

# 第十届清疏竞赛班非数学类 11:

## 微分学计算

课后阅读第九届非数学类 11 pdf

证明连续Jensen凸是凸函数.

$$\text{即 } f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \forall x, y \in \mathbb{R}$$

证明:  $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \forall x, y \in \mathbb{R}, \lambda \in [0,1]$

分析:

$$f\left(\frac{m}{n}x + \left(1 - \frac{m}{n}\right)y\right) \leq \frac{m}{n}f(x) + \left(1 - \frac{m}{n}\right)f(y), \forall 1 \leq m \leq n, m, n \in \mathbb{N}$$

$$f\left(\frac{mx}{n} + \frac{(n-m)y}{n}\right) \leq \frac{m}{n}f(x) + \left(1 - \frac{m}{n}\right)f(y), \forall 1 \leq m \leq n, m, n \in \mathbb{N},$$

Jensen不等式(直接百度搜索,运用反向归纳法可以证明)

$$f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right) \leq \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}, \forall x_1, \dots, x_n \in \mathbb{R}$$

证明:

在Jensen不等式赋予 $m$ 个 $x, n-m$ 个 $y$ , 则

$$f\left(\frac{mx}{n} + \frac{(n-m)y}{n}\right) \leq \frac{m}{n}f(x) + \left(1 - \frac{m}{n}\right)f(y)$$

因为 $f \in C(\mathbb{R})$ , 而每一个无理数 $\lambda \in (0,1)$ , 都存在 $\lambda_k \in \mathbb{Q} \cap [0,1]$ , 使得

$$\lim_{k \rightarrow \infty} \lambda_k = \lambda, \text{现在}$$

$$f(\lambda_k x + (1-\lambda_k)y) \leq \lambda_k f(x) + (1-\lambda_k)f(y)$$

两边令 $k \rightarrow \infty$ , 从而 $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ ,

完成了证明.

求下列和式

$$\sum_{k=1}^n k^2 x^{k-1} (x \neq 1), \sum_{k=1}^n C_n^k k^2 x^{k-1}, \sum_{k=1}^n \frac{1}{2^k} \tan \frac{1}{2^k}, \sum_{k=1}^n k \cos kx, \sum_{k=0}^{\infty} (-1)^k C_{2k}^k k^n$$

$$(1): \sum_{k=1}^n k^2 x^{k-1} = \left( x \left( \sum_{k=1}^n x^k \right)' \right)' = \left( x \left( \frac{x - x^{n+1}}{1-x} \right)' \right)' \dots \text{自己计算}$$

$$(2): (1+x)^n = \sum_{k=0}^n C_n^k x^k, \sum_{k=1}^n k^2 C_n^k x^{k-1} = (nx(1+x)^{n-1})' \dots \text{自己算.}$$

$$\text{如果考虑 } (1+x)^n (1+x)^m = (1+x)^{n+m}$$

$$\sum_{k=0}^n C_n^k x^k \sum_{j=0}^m C_m^j x^j = \sum_{j=0}^{m+n} C_{m+n}^j x^j$$

$$\text{如果我们约定 } C_n^p = 0, p < 0, p > n$$

对比系数就有 Vandermonde 恒等式.

$$\sum_{k=0}^n \sum_{j=0}^m C_n^k C_m^j x^{k+j} = \sum_{t=0}^{m+n} C_{m+n}^t x^t$$

$$\text{于是就有 } \sum_{j=0}^t C_n^j C_m^{t-j} = C_{m+n}^t$$

$$(3): \left( \ln \cos \frac{x}{2^k} \right)' = -\frac{1}{2^k} \tan \frac{x}{2^k}$$

$$\sum_{k=1}^n \frac{1}{2^k} \tan \frac{x}{2^k} = -\sum_{k=1}^n \left( \ln \cos \frac{x}{2^k} \right)' = -\left( \sum_{k=1}^n \ln \cos \frac{x}{2^k} \right)',$$

$$= -\left( \ln \prod_{k=1}^n \cos \frac{x}{2^k} \right)' = -\left( \ln \frac{\sin \frac{x}{2^n} \prod_{k=1}^n \cos \frac{x}{2^k}}{\sin \frac{x}{2^n}} \right)' = -\left( \ln \frac{\sin \frac{x}{2^{n-1}} \prod_{k=1}^{n-1} \cos \frac{x}{2^k}}{2 \sin \frac{x}{2^n}} \right)',$$

$$= -\left( \ln \frac{\sin \frac{x}{2^{n-2}} \prod_{k=1}^{n-2} \cos \frac{x}{2^k}}{2^2 \sin \frac{x}{2^n}} \right)' = \dots = -\left( \ln \frac{\sin x}{2^n \sin \frac{x}{2^n}} \right)', \text{再赋予 } x=1 \text{ 即可}$$

$$(4), \sum_{k=1}^n k \cos kx = \left( \sum_{k=1}^n \sin kx \right)' = \left( \sum_{k=1}^n \frac{\sin \frac{x}{2} \sin kx}{\sin \frac{x}{2}} \right)',$$

对分子积化和差就可以裂项计算.

$$\sum_{k=0}^{2n} (-1)^k C_{2n}^k e^{kx} = (e^x - 1)^{2n}$$

$$\sum_{k=0}^{2n} (-1)^k k^n C_{2n}^k e^{kx} = \frac{d}{dx^n} (e^x - 1)^{2n}$$

$$\sum_{k=0}^{2n} (-1)^k k^n C_{2n}^k e^{kx} \Big|_{x=0} = \frac{d}{dx^n} (e^x - 1)^{2n} \Big|_{x=0}$$

直观上  $e^x - 1 \sim x$ ,  $\frac{dx^{2n}}{dx^n} \Big|_{x=0} = 0$ ,

严格上:

$$\frac{d}{dx^n} (e^x - 1)^{2n} \Big|_{x=0} = \frac{d \left( \left( \frac{e^x - 1}{x} \right)^{2n} x^{2n} \right)}{dx^n} \Big|_{x=0}$$

运用乘积求导法则就有

$$\frac{d \left( \left( \frac{e^x - 1}{x} \right)^{2n} x^{2n} \right)}{dx^n} \Big|_{x=0} = \sum_{k=0}^n C_n^k (x^{2n})^{(n-k)} \left( \left( \frac{e^x - 1}{x} \right)^{2n} \right)^{(k)} \Big|_{x=0} = 0.$$

给定  $n \in \mathbb{N}$ ,  $f(x) = (1 + \sqrt{x})^{2n+2}$ , 求  $f^{(n)}(1)$ .

分析:

计算:

$$\begin{aligned} (1 + \sqrt{x})^{2n+2} + (1 - \sqrt{x})^{2n+2} &= \sum_{k=0}^{2n+2} C_{2n+2}^k x^{\frac{k}{2}} + \sum_{k=0}^{2n+2} C_{2n+2}^k (-1)^k x^{\frac{k}{2}} \\ &= 2 \sum_{k=0}^{n+1} C_{2n+2}^{2k} x^k. \end{aligned}$$

又  $(1 - \sqrt{x})^{2n+2} = \frac{(1-x)^{2n+2}}{(1+\sqrt{x})^{2n+2}}$ , 因此  $\frac{d^n}{dx^n} (1 - \sqrt{x})^{2n+2} |_{x=1} = 0$

$$f^{(n)}(1) = \left( 2 \sum_{k=0}^{n+1} C_{2n+2}^{2k} x^k \right)^{(n)} |_{x=1} = 2 [n! C_{2n+2}^{2n} + (n+1)!] = 4(n+1)(n+1)!$$

方法2(不太好算, 上课同学提出):

由 Taylor 公式,  $f^{(n)}(a) = n! \cdot f(x)$  在  $x=a$  的 Taylor 的  $(x-a)^n$  的系数.

$$\begin{aligned} f(x) &= (1 + \sqrt{1+x-1})^{2n+2} \\ &= \underbrace{\left( 1 + \sum_{k=0}^{\infty} c_k (x-1)^k \right)}_{2n+2 \text{ 个}} \left( 1 + \sum_{k=0}^{\infty} c_k (x-1)^k \right) \cdots \left( 1 + \sum_{k=0}^{\infty} c_k (x-1)^k \right) \end{aligned}$$

$f^{(n)}(1) = n! (x-1)^n$  的系数不好算.

$$P_{n,m}(x) = \frac{d^n}{dx^n} (1-x^m)^n, \text{求} P_{n,m}(1)$$

证明：

$$\begin{aligned}\frac{d^n}{dx^n} (1-x^m)^n &= \frac{d^n}{dx^n} \left[ (1-x)^n (1+x+x^2+\dots+x^{m-1})^n \right] \\ &= \sum_{k=0}^n C_n^k \left( (1-x)^n \right)^{(n-k)} \left( (1+x+x^2+\dots+x^{m-1})^n \right)^{(k)}\end{aligned}$$

把  $x=1$  代入，就有

$$P_{n,m}(1) = n! (-1)^n \left( (1+1+1+\dots+1)^n \right) = m^n n! (-1)^n$$

$$(\sin x)^{(n)} = \sin\left(x + \frac{n\pi}{2}\right), (\cos x)^{(n)} = \cos\left(x + \frac{n\pi}{2}\right).$$

$$e^{ix} = \cos x + i \sin x$$

计算  $f(x) = \frac{1}{x^2 - 4}$  的  $n$  阶导数,  $g(x) = \frac{x}{x^2 + 1}$  的  $n$  阶导数

解:

$$f(x) = \frac{1}{x^2 - 4} = \frac{1}{(x-2)(x+2)} = \frac{1}{4} \left( \frac{1}{x-2} - \frac{1}{x+2} \right)$$

$$f^{(n)}(x) = \frac{1}{4} \left( \left( \frac{1}{x-2} \right)^{(n)} - \left( \frac{1}{x+2} \right)^{(n)} \right)$$

$$= \frac{1}{4} \left( (-1)^n \frac{n!}{(x-2)^{n+1}} - (-1)^n \frac{n!}{(x+2)^{n+1}} \right)$$

$$g(x) = \frac{x}{x^2 + 1} = \frac{x}{(x+i)(x-i)} = \frac{1}{2} \left( \frac{1}{x+i} + \frac{1}{x-i} \right)$$

$$g^{(n)}(x) = \frac{(-1)^n n!}{2} \left( \frac{1}{(x+i)^{n+1}} + \frac{1}{(x-i)^{n+1}} \right)$$

当  $x > 0$ ,

$$x+i = \sqrt{x^2+1} \left( \frac{x}{\sqrt{x^2+1}} + \frac{i}{\sqrt{x^2+1}} \right) = \sqrt{x^2+1} e^{\arctan \frac{1}{x} i}$$

$$(x+i)^{-n-1} = (x^2+1)^{\frac{-n-1}{2}} e^{-(n+1)\arctan \frac{1}{x} i}$$

$$(x-i)^{-n-1} = (x^2+1)^{\frac{-n-1}{2}} e^{(n+1)\arctan \frac{1}{x} i}$$

$$g^{(n)}(x) = (-1)^n n! (x^2+1)^{\frac{-n-1}{2}} \cos\left((n+1)\arctan \frac{1}{x}\right), x > 0$$

那么利用奇偶性对称的可以写出  $x < 0$  的  $n$  阶导数.

## 幂级数方法和微分方程法(重要)

$$f(x) = \left(x + \sqrt{x^2 + 1}\right)^m, \text{求 } f^{(n)}(0)$$

解：

建立 $f$ 满足的线性微分方程，且系数多项式级别，即

$$\ln f = m \ln \left(x + \sqrt{x^2 + 1}\right), \frac{f'}{f} = \frac{m}{\sqrt{1+x^2}}$$

$$\sqrt{1+x^2} f' - mf = 0$$

$$\frac{x}{\sqrt{1+x^2}} f' + \sqrt{1+x^2} f'' - mf' = 0$$

$$xf' + (1+x^2)f'' - m\sqrt{1+x^2}f' = 0$$

$$(1+x^2)f'' + xf' - m^2f = 0$$

微分方程法：

$$\sum_{k=0}^n C_n^k \left[ (1+x^2)^{(k)} f^{(n-k+2)} \right] + \sum_{k=0}^n C_n^k \left[ (x)^{(k)} f^{(n-k+1)} \right] - m^2 f^{(n)} = 0$$

代入 $x=0$ ，就有

$$\sum_{k=0}^n C_n^k \left[ (1+x^2)^{(k)} f^{(n-k+2)} \right] = f^{(n+2)}(0) + 2C_n^2 f^{(n)}(0)$$

$$\sum_{k=0}^n C_n^k \left[ (x)^{(k)} f^{(n-k+1)} \right] = C_n^1 f^{(n)}(0)$$

$$\text{因此 } f^{(n+2)}(0) + n(n-1)f^{(n)}(0) + nf^{(n)}(0) - m^2 f^{(n)}(0) = 0$$

$$\text{即 } f^{(n+2)}(0) = (m^2 - n^2) f^{(n)}(0), n \in \mathbb{N} \cup \{0\}$$

所以 $f(0)=1, f'(0)=m$ ，所以对 $n \in \mathbb{N} \cup \{0\}$ ，有

$$f^{(2n+2)}(0) = (m^2 - 4n^2) f^{(2n)}(0)$$

$$= (m^2 - 4n^2)(m^2 - 4(n-1)^2) f^{(2n-2)}(0)$$

...

$$= (m^2 - 4n^2)(m^2 - 4(n-1)^2) \cdots m^2$$

$n \in \mathbb{N} / \{0\}$ ,

$$\begin{aligned} \text{我们有 } f^{(2n+1)}(0) &= \left(m^2 - (2n-1)^2\right) f^{(2n-1)}(0) \\ &= \left(m^2 - (2n-1)^2\right) \left(m^2 - (2n-3)^2\right) f^{(2n-3)}(0) \\ &= \dots \\ &= \left(m^2 - (2n-1)^2\right) \left(m^2 - (2n-3)^2\right) \dots \left(m^2 - 1\right) m. \end{aligned}$$

$$f \in D^2(\mathbb{R}), \frac{f(u) - f(v)}{u - v} = \alpha f'(u) + \beta f'(v), \alpha + \beta = 1, \alpha, \beta > 0$$

计算 $f$

$$\text{证明: } f(u) - f(v) = [\alpha f'(u) + \beta f'(v)](u - v)$$

$$f(v) - f(u) = [\alpha f'(v) + \beta f'(u)](v - u)$$

于是两式相加

$$[\alpha f'(u) + \beta f'(v) - \alpha f'(v) - \beta f'(u)](u - v) = 0$$

$$(\alpha - \beta)(f'(u) - f'(v))(u - v) = 0, \text{ 若 } \alpha \neq \beta, \text{ 则 } (f'(u) - f'(v))(u - v) = 0$$

故 $f'(u) - f'(v) = 0, \forall u \neq v \Rightarrow f' \text{ 常数} \Rightarrow f \text{ 线性函数}$

$$\text{若 } \alpha = \beta = \frac{1}{2}, \text{ 则 } f(u) - f(v) = \frac{1}{2}(f'(u) + f'(v))(u - v)$$

对 $u$ 求导就有

$$f'(u) = \frac{1}{2} f''(u)(u - v) + \frac{1}{2}(f'(u) + f'(v))$$

对 $v$ 求导就有

$$0 = -\frac{1}{2} f''(u) + \frac{1}{2} f''(v) \Rightarrow f''(u) = f''(v), \text{ 因此我们知道 } f'' \text{ 是常数}$$

从而 $f$ 是二次函数.

