqrt{n}\left[\sqrt{n}\left(\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}-1\right)+\frac{2}{3}\right]=\frac{1}{2}$$

证明：

分析：$\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}=\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\sqrt{\frac{k}{n^2}}}$，$k$和$n$不同阶，$taylor$型

回忆k和n同阶时属于定积分定义型（第九届数学类1最后用EM）公式处理，

证明：

$$\frac{1}{1+x}=1-x+x^2+O\left(x^3\right)$$

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{1+\sqrt{\frac{k}{n^2}}}=\frac{1}{n}\sum_{k=1}^{n}\left(1-\sqrt{\frac{k}{n^2}}+\frac{k}{n^2}+O\left(\left(\frac{k}{n^2}\right)^{\frac{3}{2}}\right)\right)$$

$$=\frac{\sum_{k=1}^{n}\left(1-\sqrt{\frac{k}{n^2}}+\frac{k}{n^2}\right)+O\left(\frac{1}{\sqrt{n}}\right)}{n}$$

$$=\frac{n-\frac{\sum_{k=1}^{n}\sqrt{k}}{n}+\sum_{k=1}^{n}\frac{k}{n^2}+O\left(\frac{1}{\sqrt{n}}\right)}{n}$$

$$=\frac{n-\frac{\sum_{k=1}^{n}\sqrt{k}}{n}+\frac{n+1}{2n}}{n}+O\left(\frac{1}{n\sqrt{n}}\right)$$

$$=1-\frac{\sum_{k=1}^{n}\sqrt{k}}{n^2}+\frac{1}{2n}+\frac{1}{2n^2}+O\left(\frac{1}{n\sqrt{n}}\right)$$

$$\int_0^1\sqrt{x}dx=\frac{2}{3}$$

$$\frac{\sum_{k=1}^{n}\sqrt{k}-\frac{2}{3}n\sqrt{n}}{\sqrt{n}}=\sum_{k=1}^{n}\sqrt{\frac{k}{n}}-\frac{2}{3}n=n\left(\frac{1}{n}\sum_{k=1}^{n}\sqrt{\frac{k}{n}}-\frac{2}{3}\right)\to\frac{\sqrt{1}-\sqrt{0}}{2}=\frac{1}{2}$$

因此$\sum_{k=1}^{n}\sqrt{k}=\frac{2}{3}n\sqrt{n}+\frac{\sqrt{n}}{2}+o\left(\sqrt{n}\right)=\frac{2}{3}n\sqrt{n}+O\left(\sqrt{n}\right)$

$$1-\frac{\sum_{k=1}^{n}\sqrt{k}}{n^2}+\frac{1}{2n}+\frac{1}{2n^2}+O\left(\frac{1}{n\sqrt{n}}\right)$$

$$=1-\frac{\frac{2}{3}n\sqrt{n}+O\left(\sqrt{n}\right)}{n^2}+\frac{1}{2n}+\frac{1}{2n^2}+O\left(\frac{1}{n\sqrt{n}}\right)$$

$$=1-\frac{\frac{2}{3}}{\sqrt{n}}+\frac{1}{2n}+O\left(\frac{1}{n\sqrt{n}}\right)$$

因此$\lim_{n\to\infty}\sqrt{n}\left[\sqrt{n}\left(\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}-1\right)+\frac{2}{3}\right]=\frac{1}{2}$

估计$\sum_{k=1}^{n}\sqrt{k}$的通法仍然是$E-M$公式,但考试中我们不到万不得已不使用

$$\sum_{k=1}^{n}\sqrt{k}=\frac{\sqrt{1}+\sqrt{n}}{2}+\int_1^n\sqrt{x}dx+\frac{1}{2}\int_1^n\frac{b_1\left(x\right)}{\sqrt{x}}dx$$

$$\left|\frac{1}{2}\int_1^n\frac{b_1\left(x\right)}{\sqrt{x}}dx\right|\leq M\int_0^n\frac{1}{\sqrt{x}}dx=2M\sqrt{n}\Rightarrow\frac{1}{2}\int_1^n\frac{b_1\left(x\right)}{\sqrt{x}}dx=O\left(\sqrt{n}\right)$$

所以$\sum_{k=1}^{n}\sqrt{k}=\frac{\sqrt{1}+\sqrt{n}}{2}+\frac{2n\sqrt{n}}{3}-\frac{2}{3}+O\left(\sqrt{n}\right)=\frac{2n\sqrt{n}}{3}+O\left(\sqrt{n}\right)$

习题:

计算$\lim_{n\to\infty}\sqrt{n}\left[\sqrt{n}\left[\sqrt{n}\left(\sum_{k=1}^{n}\frac{1}{n+\sqrt{k}}-1\right)+\frac{2}{3}\right]-\frac{1}{2}\right]$