第十届清疏竞赛班非数学类 23
多元积分计算训练

求极限

$$\lim_{R\to+\infty}\iiint_{x^2+y^2+z^2\le R^2}\frac{\cos\left(ax+by+cz\right)}{x^2+y^2+z^2}dV,$$ a,b,c不全为0.

计算:

由$possion$公式的证明,即正交变换，我们知道

$$\lim_{R\to+\infty}\iiint_{x^2+y^2+z^2\le R^2}\frac{\cos\left(ax+by+cz\right)}{x^2+y^2+z^2}dV$$
$$=\lim_{R\to+\infty}\iiint_{x^2+y^2+z^2\le R^2}\frac{\cos\left(\sqrt{a^2+b^2+c^2}z\right)}{x^2+y^2+z^2}dV$$
$$=\lim_{R\to+\infty}\int_0^R dr\oiint_{x^2+y^2+z^2=r^2}\frac{\cos\left(\sqrt{a^2+b^2+c^2}z\right)}{x^2+y^2+z^2}dS$$
$$=\lim_{R\to+\infty}\int_0^R\frac{1}{r^2}dr\oiint_{x^2+y^2+z^2=r^2}\cos\left(\sqrt{a^2+b^2+c^2}z\right)dS$$
$$=\lim_{R\to+\infty}\int_0^R\frac{1}{r^2}\cdot r^2dr\oiint_{x^2+y^2+z^2=1}\cos\left(\sqrt{a^2+b^2+c^2}rz\right)dS$$
$$=\lim_{R\to+\infty}\int_0^R\frac{1}{r^2}\cdot r^2dr\int_0^\pi\sin\psi d\psi\int_0^{2\pi}\cos\left(\sqrt{a^2+b^2+c^2}r\cos\psi\right)d\theta$$
$$=2\pi\lim_{R\to+\infty}\int_0^R dr\int_0^\pi\sin\psi\cos\left(\sqrt{a^2+b^2+c^2}r\cos\psi\right)d\psi$$
$$=2\pi\lim_{R\to+\infty}\int_0^R dr\int_{-1}^1\cos\left(\sqrt{a^2+b^2+c^2}ru\right)du$$
$$=4\pi\lim_{R\to+\infty}\int_0^R\frac{\sin\left(\sqrt{a^2+b^2+c^2}r\right)}{\sqrt{a^2+b^2+c^2}r}dr$$
$$=\frac{4\pi}{\sqrt{a^2+b^2+c^2}}\int_0^\infty\frac{\sin\left(\rho\right)}{\rho}d\rho$$
$$=\frac{2\pi^2}{\sqrt{a^2+b^2+c^2}}$$

一点小思考, 如果是:

$$\iiint_{\mathbb{R}^3}\frac{\cos\left(ax+by+cz\right)}{x^2+y^2+z^2}dV,$$ 那么值是否一样呢?

实际上发散!

计算$\iint_{[0,1]\times[0,1]}\left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\}\right)^2 dxdy, \{x\} \triangleq x-[x]$

计算：

$$\iint_{[0,1]\times[0,1]}\left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\}\right)^2 dxdy = 2\iint_{0\le y\le x\le 1}\left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{x}\right\}\right)^2 dxdy$$

$$= 2\iint_{0\le y\le x\le 1}\frac{y^2}{x^2}\left(\frac{x}{y}-\left[\frac{x}{y}\right]\right)^2 dxdy$$

$$= 2\int_0^1\int_0^x\frac{y^2}{x^2}\left(\frac{x}{y}-\left[\frac{x}{y}\right]\right)^2 dy$$

$$k\le\frac{x}{y}<k+1 \Leftrightarrow \frac{x}{k+1}<y\le\frac{x}{k}, k=1,2,\cdots$$

$$2\int_0^1\int_0^x\frac{y^2}{x^2}\left(\frac{x}{y}-\left[\frac{x}{y}\right]\right)^2 dydx = 2\sum_{k=1}^{\infty}\int_0^1\int_{\frac{x}{k+1}}^{\frac{x}{k}}\frac{y^2}{x^2}\left(\frac{x}{y}-k\right)^2 dydx$$

$$= 2\sum_{k=1}^{\infty}\int_0^1\frac{1}{x^2}\int_{\frac{x}{k+1}}^{\frac{x}{k}}\left(x-ky\right)^2 dydx$$

$$= 2\sum_{k=1}^{\infty}\frac{1}{3k\left(k+1\right)^3}\int_0^1 xdx$$

$$= \frac{1}{3}\sum_{k=1}^{\infty}\frac{1}{k\left(k+1\right)^3}$$

$$= \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+1}-\frac{1}{\left(k+1\right)^2}-\frac{1}{\left(k+1\right)^3}\right)$$

$$= 1-\frac{\pi^2}{18}-\frac{\zeta\left(3\right)}{3}$$

计算

$$\iiint_{[0,\pi]^3}\frac{1}{1-\cos x\cos y\cos z}dV$$

$$=\iiint_{[0,\pi]^3}\sum_{n=0}^{\infty}\left(\cos x\cos y\cos z\right)^n dV$$

$$=\sum_{n=0}^{\infty}\iiint_{[0,\pi]^3}\left(\cos x\cos y\cos z\right)^n dV$$

$$=\sum_{n=0}^{\infty}\left(\int_0^{\pi}\cos^n xdx\right)^3$$

$$=\sum_{n=0}^{\infty}\left(2\int_0^{\frac{\pi}{2}}\cos^n xdx\right)^3$$

$$=8\sum_{n=0}^{\infty}\left(\int_0^{\frac{\pi}{2}}\cos^{2n} xdx\right)^3+8\sum_{n=0}^{\infty}\left(\int_0^{\frac{\pi}{2}}\cos^{2n+1} xdx\right)^3$$

$$=\pi^3\sum_{n=0}^{\infty}\frac{\left(2n-1\right)!!^3}{\left(2n\right)!!^3}+8\sum_{n=0}^{\infty}\frac{\left(2n\right)!!^3}{\left(2n+1\right)!!^3}$$

后面这个级数难以继续计算.

方法2：

$$\iiint_{[0,\pi]^3}\frac{1}{1-\cos x\cos y\cos z}dV \overset{\begin{cases}u=\tan\frac{x}{2}\\ v=\tan\frac{y}{2}\\ w=\tan\frac{z}{2}\end{cases}}{=} \iiint_{\text{第一象限}}\frac{1}{1-\frac{1-u^2}{1+u^2}\frac{1-v^2}{1+v^2}\frac{1-w^2}{1+w^2}}\left|J\right|dV$$

$$=\frac{1}{2}\iiint_{\text{第一象限}}\frac{\left(u^2+1\right)\left(v^2+1\right)\left(w^2+1\right)}{u^2+v^2+w^2+u^2v^2w^2}\left|J\right|dV$$

$$\left|J\right|=\left|\det\begin{pmatrix}\frac{2}{1+u^2} & 0 & 0\\ 0 & \frac{2}{1+v^2} & 0\\ 0 & 0 & \frac{2}{1+w^2}\end{pmatrix}\right|=\frac{8}{\left(u^2+1\right)\left(v^2+1\right)\left(w^2+1\right)}$$

因此$\frac{1}{2}\iiint_{\text{第一象限}}\frac{\left(u^2+1\right)\left(v^2+1\right)\left(w^2+1\right)}{u^2+v^2+w^2+u^2v^2w^2}\left|J\right|dV$

$$=4\iiint_{\text{第一象限}}\frac{1}{u^2+v^2+w^2+u^2v^2w^2}dV$$

$$=4\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\sin\psi\,d\psi\int_0^{\infty}\frac{r^2}{r^2+r^6\sin^4\psi\cos^2\theta\sin^2\theta\cos^2\psi}dr$$

$$=4\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\sin\psi\,d\psi\int_0^{\infty}\frac{1}{1+r^4\sin^4\psi\cos^2\theta\sin^2\theta\cos^2\psi}dr$$

$$=\frac{4\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\sin\psi\cdot\left(\sin^4\psi\cos^2\theta\sin^2\theta\cos^2\psi\right)^{-\frac{1}{4}}d\psi$$

$$=\frac{4\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\left(\cos\theta\sin\theta\right)^{-\frac{1}{2}}d\theta\int_0^{\frac{\pi}{2}}\cos^{-\frac{1}{2}}\psi\,d\psi$$

$$=\frac{128\pi^2}{\Gamma^2\left(-\frac{1}{4}\right)}\Gamma^2\left(\frac{5}{4}\right)(\text{未化简})$$

计算方法见视频(上课的时候有点笔误,但是方法一致)

如果计算

$$=\frac{4\pi}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}d\theta\int_0^{\frac{\pi}{2}}\sin\psi\cdot\left(\sin^4\psi\cos^2\theta\sin^2\theta\cos^2\psi\right)^{\frac{1}{4}}d\psi$$

$$=\sqrt{2}\pi\int_0^{\frac{\pi}{2}}\sqrt{\cos\theta\sin\theta}d\theta\int_0^{\frac{\pi}{2}}\sin\psi\cdot\sqrt{\cos\psi}d\psi$$

$$\beta\left(a,b\right)=\int_0^1 t^{a-1}\left(1-t\right)^{b-1}dt,a,b>0$$

则$\beta\left(a,b\right)\overset{t=\cos^2\theta}{=}2\int_0^{\frac{\pi}{2}}\cos^{2a-1}\theta\sin^{2b-1}\theta d\theta$

$$\sqrt{2}\pi\int_0^{\frac{\pi}{2}}\sqrt{\cos\theta\sin\theta}d\theta\int_0^{\frac{\pi}{2}}\sin^2\psi\cdot\sqrt{\cos\psi}d\psi$$

$$=\frac{\sqrt{2}\pi}{4}2\int_0^{\frac{\pi}{2}}\sqrt{\cos\theta\sin\theta}d\theta\cdot 2\int_0^{\frac{\pi}{2}}\sin^2\psi\cdot\sqrt{\cos\psi}d\psi$$

$$=\frac{\sqrt{2}\pi}{4}\beta\left(\frac{3}{4},\frac{3}{4}\right)\cdot\beta\left(\frac{3}{4},\frac{3}{2}\right)$$

利用$\beta\left(a,b\right)=\frac{\Gamma\left(a\right)\Gamma\left(b\right)}{\Gamma\left(a+b\right)}$

$$\frac{\sqrt{2}\pi}{4}\beta\left(\frac{3}{4},\frac{3}{4}\right)\cdot\beta\left(\frac{3}{4},\frac{3}{2}\right)$$

$$=\frac{\sqrt{2}\pi}{4}\frac{\Gamma^2\left(\frac{3}{4}\right)}{\Gamma\left(\frac{3}{2}\right)}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{4}\right)}$$

$$=\frac{\sqrt{2}\pi}{4}\frac{\Gamma^3\left(\frac{3}{4}\right)}{\Gamma\left(\frac{9}{4}\right)}$$

$$\Gamma\left(x+1\right)=x\Gamma\left(x\right),\Gamma\left(\frac{9}{4}\right)=\frac{5}{4}\Gamma\left(\frac{5}{4}\right)=\frac{5}{16}\Gamma\left(\frac{1}{4}\right)$$

$$\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{\pi}{\sin\frac{\pi}{4}}=\sqrt{2}\pi$$

则$\frac{\sqrt{2}\pi}{4}\frac{\Gamma^3\left(\frac{3}{4}\right)}{\Gamma\left(\frac{9}{4}\right)}=\frac{\sqrt{2}\pi}{4}\frac{\Gamma^3\left(\frac{3}{4}\right)}{\frac{5}{16}\frac{\sqrt{2}\pi}{\Gamma\left(\frac{3}{4}\right)}}=\frac{4}{5}\Gamma^4\left(\frac{3}{4}\right).$

$f\left(x,y,z\right)=a_1x^4+a_2y^4+a_3z^4+3a_4x^2y^2+3a_5y^2z^2+3a_6x^2z^2$

计算 $\iint\limits_{x^2+y^2+z^2=1} f\left(x,y,z\right)dS$

证明：

考虑ζ=$x+iy$

$$\iint\limits_{x^2+y^2+z^2=1}\zeta^4dS\overset{\zeta=e^{\frac{\pi}{4}i}\zeta}{=}e^{\pi i}\iint\limits_{x^2+y^2+z^2=1}\zeta^4dS=-\iint\limits_{x^2+y^2+z^2=1}\zeta^4dS$$

因此 $\iint\limits_{x^2+y^2+z^2=1}\zeta^4dS=0$

$$\iint\limits_{x^2+y^2+z^2=1}\left(x+iy\right)^4dS=\iint\limits_{x^2+y^2+z^2=1}x^4+y^4+C_4^2x^2\left(iy\right)^2dS$$

$$=\iint\limits_{x^2+y^2+z^2=1}x^4+y^4-6x^2y^2dS$$

因此 $\iint\limits_{x^2+y^2+z^2=1}x^2y^2dS=\frac{1}{6}\iint\limits_{x^2+y^2+z^2=1}x^4+y^4dS=\frac{1}{3}\iint\limits_{x^2+y^2+z^2=1}x^4dS$

$$\iint\limits_{x^2+y^2+z^2=1}f\left(x,y,z\right)dS=\iint\limits_{x^2+y^2+z^2=1}a_1x^4+a_2y^4+a_3z^4+3a_4x^2y^2+3a_5y^2z^2+3a_6x^2z^2dS$$

$$=\iint\limits_{x^2+y^2+z^2=1}a_1x^4+a_2y^4+a_3z^4+a_4x^4+a_5x^4+a_6x^4dS$$

$$=\frac{a_1+a_2+\cdots+a_6}{3}\iint\limits_{x^2+y^2+z^2=1}x^4+y^4+z^4dS$$

其中 $\iint\limits_{x^2+y^2+z^2=1}x^4+y^4+z^4dS$

$$=\iint\limits_{x^2+y^2+z^2=1}\left(x^2+y^2+z^2\right)^2-6x^2y^2dS$$

$$=\iint\limits_{x^2+y^2+z^2=1}1-6x^2y^2dS$$

$$=4\pi-\frac{2}{3}\iint\limits_{x^2+y^2+z^2=1}x^4+y^4+z^4dS$$

因此 $\iint\limits_{x^2+y^2+z^2=1}x^4+y^4+z^4dS=\frac{12}{5}\pi$

因此原式=$\frac{4\pi}{5}\left(a_1+a_2+\cdots+a_6\right)$

$$
\begin{aligned}
&\int_0^{2\pi} d\psi \int_0^{\pi} e^{\sin\theta\left(\cos\psi - \sin\psi\right)} \sin\theta d\theta \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} e^{\sin\psi\left(\cos\theta - \sin\theta\right)} \sin\psi d\psi \\
&= \oiint_{x^2+y^2+z^2=1} e^{x-y} dS \\
&= \oiint_{x^2+y^2+z^2=1} e^{\sqrt{2}z} dS \\
&= \int_0^{2\pi} d\theta \int_0^{\pi} e^{\sqrt{2}\cos\psi} \sin\psi d\psi \\
&= -2\pi \int_0^{\pi} e^{\sqrt{2}\cos\psi} d\cos\psi \\
&= 2\pi \int_{-1}^{1} e^{\sqrt{2}t} dt \\
&= \sqrt{2}\pi \left(e^{\sqrt{2}} - e^{-\sqrt{2}}\right)
\end{aligned}
$$

计算$\iiint_{[0,1]^3}\frac{1}{\left(1+x^2+y^2+z^2\right)^2}dV$

结论：

平时要积累有关积分积出来的样子

例如对本题而言,$\int_0^\infty e^{-sx}dx=\frac{1}{s},x>0,\int_0^\infty xe^{-sx}dx=\frac{1}{s^2},x>0$

证明：

$$\iiint_{[0,1]^3}\frac{1}{\left(1+x^2+y^2+z^2\right)^2}dV=\iiint_{[0,1]^3}\int_0^\infty te^{-\left(1+x^2+y^2+z^2\right)t}dtdV$$

$$=\int_0^\infty te^{-t}dt\iiint_{[0,1]^3}e^{-\left(x^2+y^2+z^2\right)t}dV$$

$$=\int_0^\infty te^{-t}dt\left(\int_0^1 e^{-x^2t}dx\right)^3$$

$$=2\int_0^\infty\frac{\left(\int_0^{\sqrt{t}}e^{-y^2}dy\right)^3}{2\sqrt{t}}e^{-t}dt$$

$$=2\int_0^\infty\left(\int_0^{\sqrt{t}}e^{-y^2}dy\right)^3 d\left(\int_0^{\sqrt{t}}e^{-y^2}dy\right)$$

$$=\frac{1}{2}\left(\int_0^\infty e^{-y^2}dy\right)^4=\frac{\pi^2}{32}$$

相比参考答案的方法,这个方法更为自然一些.

设A是d阶实正定矩阵，记$B=\left(b_{ij}\right), b_{ij}=\int_{\mathbb{R}^d} x_i x_j e^{-x^T A x} dx$，
计算B.
证明：

$$\lambda>0, \int_{-\infty}^{\infty} e^{-\lambda x^2} dx=\frac{1}{\sqrt{\lambda}} \int_{-\infty}^{\infty} e^{-x^2} dx=\sqrt{\frac{\pi}{\lambda}}$$

$$\int_{-\infty}^{\infty} x^2 e^{-\lambda x^2} dx=\frac{1}{\lambda \sqrt{\lambda}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx=\frac{\sqrt{\pi}}{2 \lambda \sqrt{\lambda}}$$

取实正交矩阵$T=\left(t_{ij}\right)$，使得$T^T A T=\begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_d \end{pmatrix} \triangleq \Lambda$

于是做换元$x=Ty$，我们就有$x_i=\sum_{k=1}^d t_{ik} y_k$

$$b_{ij}=\int_{\mathbb{R}^d} x_i x_j e^{-x^T A x} dx=\int_{\mathbb{R}^d} \sum_{k=1}^d \sum_{l=1}^d t_{ik} t_{jl} y_k y_l e^{-y^T T^T A T y} dy$$

$$=\int_{\mathbb{R}^d} \sum_{k=1}^d \sum_{l=1}^d t_{ik} t_{jl} y_k y_l e^{-\sum_{k=1}^d \lambda_k y_k^2} dy$$

$$=\sum_{k=1}^d \sum_{l=1}^d t_{ik} t_{jl} \int_{\mathbb{R}^d} y_k y_l e^{-\sum_{k=1}^d \lambda_k y_k^2} dy$$

$$=\sum_{k=1}^d t_{ik} t_{jk} \int_{\mathbb{R}^d} y_k^2 e^{-\sum_{k=1}^d \lambda_k y_k^2} dy$$

$$=\frac{1}{2} \sqrt{\frac{\pi^d}{\lambda_1 \lambda_2 \cdots \lambda_d}} \sum_{k=1}^d t_{ik} t_{jk} \frac{1}{\lambda_k}$$

$$=\frac{1}{2} \sqrt{\frac{\pi^d}{\lambda_1 \lambda_2 \cdots \lambda_d}} \left(T \Lambda^{-1} T^T\right)_{(i,j)}$$

$$=\frac{1}{2} \sqrt{\frac{\pi^d}{\det A}} \left(A^{-1}\right)_{(i,j)}$$

因此$B=\frac{1}{2} \sqrt{\frac{\pi^d}{\det A}} A^{-1}$.