十四届决赛解析非数学类

七：

$$\sum_{n=1}^{\infty}\left[\ln\left(1+\frac{1}{2n}\right)\ln\left(1+\frac{1}{2n+1}\right)\right]$$

$$a_n=\ln\frac{n+1}{n},a_{2n}+a_{2n+1}=\ln\left(\frac{2n+1}{2n}\right)+\ln\left(\frac{2n+2}{2n+1}\right)=\ln\frac{n+1}{n}=a_n$$

$$\left(a_{2n}+a_{2n+1}\right)^2=a_n^2=a_{2n}^2+a_{2n+1}^2+2a_{2n}a_{2n+1}$$

$$\sum_{n=1}^{\infty}\left[\ln\left(1+\frac{1}{2n}\right)\ln\left(1+\frac{1}{2n+1}\right)\right]=\lim_{m\to\infty}\sum_{n=1}^{m}a_{2n}a_{2n+1}$$

$$=\frac{1}{2}\lim_{m\to\infty}\sum_{n=1}^{m}\left(a_n^2-a_{2n}^2-a_{2n+1}^2\right)=\begin{cases}\text{可以直接看出}\dfrac{1}{2}\lim\limits_{m\to\infty}\left(a_1^2-\sum\limits_{n=m+1}^{2m+1}a_n^2\right)\\ \text{或者裂项}\dfrac{1}{2}\lim\limits_{m\to\infty}\sum\limits_{n=1}^{m}\left(\sum\limits_{k=n}^{2n-1}a_k^2-\sum\limits_{k=n+1}^{2n+1}a_k^2\right)\end{cases}$$

$$\sum_{n=m+1}^{2m+1}a_n^2=\sum_{n=m+1}^{2m+1}\ln^2\left(1+\frac{1}{n}\right)\le\sum_{n=m+1}^{2m+1}\frac{1}{n^2},\text{因此由}cauchy\text{收敛准则}$$

$$\lim_{m\to\infty}\sum_{n=m+1}^{2m+1}a_n^2=0!,\text{因此}\sum_{n=1}^{\infty}\left[\ln\left(1+\frac{1}{2n}\right)\ln\left(1+\frac{1}{2n+1}\right)\right]=\frac{\ln^2 2}{2}.$$

第九届出的原题是计算$\sum_{n=1}^{\infty}\ln\left(1+\frac{1}{n}\right)\ln\left(1+\frac{1}{2n}\right)\ln\left(1+\frac{1}{2n+1}\right)$

证明：设$a_n=\ln\left(1+\frac{1}{n}\right)$，计算$\sum_{n=1}^{\infty}a_n a_{2n} a_{2n+1}$

注意到当$x+y+z=0$，有等式$x^3+y^3+z^3=3xyz$，

事实上$x^3+y^3+(-x-y)^3=-3x^2y-3xy^2=-3xy(x+y)=3xyz.$

又$a_n-a_{2n}-a_{2n+1}=0$，因此两边立方(或者直接用上一个恒等式)就有

$a_n a_{2n} a_{2n+1}=\frac{1}{3}\left(a_n^3-a_{2n}^3-a_{2n+1}^3\right)$

因此$\sum_{n=1}^{\infty}a_n a_{2n} a_{2n+1}=\frac{1}{3}\lim_{m\to\infty}\sum_{n=1}^{m}\left(a_n^3-a_{2n}^3-a_{2n+1}^3\right)=\frac{1}{3}\lim_{m\to\infty}\left(a_1^3-\sum_{n=m+1}^{2m+1}a_n^3\right)$

由$cauchy$收敛准则，显然$\lim_{m\to\infty}\sum_{n=m+1}^{2m+1}a_n^3=0$，因此$\sum_{n=1}^{\infty}a_n a_{2n} a_{2n+1}=\frac{1}{3}a_1^3=\frac{\ln^3 2}{3}.$

六.

方法1：两点之间线段最短.

方法2：变b为t，对t求导.

方法3：配$cauchy$不等式.

$$\int_a^b \frac{\sqrt{1+\left|f'(x)\right|^2}\sqrt{(x-a)^2+\left|f(x)-f(a)\right|^2}}{\sqrt{(x-a)^2+\left|f(x)-f(a)\right|^2}}dx$$

$$\geq \int_a^b \frac{x-a+f'(x)\left[f(x)-f(a)\right]}{\sqrt{(x-a)^2+\left|f(x)-f(a)\right|^2}}dx$$

$$=\frac{1}{2}\int_a^b \frac{1}{\sqrt{(x-a)^2+\left|f(x)-f(a)\right|^2}}d\left((x-a)^2+\left|f(x)-f(a)\right|^2\right)$$

$$=\sqrt{(x-a)^2+\left|f(x)-f(a)\right|^2}\ \Big|_a^b$$

$$=\sqrt{(b-a)^2+\left|f(b)-f(a)\right|^2}$$

$cauchy$不等式等号成立等价于$\dfrac{f(x)-f(a)}{f'(x)}=x-a,\forall x$

从而$f'(x)(x-a)=f(x)-f(a)\overset{\text{解微分方程}}{\Rightarrow} f(x)=f(a)+\dfrac{f(b)-f(a)}{b-a}(x-a)$.

五.

$(1): \sin tx \ge t\sin x$

$F(x,t) = \sin tx - t\sin x$

若$(x,t) \in (0,\pi)\times(0,1)$使得$F$为极值点, 则

$F_x = t\cos tx - t\cos x = 0$

$F_t = x\cos tx - \sin x = 0$

$\Rightarrow t\dfrac{\sin x}{x} = t\cos x \Rightarrow \tan x = x \Rightarrow x = 0$矛盾

因此最值在边界取到, 此时考虑

$F(x,1) = 0, F(x,0) = 0, F(0,t) = 0, F(\pi,t) = \sin \pi t \ge 0$

因此$F(x,t) \ge 0$恒成立.

$(2):$

$\int_0^{\frac{\pi}{2}} \left|\sin u\right|^p du \ge \dfrac{\pi}{2(p+1)}$

利用$jordan$不等式$\sin x \ge \dfrac{2}{\pi}x, x \in \left[0, \dfrac{\pi}{2}\right]$

此时$\int_0^{\frac{\pi}{2}} \left|\sin u\right|^p du \ge \int_0^{\frac{\pi}{2}} \left(\dfrac{2}{\pi}u\right)^p du = \dfrac{\pi}{2(p+1)}.$

$(3): \int_0^x \left|\sin u\right|^p du \ge \dfrac{x\left|\sin x\right|^p}{p+1}, \forall x \ge 0, p > 0$

$x = n\pi + r, r \in \left[0, \pi\right)$,即证明：

$$\int_0^{n\pi+r} \left|\sin u\right|^p du \ge \frac{\left(n\pi + r\right)\left|\sin r\right|^p}{p+1}$$

$$\Leftrightarrow n\int_0^{\pi} \left|\sin u\right|^p du + \int_0^{r} \left|\sin u\right|^p du \ge \frac{n\pi\left|\sin r\right|^p}{p+1} + \frac{r\left|\sin r\right|^p}{p+1}$$

$$n\int_0^{\pi} \left|\sin u\right|^p du = 2n\int_0^{\frac{\pi}{2}} \left|\sin u\right|^p du = 2n\pi\int_0^{\frac{1}{2}} \left|\sin \pi u\right|^p du \ge 2n\pi\int_0^{\frac{1}{2}} \left|2u\sin\frac{\pi}{2}\right|^p du$$

$$= 2^{p+1}n\pi\frac{\left(\frac{1}{2}\right)^{p+1}}{p+1} = \frac{n\pi}{p+1} \ge \frac{n\pi\left|\sin r\right|^p}{p+1}$$

$$\int_0^{r} \left|\sin u\right|^p du = r\int_0^{1} \left|\sin\left(ru\right)\right|^p du \ge r\int_0^{1} \left|u\sin r\right|^p du = \frac{r\left|\sin r\right|^p}{p+1}$$

结合上面两个不等式,我们证明了$\int_0^x \left|\sin u\right|^p du \ge \dfrac{x\left|\sin x\right|^p}{p+1}, \forall x \ge 0, p > 0.$

四：

由$\begin{cases}\left(x^2+z^2\right)^2=4\left(x^2-z^2\right)\\ y=0\end{cases}$

见第九届解析几何旋转曲面的求法(一般直线都可以)

或者直接背同济上面的公式(绕oz轴把x改成$\pm\sqrt{x^2+y^2}$).

知旋转曲面为$\left(x^2+y^2+z^2\right)^2=4\left(x^2+y^2-z^2\right)$

使用球坐标, $x=\rho\sin\psi\cos\theta, y=\rho\sin\psi\sin\theta, z=\rho\cos\psi$

$\rho^4=4\left(\rho^2\sin^2\psi-\rho^2\cos^2\psi\right)\Rightarrow\rho^2=4\left(\sin^2\psi-\cos^2\psi\right)$

$\Rightarrow\rho=2\sqrt{-\cos\left(2\psi\right)}, \cos\left(2\psi\right)\le 0\Rightarrow\rho=2\sqrt{-\cos\left(2\psi\right)}, \psi\in\left[\frac{\pi}{4},\frac{3\pi}{4}\right]$

因此体积为$\int_0^{2\pi}d\theta\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin\psi d\psi\int_0^{2\sqrt{-\cos\left(2\psi\right)}}\rho^2d\rho$

$$=\frac{16\pi}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin\psi\left(\sqrt{-\cos\left(2\psi\right)}\right)^3d\psi$$

$$=\frac{16\pi}{3}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}}\sin\psi\left(\sqrt{1-2\cos^2\psi}\right)^3d\psi$$

$$=\frac{16\pi}{3}\int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}}\left(\sqrt{1-2t^2}\right)^3dt$$

$$=\frac{16\pi}{3\sqrt{2}}\int_{-1}^{1}\left(\sqrt{1-y^2}\right)^3dy$$

$$=\frac{32\pi}{3\sqrt{2}}\int_0^1\left(1-y^2\right)^{\frac{3}{2}}dy$$

$$=\frac{32\pi}{3\sqrt{2}}\int_0^{\frac{\pi}{2}}\cos^4\theta d\theta$$

$$=\frac{32\pi}{3\sqrt{2}}\frac{\pi}{2}\frac{3}{8}=\sqrt{2}\pi^2$$

三.

$$2yg'+2f'=2yf+2e^{x}-8g$$

$$f=g',f'=e^{x}-4g$$

$$g''+4g=e^{x}$$

$$g=c_{1}\cos 2x+c_{2}\sin 2x+\frac{1}{5}e^{x}$$

$$f=-2c_{1}\sin 2x+2c_{2}\cos 2x+\frac{1}{5}e^{x}$$

$$g\left(0\right)=1$$

$$\Rightarrow g=\frac{4}{5}\cos 2x+c_{2}\sin 2x+\frac{1}{5}e^{x}$$

$$f\left(0\right)=1$$

$$\Rightarrow c_{2}=\frac{2}{5}$$

$$\Rightarrow \begin{cases} g=\frac{4}{5}\cos 2x+\frac{2}{5}\sin 2x+\frac{1}{5}e^{x} \\ f=-\frac{8}{5}\sin 2x+\frac{4}{5}\cos 2x+\frac{1}{5}e^{x} \end{cases}.$$

二.

$$A=\begin{pmatrix} 1 & -2 & 2 \\ -2 & -2 & 4 \\ 2 & 4 & -2 \end{pmatrix},x^{T}Ax=27$$

纯课内期末题，取$T=\begin{pmatrix} \frac{-2}{\sqrt{5}} & \frac{2}{3\sqrt{5}} & \frac{1}{3} \\ \frac{1}{\sqrt{5}} & \frac{4}{3\sqrt{5}} & \frac{2}{3} \\ 0 & \frac{5}{3\sqrt{5}} & -\frac{2}{3} \end{pmatrix},x=Ty$

$2u^{2}+2v^{2}-7z^{2}=27.$

这是单叶双曲面(可以参考丘维声解析几何二次曲面分类).

$$1.\lim_{x\to 0}\frac{\arctan x - x}{x-\sin x}=\lim_{x\to 0}\frac{-\frac{1}{3}x^3}{\frac{1}{6}x^3}=-2.$$

$$2.\int_0^{\infty}x^3e^{-ax}dx=\frac{1}{a^4}\int_0^{\infty}x^3e^{-x}dx=\frac{\Gamma\left(4\right)}{a^4}=\frac{6}{a^4}.$$

3.

设对称点$\left(a,b,c\right),\left(\frac{a+2}{2},\frac{b+2}{2},\frac{c+2}{2}\right)$落在直线上

且$\left(a-2,b-2,c-2\right)\cdot\left(3,2,1\right)=0$

$3\left(a-2\right)+2\left(b-2\right)+c-2=0\Rightarrow 3a+2b+c=12$

$$\frac{\frac{a+2}{2}-1}{3}=\frac{\frac{b+2}{2}+4}{2}=\frac{c+2}{2}-3\Rightarrow\frac{a}{6}=\frac{b+10}{4}=\frac{c}{2}-2$$

$\Rightarrow\left(a,b,c\right)=\left(6,-6,6\right)$

4.

$f_x=3y-3x^2=0, f_y=3x-3y^2=0$

$y=y^4\Rightarrow y=0$或者1

所以有二组$\left(0,0\right),\left(1,1\right)$

$f_{xx}=-6x, f_{xy}=3, f_{yy}=-6y$

$$D^2f=\begin{pmatrix}-6x & 3\\ 3 & -6y\end{pmatrix},$$

在$\left(0,0\right):\begin{pmatrix}0 & 3\\ 3 & 0\end{pmatrix}$不定（或者用$AC-B^2$看你），不是极值

在$\left(1,1\right):\begin{pmatrix}-6 & 3\\ 3 & -6\end{pmatrix}$负定, $f\left(1,1\right)$极大值$=4.$

5.

$$\sum_{n=1}^{\infty}\left(-1\right)^{n}\frac{1}{n3^{n}}x^{n},\lim_{n\to\infty}\frac{\frac{1}{n3^{n}}}{\frac{1}{\left(n+1\right)3^{n+1}}}=3$$

故收敛半径为3,$\sum_{n=1}^{\infty}\left(-1\right)^{n}\frac{1}{n3^{n}}3^{n}=\sum_{n=1}^{\infty}\left(-1\right)^{n}\frac{1}{n}$收敛

$\sum_{n=1}^{\infty}\left(-1\right)^{n}\frac{1}{n3^{n}}\left(-3\right)^{n}=\sum_{n=1}^{\infty}\frac{1}{n}$发散

因此收敛于$\left(-3,3\right]$.

$\begin{cases}x^{2}+y^{2}+z^{2}=1\\x+2y+z=0\end{cases}$,计算

$$\int\left(2x^{2}+x+y+y^{2}\right)ds$$

$$\overset{x,z\text{轮换}}{=}\int\left(x^{2}+x+y+y^{2}+z^{2}\right)ds$$

$$=\int\left(1+x+y\right)ds$$

$$=\int 1ds+\frac{1}{2}\int\left(2x+2y\right)ds$$

$$\overset{x,z\text{轮换}}{=}\int 1ds+\frac{1}{2}\int\left(x+2y+z\right)ds$$

$$=\int 1ds=\text{曲线长度}$$

注意平面过球心,因此圆半径就是球半径1, 所以周长为2π.