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Preface

This book offers a combined course of lectures on game theory which the author has delivered for several years in Russian and foreign universities.

In addition to classical branches of game theory, our analysis covers modern branches left without consideration in most textbooks on the subject (negotiation models, potential games, parlor games, best choice games, and network games). The fundamentals of mathematical analysis, algebra, and probability theory are the necessary prerequisites for reading.

The book can be useful for students specializing in applied mathematics and informatics, as well as economical cybernetics. Moreover, it attracts the mutual interest of mathematicians operating in the field of game theory and experts in the fields of economics, management science, and operations research.

Each chapter concludes with a series of exercises intended for better understanding. Some exercises represent open problems for conducting independent investigations. As a matter of fact, stimulation of reader's research is the main priority of the book. A comprehensive bibliography will guide the audience in an appropriate scientific direction.

For many years, the author has enjoyed the opportunity to discuss derived results with Russian colleagues L.A. Petrosjan, V.V. Zakharov, N.V. Zenkevich, I.A. Seregin, and A.Yu. Garnaev (St. Petersburg State University), A.A. Vasin (Lomonosov Moscow State University), D.A. Novikov (Trapeznikov Institute of Control Sciences, Russian Academy of Sciences), A.V. Kryazhimskii and A.B. Zhizhchenko (Steklov Mathematical Institute, Russian Academy of Sciences), as well as with foreign colleagues M. Sakaguchi (Osaka University), M. Tamaki (Aichi University), K. Szajowski (Wroclaw University of Technology), B. Monien (University of Paderborn), K. Avratchenkov (INRIA, Sophia-Antipolis), and N. Perrin (University of Lausanne). They all have my deep and sincere appreciation. The author expresses profound gratitude to young colleagues A.N. Rettieva, J.S. Tokareva, Yu.V. Chirkova, A.A. Ivashko, A.V. Shiptsova and A.Y. Kondratjev from Institute of Applied Mathematical Research (Karelian Research Center, Russian Academy of Sciences) for their assistance in typing and formatting of the book. Next, my frank acknowledgement belongs to A.Yu. Mazurov for his careful translation, permanent feedback, and contribution to the English version of the book.

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Introduction

"*Equilibrium arises from righteousness, and righteousness arises from the meaning of the cosmos.*"

From Hermann Hesse's *The Glass Bead Game*

Game theory represents a branch of mathematics, which analyzes models of optimal decision-making in the conditions of a conflict. Game theory belongs to operations research, a science originally intended for planning and conducting military operations. However, the range of its applications appears much wider. Game theory always concentrates on models with several participants. This forms a fundamental distinction of game theory from optimization theory. Here the notion of an optimal solution is a matter of principle. There exist many definitions of the solution of a game. Generally, the solution of a game is called an equilibrium, but one can choose different concepts of an equilibrium (a Nash equilibrium, a Stackelberg equilibrium, a Wardrop equilibrium, to name a few).

In the last few years, a series of outstanding researchers in the field of game theory were awarded Nobel Prize in Economic Sciences. They are J.C. Harsanyi, J.F. Nash Jr., and R. Selten (1994) "for their pioneering analysis of equilibria in the theory of non-cooperative games," F.E. Kydland and E.C. Prescott (2004) "for their contributions to dynamic macroeconomics: the time consistency of economic policy and the driving forces behind business cycles," R.J. Aumann and T.C. Schelling (2005) "for having enhanced our understanding of conflict and cooperation through game-theory analysis," L. Hurwicz, E.S. Maskin, and R.B. Myerson (2007) "for having laid the foundations of mechanism design theory." Throughout the book, we will repeatedly cite these names and corresponding problems.

Depending on the number of players, one can distinguish between zero-sum games (antagonistic games) and nonzero-sum games. Strategy sets are finite or infinite (matrix games and games on compact sets, respectively). Next, players may act independently or form coalitions; the corresponding models represent non-cooperative games and cooperative games. There are games with complete or partial incoming information.

Game theory admits numerous applications. One would hardly find a field of sciences focused on life and society without usage of game-theoretic methods. In the first place, it is necessary to mention economic models, models of market relations and competition, pricing models, models of seller-buyer relations, negotiation, and stable agreements, etc. The pioneering book by J. von Neumann and O. Morgenstern, the founders of game theory, was entitled *Theory of Games and Economic Behavior*. The behavior of market participants, modeling of their psychological features forms the subject of a new science known as experimental economics.

Game-theoretic methods generated fundamental results in evolutionary biology. The notion of evolutionary stable strategies introduced by British biologist J.M. Smith enabled explaining the evolution of several behavioral peculiarities of animals such as aggressiveness, migration, and struggle for survival. Game-theoretic methods are intensively used in rational nature management problems. For instance, fishing quotas distribution in the ocean, timber extraction by several participants, agricultural pricing are problems of game theory. Today, it seems even impossible to implement intergovernmental agreements on natural resources utilization and environmental pollution reduction (e.g., The Kyoto Protocol) without game-theoretic analysis. In political sciences, game theory concerns voting models in parliaments, influence assessment models for certain political factions, as well as models of defense resources distribution for stable peace achievement. In jurisprudence, game theory is applied in arbitration for assessing the behavioral impact of conflicting sides on judicial decisions.

We have recently observed a technological breakthrough in the analysis of the virtual information world. In terms of game theory, all participants of the global computer network (Internet) and mobile communication networks represent interacting players that receive and transmit information by appropriate data channels. Each player pursues individual interests (acquire some information or complicate this process). Players strive for channels with high-level capacities, and the problem of channel distribution among numerous players arises naturally. And game-theoretic methods are of assistance here. Another problem concerns the impact of user service centralization on system efficiency. The estimate of the centralization effect in a system, where each participant follows individual interests (maximal channel capacity, minimal delay, the maximal amount of received information, etc.) is known as the price of anarchy. Finally, an important problem lies in defining the influence of information network topology on the efficiency of player service. These are non-trivial problems causing certain paradoxes. We describe the corresponding phenomena in the book.

Which fields of knowledge manage without game-theoretic methods? Perhaps, medical science and finance do so, although game-theoretic methods have also recently found some applications in these fields.

The approach to material presentation in this book differs from conventional ones. We intentionally avoid a detailed treatment of matrix games, as far as they are described in many publications. Our study begins with nonzero-sum games and the fundamental theorem on equilibrium existence in convex games. Later on, this result is extended to the class of zero-sum games. The discussion covers several classical models used in economics (the models of market competition suggested by Cournot, Bertrand, Hotelling, and Stackelberg, as well as auctions). Next, we pass from normal-form games to extensive-form games and parlor games. The early chapters of the book consider two-player games, and further analysis embraces n-player games (first, non-cooperative games, and then cooperative ones).

Subsequently, we provide fundamental results in new branches of game theory, best choice games, network games, and dynamic games. The book proposes new schemes of negotiations, much attention is paid to arbitration procedures. Some results belong to the author and his colleagues. The fundamentals of mathematical analysis, algebra, and probability theory are the necessary prerequisites for reading.

This book contains an accompanying website. Please visit www.wiley.com/go/game_theory.

1

Strategic-form two-player games

Introduction

Our analysis of game problems begins with the case of two-player strategic-form (equivalently, normal-form) games. The basic notions of game theory comprise **Players, Strategies and Payoffs**. In the sequel, denote players by I and II. A normal-form game is organized in the following way. Player I chooses a certain strategy x from a set X, while player II simultaneously chooses some strategy y from a set Y. In fact, the sets X and Y may possess any structure (a finite set of values, a subset of R^n, a set of measurable functions, etc.). As a result, players I and II obtain the payoffs $H_1(x, y)$ and $H_2(x, y)$, respectively.

Definition 1.1 *A normal-form game is an object*

$$\Gamma =< I, II, X, Y, H_1, H_2 >,$$

where X, Y designate the sets of strategies of players I and II, whereas H_1, H_2 indicate their payoff functions, $H_i : X \times Y \to R, i = 1, 2$.

Each player selects his strategy regardless of the opponent's choice and strives for maximizing his own payoff. However, a player's payoff depends both on his strategy and the behavior of the opponent. This aspect makes the specifics of game theory.

How should one comprehend the solution of a game? There exist several approaches to construct solutions in game theory. Some of them will be discussed below. First, let us consider the notion of a Nash equilibrium as a central concept in game theory.

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

Definition 1.2 *A Nash equilibrium in a game* Γ *is a set of strategies* (x^*, y^*) *meeting the conditions*

$$
\begin{aligned}
H_1(x, y^*) &\leq H_1(x^*, y^*), \\
H_2(x^*, y) &\leq H_2(x^*, y^*)
\end{aligned}
\tag{1.1}
$$

for arbitrary strategies x, y *of the players.*

Inequalities (1.1) imply that, as the players deviate from a Nash equilibrium, their payoffs do decrease. Hence, deviations from the equilibrium appear non-beneficial to any player. Interestingly, there may exist no Nash equilibria. Therefore, a major issue in game problems concerns their existence. Suppose that a Nash equilibrium exists; in this case, we say that the payoffs $H_1^* = H_1(x^*, y^*)$, $H_2^* = H_2(x^*, y^*)$ are optimal. A set of strategies (x, y) is often called a **strategy profile**.

1.1 The Cournot duopoly

We mention the Cournot duopoly [1838] among pioneering game models that gained wide popularity in economic research. The term "duopoly" corresponds to a two-player game.

Imagine two companies, I and II, manufacturing some quantities of a same product (q_1 and q_2, respectively). In this model, the quantities represent the strategies of the players. The market price of the product equals an initial price p after deduction of the total quantity $Q = q_1 + q_2$. And so, the unit price constitutes $(p - Q)$. Let c be the unit cost such that $c < p$. Consequently, the players' payoffs take the form

$$
H_1(q_1, q_2) = (p - q_1 - q_2)q_1 - cq_1, \quad H_2(q_1, q_2) = (p - q_1 - q_2)q_2 - cq_2. \tag{1.2}
$$

In the current notation, the game is defined by $\Gamma = < I, II, Q_1 = [0, \infty), Q_2 = [0, \infty), H_1, H_2 >$.

Nash equilibrium evaluation (see formula (1.1)) calls for solving two problems, *viz.*, $\max_{q_1} H_1(q_1, q_2^*)$ and $\max_{q_2} H_2(q_1^*, q_2)$. Moreover, we have to demonstrate that the maxima are attained at $q_1 = q_1^*$, $q_2 = q_2^*$. The quadratic functions $H_1(q_1, q_2^*)$ and $H_2(q_1^*, q_2)$ get maximized by

$$
\begin{aligned}
q_1 &= \frac{1}{2}\left(p - c - q_2^*\right) \\
q_2 &= \frac{1}{2}\left(p - c - q_1^*\right).
\end{aligned}
$$

Naturally, these quantities must be non-negative, which dictates that

$$
q_i^* \leq p - c, \; i = 1, 2. \tag{1.3}
$$

By resolving the derived system of equations in q_1^*, q_2^*, we find

$$
q_1^* = q_2^* = \frac{p - c}{3}
$$

that satisfy the conditions (1.3). And the optimal payoffs become

$$H_1^* = H_2^* = \frac{(p-c)^2}{9}.$$

1.2 Continuous improvement procedure

Imagine that player I knows the strategy q_2 of player II. Then his **best response** lies in the strategy q_1 yielding the maximal payoff $H_1(q_1, q_2)$. Recall that $H_1(q_1, q_2)$ is a concave parabola possessing its vertex at the point

$$q_1 = \frac{1}{2}(p - c - q_2). \tag{2.1}$$

We denote the best response function by $q_1 = R(q_2) = \frac{1}{2}(p - c - q_2)$. Similarly, if the strategy q_1 of player I becomes known to player II, his best response is the strategy q_2 corresponding to the maximal payoff $H_2(q_1, q_2)$. In other words,

$$q_2 = R(q_1) = \frac{1}{2}(p - c - q_1). \tag{2.2}$$

Draw the lines of the best responses (2.1)–(2.2) on the plane (q_1, q_2) (see Figure 1.1). For any initial strategy q_2^0, construct the sequence of the best responses

$$q_2^{(0)} \to q_1^{(1)} = R(q_2^{(0)}) \to q_2^{(1)} = R(q_1^{(1)}) \to \cdots \to q_1^{(n)} = R(q_2^{(n-1)}) \to q_2^{(n)} = R(q_1^n) \to \ldots$$

The sequence (q_1^n, q_2^n) is said to be the best response sequence. Such iterative procedure agrees with the behavior of sellers on a market (each of them modifies his strategy depending on the actions of the competitors). According to Figure 1.1, the best response sequence of the players tends to an equilibrium for any initial strategy $q_2^{(0)}$. However, we emphasize that the best response sequence does not necessarily brings a Nash equilibrium.

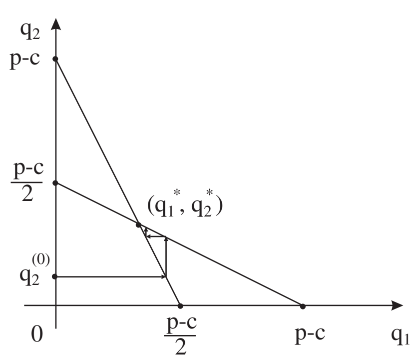


Figure 1.1 The Cournot duopoly.

1.3 The Bertrand duopoly

Another two-player game which models market pricing concerns the Bertrand duopoly [1883].

Consider two companies, I and II, manufacturing products A and B, respectively. Here the players choose product prices as their strategies. Assume that company I declares the unit prices of c_1, while company II declares the unit prices of c_2.

As the result of prices quotation, one observes the demands for each product on the market, i.e., $Q_1(c_1, c_2) = q - c_1 + kc_2$ and $Q_2(c_1, c_2) = q - c_2 + kc_1$. The symbol q means an initial demand, and the coefficient k reflects the interchangeability of products A and B.

By analogy to the Cournot model, the unit cost will be specified by c. Consequently, the players' payoffs acquire the form

$$H_1(c_1, c_2) = (q - c_1 + kc_2)(c_1 - c),\ \ H_2(c_1, c_2) = (q - c_2 + kc_1)(c_2 - c).$$

The game is completely defined by: $\Gamma =< I, II, Q_1 = [0, \infty), Q_2 = [0, \infty), H_1, H_2 >$.

Fix the strategy c_1 of player I. Then the best response of player II consists in the strategy c_2 guaranteeing the maximal payoff $\max\limits_{c_2} H_2(c_1, c_2)$. Since $H_2(c_1, c_2)$ forms a concave parabola, its vertex is at the point

$$c_2 = \frac{1}{2}(q + kc_1 + c). \tag{3.1}$$

Similarly, if the strategy c_2 of player II is fixed, the best response of player I becomes the strategy c_1 ensuring the maximal payoff $\max\limits_{c_1} H_1(c_1, c_2)$. We easily find

$$c_1 = \frac{1}{2}(q + kc_2 + c). \tag{3.2}$$

There exists a unique solution to the system of equations (3.1)–(3.2):

$$c_1^* = c_2^* = \frac{q + c}{2 - k}.$$

We seek for positive solutions; therefore, $k < 2$.

The resulting solution represents a Nash equilibrium. Indeed, the best response of player II to the strategy c_1^* lies in the strategy c_2^*; and vice versa, the best response of player I to the strategy c_2^* makes the strategy c_1^*.

The optimal payoffs of the players in the equilibrium are given by

$$H_1^* = H_2^* = \left[\frac{q - c(1 - k)}{2 - k}\right]^2.$$

Draw the lines of the best responses (3.1)–(3.2) on the plane (c_1, c_2) (see Figure 1.2). Denote by $R(c_1), R(c_2)$ the right-hand sides of (3.1) and (3.2). For any initial strategy c_2^0, construct the best response sequence

$$c_2^{(0)} \to c_1^{(1)} = R(c_2^{0)}) \to c_2^{(1)} = R(c_1^{(1)}) \to \cdots \to c_1^{(n)} = R(c_2^{(n-1)}) \to c_2^{(n)} = R(c_1^{n}) \to \cdots$$

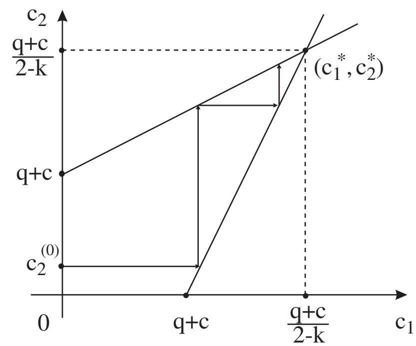


Figure 1.2 The Bertrand duopoly.

Figure 1.2 demonstrates the following. The best response sequence tends to the equilibrium (c_1^*, c_2^*) for any initial strategy $c_2^{(0)}$.

1.4 The Hotelling duopoly

This two-player game introduced by Hotelling [1929] also belongs to pricing problems but takes account of the location of companies on a market. Consider a linear market (see Figure 1.3) representing the unit segment $[0, 1]$. There exist two companies, I and II, located at points x_1 and x_2. Each company quotes its price for the same product (the parameters c_1 and c_2, respectively). Subsequently, each customer situated at point x compares his costs to visit each company, $L_i(x) = c_i + |x - x_i|, i = 1, 2$, and chooses the one corresponding to smaller costs. Within the framework of the Hotelling model, the costs $L(x)$ can be interpreted as the product price supplemented by transport costs. And all customers are decomposed into two sets, $[0, x)$ and $(x, 1]$. The former prefer company I, whereas the latter choose company II. The boundary of these sets x follows from the equality $L_1(x) = L_2(x)$:

$$x = \frac{x_1 + x_2}{2} + \frac{c_2 - c_1}{2}.$$

In this case, we understand the payoffs as the incomes of the players, i.e.,

$$H_1(c_1, c_2) = c_1 x = c_1 \left[\frac{x_1 + x_2}{2} + \frac{c_2 - c_1}{2}\right], \tag{4.1}$$

$$H_2(c_1, c_2) = c_2(1 - x) = c_2 \left[1 - \frac{x_1 + x_2}{2} - \frac{c_2 - c_1}{2}\right]. \tag{4.2}$$

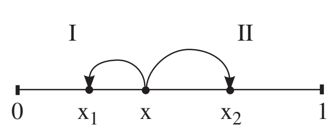


Figure 1.3 The Hotelling duopoly on a segment.

A Nash equilibrium (c_1^*, c_2^*) satisfies the equations $\frac{\partial H_1(c_1,c_2^*)}{\partial c_1} = 0$, $\frac{\partial H_2(c_1^*,c_2)}{\partial c_2} = 0$. And so,

$$\frac{\partial H_1(c_1, c_2)}{\partial c_1} = \frac{c_2 - c_1}{2} + \frac{x_1 + x_2}{2} - \frac{c_1}{2} = 0,$$
$$\frac{\partial H_2(c_1, c_2)}{\partial c_2} = 1 - \frac{c_2 - c_1}{2} - \frac{x_1 + x_2}{2} - \frac{c_2}{2} = 0.$$

Summing up the above equations yields

$$c_1^* + c_2^* = 2,$$

which leads to the equilibrium prices

$$c_1^* = \frac{2 + x_1 + x_2}{3}, \; c_2^* = \frac{4 - x_1 - x_2}{3}.$$

Substitute the equilibrium prices into (4.1)–(4.2) to get the equilibrium payoffs:

$$H_1(c_1^*, c_2^*) = \frac{[2 + x_1 + x_2]^2}{18}, \; H_2(c_1^*, c_2^*) = \frac{[4 - x_1 - x_2]^2}{18}.$$

Just like in the previous case, here the payoff functions (4.1)–(4.2) are concave parabolas. Hence, the strategy improvement procedure tends to the equilibrium.

1.5 The Hotelling duopoly in 2D space

The preceding section proceeded from the idea that a market forms a linear segment. Actually, a market makes a set in 2D space. Let a city be a unit circle S with a uniform distribution of customers (see Figure 1.4). For the sake of simplicity, suppose that companies I and II are located at diametrally opposite points $(-1, 0)$ and $(1, 0)$. Each company announces a certain product price c_i, $i = 1, 2$. Without loss of generality, we believe that $c_1 < c_2$.

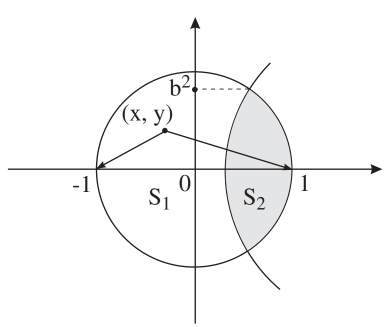


Figure 1.4 The Hotelling duopoly in 2D space.

A customer situated at point $(x, y) \in S$ compares the costs to visit the companies. Denote by $\rho_1(x, y) = \sqrt{(x+1)^2 + y^2}$ and $\rho_2(x, y) = \sqrt{(x-1)^2 + y^2}$ the distance to each company. Again, the total costs comprise a product price and transport costs: $L_i(x, y) = c_i + \rho_i(x, y),\ i = 1, 2$. The set of all customers is divided into two subsets, S_1 and S_2, whose boundary meets the equation

$$c_1 + \sqrt{(x+1)^2 + y^2} = c_2 + \sqrt{(x-1)^2 + y^2}.$$

After trivial manipulations, one obtains

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where

$$a = (c_2 - c_1)/2,\ b = \sqrt{1 - a^2}. \tag{5.1}$$

Therefore, the boundary of the sets S_1 and S_2 represents a hyperbola. The players' payoffs take the form

$$H_1(c_1, c_2) = c_1 s_1,\ H_2(c_1, c_2) = c_2 s_2,$$

with $s_i (i = 1, 2)$ meaning the areas occupied by appropriate sets.

As far as $s_1 + s_2 = \pi$, it suffices to evaluate s_2. Using Figure 1.4, we have

$$\begin{aligned} s_2 &= \frac{\pi}{2} - 2\left[a \int_0^{b^2} \sqrt{1 + \frac{y^2}{b^2}} dy + \int_{b^2}^{1} \sqrt{1 - y^2} dy\right] \\ &= \frac{\pi}{2} - 2\left[ab \int_0^{b} \sqrt{1 + y^2} dy + \int_{b^2}^{1} \sqrt{1 - y^2} dy\right]. \end{aligned} \tag{5.2}$$

The Nash equilibrium (c_1^*, c_2^*) of this game follows from the conditions

$$\frac{\partial H_1(c_1, c_2)}{\partial c_1} = \pi - s_2 - c_1 \frac{\partial s_2}{\partial c_1} = 0, \tag{5.3}$$

$$\frac{\partial H_2(c_1, c_2)}{\partial c_2} = s_2 + c_2 \frac{\partial s_2}{\partial c_2} = 0. \tag{5.4}$$

Revert to formula (5.2) to derive

$$\frac{\partial s_2}{\partial c_1} = \frac{b^2 - a^2}{b} \int_0^b \sqrt{1 + y^2} dy + a^2 \sqrt{1 + b^2}. \tag{5.5}$$

By virtue of $\frac{\partial a}{\partial c_1} = -\frac{\partial a}{\partial c_2}$, we arrive at

$$\frac{\partial s_2}{\partial c_2} = -\frac{\partial s_2}{\partial c_1}. \tag{5.6}$$

The function $s_2(c_1, c_2)$ strictly increases with respect to the argument c_1. This fact is immediate from an important observation. If player *I* quotes a higher price, then the customer from S_2 (characterized by greater costs to visit company *I* in comparison with company *II*) still benefits by visiting company *II*.

To proceed, let us evaluate the equilibrium in this game.

Owing to (5.6), the expressions (5.3)–(5.4) yield

$$s_2\left(1 + \frac{c_1}{c_2}\right) = \pi.$$

And so, if $c_1 < c_2$, then s_2 must exceed $\pi/2$. Meanwhile, this contradicts the following idea. Imagine that the price declared by company *I* appears smaller than the one offered by the opponent; in this case, the set of customers preferring this company (S_1) becomes larger than S_2, i.e., $s_2 < \pi/2$. Therefore, the solution to the system (5.3)–(5.4) (if any) exists only under $c_1 = c_2$. Generally speaking, this conclusion also follows from the symmetry of the problem.

Thus, we look for the solution in the class of identical prices: $c_1 = c_2$. Then $s_1 = s_2 = \pi/2$ and the notation $a = 0,\ b = 1$ from (5.5) brings to

$$\frac{\partial s_2}{\partial c_1} = \int_0^1 \sqrt{1 + y^2} dy = \frac{1}{2}\left[\sqrt{2} + \ln(1 + \sqrt{2})\right].$$

Formulas (5.3)–(5.4) lead to the equilibrium prices

$$c_1^* = c_2^* = \frac{\pi}{\sqrt{2} + \ln(1 + \sqrt{2})} \approx 1.3685.$$

1.6 The Stackelberg duopoly

Up to here, we have studied two-player games with equal rights of the opponents (they choose decisions simultaneously). The Stackelberg duopoly [1934] deals with a certain hierarchy of players. Notably, player *I* chooses his decision first, and then player *II* does. Player *I* is called a **leader**, and player *II* is called a **follower**.

Definition 1.3 *A Stackelberg equilibrium in a game* Γ *is a set of strategies* (x^*, y^*) *such that* $y^* = R(x^*)$ *represents the best response of player II to the strategy* x^* *which solves the problem*

$$H_1(x^*, y^*) = \max_x H_1(x, R(x)).$$

Therefore, in a Stackelberg equilibrium, a leader knows that a *follower* chooses the best response to any strategy and easily finds the strategy x^* maximizing his payoff.

Now, analyze the Stackelberg model within the Cournot duopoly. There exist two companies, I and II, manufacturing a same product. At step 1, company I announces its product output q_1. Subsequently, company II chooses its strategy q_2.

Recall the outcomes of Section 1.1; the best response of player II to the strategy q_1 is the strategy $q_2 = R(q_1) = (p - c - q_1)/2$. Knowing this, player I maximizes his payoff

$$H_1(q_1, R(q_1)) = q_1(p - c - q_1 - R(q_1)) = q_1(p - c - q_1)/2.$$

Clearly, the optimal strategy of this player lies in

$$q_1^* = \frac{p - c}{2}.$$

Accordingly, the optimal strategy of player II makes up

$$q_2^* = \frac{p - c}{4}.$$

The equilibrium payoffs of the players equal

$$H_1^* = \frac{(p - c)^2}{8},$$
$$H_2^* = \frac{(p - c)^2}{16}.$$

Obviously, the leader gains twice as much as the *follower* does.

1.7 Convex games

Nash equilibria do exist in all games discussed above. Generally speaking, the class of games admitting no equilibria appears much wider. The current section focuses on this issue. For the time being, note that the existence of Nash equilibria in the duopolies relates to the form of payoff functions (all economic examples considered employ continuous concave functions).

Definition 1.4 *A function $H(x)$ is called concave (convex) on a set $X \subseteq R^n$, if for any $x, y \in X$ and $\alpha \in [0, 1]$ the inequality $H(\alpha x + (1 - \alpha)y) \geq (\leq) \alpha H(x) + (1 - \alpha)H(y)$ holds true.*

Interestingly, this definition directly implies the following result. Concave functions also meet the inequality

$$H\left(\sum_{i=1}^{p} \alpha_i x_i\right) \geq \sum_{i=1}^{p} \alpha_i H(x_i)$$

for any convex combination of the points $x_i \in X$, $i = 1, \ldots, p$, where $\alpha_i \geq 0$, $i = 1, \ldots, p$ and $\sum \alpha_i = 1$.

The Nash theorem [1951] forms a central statement regarding equilibrium existence in such games. Prior to introducing this theorem, we prove an auxiliary result to-be-viewed as an alternative definition of a Nash equilibrium.

Lemma 1.1 *A Nash equilibrium exists in a game* $\Gamma =< I, II, X, Y, H_1, H_2 >$ *iff there is a set of strategies* (x^*, y^*) *such that*

$$\max_{x,y} \left\{H_1(x, y^*) + H_2(x^*, y)\right\} = H_1(x^*, y^*) + H_2(x^*, y^*). \tag{7.1}$$

Proof: The necessity part. Suppose that a Nash equilibrium (x^*, y^*) exists. According to Definition 1.2, for arbitrary (x, y) we have

$$H_1(x, y^*) \leq H_1(x^*, y^*), \;\; H_2(x^*, y) \leq H_2(x^*, y^*) .$$

Summing these inequalities up yields

$$H_1(x, y^*) + H_2(x^*, y) \leq H_1(x^*, y^*) + H_2(x^*, y^*) \tag{7.2}$$

for arbitrary strategies x, y of the players. And the expression (7.1) is immediate.

The sufficiency part. Assume that there exists a pair (x^*, y^*) satisfying (7.1) and, hence, (7.2). By choosing $x = x^*$ and, subsequently, $y = y^*$ in inequality (7.2), we arrive at the conditions (1.1) that define a Nash equilibrium. The proof of Lemma 1.1 is finished.

Lemma 1.1 allows to use the conditions (7.1) or (7.2) instead of equilibrium verification in formula (1.1).

Theorem 1.1 *Consider a two-player game* $\Gamma =< I, II, X, Y, H_1, H_2 >$. *Let the sets of strategies* X, Y *be compact convex sets in the space* R^n, *and the payoffs* $H_1(x, y)$, $H_2(x, y)$ *represent continuous convex functions in x and y, respectively. Then the game possesses a Nash equilibrium.*

Proof: We apply the *ex contrario* principle. Suppose that no Nash equilibria actually exist. In this case, the above lemma requires that, for any pair of strategies (x, y), there is (x', y') violating the condition (7.2), i.e.,

$$H_1(x', y) + H_2(x, y') > H_1(x, y) + H_2(x, y).$$

Take the sets

$$S_{(x',y')} = \left\{(x, y) : H_1(x', y) + H_2(x, y') > H_1(x, y) + H_2(x, y)\right\} ,$$

representing open sets due to the continuity of the functions $H_1(x, y)$ and $H_2(x, y)$. The whole space of strategies $X \times Y$ is covered by the sets $S_{(x',y')}$, i.e., $\bigcup_{(x',y')\in X\times Y} S_{(x',y')} = X \times Y$. Owing to the compactness of $X \times Y$, one can separate out a finite subcovering

$$\bigcup_{i=1,\ldots,p} S_{(x_i,y_i)} = X \times Y.$$

For each $i = 1, \ldots , p$, denote

$$\varphi_i(x, y) = \left[H_1(x_i, y) + H_2(x, y_i) - (H_1(x, y) + H_2(x, y))\right]^+ , \tag{7.3}$$

where $a^+ = \max\{a, 0\}$. All functions $\varphi_i(x, y)$ enjoy non-negativity; moreover, at least, for a single $i = 1, \dots, p$ the function $\varphi_i(x, y)$ is positive according to the definition of $S_{(x_i, y_i)}$. Hence, it appears that $\sum_{i=1}^{p} \varphi_i(x, y) > 0,\ \forall (x, y)$.

Now, we define the mapping $\varphi(x, y) : X \times Y \to X \times Y$ by

$$\varphi(x, y) = \left(\sum_{i=1}^{p} \alpha_i(x, y) x_i, \sum_{i=1}^{p} \alpha_i(x, y) y_i \right),$$

where

$$\alpha_i(x, y) = \frac{\varphi_i(x, y)}{\sum_{i=1}^{p} \varphi_i(x, y)}, \quad i = 1, \dots, p, \quad \sum_{i=1}^{p} \alpha_i(x, y) = 1.$$

The functions $H_1(x, y),\ H_2(x, y)$ are continuous, whence it follows that the mapping $\varphi(x, y)$ turns out continuous. By the premise, X and Y form convex sets; consequently, the convex combinations $\sum_{i=1}^{p} \alpha_i x_i \in X$, $\sum_{i=1}^{p} \alpha_i y_i \in Y$. Thus, $\varphi(x, y)$ makes a self-mapping of the convex compact set $X \times Y$. The Brouwer fixed point theorem states that this mapping has a fixed point $(\bar{x}, \bar{y})$ such that $\varphi(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$, or

$$\bar{x} = \sum_{i=1}^{p} \alpha_i(\bar{x}, \bar{y}) x_i, \ \ \bar{y} = \sum_{i=1}^{p} \alpha_i(\bar{x}, \bar{y}) y_i.$$

Recall that the functions $H_1(x, y)$ and $H_2(x, y)$ are concave in x and y, respectively. And so, we naturally arrive at

$$\begin{aligned} H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}) &= H_1\left(\sum_{i=1}^{p} \alpha_i x_i, \bar{y} \right) + H_2\left(\bar{x}, \sum_{i=1}^{p} \alpha_i y_i \right) \\ &\geq \sum_{i=1}^{p} \alpha_i H_1(x_i, \bar{y}) + \sum_{i=1}^{p} \alpha_i H_2(\bar{x}, y_i). \end{aligned} \tag{7.4}$$

On the other hand, by the definition $\alpha_i(x, y)$ is positive simultaneously with $\varphi_i(x, y)$. For a positive function $\varphi_i(\bar{x}, \bar{y})$ (there exists at least one such function), one obtains (see (7.3))

$$H_1(x_i, \bar{y}) + H_2(\bar{x}, y_i) > H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}). \tag{7.5}$$

Indexes j corresponding to $\alpha_j(\bar{x}, \bar{y}) = 0$ satisfy the inequality

$$\alpha_j(\bar{x}, \bar{y}) \left(H_1(x_j, \bar{y}) + H_2(\bar{x}, y_j) \right) > \alpha_j(\bar{x}, \bar{y}) \left(H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}) \right). \tag{7.6}$$

Multiply the expression (7.5) by $\alpha_i(\bar{x}, \bar{y})$ and sum up with (7.6) over all indexes $i, j = 1, \dots, p$. These manipulations yield the inequality

$$\sum_{i=1}^{p} \alpha_i H_1(x_i, \bar{y}) + \sum_{i=1}^{p} \alpha_i H_2(\bar{x}, y_i) > H_1(\bar{x}, \bar{y}) + H_2(\bar{x}, \bar{y}),$$

which evidently contradicts (7.4). And the conclusion regarding the existence of a Nash equilibrium in convex games follows immediately. This concludes the proof of Theorem 1.1.

1.8 Some examples of bimatrix games

Consider a two-player game $\Gamma =< I, II, M, N, A, B >$, where players have finite sets of strategies, $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$, respectively. Their payoffs are defined by matrices A and B. In this game, player I chooses row i, whereas player II chooses column j; and their payoffs are accordingly specified by $a(i, j)$ and $b(i, j)$. Such games will be called **bimatrix games**. The following examples show that Nash equilibria may exist or not exist in such games.

Prisoners' dilemma. Two prisoners are arrested on suspicion of a crime. Each of them chooses between two actions, *viz.*, admitting the crime (strategy "Yes") and remaining silent (strategy "No"). The payoff matrices take the form

$$A = \begin{array}{c} \\ \text{Yes} \\ \text{No} \end{array} \begin{array}{c} \begin{array}{cc} \text{Yes} & \text{No} \end{array} \\ \begin{pmatrix} -6 & 0 \\ -10 & -1 \end{pmatrix} \end{array} \qquad B = \begin{array}{c} \\ \text{Yes} \\ \text{No} \end{array} \begin{array}{c} \begin{array}{cc} \text{Yes} & \text{No} \end{array} \\ \begin{pmatrix} -6 & -10 \\ 0 & -1 \end{pmatrix} \end{array}.$$

Therefore, if the prisoners admit the crime, they sustain a punishment of 6 years. When both remain silent, they sustain a small punishment of 1 year. However, admitting the crime seems very beneficial (if one prisoner admits the crime and the other does not, the former is set at liberty and the latter sustains a major punishment of 10 years). Clearly, a Nash equilibrium lies in the strategy profile (Yes, Yes), where players' payoffs constitute $(-6, -6)$. Indeed, by deviating from this strategy, a player gains -10. Prisoners' dilemma has become popular in game theory due to the following features. It models a Nash equilibrium leading to guaranteed payoffs (however, being appreciably worse than payoffs in the case of coordinated actions of the players).

Battle of sexes. This game involves two players, a "husband" and a "wife." They decide how to pass away a weekend. Each spouse chooses between two strategies, "boxing" and "theater." Depending on their choice, the payoffs are defined by the matrices

$$A = \begin{array}{c} \\ \text{Boxing} \\ \text{Theater} \end{array} \begin{array}{c} \begin{array}{cc} \text{Boxing} & \text{Theater} \end{array} \\ \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \qquad B = \begin{array}{c} \\ \text{Boxing} \\ \text{Theater} \end{array} \begin{array}{c} \begin{array}{cc} \text{Boxing} & \text{Theater} \end{array} \\ \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \end{array}.$$

In the previous game, we have obtained a single Nash equilibrium. Contrariwise, the battle of sexes admits two equilibria (actually, there exist three Nash equilibria—see the discussion below). The list of Nash equilibria includes the strategy profiles (Boxing, Boxing) and (Theater, Theater), but spouses have different payoffs. One gains 1, whereas the other gains 4.

The Hawk-Dove game. This game is often involved to model the behavior of different animals; it proceeds from the following assumption. While assimilating some resource V (e.g., a territory), each individual chooses between two strategies, namely, aggressive strategy (Hawk) or passive strategy (Dove). In their rivalry, Hawk always captures the whole of the resource from Dove. When two Doves meet, they share the resource equally. And finally, both individuals with aggressive strategy struggle for the resource. In this case, an individual obtains the resource with the identical probability of $1/2$, but both Hawks suffer from the losses of c. Let us present the corresponding payoff matrices:

$$A = \begin{array}{c} \\ \text{Hawk} \\ \text{Dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{pmatrix} \frac{1}{2}V - c & V \\ 0 & V/2 \end{pmatrix} \end{array} \quad B = \begin{array}{c} \\ \text{Hawk} \\ \text{Dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{Hawk} & \text{Dove} \end{array} \\ \begin{pmatrix} \frac{1}{2}V - c & 0 \\ V & V/2 \end{pmatrix} \end{array}.$$

Depending on the relationship between the available quantity of the resource and the losses, one obtains a game of the above types. If the losses c are smaller than $V/2$, prisoners' dilemma arises immediately (a single equilibrium, where the optimal strategy is Hawk for both players). At the same time, the condition $c \geq V/2$ corresponds to the battle of sexes (two equilibria, (Hawk, Dove) and (Dove, Hawk)).

The Stone-Scissors-Paper game. In this game, two players assimilate 1 USD by simultaneously announcing one of the following words: "Stone," "Scissors," and "Paper." The payoff is defined according to the rule: Stone breaks Scissors, Scissors cut Paper, and Paper wraps up Stone. And so, the players' payoffs are expressed by the matrices

$$A = \begin{array}{c} \\ \text{Stone} \\ \text{Scissors} \\ \text{Paper} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Stone} & \text{Scissors} & \text{Paper} \end{array} \\ \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \end{array} \quad B = \begin{array}{c} \\ \text{Stone} \\ \text{Scissors} \\ \text{Paper} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Stone} & \text{Scissors} & \text{Paper} \end{array} \\ \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}.$$

Unfortunately, the Stone-Scissors-Paper game admits no Nash equilibria among the considered strategies. It is impossible to suggest a strategy profile such that a player would not benefit by unilateral deviation from his strategy.

1.9 Randomization

In the preceding examples, we have observed the following fact. There may exist no equilibria in finite games. The "way out" concerns randomization. For instance, recall the Stone-Scissors-Paper game; obviously, one should announce a strategy randomly, and an opponent would not guess it. Let us extend the class of strategies and seek for a Nash equilibrium among probabilistic distributions defined on the sets $M = \{1, 2, \dots, m\}$ and $N = \{1, 2, \dots, n\}$.

Definition 1.5 *A mixed strategy of player I is a vector $x = (x_1, x_2, \dots, x_m)$, where $x_i \geq 0$, $i = 1, \dots, m$ and $\sum_{i=1}^{m} x_i = 1$. Similarly, introduce a mixed strategy of player II as $y = (y_1, y_2, \dots, y_n)$, where $y_j \geq 0$, $j = 1, \dots, n$ and $\sum_{i=1}^{n} y_j = 1$.*

Therefore, x_i (y_j) represents a probability that player I (II) chooses strategy i (j, respectively). In contrast to new strategies, we call $i \in M$, $j \in N$ by pure strategies. Note that pure strategy i corresponds to the mixed strategy $x = (0, \dots, 0, 1, 0, \dots.0)$, where 1 occupies position i (in the sequel, we simply write $x = i$ for compactness). Denote by X (Y) the set of mixed strategies of player I (player II, respectively). Those pure strategies adopted with a positive probability in a mixed strategy form the **support** or **spectrum** of the mixed strategy.

Now, any strategy profile (i,j) is realized with the probability x_iy_j. Hence, the expected payoffs of the players become

$$H_1(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a(i,j)x_iy_j,\quad H_2(x,y) = \sum_{i=1}^{m}\sum_{j=1}^{n} b(i,j)x_iy_j. \tag{9.1}$$

Thus, the extension of the original discrete game acquires the form $\Gamma =< I, II, X, Y, H_1, H_2 >$, where players' strategies are probabilistic distributions of x and y, and the payoff functions have the bilinear representation (9.1). Interestingly, strategies x and y make simplexes $X = \{x : \sum_{i=1}^{m} x_i = 1, x_i \geq 0,\ i = 1,\dots,m\}$ and $Y = \{y : \sum_{j=1}^{n} y_j = 1, y_j \geq 0, j = 1,\dots,n\}$ in the spaces R^m and R^n, respectively.

The sets X and Y form convex polyhedra in R^m and R^n, and the payoff functions $H_1(x,y)$, $H_2(x,y)$ are linear in each variable. And so, the resulting game $\Gamma =< I, II, X, Y, H_1, H_2 >$ belongs to the class of convex games, and Theorem 1.1 is applicable.

Theorem 1.2 *Bimatrix games admit a Nash equilibrium in the class of mixed strategies.*

The Nash theorem establishes the existence of a Nash equilibrium, but offers no algorithm to evaluate it. In a series of cases, one can benefit by the following assertion.

Theorem 1.3 *A strategy profile (x^*, y^*) represents a Nash equilibrium iff for any pure strategies $i \in M$ and $j \in N$:*

$$H_1(i, y^*) \leq H_1(x^*, y^*),\ H_2(x^*, j) \leq H_2(x^*, y^*). \tag{9.2}$$

Proof: The necessity part is immediate from the definition of a Nash equilibrium. Indeed, the conditions (1.1) hold true for arbitrary strategies x and y (including pure strategies).

The sufficiency of the conditions (9.2) can be shown as follows. Multiply the first inequality $H_1(i, y^*) \leq H_1(x^*, y^*)$ by x_i and perform summation over all $i = 1,\dots,m$. These operations yield the condition $H_1(x, y^*) \leq H_1(x^*, y^*)$ for an arbitrary strategy x. Analogous reasoning applies to the second inequality in (9.2). The proof of Theorem 1.3 is completed.

Theorem 1.4 (on complementary slackness) *Let (x^*, y^*) be a Nash equilibrium strategy profile in a bimatrix game. If for some i: $x_i^* > 0$, then the equality $H_1(i, y^*) = H_1(x^*, y^*)$ takes place. Similarly, if for some j: $y_j^* > 0$, we have $H_2(x^*, j) = H_2(x^*, y^*)$.*

Proof is by *ex contrario*. Suppose that for a certain index i' such that $x_{i'}^* > 0$ one obtains $H_1(i', y^*) < H_1(x^*, y^*)$. Theorem 1.3 implies that the inequality $H_1(i, y^*) \leq H_1(x^*, y^*)$ is valid for the rest indexes $i \neq i'$. Therefore, we arrive at the system of inequalities

$$H_1(i, y^*) \leq H_1(x^*, y^*),\ i = 1,\dots,n, \tag{9.2$'$}$$

where inequality i' turns out strict. Multiply (9.2$'$) by x_i^* and perform summation to get the contradiction $H(x^*, y^*) < H(x^*, y^*)$. By analogy, one easily proves the second part of the theorem.

Theorem 1.4 claims that a Nash equilibrium involves only those pure strategies leading to the optimal payoff of a player. Such strategies are called **equalizing**.

Theorem 1.5 *A strategy profile (x^*, y^*) represents a mixed strategy Nash equilibrium profile iff there exist pure strategy subsets $M_0 \subseteq M$, $N_0 \subseteq N$ and values H_1, H_2 such that*

$$\sum_{j\in N_0} H_1(i,j)y_j^* \left\{ \begin{matrix} \equiv \\ \leq \end{matrix} \right\} H_1, \quad for \quad \left\{ \begin{matrix} i \in M_0 \\ i \notin M_0 \end{matrix} \right\} \tag{9.3}$$

$$\sum_{i\in M_0} H_2(i,j)x_i^* \left\{ \begin{matrix} \equiv \\ \leq \end{matrix} \right\} H_2, \quad for \quad \left\{ \begin{matrix} j \in N_0 \\ i \notin N_0 \end{matrix} \right\} \tag{9.4}$$

and

$$\sum_{i\in M_0} x_i^* = 1, \ \sum_{j\in N_0} y_j^* = 1. \tag{9.5}$$

Proof (the necessity part). Assume that (x^*, y^*) is an equilibrium in a bimatrix game. Set $H_1 = H_1(x^*, y^*)$, $H_2 = H_2(x^*, y^*)$ and $M_0 = \{i \in M : x_i^* > 0\}$, $N_0 = \{j \in N : y_j^* > 0\}$. Then the conditions (9.3)–(9.5) directly follow from Theorems 1.3 and 1.4.

The sufficiency part. Suppose that the conditions (9.3)–(9.5) hold true for a certain strategy profile (x^*, y^*). Formula (9.5) implies that (a) $x_i^* = 0$ for $i \notin M_0$ and (b) $y_j^* = 0$ for $j \notin N_0$. Multiply (9.3) by x_i^* and (9.4) by y_j^*, as well as perform summation over all $i \in M$ and $j \in N$, respectively. Such operations bring us to the equalities

$$H_1(x^*, y^*) = H_1, \ H_2(x^*, y^*) = H_2.$$

This result and Theorem 1.3 show that (x^*, y^*) is an equilibrium. The proof of Theorem 1.5 is concluded.

Theorem 1.5 can be used to evaluate Nash equilibria in bimatrix games. Imagine that we know the optimal strategy spectra M_0, N_0. It is possible to employ equalities from the conditions (9.3)–(9.5) and find the optimal mixed strategies x^*, y^* and the optimal payoffs H_1^*, H_2^* from the system of linear equations. However, this system can generate negative solutions (which contradicts the concept of mixed strategies). Then one should modify the spectra and go over them until an equilibrium appears. Theorem 1.5 demonstrates high efficiency, if all $x_i, i \in M$ and $y_j, j \in N$ have positive values in an equilibrium.

Definition 1.6 *An equilibrium strategy profile (x^*, y^*) is called completely mixed, if $x_i > 0$, $i \in M$ and $y_j > 0, j \in N$.*

Suppose that a bimatrix game admits a completely mixed equilibrium strategy profile (x, y). According to Theorem 1.5, it satisfies the system of linear equations

$$\begin{aligned} &\sum_{j\in N} H_1(i,j)y_j^* = H_1, i \in M \\ &\sum_{i\in M} H_2(i,j)x_i^* = H_2, \ j \in N \\ &\sum_{i\in M} x_i^* = 1, \ \sum_{j\in N} y_j^* = 1. \end{aligned} \tag{9.6}$$

Actually, the system (9.6) comprises $n + m + 2$ equations with $n + m + 2$ unknowns. Its solution gives a Nash equilibrium in a bimatrix game and the values of optimal payoffs.

1.10 Games 2 × 2

A series of bimatrix games can be treated via geometric considerations. The simplest case covers players choosing between two strategies. The mixed strategies of players *I* and *II* take the form $(x, 1-x)$ and $(y, 1-y)$, respectively. And their payoffs are defined by the matrices

$$A = \begin{array}{c c} & \begin{array}{cc} y & 1-y \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \end{array} \quad B = \begin{array}{c c} & \begin{array}{cc} y & 1-y \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \end{array}.$$

The mixed strategy payoffs of the players become

$$\begin{aligned} H_1(x,y) &= a_{11}xy + a_{12}x(1-y) + a_{21}(1-x)y + a_{22}(1-x)(1-y) \\ &= Axy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}, \end{aligned}$$

$$\begin{aligned} H_2(x,y) &= b_{11}xy + b_{12}x(1-y) + b_{21}(1-x)y + b_{22}(1-x)(1-y) \\ &= Bxy + (b_{12} - b_{22})x + (b_{21} - b_{22})y + b_{22}, \end{aligned}$$

where $A = a_{11} - a_{12} - a_{21} + a_{22},\ B = b_{11} - b_{12} - b_{21} + b_{22}$.

By virtue of Theorem 1.3, the equilibrium (x, y) follows from inequalities (9.2), i.e.,

$$H_1(0,y) \le H_1(x,y),\ \ H_1(1,y) \le H_1(x,y), \tag{10.1}$$

$$H_2(x,0) \le H_2(x,y),\ \ H_2(x,1) \le H_1(x,y). \tag{10.2}$$

Rewrite inequalities (10.1) as

$$\begin{aligned} (a_{21} - a_{22})y + a_{22} &\le Axy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}, \\ Ay + (a_{21} - a_{22})y + a_{12} &\le Axy + (a_{12} - a_{22})x + (a_{21} - a_{22})y + a_{22}, \end{aligned}$$

and, consequently,

$$(a_{22} - a_{12})x \le Axy, \tag{10.3}$$

$$Ay(1-x) \le (a_{22} - a_{12})(1-x). \tag{10.4}$$

Now, take the unit square $0 \le x \le 1, 0 \le y \le 1$ (see Figure 1.5) and draw the set of points (x, y) meeting the conditions (10.3)–(10.4).

If $x = 0$, then (10.3) is immediate, whereas the condition (10.4) implies the inequality $Ay \le a_{22} - a_{12}$. In the case of $x = 1$, the expression (10.4) is valid, and (10.3) leads to $Ay \ge a_{22} - a_{12}$. And finally, under $0 \le x \le 1$, the conditions (10.3)–(10.4) bring to $Ay = a_{22} - a_{12}$.

Similar analysis of inequalities (10.2) yields the following. If $y = 0$, then $Bx \le b_{22} - b_{21}$. In the case of $y = 1$, we have $Bx \ge b_{22} - b_{21}$. If $0 \le y \le 1$, then $Bx = b_{22} - b_{21}$.

Depending on the signs of A and B, these conditions result in different sets of feasible equilibria in a bimatrix game (zigzags inside the unit square).

Prisoners' dilemma. Recall the example studied above; here $A = B = -6 + 10 + 0 - 1 = 3$ and $a_{22} - a_{12} = b_{22} - b_{21} = -1$. Hence, an equilibrium represents the intersection of two lines $x = 1$ and $y = 1$ (see Figure 1.6). Therefore, the equilibrium is unique and takes the form $x = 1,\ y = 1$.

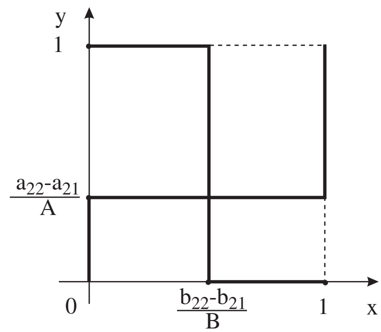


Figure 1.5 A zigzag in a bimatrix game.

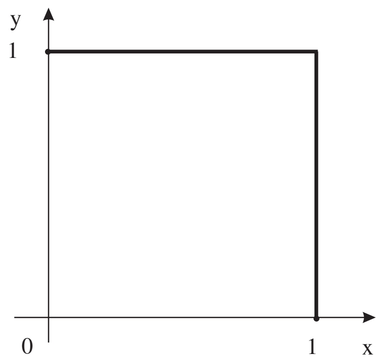


Figure 1.6 A unique equilibrium in the prisoners' dilemma game.

Battle of sexes. In this example, one obtains $A = B = 4 - 0 - 0 + 1 = 5$ and $a_{22} - a_{12} = 1$, $b_{22} - b_{21} = 4$. And so, the zigzag defining all equilibrium strategy profiles is shown in Figure 1.7. Obviously, the game under consideration includes three equilibria. Among them, two equilibria correspond to pure strategies ($x = 0$, $y = 1$), ($x = 1$, $y = 0$), while the third one has the mixed strategy type ($x = 4/5$, $y = 1/5$). The payoffs in these equilibria make up ($H_1^* = 4, H_2^* = 1$), ($H_1^* = 1, H_2^* = 4$) and ($H_1^* = 4/5, H_2^* = 4/5$), respectively.

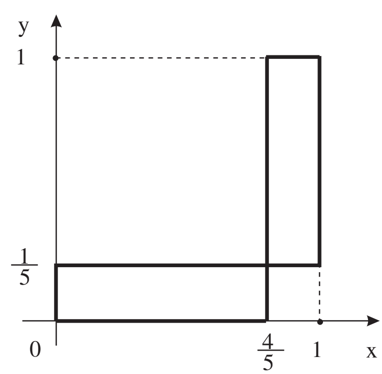


Figure 1.7 Three equilibria in the battle of sexes game.

The stated examples illustrate the following aspect. Depending on the shape of zigzags, bimatrix games may admit one, two, or three equilibria, or even the continuum of equilibria.

1.11 Games $2 \times n$ and $m \times 2$

Suppose that player I chooses between two strategies, whereas player II has n strategies available. Consequently, their payoffs are defined by the matrices

$$A = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & \dots & y_n \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \end{pmatrix} \end{array} \quad B = \begin{array}{c c} & \begin{array}{cccc} y_1 & y_2 & \dots & y_n \end{array} \\ \begin{array}{c} x \\ 1-x \end{array} & \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \end{pmatrix} \end{array}.$$

In addition, assume that player I uses the mixed strategy $(x, 1-x)$. If player II chooses strategy j, his payoff equals $H_2(x,j) = b_{1,j}x + b_{2j}(1-x)$, $j = 1, \dots, n$.

We show these payoffs (linear functions) in Figure 1.8. According to Theorem 1.3, the equilibrium (x, y) corresponds to $\max_j H_2(x,j) = H_2(x,y)$. For any x, construct the maximal envelope $l(x) = \max_j H_2(x,j)$. As a matter of fact, $l(x)$ represents a jogged line composed of at most $n+1$ segments. Denote by $x_0 = 0, x_1, \dots, x_k = 1,\ k \le n+1$ the salient points of this envelope. Since the function $H_1(x, y)$ is linear in x, its maximum under a fixed strategy of player II is attained at the points x_i, $i = 0, \dots, k$. Hence, equilibria can be focused only in these points. Imagine that the point x_i results from intersection of the straight lines $H_2(x, j_1)$ and $H_2(x, j_2)$. This means that player II optimally plays the mix of his strategies j_1 and j_2 in response to the strategy x by player I. Thus, we obtain a game 2×2 with the payoff matrices

$$A = \begin{pmatrix} a_{1j_1} & a_{1j_2} \\ a_{2j_1} & a_{2j_2} \end{pmatrix} \quad B = \begin{pmatrix} b_{1j_1} & b_{1j_2} \\ b_{2j_1} & b_{2j_2} \end{pmatrix}.$$

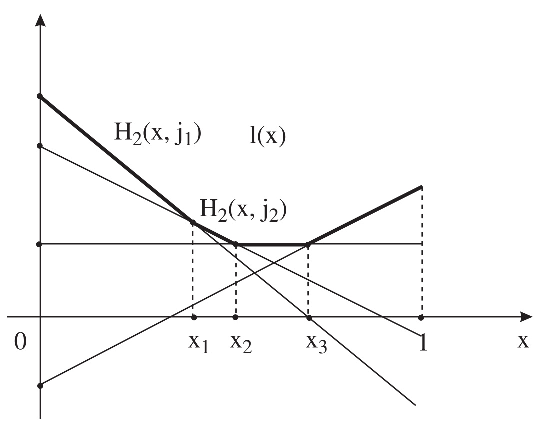


Figure 1.8 The maximal envelope l(x).

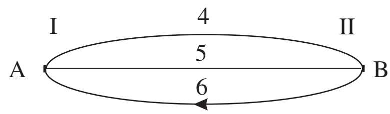


Figure 1.9 The road selection game.

It has been solved in the previous section. To verify the optimality of the strategy x_i, one can adhere to the following reasoning. The strategies x_i and the mix $(y, 1-y)$ of the strategies j_1 and j_2 form an equilibrium, if there exists $y, 0 \le y \le 1$ such that $H_1(1, y) = H_1(2, y)$. In this case, the payoff of player I is independent from x, and the best response of player II to the strategy x_i lies in mixing the strategies j_1 and j_2. Rewrite the last condition as

$$a_{1j_1}y + a_{1j_2}(1-y) = a_{2j_1}y + a_{2j_2}(1-y). \qquad (11.1)$$

Let us consider this procedure using an example.

Road selection. Points A and B communicate through three roads. One of them is one-way road right-to-left (see Figure 1.9).

A car (player I) leaves A, and another car (player II) moves from B. The journey-time on these roads varies (4, 5, and 6 hours, respectively, for a single car on a road). If both players choose the same road, the journey-time doubles. Each player has to select a road for the journey.

And so, player I (player II) chooses between two strategies (among three strategies, respectively). The payoff matrices take the form

$$A = \begin{array}{c} x \\ 1-x \end{array} \begin{pmatrix} -8 & -4 & -4 \\ -5 & -10 & -5 \end{pmatrix} \quad B = \begin{array}{c} x \\ 1-x \end{array} \begin{pmatrix} -8 & -5 & -6 \\ -4 & -10 & -6 \end{pmatrix}.$$

Find the payoffs of player II: $H_2(x, 1) = -4x - 5$, $H_2(x, 2) = 5x - 10$, $H_2(x, 3) = -6$. Draw these functions on Figure 1.10 and the maximal envelope $l(x)$ (see the thick line).

The salient points of $l(x)$ are located at $x = 0, x = 0.5, x = 0.8$, and $x = 1$. The corresponding equilibria form $(x = 0, y = (1, 0, 0))$, $(x = 1, y = (0, 1, 0))$. The point $x = 1/2$ answers

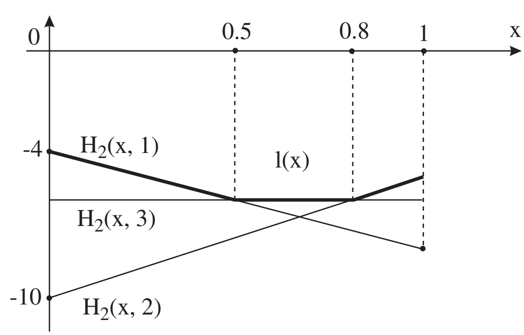


Figure 1.10 The maximal envelope in the road selection game.

for intersection of the functions $H_2(x, 1)$ and $H_2(x, 3)$. The condition (11.1) implies that $y = (1/4, 0, 3/4)$. However, this condition fails at the point $x = 0.8$. Therefore, the game in question admits three equilibrium solutions:

1. car *I* moves on the first road, and car *II* chooses the second road,
2. car *I* moves on the second road, and car *II* chooses the first road,
3. car *I* moves on the first or second road with identical probabilities, and car *II* chooses the first road with the probability of $1/4$ or the third road with the probability of $3/4$.

Interestingly, in the third equilibrium, the mean journey time of player *I* constitutes 5 h, whereas player *II* spends 6 h. It would seem that player *II* "has the cards" (owing to the additional option of using the third road). For instance, suppose that the third road is closed. In the worst case, player *II* requires just 5 h for the journey. This contradicting result is known as **the Braess paradox**. We will discuss it later. In fact, if player *I* informs the opponent that he chooses the road by coin-tossing, player *II* has nothing to do but to follow the third strategy above.

1.12 The Hotelling duopoly in 2D space with non-uniform distribution of buyers

Let us revert to the problem studied in Section 1.5. Consider the Hotelling duopoly in 2D space with non-uniform distribution of buyers in a city (according to some density function $f(x, y)$). As previously, we believe that a city represents a circle S having the radius of 1 (see Figure 1.11). It contains two companies (players *I* and *II*) located at different points P_1 and P_2. The players strive for defining optimal prices for their products depending on their location in the city.

Again, players *I* and *II* quote prices for their products (some quantities c_1 and c_2, respectively). A buyer located at a point $P \in S$ compares his costs (for the sake of simplicity, here we define them by $F(c_i, \rho(P, P_i)) = c_i + \rho^2$) and chooses the company with the minimal value.

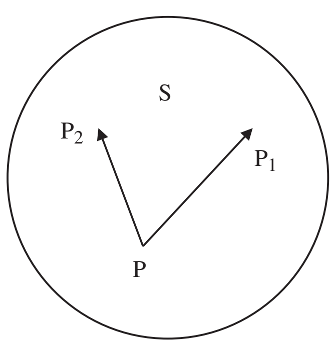


Figure 1.11 The Hotelling duopoly.

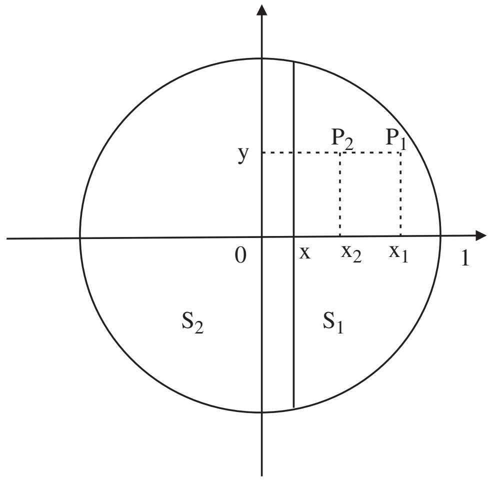


Figure 1.12 P_1, P_2 have the same ordinate y.

Therefore, all buyers in S get decomposed into two subsets (S_1 and S_2) according to their priorities of companies I and II. Then the payoffs of players I and II are specified by

$$H_1(c_1, c_2) = c_1\mu(S_1), \quad H_2(c_1, c_2) = c_2\mu(S_2), \tag{12.1}$$

where $\mu(S) = \int\limits_S f(x, y)dxdy$ denotes the probabilistic measure of the set S. First, we endeavor to evaluate equilibrium prices under the uniform distribution of buyers.

Rotate the circle S such that the points P_1 and P_2 have the same ordinate y (see Figure 1.12). Designate the abscissas of P_1 and P_2 by x_1 and x_2, respectively. Without loss of generality, assume that $x_1 \geq x_2$.

Within the framework of the Hotelling scheme, the sets S_1 and S_2 form sectors of the circle divided by the straight line

$$c_1 + (x - x_1)^2 = c_2 + (x - x_2)^2,$$

which is parallel to the axis O_y with the coordinate

$$x = \frac{1}{2}(x_1 + x_2) + \frac{c_1 - c_2}{2(x_1 - x_2)}. \tag{12.2}$$

According to (12.1), the payoffs of the players in this game constitute

$$H_1(c_1, c_2) = c_1\left(\arccos x - x\sqrt{1 - x^2}\right)/\pi, \tag{12.3}$$

$$H_2(c_1, c_2) = c_2\left(\pi - \arccos x + x\sqrt{1 - x^2}\right)/\pi, \tag{12.4}$$

where x meets (12.2). We find the equilibrium prices via the equation $\frac{\partial H_1}{\partial c_1} = \frac{\partial H_2}{\partial c_2} = 0$.

Evaluate the derivative of (12.3) with respect to c_1:

$$\pi \frac{\partial H_1}{\partial c_1} = \arccos x - x\sqrt{1-x^2} + c_1 \left[-\frac{1}{\sqrt{1-x^2}} \frac{1}{2(x_1-x_2)} - \sqrt{1-x^2} \frac{1}{2(x_1-x_2)} \right.$$
$$\left. + \frac{2x^2}{2\sqrt{1-x^2}} \frac{1}{2(x_1-x_2)} \right].$$

Using the first-order necessary optimality conditions, we get

$$c_1 = (x_1 - x_2)\left[\frac{\arccos x}{\sqrt{1-x^2}} - x\right]. \tag{12.5}$$

Similarly, the condition $\frac{\partial H_2}{\partial c_2} = 0$ brings to

$$c_2 = (x_1 - x_2)\left[x + \frac{\pi - \arccos x}{\sqrt{1-x^2}}\right]. \tag{12.6}$$

Finally, the expressions (12.2), (12.5), and (12.6) imply that the equilibrium prices can be rewritten as

$$c_1 = \frac{x_1 - x_2}{2}\left[\frac{\pi}{\sqrt{1-x^2}} - 2\left(\frac{x_1+x_2}{2} - x\right)\right], \tag{12.7}$$

$$c_2 = \frac{x_1 - x_2}{2}\left[\frac{\pi}{\sqrt{1-x^2}} + 2\left(\frac{x_1+x_2}{2} - x\right)\right], \tag{12.8}$$

where

$$x = \frac{x_1+x_2}{4} - \frac{\pi/2 - \arccos x}{2\sqrt{1-x^2}}. \tag{12.9}$$

Remark 1.1 If $x_1 + x_2 = 0$, then $x = 0$ due to (12.2). Hence, $c_1 = c_2 = \pi x_1$ according to (12.5)–(12.6), and $H_1 = H_2 = \pi x_1/2$ according to (12.3)–(12.4). The maximal equilibrium prices are achieved under $x_1 = 1$ and $x_2 = -1$; they make up $c_1 = c_2 = \pi$. The optimal payoffs take the values of $H_1 = H_2 = \pi/2 \approx 1.570$. Thus, if buyers possess the uniform distribution in the circle, the companies should be located as far as possible from each other (in the optimal solution).

To proceed, suppose that buyers are distributed non-uniformly in the circle. Analyze the case when the density function in the polar coordinates acquires the form (see Figure 1.13)

$$f(r, \theta) = 3(1-r)/\pi, \quad 0 \le r \le 1, 0 \le \theta \le 2\pi. \tag{12.10}$$

Obviously, buyers lie closer to the city center.

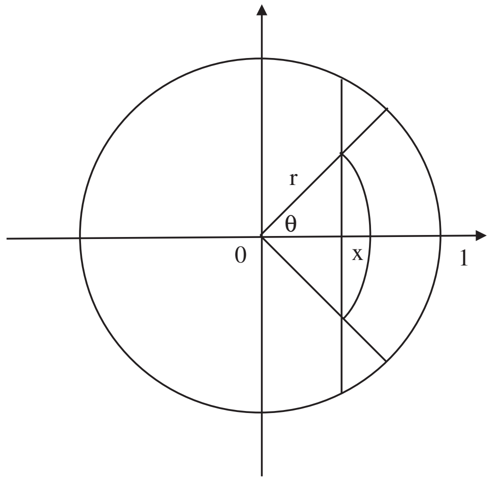


Figure 1.13 Duopoly in the polar coordinates.

Note that it suffices to consider the situation of $x_1 + x_2 \geq 0$ (otherwise, simply reverse the signs of x_1, x_2). The expected incomes of the players (12.1) are given by

$$H_1(c_1, c_2) = \frac{6}{\pi} c_1 A(x), \quad H_2(c_1, c_2) = c_2 \left(1 - \frac{6}{\pi} A(x)\right), \tag{12.11}$$

where

$$A(x) = \int_x^1 r(1-r) \arccos\left(\frac{x}{r}\right) dr = \frac{1}{6}\left[\arccos x - x\sqrt{1-x^2} - 2x \int_x^1 \sqrt{r^2 - x^2} dr\right]$$
$$= \frac{1}{6}\left[\arccos x - 2x\sqrt{1-x^2} - x^3 \log x + x^3 \log\left(1 + \sqrt{1-x^2}\right)\right],$$

such that

$$\frac{\pi}{6}\frac{\partial H_1}{\partial c_1} = A(x) + c_1 A'(x) \frac{\partial x}{\partial c_1} = A(x) - \frac{c_1}{2(x_1 - x_2)} \int_x^1 \sqrt{r^2 - x^2} dr,$$

$$\frac{\partial H_2}{\partial c_2} = 1 - \frac{6}{\pi} A(x) - c_2 \frac{6}{\pi} A'(x) \frac{\partial x}{\partial c_2} = 1 - \frac{6}{\pi} A(x) - \frac{6}{\pi}\frac{c_2}{2(x_1 - x_2)} \int_x^1 \sqrt{r^2 - x^2} dr,$$

since

$$A'(x) = -\int_x^1 \frac{r(1-r)}{\sqrt{r^2 - x^2}} dr = -\int_x^1 \sqrt{r^2 - x^2} dr. \tag{12.12}$$

The conditions $\frac{\partial H_1}{\partial c_1} = \frac{\partial H_2}{\partial c_2} = 0$ yield that

$$c_1 = 2(x_1 - x_2)A(x) / \int_x^1 \sqrt{r^2 - x^2} dr, \tag{12.13}$$

$$c_2 = 2(x_1 - x_2)\left(\frac{\pi}{6} - A(x)\right) / \int_x^1 \sqrt{r^2 - x^2} dr. \tag{12.14}$$

By substituting c_1 and c_2 into

$$x = \frac{1}{2}(x_1 + x_2) + \frac{c_1 - c_2}{2(x_1 - x_2)}, \tag{12.2$'$}$$

we arrive at

$$x - \frac{1}{2}(x_1 + x_2) = (2A(x) - \pi/6) / \int_x^1 \sqrt{r^2 - x^2} dr. \tag{12.15}$$

Remark 1.2 It follows from (12.12) that $A(x)$ represents a convex decreasing function such that $A(0) = \pi/12$ and $A(1) = 0$. The right-hand side of (12.15) is negative, which leads to

$$x \le (x_1 + x_2)/2.$$

Below we demonstrate that equation (12.15) admits a unique solution. Rewrite it as

$$B(x) = -\left[x - \frac{1}{2}(x_1 + x_2)\right] A'(x) - (2A(x) - \pi/6) = 0. \tag{12.16}$$

The derivative of the function $B(x)$, i.e.,

$$B'(x) = -3A'(x) - A''(x)\left(x - \frac{x_1 + x_2}{2}\right) = \int_x^1 \left[3\sqrt{r^2 - x^2} + \frac{x}{\sqrt{r^2 - x^2}}\left(\frac{x_1 + x_2}{2} - x\right)\right] dr,$$

possesses positive values exclusively. Hence, $B(x)$ increases within the interval $[0, \frac{x_1+x_2}{2}]$ such that $B(0) = -\frac{x_1+x_2}{4} < 0$ and $B(\frac{x_1+x_2}{2}) = \pi/6 - 2A(\frac{x_1+x_2}{2}) \ge 0$.

If $x_1 + x_2 = 0$, then $x = 0$ satisfies equation (12.15). Moreover, the conditions (12.13)–(12.14) lead to $c_1 = c_2 = \frac{2}{3}\pi x_1$, whereas formula (12.11) implies that $H_1 = H_2 = \frac{1}{3}\pi x_1$. Under $x_1 = 1, x_2 = -1$, we have $c_1 = c_2 = \frac{2}{3}\pi \approx 2.094$ and $H_1 = H_2 = \frac{1}{3}\pi \approx 1.047$.

1.13 Location problem in 2D space

In the preceding subsection, readers have observed the following fact. If the location points P_1 and P_2 are fixed, there exist equilibrium prices c_1 and c_2. In other words, c_1 and c_2 make some functions of x_1, x_2. In this context, an interesting question arises immediately. Are there equilibrium points x_1^*, x_2^* of location for these companies? Such problem often appears during infrastructure planning for regional socioeconomic systems. Consider the posed problem in the case of non-uniform distribution of companies.

Suppose that player *II* chooses some point $x_2 < 0$. Player *I* aims at finding a certain point x_1 which maximizes his income $H_1(c_1, c_2)$. Let us solve the equation $\frac{\partial H_1}{\partial x_1} = 0$. By virtue of (12.11),

$$\frac{\pi}{6}\frac{\partial H_1}{\partial x_1} = \frac{\partial c_1}{\partial x_1}A(x) + c_1 A'(x)\frac{\partial x}{\partial x_1} = 0. \tag{13.1}$$

Differentiation of (12.13) and (12.16) with respect to x_1 gives

$$\frac{1}{2}\frac{\partial c_1}{\partial x_1} = -\frac{A(x)}{A'(x)} - (x_1 - x_2)\left[1 - \frac{A'(x)A''(x)}{[A'(x)]^2}\right]\frac{\partial x}{\partial x_1} \tag{13.2}$$

and

$$-\left(\frac{\partial x}{\partial x_1} - \frac{1}{2}\right)A'(x) - \left[x - \frac{1}{2}(x_1 + x_2)\right]A''(x)\frac{\partial x}{\partial x_1} - 2A'(x)\frac{\partial x}{\partial x_1} = 0\,.$$

Therefore,

$$\frac{\partial x}{\partial x_1} = A'(x)\left[6A'(x) + 2\left(x - \frac{x_1 + x_2}{2}\right)A''(x)\right]^{-1}. \tag{13.3}$$

Equations (13.1)–(13.3) can serve for obtaining the optimal response x_1 of player *I*.

Owing to the symmetry of this problem, if an equilibrium exists, it has the form $(x_1, x_2 = -x_1)$. In this case, $x = 0$, $A(0) = \pi/12$, $A'(0) = -1/2$, $A''(0) = 0$. The expression (13.3) brings to

$$\frac{\partial x}{\partial x_1} = (-1/2)/(-3 + 0) = 1/6.$$

On the other hand, formula (13.2) yields

$$\frac{\partial c_1}{\partial x_1} = \frac{\pi}{3} - \frac{2}{3}x_1.$$

Substitute these results into (13.1) to derive

$$\left(\frac{\pi}{3} - \frac{2}{3}x_1\right)\frac{\pi}{12} + \left(\frac{2}{3}\pi x_1\right)\cdot\left(-\frac{1}{2}\right)\cdot\frac{1}{6} = 0,$$

and, finally,

$$x_1^* = \frac{\pi}{4}.$$

Thus, the optimal location points of the companies become $x_1^* = \pi/4, x_2^* = -\pi/4$; the corresponding equilibrium prices and incomes constitute $c_1 = c_2 = \pi^2/6$ and $H_1 = H_2 = \pi^2/12$, respectively.

Remark 1.3 Recall the case of the uniform distribution of buyers discussed earlier. Similar argumentation generates the following outcomes.

It appears from (12.3), (12.7), and (12.9) that

$$\pi \frac{\partial H_1}{\partial x_1} = \frac{\partial c_1}{\partial x_1}\left(\arccos x - x\sqrt{1-x^2}\right) - 2c_1\sqrt{1-x^2}\frac{\partial x}{\partial x_1},$$
$$\frac{\partial c_1}{\partial x_1} = \frac{\pi}{2\sqrt{1-x^2}} + x - x_1 + \frac{x_1 - x_2}{2}\left(2 + \pi x(1-x^2)^{-3/2}\right)\frac{\partial x}{\partial x_1},$$
$$\frac{\partial x}{\partial x_1} = \frac{1}{4}\left[1 + \frac{1}{2(1-x^2)} + \frac{x}{2(1-x^2)^{3/2}}\left(\frac{\pi}{2} - \arccos x\right)\right]^{-1}.$$

Consequently,

$$\pi\left[\frac{\partial H_1}{\partial x_1}\right]_{x=0} = \left[\frac{\partial c_1}{\partial x_1}\right]_{x=0}\frac{\pi}{2} - 2[c_1]_{x=0}\left[\frac{\partial x}{\partial x_1}\right]_{x=0}$$
$$= \left(\frac{\pi}{2} - \frac{2x_1}{3}\right)\frac{\pi}{2} - 2\pi x_1\frac{1}{6} = \frac{\pi}{4}\left(\pi - \frac{8}{3}x_1\right) > 0, \quad \forall x_1 \in (0,1).$$

And so, the maximal incomes are attained at the points $x_1^* = -x_2^* = 1$. According to (12.3) and (12.7), these points correspond to

$$c_i^* = \pi \approx 3.1415 \text{ and } H_i^* = \pi/2 \approx 1.5708, \quad i = 1, 2. \tag{13.4}$$

Exercises

1. The crossroad problem.

   Two automobilists move along two mutually perpendicular roads and simultaneously meet at a crossroad. Each of them may stop (strategy I) or continue motion (strategy II).

   By assumption, a player would rather stop than suffer a catastrophe; on the other hand, a player would rather continue motion if the opponent stops. This conflict can be represented by a bimatrix game with the payoff matrix

$$\begin{pmatrix} (1,1) & (1-\varepsilon,2) \\ (2,1-\varepsilon) & (0,0) \end{pmatrix}.$$

   Here $\varepsilon \geq 0$ is a number characterizing player's displeasure of his stop to let the opponent pass.

   Find pure strategy Nash equilibria and mixed strategy Nash equilibria in the crossroad problem.

2. Games 2×2. Evaluate Nash equilibria in the bimatrix games below:

$$A = \begin{pmatrix} -6 & 0 \\ -9 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -6 & -9 \\ 0 & -1 \end{pmatrix}.$$

$$A = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}.$$

3. Find a Nash equilibrium in a bimatrix game defined by

$$A = \begin{pmatrix} 3 & 6 & 8 \\ 4 & 3 & 2 \\ 7 & -5 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 7 & 4 & 3 \\ 7 & 7 & 3 \\ 4 & 6 & 6 \end{pmatrix}.$$

4. Evaluate a Stackelberg equilibrium in a two-player game with the payoff functions

$$H_1(x_1, x_2) = bx_1(c - x_1 - x_2) - d,$$
$$H_2(x_1, x_2) = bx_2(c - x_1 - x_2) - d.$$

5. Consider a general bimatrix game and demonstrate that (x, y) is a mixed strategy equilibrium profile iff the following inequalities hold true:

$$(x-1)(ay-\alpha) \geq 0, \quad x(ay-\alpha) \geq 0,$$
$$(y-1)(bx-\beta) \geq 0, \quad y(bx-\beta) \geq 0,$$

where

$$a = a_{11} - a_{12} - a_{21} + a_{22}, \quad \alpha = a_{22} - a_{12},$$
$$b = b_{11} - b_{12} - b_{21} + b_{22}, \quad \beta = a_{22} - a_{21}.$$

6. Prove the following result. If a bimatrix game admits a completely mixed Nash equilibrium strategy profile, then $n = m$.

7. Find an equilibrium in a game $2 \times n$ described by the payoff matrices

$$A = \begin{pmatrix} 2 & 0 & 5 \\ 2 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 7 & 8 \end{pmatrix}.$$

8. Find an equilibrium in a game $m \times 2$ described by the payoff matrices

$$A = \begin{pmatrix} 8 & 2 \\ 2 & 7 \\ 3 & 9 \\ 6 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 4 \\ 8 & 4 \\ 7 & 2 \\ 2 & 9 \end{pmatrix}.$$

9. Consider the company allocation problem in 2D space. Evaluate equilibrium prices (p_1, p_2) under the cost function $F_2 = p^2 + \rho^2$.

10. Consider the company allocation problem in 2D space. Find the optimal allocation of companies (x_1, x_2) under the cost function $F_2 = p^2 + \rho^2$.

2

Zero-sum games

Introduction

The previous chapter has been focused on games $\Gamma =< I, II, X, Y, H_1, H_2 >$, where players' payoffs $H_1(x, y)$ and $H_2(x, y)$ represent arbitrary functions defined on the set product $X \times Y$. However, there exists a special case of normal-form games when $H_1(x, y) + H_2(x, y) = 0$ for all (x, y). Such games are called **zero-sum games** or **antagonistic games**. Here players have opposite goals—the payoff of a player equals the loss of the opponent. It suffices to specify the payoff function of player *II* for complete description of a game.

Definition 2.1 *A zero-sum game is a normal-form game* $\Gamma =< I, II, X, Y, H >$*, where* X, Y *indicate the strategy sets of players I and II, and* $H(x, y)$ *means the payoff function of player I,* $H : X \times Y \to R$*.*

Each player chooses his strategy regardless of the opponent. Player *I* strives for maximizing the payoff $H(x, y)$, whereas player *II* seeks to minimize this function. Zero-sum games satisfy all properties established for normal-form games. Nevertheless, the former class of games enjoys a series of specific features. First, let us reformulate the notion of a Nash equilibrium.

Definition 2.2 *A Nash equilibrium in a game* Γ *is a set of strategies* (x^*, y^*) *meeting the conditions*

$$H(x, y^*) \leq H(x^*, y^*) \leq H(x^*, y) \tag{1.1}$$

for arbitrary strategies x, y *of the players.*

Inequalities (1.1) imply that, as player *I* deviates from a Nash equilibrium, his payoff goes down. If player *II* deviates from the equilibrium, his opponent gains more (accordingly, player *II* loses more). Hence, none of the players benefit by deviating from a Nash equilibrium.

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

Remark 2.1 Constant-sum games ($H_1(x, y) + H_2(x, y) = c = const$ for arbitrary strategies x, y) can be reduced to zero-sum games. Notably, find a solution to a zero-sum game with the payoff function $H_1(x, y)$. Then any Nash equilibrium (x^*, y^*) in this game also acts as a Nash equilibrium in the corresponding constant-sum game. Indeed, according to Definition (1.1), for any x, y we have

$$H_1(x, y^*) \le H_1(x^*, y^*) \le H_1(x^*, y).$$

At the same time, $H_1(x, y) = c - H_2(x, y)$, and the second inequality can be rewritten as

$$c - H_2(x^*, y^*) \le c - H_2(x^*, y),$$

or

$$H_2(x^*, y) \le H_2(x^*, y^*), \forall y.$$

Thus, (x^*, y^*) is also a Nash equilibrium in the zero-sum game.

By analogy to the general class of normal-form games, zero-sum games may admit no Nash equilibria. A major role in zero-sum games analysis belongs to the concepts of minimax and maximin.

2.1 Minimax and maximin

Suppose that player I employs some strategy x. In the worst case, he has the payoff $\inf_y H(x, y)$. Naturally, he would endeavor to maximize this quantity. In the worst case, the guaranteed payoff of player I makes up $\sup_x \inf_y H(x, y)$. Similarly, player II can guarantee the maximum loss of $\inf_y \sup_x H(x, y)$.

Definition 2.3 *The minimax* $\bar{v} = \inf_y \sup_x H(x, y)$ *is called the upper value of a game* Γ, *and the maximin* $\underline{v} = \sup_x \inf_y H(x, y)$ *is called the lower value of this game.*

The lower value of any game does not exceed its upper value.

Lemma 2.1 $\underline{v} \le \bar{v}$.

Proof: For any (x, y), the inequality $H(x, y) \le \sup_x H(x, y)$ holds true. By evaluating inf in both sides, we obtain $\inf_y H(x, y) \le \inf_y \sup_x H(x, y)$. This inequality involves a function of x in the left-hand side; this function is bounded above by the quantity $\inf_y \sup_x H(x, y)$. Therefore,

$$\sup_x \inf_y H(x, y) \le \inf_y \sup_x H(x, y).$$

Now, we provide a simple criterion to verify Nash equilibrium existence in this game.

Theorem 2.1 *A Nash equilibrium* (x^*, y^*) *in the zero-sum game exists iff* $\inf_y \sup_x H(x, y) = \min_y \sup_x H(x, y)$ *and* $\sup_x \inf_y H(x, y) = \max_x \inf_y H(x, y)$. *Moreover,*

$$\underline{v} = \bar{v}. \tag{1.2}$$

Proof: Assume that (x^*, y^*) forms a Nash equilibrium. Definition (1.1) implies that $H(x, y^*) \leq H(x^*, y^*), \forall x$. Then it follows that $\sup_x H(x, y^*) \leq H(x^*, y^*)$; hence,

$$\bar{v} = \inf_y \sup_x H(x, y) \leq \sup_x H(x, y^*) \leq H(x^*, y^*). \tag{1.3}$$

Similarly,

$$H(x^*, y^*) \leq \inf_y H(x^*, y) \leq \sup_x \inf_y H(x, y) = \underline{v}. \tag{1.4}$$

However, Lemma 2.1 claims that $\underline{v} \leq \bar{v}$. Therefore, all inequalities in (1.3)–(1.4) become strict equalities, i.e., the values corresponding to the external operators sup, inf are achieved and $\underline{v} = \bar{v}$. This proves necessity.

The sufficiency part. Denote by x^* a point, where $\max_x \inf_y H(x, y) = \inf_x H(x^*, y)$. By analogy, y^* designates a point such that $\min_y \sup_x H(x, y) = \sup_x H(x, y^*)$.

Consequently,

$$H(x^*, y^*) \geq \inf_y H(x^*, y) = \underline{v}.$$

On the other hand,

$$H(x^*, y^*) \leq \sup_x H(x, y^*) = \bar{v}.$$

In combination with the condition (1.2) $\underline{v} = \bar{v}$, this fact leads to

$$H(x^*, y^*) = \inf_y H(x^*, y) = \sup_x H(x, y^*).$$

The last expression immediately shows that for all (x, y):

$$H(x, y^*) \leq \sup_x H(x, y^*) = H(x^*, y^*) = \inf_y H(x^*, y) \leq H(x^*, y),$$

i.e., (x^*, y^*) makes a Nash equilibrium.

Theorem 2.1 implies that, in the case of several equilibria, optimal payoffs do coincide. This value $v = H(x^*, y^*)$ (identical for all equilibria) is said to be **game value**.

Moreover, readers would easily demonstrate the following circumstance. Any combination of optimal strategies also represents a Nash equilibrium.

Theorem 2.2 *Suppose that* (x_1, y_1) *and* (x_2, y_2) *are Nash equilibria in a zero-sum game. Then* (x_1, y_2) *and* (x_1, y_2) *is also a Nash equilibrium.*

Proof: According to the definition of a Nash equilibrium, for any (x, y):

$$H(x, y_1) \le H(x_1, y_1) \le H(x, y) \tag{1.5}$$

and

$$H(x, y_2) \le H(x_2, y_2) \le H(x_2, y). \tag{1.6}$$

Set $x = x_1, y = y_2$ in inequality (1.5) and $x = x_2, y = y_1$ in equality (1.6). This generates a chain of inequalities with the same quantity $H(x_2, y_1)$ in their left- and right-hand sides. Therefore, all inequalities in (1.5)–(1.6) appear strict equalities. And (x_1, y_2) becomes a Nash equilibrium, since for any (x, y):

$$H(x, y_2) \le H(x_2, y_2) = H(x_1, y_2) = H(x_1, y_1) \le H(x_1, y).$$

Similar result applies to (x_2, y_1).

These properties express distinctive features of the games in question from nonzero-sum games. We have observed that, in nonzero-sum games, different combinations of optimal strategies may form no equilibria, and players' payoffs in equilibria may vary appreciably. In addition, the values of minimaxes and maximins play a considerable role in antagonistic games. It is possible to evaluate maximins for each player in nonzero-sum games; they give the guaranteed payoff if the opponent plays against a given player (by ignoring his own payoff). This approach will be adopted in negotiations analysis.

2.2 Randomization

Imagine that Nash equilibria do not exist. In this case, one can employ randomization, i.e., extend the strategy set by mixed strategies.

Definition 2.4 *Mixed strategies of players I and II are probabilistic measures* μ *and* ν *defined on the sets* X*,* Y*.*

Randomization generates a new game, where players' strategies represent distribution functions and the payoff function is the expected value of the payoff

$$H(\mu, \nu) = \int_X \int_Y H(x, y) d\mu(x) d\nu(y).$$

This formula contains the Lebesgue–Stieltjes integral. In the sequel, we also write

$$H(\mu, y) = \int_X H(x, y) d\mu(x) \quad H(x, \nu) = \int_Y H(x, y) d\nu(y).$$

Find a Nash equilibrium in the stated extension of the game.

Definition 2.5 *A mixed strategy Nash equilibrium in the game* Γ *are measures* (μ^*, ν^*) *meeting the inequalities*

$$H(\mu, \nu^*) \leq H(\mu^*, \nu^*) \leq H(\mu^*, \nu)$$

for arbitrary measures μ, ν.

We begin with the elementary case when each player chooses his strategy from a finite set ($X = \{1, \dots, m\}$ and $Y = \{1, \dots, n\}$). Consequently, the payoff of player II can be defined using some matrix $A = [a(i, j)], i = 1, \dots, m, \ j = 1, \dots, n$. Such games are referred to as **matrix games**. Mixed strategies form vectors $x = (x_1, \dots, x_m) \in R^m$ and $y = (y_1, \dots, y_m) \in R^n$. In terms of new strategies, the payoff acquires the form $H(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i y_j$. Note that matrix games make a special case of bimatrix games discussed in Chapter 1. Therefore, they enjoy the following property.

Theorem 2.3 *Matrix games always admit a mixed strategy Nash equilibrium, i.e., a strategy profile* (x^*, y^*) *such that*

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i y_j^* \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i^* y_j^* \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i^* y_j \ \forall x, y.$$

Interestingly, games with continuous payoff functions always possess a Nash equilibrium. Prior to demonstrating this fact rigorously, we establish an intermediate result.

Lemma 2.2 *If the function* $H(x, y)$ *is continuous on a compact set* $X \times Y$*, then* $H(\mu, y) = \int_X H(x, y) d\mu(x)$ *turns out continuous in* y.

Proof: Let $H(x, y)$ be continuous on a compact set $X \times Y$; hence, this function enjoys uniform continuity. Notably, $\forall \epsilon > 0 \ \exists \delta$ such that, if $\rho(y_1, y_2) < \delta$, then $|H(x, y_1) - H(x, y_2)| < \epsilon$ for all $x \in X$. And so, it follows that

$$|H(\mu, y_1) - H(\mu, y_2)| \leq |\int_X [H(x, y_1) - H(x, y_2)] d\mu(x)|$$

$$\leq \int_X |H(x, y_1) - H(x, y_2)| d\mu(x) \leq \epsilon \int_X d\mu(x) \leq \epsilon.$$

Theorem 2.4 *Consider a zero-sum game* $\Gamma =< I, II, X, Y, H >$. *Suppose that the strategy sets* X, Y *form compact sets in the space* $R^m \times R^n$*, while the function* $H(x, y)$ *is continuous. Then this game has a mixed strategy Nash equilibrium.*

Proof: According to Theorem 2.1, it suffices to show that minimax and maximin are attainable and do coincide. First, we prove that

$$\underline{v} = \sup_{\mu} \inf_{\nu} H(\mu, \nu) = \max_{\mu} \min_{\nu} H(\mu, \nu).$$

Lemma 2.2 claims that, for an arbitrary strategy μ, the function $H(\mu, y) = \int_X H(x, y) d\mu(x)$ is continuous in y. By the hypothesis, Y represents a compact set; therefore, the function $H(\mu, y)$ reaches its maximum. Consequently, $\sup_\mu \inf_\nu = \sup_\mu \min_\nu$. By definition of sup, for any n there exists a measure μ_n such that

$$\min_y H(\mu_n, y) > \underline{v} - \frac{1}{n}. \tag{2.1}$$

Recall that X is a compact set. By virtue of Helly's theorem (see Shiryaev, 1996), take the sequence $\{\mu_n, n = 1, 2 \ldots\}$ and choose a subsequence $\mu_{n_k}, k = 1, 2, \ldots$ which converges to the probabilistic measure μ^*. Moreover, for an arbitrary continuous function $f(x)$, the sequence of integrals $\int_X f(x) d\mu_{n_k}(x)$ will tend to the integral $\int_X f(x) d\mu^*(x)$.

Then for any y we obtain

$$\int_X H(x, y) d\mu_{n_k}(x) \to \int_X H(x, y) d\mu^*(x) = H(\mu^*, y).$$

The inequality $\underline{v} \le \min_y H(\mu^*, y)$ and the condition (2.1) yield

$$\underline{v} = \min_y H(\mu^*, y) = \max_\mu \min_y H(\mu^*, y).$$

By analogy, one can demonstrate that minimax is also achieved.

Now, we should vindicate that $\underline{v} = \bar{v}$. Owing to the compactness of X and Y, for any n there exists a finite $1/n$-network (i.e., finite set points $X_n = \{x_1, \ldots, x_k\} \in X$ and $Y_n = \{y_1, \ldots, y_m\} \in Y$) such that for any $x \in X, y \in Y$ it is possible to find points $x_i \in X_n$ and $y_j \in Y_n$ satisfying the conditions $\rho(x, x_i) < 1/n$ and $\rho(y, y_j) < 1/n$.

Fix some positive number ϵ. Select a sufficiently small quantity n such that, if for arbitrary $(x, y), (y, y')$ we have $\rho(x, x') < 1/n$ and $\rho(y, y') < 1/n$, then $|H(x, y) - H(x', y')| < \epsilon$. This is always feasible due to the continuity of $H(x, y)$ on the corresponding compact set (*ergo*, due to the uniform continuity of this function).

To proceed, construct the payoff matrix $[H(x_i, y_j)], i = 1, \ldots, k, j = 1, \ldots, m$ at the nodes of the $1/n$-network and solve the resulting matrix game. Denote by $p(n) = (p_1(n), \ldots, p_k(n))$ and $q(n) = (q_1(n), \ldots, q_m(n))$ the optimal mixed strategies in this game, and let the game value be designated by v_n.

The mixed strategy $p(n)$ corresponds to the probabilistic measure μ_n, where for $A \subset X$:

$$\mu_n(A) = \sum_{i: x_i \in A} p_i(n).$$

In this case, for any $y_j \in Y_n$ we have

$$H(\mu_n, y_j) = \sum_{i=1}^{k} H(x_i, y_j) p_i(n) \ge v_n. \tag{2.2}$$

According to Lemma 2.2, for any $y \in Y$ $\exists\, y_j \in Y_n$ such that $\rho(y, y_j) < 1/n$; and so, $|H(x, y) - H(x, y_j)| < \epsilon$. This immediately leads to

$$|H(\mu_n, y) - H(\mu_n, y_j)| \le \epsilon.$$

In combination with (2.2), the above condition yields the inequality

$$H(\mu_n, y) > v_n - \epsilon,$$

for any $y \in Y$. Therefore,

$$\underline{v} = \max_{\mu} \min_{y} H(\mu_n, y) \geq \min_{y} H(\mu_n, y) > v_n - \epsilon. \tag{2.3}$$

Similar reasoning gives

$$\bar{v} < v_n + \epsilon. \tag{2.4}$$

It appears from (2.3)–(2.4) that

$$\bar{v} < \underline{v} + 2\epsilon.$$

So long as ϵ is arbitrary, we derive

$$\bar{v} \leq \underline{v}.$$

This result and Lemma 2.1 bring to the equality $\bar{v} = \underline{v}$. The proof of Theorem 2.4 is completed.

2.3 Games with discontinuous payoff functions

The preceding section has demonstrated that games with continuous payoff functions and compact strategy sets admit mixed strategy equilibria. Here we show that, if a payoff function suffers from discontinuities, there exist no equilibria in the class of mixed strategies.

The Colonel Blotto game. Colonel Blotto has to capture two passes in mountains (see Figure 2.1).

His forces represent some unit resource to-be-allocated between two passes. His opponent performs similar actions. If the forces of a player exceed those of the opponent at a given pass, then his payoff equals unity (and vanishes otherwise). Furthermore, at a certain pass Colonel Blotto's opponent has already concentrated additional forces of size $1/2$.

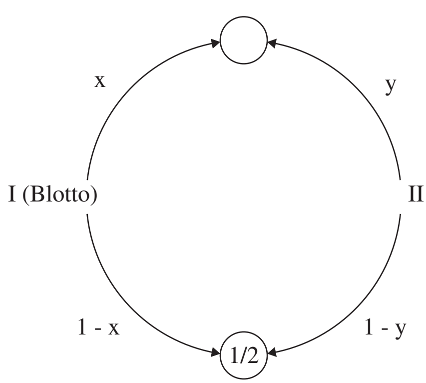


Figure 2.1 The Colonel Blotto game.

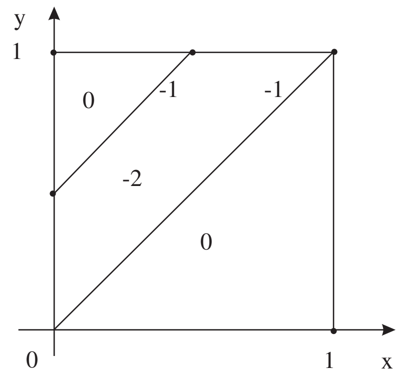


Figure 2.2 The payoff function of player I.

Therefore, we face a constant-sum game $\Gamma = < I, II, X, Y, H >$, where $X = [0, 1]$, $Y = [0, 1]$ indicate the strategy sets of players I and II. Suppose that Colonel Blotto and his opponent have allocated their forces $(x, 1 - x)$ and $(y, 1 - y)$ between the passes. Subsequently, the payoff function of player I takes the form

$$H(x, y) = sgn(x - y) + sgn\left(1 - x - \left(\frac{1}{2} + 1 - y\right)\right).$$

For its curve, see Figure 2.2. The function $H(x, y)$ possesses discontinuities at $x = y$ and $y = x + 1/2$.

Evaluate maximin and minimax in this game. Assume that the measure μ is concentrated at the points $x = \{0, 1/2, 1\}$ and has identical weights of $1/3$. Then for any $y \in [0, 1]$ the inequality

$$H(\mu, y) = \frac{1}{3}H(0, y) + \frac{1}{3}H(1/2, y) + \frac{1}{3}H(1, y) \geq -\frac{2}{3}$$

holds true. And it appears that $\underline{v} = \sup_{\mu} \inf_{\nu} \geq -2/3$.

On the other hand, choose strategy y by the following rule: if $\mu[1/2, 1] \geq 2/3$, set $y = 1$, and then $H(\mu, 1) \leq -2/3$; if $\mu[1/2, 1] < 2/3$, then $\mu[0, 1/2) > 1/3$, and there exists δ such that $\mu[0, 1/2 - \delta) \geq 1/3$, set $y = 1/2 - \delta$. Obviously, we also obtain $H(\mu, 1) \leq -2/3$ in this case. Hence, for any μ: $\inf_{y} H(\mu, y) \leq -2/3$, which means that $\sup_{\mu} \inf_{\nu} \leq -2/3$. We have successfully evaluated the lower value of the game:

$$\underline{v} = \sup_{\mu} \inf_{\nu} = -2/3. \tag{3.1}$$

Now, calculate the upper value $\bar{v}$ of the game. Suppose that the measure ν is concentrated at the points $y = \{1/4, 1/2, 1\}$ and has the weights $\nu(1/4) = 1/7, \nu(1/2) = 2/7$, and $\nu(1) = 4/7$. Using Figure 2.2, we find $H(0, \nu) = H(1, \nu) = -4/7$ $H(1/4, \nu) = -5/7, H(1/2, \nu) = -6/7$, and $H(x, \nu) = -6/7$ for $x \in (0, 1/4)$, $H(x, \nu) = -4/7$ for $x \in (1/4, 1/2)$ and $H(x, \nu) = -8/7$ for $x \in (1/2, 1)$. And so, this strategy of player II leads to $H(x, \nu) \leq -4/7$ for all x. This gives $\bar{v} = \inf_{\nu} \sup_{\mu} H(\mu, \nu) \leq -4/7$.

To prove the inverse inequality, select player I strategy x according to the following rule. If $\nu(1) \le 4/7$, set $x = 1$; then player I guarantees his payoff $H(1, \nu) \ge -4/7$. Now, assume that $\nu(1) > 4/7$, i.e., $\nu[0, 1) < 3/7$. Two alternatives appear here, *viz*., either $\nu[0, 1/2) \le 2/7$, or $\nu[0, 1/2) > 2/7$. In the first case, set $x = 0$, and player I ensures his payoff $H(0, \nu) \ge -4/7$. In the second case, there exists $\delta > 0$ such that $\nu[0, 1/2 - \delta) \ge 2/7$. In combination with the condition $\nu[0, 1) < 3/7$, this yields $\nu[1/2 - \delta, 1) < 1/7$. By choosing the strategy $x = 1/2 - \delta$, we obtain $H(1/2 - \delta, \nu) > -2/7 > -4/7$.

Evidently, under an arbitrary mixed strategy ν, player I guarantees his payoff $\sup_x H(x, \nu) \ge -4/7$. Hence it follows that $\bar{v} = \inf_\nu \sup_\mu H(\mu, \nu) \ge -4/7$.

We have found the exact upper value of the game:

$$\bar{v} = \inf_\nu \sup_\mu H(\mu, \nu) = -4/7. \tag{3.2}$$

Direct comparison of the expressions (3.1) and (3.2) indicates the following. In the Colonel Blotto game, the lower and upper values do differ—this game admits no equilibria.

However, in a series of cases, equilibria may exist under discontinuous payoff functions. The general equilibria evaluation scheme for such games is described below.

Theorem 2.5 *Consider an infinite game $\Gamma =< I, II, X, Y, H >$. Suppose that there exists a Nash equilibrium (μ^*, ν^*), while the payoff functions $H(\mu^*, y)$ and $H(x, \nu^*)$ are continuous in y and x, respectively. Then the following conditions take place:*

$$H(\mu^*, y) = v, \ \forall y \text{ on the support of the measure } \nu^*, \tag{3.3}$$

$$H(x, \nu^*) = v, \ \forall x \text{ on the support of the measure } \mu^*, \tag{3.4}$$

where v corresponds to the value of the game Γ.

Proof: Let μ^* be the optimal mixed strategy of player I. In this case, $H(\mu^*, y) \ge v$ for all $y \in Y$. Assume that (3.3) fails, i.e., $H(\mu^*, y') > v$ at a certain point y'. Due to the continuity of the function $H(\mu^*, y)$, this inequality is then valid in some neighborhood $U_{y'}$ of the point y'. The point y' belongs to the support of the measure ν^*, which means that $\nu^*(U_{y'}) > 0$. And we arrive at the contradiction:

$$H(\mu^*, \nu^*) = \int_Y H(\mu^*, y) d\nu^*(y) = \int_{U_{y'}} H(\mu^*, y) d\nu^*(y) + \int_{Y \setminus U_{y'}} H(\mu^*, y) d\nu^*(y) > v.$$

This proves (3.3). A similar line of reasoning demonstrates validity of the condition (3.4).

By performing differentiation in (3.3)–(3.4), we obtain the differential equations

$$\frac{\partial H(\mu^*, y)}{\partial y} = 0, \ \forall y \text{ on the support of the measure } \nu^*,$$

and

$$\frac{\partial H(x, \nu^*)}{\partial x} = 0, \ \forall x \text{ on the support of the measure } \mu^*.$$

They serve to find optimal strategies. We will illustrate their application by discrete arbitration procedures.

Note that Theorem 2.5 provides necessary conditions for mixed strategy equilibrium evaluation in games with discontinuous payoff functions $H(x, y)$. Moreover, it is possible to obtain optimal strategies even if the functions $H(\mu^*, y)$ and $H(x, \nu^*)$ appear discontinuous. Most importantly, we need the conditions (3.3)–(3.4) on the supports of distributions (the rest x and y must meet the inequalities $H(x, \nu^*) \leq v \leq H(\mu^*, x)$).

2.4 Convex-concave and linear-convex games

Games, where strategy sets $X \subset R^m, Y \subset R^n$ represent compact convex sets and the payoff function $H(x, y)$ is continuous, concave in x and convex in y, are called **concave-convex games**. According to Theorem 1.1, we can formulate the following result.

Theorem 2.6 *Concave-convex games always admit a pure strategy Nash equilibrium.*

A special case of concave-convex games concerns **linear convex games** $\Gamma =< X, Y, H(x, y) >$, where strategies are points from the simplexes $X = \{(x_1, \dots, x_m) : x_i \geq 0, i = 1, \dots, m; \sum_{i=1}^{m} x_i = 1\}$ and $Y = \{(y_1, \dots, y_n) : y_j \geq 0,\ j = 1, \dots, n; \sum_{j=1}^{n} y_j = 1\}$, while the payoff function is described by some matrix $A = [a(i, j)], i = 1, \dots, m,\ j = 1, \dots, n$:

$$H(x, y) = \sum_{i=1}^{m} x_i f\left(\sum_{j=1}^{n} a(i, j) y_j\right). \tag{4.1}$$

In formula (4.1), we believe that f is a non-decreasing convex function. Interestingly, there exists a connection between equilibria in such games and equilibria in a matrix game defined by A.

Theorem 2.7 *Any Nash equilibrium in a matrix game defined by the matrix A gives an equilibrium for a corresponding linear convex game.*

Proof: Let (x^*, y^*) be a Nash equilibrium in the matrix game. By the definition,

$$\sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i y_j^* \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i^* y_j^* \leq \sum_{i=1}^{m} \sum_{j=1}^{n} a(i, j) x_i^* y_j \ \forall x, y. \tag{4.2}$$

The convexity of f implies that

$$H(x^*, y) = \sum_{i=1}^{m} x_i^* f\left(\sum_{j=1}^{n} a(i, j) y_j\right) \geq f\left(\sum_{i=1}^{m} x_i^* \sum_{j=1}^{n} a(i, j) y_j\right).$$

This result, the monotonicity of f and inequality (4.2) lead to

$$f\left(\sum_{i=1}^{m} x_i^* \sum_{j=1}^{n} a(i, j) y_j\right) \geq f\left(\sum_{i=1}^{m} x_i^* \sum_{j=1}^{n} a(i, j) y_j^*\right). \tag{4.3}$$

Now, notice that the left-hand side of (4.2) holds true for arbitrary x, particularly, for all pure strategies of player I:

$$\sum_{j=1}^{n} a(i,j)y_j^* \leq \sum_{i=1}^{m}\sum_{j=1}^{n} a(i,j)x_i^* y_j^*, i = 1, \dots, m.$$

The monotonous property of f brings to

$$f\left(\sum_{j=1}^{n} a(i,j)y_j^*\right) \leq f\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a(i,j)x_i^* y_j^*\right), i = 1, \dots, m.$$

By multiplying these inequalities by x_i and summing up the resulting expressions, we arrive at

$$\sum_{i=1}^{m} x_i f\left(\sum_{j=1}^{n} a(i,j)y_j^*\right) \leq f\left(\sum_{i=1}^{m}\sum_{j=1}^{n} a(i,j)x_i^* y_j^*\right). \tag{4.4}$$

It follows from (4.3) and (4.4) that the inequalities

$$H(x^*, y) \geq H(x, y^*)$$

take place for arbitrary x, y. This immediately implies that (x^*, y^*) makes an equilibrium in the game under consideration.

We underline that the inverse proposition fails. For instance, if f is a constant function, then any strategy profile (x, y) forms an equilibrium in this game (but a matrix game may have a specific set of equilibria). Linear convex games arise in resource allocation problems. As an example, study the city defense problem.

The city defense problem. Imagine the following situation. Player I (Colonel Blotto) attacks a city using tanks, whereas player II conducts a defense by anti-tank artillery. Consider a game, where player I must allocate his resources between light and heavy tanks, and player II distributes his resources between light and heavy artillery. For simplicity, suppose that the resources of both players equal unity.

Define the efficiency of different weaponry. Let the rate of fire of heavy artillery units be three times higher than of light artillery ones. In addition, light artillery units must open fire on heavy tanks five times more quickly than heavy artillery units do. The survival probabilities of tanks are specified by the table below.

$$\begin{array}{l c} & \begin{array}{cc} \text{Light artillery} & \text{Heavy artillery} \end{array} \\ \begin{array}{l} \text{Light tanks} \\ \text{Heavy tanks} \end{array} & \begin{pmatrix} 1/2 & 1/4 \\ 3/4 & 1/2 \end{pmatrix} \end{array}$$

Suppose that player I has x light and $1 - x$ heavy tanks. On the other hand, player II organizes city defense by y light and $1 - y$ heavy artillery units. After a battle, Colonel Blotto will possess the following average number of tanks:

$$H(x,y) = x\left(\frac{1}{2}\right)^{\alpha y}\left(\frac{1}{4}\right)^{\beta(1-y)} + (1-x)\left(\frac{3}{4}\right)^{5\alpha y}\left(\frac{1}{2}\right)^{\beta(1-y)/3}.$$

Here α and β are certain parameters of the problem. Rewrite the function $H(x,y)$ as

$$H(x,y) = x\exp\left[-\ln 2(\alpha y + 2/3\beta(1-y))\right] + (1-x)\exp\left[-5\alpha y\ln 4/3 - 1/3\beta(1-y)\ln 2\right].$$

Clearly, this game is linear convex with the payoff function $f(x) = \exp[x]$. An equilibrium follows from solving the matrix game described by the matrix

$$-\ln 2\begin{pmatrix} \alpha & 2/3\beta \\ 5\alpha(2-\ln 2/\ln 3) & 1/3\beta \end{pmatrix}.$$

For instance, if $\alpha = 1, \beta = 2$, the optimal strategy of player *I* becomes $x^* \approx 0.809$ (accordingly, the optimal strategy of player *II* is $y^* \approx 0.383$). In this case, the game receives the value $v \approx 0.433$. Thus, under optimal behavior, Colonel Blotto will have less than 50% tanks available after a battle.

2.5 Convex games

Assume that a payoff function is continuous and concave in x or convex in y. According to the theory of continuous games, generally an equilibrium exists in the class of mixed strategies. However, in the convex case, we can establish the structure of optimal mixed strategies. The following theorem (Helly's theorem) from convex analysis would facilitate here.

Theorem 2.8 *Let S be a family of compact convex sets in R^m whose number is not smaller than $m+1$. Moreover, suppose that the intersection of any $m+1$ sets from this family appears non-empty. Then there exists a point belonging to all sets.*

Proof: First, suppose that S contains a finite number of sets. We argue by induction. If S consists of $m+1$ sets, the above assertion clearly holds true. Next, admit its validity for any family of $k \geq m+1$ sets; we have to demonstrate this result for S comprising $k+1$ sets. Denote $S = \{X_1, \ldots, X_{k+1}\}$ and consider $k+1$ new families of the form $S \setminus X_i, i = 1, \ldots, k+1$. Each family $S \setminus X_i$ consists of k sets; owing to the induction hypothesis, there exists a certain point x_i belonging to all sets from the family S except the set X_i, $i = 1, \ldots, k+1$. The number of such points is $k+1 \geq m+2$.

Now, take the system of $m+1$ linear equations

$$\sum_{i=1}^{k+1}\lambda_i x_i = 0, \sum_{i=1}^{k+1}\lambda_i = 0. \tag{5.1}$$

The number of unknowns $\lambda_i, i = 1, \ldots, k+1$ exceeds the number of equations; hence, this system possesses a non-zero solution. Decompose them into two groups, depending on their signs. Without loss of generality, we believe that $\lambda_i > 0, i = 1, \dots, l$ and $\lambda_i \le 0, i = l+1, \dots k+1$. By virtue of formula (5.1),

$$\sum_{i=1}^{l} \lambda_i = -\sum_{i=l+1}^{k+1} \lambda_i = \lambda > 0,$$

and

$$x = \sum_{i=1}^{l} \frac{\lambda_i}{\lambda} x_i = \sum_{i=l+1}^{k+1} \frac{-\lambda_i}{\lambda} x_i. \tag{5.2}$$

According to the construction procedure, for $i = 1, \dots, l$ all $x_i \in X_{l+1}, \dots, X_{k+1}$. The convexity of these sets implies that the convex combination $x = \sum_{i=1}^{l} \frac{\lambda_i}{\lambda} x_i$ belongs to the intersection of these sets. Similarly, for all $i = l+1, \dots, k+1$ we have $x_i \in X_1, \dots, X_l$, *ergo*, $x \in \bigcap_{i=1}^{l} X_i$. And so, there exists a point x belonging to all sets $X_i, i = 1, \dots, k+1$.

Thus, if S represents a finite set, Theorem 2.8 is proved. In what follows, we illustrate its correctness for an arbitrary family. Let a family $S = S_\alpha$ be such that its any finite subsystem possesses a non-empty intersection. Choose some set X from this family and consider the new family $S_\alpha^x = S_\alpha \bigcap X$. It also consists of compact convex sets.

Assume that $\bigcap_\alpha S_\alpha^x = \emptyset$. Then its complement becomes $\overline{\bigcap_\alpha S_\alpha^x} = \bigcup_\alpha \overline{S_\alpha^x} = R^m$. Due to the compactness of X, a finite subcovering $\{\overline{S_{\alpha_i}^x}, i = 1, \dots, r\}$ of the set X can be extracted from the finite covering $\{\overline{S_\alpha^x}\}$. However, in this case, the finite family $\{S_{\alpha_i}^x, i = 1, \dots, r\}$ has an empty intersection, which contradicts the premise. Consequently, $\bigcap_\alpha S_\alpha^x = \bigcap_\alpha S_\alpha$ appears non-empty.

Theorem 2.9 *Let $X \subset R^m$ and $Y \subset R^n$ be compact sets, Y enjoy convexity and the function $H(x, y)$ appear continuous in both arguments and convex in y. Then player II possesses an optimal pure strategy, whereas the optimal strategy of player I belongs to the class of mixed strategies and is concentrated at most in $(m + 1)$ points of the set X. Moreover, the game has the value*

$$v = \max_{x_1,\dots,x_{m+1}} \min_y \max\{H(x_1, y), \dots, H(x_{m+1}, y)\} = \min_y \max_x H(x, y).$$

Proof: Introduce the function

$$h(x_1, \dots, x_{m+1}, y) = \max\{H(x_1, y), \dots, H(x_{m+1}, y)\}, x_i \in R^m, i = 1, \dots, m+1, y \in R^n.$$

It is continuous in both arguments. Indeed,

$$|h(x_1', \dots, x_{m+1}', y') - h(x_1'', \dots, x_{m+1}'', y'')| = |H(x_{i_1}', y') - H(x_{i_2}'', y'')|, \tag{5.3}$$

where

$$H(x'_{i_1}, y') = \max\{H(x'_1, y'), \dots, H(x'_{m+1}, y')\}, H(x''_{i_2}, y'') = \max\{H(x''_1, y''), \dots, H(x''_{m+1}, y'')\}.$$

In formula (5.3), we have either

$$H(x'_{i_1}, y') \geq H(x''_{i_2}, y''), \tag{5.4}$$

or the inverse inequality. For definiteness, suppose that (5.4) holds true (in the second case, reasoning is by analogy).

The function $H(x, y)$ is continuous on the compact set $X^{m+1} \times Y$, *ergo*, uniformly continuous. And so, for any $\epsilon > 0$ there exists $\delta > 0$ such that, if $||x'_i - x''_i|| < \delta, i = 1, \dots, n+1$ and $||y' - y''|| < \delta$, then

$$0 \geq H(x'_{i_1}, y') - H(x''_{i_2}, y'') \leq H(x'_{i_1}, y') - H(x''_{i_1}, y'') < \epsilon.$$

This proves continuity of the function h.

The established fact directly implies the existence of

$$w = \max_{x_1,\dots,x_{m+1}} \min_y h(x_1, \dots, x_{m+1}, y) = \max_{x_1,\dots,x_{m+1}} \min_y \max\{H(x_1, y), \dots, H(x_{m+1}, y)\}.$$

So long as $\min_y H(x, y) \leq \max_\mu \min_y H(\mu, y)$ for any x, we obtain the inequality

$$w \leq v.$$

To derive the inverse inequality, consider the infinite family of sets $S_x = \{y : H(x, y) \leq w\}$. Since $H(x, y)$ is convex in y and continuous, all sets from this family become convex and compact. Note that any finite subsystem of this family, consisting of $m + 1$ sets $S_{x_i} = \{y : H(x_i, y) \leq w\}, i = 1, \dots, m + 1$, possesses a common point.

Really, if $x_1, \dots, x_{m+1}$ are fixed, the function $h(x_1, \dots, x_{m+1}, y)$ attains its maximum at some point $\bar{y}$ (due to continuity of h and compactness of Y). Consequently,

$$w = \max_{x_1,\dots,x_{m+1}} \min_y h(x_1, \dots, x_{m+1}, y) \geq \max\{H(x_1, \bar{y}), \dots, H(x_{m+1}, \bar{y}) \geq H(x_i, \bar{y}),$$
$$i = 1, \dots, m + 1,$$

i.e., $\bar{y}$ belongs to all $S_{x_i}, i = 1, \dots, m + 1$.

Helly's theorem claims the existence of a point y^* such that

$$H(x, y^*) \leq w, \forall x \in X.$$

Hence, $\max_x H(x, y^*) \leq w$, which means that $v \leq w$. Therefore, we have shown that $v = w$ or

$$v = \max_{x_1,\dots,x_{m+1}} \min_y \max\{H(x_1, y), \dots, H(x_{m+1}, y)\}.$$

Suppose that maximal value corresponds to the points $\bar{x}_1, \dots, \bar{x}_{m+1}$. Then

$$v = \min_y \max\{H(\bar{x}_1, y), \dots, H(\bar{x}_{m+1}, y)\} = \min_y \max_{\mu_0} \sum_{i=1}^{m+1} H(\bar{x}_i, y)\mu_i,$$

where $\bar{\mu} = (\mu_1, \dots, \mu_{m+1})$ is a discrete distribution located at the points $\bar{x}_1, \dots, \bar{x}_{m+1}$.

A concave-convex game on a compact set with the payoff function

$$H(\bar{\mu}, y) = \sum_{i=1}^{m+1} H(\bar{x}_i, y)\mu_i$$

admits a Nash equilibrium $(\bar{\mu}^*, y^*)$ such that

$$\max_{\bar{\mu}} \min_y H(\bar{\mu}, y) = \min_y \max_{\bar{\mu}} H(\bar{\mu}, y) = v.$$

This expression confirms the optimal character of the pure strategy y^* and the mixed strategy $\bar{\mu}$ located at $m + 1$ points, since

$$H(\bar{\mu}^*, y^*) = \min_y H(\bar{\mu}^*, y) \leq H(\bar{\mu}^*, y), \ \forall y$$
$$H(\bar{\mu}^*, y^*) = \max_{\bar{\mu}} H(\bar{\mu}, y^*) = \max_{\mu} H(\mu, y^*) \geq H(x, y), \ \forall x.$$

The proof of Theorem 2.9 is finished.

Corollary 2.1 *Consider a convex game, where the strategy sets of players represent linear sections. Player II has an optimal pure strategy, whereas the optimal strategy of player I is either mixed or a probabilistic compound of two pure strategies.*

A similar result applies to a concave game.

Theorem 2.10 *Let $X \subset R^m$ and $Y \subset R^n$ be compact sets, X enjoy convexity and the function $H(x, y)$ appear continuous in both arguments and concave in x. Then player I possesses an optimal pure strategy, whereas the optimal strategy of player II belongs to the class of mixed strategies and is concentrated at most in $(n + 1)$ points of the set Y. Moreover, the game has the value*

$$v = \min_{y_1,\dots,y_{n+1}} \max_x \max\{H(x, y_1), \dots, H(x, y_{n+1})\} = \max_x \min_y H(x, y).$$

2.6 Arbitration procedures

Consider a two-player game involving player *I* (Company Trade Union) and player *II* (Company Manager). The participants of this game have to negotiate a raise for company employees. Each player submits some offer (x and y, respectively). In the case of a conflict (i.e., $x > y$), both sides go to an arbitration court. The latter must support a certain player. There exist various arbitration procedures, namely, final-offer arbitration, conventional arbitration, bonus/penalty arbitration, as well as their combinations.

We begin analysis with **final-offer arbitration**. Without a conflict (if $x \le y$), this procedure leads to successful raise negotiation in the interval between x and y. For the sake of definiteness, suppose that the negotiated raise makes up $(x+y)/2$. In the case of $x < y$, the sides address the third party (call it an arbitrator). An arbitrator possesses a specific opinion α, and he takes the side whose offer is closer to α.

Actually, we have described a game $\Gamma =< I, II, R^1, R^1, H_\alpha >$ with the payoff function

$$H_\alpha(x,y) = \begin{cases} \frac{x+y}{2}, & \text{if} \quad x \le y \\ x, & \text{if} \quad x > y, |x-\alpha| < |y-\alpha| \\ y, & \text{if} \quad x > y, |x-\alpha| > |y-\alpha| \\ \alpha, & \text{if} \quad x > y, |x-\alpha| = |y-\alpha| \end{cases} \tag{6.1}$$

The parameter α being fixed, an equilibrium lies in the pair of strategies (α, α). However, the problem becomes non-trivial when the arbitrator may modify his opinion.

Consider the non-deterministic case, i.e., α represents a random variable with some continuous distribution function $F(a), a \in R^1$. Imagine that both players know $F(a)$ and there exists the corresponding density function $f(a)$.

If $y < x$, arbitrator accepts the offer y under $\alpha < (x+y)/2$ (see the payoff function formula and Figure 2.3). Otherwise, he accepts the offer x. The player I has the expected payoff $H(x,y) = EH_\alpha(x,y)$

$$H(x,y) = F\left(\frac{x+y}{2}\right)y + \left(1 - F\left(\frac{x+y}{2}\right)\right)x. \tag{6.2}$$

With the aim of evaluating minimax strategies, we perform differentiation in (6.2):

$$\frac{\partial H}{\partial x} = 1 - F\left(\frac{x+y}{2}\right) + \frac{y-x}{2} f\left(\frac{x+y}{2}\right) = 0,$$

$$\frac{\partial H}{\partial y} = F\left(\frac{x+y}{2}\right) + \frac{y-x}{2} f\left(\frac{x+y}{2}\right) = 0.$$

The difference of these equations yields

$$F\left(\frac{x+y}{2}\right) = \frac{1}{2},$$

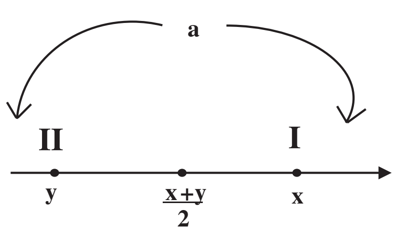


Figure 2.3 An arbitration game.

i.e., the point $(x+y)/2$ coincides with the median of the distribution F. Hence, it appears that $(x+y)/2 = m_F$. On the other part, summing up these equations gives

$$(y-x)f(m_F) = 1.$$

Therefore, if a pure strategy equilibrium does exist, it acquires the form

$$x = m_F + \frac{1}{2f(m_F)},\ y = m_F - \frac{1}{2f(m_F)}, \tag{6.3}$$

and the game has the value of m_F.

The following sufficient condition guarantees that (6.3) is an equilibrium:

$$H\left(x, m_F - \frac{1}{2f(m_F)}\right) \leq m_F, \forall x \geq m_F \tag{6.4}$$

$$H\left(m_F + \frac{1}{2f(m_F)}, y\right) \geq m_F, \forall y \leq m_F. \tag{6.5}$$

Recall that m_F makes the median of the distribution F and rewrite (6.4) as

$$\left(\frac{1}{2} + \int\limits_{m_F}^{U(x)} f(a)da\right)(x - m + \frac{1}{2f(m_F)} \geq x - m_F,$$

where $U(x) = (x + m_F - \frac{1}{2f(m_F)})/2$, or

$$\int\limits_{m_F}^{U(x)} f(a)da \geq \frac{x - m_F - 1/(2f(m_F))}{2(x - m_F + 1/(2f(m_F))}, \ \forall x > m_F. \tag{6.6}$$

By analogy, the condition (6.5) can be reexpressed as

$$\int\limits_{V(y)}^{m_F} f(a)da \geq \frac{y - m_F + 1/(2f(m_F))}{2(y - m_F - 1/(2f(m_F))}, \ \forall y < m_F. \tag{6.7}$$

Here $V(y) = (y + m_F + \frac{1}{2f(m_F)})/2$.

Theorem 2.11 *Consider the final-offer arbitration procedure and let the distribution $F(a)$ satisfy the conditions (6.6)–(6.7). Then a Nash equilibrium exists in the class of pure strategies and takes the form $x = m_F + \frac{1}{2f(m_F)}, y = m_F - \frac{1}{2f(m_F)}$.*

For instance, suppose that $F(a)$ is the **Gaussian distribution** with the parameters $\bar{a}$ and σ. The median coincides with $\bar{a}$; according to (6.3), the optimal offers of the players become

$$x = \bar{a} + \sqrt{\pi/2}, y = \bar{a} - \sqrt{\pi/2}.$$

Under the **uniform** distribution $F(a)$ on a segment $[c, d]$, the median is $(c+d)/2$ and the optimal offers of the players correspond to the ends of this segment:

$$x = \frac{c+d}{2} + \frac{d-c}{2} = d, \quad y = \frac{c+d}{2} - \frac{d-c}{2} = c.$$

Interestingly, games with the payoff function (6.2) can be concave-convex. Moreover, if the distribution F turns out discontinuous, the payoff function (6.2) suffers from discontinuities, as well. Therefore, this game may have no pure strategy equilibrium. And so, the strategies (6.3) being evaluated, one should verify the equilibria condition. Below we will demonstrate that, even for simple distributions, the final-offer arbitration procedure admits a mixed strategy equilibrium.

Conventional arbitration. In contrast to final-offer arbitration (an arbitrator chooses one of submitted offers), conventional arbitration settles a conflict through an arbitrator's specific opinion. Of course, his opinion depends on the offers of both players. In what follows, we study the combined arbitration procedure proposed by S.J. Brams and S. Merrill.

In this procedure, arbitrator's opinion α again represents a random variable with a known distribution function $F(a)$ and density function $f(a)$. Company Trade Union (player I) and Company Manager (player II) submit their offers, x and y. If α belongs to the offer interval, arbitration is performed by the last offer; otherwise, arbitrator makes his decision.

Therefore, we obtain a game $\Gamma = < I, II, R^1, R^1, H >$ with the payoff function $H(x, y) = EH_\alpha(x, y)$, where

$$H_\alpha(x,y) = \begin{cases} \frac{x+y}{2,} & \text{if} \quad x \le y \\ x, & \text{if} \quad x > \alpha > y, x-\alpha < \alpha - y \\ y, & \text{if} \quad x > \alpha > y, x - \alpha > \alpha - y \\ \alpha, & \text{otherwise.} \end{cases} \tag{6.8}$$

Suppose that the density function $f(a)$ is a symmetrical unimodal function (i.e., it possesses a unique maximum). To proceed, we demonstrate that (a) an equilibrium exists in the class of pure strategies and (b) the optimal strategy of both players lies in m_F—the median of the distribution $F(a)$.

Let player II apply the strategy $y = m_F$. According to the arbitration rules, his payoff $H(x, m_F)$ constitutes $(x + m_F)/2$, if $x < m_F$. This is smaller than the payoff gained by the strategy $x = m_F$. In the case of $x \ge m_F$, formula (6.8) implies that the payoff becomes

$$H(x, m_F) = \int_{-\infty}^{m_F} adF(a) + \int_{x}^{\infty} adF(a) + \int_{m_F}^{\frac{x+m_F}{2}} mdF(a) + \int_{\frac{x+m_F}{2}} xdF(a)$$

$$= m_F - \int_{m_F}^{\frac{x+m_F}{2}} (a - m_F)dF(a) + \int_{\frac{x+m_F}{2}} (x-a)dF(a).$$

The last expression does not exceed m_F, since the function

$$g(x) = -\int_{m_F}^{\frac{x+m_F}{2}} (a - m_F)dF(a) + \int_{\frac{x+m_F}{2}} (x - a)dF(a)$$

is non-increasing. Really, its derivative takes the form

$$g'(x) = -\frac{x - m_F}{2} f\left(\frac{x + m_F}{2}\right) + \int_{\frac{x+m_F}{2}} f(a)da \le 0 - \int_{m_F}^{\frac{x+m_F}{2}} (a - m_F)dF(a) + \int_{\frac{x+m_F}{2}} (x - a)dF(a)$$

(the value of $f(a)$ at the point $(x + m_F)/2$ is not smaller than at the points $a \in [(x + m_f)/2), x]$).

Consequently, we have $H(x, m_F) \le m_F$ for all $x \in R^1$. And so, the best response of player I also lies in m_F. Similar arguments cover the behavior of player II. Thus, the arbitration procedure in question also admits a pure strategy equilibrium coinciding with the median of the distribution $F(a)$.

Theorem 2.12 *Consider the conventional arbitration procedure. If the density function $f(a)$ is symmetrical and unimodal, the arbitration game has a Nash equilibrium consisting of identical pure strategies m_F.*

Penalty arbitration. Earlier, we have studied arbitration procedures, where each player may submit any offers (including the ones discriminating against the opponent). To avoid these situations, an arbitrator can apply penalty arbitration procedures. To proceed, analyze such scheme introduced by Zeng (2003).

Arbitrator's opinion α represents a random variable with a known distribution function $F(a)$ and density function $f(a)$. Denote by E the expected value $E\alpha = \int_{R^1} adF(a)$. Company Trade Union (player I) and Company Manager (player II) submit their offers x and y, respectively. Arbitrator follows the conventional mechanism, but adds some quantity to his decision. This quantity depends on the announced offers. As a matter of fact, it can be interpreted as player's penalty. Imagine that an arbitrator has the decision a. If $|x - a| < |y - a|$, an arbitrator supports player I and "penalizes" player II by the quantity $a - y$. In other words, the arbitrator's decision becomes $a + (a - y) = 2a - y$. The penalty is higher for greater differences between the arbitrator's opinion and the offer of player II. In the case of $|x - a| > |y - a|$, arbitrator "penalizes" player I, and his decision makes up $a - (x - a) = 2a - x$.

Hence, this arbitration game has the payoff function $H(x, y) = EH_\alpha(x, y)$, where

$$H_\alpha(x, y) = \begin{cases} \frac{x+y}{2}, & \text{if} \quad x \le y \\ 2\alpha - y, & \text{if} \quad x > y, |x - \alpha| < |\alpha - y| \\ 2\alpha - x, & \text{if} \quad x > y, |x - \alpha| > |\alpha - y| \\ \alpha, & \text{if} \quad x > y, |x - \alpha| = |\alpha - y|. \end{cases} \tag{6.9}$$

Theorem 2.13 *Consider the penalty arbitration procedure with the payoff function (6.9). It admits a unique pure strategy Nash equilibrium which consists of identical pure strategies E.*

Proof: First, we demonstrate that the strategy profile (E, E) forms an equilibrium. Suppose that the players choose pure strategies x and y such that $x > y$. Consequently, the arbitration game leads to the payoff

$$H(x,y) = \int_{-\infty}^{\frac{x+y}{2}-0} (2a-x)dF(a) + \int_{\frac{x+y}{2}+0}^{\infty} (2a-y)dF(a) + \frac{x+y}{2}\left[F\left(\frac{x+y}{2}\right) - F\left(\frac{x+y}{2}-0\right)\right]. \tag{6.10}$$

This formula takes into account the following aspect. The distribution function is right continuous and the point $(x+y)/2$ may correspond to a discontinuity of $F(a)$. Rewrite (6.10) as

$$\begin{aligned} H(x,y) &= \int_{-\infty}^{\infty} 2adF(a) - 2\frac{x+y}{2}\left(F\left(\frac{x+y}{2}\right) - F\left(\frac{x+y}{2}-0\right)\right) \\ &\quad - xF\left(\frac{x+y}{2}-0\right) - y\left(1 - F\left(\frac{x+y}{2}\right)\right) + \frac{x+y}{2}\left[F\left(\frac{x+y}{2}\right) - F\left(\frac{x+y}{2}-0\right)\right] \\ &= 2E - y - \frac{x-y}{2}\left[F\left(\frac{x+y}{2}\right) + F\left(\frac{x+y}{2}-0\right)\right]. \end{aligned} \tag{6.11}$$

Assume that player II employs the pure strategy $y = E$. The expression (6.11) implies that, under $x \geq E$, the payoff of player I is defined by

$$H(x,E) = E - \frac{x-E}{2}\left[F\left(\frac{x+E}{2}\right) + F\left(\frac{x+E}{2}-0\right)\right] \leq E.$$

In the case of $x < E$, we have

$$H(x,E) = \frac{x+E}{2} < E.$$

By analogy, readers would easily verify that $H(E,y) \geq E$ for all $y \in R^1$.

Second, we establish the uniqueness of the equilibrium (E, E). Conjecture that there exists another pure strategy equilibrium (x, y). In an equilibrium, the condition $x < y$ takes no place (otherwise, player I would raise his payoff by increasing the offer within the interval $[x, y]$).

Suppose that $x > y$. Let z designate the bisecting point of the interval $[y, x]$. If $F(z) = 0$, formula (6.11) requires that

$$H(x,y) - H(x,z) = (z-y) + \frac{x-z}{2}\left[F\left(\frac{x+z}{2}\right) + F\left(\frac{x+z}{2}-0\right)\right] - \frac{x-y}{2}\left[F(z) + F(z-0)\right].$$

However, $F(z) = 0$, which immediately gives $F(z - 0) = 0$. Therefore,

$$H(x, y) - H(x, z) = (z - y) + \frac{x - z}{2}\left[F\left(\frac{x + z}{2}\right) + F\left(\frac{x + z}{2} - 0\right)\right] > 0.$$

And so, the strategy z dominates the strategy y, and y is not the optimal strategy of player II.

In the case of $F(z) > 0$, study the difference

$$H(z, y) - H(x, y) = \frac{x - y}{2}\left[F(z) + F(z - 0)\right] - \frac{z - y}{2}\left[F\left(\frac{z + y}{2}\right) + F\left(\frac{z + y}{2} - 0\right)\right].$$

So far as $x - y = 2(z - y)$, we obtain

$$\begin{aligned} H(z, y) - H(x, y) &= (z - y)\left[F(z) + F(z - 0)\right] - \frac{z - y}{2}\left[F\left(\frac{z + y}{2}\right) + F\left(\frac{z + y}{2} - 0\right)\right] \\ &= \frac{z - y}{2}\left[2F(z) - F\left(\frac{z + y}{2}\right) + 2F(z - 0) - F\left(\frac{z + y}{2} - 0\right)\right] > 0, \end{aligned}$$

since $F(z) > 0$ and $F(z) \geq F(\frac{z+y}{2})$, $F(z - 0) \geq F(\frac{z+y}{2} - 0)$. Thus, here the strategy z of player I dominates x, and (x, y) is not an equilibrium. The equilibrium (E, E) turns out to be unique. This concludes the proof of Theorem 2.13.

The above arbitration schemes bring pure strategy equilibria. Such result is guaranteed by several assumptions concerning the distribution function $F(a)$ of an arbitrator. In what follows, we demonstrate the non-existence of pure strategy equilibria in a wider class of distributions. We will look for Nash equilibria among mixed strategies. At the same time, this approach provides new interesting applications of arbitration procedures. Both sides must submit random offers, which generate a series of practical benefits.

2.7 Two-point discrete arbitration procedures

For the sake of simplicity, let α be a random variable taking the values of -1 and 1 with identical probabilities $p = 1/2$. The strategies of players I and II also represent arbitrary values $x, y \in R^1$. The payoff function in this game acquires the form (6.1). This game has an equilibrium in the class of mixed strategies. We prove this fact rigorously. Here, define mixed strategies either through distribution functions or through their density functions.

Owing to its symmetry, the game under consideration has zero value. And so, the optimal strategies of players must be symmetrical with respect to the origin. Therefore, it suffices to construct an optimal strategy for one player (e.g., player I). Denote by $F(y)$ the mixed strategy of player II. Assume that the support of the distribution F lies on the negative semiaxis. Then the conditions of the game imply the following. For all $x < 0$, the payoff of player I satisfies $H(x, F) \leq 0$. In the case of $x \geq 0$, his payoff becomes

$$H(x, F) = \frac{1}{2}\left[F(-2 - x)x + \int_{-2-x}^{\infty} ydF(y)\right] + \frac{1}{2}\left[F(2 - x)x + \int_{2-x}^{\infty} ydF(y)\right]. \tag{7.1}$$

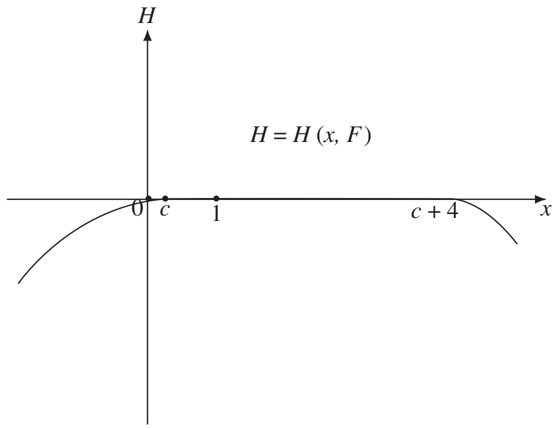


Figure 2.4 The payoff function $H(x, F)$.

We search for a distribution function $F(y)$ such that (a) its support belongs to the interval $[-c-4, -c]$, where $0 < c < 1$, and (b) the payoff function $H(x, F)$ vanishes on the interval $[c, c+4]$ and is negative-valued for the rest x. Figure 2.4 illustrates the idea.

According to (7.1), we have

$$H(x,F) = \frac{1}{2}\left[F(-2-x)x + \int_{-2-x}^{-c} ydF(y)\right] + \frac{1}{2}x, x \in [c, c+2]. \tag{7.2}$$

Since $H(x, F)$ is fixed on the interval $[c, c+2]$, we uniquely determine the distribution function $F(y)$. For this, perform differentiation in (7.2) and equate the result to zero:

$$\frac{dH}{dx} = \frac{1}{2}\left[-F'(-2-x)x + F(-2-x) + (-2-x)F'(-2-x)\right] + \frac{1}{2} = 0. \tag{7.3}$$

Substitute $-2-x = y$ into (7.3) to get the differential equation

$$2F'(y)(y+1) = -[F(y)+1], y \in [-c-4, -c-2].$$

Its solution yields the distribution function $F(y)$ on the interval $[-c-4, -c-2]$:

$$F(y) = -1 + \frac{const}{\sqrt{-y-1}}, y \in [-c-4, -c-2].$$

And finally, the condition $F(-c-4) = 0$ brings to

$$F(y) = -1 + \frac{\sqrt{3+c}}{\sqrt{-y-1}}, y \in [-c-4, -c-2]. \tag{7.4}$$

On the interval $[c+2, c+4]$, the function $H(x, F)$ takes the form

$$H(x, F) = \frac{1}{2} \int_{-c-4}^{-c} ydF(y) + \frac{1}{2} \left[F(2-x)x + \int_{2-x}^{-c} ydF(y) \right], x \in [c+2, c+4]. \tag{7.5}$$

By requiring its constancy, we find

$$\frac{dH}{dx} = \frac{1}{2} \left[-F'(2-x)x + F(2-x) + (2-x)F'(2-x) \right] = 0.$$

Next, set $2 - x = y$ and obtain the differential equation

$$2F'(y)(y-1) = -F(y), y \in [-c-2, -c].$$

The condition $F(-c) = 1$ leads to

$$F(y) = \frac{\sqrt{1+c}}{\sqrt{1-y}}, y \in [-c-2, -c]. \tag{7.6}$$

Let us demand continuity of the function $F(y)$. For this, paste together the functions (7.4) and (7.6) at the point $y = -c - 2$. This condition

$$\frac{\sqrt{1+c}}{\sqrt{3+c}} = -1 + \frac{\sqrt{3+c}}{\sqrt{1+c}}$$

generates the quadratic equation

$$(1+c)(3+c) = 4. \tag{7.7}$$

Its solution can be represented as

$$c = 2z - 1 \approx 0.236,$$

where z indicates the "golden section" of the interval $[0, 1]$ (a solution of the quadratic equation $z^2 + z - 1 = 0$).

Therefore, we have constructed a continuous distribution function $F(y), y \in [-c-4, -c]$ such that the payoff function $H(x, F)$ of player I possesses a constant value on the interval $[c, c+4]$. It forms the optimal strategy of player II, if we prove the following. The function $H(x, F)$ has the shape illustrated by the figure (its curve is below abscissa axis).

The solution to this game is provided by

Theorem 2.14 *In the discrete arbitration procedure, optimal strategies acquire the form*

$$G(x) = \begin{cases} 0, & x \in (-\infty, c] \\ 1 - \frac{\sqrt{1+c}}{\sqrt{x+1}}, & x \in (c, c+2] \\ 2 - \frac{\sqrt{3+c}}{\sqrt{x-1}}, & x \in (c+2, c+4] \\ 1, & x \in (c+4, \infty) \end{cases} \tag{7.8}$$

$$F(y) = \begin{cases} 0, & y \in (-\infty, -c-4] \\ -1 + \frac{\sqrt{3+c}}{\sqrt{-y-1}}, & y \in (-c-4, -c-2] \\ \frac{\sqrt{1+c}}{\sqrt{1-y}}, & y \in (-c-2, -c] \\ 1, & y \in (-c, \infty) \end{cases} \tag{7.9}$$

where $c = \sqrt{5} - 2$.

For *proof*, it suffices to show that $H(x, F) \leq 0$ for all $x \in R^1$. In the case of $x \leq 0$, this inequality is obvious ($y < 0$ holds true almost surely and, due to (6.3), $H(x, F)$ is negative).

Recall that, within the interval $[c, c+4]$, the function $H(x, F)$ possesses a constant value. Let us find the latter. Formula (7.2) yields

$$\begin{aligned} H(x, F) = H(c+2) &= \frac{1}{2}\left[F(-c-4)(c+2) + \int_{-c-4}^{-c} y dF(y)\right] + \frac{1}{2}(c+2) \\ &= \frac{1}{2}(\bar{y} + c + 2), x \in [c, c+2], \end{aligned} \tag{7.10}$$

where $\bar{y}$ indicates the mean value of the random variable y having the distribution (7.4), (7.6).

By performing simple computations, we arrive at

$$\bar{y} = \int_{-c-4}^{-c-2} y d\frac{\sqrt{3+c}}{\sqrt{-y-1}} + \int_{-c-2}^{-c} y d\frac{\sqrt{1+c}}{\sqrt{1-y}} = -c - 2.$$

It follows from (6.3) that

$$H(x, F) = 0, x \in [c, c+2].$$

Similarly, one obtains $H(x, F) = 0$ on the interval $[c+2, c+4]$.

If $x \geq c + 4$, the function $H(x, F)$ acquires the form (7.5). After the substitution $2 - x = y$, its derivative becomes

$$\frac{dH}{dx} = \frac{1}{2}[F'(2-x)(2-2x) + F(2-x)] = \frac{1}{2}[F'(y)(2y-2) + F(y)].$$

Using the expression (7.4) for F, it is possible to get

$$\frac{dH}{dx} = \frac{1}{2}\left[\frac{\sqrt{3+c}}{\sqrt{-y-1}}\frac{2y}{y+1} - 1\right], y \leq -c-2. \tag{7.11}$$

The function (7.11) increases monotonically with respect to y on the interval $[-c-4, -c-2]$, reaching its maximum at the point $y = c - 2$. By virtue of (7.7), the maximal value appears

$$\frac{1}{2}\left[\frac{\sqrt{3+c}}{\sqrt{1+c}}\frac{2(-c-2)}{-c-1} - 1\right] = -\frac{1}{2}\frac{(c+3)^2}{(1+c)^2} < 0.$$

This testifies that the function $H(x, F)$ is decreasing on the set $x \geq c + 4$ and vanishing at the point $x = c + 4$. Consequently,

$$H(x, F) \leq 0, x \geq c + 4.$$

If $x \in [0, c]$, then $H(x, F)$ takes the form (7.2). After the substitution $-2 - x = y$, its derivative (7.3) is determined by

$$\frac{dH}{dx} = \frac{1}{2}\left[F'(y)(2y+2) + F(y) + 1\right], y \geq -c-2. \tag{7.12}$$

Again, employ the expression (7.12) for F to obtain

$$\frac{dH}{dx} = \frac{1}{2}\left[\frac{\sqrt{1+c}}{\sqrt{(1-y)^3}}(2y+2) + \frac{\sqrt{1+c}}{\sqrt{1-y}} + 1\right].$$

This function increases monotonically with respect to y on the interval $[-c-2, -2]$, reaching its minimum at the point $y = -c - 2$. Furthermore, the minimal value

$$\frac{1}{2}\left[\frac{\sqrt{1+c}}{\sqrt{(3+c)^3}}(-2c-2) + \frac{\sqrt{1+c}}{\sqrt{3+c}} + 1\right] = \frac{1}{2}\left[\frac{2(1-c)}{(3+c)^2} + 1\right]$$

is positive. And so, the function $H(x, F)$ increases under $x \in [0, c]$ and vanishes at the point $x = c$. This fact dictates that $H(x, F) < 0, x \leq c$. Therefore, we have demonstrated that

$$H(x, F) \leq 0, x \in R.$$

Hence, any mixed strategy G of player L satisfies the condition

$$H(G, F) \leq 0.$$

This directly implies that F—see (7.9)—is the optimal strategy of player II. And finally, we take advantage of the problem's symmetry. Being symmetrical with respect to the origin, the strategy G defined by (7.8) is optimal for player I. This finishes the proof of Theorem 2.14.

Remark 2.2 Interestingly, the optimal rule in the above arbitration game relates to the golden section. The optimal rule proper guides player II to place his offers within the interval $[-3+2z, 1-2z] \approx [-4.236, -0.236]$ on the negative semiaxis. On the other hand, player I must submit offers within the interval $[2z-1, 3-2z] \approx [0.236, 4.236]$. Therefore, the game avoids strategy profiles when the offers of player II exceed those of player I. In addition, we emphasize that the mean value of the distributions F and G coincides with the bisecting point of the interval corresponding to the support of the distribution.

2.8 Three-point discrete arbitration procedures with interval constraint

Suppose that the random variable α is concentrated in the points $a_1 = -1, a_2 = 0$ and $a_3 = 1$ with identical probabilities of $p = 1/3$. Contrariwise to the model scrutinized in Section 2.6, we believe that players submit offers within the interval $x, y \in [-a, a]$.

Let us search for an equilibrium in the class of mixed strategies. Denote by $f(x)$ and $g(y)$ the strategies of players I and II, respectively. Assume that the support of the distribution $g(y)(f(x))$ lies on the negative semiaxis (positive semiaxis, respectively). In other words,

$$f(x) \geq 0, x \in [0, a], \int_0^a f(x)dx = 1, \quad g(y) \geq 0, y \in [-a, 0], \int_{-a}^0 g(y)dy = 1.$$

Owing to the symmetry, the game has zero value, and the optimal strategies must be symmetrical with respect to ordinate axis: $g(y) = f(-y)$. This condition serves for constructing the optimal strategy of some player (e.g., player I).

Theorem 2.15 *For $a \in (0, 8/3]$, the optimal strategy acquires the form*

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{a}{4} \\ \frac{\sqrt{a}}{2\sqrt{x^3}}, & \frac{a}{4} \leq x \leq a. \end{cases} \tag{8.1}$$

In the case of $a \in (8/3, \infty)$, it becomes

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{2}{3} \\ \sqrt{\frac{2}{3}} \frac{1}{\sqrt{x^3}}, & \frac{2}{3} \leq x \leq \frac{8}{3}, \\ 0, & \frac{8}{3} < x \leq a \end{cases} \tag{8.2}$$

Proof: We begin with the case when $a \in (0, 2]$. According to the rules of this game, for $y \in [-a, 0]$ player *II* gains the payoff

$$H(f, y) = \frac{1}{3}\int_0^a yf(x)dx + \frac{1}{3}\left(\int_0^{-y} xf(x)dx + \int_{-y}^{a} yf(x)dx\right) + \frac{1}{3}\int_0^a xf(x)dx.$$

Seek for f in the following class of strategies:

$$f(x) = \begin{cases} 0, & 0 \le x < \alpha \\ \varphi(x), & \alpha \le x \le \beta, \\ 0, & \beta < x \le a, \end{cases} \tag{8.3}$$

where $\varphi(x) > 0, \quad x \in [\alpha, \beta]$ and φ is continuously differentiable with respect to (α, β).

The strategy (8.3) enjoys optimality, if $H(f, y) = 0$ for $y \in [-\beta, -\alpha]$ and $H(f, y) \ge 0$ for $y \in [-a, -\beta) \cup (-\alpha, 0]$. Note that $H(f, 0) = \frac{1}{3}\int_0^a xf(x)dx > 0$.

The condition $H(f, -\alpha) = H(f, -\beta) = 0$ implies that $\beta = 4\alpha$ and $\int_\alpha^\beta x\varphi(x)dx = 2\alpha$. Clearly, $0 < \alpha \le \frac{a}{4}$. At the same time, $H(f, -a) = \frac{1}{3}[-a + 4\alpha]$. Therefore, $H(f, a) \ge 0$ iff $a \le 4\alpha$. And so, $\alpha = \frac{a}{4}$ and $\beta = a$.

Let us find the function $\varphi(x)$. The condition $H(f, y) = 0, \quad y \in [\beta, -\alpha]$ yields $H'(f, y) = H''(f, y) = 0$. Consequently,

$$H'(f, y) = 1 + 2yf(-y) + \int_{-y}^{2} f(x)dx = 0,\; H''(y) = 3f(-y) - 2yf'(-y) = 0.$$

By setting $y = -x$, we arrive at the differential equation

$$3f(x) + 2xf'(x) = 0. \tag{8.4}$$

It has the solution

$$f(x) = \frac{c}{\sqrt{x^3}}. \tag{8.5}$$

So long as

$$1 = \int_0^a f(x)dx = \int_{a/4}^a \frac{c}{\sqrt{x^3}} = \frac{2c}{\sqrt{a}},$$

we evaluate

$$c = \frac{\sqrt{a}}{2}.$$

Thus,

$$f(x) = \begin{cases} 0, & 0 \le x < a/4 \\ \frac{\sqrt{a}}{2\sqrt{x^3}}, & a/4 \le x \le a. \end{cases}$$

To proceed, verify the optimality condition. Under $y \in [-a, -a/4]$, we have

$$\begin{aligned} 3H(f,y) &= y + \int_{a/4}^{-y} \frac{\sqrt{a}}{2\sqrt{x}} dx + y \int_{-y}^{a} \frac{\sqrt{a}}{2\sqrt{x^3}} dx + \int_{a/4}^{a} \frac{\sqrt{a}}{2\sqrt{x}} dx \\ &= y + \sqrt{a}\sqrt{-y} - \frac{a}{2} - y - \sqrt{a}\sqrt{-y} + \frac{a}{2} = 0. \end{aligned}$$

In the case of $y \in (-a/4, 0]$,

$$H(f,y) = y + y \int_{a/4}^{a} \frac{\sqrt{a}}{2\sqrt{x^3}} dx + \frac{a}{2} = 2\left(y + \frac{a}{4}\right) > 0.$$

This guarantees the optimal character of the strategy (8.1).

Now, let $a \in (2, \frac{8}{3}]$. Consider $H(f, y)$ for $y \in [-a, -a/4]$, where f meets (2). The support of the distribution f is $[a/4, a]$ and $a \le \frac{8}{3}$. And so, $-1 + (-1 - y) \le a/4$ and $-y \ge a/4$ for all $y \in [-a, -a/4]$.

This means that, for $y \in [-a, -a/4]$, we get

$$3H(f,y) = \int_{\frac{a}{4}}^{a} yf(x)dx + \left(\int_{\frac{a}{4}}^{-y} xf(x)dx + \int_{-y}^{a} yf(x)dx \right) + \int_{\frac{a}{4}}^{a} xf(x)dx.$$

Differentiation again brings to equation (8.4). Its solution $f(x)$ is given by (8.1). Follow the same line of reasoning as above to establish that $H(f, y) > 0$ for $y \in (-a/4, 0]$. Therefore, the strategy (8.1) is also optimal for $a \in (2, \frac{8}{3}]$.

And finally, assume that $a \in (\frac{8}{3}, \infty)$. In this case, the function $H(f, y)$ becomes somewhat more complicated. As an example, consider the situation when $a = 4$. Under $y \in [-4, -2]$, we have

$$3H(f,y) = \left[\int_{0}^{-2-y} xf(x)dx + \int_{-2-y}^{4} yf(x)dx \right] + \left[\int_{0}^{-y} xf(x)dx + \int_{-y}^{4} yf(x)dx \right] + \int_{0}^{4} xf(x)dx. \quad (8.6)$$

For $y \in [-2, 0]$,

$$3H(f,y) = \int_{0}^{4} yf(x)dx + \left[\int_{0}^{-y} xf(x)dx + \int_{-y}^{4} yf(x)dx \right] + \left[\int_{0}^{2-y} xf(x)dx + \int_{2-y}^{4} yf(x)dx \right]. \quad (8.7)$$

Find f in the form (8.3), where $\beta = \alpha + 2$. The conditions $H(f, \beta) = H(f, \alpha) = 0$ yield

$$\int_{\alpha}^{\beta} xf(x)dx = 2\alpha = \frac{\beta}{2}.$$

Consequently, $\beta = 4\alpha$; in combination with $\beta = \alpha + 2$, this result leads to $\alpha = 2/3$, $\beta = 8/3$.

According to (8.6), on the interval $[-\beta, -2]$ the condition $H''(f, y) = 0$ turns out equivalent to

$$[3f(-y) - 2yf'(-y)] + [3f(-2 - y) - (2 + 2y)f'(-2 - y)] = 0. \tag{8.8}$$

If $y \in [-\beta, -2]$, then $x = -y \in [2, \beta]$ and $-2 - y \in [0, \beta - 2]$ or $-2 - y \in [0, \alpha]$. However, for $x \in [0, \alpha]$ we have $f(x) = 0, \quad f'(x) = 0$. Hence, the second expression in square brackets (see (8.8)) equals zero. And the following equation in $f(-y)$ arises immediately:

$$[3f(-y) - 2yf'(-y)] = 0.$$

As a matter of fact, it completely matches (8.4) under $x = -y$.

Similarly, it is possible to rewrite $H''(f, y)$ for $y \in [-2, -\alpha]$ as

$$[3f(-y) - 2yf'(-y)] + [3f(2 - y) + (2 - 2y)f'(2 - y)] = 0.$$

Here $-y \in [\alpha, 2]$ and $2 - y \in [\alpha + 2, 4]$. Therefore, $f(2 - y) = f'(2 - y) = 0$ and we derive the same equation (8.4) in $f(x)$.

Within the interval $(2/3, 8/3)$, the solution to (8.4) has the form

$$f(x) = \begin{cases} 0, & 0 \le x < 2/3 \\ \dfrac{\sqrt{\frac{2}{3}}}{\sqrt{x^3}}, & 2/3 \le x \le 8/3 \\ 0, & 8/3 < x \le 4 \end{cases} \tag{8.9}$$

The optimality of (8.9) can be verified by analogy to the case when $a \in (0, \frac{8}{3}]$.

For $a \in (\frac{8}{3}, \infty)$, the complete proof of this theorem lies in thorough analysis of the intervals $a \in (8/3, 4]$, $a \in [4, 14/3]$ and $a \in (14/3, \infty)$. The function $H(f, y)$ has the same form as (8.6) and (8.7).

2.9 General discrete arbitration procedures

Consider the case when arbitrator's offer is a random variable α, taking values $\{-n, -(n - 1), \dots, -1, 0, 1, \dots, n - 1, n\}$ with identical probabilities $p = 1/(2n + 1)$. The offers submitted by players must belong to the interval $x, y \in [-a, a]$ (see Figure 2.5).

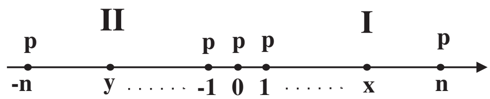


Figure 2.5 The discrete distribution of offers by an arbitrator, $p = \frac{1}{2n+1}$.

As earlier, we seek for a mixed strategy equilibrium. Denote by $f(x)$ and $g(y)$ the mixed strategies of players I and II. Suppose that the supports of the distributions $g(y)(f(x))$ lie in the negative (positive) domain, i.e.,

$$f(x) \geq 0, x \in [0, a], \int_0^a f(x)dx = 1, \quad g(y) \geq 0, y \in [-a, 0], \int_{-a}^0 g(y)dy = 1. \tag{9.1}$$

Let us search for the strategy of some player (say, player I).

Theorem 2.16 *Under $a \in (0, \frac{2(n+1)^2}{2n+1}]$, the optimal strategy of player I takes the form*

$$f(x) = \begin{cases} 0, & 0 \leq x < \left(\frac{n}{n+1}\right)^2 a, \\ \frac{n\sqrt{a}}{2\sqrt{x^3}}, & \left(\frac{n}{n+1}\right)^2 a \leq x \leq a. \end{cases} \tag{9.2}$$

In the case of $a \in (\frac{2(n+1)^2}{2n+1}, +\infty)$, it becomes

$$f(x) = \begin{cases} 0, & 0 \leq x < \frac{2n^2}{2n+1}, \\ \frac{n(n+1)}{\sqrt{2(2n+1)}} \frac{1}{\sqrt{x^3}} & \frac{2n^2}{2n+1} \leq x \leq \frac{2(n+1)^2}{2n+1}, \\ 0, & \frac{2(n+1)^2}{2n+1} < x \leq a. \end{cases} \tag{9.3}$$

Proof: First, consider the case of $a \in (0, 2]$. Under $y \in [-a, 0]$, the payoff of player I equals

$$H(f, y) = \frac{1}{2n+1}\left[n\int_0^a yf(x)dx + \left(\int_0^{-y} xf(x)dx + \int_{-y}^a yf(x)dx\right) + n\int_0^a xf(x)dx\right].$$

Find the strategy f in the form

$$f(x) = \begin{cases} 0, & 0 \leq x < \alpha, \\ \varphi(x), & \alpha \leq x \leq \beta, \\ 0, & \beta < x \leq a, \end{cases} \tag{9.4}$$

where $\varphi(x) > 0, \quad x \in [\alpha, \beta]$ and φ is continuously differentiable on (α, β).

The strategy (9.4) appears optimal, if $H(f, y) = 0$ for $y \in [-\beta, -\alpha]$ and $H(f, y) \geq 0$ for $y \in [-a, -\beta) \cup (-\alpha, 0]$. Note that $H(f, 0) = \frac{n}{2n+1} \int_0^a xf(x)dx > 0$.

It follows from $H(f, -\alpha) = H(f, -\beta) = 0$ that

$$H(f, -\alpha) = \frac{1}{2n+1}\left[-(n+1)\alpha + n\int_\alpha^\beta x\varphi(x)dx\right] = 0,$$

$$H(f, -\beta) = \frac{1}{2n+1}\left[-n\beta + (n+1)\int_\alpha^\beta x\varphi(x)dx\right] = 0.$$

This system yields

$$\int_\alpha^\beta x\varphi(x)dx = \frac{n+1}{n}\alpha = \frac{n}{n+1}\beta$$

and $\beta = (\frac{n+1}{n})^2\alpha$ or $\alpha = (\frac{n}{n+1})^2\beta$.

For $y = -a$, we have $H(f, -a) = \frac{1}{2n+1}[-na + n\beta] = \frac{n}{2n+1}(\beta - a)$. Hence, if $\beta < a$, then $H(f, -a) < 0$. Therefore, $\beta = a$, $\alpha = (\frac{n}{n+1})^2 a$, and

$$\int_0^a xf(x)dx = \int_\alpha^\beta x\varphi(x)dx = \frac{n}{n+1}a. \tag{9.5}$$

Now, obtain the explicit formula of $\varphi(x)$. The condition $H(f, y) = 0, y \in [\beta, -\alpha]$ brings to $H'(f, y) = H''(f, y) = 0$. And so,

$$H'(f, y) = 1 + 2yf(-y) + \int_{-y}^a f(x)dx = 0, H''(y) = 3f(-y) - 2yf'(-y) = 0.$$

By setting $y = -x$, we get the differential equation

$$3f(x) + 2xf'(x) = 0, \tag{9.6}$$

which admits the solution

$$f(x) = \frac{c}{\sqrt{x^3}}. \tag{9.7}$$

So far as,

$$1 = \int_0^a f(x)dx = \int_{\left(\frac{n}{n+1}\right)^2 a}^a \frac{c}{\sqrt{x^3}} = \frac{2c}{n\sqrt{a}},$$

it is possible to evaluate

$$c = \frac{n\sqrt{a}}{2}.$$

And finally,

$$f(x) = \begin{cases} 0, & 0 \le x < \left(\frac{n}{n+1}\right)^2 a, \\ n\frac{\sqrt{a}}{2\sqrt{x^3}}, & \left(\frac{n}{n+1}\right)^2 a \le x \le a. \end{cases}$$

Verify the optimality conditions. Under $y \in [-a, -(\frac{n}{n+1})^2 a]$, one obtains

$$\begin{aligned}(2n+1)H(f,y) &= ny + \int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{-y} n\frac{\sqrt{a}}{2\sqrt{x}}dx + y\int\limits_{-y}^{a} n\frac{\sqrt{a}}{2\sqrt{x^3}}dx + \frac{n^2}{n+1}a \\ &= ny + n\sqrt{a}\left(\sqrt{-y} - \frac{n}{n+1}\sqrt{a}\right) - n\sqrt{a}y\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{-y}}\right) + \frac{n^2}{n+1}a = 0.\end{aligned}$$

If $y \in (-(\frac{n}{n+1})^2 a, 0]$,

$$(2n+1)H(f,y) = ny + y + \frac{n^2}{n+1}a = (n+1)\left[y + \left(\frac{n}{n+1}\right)^2 a\right] > 0.$$

These conditions lead to the optimality of (9.2).

To proceed, analyze the case of $2 < a \le \frac{2(n+1)^2}{2n+1}$.

Consider $H(f,y)$ provided that $y \in [-a, -(\frac{n}{n+1})^2 a]$, where f is defined by (9.2). Recall that the distribution f possesses the support $[(\frac{n}{n+1})^2 a, a]$ and $a \le \frac{2(n+1)^2}{2n+1}$. These facts imply that $a - (\frac{n}{n+1})^2 a \le 2$.

Consequently, for $y \in [-a, -(\frac{n}{n+1})^2 a]$, we have

$$(2n+1)H(f,y) = n\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} yf(x)dx + \left(\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{-y} xf(x)dx + \int\limits_{-y}^{a} yf(x)dx\right) + n\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} xf(x)dx.$$

Again, differentiation yields equation (9.5). Its solution $f(x)$ acquires the form (9.2). Thus, $H(f,y) \equiv 0$ under $y \in [-a, -(\frac{n}{n+1})^2 a]$.

Now, it is necessary to show that $H(f,y) > 0$ for $y \in (-(\frac{n}{n+1})^2 a, 0]$. Find out the sign of $H(f,y)$ within the interval $[-(\frac{n}{n+1})^2 a, -(\frac{n}{n+1})^2 a + 2]$.

If $y \in [-(\frac{n}{n+1})^2 a, -a+2]$, then

$$H(f,y) = \frac{n+1}{2n+1}y + \frac{n}{2n+1}\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} xf(x)\,dx = \frac{n+1}{2n+1}\left[y + \left(\frac{n}{n+1}\right)^2 a\right] > 0.$$

On the other hand, under $y \in [-a+2, -(\dfrac{n}{n+1})^2 a + 2]$, we obtain

$$H(f,y) = \frac{n+1}{2n+1}y + \frac{1}{2n+1}\left(\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{2-y} xf(x)\,dx + \int\limits_{2-y}^{a} yf(x)\,dx\right) + \frac{n-1}{2n+1}\int\limits_{\left(\frac{n}{n+1}\right)^2 a}^{a} xf(x)\,dx.$$

Then

$$H'(f,y) = \frac{1}{2n+1}\left[n+1+(2y-2)f(2-y) + \int\limits_{2-y}^{a} f(x)\,dx\right]$$

$$= \frac{1}{2n+1}\left[n+1+\frac{(y-1)n\sqrt{a}}{\sqrt{(2-y)^3}} - n + \frac{n\sqrt{a}}{\sqrt{2-y}}\right] = \frac{1}{2n+1}\left(1 + \frac{n\sqrt{a}}{\sqrt{(2-y)^3}}\right) > 0.$$

Hence, $H(f,y) > 0$ for $y \in (-(\frac{n}{n+1})^2 a, -(\frac{n}{n+1})^2 a + 2]$.

If $-(\frac{n}{n+1})^2 a + 2 \geq 0$, the proof is completed. Otherwise, shift the interval to the right and demonstrate that $H(f,y) > 0$ for $y \in (-(\frac{n}{n+1})^2 a + 2, -(\frac{n}{n+1})^2 a + 4]$, etc. Thus, we have established the optimality of the strategy (9.2) under $a \in (2, \frac{2(n+1)^2}{2n+1}]$.

And finally, investigate the case when $\frac{2(n+1)^2}{2n+1} < a \leq \infty$. Here the function $H(f,y)$ becomes more complicated. Let us analyze the infinite horizon case $a = \infty$ only.

Assume that player I employs the strategy (9.3) and find the payoff function $H(f,y)$. For the sake of simplicity, introduce the notation $\alpha = \frac{2n^2}{2n+1}$ and $\beta = \alpha + 2 = \frac{2(n+1)^2}{2n+1}$. For $y \in (-\infty, -2n-\beta]$, we accordingly have

$$H(f,y) = \int\limits_{\alpha}^{\beta} xf(x)dx = \frac{2n(n+1)}{2n+1} > 0.$$

Set $k = 3[\frac{n}{2}] + 2$, if n is an odd number and $k = 3\frac{n}{2}$, otherwise. For $y \in [-2n+2r-\beta, -2n+2r-\alpha]$, where $r = 0, 1, \dots, n, \dots, k-1$, and for $y \in [-2n+2r-\beta, 0]$, where $r = k$, one has the following chain of calculations:

$$H(f,y) = \frac{r}{2n+1}y + \frac{1}{2n+1}\left[\int\limits_{\alpha}^{-2n+2r-y} xf(x)dx + \int\limits_{-2n+2r-y}^{\beta} yf(x)dx\right] + \frac{2n-r}{2n+1}\int\limits_{\alpha}^{\beta} xf(x)dx$$

$$= \int\limits_{\alpha}^{\beta} xf(x)dx - \frac{r}{2n+1}\int\limits_{\alpha}^{\beta}(x-y)f(x)dx - \frac{1}{2n+1}\int\limits_{-2n+2r-y}^{\beta}(x-y)f(x)dx. \tag{9.8}$$

Perform differentiation in formula (9.8), where f is determined by (9.2), to derive the equation

$$H'(f,y) = \frac{r}{2n+1} + \frac{1}{2n+1}\int\limits_{-2n+2r-y}^{\beta} f(x)dx + \frac{1}{2n+1}(2y+2n-2r)f(-2n+2r-y)$$

$$= \frac{r-n}{2n+1}\left(1 + \frac{2n(n+1)}{\sqrt{2(2n+1)(-2n+2r-y)^3}}\right). \tag{9.9}$$

According to (9.9), the expected payoff $H(f,y)$ is constant within the interval $y \in [-\beta, -\alpha]$, where $r = n$. Furthermore, since

$$H(f,\beta) = \int\limits_{\alpha}^{\beta} xf(x)dx - \frac{n}{2n+1}\int\limits_{\alpha}^{\beta}(x+\beta)f(x)dx$$

$$= \frac{n+1}{2n+1}\int\limits_{\alpha}^{\beta} xf(x)dx - \frac{n}{2n+1}\beta = \frac{n+1}{2n+1}\frac{2n(n+1)}{2n+1} - \frac{n}{2n+1}\frac{2(n+1)^2}{2n+1} = 0,$$

we have $H(f,y) \equiv 0$ for $y \in [-\beta, -\alpha]$.

In the case of $r < n$ $(r > n)$, formula (9.9) brings to $H'(f,y) < 0$ $(H'(f,y) > 0$, respectively) on the interval $y \in [-2n+2r-\beta, -2n+2r-\alpha]$.

Hence, $H(f,y) \geq 0$ for all y. This testifies to the optimality of the strategy (9.3).

For $a \in (\frac{2(n+1)^2}{2n+1}, \infty)$, the complete proof of Theorem 2.16 is by similar reasoning as for $a = \infty$.

Obviously, optimal strategies in the discrete arbitration scheme with uniform distribution appear randomized. This result differs from the continuous setting discussed in Section 2.6 (players have optimal strategies in the class of optimal strategies). By analogy to the uniform case, optimal strategies of players are concentrated in the boundaries of the interval $[-a, a]$. Note the following aspect. According to Theorem 2.14, the optimal strategy (9.2) in the discrete scheme with $a = n$ possesses a non-zero measure only on the interval $[(\frac{n}{n+1})^2 a, a]$. Actually, its length tends to zero for large n. In other words, the solutions to the discrete and continuous settings of the above arbitration game do coincide for sufficiently large n.

Exercises

1. Find a pure strategy solution of a convex-concave game with the payoff function

$$H(x, y) = -5x^2 + 2y^2 + xy - 3x - y.$$

and a mixed strategy solution of a convex game with the payoff function

$$H(x, y) = y^3 - 4xy + x^3.$$

2. Obtain a mixed strategy solution of a duel with the payoff function

$$H(x, y) = \begin{cases} 2x - y + xy, & x < y, \\ 0, & x = y, \\ x - 2y - xy, & x > y. \end{cases}$$

3. Find a solution to the following duel. Player *I* has two bullets, whereas player *II* disposes of one bullet.
4. The birthday game. Peter goes home from his work and suddenly remembers that today Kate celebrates her birthday. Is that the case? Peter chooses between two strategies, namely, visiting Kate with or without a present. Suppose that today Kate celebrates no birthday; if Peter visits Kate with a present, his payoff makes 1 (and 0, otherwise). Assume that today Kate celebrates her birthday; if Peter visits Kate with a present, his payoff equals 1.5 (and -10, otherwise). Construct the payoff matrix and evaluate an equilibrium in the stated game.
5. The high-quality amplifier game.
A company manufactures amplifiers. Their operation strongly depends on some parameters of a small (yet, scarce) capacitor. The standard price of this capacitor is 100 USD. However, the company's costs for warranty return of a failed capacitor constitute 1000 USD. The company chooses between the following strategies: 1) applying an inspection method for capacitors, which costs 100 USD and guarantees failure identification three times out of four; 2) applying a reliable and cheap inspection method, which causes breakdown of an operable capacitor nine times out of ten; 3) purchasing the desired capacitors at the price of 400 USD with full warranty. Construct the payoff matrix for the game involving nature (possible failures of a capacitor) and the company. Evaluate an equilibrium in this game.
6. Obtain a solution of a game 3×3 with the payoff matrix

$$A = \begin{pmatrix} 3 & 6 & 8 \\ 4 & 3 & 2 \\ 7 & -5 & -1 \end{pmatrix}.$$

7. The game of words.
Two players announce a letter as follows. Player *I* chooses between letters "a" and "i," whereas player *II* chooses between letters "f," "m," and "t." If the selected letters form a word, player *I* gains 1; moreover, player *I* receives the additional reward of 3 if this

word corresponds to an animate noun or pronoun. When the announced letters form no word, player *II* gains 2. Therefore, the payoff matrix takes the form

$$
\begin{array}{c|ccc|}
 & \text{f} & \text{m} & \text{t} \\
\hline
\text{a} & -2 & 1 & 1 \\
\text{i} & 1 & -2 & 4 \\
\hline
\end{array}
$$

Find a solution in this game.

8. Demonstrate that a game with the payoff function

$$
H(x, y) = \begin{cases} -1, & x = 1, y < 1 \text{ and } x < y < 1, \\ 0, & x = y, \\ 1, & y = 1, x < 1 \text{ and } y < x < 1 \end{cases}
$$

admits no solution.

9. Provide the complete proof to Theorem 2.15 in the case of $a \in (\frac{8}{3}, \infty)$.

10. Find an equilibrium in the arbitration procedure provided that arbitrator's offers are located at the points $-n, -n-1, \ldots, -1, 1, \ldots, n$.

3

Non-cooperative strategic-form *n*-player games

Introduction

In Chapter 1, we have explored nonzero-sum games of two players. The introduced definitions of strategies, strategy profiles, payoffs, and the optimality principle are naturally extended to the case of n players. Let us give the basic definitions for n-player games.

Definition 3.1 *A normal-form n-player game is an object*

$$\Gamma =< N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N} >,$$

where $N = \{1, 2, \dots, n\}$ *indicates the set of players,* X_i *represents the strategy set of player i, and* $H_i : \prod_{i=1}^{n} X_i \to R$ *means the payoff function of player i,* $i = 1, \dots, n$.

As previously, player i chooses some strategy $x_i \in X_i$, being unaware of the opponents' choice. Player i strives for maximizing his payoff $H_i(x_1, \dots, x_n)$ which depends on the strategies of all players. A set of strategies of all players is called a **strategy profile** of a game.

Consider some strategy profile $x = (x_1, \dots, x_n)$. For this profile, the associated notation

$$(x_{-i}, x_i') = (x_1, \dots, x_{i-1}, x_i', x_{i+1}, \dots, x_n)$$

designates a strategy profile, where player i has modified his strategy from x_i to x_i', while the rest of the players use the same strategies as before. The major solution approach to n-player games still consists in the concept of Nash equilibria.

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

Definition 3.2 *A Nash equilibrium in a game Γ is a strategy profile $x^* = (x_1^*, \ldots, x_n^*)$ such that the following conditions hold true for any player $i \in N$:*

$$H_i(x_{-i}^*, x_i) \le H_i(x^*), \forall x_i.$$

All strategies in such equilibrium are called optimal.

Definition 3.2 implies that, as any player deviates from a Nash equilibrium, his payoff goes down. Therefore, none of players benefit by a unilateral deviation from a Nash equilibrium. Of course, the matter does not concern two or three players simultaneously deviating from a Nash equilibrium. Later on, we will study such games.

3.1 Convex games. The Cournot oligopoly

Our analysis of n-player games begins with the following case. Suppose that payoff functions are concave, whereas strategy sets form convex sets. The equilibrium existence theorem for two-player games can be naturally extended to the general case.

Theorem 3.1 *Consider an n-player game $\Gamma =< N, \{X_i\}_{i \in N}, \{H_i\}_{i \in N} >$. Assume that strategy sets X_i are compact convex sets in the space R^n, and payoff functions $H_i(x_1, \ldots, x_n)$ are continuous and concave in x_i. Then the game always admits a Nash equilibrium.*

Convex games comprise different oligopoly models, where n companies compete on a market. Similarly to duopolies, one can discriminate between Cournot oligopolies and Bertrand oligopolies. Let us be confined with the Cournot oligopoly. Imagine that there exist n companies $1, 2, \ldots, n$ on a market. They manufacture some amounts of a product ($x_1, x_2, \ldots, x_n$, respectively) that correspond to their strategies. Denote by x the above strategy profile. Suppose that product price is a linear function, *viz.*, an initial price p minus the total amount of products $Q = \sum_{i=1}^{n} x_i$ multiplied by some factor b. Therefore, the unit price of the product makes up $p - bQ$. The cost prices of unit product will be indicated by $c_i, i = 1, \ldots, n$. The payoff functions of the players acquire the form

$$H_i(x) = (p - b\sum_{j=1}^{n} x_j)x_i - c_i x_i, \quad i = 1, \ldots, n.$$

Recall that the payoff functions $H_i(x)$ enjoy concavity in x_i, and the strategy set of player i is convex. Consequently, oligopolies represent an example of convex games with pure strategy equilibria. A Nash equilibrium satisfies the following system of equations:

$$\frac{\partial H_i(x_i^*)}{\partial x_i} = 0, \quad i = 1, \ldots, n. \tag{1.1}$$

Equations (1.1) bring to the expressions

$$p - c_i - b\sum_{j=1}^{n} x_j - bx_i = 0, \quad i = 1, \ldots, n. \tag{1.2}$$

By summing up these equalities, we arrive at

$$np - \sum_{j=1}^{n} c_j - b(n+1)\sum_{j=1}^{n} x_j = 0,$$

and it appears that

$$\sum_{j=1}^{n} x_j = \frac{np - \sum_{j=1}^{n} c_j}{b(n+1)}.$$

Thus, an equilibrium in the oligopoly model is given by

$$x_i^* = \frac{1}{b}\left(\frac{p}{n+1} - \left(c_i - \frac{\sum_{j=1}^{n} c_j}{n+1}\right)\right), \ i = 1, \dots, n. \tag{1.3}$$

The corresponding optimal payoffs become

$$H_i^* = b{x_i^*}^2, \ i = 1, \dots, n.$$

3.2 Polymatrix games

Consider n-player games $\Gamma =< N, \{X_i = \{1, 2, \dots, m_i\}\}_{i\in N}, \{H_i\}_{i\in N} >$, where players' strategies form finite sets and payoffs are defined by a set of multi-dimensional matrices $H_i = H_i(j_1, \dots, j_n), i \in N$. Such games are known as polymatrix games. They may have no pure strategy Nash equilibrium.

As previously, perform randomization and introduce the class of mixed strategies $\bar{X}_i = \{x^{(i)} = (x_1^i, \dots, x_{m_i}^{(i)})\}$. Here $x_j^{(i)}$ gives the probability that player i chooses strategy $j \in \{1, \dots, m_i\}$. Under a strategy profile $x = (x^{(1)}, \dots, x_n^{(n)})$, the expected payoff of player i takes the form

$$H_i(x^{(1)}, \dots, x^{(n)}) = \sum_{j_1=1}^{m_1} \dots \sum_{j_n=1}^{m_n} H_i(j_1, \dots, j_n) x_{j_1}^{(1)} \dots x_{j_n}^{(n)}, \ i = 1, \dots, n. \tag{2.1}$$

The payoff functions (2.1) appear concave, while the strategy set of players enjoys compactness and convexity. According to Theorem 3.1, the stated game always has a Nash equilibrium.

Theorem 3.2 *There exists a mixed strategy Nash equilibrium in an n-player polymatrix game* $\Gamma =< N, \{X_i = \{1, 2, \dots, m_i\}\}_{i\in N}, \{H_i\}_{i\in N} >$.

In a series of cases, it is possible to solve polymatrix games by analytic methods. For the time being, we explore the case when each player chooses between two strategies. Then equilibrium evaluation proceeds from geometric considerations. For simplicity of expositions, let us study three-player games $\Gamma =< N = \{I, II, III\}, \{X_i = \{1,2\}\}_{i=1,2,3}, \{H_i\}_{i=1,2,3} >$ with payoff matrices $H_1 = \{a_{ijk}\}_{i,j,k=1}^2$, $H_2 = \{b_{ijk}\}_{i,j,k=1}^2$, and $H_3 = \{c_{ijk}\}_{i,j,k=1}^2$. Recall that each player possesses just two strategies. And so, we comprehend mixed strategies as x_1, x_2, x_3—the probabilities of choosing strategy 1 by players I, II, and III, respectively. The opposite event occurs with the probability $\bar{x} = 1 - x$.

A strategy profile (x_1^*, x_2^*, x_3^*) is an equilibrium strategy profile, if for any strategies x_1, x_2, x_3 the following conditions take place:

$$
\begin{aligned}
H_1(x_1, x_2^*, x_3^*) &\le H_1(x^*),\\
H_2(x_1^*, x_2, x_3^*) &\le H_2(x^*),\\
H_3(x_1^*, x_2^*, x_3) &\le H_3(x^*).
\end{aligned}
$$

Particularly, the equilibrium conditions hold true under $x_i = 0$ and $x_i = 1$, $i = 1,2,3$. For instance, consider these inequalities for player III:

$$H_3(x_1^*, x_2^*, 0) \le H_3(x^*), \;\; H_3(x_1^*, x_2^*, 1) \le H_3(x^*).$$

According to (2.1), the first inequality acquires the form

$$
\begin{aligned}
&c_{112}x_1x_2 + c_{122}x_1\bar{x}_2 + c_{212}\bar{x}_1x_2 + c_{222}\bar{x}_1\bar{x}_2 \le c_{112}x_1x_2\bar{x}_3 + c_{122}x_1\bar{x}_2\bar{x}_3\\
&\quad + c_{212}\bar{x}_1x_2\bar{x}_3 + c_{222}\bar{x}_1\bar{x}_2\bar{x}_3 + c_{111}x_1x_2x_3 + c_{211}\bar{x}_1x_2x_3 + c_{121}x_1\bar{x}_2x_3 + c_{221}\bar{x}_1\bar{x}_2x_3.
\end{aligned}
$$

Rewrite it as

$$
\begin{aligned}
&x_3(c_{111}x_1x_2 + c_{211}\bar{x}_1x_2 + c_{121}x_1\bar{x}_2 + c_{221}\bar{x}_1\bar{x}_2\\
&\quad - c_{112}x_1x_2 - c_{122}x_1\bar{x}_2 - c_{212}\bar{x}_1x_2 - c_{222}\bar{x}_1\bar{x}_2) \ge 0.
\end{aligned}
\tag{2.2}
$$

Similarly, the second inequality becomes

$$
\begin{aligned}
&(1 - x_3)(c_{111}x_1x_2 + c_{211}\bar{x}_1x_2 + c_{121}x_1\bar{x}_2 + c_{221}\bar{x}_1\bar{x}_2\\
&\quad - c_{112}x_1x_2 - c_{122}x_1\bar{x}_2 - c_{212}\bar{x}_1x_2 - c_{222}\bar{x}_1\bar{x}_2) \le 0.
\end{aligned}
\tag{2.3}
$$

Denote by $C(x_1, x_2)$ the bracketed expression in (2.2) and (2.3). For $x_3 = 0$, inequality (2.2) is immediate, whereas (2.3) brings to

$$C(x_1, x_2) \le 0. \tag{2.4}$$

Next, for $x_3 = 1$, we directly have (2.3), while inequality (2.2) requires that

$$C(x_1, x_2) \ge 0. \tag{2.5}$$

And finally, for $0 < x_3 < 1$, inequalities (2.2)–(2.3) dictate that

$$C(x_1, x_2) = 0. \tag{2.6}$$

The conditions (2.4)–(2.6) determine the set of strategy profiles acceptable for player *III*. By analogy, we can derive the conditions and corresponding sets of acceptable strategy profiles for players *I* and *II*. Subsequently, all equilibrium strategy profiles can be defined by their intersection. Let us provide an illustrative example.

Struggle for markets. Imagine that companies *I*, *II*, and *III* manufacture some product and sell it on market *A* or market *B*. In comparison with the latter, the former market is characterized by doubled product price. However, the payoff on any market appears inversely proportional to the number of companies that have selected this market. Notably, the payoff of a company on market *A* makes up 6, 4, or 2, if one, two, or three companies, respectively, do operate on this market; the corresponding payoffs on market *B* are 3, 2, and 1, respectively.

Construct the set of strategy profiles acceptable for player *III*. In the present case, $C(x_1, x_2)$ is given by

$$2x_1x_2 + 4\bar{x}_1x_2 + 4x_1\bar{x}_2 + 6\bar{x}_1\bar{x}_2 - 3x_1x_2 - 2x_1\bar{x}_2 - 2\bar{x}_1x_2 - \bar{x}_1\bar{x}_2 = -3x_1 - 3x_2 + 5.$$

The conditions (2.4)–(2.6) take the form

$$x_3 = 0,\ x_1 + x_2 \geq \frac{5}{3},$$

$$x_3 = 1,\ x_1 + x_2 \leq \frac{5}{3},$$

$$0 < x_3 < 1,\ x_1 + x_2 = \frac{5}{3}.$$

Figure 3.1 demonstrates the set of strategy profiles acceptable for player *III*. Owing to problem symmetry, similar conditions and sets of acceptable strategy profiles apply to players *I* and *II*. The intersection of these sets yields four equilibrium strategy profiles. Three of them represent pure strategy equilibria, *viz*., (A,A,B), (A,B,A), (B,A,A). And the fourth one is a mixed strategy profile: $x_1 = x_2 = x_3 = \frac{5}{6}$. Under the first free equilibria, players *I* and *II* have

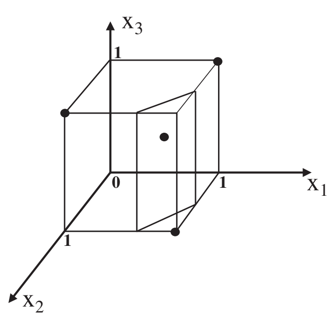


Figure 3.1 The set of strategy profiles acceptable for player *III*.

the payoff of 4, whereas player *III* gains 3. With the fourth equilibrium, all players receive the identical payoff:

$$H^* = 2(5/6)^3 + 4(1/6)(5/6)^2 + 4(1/6)(5/6)^2 + 6(1/6)^2(5/6) \\ + 3(5/6)^2(1/6) + 2(1/6)(5/6)^2 + 2(1/6)(5/6)^2 + (1/6)^3 = 8/3.$$

In a pure strategy equilibrium above, some player gets the worst of the game (his payoff is smaller as against the opponents). The fourth equilibrium is remarkable for the following. All players enjoy equal rights; nevertheless, their payoff appears smaller even than the minimal payoff of the players under a pure strategy equilibrium.

3.3 Potential games

Games with potentials were studied by Monderer and Shapley [1996]. Consider a normal-form n-player game $\Gamma = < N, \{X_i\}_{i\in N}, \{H_i\}_{i\in N} >$. Suppose that there exists a certain function $P: \prod_{i=1}^{n} X_i \to R$ such that for any $i \in N$ we have the inequality

$$H_i(x_{-i}, x_i') - H_i(x_{-i}, x_i) = P(x_{-i}, x_i') - P(x_{-i}, x_i) \tag{3.1}$$

for arbitrary $x_{-i} \in \prod_{j\neq i} X_j$ and any strategies $x_i, x_i' \in X_i$. If this function exists, it is called the **potential** of the game Γ, whereas the game proper is referred to as a **potential game**.

Traffic jamming. Suppose that companies *I* and *II*, each possessing two trucks, have to deliver some cargo from point A to point B. These points communicate through two roads (see Figure 3.2), and one road allows a two times higher speed than the other. Moreover, assume that the journey time on any road is proportional to the number of trucks moving on it. Figure 3.2 indicates the journey time on each road depending on the number of moving trucks.

Therefore, players choose the distribution of their trucks by roads as their strategies. And so, the possible strategies of players are one of the combinations $(2, 0), (1, 1), (0, 2)$. The costs of a player equal the total journey time of both his trucks. Consequently, the payoff matrix is determined by

$$\begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} \quad(2,0) & \quad(1,1) & \quad(0,2) \end{array} \\ \begin{pmatrix} (-8,-8) & (-6,-5) & (-4,-8) \\ (-5,-6) & (-6,-6) & (-7,-12) \\ (-8,-4) & (-12,-7) & (-16,-16) \end{pmatrix} \end{array}.$$

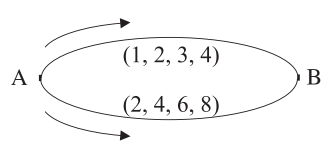


Figure 3.2 Traffic jamming.

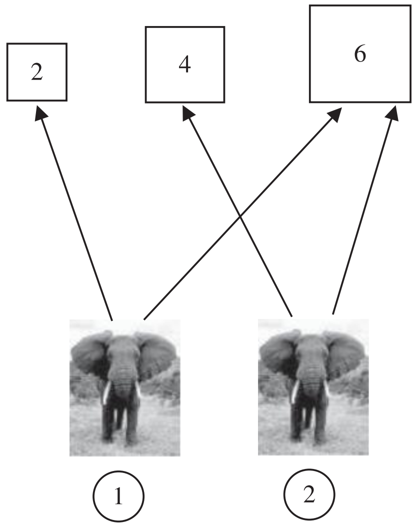


Figure 3.3 Animal foraging.

Obviously, the described game admits three pure strategy equilibria. These are strategy profiles, where (a) the trucks of one player move on road 1, whereas the other player chooses different roads for his trucks, and (b) both players select different roads for their trucks.

The game in question possesses the potential

$$
P = \begin{array}{c} \\ (2,0) \\ (1,1) \\ (0,2) \end{array} \begin{array}{c} \begin{array}{ccc} (2,0) & (1,1) & (0,2) \end{array} \\ \begin{pmatrix} 13 & 16 & 13 \\ 16 & 16 & 10 \\ 13 & 10 & 1 \end{pmatrix} \end{array}.
$$

Animal foraging. Two animals choose one or two areas among three areas for their foraging (see Figure 3.3). These areas provide 2, 4 and 6 units of food, respectively. If both animals visit a same area, they equally share available food. The payoff of each player is the total units of food gained at each area minus the costs to visit this area (we set them equal to 1).

Therefore, the strategies of players lie in choosing areas for their foraging: $(1), (2), (3), (1,2), (1,3)$, and $(2,3)$. And the payoff matrix becomes

$$
\begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\ \begin{pmatrix} (0,0) & (1,3) & (1,5) & (0,3) & (0,5) & (1,8) \\ (3,1) & (1,1) & (3,5) & (1,2) & (3,6) & (1,6) \\ (5,1) & (5,3) & (2,2) & (5,4) & (2,3) & (2,5) \\ (3,0) & (2,1) & (4,5) & (1,1) & (3,5) & (2,6) \\ (5,0) & (6,3) & (3,2) & (5,3) & (2,2) & (3,5) \\ (8,1) & (6,1) & (5,2) & (6,2) & (5,3) & (3,3) \end{pmatrix} \end{array}.
$$

This game has three pure strategy equilibria. In the first one, both animals choose areas 2 and 3. In the second and third pure strategy equilibria, one player selects areas 1 and 3, while the other chooses areas 2 and 3. The game under consideration also admits the potential

$$P = \begin{array}{c} \\ (1) \\ (2) \\ (3) \\ (1,2) \\ (1,3) \\ (2,3) \end{array} \begin{array}{c} \begin{array}{cccccc} (1) & (2) & (3) & (1,2) & (1,3) & (2,3) \end{array} \\ \begin{pmatrix} 1 & 4 & 6 & 4 & 6 & 9 \\ 4 & 4 & 8 & 5 & 9 & 9 \\ 6 & 8 & 7 & 9 & 8 & 10 \\ 4 & 5 & 9 & 5 & 9 & 10 \\ 6 & 9 & 8 & 9 & 8 & 11 \\ 9 & 9 & 10 & 10 & 11 & 11 \end{pmatrix} \end{array}.$$

Theorem 3.3 *Let an n-player game $\Gamma =< N, \{X_i\}_{i\in N}, \{H_i\}_{i\in N} >$ have a potential P. Then a Nash equilibrium in the game Γ represents a Nash equilibrium in the game $\Gamma' =< N, \{X_i\}_{i\in N}, P >$, and vice versa. Furthermore, the game Γ admits at least one pure strategy equilibrium.*

Proof: The first assertion follows from the definition of a potential. Indeed, due to (3.1), the conditions

$$H_i(x^*_{-i}, x_i) \le H_i(x^*), \forall x_i,$$

and

$$P(x^*_{-i}, x_i) \le P(x^*), \forall x_i$$

do coincide. Hence, if x^* is a Nash equilibrium in the game Γ, it forms a Nash equilibrium in the game Γ', and vice versa.

Now, we argue that the game Γ' always has a pure strategy equilibrium. Let x^* be the pure strategy profile maximizing the potential $P(x)$ on the set $\prod_{i=1}^{n} X_i$. For any $x \in \prod_{i=1}^{n} X_i$, the inequality $P(x) \le P(x^*)$ holds true at this point, particularly,

$$P(x^*_{-i}, x_i) \le P(x^*), \forall x_i.$$

Therefore, x^* represents a Nash equilibrium in the game Γ' and, hence, in the game Γ.

And so, if the game admits a potential, it necessarily has a pure strategy equilibrium. For instance, revert to the examples of traffic jamming and animal foraging.

The Cournot oligopoly. In the previous section, we have considered the Cournot oligopoly with the payoff functions

$$H_i(x) = (p - b\sum_{j=1}^{n} x_j)x_i - c_i x_i, \quad i = 1, \dots, n.$$

This game is potential, as well. Here potential makes the function

$$P(x_1,\dots,x_n)=\sum_{j=1}^{n}(p-c_j)x_j-b\left(\sum_{j=1}^{n}x_j^2+\sum_{1\le i<j\le n}x_ix_j\right). \tag{3.2}$$

Really, the functions $H_i(x)$ and $P(x)$ appear quadratic in the variable x_i, and their derivatives coincide:

$$\frac{\partial H_i}{\partial x_i}=\frac{\partial P}{\partial x_i}=p-c_i-2bx_i-b\sum_{j\neq i}x_j,\ i=1,\dots,n. \tag{3.3}$$

Consequently, the functions $H_i(x)$ and $P(x)$ possess the same values with respect to each variable x_i (to some constant), i.e.,

$$H_i(x_{-i},x_i)-H_i(x_{-i},x_i')=P(x_{-i},x_i)-P(x_{-i},x_i'),\forall x_i.$$

Thus, the function (3.2) gives potential in the oligopoly model. According to Theorem 3, an equilibrium follows from maximization of the function (3.2). The first-order necessary optimality conditions

$$\frac{\partial P}{\partial x_i}=p-c_i-2bx_i-b\sum_{j\neq i}x_j=0,\ i=1,\dots,n$$

bring to the expressions (1.2); in the preceding section, they have yielded an equilibrium in the oligopoly model (1.3).

A game without potential. Note that a game may have no potential, even if a pure strategy equilibrium does exist. Get back to the example of traffic jamming (see Figure 3.2). Within the same framework, suppose that the costs of players are defined by the maximal journey time of their trucks on both roads. In this case, the payoff matrix becomes

$$\begin{array}{cc} & \begin{array}{ccc}(2,0) & (1,1) & (0,2)\end{array} \\ \begin{array}{c}(2,0)\\(1,1)\\(0,2)\end{array} & \begin{pmatrix}(-4,-4) & (-3,-3) & (-2,-4)\\(-3,-3) & (-4,-4) & (-6,-6)\\(-4,-2) & (-6,-6) & (-8,-8)\end{pmatrix}.\end{array}$$

Such game admits two pure strategy equilibria. These are strategy profiles, where the trucks of one player move on the first road, whereas the other player chooses different roads for his trucks. Nevertheless, the described game has no potential. We demonstrate this fact rigorously. Assume that a potential P exists; then definition (3.1) implies that

$$P(1,1)-P(3,1)=H_1(1,1)-H_1(3,1)=-4-(-4)=0,$$

$$P(1,1)-P(1,2)=H_2(1,1)-H_2(1,2)=-4-(-3)=-1.$$

And so,

$$P(3,1) - P(1,2) = -1. \tag{3.4}$$

On the other hand,

$$P(1,2) - P(3,2) = H_1(1,2) - H_1(3,2) = -3 - (-6) = 3,$$

and

$$P(3,1) - P(3,2) = H_2(3,1) - H_2(3,2) = -2 - (-6) = 4,$$

whence it follows that

$$P(3,1) - P(1,2) = 1.$$

This evidently contradicts the expression (3.4), and the game possesses no potential. Interestingly, in contrast to the earlier example, here the costs are not in the additive form.

3.4 Congestion games

Congestion games have been pioneered by Rosenthal [1973]. The term "congestion game" has the following origin. In such games, payoff functions depend only on the number of players choosing identical strategies. For instance, this class comprises routing games during transition between two points (journey time depends on the number of automobiles on a given route). The animal foraging game also belongs to congestion games—in a given area, the amount of food resources acquired by an animal depends on the total number of animals occupying this area.

Definition 3.3 *A symmetrical congestion game is an n-player game*

$$\Gamma =< N, M, \{S_i\}_{i\in N}, \{c_i\}_{i\in N} >,$$

where $N = \{1, \dots, n\}$ stands for the set of players, and $M = \{1, \dots, m\}$ means the set of some objects for strategy formation. A strategy of player i is the choice of a certain subset from M. The set of all feasible strategies makes the strategy set of player i, denoted by S_i, $i = 1, \dots, n$. Each object $j \in M$ is associated with a function $c_j(k)$, $1 \le k \le n$, which represents the payoff (or costs) of each player from k players that have selected strategies containing j. This function depends only on the total number k of such players.

Imagine that players have chosen strategies $s = (s_1, \dots, s_n)$. Each s_i forms a set of objects from M. Then the payoff function of player i is determined by the total payoff on each object:

$$H_i(s_1, \dots, s_n) = \sum_{j\in s_i} c_j(k_j(s_1, \dots, s_n)).$$

Here $k_j(s_1, \dots, s_n)$ gives the number of players whose strategies incorporate object j, $i = 1, \dots, n$.

Theorem 3.4 *A symmetrical congestion game is potential, ergo admits a pure strategy equilibrium.*

Proof: Consider the function

$$P(s_1, \dots, s_n) = \sum_{j \in \cup_{i \in N} s_i} \left(\sum_{k=1}^{k_j(s_1, \dots, s_n)} c_j(k) \right)$$

and demonstrate that this is a potential of the game. Let us verify the conditions (2.1). On the one part,

$$H_i(s_{-i}, s_i') - H_i(s_{-i}, s_i) = \sum_{j \in s_i'} c_j(k_j(s_{-i}, s_i')) - \sum_{j \in s_i} c_j(k_j(s_{-i}, s_i)).$$

For all $j \in s_i \cap s_i'$, the payoffs c_j in the first and second sums are identical. Therefore,

$$H_i(s_{-i}, s_i') - H_i(s_{-i}, s_i) = \sum_{j \in s_i' \setminus s_i} c_j(k_j(s) + 1) - \sum_{j \in s_i \setminus s_i'} c_j(k_j(s)).$$

Accordingly, we find

$$P(s_{-i}, s_i') - P(s_{-i}, s_i) = \sum_{j \in \cup_{l \neq i} s_l \cup s_i'} \left(\sum_{k=1}^{k_j(s_{-i}, s_i')} c_j(k) \right) - \sum_{j \in \cup_{l \in N} s_l} \left(\sum_{k=1}^{k_j(s_{-i}, s_i)} c_j(k) \right).$$

Under $j \notin s_i \cup s_i'$, the corresponding summands in these expressions coincide, which means that

$$P(s_{-i}, s_i') - P(s_{-i}, s_i) = \sum_{j \in s_i \cup s_i'} \left(\sum_{k=1}^{k_j(s_{-i}, s_i')} c_j(k) - \sum_{k=1}^{k_j(s_{-i}, s_i)} c_j(k) \right)$$
$$= \sum_{j \in s_i' \setminus s_i} \left(\sum_{k=1}^{k_j(s_{-i}, s_i')} c_j(k) - \sum_{k=1}^{k_j(s_{-i}, s_i)} c_j(k) \right) + \sum_{j \in s_i \setminus s_i'} \left(\sum_{k=1}^{k_j(s_{-i}, s_i')} c_j(k) - \sum_{k=1}^{k_j(s_{-i}, s_i)} c_j(k) \right).$$

In the case of $j \in s_i' \setminus s_i$, we have $k_j(s_{-i}, s_i') = k_j(s) + 1$; if $j \in s_i \setminus s_i'$, the equality $k_j(s_{-i}, s_i') = k_j(s) - 1$ takes place. Consequently,

$$P(s_{-1}, s_i') - P(s_{-1}, s_i) = \sum_{j \in s_i' \setminus s_i} \left(\sum_{k=1}^{k_j(s)+1} c_j(k) - \sum_{k=1}^{k_j(s)} c_j(k) \right) + \sum_{j \in s_i \setminus s_i'} \left(\sum_{k=1}^{k_j(s)-1} c_j(k) - \sum_{k=1}^{k_j(s)} c_j(k) \right)$$
$$= \sum_{j \in s_i' \setminus s_i} c_j(k_j(s) + 1) - \sum_{j \in s_i \setminus s_i'} c_j(k_j(s).$$

This result matches the expression of $H_i(s_{-i}, s_i') - H_i(s_{-i}, s_i)$. The proof of Theorem 3.4 is finished.

Thus, symmetrical congestion games admit (at least) one pure strategy equilibrium. Generally speaking, a mixed strategy equilibrium may exist, as well. In applications, the major role belongs to the existence of pure strategy equilibria. We also acknowledge that the existence of such equilibria is connected with (a) the additive form of payoff functions and (b) the homogeneous form of players' payoffs in symmetrical games. To continue, let us explore player-specific congestion games, where different players may have different payoffs. Our analysis focuses on the case of simple strategies—each player chooses merely one object from the set $M = \{1, \dots, m\}$.

3.5 Player-specific congestion games

Definition 3.4 *A player-specific congestion game is an n-player game*

$$\Gamma =< N, M, \{c_{ij}\}_{i\in N, j\in M} >,$$

where $N = \{1, \dots, n\}$ designates the set of players and $M = \{1, \dots, m\}$ specifies the finite set of objects. The strategy of player i is choosing some object from M. Therefore, M can be interpreted as the set of players' strategies. The payoff of player i selecting strategy j is defined by a function $c_{ij} = c_{ij}(k_j)$, where k_j denotes the number of players employing strategy j, $0 \le k_j \le n$. For the time being, suppose that c_{ij} represent non-increasing functions. In other words, the more players have chosen a given strategy, the smaller is the payoff.

Denote by $s = (s_1, \dots, s_n)$ the strategy profile composed of strategies selected by players. Each strategy profile s corresponds to the **congestion vector** $k = (k_1, \dots, k_m)$, where k_j makes up the number of players choosing strategy j. Then the payoff function of player i is defined by

$$H_i(s_1, \dots, s_n) = c_{is_i}(k_{s_i}),\ \ i = 1, \dots, n.$$

We are concerned with pure strategy profiles. For such games, the definition of a pure strategy Nash equilibrium can be reformulated as follows. In an equilibrium s^*, any player i does not benefit by deviating from the optimal strategy s_i^*. And so, the optimal strategy payoff $c_{is_i^*}(k_{s_i^*})$ is not smaller than the one ensured by any other strategy j of all players, i.e., $c_{ij}(k_j + 1)$.

Definition 3.5 *A Nash equilibrium in a game $\Gamma =< N, M, \{c_{ij}\}_{i\in N, j\in M} >$ is a strategy profile $s^* = (s_1^*, \dots, s_n^*)$, where the following conditions hold true for any player $i \in N$:*

$$c_{is_i^*}(k_{s_i^*}) \ge c_{ij}(k_j + 1), \forall j \in M. \tag{5.1}$$

We provide another constructive conception proposed by Monderer and Shapley [1996]. It will serve to establish equilibrium existence in congestion games.

Definition 3.6 *Suppose that a game $\Gamma =< N, M, \{c_{ij}\}_{i\in N, j\in M} >$ has a sequence of strategy profiles $s(t), t = 0, 1, \dots,$ where (a) each profile differs from the preceding one in a single* *component and (b) the payoff of a player that has modified his strategy is strictly higher. Then such sequence is called an improvement sequence. If any improvement sequence in* Γ *is finite, we say that this game meets the final improvement property (FIP).*

Clearly, if an improvement sequence is finite, then the terminal strategy profile represents a Nash equilibrium (it meets the conditions (4.1)). However, there exist games with Nash equilibria, which do not enjoy the FIP. In such games, improvement sequences can be infinite and have cyclic repetitions. This fact follows from finiteness of strategy sets.

Nevertheless, games Γ with two-element strategy sets M demonstrate the FIP.

Theorem 3.5 *A player-specific congestion game* $\Gamma =< N, M, \{c_{ij}\}_{i\in N, j\in M} >$*, where* $M = \{1, 2\}$*, admits a pure strategy Nash equilibrium.*

Proof: We show that a congestion game with two strategies possesses the FIP. Suppose that this is not the case. In other words, there exists an infinite improvement sequence $s(0), s(1), \ldots$. Extract its cyclic subsequence $s(0), \ldots, s(T)$, i.e., $s(0) = s(T)$ and $T > 1$. In this chain, each strategy profile $s(t), t = 1, \ldots, T$ corresponds to a congestion vector $k(t) = (k_1(t), k_2(t)), t = 0, \ldots, T$. Obviously, $k_2 = n - k_1$. Find the element with the maximal value of $k_2(t)$. Without loss of generality, we believe that such element is $k_2(1)$. Otherwise, just renumber the elements of the sequence owing to its cyclic character. Then $k_1(1) = n - k_2(1)$ makes the minimal element in the chain. And so, at the initial instant player i switches from strategy 1 to strategy 2, i.e.,

$$c_{i2}(k_2(1)) > c_{i1}(k_1(1) + 1). \tag{5.2}$$

Since $k_2(1) \geq k_2(t), \forall t$, the monotonous property of the payoff function implies that

$$c_{i2}(k_2(1)) \leq c_{i2}(k_2(t)),\ t = 0, \ldots, T.$$

On the other hand,

$$c_{i1}(k_1(1) + 1) \geq c_{i1}(k_1(t) + 1),\ t = 0, \ldots, T.$$

In combination with (5.2), this leads to the inequality $c_{i2}(k_2(t)) > c_{i1}(k_1(t) + 1), t = 0, \ldots T$, i.e., player i strictly follows strategy 2. But at the initial instant $t = 0$, *ergo* at the instant $t = T$, he applied strategy 1.

The resulting contradiction indicates that a congestion game with two strategies necessarily enjoys the FIP. Consequently, such a game has a pure strategy equilibrium profile.

We emphasize a relevant aspect. The proof of Theorem 3.5 is based on the maximal congestion of strategy 1 corresponding to the minimal congestion of strategy 2. Generally speaking, this fails even for games with three strategies. Congestion games with three and more strategies may disagree with the FIP.

A congestion game without the FIP. Consider a two-player congestion game with two strategies and the payoff matrix

$$\begin{pmatrix} (0,4) & (5,6) & (5,3) \\ (4,5) & (3,1) & (4,3) \\ (2,5) & (2,6) & (1,2) \end{pmatrix}$$

This game has the infinite cyclic improvement sequence

$$(1,1) \to (3,1) \to (3,2) \to (2,2) \to (2,3) \to (1,3) \to (1,1).$$

Therefore, it does not satisfy the FIP. Still, there exist two (!) pure strategy equilibrium profiles: $(1,2)$ and $(2,1)$.

To establish equilibrium existence in the class of pure strategies in the general case, we introduce a stronger solution improvement condition. Notably, assume that each player in an improvement sequence chooses the best response under a given strategy profile (ensuring his maximal payoff). If there are several best responses, a player selects one of them. Such improvement sequence will be called a best response sequence.

Definition 3.7 *Suppose that a game $\Gamma =< N, M, \{c_{ij}\}_{i\in N, j\in M} >$ admits a sequence of strategy profiles $s(t), t = 0, 1, \dots$ such that (a) each profile differs from the preceding one in a single component and (b) the payoff of a player that has modified his strategy is strictly higher, gaining the maximal payoff to him in this strategy profile. Such sequence is called a best response sequence. If any improvement sequence in the game Γ appears finite, we say that this game meets the final best-reply property (FBRP).*

Evidently, any best response sequence forms an improvement sequence. The opposite statement is false. Now, we prove the basic result.

Theorem 3.6 *A player-specific congestion game $\Gamma =< N, M, \{c_{ij}\}_{i\in N, j\in M} >$ has a pure strategy Nash equilibrium.*

Proof: Apply induction by the number of players. For $n = 1$, the assertion becomes trivial, player chooses the best strategy from M. Hypothesize this result for player $n - 1$ and prove it for player n.

Consider an n-player game $\Gamma =< N, M, \{c_{ij}\}_{i\in N, j\in M} >$. First, eliminate player n from further analysis. In the reduced game Γ' with $n - 1$ players and m strategies, the induction hypothesis implies that there exists an equilibrium $s' = (s_1(0), \dots, s_{n-1}(0))$. Denote by $k'(0) = (k_1'(0), \dots, k_m'(0))$ the corresponding congestion vector. Then $c_{is_i(0)}(k'_{s_i(0)}) \geq c_{ij}(k_j' + 1), \forall j \in M, i = 1, \dots, n-1$. Revert to the game Γ and let player n choose the best response (the strategy $j(0) = s_n(0)$). Consequently, just one component varies in the congestion vector $k' = (k_1', \dots, k_m')$ (component $j(0)$ increases by unity).

Now, construct the best response sequence. The initial term of the sequence takes the form $s(0) = (s_1(0), \dots, s_{n-1}(0), s_n(0))$. The corresponding congestion vector will be designated by $k(0) = (k_1(0), \dots, k_m(0))$. In the strategy profile $s(0)$, payoffs possibly decrease only for players having the strategy $j(0)$; the rest of the players obtain the same payoff and do not benefit by modifying their strategy. Suppose that player i_1 (actually applying the strategy $j(0)$) can guarantee a higher payoff by another strategy. If such a player does not exist, an equilibrium in the game Γ is achieved. Select his best response $j(1)$ and denote by $s(1)$ the new strategy profile. In the corresponding congestion vector $k(1)$, component $j(0)$ (component $j(1)$) decreases (increases, respectively) by unity. Under the new strategy profile $s(1)$, payoffs can be improved only by players adhering to the strategy $j(1)$. The rest players gain the same payoffs (in comparison with the original strategy profile). Assume that player i_2 can improve his payoff. Choose his best response $j(2)$, and continue the procedure.

Therefore, we have built the best response sequence $s(t)$. It corresponds to a sequence of congestion vectors $k(t)$, where any component $k_j(t)$ either equals the value at the initial instant $k'_j(0)$, or exceeds it by unity. The last situation occurs, if at the instant $t-1$ player i_t switches to the strategy $j(t)$. In other cases, the number of players employing the strategy $j \neq j(t)$ constitutes $k'_j(0)$. Interestingly, each player can switch to another strategy just once. Indeed, imagine that at the instant $t-1$ player i_t switches to the strategy j; then at the instant t this strategy is adopted by the maximal number of players. Hence, at the subsequent instants the number of players with such strategy remains the same or even goes down (accordingly, the number of players choosing other strategies appears the same or goes up). Due to the monotonicity of payoff functions, player i_t is unable to get higher payoff. The game involves a finite number of players; hence, the resulting best response sequence is finite: $s(t), t = 1, 2, \dots, T$, where $T \leq n$.

There may exist several sequences of this form. Among them, take the one $s(t), t = 1, \dots, T$ with the maximal value of T. Finally, demonstrate that the last strategy profile $s(T) = (s_1(T), \dots, s_n(T))$ is a Nash equilibrium.

We have mentioned that players deviating from their original strategies would not increase their payoffs in the strategy profile $s(T)$. And so, consider players preserving their strategies during the period of T. Suppose that, among them, there is a player belonging to the group with the strategy $j(T)$; if he improves his payoff, we would extend the best response sequence to the instant $T+1$. However, this contradicts the maximality of T. Assume that, among them, there is a player belonging to the group with a strategy $j \neq j(T)$. The number of players in this group is the same as at the initial instant (see the discussion above). And this player is unable to increase his payoff, as well.

Therefore, we have argued that, under the strategy profile $s(T) = (s_1(T), \dots, s_n(T))$, any player $i = 1, \dots, n$ meets the conditions

$$c_{is_i(T)}(k_{s_i(T)}(T)) \geq c_{ij}(k_j(T)+1), \forall j \in M.$$

Consequently, $s(T)$ represents a Nash equilibrium for n players. The proof of Theorem 3.6 is finished.

3.6 Auctions

We analyze non-cooperative n-player games in the class of mixed strategies. The present section deals with models of auctions. For simplicity, consider the symmetrical case when all

n players are in identical conditions. An auction offers for sale some item possessing a same value V for all players. Players simultaneously bid for the item (suggest prices $(x_1, \dots, x_n)$, respectively). The item passes to a player announcing the highest price. As a matter of fact, there are different schemes of auctions. We will study first-price auctions and second-price auctions.

First-price auction. Imagine the following auction rules. A winner (a player suggesting the maximal price) gets the item and actually pays nothing. The rest players have to pay the price they have announced (for participation). If several players bid the maximal price, they equally share the payoff. And so, the payoff function of this game acquires the form

$$H_i(x_1, \dots, x_n) = \begin{cases} -x_i, & \text{if } x_i < y_{-i}, \\ \frac{V}{m_i(x)} - x_i, & \text{if } x_i = y_{-i}, \\ V, & \text{if } x_i > y_{-i}, \end{cases} \tag{6.1}$$

where $y_{-i} = \max_{j \neq i}\{x_j\}$ and $m_i(x)$ is the number of players whose bids coincide with x_i, $i = 1, \dots, n$. Obviously, the game admits no pure strategy equilibrium, and we search in the class of mixed strategies. By virtue of symmetry, consider player 1 only.

Suppose that players $\{2, \dots, n\}$ apply a same mixed strategy with a distribution function $F(x), x \in [0, \infty)$. The payoff of player 1 depends on the distribution of $y_{-1} = \max\{x_2, \dots, x_n\}$. Clearly, the distribution of this maximum is simply $F_{n-1}(x) = F^{n-1}(x)$. Accordingly, the bid of player 1 turns out maximal with the probability of $[F(x)]^{n-1}$, and he gains the payoff V. Another player announces a higher price with the probability of $1 - [F(x)]^{n-1}$, and player 1 would have to pay x. Under his pure strategy x, player 1 has the payoff function

$$H_1(x, \overbrace{F, \dots, F}^{n-1}) = V[F(x)]^{n-1} - x\left(1 - [F(x)]^{n-1}\right) = (V + x)[F(x)]^{n-1} - x. \tag{6.2}$$

The following sufficient condition guarantees that the strategy profile $(F(x), \dots, F(x))$ forms an equilibrium:

$$H_1(x, \overbrace{F, \dots, F}^{n-1}) = const \text{ or } \partial H_1(x, \overbrace{F, \dots, F}^{n-1})/\partial x = 0.$$

The last expression brings to the differential equation

$$\frac{dF_{n-1}(x)}{dx} = \frac{1 - F_{n-1}(x)}{V + x}, \quad 0 \le x < \infty$$

with the boundary condition $F_{n-1}(0) = 0$. Here integration yields

$$F_{n-1}(x) = \frac{x}{V + x}.$$

Hence, the optimal mixed strategy is defined by

$$F^*(x) = \left(\frac{x}{V+x}\right)^{1/(n-1)},$$

while the density function of this distribution becomes

$$f^*(x) = \frac{1}{n-1}\left(\frac{x}{V+x}\right)^{-\frac{n-2}{n-1}}.$$

Substitute the derived distribution into (6.2) to find $H_1(x, \overbrace{F^*, \dots, F^*}^{n-1}) = 0$ for any $x \geq 0$. Therefore, player 1 receives zero payoff regardless of his mixed strategy. And so, the game has zero value.

Theorem 3.7 *A first-price auction with the payoff function (6.1) admits the mixed strategy equilibrium*

$$F^*(x) = \left(\frac{x}{V+x}\right)^{1/(n-1)},$$

and the game value is zero.

Second-price auction. Here all players pay their announced prices for participation in an auction, while a winner pays merely the second highest price. Such auctions are called Vickrey auctions. If several players make the maximal bid, they share V equally.

Therefore, the payoff function takes the form

$$H_i(x_1, \dots, x_n) = \begin{cases} -x_i, & \text{if } x_i < y_{-i}, \\ \frac{V}{m_i} - x_i, & \text{if } x_i = y_{-i}, \\ V - y_{-i}, & \text{if } x_i > y_{-i}, \end{cases} \tag{6.3}$$

where $y_{-i} = \max_{j \neq i}\{x_j\}$ and m_i have the same interpretations as in the first-price auction model. Unfortunately, Vickrey auctions admit no pure strategy equilibria. If all bids do not exceed V, one should maximally increase the bid; however, if at least one bid is higher than V, it is necessary to bid zero price. Let us evaluate a mixed strategy equilibrium. Again, symmetry enables considering just player 1.

Suppose that players $\{2, \dots, n\}$ adopt a same mixed strategy with some distribution function $F(x), x \in [0, \infty)$. The payoff of player 1 depends on the distribution of the variable $y_{-1} = \max\{x_2, \dots, x_n\}$. Recall that its distribution is simply $F_{n-1}(x) = F^{n-1}(x)$ (see the discussion above). Now, we express the payoff of player 1 under the mixed strategy x:

$$H_1(x, \overbrace{F, \dots, F}^{n-1}) = \int_0^x (V - t)dF_{n-1}(t) - \int_x^\infty x dF_{n-1}(t).$$

Since the support of $F(x)$ is $[0, \infty)$, the sufficient condition of equilibrium existence $(H_1(x, \overbrace{F, \dots, F}^{n-1}) = const$ or, equivalently, $\partial H_1(x, \overbrace{F, \dots, F}^{n-1}))/\partial x = 0)$ naturally leads to the differential equation

$$\frac{dF_{n-1}(x)}{dx} = \frac{1 - F_{n-1}(x)}{V}.$$

Its general solution possesses the form

$$F_{n-1}(x) = 1 - c \exp\left(-\frac{x}{V}\right).$$

So long as $F(0) = 0$, we find $F_{n-1}(x) = 1 - \exp(-\frac{x}{V})$. And so, the function $F^*(x)$ is defined by

$$F^*(x) = \left(1 - \exp\left(-\frac{x}{V}\right)\right)^{\frac{1}{n-1}}. \tag{6.4}$$

Therefore, if players $\{2, \dots, n\}$ adhere to the mixed strategy $F^*(x)$, the payoff of player 1 turns out constant: $H_1(x, \overbrace{F^*, \dots, F^*}^{n-1}) = 0$. Regardless of the strategy selected by player 1, his payoff in this strategy profile equals zero. This means optimality of the strategies $F^*(x)$.

Theorem 3.8 *Consider a second-price auction with the payoff function (6.3). An equilibrium consists in the mixed strategies*

$$F^*(x) = \left(1 - \exp\left(-\frac{x}{V}\right)\right)^{\frac{1}{n-1}}.$$

For $n = 2$, the density function of (6.4) takes the form $f^*(x) = V^{-1}e^{-x/V}$. In the case of $n \geq 3$, we obtain

$$f^*(x) = \frac{1}{n-1}\left(1 - e^{-x/V}\right)^{\frac{1}{n-1}-1} \cdot \frac{1}{V}e^{-x/V} \to \begin{cases} +\infty, & \text{if } \ x \downarrow 0 \\ 0, & \text{if } \ x \uparrow \infty. \end{cases}$$

Despite slight difference in the conditions of the above auctions, the corresponding optimal strategies vary appreciably. In the former case, the matter concerns power functions, whereas the latter case yields an exponential distribution. Surprisingly, both optimal strategies can bring to bids exceeding a given value of V if $n = 2$. In first-price auctions, this probability makes up $1 - F^*(V) = 1 - (1/2)^{-1} = 0.5$. In second-price auctions, it is smaller: $1 - F^*(V) = 1 - (1 - \exp(-1))^{1/(n-1)} \approx 0.368$.

3.7 Wars of attrition

Actually, there exists another biological interpretation of the game studied in Section 3.5. This model is close to the animal competition model for some resource V, suggested by British biologist M. Smith.

Assume that $V = V(x)$, a positive decreasing function of x, represents a certain resource on a given area. Next, n animals (players) struggle for this resource. The game runs on unit interval. Animal i shows its strength for a specific period $x_i \in [0, 1]$, $i = 1, \dots, n$. The resource is captured by the animal with the longest strength period. The costs of players are proportional to their strength periods, and winner's costs coincide with the period when the last competitor "leaves the battlefield."

We seek for a mixed strategy equilibrium as the distribution functions

$$F(x) = I(0 \le x < a) \int_0^x h(t)dt + I(a \le x \le 1),$$

where a is some value from $[0, 1]$ and I_A denotes the indicator of event A. Imagine that all players $\{2, \dots, n\}$ adopt a same strategy F, while player 1 chooses a pure strategy $x \in [0, 1]$. His expected payoff becomes

$$H_1(x, \overbrace{F, \dots, F}^{n-1}) = \begin{cases} \displaystyle\int_0^x (V(x) - t)d\,(F(t))^{n-1} - x\left\{1 - (F(x))^{n-1}\right\}, & \text{if } 0 \le x < a, \\ \displaystyle\int_0^a (V(x) - t)\,d\,(F(t))^{n-1}, & \text{if } a < x \le 1, \end{cases} \tag{7.1}$$

where t indicates the instant of leaving the battlefield by second strongest player. Let

$$Q(x) = V(x)\,(F(x))^{n-1}, \quad \text{for} \quad 0 < x < a. \tag{7.2}$$

Under $0 < x < a$, formula (7.1) can be rewritten as

$$\begin{aligned} H_1(x, F, \dots, F) &= Q(x) - \int_0^x t d\,(F(t))^{n-1} - x\left\{1 - \frac{Q(x)}{V(x)}\right\} \\ &= Q(x) + \int_0^x \frac{Q(t)}{V(t)}dt - x. \end{aligned} \tag{7.3}$$

The condition $\frac{\partial H_1}{\partial x} = 0$ yields the linear differential equation

$$Q'(x) + \frac{Q(x)}{V(x)} = 1, \qquad Q(0) = 0. \tag{7.4}$$

Its solution is determined by

$$Q(x) = e^{-\int (V(x))^{-1} dx} \left[\int e^{\int (V(x))^{-1} dx} dx + c \right], \tag{7.5}$$

where c designates an arbitrary constant.

For instance, set $V(x) = \bar{x}, 0 \le x \le 1$. In this case, we have

$$Q(x) = \bar{x} \left[\int_0^x dt/\bar{t} + c \right] = \bar{x}(-\log \bar{x} + c).$$

The boundary conditions $Q(0) = 0$ imply that $c = 0$; hence,

$$Q(x) = -\bar{x} \log \bar{x}. \tag{7.6}$$

In combination with (7.2), this brings to

$$F(x) = (-\log \bar{x})^{\frac{1}{n-1}}, \quad 0 \le x \le a. \tag{7.7}$$

The above function is increasing such that $F(0) = 0$ and $F(a) = \left(-\log \bar{a}\right)^{\frac{1}{n-1}}$.

The condition $F(a) = 1$ yields $a = 1 - e^{-1} \approx 0.63212$.

For $F(x)$ defined by (7.4), the payoff (7.1)–(7.3) of player 1 equals

$$H_1(x, F, \dots, F) = -\bar{x} \log \bar{x} + \int_0^x (-\log \bar{t}) dt - x = 0$$

within the interval $0 < x < a$. Indeed, the second term in the right-hand side is given by $\bar{x} \log \bar{x} + x$ (see the expression $\int (1 + \log \bar{t}) dt = -\bar{t} \log \bar{t}$).

In the case of $a < x \le 1$, the function $H_1(x, F, \dots, F)$ decreases in x according to (7.1).

Consequently, if the choice of $F^*(x)$ meets (7.4), then

$$H_1(F, F^*, \dots, F^*) \le H_1(F^*, F^*, \dots, F^*) = 0, \quad \forall \text{ distribution function } F(x).$$

In other words, we finally arrive at the following result.

Theorem 3.9 *Consider a war of attrition with the resource $V(x) = \bar{x}$. A Nash equilibrium is achieved in the class of mixed strategies*

$$F^*(x) = I(0 \le x \le a)(-\log \bar{x})^{\frac{1}{n-1}} + I(a < x \le 1),$$

with zero payoff for each player. Here $a = 1 - e^{-1} (\approx 0.632)$.

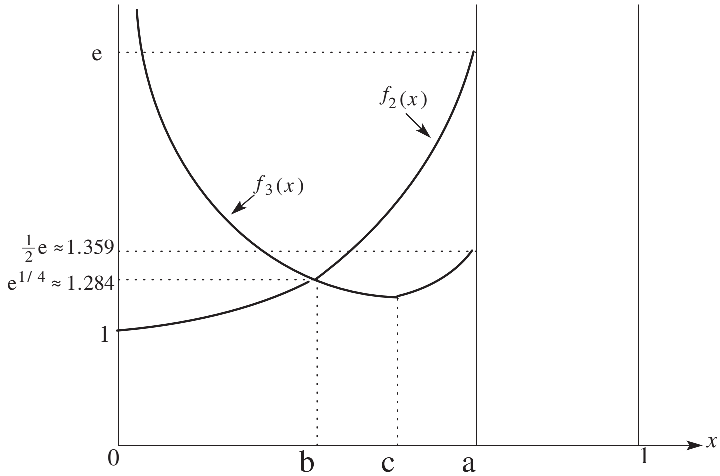


Figure 3.4 The solution under $n = 2$ and $n = 3$, $V(x) = \overline{x}$. Notation: b $= 1 - e^{-1/4} \approx 0.221$, c $= 1 - e^{-1/2} \approx 0.393$, and a ≈ 0.632.

For instance, under $n = 2$, the optimal density function becomes $f_2^*(x) = (-\log \overline{x})$. In the case of $n = 3$, we accordingly obtain

$$f_3^*(x) = \frac{1}{2\overline{x}\left(-\log \overline{x}\right)^{1/2}} \to \begin{cases} +\infty, & \text{if } x \downarrow 0 \\ e/2 \approx 1.359, & \text{if } x \uparrow a. \end{cases}$$

Their curves are illustrated in Figure 3.4. Interestingly, the form of mixed strategies changes drastically. If $n = 2$, with higher probability one should struggle for the resource as long as possible. As the number of opponents grows, with higher probability one should immediately leave the battlefield.

Similar argumentation serves to establish a more general result.

Theorem 3.10 *For* $V(x) = \frac{1}{k}\overline{x}$, $(0 < k \leq 1)$*, a Nash equilibrium is achieved in the class of mixed strategies*

$$F^*(x) = \left[\left(k/\overline{k}\right)\left\{(\overline{x})^{k-1} - 1\right\}\right]^{\frac{1}{n-1}}, \quad 0 \leq x < a,$$

where a stands for the unique root of the equation

$$-\overline{k}\log \overline{a} = -\log k$$

within the interval $(0, 1)$*. Furthermore, each player has zero optimal payoff.*

Note that $\lim\limits_{k \to 1-0} \dfrac{(\overline{x})^{k-1} - 1}{\overline{k}} = -\log \overline{x}$ and, hence, $\lim\limits_{k \to 1-0} F^*(x) = (-\log \overline{x})^{\frac{1}{n-1}}$.

3.8 Duels, truels, and other shooting accuracy contests

Consider shooting accuracy contests involving n players. It is required to hit some target (in the special case, an opponent). Each player has one bullet and can shoot at any instant from the interval $[0, 1]$. Starting at the instant $t = 0$, he moves to the target and can reach it at the instant $t = 1$; the player must shoot at the target at some instant. Let $A(t)$ be the probability of target hitting provided that shooting occurs at instant $t \in [0, 1]$. We believe that the function $A(t)$ is differentiable, $A'(t) > 0, A(0) = 0$ and $A(1) = 1$.

The payoff of a player makes up 1, if he successfully hits the target earlier than the opponents (and 0, otherwise). The payoff of several players simultaneously hitting the target equals 0. Each player strives for a strategy maximizing the mathematical expectation of target hitting.

The following assumption seems natural owing to problem symmetry. All optimal strategies of players do coincide in an equilibrium. Suppose that all players choose the same mixed strategies with a distribution function $F(t)$ and density function $f(t), a \leq t \leq 1$, where $a \in [0, 1]$ is a parameter. If player 1 shoots at instant x and other players apply mixed strategies $F(t)$, his expected payoff becomes

$$H_1(x, \overbrace{F, \dots, F}^{n-1}) = \begin{cases} A(x), & \text{if } 0 \leq x < a, \\ A(x)\left[1 - \int\limits_a^x A(t)f(t)dt\right]^{n-1}, & \text{if } a \leq x \leq 1. \end{cases} \tag{8.1}$$

Really, under $a \leq x \leq 1$, player 1 obtains the payoff of 1 only if all opponents $2 \sim n$ did not shoot or shot before the instant x but missed the target.

Let v be the optimal payoff common for all players. Then the sufficient condition of an equilibrium takes the form

$$H_1(x, F, \dots, F) \left\{ \begin{matrix} \leq \\ = \end{matrix} \right\} v, \text{ for } \left\{ \begin{matrix} 0 \leq x < a \\ a \leq x \leq 1 \end{matrix} \right\}. \tag{8.2}$$

In the case of $a \leq x \leq 1$, apply the first-order necessary optimality conditions to (8.1) to obtain the differential equation

$$\frac{f'(x)}{f(x)} = -\frac{2n-1}{n-1}\left[\frac{A'(x)}{A(x)} - \frac{A''(x)}{A'(x)}\right]. \tag{8.3}$$

Integration from a to x yields

$$\frac{f(x)}{f(a)} = \frac{A'(x)}{A'(a)}\left(\frac{A(x)}{A(a)}\right)^{-\frac{2n-1}{n-1}}, \tag{8.4}$$

whence it appears that

$$f(x) = c\,(A(x))^{-\frac{2n-1}{n-1}} A'(x). \tag{8.5}$$

The condition $\int_a^1 f(t)dt = 1$ gives

$$c^{-1} = \int_a^1 (A(x))^{-\frac{2n-1}{n-1}} A'(x)dx = \left(\frac{n-1}{n}\right)\left[(A(a))^{-\frac{n}{n-1}} - 1\right]. \tag{8.6}$$

The condition (8.2) on the interval $a \le x \le 1$ requires that

$$A(x)\left[1 - \int_a^x A(t)f(t)dt\right]^{n-1} \equiv v.$$

After some simplifications, this result and formula (8.3) bring to the equality

$$c(n-1)\left[(A(a))^{-\frac{1}{n-1}} - (A(x))^{-\frac{1}{n-1}}\right] = 1 - v^{\frac{1}{n-1}}(A(x))^{-\frac{1}{n-1}}, \quad \forall x \in [a, 1]. \tag{8.7}$$

Eliminate c according to (8.4) to derive the equality

$$(A(a))^{-\frac{1}{n-1}} - (A(x))^{-\frac{1}{n-1}} = \frac{1}{n}\left[1 - \left(\frac{v}{A(x)}\right)^{\frac{1}{n-1}}\right]\left[(A(a))^{-\frac{n}{n-1}} - 1\right], \quad \forall x \in (a, 1). \tag{8.8}$$

Hence, the following expressions must be valid:

$$(A(a))^{-\frac{n}{n-1}} - n(A(a))^{-\frac{1}{n-1}} - 1 = 0 \quad \text{and} \quad v^{\frac{1}{n-1}}\left[(A(a))^{-\frac{n}{n-1}} - 1\right] = n. \tag{8.9}$$

These equations yield $v^{-\frac{1}{n-1}} = (A(a))^{-\frac{1}{n-1}}$, and $v = A(a)$.

Moreover, by multiplying both sides of the first equation in (8.9) by $(A(a))^{\frac{n}{n-1}}$, we arrive at the equation

$$(A(a))^{\frac{n}{n-1}} + nA(a) - 1 = 0. \tag{8.10}$$

And finally, it suffices to establish the condition $H_1(x, F, \dots, F) \le v, \quad \forall x \in [0, a]$. It holds true, since $A(x) \le A(a) = v, \quad \forall x \in [0, a)$ due to the above assumptions.

The stated reasoning immediately generates

Theorem 3.11 *Let α_n be a unique root of the equation*

$$\alpha^{\frac{n}{n-1}} + n\alpha - 1 = 0 \tag{8.11}$$

within the interval $[0, 1]$.

Then the game admits the mixed strategy Nash equilibrium

$$f^*(x) = \frac{1}{n-1}\left(\alpha_n\right)^{\frac{1}{n-1}}(A(x))^{-\frac{2n-1}{n-1}}A'(x), \quad \textit{for } A^{-1}(\alpha_n) = a_n \le x \le 1. \tag{8.12}$$

In the equilibrium, the optimal payoffs of players constitute α_n.

Readers can make a series of observations. First, the optimal payoff of players (α_n) is independent from the shooting accuracy function $A(t)$. Second, the initial point of the optimal strategy support a depends on $A(t)$. Furthermore, formula (8.12) implies that the probability of draw (i.e., all players gain nothing) becomes $\left(\alpha_n\right)^{\frac{n}{n-1}}$.

In the case of $n = 2$ (a duel), the expected payoff equals $\alpha_n = \sqrt{2} - 1 \approx 0.414$. In a truel ($n = 3$), this quantity is $\alpha_n \approx 0.283$. The interval of the distribution support depends on the form of the shooting accuracy function.

Example 3.1 Select $A(x) = x^\gamma, \gamma > 0$. Then

$$a_n = A^{-1}(\alpha_n) = \alpha_n^{1/\gamma},$$

and the optimal strategy possesses the density function

$$f^*(x) = \frac{\gamma}{n-1}(\alpha_n)^{\frac{1}{n-1}}x^{-\left(\frac{n}{n-1}\gamma+1\right)}, \quad \text{for } \alpha_n^{1/\gamma} \le x \le 1.$$

If $\gamma = 1$ and $n = 2$ (a duel), we have $a_n = \alpha_n = \sqrt{2} - 1$. In other words, players should shoot after the instant of 0.414. For any $n \ge 2$, the quantity a_n increases if the parameter γ does. This fact agrees with the following intuitive expectation. The lower is the shooting accuracy of a player, the later he should shoot.

Example 3.2 Now, choose $A(x) = \dfrac{e^x - 1}{e - 1}$. Consequently,

$$a_n = A^{-1}(\alpha_n) = \log\left\{1 + (e-1)\alpha_n\right\}.$$

Hence, a_n decreases if n goes up. In the case of a duel ($n = 2$),

$$a_n = \log\left\{(\sqrt{2}-1)(e+\sqrt{2})\right\} \approx 0.537.$$

For a truel ($n = 3$), we obtain

$$a_n \approx 0.396.$$

The optimal strategies are defined by the density function

$$f^*(x) = \frac{1}{n-1}\left(\alpha_n\right)^{\frac{1}{n-1}}(e-1)^{-1}(e^x-1)^{-\frac{2n-1}{n-1}}e^x, \quad \text{for } a_n \le x \le 1.$$

3.9 Prediction games

Imagine that n players endeavor to predict the value u of a random variable U which has the uniform distribution $U_{[0,1]}$ on the interval $[0, 1]$. The game is organized as follows. A winner is a player who predicts a value closest to u (but not exceeding the latter). His payoff makes up 1, whereas the rest $n - 1$ players benefit nothing. Each player strives for maximizing his expected payoff.

We search for an equilibrium in the form of distributions whose support belongs to some interval $[0, a]$, $a \le 1$. Notably, let

$$G(x) = I(x < a) \int_0^x g(t)dt + I(x \ge a).$$

Suppose that player 1 predicts x and his opponents choose the mixed strategies with the distribution function $G(t)$ and density function $g(t)$. Then the expected payoff of player 1 is defined by

$$H_1(x, \overbrace{G, \dots, G}^{n-1}) = \bar{x}, \quad \text{if } a < x < 1. \tag{9.1}$$

According to the conditions, for $0 < x < a$ we have

$$H_1(x, G, \dots, G) = (G(x))^{n-1}\,\bar{x} + \sum_{k=1}^{n-1} \binom{n-1}{k} k\,(G(x))^{n-1-k} \int_x^a g(t)\left(\overline{G}(t)\right)^{k-1}(t - x)dt, \tag{9.2}$$

since k players $(1 \le k \le n - 1)$ can predict higher values than x, and the rest $n - 1 - k$ players predict smaller values than x. The density function of the random variable $\min(X_1, \dots, X_k)$ takes the form $k(\overline{G}(t))^{k-1}g(t)$.

Partial integration yields the equality

$$\int_x^a (t - x)\left(\overline{G}(t)\right)^{k-1} g(t)dt = \frac{1}{k}\int_x^a \left(\overline{G}(t)\right)^k dt.$$

And so, we rewrite (9.2) as

$$H_1(x, \overbrace{G, \dots, G}^{n-1}) = (G(x))^{n-1}\,\bar{x} + \sum_{k=1}^{n-1} \binom{n-1}{k} (G(x))^{n-1-k} \int_x^a \left(\overline{G}(t)\right)^k dt, \tag{9.3}$$

provided that $0 < x < a$. Denote by v the optimal expected payoff of each player. We address the mixed equilibrium condition for $G(x)$:

$$H_1(x, G, \dots, G) \left\{ \begin{matrix} \equiv \\ \leq \end{matrix} \right\} v, \quad \text{for} \quad \left\{ \begin{matrix} 0 \leq x < a \\ a < x \leq 1 \end{matrix} \right\}. \tag{9.4}$$

Using (9.3)–(9.4), transform the equation $\frac{\partial}{\partial x} H_1(x, g, \dots, g) = 0$ on the interval $0 \leq x < a$. Divide both sides of the equation by $(G(x))^{n-1}$ and perform some simplifying operations to get the equation

$$1 + \sum_{k=1}^{n-1} \binom{n-1}{k} \left(\frac{\overline{G}(x)}{G(x)} \right)^k = \frac{g(x)}{G(x)} \left[(n-1)\overline{x} + \sum_{k=1}^{n-1} \binom{n-1}{k} (n-1-k) \int_x^a \left(\frac{\overline{G}(t)}{G(x)} \right)^k dt \right]. \tag{9.5}$$

The left-hand side of (9.5) equals $[G(x)]^{-(n-1)}$, whereas its right-hand counterpart can be reexpressed by

$$\frac{g(x)}{G(x)} \cdot (n-1) \left[\overline{a} + \int_x^a \left\{ 1 + \frac{\overline{G}(t)}{G(x)} \right\}^{n-2} dt \right].$$

Therefore, we rewrite (9.5) as

$$\overline{a}[G(x)]^{n-2} + \int_x^a (G(x) + \overline{G}(t))^{n-2} dt = [(n-1)g(x)]^{-1}, \ \ 0 < x < a, \ \ \forall n \geq 2. \tag{9.6}$$

Undoubtedly, $g(x), G(x)$ and a depend on n. For compact notation, we omit the subscript n.

Consider the sequence of functions

$$s_k(x) = \left[\overline{a}[G(x)]^k + \int_x^a (G(x) + \overline{G}(t))^k dt \right] \Big/ \overline{x}, \ \ \forall k = 1, 2, \dots, n-2. \tag{9.7}$$

Obviously, the following inequalities hold true:

$$1 \equiv s_0(x) \geq s_1(x) \geq s_2(x) \geq \cdots \geq s_{n-2}(x) \geq 0, \ \ \forall x \in [0, a]. \tag{9.8}$$

Multiply both sides of (9.7) by $\overline{x}$ and perform differentiation. Such manipulations yield the recurrent differential equations

$$\overline{x} s_k'(x) - s_k(x) = k g(x) \overline{x} s_{k-1}(x) - 1,$$

or, equivalently,

$$s_k'(x) + \left(1 - s_k(x)\right)/\bar{x} = kg(x)s_{k-1}(x), \quad \forall k = 1, 2, \dots, n-2, \tag{9.9}$$

with the boundary conditions

$$s_k(a) = 1, \quad \forall k = 1, 2, \dots, n-2.$$

Formulas (9.6)–(9.7) imply that

$$s_{n-2}(x) = [(n-1)\bar{x}g(x)]^{-1}, \tag{9.10}$$

which is equivalent to

$$g(x) = \left[(n-1)\bar{x}s_{n-2}(x)\right]^{-1} \geq \left[(n-1)\bar{x}\right]^{-1} \quad \text{(from (9.8))}.$$

The mean value of this distribution is defined by

$$\int_0^a xg(x)dx = \int_0^a \frac{xdx}{(n-1)\bar{x}s_{n-2}(x)}. \tag{9.11}$$

Theorem 3.12 *Let $\{s_1, \dots, s_{n-2}\}$ be the solution to the system of differential equations (9.9) and $g(x) = \frac{1}{(n-1)(1-x)s_{n-2}(x)}$. Choose a according to the condition $\int_0^a g(x)dx = 1$. Then $g(x)$ gives the optimal mixed strategy in the prediction game.*

The system (9.9) and formula (9.10) can serve to solve the problem. We describe the corresponding solution algorithm. First, fix the initial value of the parameter a and consider the system of differential equations (9.9) on the interval $[0, a]$. As soon as the solution with the boundary condition $s_k(a) = 1, k = 1, \dots, n-2$ is found, define the density function $g(x) = [(n-1)s_{n-2}(x)(1-x)]^{-1}, x \in [0, a]$. Next, evaluate a from the condition $\int_0^a g(x) = 1$.

The case of n = 2.
It appears from (9.1)–(9.3) that

$$H_1(x, G) = \begin{cases} G(x)\bar{x} + \int_x^a (t-x)g(t)dt, & \text{for } 0 < x < a \\ \bar{x}, & \text{for } a < x < 1. \end{cases}$$

Under $0 < x < a$, equation (9.6) yields $g(x) = 1/\bar{x}$, whence it follows that $G(x) = -\log \bar{x}$, $a = 1 - e^{-1} \approx 0.632$. If $a < x < 1$, we obtain $H_1(x, g^*) = \bar{x} \leq \bar{a} = H_1(a, g^*)$; hence, the condition (9.4) is satisfied. The total value of the game constitutes $e^{-1} \approx 0.367$.

The case of n = 3.

$$H_1(x, G, G) = \begin{cases} (G(x))^2\,\bar{x} + 2G(x)\int\limits_x^a (t-x)g(t)dt + 2\int\limits_x^a (t-x)\overline{G}(t)g(t)dt, & \text{if } 0 < x < a \\ \bar{x}, & \text{if } a < x < 1. \end{cases} \tag{9.12}$$

Under $n = 3$, equation (9.9) leads to the differential equation

$$s_1'(x) + \left(1 - s_1(x)\right)/\bar{x} = g(x)s_0(x) = g(x), \text{ and } s_1(a) = 1. \tag{9.13}$$

After some simplifications, formula (2.7) yields

$$\bar{x}s_1(x) = \bar{a}G(x) + \int\limits_x^a \left(G(x) + \overline{G}(t)\right) dt = \frac{1}{2g(x)} \text{ (from (9.6) under } n = 3). \tag{9.14}$$

By eliminating $g(x)$ from (9.13)–(9.14), we arrive at the differential equation

$$\frac{s_1 s_1'}{s_1^2 - s_1 + \frac{1}{2}} = \frac{1}{\bar{x}}, \quad 0 < x < a, \quad s_1(a) = 1. \tag{9.15}$$

The function $g(x) = \left(2s_1(x)\bar{x}\right)^{-1}$ is positive and continuous; it represents a density function if $\int\limits_0^a g(x)dx = 1$. And so,

$$1 = \int\limits_0^a g(x)dx = \int\limits_0^a \left\{ s_1'(x) + \frac{1 - s_1(x)}{1 - x} \right\} dx = 1 - s_1(0) + \int\limits_0^a \frac{1 - s_1(x)}{1 - x} dx,$$

leading to

$$s_1(0) = \int\limits_0^a \frac{1 - s_1(x)}{1 - x} dx = \int\limits_{s_1(0)}^1 \frac{s_1 \bar{s}_1}{s_1^2 - s_1 + \frac{1}{2}} ds_1 \quad \text{(from (2.15))}$$

$$= -1 + s_1(0) + \frac{\pi}{4} - \tan^{-1}(2s_1(0) - 1) \quad \left(\text{since } \int \frac{ds_1}{s_1^2 - s_1 + r_2} = 2\tan^{-1}2x\right).$$

Therefore,

$$s_1(0) = \frac{1}{2}\left\{1 - \tan\left(1 - \frac{\pi}{4}\right)\right\} \approx 0.391. \tag{9.16}$$

Perform integration in both sides of (9.15) from x to a to obtain the control law

$$\left(s_1^2 - s_1 + \frac{1}{2}\right)^{\frac{1}{2}} e^{\tan^{-1}(2s_1-1)} = \frac{1}{\sqrt{2}} e^{\pi/4}\overline{a}/\overline{x}. \tag{9.17}$$

Here, substitution of $x = 0$ and $s_1(0) \approx 0.391$ from (2.16) yields

$$a = 1 - \left\{2\left(s_1(0)\right)^2 - 2s_1(0) + 1\right\}^{1/2} e^{-1} \approx 0.7156. \tag{9.18}$$

By virtue of (9.12), the condition (9.4) holds true with $v = \overline{a} = 0.284$.

The corresponding solutions under $n = 2$ and $n = 3$ are shown in Figure 3.5.

The case of n = 4.

If $n = 4$, formulas (9.1)–(9.3) bring to

$$H_1(x, G, G, G, G) = \begin{cases} (G(x))^3\,\overline{x} + \sum_{k=1}^{3} \begin{pmatrix} 3 \\ k \end{pmatrix} (G(x))^{3-k} \int\limits_x^a \left(\overline{G}(t)\right)^k dt, & 0 < x < a \\ \overline{x}, & a < x < 1. \end{cases} \tag{9.19}$$

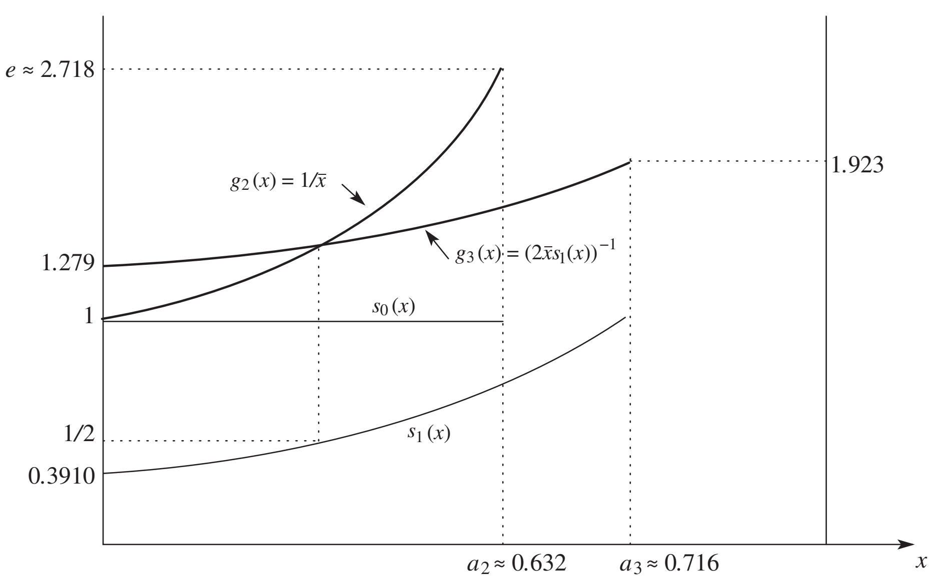


Figure 3.5 The solutions under $n = 2$ and $n = 3$.

The system (9.9)–(9.10) acquires the form

$$\begin{cases} s_1'(x) + \left(1 - s_1(x)\right)/\bar{x} = g(x)s_0(x) = g(x), & s_1(0) = 1 \\ s_2'(x) + \left(1 - s_2(x)\right)/\bar{x} = 2g(x)s_1(x), & s_2(0) = 1 \\ s_2(x) = \left(3g(x)\bar{x}\right)^{-1}. \end{cases} \tag{9.20}$$

The density function is defined by $g(x) = \left(3\bar{x}s_2(x)\right)^{-1}$, and we can choose a such that $1 = \int_0^a \frac{dx}{3\bar{x}s_2(x)}$. Since the right-hand side meets the inequality $\geq \frac{2}{3}\int_0^a \frac{dx}{2\bar{x}s_1(x)}$, it is possible to adopt the solution under $n = 3$.

Elimination of $g(x)$ from (9.20) yields the ordinary differential equation

$$\begin{cases} s_1'(x) + \left(1 - s_1(x)\right)/\bar{x} = \left(3\bar{x}s_2(x)\right)^{-1} \\ s_2'(x) + \left(1 - s_2(x)\right)/\bar{x} = \frac{2}{3}s_1(x)/\left(\bar{x}s_2(x)\right). \end{cases} \tag{9.21}$$

Computations lead to $a \approx 0.791$. The condition (9.4) is valid with $v = \bar{a} \approx 0.208$.

Other examples.

If $n \geq 5$, computations result in the following:

$n = 2;$	$a \approx 0.632,$	$v \approx 0.367,$	$\int_0^a xg(x)dx \approx 0.367$
3;	0.715,	0.284,	
4;	0.791,	0.208,	
5;	0.828,	0.171,	0.425
7;	0.873,	0.126,	0.442
10;	0.908,	0.091,	0.457

Apparently, as n increases, $a \uparrow 1$; at the same time, the optimal payoffs $\downarrow 0$. In an equilibrium, the density functions asymptotically tend to the uniform distributions $U_{[0,1]}$.

Exercises

1. The city transport game.

This game involves n players. Each player chooses transport for today's trip, namely, $x_i = 0$ (private automobile) or $x_i = 1$ (public transport). The payoff of player i depends on the number of other players choosing the same transport as he does. Notably, the payoff function takes the form

$$H_i(x_1, \ldots, x_n) = \begin{cases} a(t), & x_i = 1, \\ b(t), & x_i = 0, \end{cases}$$

where $t = 1/n \sum_{j=1}^{n} x_j$, a(t) and b(t) are demonstrated in Figure 3.6.

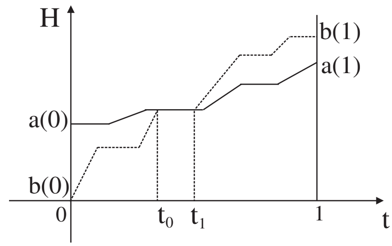


Figure 3.6 The payoff function in the city transport game.

This figure illustrates the following aspect. If the share of players choosing 1 exceeds t_1, city traffic is less intensive, and automobilists feel better than public passengers. However, if the share of automobilists is higher than $1 - t_0$, city traffic gets intensified such that public transport becomes preferable.

Prove that solution to this game lies in a set $x^* = (x_1^*, \dots, x_n^*)$ such that $t_0 + \frac{1}{n} \leq \frac{1}{n}\sum_{j=1}^{n} x_j^* \leq t_1 - \frac{1}{n}$.

2. The commune problem.

Imagine that n dwellers keep sheep on a farm. Each dweller has q_i sheep. Denote by $G = q_1 + \cdots q_n$ the total number of sheep. The maximal number of sheep kept by dwellers is G_{max}. Each sheep gains some profits $v(G)$ and requires costs c for keeping. Suppose that $v(G) > 0$ under $G < G_{max}$ and $v(G) = 0$ under $G > G_{max}$.

Find the payoff of the commune provided that dwellers keep same numbers of sheep. Construct a Nash equilibrium and show that the number of sheep is higher than in the case of their uniform distribution among dwellers.

3. The environmental protection problem.

Three enterprises (players I, II, and III) exploit water resources from a natural reservoir. Each of them chooses between the following pure strategies: building water purification facilities or releasing polluted water. By assumption, water in the reservoir remains usable if just one enterprise releases polluted water. In this case, enterprises incur no costs. However, if at least two enterprises release polluted water in the reservoir, then each player has the costs of 3. The maintenance of water purification facilities requires the costs of 1 from each enterprise. Draw the strategy profile cube and players' payoffs at corresponding nodes. Find the set of equilibrium strategy profiles by intersection of the sets of acceptable strategy profiles of each player.

4. Bayesian games.

A Bayesian game is a game of the form

$$G =< N, \{x_i\}_{i=1}^n, T_i, H_i > ,$$

where $N = 1, 2, \dots, n$ is the set of players, and $t_i \in T_i$ denote the types of players (unknown to their opponents).

The game is organized as follows. Nature reports to player i his type, players choose their strategies $x_i(t_i)$ and receive the payoffs $H_i(x_1, \dots, x_n, t_i)$.

An equilibrium in a Bayesian game is a set of strategies $(x_1^*(t_1), \dots, x_n^*(t_n))$ such that for any player i and any type t_i the function

$$\sum_{t_{-i}} H_i(x_1^*(t_1), \dots, x_{i-1}^*(t_{i-1}), x, x_{i+1}^*(t_{i+1}), \dots, x_n^*(t_n), t_i) P(t_{-i}|t_i),$$

where $t_{-i} = (t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n)$, attains its maximum at the point x_i^*.

Reexpress the environmental protection problem as a Bayesian game provided that the payoff of -4 is supplemented by random variables t_i with the uniform distribution on $[0, 1]$.

5. Auction.

This game employs two players bidding for an item at an auction. Player I offers a price b_1 and estimates item's value by $v_1 \in [0, 1]$. Player II offers a price b_2, and his value of the item is $v_2 \in [0, 1]$.

The payoff of player i takes the form

$$H_i(b_1, b_2, v) = \begin{cases} v - b_i, & b_i > b_j, \\ 0, & b_i < b_j, \\ (v - b_i)/2, & b_i = b_j, \end{cases}$$

where $i, j = 1, 2$, $i \neq j$.

Evaluate equilibrium prices in this auction.

6. Demonstrate that x^* represents a Nash equilibrium in an n-player game iff x^* makes the global maximum point of the function

$$F(x) = \sum_{i=1}^{n} (H_i(x) - \max_{y_i \in X_i} H_i(x_{-i}, y_i))$$

on the set X.

7. Consider the Cournot oligopoly model.

Find optimal strategy profiles in the sense of Nash for n players with the payoff functions

$$H_i(x) = x_i \left(a - b \sum_{j=1}^{n} x_j - c_j \right).$$

8. The traffic jamming game with three players.

The game engages three companies each possessing two automobiles. They have to move from point A to point B along one of two roads. Delay on the first (second) road equals $2k$ ($3k$, respectively), where k indicates the number of moving automobiles. Evaluate a Nash equilibrium in this game.

9. Prove that the Bertrand oligopoly is a potential game.

10. Find solutions to the duel problem and truel problem in the following case. The target hitting probability is given by $A(x) = \frac{\ln(x+1)}{\ln 2}$.

4

Extensive-form n-player games

Introduction

Chapters 1–3 have studied normal-form games, where players make their offers at the very beginning of a game and their payoffs are determined accordingly. However, real games evolve in time—players can modify their strategies depending on opponents' strategy profiles and their own interests. Therefore, we naturally arrive at the concept of dynamic games that vary depending on the behavior of players. Furthermore, a game may incorporate uncertainties occurring by chance. In such positions, a game evolves in a random way. The mentioned factors lead to extensive-form games (also known as positional games).

Definition 4.1 *An extensive-form game with complete information is a pair $\Gamma =< N, G >$, where $N = \{1, 2, ..., n\}$ indicates the set of players and $G = \{X, Z\}$ represents a directed graph without cycles (a finite tree) having the initial node x_0, the set of nodes (positions) X and $Z(x)$ as the set of nodes directly following node x.*

Figure 4.1 demonstrates the tree of such a game with the initial state x_0. For each player, it is necessary to define the position of his decision making.

Definition 4.2 *A partition of the position set X into $n + 1$ non-intersecting subsets $X = X_1 \cup X_2 \cup ... \cup X_n \cup T$ is called a partition into the personal position sets of the players. Player i moves in positions belonging to the set X_i, $i = 1, ..., n$. The set T contains terminal nodes, where the game ends.*

Terminal nodes $x \in T$ satisfy the property $Z(x) = \emptyset$. The payoffs of all players are specified in terminal positions: $H(x) = (H_1(x), ..., H_n(x)),\ x \in T$. In each position x from the personal position set X_i, player i chooses a node from the set $Z(x) = \{y_1, ..., y_k\}$ (referred to as an **alternative** in the position x), and the game passes to a new position. Sometimes, it appears convenient to identify alternatives with arcs incident to x. Thus, each player has to choose a next position in each set of his personal positions.

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

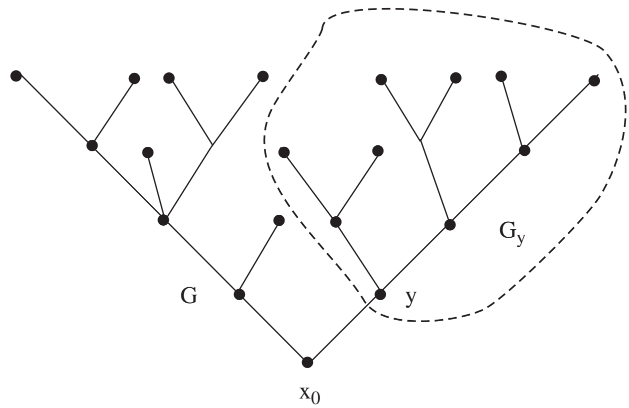


Figure 4.1 The tree of an extensive-form game G and a subtree G_y.

Definition 4.3 *A strategy of player i is a function $u_i(x)$ defined on the personal position set X_i, $i = 1, ..., n$, whose values are alternatives of the position x. A set of all strategies $u = (u_1, ..., u_n)$ is a strategy profile in the game.*

For each strategy profile, one can uniquely define a corresponding **play** in an extensive-form game. Indeed, this game begins in the position x_0. Suppose that $x_0 \in X_{i_1}$. Hence, player i_1 makes his move. Following his strategy $u_{i_1}(x_0) = x_1 \in Z(x_0)$, the play passes to the position x_1. Next, x_1 belongs to the personal position set of some player i_2. His strategy $u_{i_2}(x_1)$ shifts the play to the position $x_2 \in Z(x_1)$. The play continues until it reaches an end position $x_k \in T$. The game ends in a terminal position, and player i obtains the payoff $H_i(x_k)$, $i = 1, ..., n$. Therefore, each strategy profile in an extensive-form game corresponds to a certain payoff of each player. It is possible to comprehend payoffs as functions of players' strategies, i.e., $H = H(u_1, ..., u_n)$. Now, we introduce the notion of a solution in such games.

4.1 Equilibrium in games with complete information

For convenience, set $u = (u_1, ..., u_i', ..., u_n)$ as a strategy profile, where just one strategy u_i is replaced by u_i'. Denote the new strategy by (u_{-i}, u_i').

Definition 4.4 *A strategy profile $u^* = (u_1^*, ..., u_n^*)$ is a Nash equilibrium, if for each player the following condition holds true:*

$$H_i(u_{-i}^*, u_i) \le H_i(u^*), \forall u_i,\ i = 1, ..., n. \tag{1.1}$$

Inequalities (1.1) imply that, as player i deviates from an equilibrium, his payoff goes down. We will show that the game Γ may have many equilibria. But one of them is special. To extract it, we present the notion of a subgame of Γ.

Definition 4.5 *Let $y \in X$. A subgame $\Gamma(y)$ of the game Γ, which begins in the position y, is a game $\Gamma(y) =< N, G_y >$, where the subgraph $G_y = \{X_y, Z\}$ contains all nodes following y, the personal position sets of players are defined by the intersection $Y_i = X_i \cap X_y, i = 1, ..., n$,* *the set of terminal positions is* $T_y = T \cap X_y$*, and the payoff of player i in the subgame is given by* $H_i^y(x) = H_i(x),\ x \in T_y$.

Figure 4.1 illustrates the subtree corresponding to a subgame with the initial position y. We understand a strategy of player i in a subgame $\Gamma(y)$ as a restriction of the strategy $u_i(x)$ in the game Γ to the set Y_i^y. Designate such strategies by $u_i^y(x)$. A set of strategies $u = (u_1^y, .., u_n^y)$ is a strategy profile in the subgame. Each strategy profile in the subgame corresponds to a play in the subgame and the payoff of each player $H^y(u_1^y, ..., u_n^y)$.

Definition 4.6 *A Nash equilibrium strategy profile* $u^* = (u_1^*, ..., u_n^*)$ *in the game* Γ *is called a subgame-perfect equilibrium, if for any* $y \in X$ *the strategy profile* $(u^*)^y$ *forms a Nash equilibrium in the subgame* $\Gamma(y)$.

Below we argue that any finite game with complete information admits a subgame-perfect equilibrium.

For this, separate out all positions preceding terminal positions and denote the resulting set by Z_1. Assume that a position $x \in Z_1$ belongs to the personal position set of player i. Consider the set of terminal positions $Z(x) = \{y_1, ..., y_{k_i}\}$ that follow the position x, and select the one maximizing the payoff of player i: $H_i(y_j) = \max\{H_i(y_i), ..., H_i(y_{k_i})\}$. Subsequently, shift the payoff vector $H(y_j)$ to the position y_j and make it terminal. Proceed in this way for all positions $x \in Z_1$. And the game tree decreases its length by unity.

Similarly, extract the set Z_2 of preterminal positions followed by positions from Z_1. Take a position $x \in Z_2$ and suppose that $x \in X_l$ (i.e., player l moves in this position). Consider the set of positions $Z(x) = \{y_1, ..., y_{k_l}\}$ that follow the position x and separate out the one (e.g., y_m) maximizing the payoff of player l. Transfer the payoff vector $H(y_j)$ to the position y_m and make it terminal. Repeat the procedure $T \to Z_1 \to Z_2 \to ...$ until the initial state x_0 is reached. This surely happens, since the game possesses a finite tree.

At each step, this algorithm yields equilibrium strategies in each subgame. In the final analysis, it brings to a subgame-perfect equilibrium. Actually, we have proven Kuhn's theorem.

Theorem 4.1 *An extensive-form game with complete information possesses a subgame-perfect equilibrium.*

Figure 4.2 presents an extensive-form two-player game. The set of personal positions of player *I* is indicated by circles, while boxes mark positions corresponding to player *II* moves. Therefore, $X_1 = \{x_0, x_1, ..., x_4\}$ and $X_2 = \{y_1, y_2\}$. The payoffs of both players are specified in terminal positions $T = \{t_1, ..., t_8\}$. And so, the strategy of player *I* lies in the vector $u = (u_0, ..., u_4)$, whereas the strategy of player *II* is the vector $v = (v_1, v_2)$. Their components can be rewritten as "l" (left alternative) and "r" (right alternative). For instance, the strategy profiles $u =$ (l, r, l, r, l), $v =$ (r, l) correspond to a play bringing to the terminal node t_3 with players' payoffs $H_1 = 2, H_2 = 4$.

Figure 4.3 illustrates the backward induction method that leads to the subgame-perfect equilibrium $u^* =$ (l, r, r, r, r),$v^* =$ (l, r), where players' payoffs are $H_1 = 6, H_2 = 3$. These strategies yield a Nash equilibrium in any subgame shown by Figure 4.2.

At the same time, we underline a relevant aspect. There exist other strategy profiles representing a Nash equilibrium but not a subgame-perfect equilibrium. For instance, consider the following strategies of the players: $\bar{u} =$ (r, r, r, r, l), $\bar{v} =$ (l, l). This strategy profile corresponds to a play bringing to the terminal node t_6 with players' payoffs $H_1 = 7, H_2 = 1$.

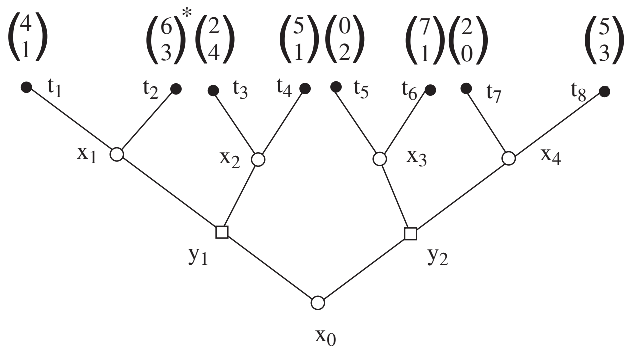


Figure 4.2 An extensive-form game of length 3. Notation: ∘—personal positions of player *I*; □—personal positions of player *II*; (∗)—subgame-perfect equilibrium.

The strategy profile $\bar{u}, \bar{v}$ forms a Nash equilibrium. This is obvious for player *I*, since he gains the maximal payoff $H_1 = 7$. If player *II* deviates from his strategy $\bar{v}$ and chooses alternative "r" in the position y_2, the game ends in the terminal position t_7 and player *II* receives the payoff $H_2 = 0$ (smaller than in an equilibrium). Thus, we have argued that $\bar{u}, \bar{v}$ is a Nash equilibrium. However, this strategy profile is not a subgame-perfect equilibrium, since it is not a Nash equilibrium in the subgame with initial node x_4. In this position, player *I* moves left and gains 2 (instead of choosing the right alternative and obtain 5). Such situation can be treated as player *I* pressure on the opponent in order to reach the terminal position t_7.

4.2 Indifferent equilibrium

A subgame-perfect equilibrium may appear non-unique in an extensive-form game. This happens when the payoffs of some player do coincide in terminal positions. Then his behavior depends on his attitude to the opponent. And the concept of **player's type** arises naturally. We will distinguish between benevolent and malevolent attitude of players to each other.

For instance, consider an extensive-form game in Figure 4.4. To evaluate a subgame-perfect equilibrium, apply the backward induction method. In the position x_1, the payoff of player *I* is independent from his behavior (in both cases, it equals 6). However, the choice of player *I* appears important for his opponent: if player *I* selects "l," player *II* gains 1 (in the case of "r," the latter gets 2). Imagine that player *I* is **benevolent** to player *II*. Consequently, he chooses "r" in the position x_1. A similar situation occurs in the position x_3. Being benevolent,

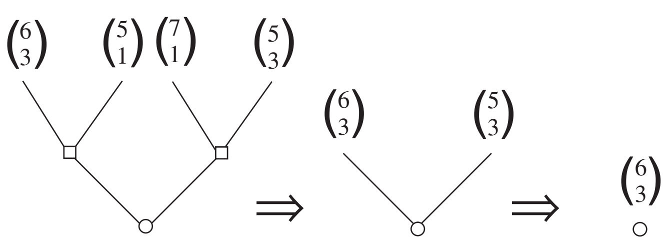


Figure 4.3 Subgame-perfect equilibrium evaluation by the backward induction method.

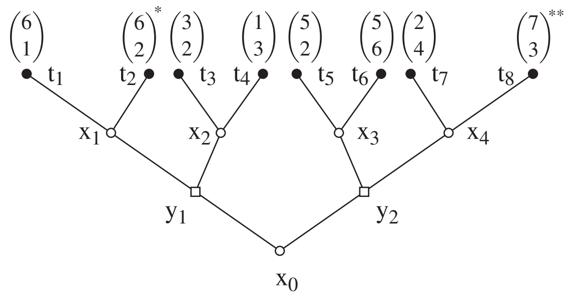


Figure 4.4 An extensive-form game. Notation: $(*)$—an equilibrium for benevolent players; $(**)$—an equilibrium for malevolent players.

player I also chooses the alternative "r." In the positions x_2 and x_4, further payoffs of player I do vary. And so, he moves in a common way by maximizing his payoff. The backward induction method finally brings to the subgame-perfect equilibrium $\bar{u} = (\mathrm{l, r, l, r, r})$, $\bar{v} = (\mathrm{l, l})$ and the payoffs $H_1 = 6, H_2 = 2$.

Now, suppose that players demonstrate **malevolent** attitude to each other. In the positions x_1 and x_3, player I then chooses the alternative "l." The backward induction method yields the subgame-perfect equilibrium $\bar{u} = (\mathrm{r, l, l, l, r})$, $\bar{v} = (\mathrm{r, r})$ with players' payoffs $H_1 = 7, H_2 = 3$.

Therefore, we have faced a paradoxical situation—the malevolent attitude of players to each other results in higher payoffs of both players (in comparison with their benevolent attitude). No doubt, the opposite situation is possible as well (when benevolence increases the payoff of players). But this example elucidates the non-trivial character of benevolence.

As a matter of fact, there exists another approach to avoid ambiguity in a player's behavior when any continuation of a game yields the same payoffs. Such an approach was proposed by L. Petrosjan [1996]. It utilizes the following idea: in a given position, a player randomizes feasible alternatives with identical probabilities. A corresponding equilibrium is called an "indifferent equilibrium."

Let us pass from the original game Γ to a new game described below. In positions $x \in X$, where player $i(x)$ appears indifferent to feasible alternatives $y \in Z_{i(x)}$, he chooses each of them $y_k \in Z_{i(x)}$, $k = 1, ..., |Z_{i(x)}|$ with an identical probability of $1/|Z_{i(x)}|$. Denote the new game by $\bar{\Gamma}$.

Definition 4.7 *A Nash equilibrium strategy profile $u^* = (u_1^*, ..., u_n^*)$ in the game Γ is an indifferent equilibrium, if it forms a subgame-perfect equilibrium in the game $\bar{\Gamma}$.*

For instance, evaluate an indifferent equilibrium for the game in Figure 4.4. We employ the backward induction method. In the position x_1, player 1 turns out indifferent to both feasible alternatives. Therefore, he chooses any with the probability of $1/2$. Such behavior guarantees the expected payoffs $H_1 = 6, H_2 = 3/2$ to the players. In the position x_2, the alternative "l" yields a higher payoff. In the position x_3, player I is again indifferent—he selects any of the two feasible alternatives with the probability of $1/2$. And the expected payoffs of the players in this position become $H_1 = 5, H_2 = 4$. Finally, the alternative "r" is optimal in the position x_4.

Now, analyze the subgame $\Gamma(y_1)$. In the position y_1, player II moves. The alternative "l" corresponds to the payoff $3/2$, which is smaller than 2 gained by the alternative "r." Thus, his optimal strategy in the position y_1 consists in the alternative "r." Consider the subgame

$\Gamma(y_2)$ and the position y_2. The alternative "r" ensures the payoff of 7 to player II, whereas the alternative "l" leads to 4. Hence, the optimal strategy of player II in the position y_2 is "r."

And finally, study the subgame $\Gamma(x_0)$. The first move belongs to player I. The alternative "l" yields his payoff of 3, while the alternative "r" gives 5. Evidently, the optimal strategy of player I becomes "r."

Therefore, we have established the indifferent equilibrium $u^* = (\text{r}, \frac{1}{2}\text{l}+\frac{1}{2}\ \text{r}, \text{l}, \frac{1}{2}\ \text{l}+\frac{1}{2}\ \text{r}, \text{r})$, $\bar{v} = (\text{r}, \text{l})$ and the corresponding payoffs $H_1 = 5, H_2 = 4$.

4.3 Games with incomplete information

Some extensive-form games incorporate positions, where a play may evolve randomly. For instance, in parlor games, players first receive cards (in a random way), and a play continues according to the strategies selected by players. Therefore, players do not know for sure the current position of a play. They can merely make certain assumptions on it. In this case, we have the so-called games with incomplete information. A key role here belongs to the concept of an information set.

Definition 4.8 *An extensive-form game with incomplete information is an n player game $\Gamma =< N, G >$ on a tree graph $G = \{X, Z\}$ with an initial node x_0 and a set of nodes (positions) X such that*

1. There is a given partition of the position set X into $n + 2$ non-intersecting subsets $X = X_0 \cup X_1 \cup ... \cup X_n \cup T$, where X_i indicates the personal position set of player i, $i = 1, ..., n$ and X_0 designates the position set of random moves, the set T contains terminal nodes with defined payoffs of all players $H(x) = (H_1(x), ..., H_n(x))$.

2. There is a given partition of each set $X_i, i = 1, ..., n$ into non-intersecting subsets $X_i^j, j = 1, ..., J_i$ (the so-called **information** *sets of player i) with the following property: all nodes entering a same information set have an identical number of alternatives, and none of them follows a certain node from the same information set.*

Each position $x \in X_0$ with random moves has a given probability distribution on the set of alternatives of the node x. For instance, if $Z(x) = \{y_1, ..., y_k\}$, then the probabilities of play transition to the next position, $p(y|x), y \in Z(x)$, are defined in the node x. We provide a series of examples to show possible informational partitions and their impact on optimal solution.

Example 4.1 A non-cooperative game with complete information.

Move 1. Player I chooses between the alternatives "l" and "r."

Move 2. Random move; one of the alternatives, "l" or "r," is selected equiprobably.

Move 3. Being aware of the choice of player I and the random move, player II chooses between the alternatives "l" and "r."

The payoff of player I in the terminal positions makes up H(l,l,l) = 1, H(l,l,r) = -1, H(l,r,l) = 0, H(l,r,r) = 1, H(r,l,l) = -2, H(r,l,r) = 3, H(r,r,l) = 1, H(r,r,r) = -2, and Figure 4.5 demonstrates the corresponding tree of the game.

The strategy u of player I possesses two values, "l" and "r." The strategy $v = (v_1, v_2, v_3, v_4)$ of player II has $2^4 = 16$ feasible values. However, to avoid complete enumeration of all strategies, let us simplify the game. In the position y_1, y_2, y_3, and y_4, player II optimally chooses the alternative "r," "l," "l," "r," respectively.

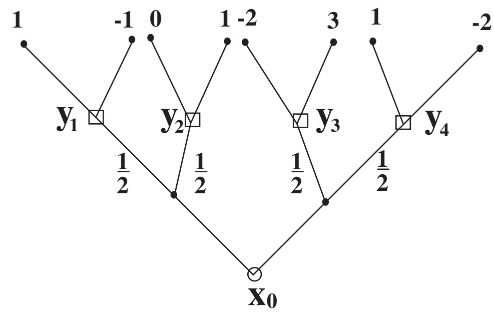


Figure 4.5 A game with complete information.

Therefore, in the initial position x_0, player I obtains the payoff $\frac{1}{2}(-1) + \frac{1}{2}0 = -1/2$ (by choosing "l") or $\frac{1}{2}(-2) + \frac{1}{2}(-2) = -2$ (by choosing "r"). The resulting equilibrium is $u = (\text{l})$, $v = (\text{r,l,l,r})$ and the game has the value of $-1/2$, i.e., it is beneficial to player II.

Example 4.2 A non-cooperative game without information.

Move 1. Player I chooses between the alternatives "l" and "r."

Move 2. Random move; one of the alternatives, "l" or "r," is selected equiprobably.

Move 3. Being aware of the choice of player I only, player II chooses between the alternatives "l" and "r."

The payoff of player I in the terminal positions turns out the same as in the previous example. Figure 4.6 demonstrates the corresponding tree of the game (dashed line highlights the information set of player II).

Again, here the strategy u of player I takes two values, "l" and "r." This is also the case for player II (the strategy v), since he does not know the current position of the play. The payoff matrix of the game is described by

$$\begin{array}{c c} & \begin{array}{cc} \text{l} & \text{r} \end{array} \\ \begin{array}{c} \text{l} \\ \text{r} \end{array} & \begin{pmatrix} \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \end{array}.$$

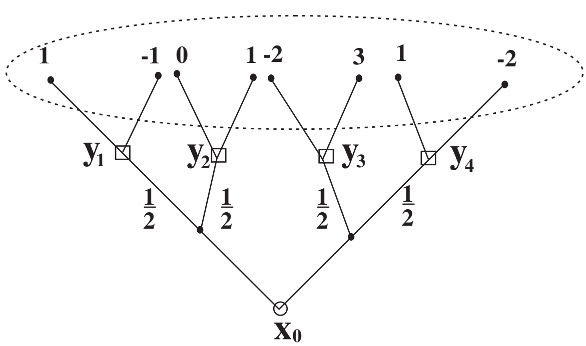


Figure 4.6 A game without information.

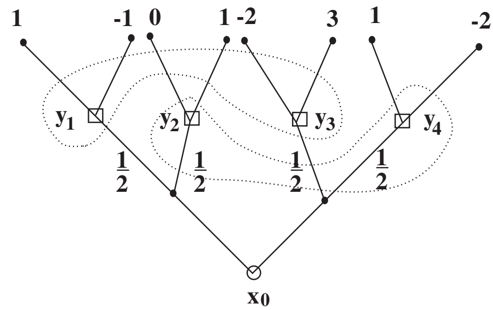


Figure 4.7 A game with incomplete information.

Indeed, H(l,l) $= \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}$, H(l,r) $= \frac{1}{2}(-1) + \frac{1}{2}1 = 0$, H(r,l) $= \frac{1}{2}(-2) + \frac{1}{2}1 = -\frac{1}{2}$, H(r,r) $= \frac{1}{2}3 + \frac{1}{2}(-2) = \frac{1}{2}$.

And the equilibrium of this game is attained in the mixed strategies $(\frac{2}{3}, \frac{1}{3})$ and $(\frac{1}{3}, \frac{2}{3})$, the value of this game constitutes $1/6$. Apparently, the absence of information for player *II* makes the game non-beneficial to him.

Example 4.3 A non-cooperative game with incomplete information.

Move 1. Player *I* chooses between the alternatives "l" and "r."

Move 2. Random move; one of the alternatives, "l" or "r," is selected equiprobably.

Move 3. Being aware of the random move only, player *II* chooses between the alternatives "l" and "r."

The payoff of player *I* in the terminal positions coincides with Example 4.1. And Figure 4.7 presents the corresponding tree of the game (dashed line indicates the information set of player *II*). It differs from Example 4.2, since it comprises two subset X_2^1 and X_2^2.

Here, the strategy u of player I takes two values, "l" and "r." The strategy $v = (v_1, v_2)$ of player *II* consists of two components (for each information set X_2^1 and X_2^2) and has four possible values. The payoff matrix of this game is defined by

$$\begin{array}{c} \\ \text{l} \\ \text{r} \end{array} \begin{array}{c} \begin{array}{cccc} \text{ll} & \text{lr} & \text{rl} & \text{rr} \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & -2 & 2 & \frac{1}{2} \end{pmatrix} \end{array}.$$

Really, H(l,ll) $= \frac{1}{2}1 + \frac{1}{2}0 = \frac{1}{2}$, H(l,lr) $= \frac{1}{2}1 + \frac{1}{2}1 = 1$, H(l,rl) $= \frac{1}{2}(-1) + \frac{1}{2}0 = -\frac{1}{2}$, H(l,rr) $= \frac{1}{2}(-1) + \frac{1}{2}1 = 0$, H(r,ll) $= \frac{1}{2}(-2) + \frac{1}{2}1 = -\frac{1}{2}$, H(r,lr) $= \frac{1}{2}(-2) + \frac{1}{2}(-2) = -2$, H(r,rl) $= \frac{1}{2}3 + \frac{1}{2}1 = 2$, H(r,rr) $= \frac{1}{2}3 + \frac{1}{2}(-2) = \frac{1}{2}$.

The game admits the mixed strategy equilibria $(\frac{5}{7}, \frac{2}{7})$ and $(0, \frac{1}{7}, 0, \frac{6}{7})$ and has the value of $1/7$. Obviously, some information available to player *II* allows to reduce his loss (in comparison with the previous example).

Examples 4.1–4.3 show the relevance of informational partitions for game trees. Being in an information set, a player does not know the current position of a play. All positions in a given information set appear identical for a player. Thus, his strategy depends on a given information set only. Let the personal position set of player i be decomposed into information sets $X_i^1 \cup \ldots \cup X_i^{J_i}$. Here we comprehend alternatives as arcs connecting nodes x and $y \in Z(x)$.

Definition 4.9 *Suppose that, in a position $x \in X_i^j$, player i chooses among k_j alternatives, i.e., $Z(x) = \{y_1, \ldots, y_{k_j}\}$. A pure strategy of player i in a game with incomplete information is a function $u_i = u_i(X_i^j), j = 1, \ldots, J_i$, which assigns some alternative $k \in \{1, \ldots, k_j\}$ to each information set.*

Similar to games with complete information, specification of a pure strategy profile $(u_1, \ldots, u_n)$ and random move alternatives uniquely define a play of the game and the payoffs of any player. Actually, each player possesses a finite set of strategies—their number makes up $k_1 \times \ldots \times k_{J_i}$, $i = 1, \ldots, n$.

Definition 4.10 *A mixed strategy of player i in a game with incomplete information is a probability distribution $\mu_i = \mu_i(u_i)$ on the set of pure strategies of player i.*

Here $\mu_i(u_i)$ means the realization probability of the pure strategy $(u_i(X_i^1) = k_1, \ldots, u_i(X_i^{J_i}) = k_{J_i})$.

Definition 4.11 *A position $x \in X$ is feasible for a pure strategy u_i (μ_i), if there exists a strategy profile $u = (u_1, \ldots, u_i, \ldots, u_n)$ ($\mu = (\mu_1, \ldots, \mu_i, \ldots, \mu_n)$) such that a play passes through the position x with a positive probability. Denote by $Possu_i$ ($Poss\mu_i$) the set of such positions. An information set X_i^j is relevant for u_i (μ_i), if it contains, at least, one feasible position for u_i (μ_i). The collection of sets relevant for u_i (μ_i) will be designated by $Relu_i$ ($Rel\mu_i$).*

Consider some terminal node $t \in T$; denote by $[x_0, t]$ a play beginning at x_0 and ending at t. Assume that player i possesses a certain position in the play $[x_0, t]$. Let x indicate his last position in the play, $x \in X_i^j \cap [x_0, t]$, and k be an alternative in this position, which belongs to the play $[x_0, t]$, $i = 1, \ldots, n$. Under a given strategy profile $\mu = (\mu_1, \ldots, \mu_n)$, the realization probability of such play in the game becomes

$$P_\mu[x_0, t] = \left(\sum_{u:\, X_i^j \in Relu_i, u_i(X_i^j) = k} \prod_{i=1}^{n} \mu_i(u_i) \right) \prod_{x \in X_0 \cap [x_0, t], y \in Z(x) \cap [x_0, t]} p(y|x). \tag{3.1}$$

Formula (3.1) implies summation over all pure strategy profiles realizing a given play and multiplication by the probabilities of alternatives belonging to this play (for random moves). Under a given mixed strategy profile, the payoffs of players are the mean values

$$H_i(\mu_1, \ldots, \mu_n) = \sum_{t \in T} H_i(t) P_\mu[x_0, t], \;\; i = 1, \ldots, n. \tag{3.2}$$

Recall that the number of pure strategies appears finite. And so, this extensive-form game is equivalent to some non-cooperative normal-form game. The general theory of non-cooperative games claims the existence of a mixed strategy Nash equilibrium.

Theorem 4.2 *An extensive-form game with incomplete information has a mixed strategy Nash equilibrium.*

4.4 Total memory games

Although extensive-form games with incomplete information possess solutions in the class of mixed strategies, they do not seem practicable due to high dimensionality. Subsequent models of real games involve the so-called behavioral strategies.

Definition 4.12 *A behavioral strategy of player i is a vector function β_i defining for each information set X_i^j a probability distribution on the alternative set $(1, ..., k_j)$ for positions $x \in X_i^j$, $j = 1, ..., J_i$. Clearly,*

$$\sum_{k=1}^{k_j} \beta_i(X_i^j, k) = 1, \; j = 1, ..., J_i.$$

Consider some terminal position t and the corresponding play $[x_0, t]$. Under a given behavioral strategy profile $\beta = (\beta_1, ..., \beta_n)$, the realization probability of the play $[x_0, t]$ takes the form

$$P_\beta[x_0, t] = \prod_{i \in N, j=1,...,J_i, k \in [x_0,t]} \beta_i(X_i^j, k) \prod_{x \in X_0 \cap [x_0,t], y \in Z(x) \cap [x_0,t]} p(y|x). \tag{4.1}$$

And the expected payoff of players is described by

$$H_i(\beta_1, ..., \beta_n) = \sum_{t \in T} H_i(t) P_\beta[x_0, t], \; i = 1, ..., n. \tag{4.2}$$

Naturally enough, each behavioral strategy corresponds to a certain mixed strategy (the converse statement fails). Still, one can look for behavioral strategy equilibria in a wide class of games known as total memory games.

Definition 4.13 *A game Γ is a total memory game for player i, if for any pure strategy u_i and any information set X_i^j such that $X_i^j \in Relu_i$ it follows that any position $x \in X_i^j$ is feasible.*

According to this definition, any position from a relevant information set is feasible in a total memory game. Moreover, any player can exactly recover his alternatives at preceding moves.

Theorem 4.3 *In the total memory game Γ, any mixed strategy μ corresponds to some behavioral strategy ensuring the same probability distribution on the set of plays.*

Proof: Consider the total memory game Γ and a mixed strategy μ. Using it, we construct a special behavioral strategy for each player. Let X_i^j be the information set of player i and k represent a certain alternative in the position $x \in X_i^j$, $k = 1, ..., k_j$. Introduce

$$P_\mu(X_i^j) = \sum_{u_i : X_i^j \in Relu_i} \mu_i(u_i) \tag{4.3}$$

as the choice probability of the pure strategy u_i admitting the information set X_i^j, and

$$P_\mu(X_i^j, k) = \sum_{u_i:\, X_i^j \in Relu_i, u_i(X_i^j)=k} \mu_i(u_i) \tag{4.4}$$

as the choice probability of the pure strategy u_i admitting the information set X_i^j and the alternative $u_i(X_i^j) = k$. The following equality holds true:

$$\sum_{i=1}^{k_j} P_\mu(X_i^j, k) = P_\mu(X_i^j).$$

Evidently, a total memory game enjoys the following property. If the play $[x_0, t]$ with the terminal position t passes through the position $x_1 \in X_i^j$ of player i, alternative k, and the subsequent position of player i is $x_2 \in X_i^l$ (see Figure 4.8), the pure strategy sets

$$\{u_i : X_i^j \in Relu_i, u_i(X_i^j) = k\} \text{ and } \{u_i : X_i^l \in Relu_i\}$$

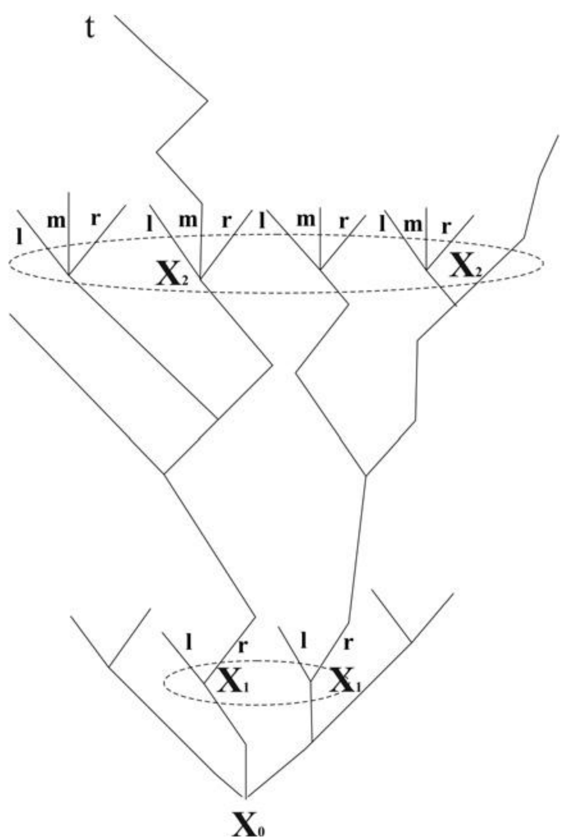


Figure 4.8 A total memory game.

do coincide. Therefore,

$$P_\mu(X_i^j, k) = P_\mu(X_i^l). \tag{4.5}$$

For each player $i = 1, ..., n$, define a behavioral strategy as follows. If X_i^j turns out relevant for μ_i, then

$$\beta_i(X_i^j, k) = \frac{P_\mu(X_i^j, k)}{P_\mu(X_i^j)}, \; k = 1, ..., k_j. \tag{4.6}$$

Otherwise, the denominator in (4.6) vanishes. Let us set

$$\beta_i(X_i^j, k) = \sum_{u_i:\, u_i(X_i^j)=k} \mu_i(u_i).$$

For instance, analyze the play $[x_0 t]$ in Figure 4.8. It passes through two information sets of player i, his pure strategy makes a pair of alternatives u = (ll, lm, lr, rl, rm, rr). In this case, his mixed strategy can be rewritten as the vector $\mu = (\mu_1, ..., \mu_6)$. By virtue of (4.6), the corresponding behavioral strategy acquires the following form. In the first information set, we obtain

$$\beta(X_1, \mathrm{l}) = \mu_1 + \mu_2 + \mu_3, \;\; \beta(X_1, \mathrm{r}) = \mu_4 + \mu_5 + \mu_6.$$

In the second information set, we obtain

$$\beta(X_2, \mathrm{l}) = \frac{\mu_4}{\mu_4 + \mu_5 + \mu_6}, \;\; \beta(X_2, \mathrm{m}) = \frac{\mu_5}{\mu_4 + \mu_5 + \mu_6}, \;\; \beta(X_2, \mathrm{r}) = \frac{\mu_6}{\mu_4 + \mu_5 + \mu_6}.$$

Obviously, the behavioral strategy in this play

$$\beta(X_1, \mathrm{r})\beta(X_2, \mathrm{m}) = \mu_5$$

completely matches the mixed strategy of the realization (r,m).

Now, we demonstrate that the behavioral strategy (4.6) yields precisely the same probability distribution on all plays as the mixed strategy μ.

Select the play $[x_0, t]$, where t is a terminal node. Suppose that the play $[x_0, t]$ sequentially intersects the information sets $X_i^1, ..., X_i^{J_i}$ of player i and alternatives $k_1, ..., k_{J_i}$ belonging to the path $[x_0, t]$ are chosen. If, at least, one of these sets appears irrelevant for μ_i, then $P_\mu[x_0, t] = P_\beta[x_0, t] = 0$. Therefore, suppose that all $X_i^j \in Rel\, \mu_i, j = 1, ..., J_i$.

It follows from (4.5) that, for β determined by (4.6) and any i, we have the equality

$$\prod_{j=1,...,J_i,\, k\in[x_0,t]} \beta_i(X_i^j, k) = \prod_{j=1,...,J_i,\, k\in[x_0,t]} \frac{P_\mu(X_i^j, k)}{P_\mu(X_i^j)} = P_\mu(X_i^{J_i}, k_{J_i}).$$

Evaluate $P_\beta[x_0, t]$ for β defined by (4.6). To succeed, transform the first product in formula (4.1):

$$\prod_{i\in N, j=1,\ldots,J_i, k\in[x_0,t]} \beta_i(X_i^j, k) = \prod_{i\in N, k_{J_i}\in[x_0,t]} P_\mu(X_i^{J_i}, k_{J_i}) =$$

$$\prod_{i\in N}\left(\sum_{u_i: X_i^{J_i}\in Relu_i, u_i(X_i^{J_i})=k_{J_i}} \mu_i(u_i)\right) = \left(\sum_{u: X_i^{J_i}\in Relu_i, u_i(X_i^{J_i})=k_{J_i}} \prod_{i=1}^{n} \mu_i(u_i)\right).$$

Thus,

$$P_\beta[x_0, t] = \left(\sum_{u: X_i^{J_i}\in Relu_i, u_i(X_i^{J_i})=k_{J_i}} \prod_{i=1}^{n} \mu_i(u_i)\right) \prod_{x\in X_0\cap[x_0,t], y\in Z(x)\cap[x_0,t]} p(y|x).$$

The last expression coincides with the representation (3.1) for $P_\mu[x_0, t]$.

We have argued that total memory games possess identical distributions of mixed strategies and corresponding behavioral strategies. Hence, the expected payoffs also coincide for such strategies. And so, while searching for equilibrium strategy profiles in such games, one can be confined to a wider class of behavioral strategies. The application of behavioral strategies will be illustrated in forthcoming chapters of the book.

Exercises

1. Consider a game with complete information described by the tree in Figure 4.9.

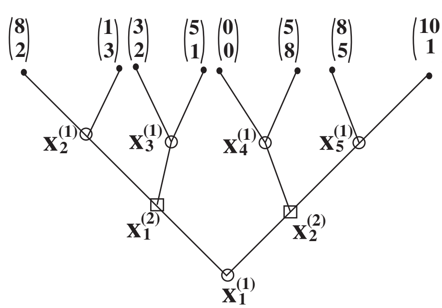


Figure 4.9 A game with complete information.

Find a subgame-perfect equilibrium in this game.

2. Evaluate an equilibrium in a game described by the tree in Figure 4.10: (a) under benevolent behavior and (b) under malevolent behavior of players.

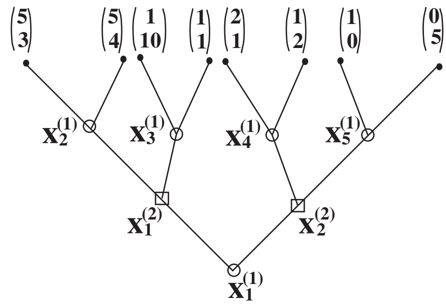


Figure 4.10 A game with complete information.

3. Establish a condition when malevolent behavior yields higher payoffs to both players (in comparison with their benevolent behavior).

4. Find an indifferent equilibrium in game no. 2.

5. Reduce the following game (see the tree in Figure 4.11) to the normal form.

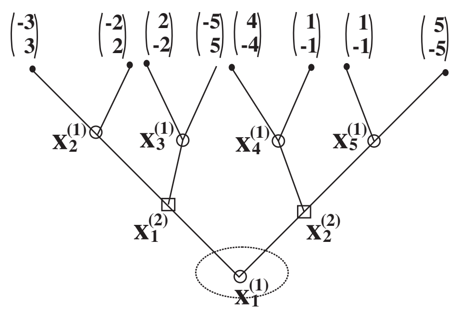


Figure 4.11 Zero-sum game in extensive form.

6. Consider a game with incomplete information described by the tree in Figure 4.12.

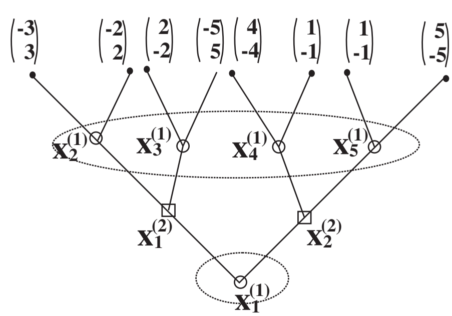


Figure 4.12 A game with incomplete information.

Find a subgame-perfect equilibrium in this game.

7. Evaluate an equilibrium in a game with incomplete information described by the tree in Figure 4.13.

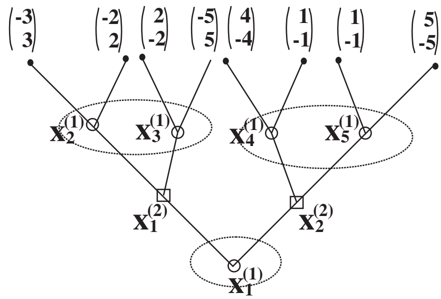


Figure 4.13 A game with incomplete information.

8. Give an example of a partial memory game in the extensive form.

9. A card game.

This game involves two players. Each of them has two cards: $x_1 = 0, x_2 = 1$ (player I) and $y_1 = 0, y_2 = 1$ (player II). Two additional cards lie on the table: $z_1 = 0, z_2 = 1$. The top card on the table is turned up. Each player chooses one of his cards and puts it on the table. The winner is the player putting a higher card; his payoff is the value of the opponent's card. If both players put identical cards, the game is drawn. Construct the corresponding tree and specify information sets.

10. Construct the tree for game no. 9 provided that x, y, z represent independent random variables with the uniform distribution on $[0, 1]$.

5

Parlor games and sport games

Introduction

Parlor games include various card games, chess, draughts, etc. Many famous mathematicians (J. von Neumann, R. Bellman, S. Karlin, T. Ferguson, M. Sakaguchi, to name a few) endeavored to apply game theory methods to parlor games. We have mentioned that chess analysis provokes small interest (this is a finite game with complete information—an equilibrium does exist). Recent years have been remarkable for the development of very powerful chess computers (e.g., *Junior*, *Hydra*, *Pioneer*) that surpass human capabilities.

On the other hand, card games represent games with incomplete information. Therefore, it seems attractive to model psychological effects (risk, bluffing, etc.) by game theory methods. Here we will search for equilibria in the class of behavioral strategies.

Our investigation begins with poker. Let us describe this popular card game. A poker pack consists of 52 cards of four suits (spades, clubs, diamonds, and hearts). Cards within a suit differ by their denomination. There exist 13 denominations: 2, 3, … , 10, jack, queen, king, and ace. In poker, each player is dealt five cards. Different combinations of cards (called hands) have specific rankings. A typical hand ranking system is as follows. The highest ranking belongs to

(a) royal flush (10, jack, queen, king, ace—all having a same suit); the corresponding probability is approximately $1.5 \cdot 10^{-6}$.

Lower rankings (in the descending order of their probabilities) are assigned to the following hands:

(b) four of a kind or quads (all four cards of one denomination and any other (unmatched) card); the probability makes up 0.0002;

(c) full house (three matching cards of one denomination and two matching cards of another denomination); the probability equals 0.0014;

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

(d) straight flush (five cards of sequential denomination in at least two different suits); the probability is 0.0035;

(e) three of a kind or trips (three cards of the same denomination, plus two cards which are not of this denomination nor the same as each other); the probability constitutes 0.0211;

(f) two pairs (two cards of the same denomination, plus two cards of another denomination (that match each other but not the first pair), plus any card not of either denomination); the probability makes up 0.0475;

(g) one pair (two cards of one denomination, plus three cards which are not of this denomination nor the same as each other); the probability is given by 0.4225.

After cards selection, players make bets. Subsequently, they open the cards and the one having the highest ranking hand breaks the bank. Players do not know the cards of their opponents—poker is a game with incomplete information.

5.1 Poker. A game-theoretic model

As a mathematical model of this card game, let us consider a two-player game. In the beginning of a play, both players (e.g., Peter and Paul) contribute the buy-ins of 1. Afterwards, they are dealt two cards of denominations x and y, respectively (a player has no information on the opponent's card). Peter moves first. He either passes (losing his buy-in), or makes a bet $c > 1$. In this case, the move is given to Paul who chooses between the same alternatives. If Paul passes, he loses his buy-in; otherwise, the players open up the cards and the one having a higher denomination becomes the winner.

Note that the cards of players possess random denominations. It is necessary to define the probabilistic character of all possible outcomes. Assume that the denominations of cards lie within the interval from 0 to 1 and appear equiprobable. In other words, the random variables x and y obey the uniform distribution on the interval $[0, 1]$.

Now, specify strategies in this game. Each player merely knows his card; hence, his decision is based on this knowledge. Therefore, we understand Peter's strategy as a function $\alpha(x)$—the probability of betting under the condition that he disposes of the card x. Since α represents a probability, its values satisfy $0 \le \alpha \le 1$ and the function $\bar{\alpha} = 1 - \alpha$ corresponds to the probability of passing. Similarly, if Peter bets something, Paul's strategy consists in a function $\beta(y)$—the probability of calling provided that he has the card y. Obviously, $0 \le \beta \le 1$.

Different combinations of cards (hands) appear in the course of a play. Thus, the payoff of each player represents a random quantity. As a criterion, we adopt the expected value of the payoff. Imagine that the players have selected their strategies (α and β). By virtue of the game conditions, the expected payoff of player I makes up

-1, with the probability of $\bar{\alpha}(x)$,

$+1$, with the probability of $\alpha(x)\bar{\beta}(y)$,

$(c + 1)sgn(x - y)$, with the probability of $\alpha(x)\beta(y)$.

Here the function $sgn(x - y)$ equals 1, if $x > y$; -1, if $x < y$ and 0, if $x = y$.

Due to these expressions, the expected payoff of Peter becomes

$$H(\alpha, \beta) = \int_0^1 \int_0^1 \left[-\bar{\alpha}(x) + \alpha(x)\bar{\beta}(y) + (c+1)sgn(x-y)\alpha(x)\beta(y)\right] dxdy. \qquad (1.1)$$

For the time being, the game is completely defined. Actually, we have described the strategies and payoffs of both players. Player I strives for maximizing the expected payoff (1.1), whereas player II seeks to minimize it.

5.1.1 Optimal strategies

Readers would easily guess the form of optimal strategies. By extracting the terms containing $\alpha(x)$, rewrite the payoff (1.1) as

$$H(\alpha, \beta) = \int_0^1 \alpha(x)\left[1 + \int_0^1 \left(\bar{\beta}(y) + (c+1)sgn(x-y)\beta(y)\right) dy\right] dx - 1. \qquad (1.2)$$

Denote by Q(x) the bracketed expression in (1.2). It follows from (1.2) that Peter's optimal strategy $\alpha^*(x)$, maximizing his payoff, takes the following form. If $Q(x) > 0$, then $\alpha^*(x) = 1$ and, if $Q(x) < 0$, then $\alpha^*(x) = 0$. In the case of $Q(x) = 0$, the function $\alpha^*(x)$ possesses any values.

The function $sgn(x - y)$, as well as the function $Q(x)$ proper, are non-decreasing. Figure 5.1 illustrates that the optimal strategy $\alpha^*(x)$ must be defined by some threshold a. If the dealt card x has a denomination smaller than a, the player should pass (and bet otherwise).

Similarly, we reexpress the payoff $H(\alpha, \beta)$ as

$$H(\alpha, \beta) = \int_0^1 \beta(y)\left[\int_0^1 \alpha(x)\left(-(c+1)sgn(y-x) - 1\right) dx\right] dy + \int_0^1 (2\alpha(x) - 1)dx. \quad (1.3)$$

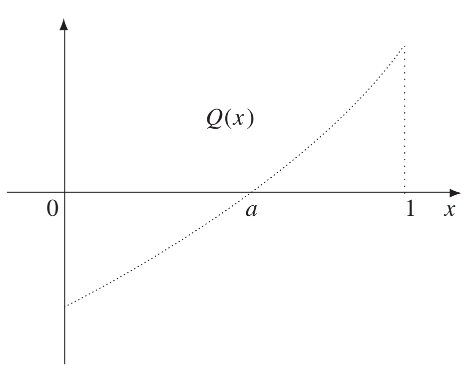


Figure 5.1 The function $Q(x)$.

And Paul's optimal strategy $\beta^*(y)$ also gets conditioned by a certain threshold b. If his card's denomination exceeds this threshold, Paul makes a bet (and passes otherwise).

Let us evaluate the stated optimal thresholds a^*, b^*.

Suppose that Peter employs the strategy α with a threshold a. According to (1.3), Paul's payoff makes up

$$H(\alpha, \beta) = \int_0^1 \beta(y)G(y)dy + 2(1-a) - 1, \tag{1.4}$$

where $G(y) = \int_a^1 [-(c+1)sgn(y-x) - 1]dx$.

A series of standard calculations lead to

$G(y) = \int_a^1 cdx = c(1-a)$, if $y < a$,

$G(y) = \int_a^y (-c-2)dx + \int_y^1 cdx = -2(c+1)y + a(c+2) + c$, if $y \geq a$.

Figure 5.2 shows the curve of $G(y)$. Obviously, the optimal threshold b is defined by $-2(c+1)b + a(c+2) + c = 0$, whence it appears that

$$b = \frac{1}{2(c+1)}[a(c+2) + c]. \tag{1.5}$$

Therefore, the optimal threshold of player II is uniquely defined by the corresponding threshold of the opponent. And the minimal value of Paul's loss equals

$$\begin{aligned} H(\alpha, \beta) &= \int_b^1 G(y)dy + 2(1-a) - 1 \\ &= \int_b^1 [-2(c+1)y + a(c+2) + c]dy + 2(1-a) - 1. \end{aligned}$$

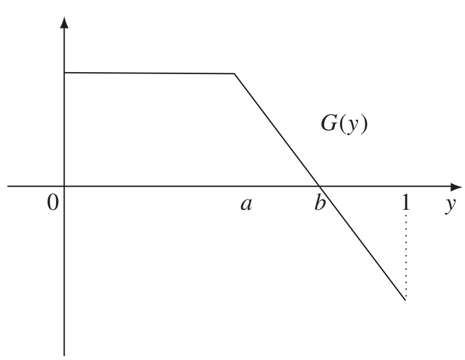


Figure 5.2 The function $G(y)$.

Integration yields

$$H(\alpha, \beta) = -(c+1)(1-b^2) + [a(c+2)+c](1-b) - 2a + 1$$
$$= (c+1)b^2 - b[a(c+2)+c] + ac. \qquad (1.6)$$

Substitute the optimal value of b (see (1.5)) into formula (1.6) to represent Paul's minimal loss as a function of argument a:

$$H(a) = \frac{1}{4(c+1)}[a(c+2)+c]^2 - \frac{1}{2(c+1)}[a(c+2)+c]^2 + ac.$$

Some uncomplicated manipulations bring to

$$H(a) = \frac{(c+2)^2}{4(c+1)}\left[-a^2 + 2a\frac{c^2}{(c+2)^2} - \frac{c^2}{(c+2)^2}\right]. \qquad (1.7)$$

Recall that a forms Peter's strategy—he strives for maximizing the minimal loss of Paul, see (1.7). Therefore, we finally arrive at the maximization problem for the parabola

$$h(a) = -a^2 + 2a\frac{c^2}{(c+2)^2} - \frac{c^2}{(c+2)^2}.$$

This function is demonstrated in Figure 5.3. Its maximum lies at the point

$$a^* = \left(\frac{c}{c+2}\right)^2$$

within the interval $[0, 1]$. Substitute this value into (1.5) to find the optimal threshold of player II:

$$b^* = \frac{c}{c+2}.$$

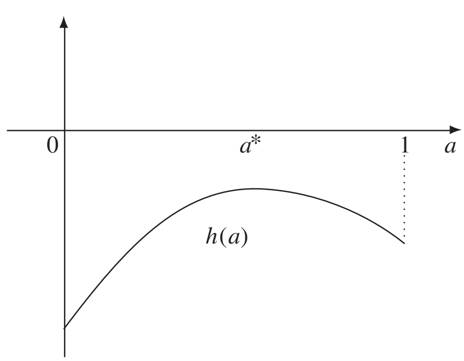


Figure 5.3 The parabola $h(a)$.

The payoff of player I (being the best for Peter and Paul) results from substituting the optimal threshold a^* into (1.7):

$$H^* = H(a^*, b^*) = \frac{(c+2)^2}{4(c+1)}\left[\left(\frac{c}{c+2}\right)^4 - \left(\frac{c}{c+2}\right)^2\right] = -\left(\frac{c}{c+2}\right)^2.$$

Apparently, the game has negative value, i.e., player I (Peter) is disadvantaged.

5.1.2 Some features of optimal behavior in poker

We have evaluated the optimal payoffs of both players and the game value. The optimal threshold of player I is smaller than that of his opponent. In other words, Peter should be more careful. The game possesses a negative value. This aspect admits the following explanation. The move of player I provides some information on his card to player II.

Now, we discuss uniqueness of optimal strategies. Figure 5.2 elucidates the following. If player I (Peter) employs the optimal strategy $\alpha^*(x)$ with the threshold $a^* = \left(\frac{c}{c+2}\right)^2$, then the best response of player II (Paul) is also the threshold strategy $\beta^*(y)$ with the threshold $b^* = \frac{c}{c+2}$. Notably, Paul's optimal strategy appears uniquely defined.

Fix Paul's strategy with the threshold b^* and find the best response of Peter. For this, address the expression (1.2) and compute the function $Q(x)$. Under the given b^*, we establish that, if $x < b^*$, then

$$Q(x) = 1 + \int_0^1 \left(\bar{\beta}(y) + (c+1)sgn(x-y)\beta(y)\right) dy$$
$$= 1 + \int_0^{b^*} dy - \int_{b^*}^1 (c+1)dy = 1 + b^* - (c+1)(1-b^*) = 0.$$

On the other hand, if $x \geq b^*$,

$$Q(x) = 1 + b^* + \int_{b^*}^{x} (c+1)dy - \int_x^1 (c+1)dy$$
$$= 1 + b^* + (c+1)(x-b^*) - (c+1)(1-x) = 2(c+1)x + (c+2)(b^*+1).$$

Figure 5.4 demonstrates that the function $Q(x)$ is positive on the interval $(b^*, 1]$. If Peter has a card $x > b^*$, his best response consists in betting. However, if x lies within the interval $[0, b^*]$, then $Q(x) = 0$ and $\alpha^*(x)$ may possess any values (this does not affect the payoff (1.2)). Of course, the evaluated strategy with the threshold a^* meets this condition. Is there another Peter's strategy $\alpha(x)$ such that Paul's optimal strategy coincides with $\beta^*(y)$?

Such strategies do exist. For instance, consider the following strategy $\alpha(x)$. If $x \geq b^*$, player I makes a bet; in the case of $x < b^*$, he makes a bet with the probability of $p = \frac{2}{c+2}$ (and passes with the probability $\bar{p} = 1 - p = \frac{c}{c+2}$, accordingly). Find the best response of

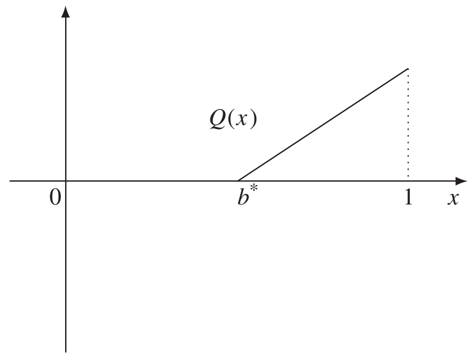


Figure 5.4 The function $Q(x)$.

player *II* to the described strategy of the opponent. Again, rewrite the payoff in the form (1.4). The function $G(y)$ is then defined by

$$G(y) = \int_0^{b^*} p(-(c+1)sgn(y-x) - 1)dx + \int_{b^*}^1 (-(c+1)sgn(y-x) - 1)dx,$$

whence it follows that, under $y < b^*$,

$$\begin{aligned} G(y) &= p\int_0^y (-(c+1) - 1)dx + p\int_y^{b^*} (c+1-1)dx + \int_{b^*}^1 (c+1-1)dx \\ &= -2p(c+1)y + pcb^* + c(1-b^*), \end{aligned}$$

and, under $y \geq b^*$,

$$\begin{aligned} G(y) &= p\int_0^{b^*} (-c-2)dx + \int_{b^*}^y (-c-2)dx + \int_y^1 cdx \\ &= -2(c+1)y + (c+2)b^*(1-p) + c. \end{aligned}$$

The curve of $G(y)$ can be observed in Figure 5.5. Interestingly, the choice of p leads to $G(b^*) = 0$ and the best strategy of Paul is still $\beta^*(y)$.

Therefore, we have obtained another solution to this game. It fundamentally differs from the previous one for player I. Now, Peter can make a bet even having a card of a small denomination. Such effect in card games is well-known as bluffing. A player feigns that he has a high-rank card, thus compelling the opponent to pass. However, the probability of bluffing is smaller for larger bets c. For instance, if $c = 100$, the probability of bluffing must be less than 0.02.

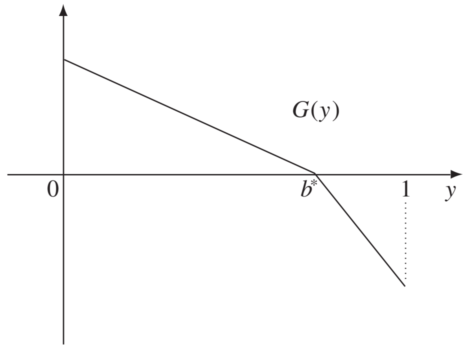


Figure 5.5 The function $G(y)$.

5.2 The poker model with variable bets

The above poker model proceeds from fixed bets c. In real games, bets vary. Consider a variable-bet modification of the model.

As usual, Peter and Paul contribute the buy-ins of 1 in the beginning of a play. Afterwards, they are dealt two cards of denominations x and y, respectively (a player has no information on the opponent's card). At first shot, Peter makes a bet $c(x)$ depending on the denomination x of his card. The move is given to Paul who chooses between the same alternatives. If Paul passes, he loses his buy-in; otherwise, he calls the opponent's bet and adds $c(x)$ to the bank. The players open up the cards and the one having a higher denomination becomes the winner. In this model, Peter wins 1 or $(1 + c(x))sgn(x - y)$. The problem is to find the optimal function $c(x)$ and optimal response of player II. It was originally formulated by R. Bellman in the late 1950s [Bellman et al. 1958].

Our analysis starts with the discrete model of this game. Suppose that Peter's bet takes any value from a finite set $0 < c_1 < c_2 < \cdots < c_n$. Then the strategy of player I is a mixed strategy $\alpha(x) = (\alpha_1(x), \dots, \alpha_n(x))$, where $\alpha_i(x)$ denotes the probability of betting $c_i, i = 1, \dots, n$, provided that his card has the denomination of x. Consequently, $\sum_{i=1}^n \alpha_i = 1$. The strategy of player II lies in a behavioral strategy $\beta(y) = (\beta_1(y), \dots, \beta_n(y))$, where $\beta_i(y)$ designates the probability of calling the bet c_i, $0 \le \beta_i \le 1, i = 1, \dots, n$, under the selected card y. Accordingly, $\bar{\beta}_i(y) = 1 - \beta_i(y)$ gives the probability of passing under the bet c_i and the card y.

The expected payoff of player I acquires the form

$$H(\alpha, \beta) = \int_0^1 \int_0^1 \sum_{i=1}^n \left[\alpha_i(x)\bar{\beta}_i(y) + (1 + c_i)sgn(x - y)\alpha_i(x)\beta_i(y)\right] dxdy. \tag{2.1}$$

First, consider the case of $n = 2$.

5.2.1 The poker model with two bets

Assume that player I can bet c_1 or c_2 $(c_1 < c_2)$ depending on a selected card x. Therefore, his strategy can be defined via the function $\alpha(x)$—the probability of the bet c_1. The quantity

$\bar{\alpha}(x) = 1 - \alpha(x)$ indicates the probability of the bet c_2. Player *II* strategy is completely described by two functions, $\beta_1(y)$ and $\beta_2(y)$, that specify the probabilities of calling the bets c_1 and c_2, respectively. The payoff function (2.1) becomes

$$H(\alpha, \beta) = \int_0^1 \int_0^1 \big[\alpha(x)\bar{\beta}_1(y) + (1 + c_1)sgn(x - y)\alpha(x)\beta_1(y)$$
$$+ (1 - \alpha(x))\bar{\beta}_2(y) + (1 + c_2)sgn(x - y)(1 - \alpha(x))\beta_2(y)\big]\, dxdy. \tag{2.2}$$

We evaluate the optimal strategy of player *II*. Take formula (2.2) and extract terms with β_1 and β_2. They are

$$\int_0^1 \beta_1(y)dy\left[\int_0^1 \alpha(x)(-1 + (1 + c_1)sgn(x - y))dx\right] \tag{2.3}$$

and

$$\int_0^1 \beta_2(y)dy\left[\int_0^1 (1 - \alpha(x))(-1 + (1 + c_2)sgn(x - y))dx\right]. \tag{2.4}$$

The function $sgn(x - y)$ is non-increasing in y. Hence, the bracketed expressions in (2.3)–(2.4) (denote them by $G_i(y), i = 1, 2$) represent non-increasing functions of y (see Figure 5.6). Suppose that the functions $G_i(y)$ intersect axis O_y within the interval $[0, 1]$ at some points $b_i, i = 1, 2$.

Player *II* aims at minimizing the functionals (2.3)–(2.4). The integrals in these formulas possess minimal values under the following necessary condition. The function $\beta_i(y)$ vanishes for $G_i(y) > 0$ and equals 1 for $G_i(y) < 0, i = 1, 2$. And so, the optimal strategy of player *II* has the form $\beta_i(y) = I(y \geq b_i), i = 1, 2$, where $I(A)$ means the indicator of the set A. In other words, player *II* calls the opponent's bet under sufficiently high denominations of cards (exceeding

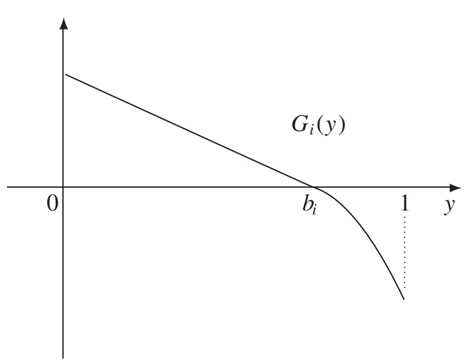


Figure 5.6 The function $G_i(y)$.

the threshold $b_i, i = 1, 2$). So long as $c_1 < c_2$, a natural supposition is the following. The threshold for calling a higher bet must be greater, as well: $b_1 < b_2$.

The thresholds b_1, b_2 are defined by the equations $G_i(b_i) = 0, i = 1, 2$, or, according to (2.3)–(2.4),

$$\int_0^{b_1} (-2 - c_1)\alpha(x)dx + \int_{b_1}^1 c_1\alpha(x)dx = 0 \tag{2.5}$$

and

$$\int_0^{b_2} (-2 - c_2)\bar{\alpha}(x)dx + \int_{b_2}^1 c_2\bar{\alpha}(x)dx = 0. \tag{2.6}$$

Now, construct the optimal strategy of player I—the function $\alpha(x)$. In the payoff (2.2), extract the expression containing $\alpha(x)$:

$$\int_0^1 \alpha(x)dx\left[\int_0^1 \beta_2(y) - \beta_1(y) + sgn(x - y)\left((1 + c_1)\beta_1(y) - (1 + c_2)\beta_2(y)\right) dy\right].$$

Designate by $Q(x)$ the bracketed expression above. For x such that $Q(x) < 0$ ($Q(x) > 0$), the optimal strategy $\alpha(x)$ equals zero (unity, respectively). In the case of $Q(x) = 0$, the function $\alpha(x)$ takes arbitrary values. After some transformations, we obtain

$$Q(x) = \int_0^x \left(c_1\beta_1(y) - c_2\beta_2(y)\right) dy + \int_x^1 \left((2 + c_2)\beta_2(y) - (2 + c_1)\beta_1(y)\right) dy.$$

Recall the form of the strategies $\beta_i(x), i = 1, 2$. The derivative of the function $Q(x)$,

$$Q'(x) = (2 + 2c_1)\beta_1(x) - (2 + 2c_2)\beta_2(x),$$

allows being represented by

$$Q'(x) = \begin{cases} 0, & \text{if } x \in [0, b_1] \\ 2 + 2c_1, & \text{if } x \in (b_1, b_2) \\ -2(c_2 - c_1), & \text{if } x \in [b_2, 1]. \end{cases}$$

Therefore, the function $Q(x)$ appears constant on the interval $[0, b_1]$, increases on the interval (b_1, b_2), and decreases on the interval $[b_2, 1]$.

Require that the function $Q(x)$ vanishes on the interval $[0, b_1]$ and crosses axis O_x at some point a on the interval $[b_2, 1]$ (see Figure 5.7). Consequently, we will have $b_1 < b_2 < a$.

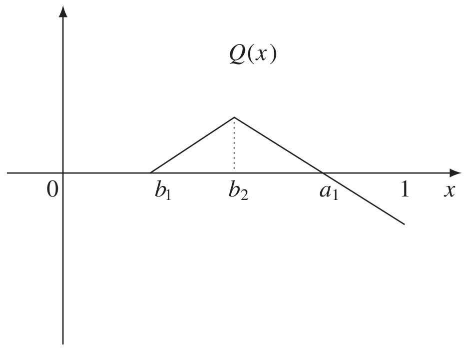


Figure 5.7 The function $Q(x)$.

For it is necessary that

$$Q(0) = \int_{b_2}^{1} (2 + c_2)dy - \int_{b_1}^{1} (2 + c_1)dy = 0$$

and

$$Q(a) = \int_{b_1}^{a} c_1 dy - \int_{b_2}^{a} c_2 dy + \int_{a}^{1} (c_2 - c_1)dy = 0.$$

Further simplification of the above conditions yields

$$(1 - b_1)(2 + c_1) = (1 - b_2)(2 + c_2),$$
$$(c_2 - c_1)(2a - 1) = c_2 b_2 - c_1 b_1.$$

Under these conditions, the optimal strategy of player I acquires the form:

$$\alpha(x) = \begin{cases} \text{an arbitrary value,} & \text{if } x \in [0, b_1] \\ 1, & \text{if } x \in (b_1, a) \\ 0, & \text{if } x \in [a, 1]. \end{cases}$$

And the conditions (2.5)–(2.6) are rewritten as

$$\int_{0}^{b_1} \alpha(x)dx = \frac{c_1(a - b_1)}{2 + c_1} = b_1 - \frac{c_2(1 - a)}{2 + c_2}. \tag{2.7}$$

Therefore, the parameters of players' optimal strategies meet the system of equations

$$\begin{aligned} (1-b_1)(2+c_1) &= (1-b_2)(2+c_2), \\ (c_2-c_1)(2a-1) &= c_2b_2 - c_1b_1, \\ \frac{c_1(a-b_1)}{2+c_1} &= b_1 - \frac{c_2(1-a)}{2+c_2}. \end{aligned} \tag{2.8}$$

The system of equations (2.8) possesses a solution $0 \le b_1 < b_2 \le a \le 1$. We demonstrate this fact in the general case.

The optimal strategy of player I is remarkable for the following reasons. It takes arbitrary values on the interval $[0, b_1]$ such that the condition (2.7) holds true. This corresponds to a bluffing strategy, since player I can make a high bet for small rank cards. The optimal strategy of player II dictates to escape a play under small rank cards (and call a certain bet of the opponent under sufficiently high denominations of the cards).

For instance, we select $c_1 = 2, c_2 = 4$ to obtain the following optimal parameters: $b_1 = 0.345, b_2 = 0.563, a = 0.891$. If the rank of his cards is less than 0.345, player I bluffs. He bets 2, if the card rank exceeds the above threshold yet is smaller than 0.891. And finally, for cards whose denomination is higher than 0.891, player I bets 4. Player II calls the bet of 2, if the rank of his cards belongs to the interval $[0.345, 0.563]$ and calls the bet of 4, if the card rank exceeds 0.563. In the rest situations, player II prefers to pass.

5.2.2 The poker model with *n* bets

Now, assume that player I is dealt a card x and can bet any value from a finite set $0 < c_1 < \cdots < c_n$. Then his strategy lies in a mixed strategy $\alpha(x) = (\alpha_1(x), \dots, \alpha_n(x))$, where $\alpha_i(x)$ represents the probability of making the bet c_i. The next shot belongs to player II. Depending on a selected card y, he either passes (losing his buy-in in the bank), or continues the play. In the latter case, player II has to call the opponent's bet. Subsequently, both players open up their cards; the winner is the one whose card possesses a higher denomination. The strategy of player II is a behavioral strategy $\beta(y) = (\beta_1(y), \dots, \beta_n(y))$, where $\beta_i(y)$ indicates the probability of calling the bet of player I (the quantity c_i, $i = 1, \dots, n$). And the payoff function takes the form

$$H(\alpha, \beta) = \int_0^1 \int_0^1 \sum_{i=1}^n \left[\alpha_i(x)\bar{\beta}_i(y) + (1+c_i)sgn(x-y)\alpha_i(x)\beta_i(y)\right] dxdy. \tag{2.9}$$

First, we evaluate the optimal strategy of player II. To succeed, rewrite the function (2.9) as

$$H(\alpha, \beta) = \sum_{i=1}^n \int_0^1 \beta_i(y)dy \left[\int_0^1 \alpha_i(x)\left(-1 + (1+c_i)sgn(x-y)\right) dx\right] + 1. \tag{2.10}$$

Denote by $G_i(y)$ the bracketed expression in the previous formula. For each fixed strategy $\alpha(x)$ and bet c_i, player II strives to minimize (2.10). Therefore, for any $i = 1, \dots, n$, his optimal strategy is given by

$$\beta_i(y) = \begin{cases} 0, & \text{if } G_i(y) > 0 \\ 1, & \text{if } G_i(y) < 0. \end{cases}$$

Obviously, the function

$$G_i(y) = -(2 + c_i) \int_0^y \alpha_i(x)dx + c_i \int_y^1 \alpha_i(x)dx$$

does not increase in y. Furthermore, $G_i(0) = c_i \int_0^1 \alpha_i(x)dx \geq 0$ and $G_i(1) = -(2 + c_i) \int_0^1 \alpha_i(x)dx \leq 0$. Hence, the equation $G_i(y) = 0$ always admits a root b_i (see Figure 5.6). The quantity b_i satisfies the equation

$$\int_0^{b_i} \alpha_i(x)dx = \frac{c_i}{2 + c_i} \int_{b_i}^1 \alpha_i(x)dx. \tag{2.11}$$

And so, the optimal strategy of player II becomes

$$\beta_i(y) = \begin{cases} 0, & \text{if } 0 \leq y < b_i \\ 1, & \text{if } b_i \leq y \leq 1 \end{cases}$$

$i = 1, \dots, n$. Interestingly, the values $b_i, i = 1, \dots, n$ meeting (2.11) do exist for any strategy $\alpha(x)$.

Construct the optimal strategy of player I. Reexpress the payoff function (2.9) as

$$H(\alpha, \beta) = \sum_{i=1}^n \int_0^1 \alpha_i(x)Q_i(x)dx, \tag{2.12}$$

where

$$Q_i(x) = \int_0^1 \left(\bar{\beta}_i(y) + (1 + c_i)sgn(x - y)\beta_i(y)\right) dy$$

or

$$Q_i(x) = b_i + (1 + c_i)\left(\int_0^x \beta_i(y)dy - \int_x^1 \beta(y)dy \right). \tag{2.13}$$

For each x, player I seeks a strategy $\alpha(x)$ maximizing the payoff (2.12). Actually, this is another optimization problem which differs from the one arising for player II.

Here $\alpha(x)$ forms a mixed strategy, $\sum_{i=1}^{n} \alpha_i(x) = 1$. The maximal value of the payoff (2.12) is attained by $\alpha(x)$ such that $\alpha_i(x) = 1$, if for a given x the function $Q_i(x)$ takes greater values than other functions $Q_j(x), j \neq i$, or $\alpha_i(x) = 0$ (otherwise). The function $\alpha(x)$ may possess arbitrary values, if all values $Q_i(x)$ coincide for a given value x.

We search for the optimal strategy $\alpha(x)$ in a special class. Let all functions $Q_i(x)$ coincide on the interval $[0, b_1)$, i.e., $Q_1(x) = \ldots = Q_n(x)$. This agrees with bluffing by player I. Set $a_1 = b_1$ and suppose that $Q_1(x) > \max\{Q_j(x), j \neq 1\}$ on the interval $[a_1, a_2)$, $Q_2(x) > \max\{Q_j(x), j \neq 2\}$ on the interval $[a_2, a_3)$, and so on. Moreover, assume that the maximal value on the interval $[a_n, 1]$ belongs to $Q_n(x)$. Then the optimal strategy of player I acquires the form:

$$\alpha_i(x) = \begin{cases} \text{an arbitrary value}, & \text{if } x \in [0, b_1] \\ 1, & \text{if } x \in [a_i, a_{i+1}) \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

We specify the function $Q_i(x)$. Further simplification of (2.13) yields

$$Q_i(x) = \begin{cases} b_i - (1 + c_i)(1 - b_i), & \text{if } 0 \leq x < b_i \\ (1 + c_i)(2x - 1) - c_i b_i, & \text{if } b_i \leq x \leq 1. \end{cases}$$

The function $Q_i(x)$ has constant values on the interval $[0, b_i]$. Require that these values are identical for all functions $Q_i(x), i = 1, \ldots, n$:

$$b_i - (1 + c_i)(1 - b_i) = k, \ \ i = 1, \ldots, n.$$

In this case, all $b_i, i = 1, \ldots, n$ satisfy the formula

$$b_i = \frac{1 + k + c_i}{2 + c_i} = 1 - \frac{1 - k}{2 + c_i}, \ \ i = 1, \ldots, n. \tag{2.15}$$

It is immediate from (2.15) that $b_1 < b_2 < \cdots < b_n$. This fact conforms with intuitive reasoning that player II must call a higher bet under higher-rank cards.

The function $Q_i(x)$ is linear on the interval $[b_i, 1]$. Let a_i, $i = 2, \ldots, n$ designate the intersection points of the functions $Q_{i-1}(x)$ and $Q_i(x)$. In addition, $a_1 = b_1$. To assign the form (2.14) to the optimal strategy $\alpha(x)$, we require that $a_1 < a_2 < \cdots < a_n$. Then the function $Q_i(x) (i = 1, \ldots, n)$ is maximal on the interval $[a_i, a_{i+1})$. Figure 5.8 demonstrates the functions $Q_i(x), i = 1, \ldots, n$.

The intersection points a_i result from the equations

$$(1 + c_{i-1})(2a_i - 1) - c_{i-1} b_{i-1} = (1 + c_i)(2a_i - 1) - c_i b_i, \ \ i = 2, \ldots, n,$$

or, after simple transformations,

$$a_i = 1 - \frac{\bar{k}}{(2 + c_{i-1})(2 + c_i)}, i = 2, \ldots, n, \tag{2.16}$$

where $\bar{k} = 1 - k$.

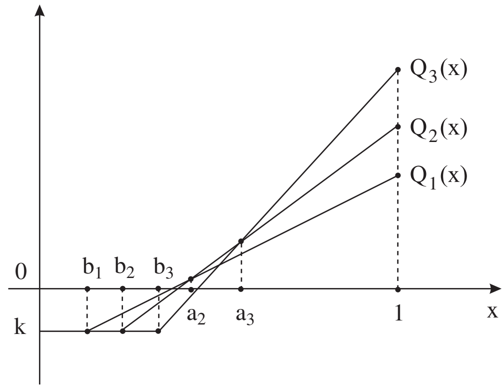


Figure 5.8 Optimal strategies.

It remains to find k. Recall that the optimal thresholds b_i of player *II* strategies satisfy equation (2.11). By virtue of (2.14), this equation takes the form

$$\int_0^{b_1} \alpha_i(x)dx = \frac{c_i}{2+c_i}(a_{i+1} - a_i),\ i = 1, \dots, n. \tag{2.17}$$

By summing up all equations (2.17) and considering the condition $\sum_{i=1}^{n} \alpha_i(x) = 1$, we arrive at

$$b_1 = \sum_{i=1}^{n} \frac{c_i}{2+c_i}(a_{i+1} - a_i).$$

Hence, it follows that

$$\frac{1+k+c_1}{2+c_1} = \bar{k}A,$$

where

$$A = \sum_{i=1}^{n} \frac{c_i(c_{i+1} - c_{i-1})}{(2+c_{i-1})(2+c_i)^2(2+c_{i+1})}.$$

We believe that $c_0 = -1, c_{n+1} = \infty$ in the sum above. Consequently,

$$k = 1 - \frac{2+c_1}{A(2+c_1)+1}.$$

Clearly, A and $\bar{k}$ are both positive. Therefore, the sequence a_i appears monotonous, $a_1 < a_2 < \cdots < a_n$. And all thresholds a_i lie within the interval $[0, 1]$.

Let us summarize the outcomes. The optimal strategy of player *I* is defined by

$$\alpha_i^*(x) = \begin{cases} \text{an arbitrary function meeting the condition (2.17)} & \text{if } x \in [0, b_1] \\ 1, & \text{if } x \in [a_i, a_{i+1}) \\ 0, & \text{otherwise} \end{cases}$$

where $a_i = 1 - \frac{\bar{k}}{(2+c_{i-1})(2+c_i)}, i = 2, \dots, n$. Note that player *I* bluffs on the interval $[0, b_1)$. Under small denominations of cards, he can bet anything. For definiteness, it is possible to decompose the interval $[0, b_1)$ into successive subintervals of the length $\frac{c_i}{2+c_i}(a_{i+1} - a_i), i = 1, \dots, n$ (by construction, their sum equals b_1) and set $\alpha_i^*(x) = c_i$ on a corresponding interval. For $x > b_1$, player *I* has to bet c_i on the interval $[a_i, a_{i+1})$.

The optimal strategy of player *II* is defined by

$$\beta_i^*(y) = \begin{cases} 0, & \text{if } 0 \le y < b_i \\ 1, & \text{if } b_i \le y \le 1 \end{cases}$$

where $b_i = \frac{1+k+c_i}{2+c_i},\ i = 1, \dots, n.$

Find the value of this game from (2.12):

$$H(\alpha^*, \beta^*) = \sum_{i=1}^{n} \int_0^1 \alpha_i^*(x) Q_i(x) dx = \int_0^{b_1} k \sum_{i=1}^{n} \alpha_i^*(x) dx + \sum_{i=1}^{n} \int_{a_i}^{a_{i+1}} Q_i(x) dx,$$

whence it appears that

$$H(\alpha^*, \beta^*) = kb_1 + \sum_{i=1}^{n} (a_{i+1} - a_i) \left[(1 + c_i)(a_i + a_{i+1}) - (1 + c_i + c_i b_i\right]. \tag{2.18}$$

As an illustration, select bets $c_1 = 1, c_2 = 3$, and $c_3 = 6$. In this case, readers easily obtain the following values of the parameters:

$$A = \frac{c_1(c_2 + 1)}{(2 + c_1)^2(2 + c_2)} + \frac{c_2(c_3 - c_2)}{(2 + c_1)(2 + c_2)^2(2 + c_3)} + \frac{c_3}{(2 + c_2)(2 + c_3)^2} \approx 0.122,$$
$$k = 1 - \frac{2 + c_1}{A(2 + c_1) + 1} \approx -1.193,$$

and the corresponding optimal strategies

$$b_1 = \frac{1 + k + c_1}{2 + c_1} \approx 0.269, \quad b_2 = \frac{1 + k + c_2}{2 + c_2} \approx 0.561, \quad b_3 = \frac{1 + k + c_3}{2 + c_3} \approx 0.725,$$
$$a_1 = b_1 \approx 0.269, \quad a_2 = 1 - \frac{1 - k}{(2 + c_1)(2 + c_2)} \approx 0.854, \quad a_3 = 1 - \frac{1 - k}{(2 + c_2)(2 + c_3)} \approx 0.945.$$

Finally, the value of the game constitutes

$$H(\alpha^*, \beta^*) \approx -0.117.$$

This quantity is negative—the game is non-beneficial for player I.

5.2.3 The asymptotic properties of strategies in the poker model with variable bets

Revert to the problem formulated in the beginning of Section 5.2. Suppose that, being dealt a card x, player I can make a bet $c(x)$ possessing an arbitrary value from R. Our analysis employs the results established in subsection 5.2.2.

Choose a positive value B and draw a uniform net $\{B/n, B/n, \dots, Bn/n\}$ on the segment $[0, B]$, where n is a positive integer. Imagine that nodes of this net represent bets in a play, i.e., $c_i = Bi/n$. Moreover, increase n and i infinitely such that the equality $Bi/n = c$ holds for some c. Afterwards, we will increase B infinitely.

Find the limit values of the parameters determining the optimal strategies of players in a play with such bets. First, evaluate the limit of A.

$$A = \sum_{i=1}^{n} \frac{c_i(c_{i+1} - c_{i-1})}{(2 + c_{i-1})(2 + c_i)^2(2 + c_{i+1})} = \sum_{i=1}^{n} \frac{B\frac{i}{n}2\frac{B}{n}}{\left(2 + B\frac{i-1}{n}\right)\left(2 + B\frac{i}{n}\right)^2\left(2 + B\frac{i+1}{n}\right)}.$$

As $n \to \infty$, the above integral sum tends to the integral:

$$A \to \int_0^B \frac{2c}{(2 + c)^4} dc = \frac{1}{12} - \frac{2B}{3(2 + B)^3} - \frac{1}{3(2 + B)^2},$$

which has the limit $A = 1/12$ as $B \to \infty$. This immediately brings to the limit value $k = 1 - \frac{2}{2A+1} = -5/7$.

It is possible to compute the threshold for player I bluffing: $b_1 = a_1 = 1 - (1 - k)/(2 + B/n) \to 1 - (1 + 5/7)/2 = 1/7$. Thus, if player I receives cards with denominations less than $1/7$, he should bluff.

Now, we define the bet of player I depending on the rank x of his card. According to the above optimal strategy $\alpha^*(x)$ of player I, he make a bet c_i within the interval $[a_i, a_{i+1})$, where

$$a_i = 1 - \frac{1 - k}{(2 + c_{i-1})(2 + c_i)}.$$

Therefore, the bet $c = c(x)$ corresponding to the card x satisfies the equation

$$x = 1 - \frac{1 - k}{(2 + c)^2}.$$

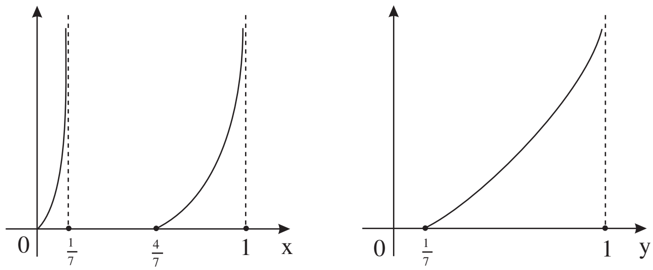


Figure 5.9 Optimal strategies in poker.

And so,

$$c(x) = \sqrt{\frac{12}{7(1-x)}} - 2. \tag{2.19}$$

The expression (2.19) is non-negative if $x \geq 4/7$; hence, player I bets nothing under $1/7 <\leq x < 4/7$. In the case of $x \geq 4/7$, his bet obeys formula (2.19).

Let us explore the asymptotic behavior of player II. Under $y < 1/7$, he should pass. If $y \geq 1/7$, the expression (2.15) states that player II should call the bet c of the opponent provided that

$$y \geq 1 - \frac{1-k}{2+c} = 1 - \frac{12}{7(2+c)}.$$

The optimal behavior of both players can be observed in Figure 5.9.

It remains to compute the limit value of the game. Take advantage of the expression (2.18). Passing to the limit yields

$$kb_1 + \int_0^\infty \frac{2(1-k)}{(2+c)^3}(1+c)\left[2(1 - \frac{1-k}{(2+c)^2} - \left(1 + \frac{c(1+k+c)}{(1+c)(2+c)}\right)\right] dc =$$

$$-\frac{5}{49} + \frac{24}{7}\int_0^\infty \frac{1+c}{(2+c)^3}\left(1 - \frac{24}{7(2+c)^2} - \frac{c\left(\frac{2}{7}+c\right)}{(1+c)(2+c)}\right) dc = -\frac{5}{49} + \frac{18}{49}. \tag{2.20}$$

We emphasize an important feature. The limit value (2.20) must be added to the payoff resulting from player I cards from the interval $[1/7, 4/7]$, i.e.,

$$\int_{\frac{1}{7}}^{\frac{4}{7}} \left(\frac{1}{7} + \int_{\frac{1}{7}}^{1} sgn(x-y)dy\right) dx = \frac{3}{49}\int_{\frac{1}{7}}^{\frac{4}{7}} \left(2x - \frac{8}{7}\right) dx = -\frac{6}{49}. \tag{2.21}$$

Summing up (2.20) and (2.21) yields

$$H(\alpha^*, \beta^*) = -\frac{5}{49} + \frac{18}{49} - \frac{6}{49} = \frac{1}{7}.$$

Therefore, the limit value of the game equals 1/7. It turns out beneficial for player I. Optimal strategies are completely defined. Interestingly, the payoff and strategies are expressed via 7—this number has repeatedly emerged in analytical formulas.

5.3 Preference. A game-theoretic model

Preference represents another popular card game. It engages two, three, four, or five players. A preference pack consists of 32 cards of four suits. Suits have the following ranking (in the ascending order): spades, clubs, diamonds, and hearts. Cards within a suit differ by their denomination. There exist eight denominations: 7, 8, 9, 10, jack, queen, king, and ace. In the beginning of a play, each player is dealt a certain number of cards. And the remaining cards form a talon. Next, players bid for the privilege of gaining the talon (declaring the contract and trump suit and playing as the soloist). Players gradually increase their bids; accordingly, they choose whisting or passing. As soon as bidding is finished, players start the game proper, revealing their cards one by one. Preference has many rules, and we do not pretend to their complete coverage. Instead, we describe the elementary model of the preference card game and endeavor to identify the characteristic features of this game. When should a player choose whisting or passing? When should a player take a talon? What is bluffing in preference?

As a game-theoretic model of preference, consider a two-player game P played Peter and Paul. In the beginning of a play, players are dealt cards of denominations x and y. Another card of rank z forms a talon. Suppose that card denominations represent random variables within the interval $[0, 1]$ and any values appear equiprobable. In this case, we say that the random variables x, y, z possess the uniform distribution on the interval $[0, 1]$.

Peter moves first. He chooses between whisting (i.e., improving his cards) or passing. In the former case, he may get the talon z, select the higher-rank card and discard the one having a smaller denomination. Subsequently, players open up their cards. If Peter's card is higher than Paul's one, he receives some payment A from Paul. Similarly, if Paul's card ranks over Peter's one, Peter pays a sum B to Paul, where $B > A$. If the cards have identical denominations, the play is drawn.

Imagine that Peter passes at the first shot. Then the move belongs to the opponent, and Paul chooses between the same alternatives. If Paul chooses whisting, he may get the talon z and discard the smaller-rank card. Next, both players open up the cards. Paul receives some payment A from Peter, if his card is higher than that of the opponent. Otherwise, he pays a sum B to Peter. The play is drawn when the cards have the same ranks.

A distinctive feature of preference is the so-called all-pass game. If both players pass, the talon remains untouched and they immediately open up the cards. But the situation reverses totally. The winner is the player having the lower-rank card. If $x < y$, Peter receives a payment C from Paul; if $x > y$, Paul pays an amount C to Peter. Otherwise, the play is drawn.

All possible situations can be described by the following table. For compactness, we use the notation $\max(x, z) = x \vee z$.

Table 5.1 The payoffs of players.

Peter (shot 1)	Paul (shot 2)	Peter's payoff
Talon		A, if $x \vee z > y$ 0, if $x \vee z = y$ $-B$, if $x \vee z < y$
Pass	Talon	$-A$, if $y \vee z > x$ 0, if $y \vee z = x$ B, if $y \vee z < x$
Pass	Pass	C, if $x < y$ 0, if $x = y$ $-C$, if $x > y$

5.3.1 Strategies and payoff function

Let us define strategies in this game. Each player is aware of his card only; hence, his decision bases on such knowledge exclusively. Therefore, we comprehend Peter's strategy as a function $\alpha(x)$—the probability of whisting provided that his card possesses the rank x. So long as α represents a probability, it takes values between 0 and 1. Accordingly, the quantity $\bar{\alpha} = 1 - \alpha$ specifies the probability of passing. If Peter passes, Paul's strategy consists in a function $\beta(y)$—the probability of whisting provided that his card has the denomination y. Obviously, $0 \le \beta \le 1$.

Recall that the payoff in this game forms a random variable. As a criterion, we involve the mean payoff (i.e., the expected value of the payoff). Suppose that players have chosen their strategies α, β. By virtue of the definition of this game (see Table 5.1), the payoff of player I has the following formula depending on a given combination of cards x, y, and z:

1. with the probability of $\alpha(x)$, the payoff equals A on the set $x \vee z > y$ and $-B$ on the set $x \vee z < y$;
2. with the probability of $\bar{\alpha}(x)\beta(y)$, the payoff equals $-A$ on the set $y \vee z > x$ and B on the set $y \vee z < x$;
3. with the probability of $\bar{\alpha}(x)\bar{\beta}(y)$, the payoff equals C on the set $x < y$ and $-C$ on the set $x > y$.

Since x, y, and z take any values from the interval $[0, 1]$, the expected payoff of player I represents the triple integral

$$
\begin{aligned}
H(\alpha, \beta) = \int_0^1 \int_0^1 \int_0^1 \Big\{ & \alpha(x) \left[AI_{\{x \vee z > y\}} - BI_{\{x \vee z < y\}} \right] \\
& + \bar{\alpha}(x)\beta(y) \left[-AI_{\{y \vee z > x\}} + BI_{\{y \vee z < x\}} \right] \\
& + \bar{\alpha}(x)\bar{\beta}(y) \left[CI_{\{x < y\}} - CI_{\{x > y\}} \right] \Big\} \, dxdydz.
\end{aligned} \tag{3.1}
$$

For convenience, this expression incorporates the function $I_A(x, y, z)$ (the so-called indicator of set A), which is 1, if (x, y, z) belongs to A and 0, otherwise.

We illustrate calculation of the first integral:

$$\int_0^1 \int_0^1 \int_0^1 \alpha(x) I_{\{x \vee z > y\}} dxdydz = \int_0^1 \alpha(x)dx \left[\int_0^1 \int_0^1 I_{\{x \vee z > y\}} dydz \right]. \tag{3.2}$$

The double integral in brackets can be computed as the iterated integral $J(x) = \int_0^1 dz \int_0^1 I_{\{x \vee z > y\}} dy$ by dividing into two integrals. Notably,

$$J(x) = \int_0^x dz \int_0^1 I_{\{x \vee z > y\}} dy + \int_x^1 dz \int_0^1 I_{\{x \vee z > y\}} dy.$$

In the first integral, we have $x \geq z$; hence, $I_{\{x \vee z > y\}} = I_{\{x > y\}}$. On the contrary, the second integral is remarkable for that $I_{\{x \vee z > y\}} = I_{\{z > y\}}$. And it follows that

$$\begin{aligned} J(x) &= \int_0^x dz \int_0^1 I_{\{x > y\}} dy + \int_x^1 dz \int_0^1 I_{\{z > y\}} dy = x \int_0^1 I_{\{x > y\}} dy + \int_x^1 dz \int_0^1 I_{\{z > y\}} dy \\ &= x \int_0^x dy + \int_x^1 dz \int_0^z dy = x^2 + \int_x^1 zdz = \frac{x^2 + 1}{2}. \end{aligned}$$

Therefore, the triple integral in (3.2) is transformed into the integral $\int_0^1 \alpha(x) \frac{x^2+1}{2} dx$. Proceeding by analogy, readers can easily calculate the rest integrals in (3.1).

After certain manipulations, we rewrite (3.1) as

$$\begin{aligned} H(\alpha, \beta) = &\int_0^1 \alpha(x) \left[x^2(A + B)/2 + (A - B)/2 - C(1 - 2x) \right] dx \\ &+ \int_0^1 \beta(y) \left[-y^2(A + B)/2 - (A - B)/2 + C(1 - 2y) \right] dy \\ &+ \int_0^1 \alpha(x)dx \left[(A - x(A + B) - C) \int_0^x \beta(y)dy + (A + C) \int_x^1 \beta(y)dy \right]. \end{aligned} \tag{3.3}$$

The payoff $H(\alpha, \beta)$ in formula (3.3) has the following representation. The first and second rows contain expressions with α and β separately, whereas the third row includes their product.

Now, find the strategies $\alpha^*(x)$ and $\beta^*(y)$ that meet the equations

$$\max_{\alpha} H(\alpha, \beta^*) = \min_{\beta} H(\alpha^*, \beta) = H(\alpha^*, \beta^*). \tag{3.4}$$

Then for any other strategies α and β we have the inequalities

$$H(\alpha, \beta^*) \leq H(\alpha^*, \beta^*) \leq H(\alpha^*, \beta),$$

i.e., the strategies α^*, β^* form an equilibrium.

5.3.2 Equilibrium in the case of $\frac{B-A}{B+C} \leq \frac{3A-B}{2(A+C)}$

Assume that, at shot 1, Peter always adheres to whisting: $\alpha^*(x) = 1$. In this case, nothing depends on Paul's behavior. Formula (3.1) implies that Peter's payoff $H(\alpha^*, \beta)$ constitutes the quantity

$$\int_0^1 \int_0^1 \int_0^1 \left[A I_{\{x \vee z > y\}} - B I_{\{x \vee z < y\}}\right] dx dy dz.$$

Recall that, in formula (3.2),

$$H(\alpha^*, \beta) = A \int_0^1 \frac{x^2+1}{2} dx - B\left[1 - \int_0^1 \frac{x^2+1}{2} dx\right] = \frac{2A-B}{3}. \tag{3.5}$$

Thus, Peter definitely guarantees the payoff $\frac{2A-B}{3}$.

Now, suppose that Paul applies the strategy

$$\beta^*(y) = \begin{cases} 1, & \text{if } y \geq a, \\ 0, & \text{if } y < a, \end{cases} \tag{3.6}$$

where a indicates some number from the interval $[0, 1]$. In this case, the expression (3.3) leads to

$$H(\alpha, \beta^*) = \int_0^1 \alpha(x) G(x) dx + \int_0^1 \beta^*(y) \left[-\frac{y^2(A+B)}{2} - \frac{A-B}{2} + C(1-2y)\right] dy, \tag{3.7}$$

with the function

$$G(x) = \begin{cases} x^2(A+B)/2 + 2Cx + (A-B)/2 - C + (A+C)(1-a), & \text{if } x < a, \\ -x^2(A+B)/2 + x(A+B)a + (3A-B)/2 - a(A-C), & \text{if } x \geq a. \end{cases} \tag{3.8}$$

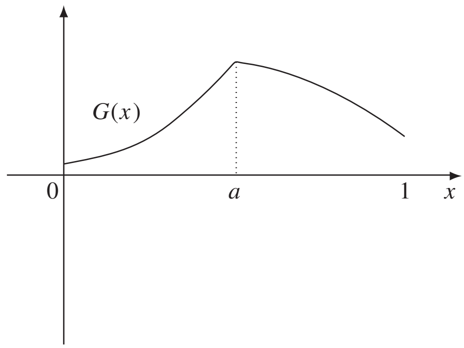


Figure 5.10 The function $G(x)$.

The second term in (3.7) is independent from $\alpha(x)$. And so, Peter's optimal strategy (which maximizes the payoff $H(\alpha, \beta^*)$) takes the form

$$\alpha^*(x) = \begin{cases} 1, & \text{if } G(x) > 0, \\ 0, & \text{if } G(x) < 0, \\ \text{an arbitrary value} & \\ \text{in the interval } [0,1], & \text{if } G(x) = 0. \end{cases} \tag{3.9}$$

Clearly, the function $G(x)$ consists of two parabolas, see (3.8). Below we demonstrate that, if a is an arbitrary value from the interval $\mathbf{U} = [0,1] \cap \left[\frac{B-A}{B+C}, \frac{3A-B}{2(A+C)}\right]$, then the function $G(x)$ possesses the curve in Figure 5.10 (for given values of the parameters A, B, C, a).

Indeed, for any $a \in \mathbf{U}$, formula (3.8) implies that

$$G(0) = \frac{A-B}{2} - C + (A+C)(1-a) == \frac{3A-B}{2} - (A+C)a \geq 0,$$

since $a \leq \frac{3A-B}{2(A+C)}$, and

$$G(1) = A - B + a(B+C) \geq 0$$

due to $a \geq \frac{B-A}{B+C}$. The function $G(x)$ has the curve presented in the figure. And it appears from (3.9) that the strategy $\alpha^*(x)$ maximizing $V(\alpha, \beta^*)$ takes the form $\alpha^* \equiv 1$.

Therefore, if Peter and Paul choose the strategies $\alpha^* \equiv 1$ and $\beta^*(y) = I_{\{y \geq a\}}$, respectively, where $a \in \mathbf{U}$, the players arrive at the following outcome. Peter guarantees the payoff of $\frac{2A-B}{3}$, and Paul will not let him gain more. In other words,

$$\max_{\alpha} H(\alpha, \beta^*) = \min_{\beta} H(\alpha^*, \beta) = H(\alpha^*, \beta^*) = \frac{2A-B}{3},$$

which proves optimality of the strategies α^*, β^*.

And so, when $\frac{B-A}{B+C} \le \frac{3A-B}{2(A+C)}$, the optimal strategies in the game P are defined by $\alpha^* \equiv 1$ and $\beta^*(y) = I_{\{y \ge a\}}$, where a represents an arbitrary value from the interval $\mathbf{U} = [0,1] \cap \left[\frac{B-A}{B+C}, \frac{3A-B}{2(A+C)}\right]$. Moreover, the game has the value $H^* = \frac{2A-B}{3}$.

5.3.3 Equilibrium in the case of $\frac{3A-B}{2(A+C)} < \frac{B-A}{B+C}$

Now, suppose that

$$\frac{3A-B}{2(A+C)} < \frac{B-A}{B+C}.$$

If Paul adopts the strategy $\beta(y)$ (3.6), formula (3.8) shows the following. For a belonging to the interval $\mathbf{U} = \left[\frac{3A-B}{2(A+C)}, \frac{B-A}{B+C}\right]$, one obtains

$$G(0) = \frac{3A-B}{2} - (A+C)a \le 0,$$
$$G(1) = (A-B) + a(B+C) \le 0.$$

The curve of $y = G(x)$—see (3.8)—intersects axis x according to Figure 5.11.

Thus, when Paul prefers the strategy $\beta(y) = I_{\{y \ge a\}}$ with some $a \in \mathbf{U}$, the best response of Peter is the strategy

$$\alpha^*(x) = \begin{cases} 1, & \text{if } b_1 \le x \le b_2, \\ 0, & \text{if } x < b_1, x > b_2 \end{cases}. \tag{3.10}$$

This function admits the compact form $\alpha^*(x) = I_{\{b_1 \le x \le b_2\}}$, where b_1, b_2 solve the system of equations $G(b_1) = 0, G(b_2) = 0$. Departing from (3.8), we derive the system of equations

$$b_1^2 \frac{A+B}{2} + \frac{A-B}{2} - C(1-2b_1) + (A+C)(1-a) = 0, \tag{3.11}$$

$$-b_2^2 \frac{A+B}{2} + b_2(A+B)a + \frac{A-B}{2} + A - a(A-C) = 0. \tag{3.12}$$

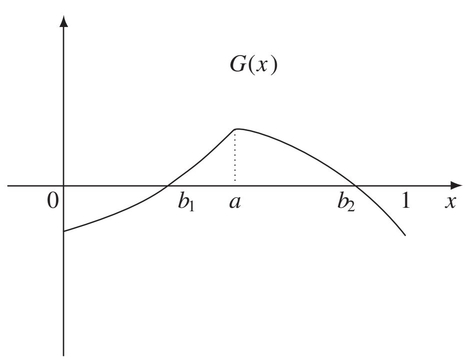


Figure 5.11 The function $G(x)$.

To proceed, assume that Peter follows the strategy (3.10) $\alpha^*(x) = I_{\{b_1 \le x \le b_2\}}$ and find Paul's best response $\beta^*(y)$, which minimizes $H(\alpha^*, \beta)$ in β. By substituting $\alpha^*(x)$ into (3.1), one can rewrite the function $H(\alpha^*, \beta)$ as

$$H(\alpha^*, \beta) = \int_0^1 \beta(y)R(y)dy + \int_{b_1}^{b_2} \left[\frac{x^2(A+B)}{2} + \frac{A-B}{2} - c(1-2x) \right] dx. \tag{3.13}$$

Here the second component is independent from $\beta(y)$, and $R(y)$ acquires the form

$$R(y) = \begin{cases} -y^2(A+B)/2 - 2Cy - (A-B)/2 + C+ & \\ +(A-C)(b_2-b_1) - (A+B)(b_2^2 - b_1^2)/2, & \text{if } y < b_1 \\ -(A-B)/2 + C + b_2(A-C) - b_2^2(A+B)/2- & \\ -b_1(A+C), & \text{if } b_1 \le y \le b_2 \\ -y^2(A+B)/2 - 2Cy - (A-B)/2 + C+ & \\ +(A+C)(b_2-b_1), & \text{if } b_2 < y \le 1. \end{cases} \tag{3.14}$$

The representation (3.13) immediately implies that the optimal strategy $\beta^*(y)$ is defined by the expressions

$$\beta^*(y) = \begin{cases} 1, & \text{if } R(y) < 0, \\ 0, & \text{if } R(y) > 0, \\ \text{an arbitrary value} & \\ \text{within the interval } [0,1], & \text{if } R(y) = 0. \end{cases} \tag{3.15}$$

Interestingly, the function $R(y)$ possesses constant values on the interval $[b_1, b_2]$. Set it equal to zero:

$$-b_2^2\frac{A+B}{2} + b_2(A-C) - b_1(A+C) - \frac{A-B}{2} + C = 0. \tag{3.16}$$

Then it takes the form demonstrated in Figure 5.12.

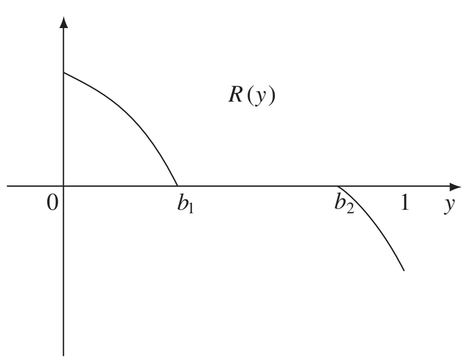


Figure 5.12 The function $R(y)$.

Table 5.2 The optimal strategies of players.

			$\alpha^*(x) = I_{\{b_1 \le x \le b_2\}}$		$\beta^*(y) = I_{\{y \ge a\}}$	
A	B	C	b_1	b_2	a	H^*
5	6	8	0	1	0.071	1.333
3	4	1	0	1	0.2	0.666
3	5	0	0	1	0.4	0.333
1	20	0	0.948	0.951	0.948	−0.024
1	2	1	0.055	0.962	0.307	−0.344
1	2	2	0.046	0.964	0.229	−0.417
1	2	3	0.039	0.968	0.184	−0.467
1	2	4	0.034	0.971	0.154	−0.504
1	4	3	0.288	0.824	0.359	−0.519
1	3	4	0.146	0.892	0.242	−0.592
1	2	20	0.010	0.989	0.044	−0.669
1	10	10	0.356	0.792	0.392	−1.366
3	8	2	0.282	0.845	0.413	−2.058

According to (3.15), Paul's optimal strategy is, in particular, $\beta^*(y) = I_{\{y \ge a\}}$, where a means an arbitrary value from the interval $[b_1, b_2]$. The system of equations (3.11), (3.12), (3.16) yields the solution to the game.

5.3.4 Some features of optimal behavior in preference

Let us analyze the obtained solution. In the case of $\frac{B-A}{B+C} \le \frac{3A-B}{2(A+C)}$, Peter should choose whisting, get the talon and open up the cards. His payoff makes up $\frac{2A-B}{3}$. Interestingly, if $B < 2A$, the game becomes beneficial for him.

However, under $\frac{3A-B}{2(A+C)} < \frac{B-A}{B+C}$, the optimal strategy changes. Peter should adhere to whisting, when his card possesses an intermediate rank (between low and high ones). Otherwise, he should pass. Paul's optimal strategy seems easier. He selects whisting, if his card has a high denomination (otherwise, Paul passes). This game incorporates the bluffing effect. As soon as Peter announces passing, Paul has to guess the rank of Peter's card (low or high).

Table 5.2 combines the optimal strategies of players and the value of this game under different values of A, B, and C. The value of this game can be computed by formula (3.13) through substituting $\beta^*(y) = I_{\{y \ge a\}}$, with the values a_1, b_1, b_2 resulting from the system (3.11), (3.12), (3.16).

5.4 The preference model with cards play

In the preceding sections, we have modeled different combinations of cards in poker and preference by a single random variable from the unit interval. For instance, many things in preference depend on specific cards entering a combination, as well as on a specific sequence players open up their cards. Notably, cards are revealed one by one, a higher-rank card beats a lower-rank one, and the move comes to a corresponding player. Here we introduce a natural generalization of the previous model when a set of cards of each player is modeled by two random variables.

Consider a two-player game engaging Peter and Paul. Both players contribute the buy-ins of 1. In the beginning of a play, each of them is dealt two cards whose denominations represent random variables within the interval $[0, 1]$. A player chooses between two alternatives, *viz.*, passing or making a bet $A > 1$. If a player passes, the opponent gets the bank. When Peter and Paul pass simultaneously, the game is drawn. If both players make bets, they open up the cards and the winner is the player whose lowest-rank card exceeds the highest-rank card of the opponent. And the winner sweeps the board.

5.4.1 The preference model with simultaneous moves

First, analyze the preference model with simultaneous moves of players. Suppose that their cards $x_i, y_i (i = 1, 2)$ make independent random variables uniformly distributed on the interval $[0, 1]$. Without loss of generality, we believe that $x_1 \le x_2, y_1 \le y_2$.

Reexpress the payoff of player I as the matrix

$$\begin{array}{c} \\ \text{betting} \\ \text{passing} \end{array} \begin{array}{c} \begin{array}{cc} \quad\text{betting} & \qquad\text{passing} \end{array} \\ \begin{pmatrix} A(I_{\{x_1>y_2\}} - I_{\{y_1>x_2\}}) & 1 \\ -1 & 0 \end{pmatrix} \end{array},$$

where $I_{\{A\}}$ designates the indicator of A.

Define strategies in this game. Denote by $\alpha(x_1, x_2)$ the strategy of player I. Actually, this is the probability that player I makes a bet under given cards x_1, x_2. Then the quantity $\overline{\alpha} = 1 - \alpha$ characterizes the probability of his passing. Similarly, for player II, the function $\beta(y_1, y_2)$ specifies the probability of making a bet under given cards y_1, y_2, whereas $\overline{\beta} = 1 - \beta$ equals the probability of his passing.

The expected payoff of player I becomes

$$\begin{aligned}
H(\alpha, \beta) &= \int_0^1 \int_0^1 \int_0^1 \int_0^1 \{\alpha\overline{\beta} - \overline{\alpha}\beta + A\alpha\beta[I_{\{x_1>y_2\}} - I_{\{y_1>x_2\}}]\} dx_1 dx_2 dy_1 dy_2 \\
&= 2\int_0^1 dx_1 \int_{x_1}^1 \alpha(x_1, x_2) dx_2 - 2\int_0^1 dy_1 \int_{y_1}^1 \beta(y_1, y_2) dy_2 \\
&\quad + 4A \int_0^1 \int_{x_1}^1 \int_0^1 \int_{y_1}^1 \alpha(x_1, x_2)\beta(y_1, y_2)[I_{\{x_1>y_2\}} - I_{\{y_1>x_2\}}] dx_1 dx_2 dy_1 dy_2 \\
&= 2\int_0^1 dx_1 \int_{x_1}^1 \alpha(x_1, x_2) dx_2 - 2\int_0^1 dy_1 \int_{y_1}^1 \beta(y_1, y_2) dy_2 \\
&\quad + 4A \int_0^1 \int_{y_1}^1 \beta(y_1, y_2)[\int_{y_2}^1 dx_1 \int_{x_1}^1 \alpha(x_1, x_2) dx_2 - \int_0^{y_1} dx_1 \int_{x_1}^{y_1} \alpha(x_1, x_2) dx_2] dy_1 dy_2
\end{aligned}$$

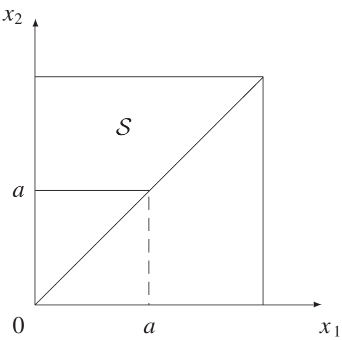


Figure 5.13 The strategy of player I.

Theorem 5.1 *In the game with the payoff function $H(\alpha, \beta)$, the optimal strategies take the form*

$$\alpha^*(x_1, x_2) = I_{\{x_2 \geq a\}}, \beta^*(y_1, y_2) = I_{\{y_2 \geq a\}},$$

where $a = 1 - \frac{1}{\sqrt{A}}$. The game has zero values.

Proof: Assume that player I applies the strategy $\alpha^*(x_1, x_2) = I_{\{x_2 \geq a\}}$ (see Figure 5.13), where $a = 1 - \frac{1}{\sqrt{A}}$. Find the best response of player II. Rewrite the payoff function as

$$H(\alpha^*, \beta) = 2\int_0^1 \int_{x_1}^1 \alpha^*(x_1, x_2) dx_1 dx_2 + 2\int_0^1 \int_{y_1}^1 \beta(y_1, y_2) \cdot R(y_1, y_2) dy_1 dy_2, \tag{4.1}$$

with the function

$$R(y_1, y_2) = \begin{cases} -2Ay_2(1-a) - Aa^2 + A - 1, & \text{if} \quad y_1 \leq y_2 < a \\ Ay_2^2 - 2Ay_2 + A - 1, & \text{if} \quad y_1 < a \leq y_2 \\ A(y_2^2 - y_1^2) - 2Ay_2 + Aa^2 + A - 1, & \text{if} \quad a \leq y_1 \leq y_2. \end{cases} \tag{4.2}$$

The first summand in (4.1) appears independent from $\beta(y_1, y_2)$. Hence, the optimal strategy of player II, which minimizes the payoff $H(\alpha^*, \beta)$, is given by

$$\beta^*(y_1, y_2) = \begin{cases} 1, & \text{if} \quad R(y_1, y_2) < 0 \\ 0, & \text{if} \quad R(y_1, y_2) > 0 \\ \text{an arbitrary value from } [0, 1], & \text{if} \quad R(y_1, y_2) = 0. \end{cases}$$

Formula (4.2) implies that the function $R(y_1, y_2)$ depends on y_2 only in the domains $y_1 \le y_2 < a$ and $y_1 < a \le y_2$. Furthermore, the function $R(y_1, y_2)$ decreases on this set, vanishing at the point $y_2 = a$ due to the choice of a. And so, the first (second) row in (4.2) is always positive (non-positive, respectively).

The expression in the third row possesses non-positive values in the domain $a \le y_1 \le y_2$. Really, the function $R(y_1, y_2)$ decreases with respect to both variables y_2 and y_1, *ergo* reaches its maximum under $y_2 = a$, $y_1 = a$ (the maximal value equals $R(a, a) = Aa^2 - 2Aa + A - 1 = 0$ under $a = 1 - \frac{1}{\sqrt{A}}$).

Therefore, $R(y_1, y_2) > 0$ for $y_2 < a$, and $R(y_1, y_2) \le 0$ for $y_2 \ge a$.

Consequently, the best strategy of player *II* lies in $\beta^*(y_1, y_2) = I_{\{y_2 \ge a\}}$, where $a = 1 - \frac{1}{\sqrt{A}}$.

By virtue of the problem symmetry, the best response of player *I* to the strategy $\beta^*(y_1, y_2) = I_{\{y_2 \ge a\}}$ is the strategy $\alpha^*(x_1, x_2) = I_{\{x_2 \ge a\}}$.

Hence it appears that $\max\limits_{\alpha} V(\alpha, \beta^*) = \min\limits_{\beta} V(\alpha^*, \beta)$, which immediately brings to the following. The strategies α^*, β^* form an equilibrium in the game.

5.4.2 The preference model with sequential moves

Now, imagine that the players announce their decisions sequentially.

Player	cards	move 1	move 2	payment
I	x_1, x_2	whisting		
		passing		
II	y_1, y_2		whisting	$A[I_{\{x_1 > y_2\}} - I_{\{y_1 > x_2\}}]$
			passing	1
			whisting	−1

In the beginning of a play, both players contribute the buy-ins of 1. The first move belongs to player *I*. He selects between two alternatives, *viz*., passing or making a bet $A > 1$. In the latter case, the move goes to player *II* who may choose passing (and the game is over) or calling the bet. If he calls the opponent's bet, both players open up the cards. The winner is the player whose lowest-rank card exceeds the highest-rank card of the opponent. And the winner gains some payoff $A > 1$. Otherwise, the game is drawn.

In such setting of the game, the payoff function of player I acquires the form

$$H(\alpha,\beta)=\int_0^1\int_0^1\int_0^1\int_0^1\{-\overline{\alpha}+\alpha\overline{\beta}+A\alpha\beta[I_{\{x_1>y_2\}}-I_{\{y_1>x_2\}}]\}dx_1dx_2dy_1dy_2$$
$$=2\int_0^1\int_0^1\alpha(x_1,x_2)dx_1dx_2-1-\int_0^1\int_0^1\int_0^1\int_0^1\alpha(x_1,x_2)\beta(y_1,y_2)dx_1dx_2dy_1dy_2$$
$$+4A\int_0^1\int_{x_1}^1\int_0^1\int_{y_1}^1\alpha(x_1,x_2)\beta(y_1,y_2)[I_{\{x_1>y_2\}}-I_{\{y_1>x_2\}}]dx_1dx_2dy_1dy_2$$
$$=4\int_0^1 dx_1\int_{x_1}^1\alpha(x_1,x_2)dx_2-1+2\int_0^1\int_{x_1}^1\alpha(x_1,x_2)[2A\int_0^{x_1}dy_1\int_{y_1}^{x_1}\beta(y_1,y_2)dy_2$$
$$-2A\int_{x_2}^1 dy_1\int_{y_1}^1\beta(y_1,y_2)dy_2-2\int_0^1 dy_1\int_{y_1}^1\beta(y_1,y_2)dy_2]dx_1dx_2.$$

Certain simplifications bring us to the expression

$$H(\alpha,\beta)=4\int_0^1 dx_1\int_{x_1}^1\alpha(x_1,x_2)dx_2-1+2\int_0^1\int_{y_1}^1\beta(y_1,y_2)[2A\int_{y_2}^1 dx_1\int_{x_1}^1\alpha(x_1,x_2)dx_2$$
$$-2A\int_0^{y_1}dx_1\int_{x_1}^{y_1}\alpha(x_1,x_2)dx_2-2\int_0^1 dx_1\int_{x_1}^1\alpha(x_1,x_2)dx_2]dy_1dy_2.$$

Suppose that player I uses the strategy $\alpha^*(x_1,x_2)=I_{\{x_2\ge a\}}$ with some threshold a such that

$$a\le\frac{A-1}{A+1}. \tag{4.3}$$

Find the best response of player II. Rewrite the payoff function $H(\alpha,\beta)$ as

$$H(\alpha^*,\beta)=4\int_0^1\int_{x_1}^1\alpha^*(x_1,x_2)dx_1dx_2-1+2\int_0^1\int_{y_1}^1\beta(y_1,y_2)\cdot R(y_1,y_2)dy_1dy_2,$$

where the function $R(y_1, y_2)$ takes the form

$$R(y_1, y_2) = \begin{cases} -2Ay_2(1-a) + (A-1)(1-a^2), & \text{if} \quad y_1 \le y_2 \le a \\ A(1-y_2)^2 - (1-a^2), & \text{if} \quad y_1 \le a < y_2 \\ A[(1-y_2)^2 - (y_1^2 - a^2)] - (1-a^2), & \text{if} \quad a < y_1 \le y_2. \end{cases}$$

Let us repeat the same line of reasoning as above. Consequently, we obtain that the optimal strategy β^* depends only on the sign of $R(y_1, y_2)$.

Interestingly, the function $R(y_1, y_2)$ in the domain $y_1 \le y_2 \le a$ depends merely on $y_2 \in [0, a]$. Thus, for a fixed quantity y_1, the function $R(y_1, y_2)$ decreases on this interval and $R(y_1, 0) > 0$. Owing to the choice of a (see formula (4.3)), we have $R(y_1, a) = A(1-a)^2 - (1-a^2) \ge 0$, i.e., the function $R(y_1, y_2)$ is non-negative in the domain $y_1 \le y_2 \le a$. This means that, in the domain under consideration, the best response of player *II* consists in $\beta(y_1, y_2) = 0$.

In the domain $y_1 \le a < y_2$, the function $R(y_1, y_2) = A(1-y_2)^2 - (1-a^2)$ depends on y_2 only; moreover, it represents a continuous decreasing function such that $R(y_1, y_2 = a) = A(1-a)^2 - (1-a^2) \ge 0$ and $R(y_1, y_2 = 1) = -(1-a^2) < 0$. Hence, there exists a point $b \in [a, 1)$ meeting the condition $R(y_1, b) = 0$. This point b is a root of the equation $A(1-b)^2 = 1 - a^2$, i.e.,

$$b = 1 - \sqrt{\frac{1-a^2}{A}}. \tag{4.4}$$

And so, the best response $\beta^*(y_1, y_2)$ of player *II* in the domain $y_1 \le a < y_2$ takes the form $\beta^*(y_1, y_2) = I_{\{y_2 \ge b\}}$, where b is defined by (4.4).

Let us partition the domain $a < y_1 \le y_2$ into two subsets, $\{a < y_1 \le c, y_1 \le y_2\}$ and $\{c < y_1 \le y_2\}$, where $c = a^2\frac{A+1}{2A} + \frac{A-1}{2A}$. Evidently, $a \le c \le b$, where $a \in \left[0, \frac{A-1}{A+1}\right]$.

Consider the equation $R(y_1, y_2) = 0$ in the domain $\{a < y_1 \le c, y_1 \le y_2\}$. It can be rewritten as

$$y_1 = f(y_2) = \sqrt{y_2^2 - 2y_2 + a^2\frac{A+1}{A} + \frac{A-1}{A}}. \tag{4.5}$$

We see that the function $f(y_2)$ is continuous and decreases on the interval $y_2 \in [c, b]$; in addition, $f(c) = c$ and $f(b) = a$.

Therefore, the optimal strategy of player *II* in the domain $\{a < y_1 \le c, y_1 \le y_2\}$ has the following form: $\beta^*(y_1, y_2) = 1$ for $y_1 \ge f(y_2)$, and $\beta^*(y_1, y_2) = 0$ for $y_1 < f(y_2)$.

The set $\{c < y_1 \le y_2\}$ corresponds to $R(y_1, y_2) < 0$; thus, the best response lies in $\beta^*(y_1, y_2) = 1$.

The above argumentation draws an important conclusion. The optimal strategy of player *II* is given by $\beta^*(y_1, y_2) = I_{\{(y_1, y_2) \in S\}}$, see Figure 5.14 which demonstrates the set S. Recall that the boundary of the domain S on the set $[a, c] \times [c, b]$ obeys equation (4.5).

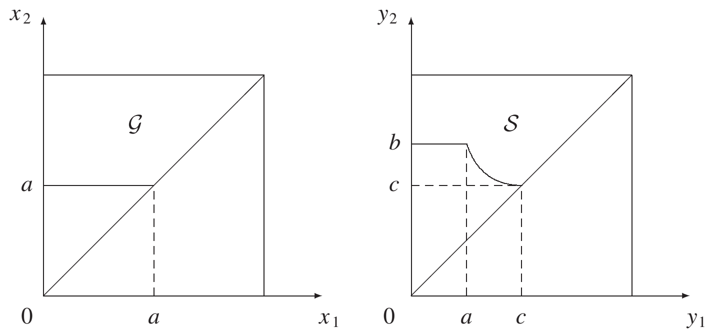


Figure 5.14 The optimal strategies of players *I* and *II*.

The parameters a, b, and c specifying this domain satisfy the conditions

$$\begin{cases} 0 \le a \le c \le b \le \frac{A-1}{A+1}, \\ c = a^2 \frac{A+1}{2A} + \frac{A-1}{2A}, \\ b = 1 - \sqrt{\frac{1-a^2}{A}}. \end{cases} \tag{4.6}$$

Now, suppose that player *II* applies the strategy $\beta^*(y_1, y_2) = I_{\{(y_1,y_2)\in\mathcal{S}\}}$, where the domain $\mathcal{S}$ has the above shape with the parameters a, b, and c. According to the definition, the payoff of player *I* becomes

$$H(\alpha, \beta^*) = 2\int_0^1 \int_{x_1}^1 \alpha(x_1, x_2) G(x_1, x_2) dx_1 dx_2 - 1,$$

where the function

$$G(x_1, x_2) = 2A \int_0^{x_1} dy_1 \int_{y_1}^{x_1} \beta(y_1, y_2) dy_2 - 2A \int_{x_2}^1 dy_1 \int_{y_1}^1 \beta(y_1, y_2) dy_2$$
$$-2 \int_0^1 dy_1 \int_{y_1}^1 \beta(y_1, y_2) dy_2 + 2. \tag{4.7}$$

Formula (4.7) implies the following. The function $G(x_1, x_2)$ is non-increasing in both arguments and, in the domain $x_1 \le x_2 \le a$, depends on x_2 only:

$$G(x_1, x_2) = 2Ax_2(1 - b) - 2AS + 2(1 - S), \tag{4.8}$$

where $S = \int_0^1 \int_{y_1}^1 \beta^*(y_1, y_2) dy_1 dy_2$ gives the area of the domain $\mathcal{S}$.

This quantity can be reexpressed by

$$S = \frac{1-b^2}{2} + \int_c^b \left(y_2 - \sqrt{y_2^2 - 2y_2 + a^2\frac{A+1}{A} + \frac{A-1}{A}} \right) dy_2. \tag{4.9}$$

Due to the conditions imposed on a, b, and c, we have

$$2c = a^2\frac{A+1}{A} + \frac{A-1}{A} \quad \text{and} \quad b^2 - 2b + 2c = a^2.$$

By virtue of these relations, S takes the form

$$S = \frac{1-c}{2} + a\frac{1-b}{2} + \frac{2c-1}{2} ln \left| \frac{2c-1}{a+b-1} \right|.$$

Choose a as

$$a = \frac{S}{1-b} - \frac{1-S}{A(1-b)}. \tag{4.10}$$

In this case, the function $G(x_1, x_2)$ (4.8) possesses negative values in the domain $x_1 \le x_2 \le a$, and vanishes on its boundary $x_2 = a$: $G(x_1, a) = 0$.

We have mentioned that the function $G(x_1, x_2)$ is non-decreasing in both arguments, *ergo* non-negative in the residual domain $x_2 > a$. This fact immediately brings to the following. The best response α^* of player II, which maximizes the payoff $H(\alpha, \beta^*)$, has the form $\alpha^* = I_{\{x_2 \ge a\}}$.

Finally, it is necessary to establish the existence of a solution to the system of equations (4.6) and (4.10). Earlier, we have shown that any $a \in \left[0, \frac{A-1}{A+1}\right]$ corresponds to unique b, c and S meeting (4.6) and (4.9). Now, argue that there exists a^* such that the condition (4.10) holds true.

Introduce the function $\Delta(a) = \Delta(a, b(a), c(a)) = \frac{S}{1-b} - \frac{1-S}{A(1-b)} - a$. The equation $\Delta(a) = 0$ admits the solution a^*, since $\Delta(a)$ is continuous and takes values of different signs in the limits of the interval $\left[0, \frac{A-1}{A+1}\right]$.

Indeed, $\Delta(a=0) = \frac{(A-1)^2 + (A+1)lnA}{4A\sqrt{A}} \ge 0$ under $A \ge 1$, since this function increases in A and vanishes if $A = 1$. Furthermore, $\Delta\left(a = \frac{A-1}{A+1}\right) = -\frac{(A-1)^2}{2A(A+1)} < 0$.

The resulting solution is formulated as

Theorem 5.2 *The optimal solution of the game with the payoff function $H(\alpha, \beta)$ is defined by*

$$\alpha^*(x_1, x_2) = I_{\{x_2 \ge a^*\}}, \beta^*(y_1, y_2) = I_{\{(y_1, y_2) \in S\}},$$

where a, b, c, and S follow from the system of equations

$$\begin{cases} \frac{S}{1-b} - \frac{1-S}{A(1-b)} - a = 0, \\ S = \frac{1-c}{2} + a\frac{1-b}{2} + \frac{2c-1}{2} ln\left|\frac{2c-1}{a+b-1}\right|, \\ c = a^2\frac{A+1}{2A} + \frac{A-1}{2A}, \\ b = 1 - \sqrt{\frac{1-a^2}{A}}, \end{cases}$$

and the set S is described above.

Table 5.3 provides the parameters of optimal strategies in different cases. Obviously, the game has negative value. The unfairness of this game for player *I* can be explained as follows. He moves first and gives some essential information to the opponent; player *II* uses such information in his optimal game.

$$H(\alpha^*, \beta^*) = -1 + 4\int_0^1\int_{x_1}^1 \alpha^*(x_1, x_2)dx_1dx_2 + 2\int_0^1\int_{y_1}^1 \beta^*(y_1, y_2)R(y_1, y_2)dx_1dx_2dy_1dy_2$$

$$= 1 - 2a^2 + \frac{4}{3}Aa^3(b-c) + 2\left\{ a\int_c^1 [A(1-y)^2 - (1-a^2)]dy + \right.$$

$$+ \frac{A}{3}\int_c^b (y^2 - 2y + 2c)^{3/2}dy + \int_c^1 dy_2 \int_a^{y_2} [A(1-y_2)^2 - A(y_1^2 - a^2) - (1-a^2)]dy_1$$

$$\left. - \int_c^b \sqrt{y^2 - 2y + 2c}[A(1-y)^2 + Aa^2 - (1-a^2)]dy \right\}.$$

Table 5.3 The parameters of optimal strategies.

A	a	b	c	S	$H(\alpha^*, \beta^*)$
1.00	0.0000	0.0000	0.0000	0.5000	0.0000
2.00	0.2569	0.3166	0.2995	0.4503	−0.0070
3.00	0.3604	0.4614	0.4199	0.3956	−0.0302
4.00	0.4248	0.5473	0.4878	0.3538	−0.0590
5.00	0.4711	0.6055	0.5331	0.3215	−0.0883
6.00	0.5069	0.6481	0.5665	0.2957	−0.1163
7.00	0.5359	0.6809	0.5927	0.2746	−0.1424
8.00	0.5600	0.7071	0.6139	0.2569	−0.1667
9.00	0.5806	0.7286	0.6317	0.2418	−0.1892
10.00	0.5984	0.7466	0.6469	0.2287	−0.2100
100.00	0.8600	0.9489	0.8685	0.0533	−0.6572
1000.00	0.9552	0.9906	0.9561	0.0099	−0.8833

Remark 5.1 In the symmetrical model with two cards, the optimal strategies of both players are such that calling a bet appears reasonable only while having a sufficiently high card. The second card may possess a very small denomination. In the model with sequential moves, the shape of the domain S demonstrates an important aspect. The optimal strategy of player *II* prescribes calling a bet if one of his cards has a sufficiently high rank or both cards have an intermediate denomination (in some cases only).

5.5 Twenty-one. A game-theoretic model

Twenty-one is a card game of two players. A pack of 36 cards is sequentially dealt to players, one card by one. In the Russian version of the game, each card possesses a certain denomination (jack—2, queen—3, king—4, ace—11; all other cards are counted as the numeric value shown on the card). Players choose a certain number of cards and calculate their total denomination. Then all cards are opened up, and the winner is the player having the maximal sum of values (yet, not exceeding the threshold of 21). If the total denomination of cards held by a player exceeds 21 and the opponent has a smaller combination, the latter wins anyway. In the rest cases, the game is drawn. Twenty-one (also known as blackjack) is the most widespread casino banking game in the world. A common strategy adopted by bankers lies in choosing cards until their sum exceeds 17.

5.5.1 Strategies and payoff functions

Let us suggest the following structure as a game-theoretic model of twenty-one. Suppose that each player actually observes the sum of independent identically distributed random variables

$$s_n^{(k)} = \sum_{i=1}^{n} x_i^{(k)}, \ k = 1, 2.$$

For simplicity, assume that the cards $\{x_i^{(k)}\}, i = 1, 2, \dots$ have the uniform distribution on the interval $[0, 1]$. Set the threshold (the maximal admissible sum of cards) equal to 1.

Imagine that a certain player employs a threshold strategy $u, 0 < u < 1$, i.e., stops choosing cards when the total denomination s_n of his cards exceeds u. Denote by τ the stopping time. Therefore,

$$\tau = \min\{n \geq 1 : s_n \geq u\}.$$

To define the payoff function in this game, we should find the distribution of the stopping sum s_τ.

Reexpress $u \leq x \leq 1$ as

$$P\{s_\tau \leq x\} = \sum_{n=1}^{\infty} P\{s_n \leq x, \tau = n\} = \sum_{n=1}^{\infty} P\{s_1 \leq u, \dots, s_{n-1} \leq u, s_n \in [u, x]\}.$$

Then

$$P\{s_\tau \le x\} = \sum_{n=1}^{\infty} \frac{u^{n-1}}{(n-1)!}(x-u) = \exp(u)(x-u). \tag{5.1}$$

Hence, the stopping probability takes the form

$$P\{s_\tau > 1\} = 1 - P\{s_\tau \le 1\} = 1 - \exp(u)(1-u). \tag{5.2}$$

Now, it is possible to construct an equilibrium in twenty-one. Assume that player *II* uses the threshold strategy u. Find the best response of player *I*.

Let $s_n^{(1)} = x$ be the current value of player *I* sum. If $x \le u$, the expected payoff of player *I* at stoppage becomes

$$h(x|u) = +P\{s_{\tau_2}^{(2)} > 1\} - P\{s_{\tau_2}^{(2)} \le 1\} = 2P\{s_{\tau_2}^{(2)} > 1\} - 1.$$

In the case of $x > u$, the expected payoff is given by

$$\begin{aligned} h(x|u) &= +P\{s_{\tau_2}^{(2)} < x\} + P\{s_{\tau_2}^{(2)} > 1\} - P\{x < s_{\tau_2}^{(2)} \le 1\} \\ &= 2\left[P\{s_{\tau_2}^{(2)} < x\} + P\{s_{\tau_2}^{(2)} > 1\}\right] - 1. \end{aligned}$$

Taking into account (5.1) and (5.2), we obtain the following. Under stoppage in the state x, the payoff becomes

$$h(x|u) = 2\left[\exp(u)(x-u) + 1 - \exp(u)(1-u)\}\right] - 1 = 1 - 2\exp(u)(1-x).$$

If player I continues and stops at next shot (receiving some card y), his payoff constitutes $1 - 2\exp(u)(1-x-y)$ for $x+y \le 1$ and $-P\{s_{\tau_2}^{(2)} \le 1\} = -\exp(u)(1-u)$ for $x+y > 1$. Hence, the expected payoff in the case of continuation is

$$Ph(x|u) = \int_0^{1-x} (1 - 2\exp(u)(1-x-y))\,dy - \int_{1-x}^{1} \exp(u)(1-u)dy.$$

Certain simplifications yield

$$Ph(x|u) = 1 - x - \exp(u)\left(1 - x(1+u) + x^2\right).$$

Obviously, the function $h(x|u)$ increases monotonically in x under $x \ge u$, whereas $Ph(x|u)$ decreases monotonically. This fact follows from negativity of the derivative

$$\frac{dPh(x|u)}{dx} = -1 + \exp(u)(1 + u - 2x),$$

since for $u > 0$ and $x \ge u$ we have

$$\exp(-u) > 1 - u \ge 1 + u - 2x.$$

Therefore, being aware of player *II* strategy u, his opponent can evaluate the best response by comparing the payoffs in the case of stoppage and continuation. The optimal threshold x_u satisfies the equation

$$h(x|u) = Ph(x|u),$$

or

$$1 - 2\exp(u)(1-x) = 1 - x - \exp(u)\left(1 - x(1+u) + x^2\right).$$

Rewrite the last equation as

$$x = \exp(u)\left(1 - x(1-u) - x^2\right). \tag{5.3}$$

Due to monotonicity of the functions $h(x|u)$ and $Ph(x|u)$, such threshold is unique (if exists).

By virtue of game symmetry, an equilibrium must comprise identical strategies. And so, we set $x_u = u$. According to (5.3), such strategy obeys the equation

$$\exp(u) = \frac{u}{1-u}. \tag{5.4}$$

The solution to (5.4) exists and is $u^* \approx 0.659$.

Thus, both players have the following optimal behavior. They choose cards until the total denomination exceeds the threshold $u^* \approx 0.659$. Subsequently, they stop and open up the cards. The probability of exceeding this threshold becomes

$$P\{s_\tau > 1\} = 1 - \exp(u)(1-u) = 1 - u \approx 0.341.$$

Remark 5.2 To find the optimal strategy of a player, we have compared the payoffs in the case of stoppage and continuation by one more shot. However, rigorous analysis requires comparing the payoffs in the case of continuation by an arbitrary number of steps. Chapter 9 treats the general setting of optimal stoppage games to demonstrate the following fact. In the monotonous case (the payoff under stoppage does not decrease and the payoff under continuation by one more shot does not increase), it suffices to consider these payoff functions.

5.6 Soccer. A game-theoretic model of resource allocation

Imagine coaches of teams *I* and *II*, allocating their players on a football ground. In a real match, each team has 11 footballers in the starting line-up. Suppose that the goal of player *I* (player *II*) is located on the right (left, respectively) half of the ground. Let us partition the ground into n sectors. For convenience, assume that n is an odd number. A match starts in the center belonging to sector $m = (n+1)/2$ and continues until the ball crosses either the left or right boundary of the ground—the goal line (player *I* or player *II* wins, respectively).

Represent movements of the ball as random walks in the state set $\{0, 1, \dots, n, n+1\}$, where 0 and $n+1$ indicate absorbing states (see Figure 5.15). Assume that each state $i \in \{1, \dots, n\}$ is associated with transition rates (probabilities) p_i and $q_i = 1 - p_i$ to state $i - 1$

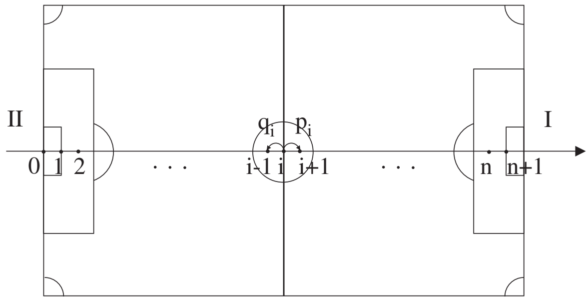


Figure 5.15 Random walks on a football ground.

(to the right) and to state $i+1$ (to the left) in a given sector, respectively. These probabilities depend on the ratio of players in the sector. A natural conjecture lies in the following. The higher is the number of team I players in sector i, the greater is the probability of ball transition to the left. Denote by x_i (y_i) the number of team I players (team II players, respectively) in sector i. In this case, we believe that the probability p_i is some non-increasing differentiable function $g(x_i/y_i)$ meeting the condition $g(1) = 1/2$. In other words, random walks appear symmetrical if the teams accommodate identical resources in this sector. For instance, g can be defined by $g(x_i/y_i) = y_i/(x_i + y_i)$). In contrast to the classical random walks, transition probabilities depend on a state and strategies of players.

Let π_i designate the probability of player I win provided that the ball is in sector i. With the probability q_i, the ball moves either to the left, where player I wins with the probability π_{i-1}, or to the right, where player I wins with the probability π_{i+1}. Write down the system of Kolmogorov equations in the probabilities $\{\pi_i\}$, $i = 0, \dots, n+1$:

$$\begin{aligned} \pi_1 &= q_1 + p_1\pi_2, \\ \pi_i &= q_i\pi_{i-1} + p_i\pi_{i+1}, \ i = 1, \dots, n, \\ \pi_n &= q_n\pi_{n-1}. \end{aligned} \tag{6.1}$$

The first and last equations take into account that $\pi_0 = 1, \pi_{n+1} = 0$. Set $s_i = q_i/p_i, i = 1, \dots, n$ and redefine the system (6.1) as

$$\begin{aligned} \pi_1 - \pi_2 &= s_1(1 - x_1), \\ \pi_i - \pi_{i+1} &= s_i(\pi_{i-1} - \pi_i), \ i = 1, \dots, n, \\ \pi_n &= s_n(\pi_{n-1} - \pi_n). \end{aligned} \tag{6.2}$$

Now, denote $c_i = s_1 \dots s_i, i = 1, \dots, n$ and, using (6.2), find $\pi_n = c_n(1 - \pi_1)$ and $\pi_i - \pi_{i+1} = c_i(1 - \pi_i)$ for $i = 1, \dots, n-1$. Summing up these formulas gives

$$\pi_1 = (c_1 + \dots + c_n)(1 - \pi_1),$$

whence it follows that $\pi_1 = (c_1 + \dots + c_n)/(1 + c_1 + \dots + c_n)$ and

$$\pi_i = \frac{c_i + c_{i+1} + \dots + c_n}{1 + c_1 + \dots + c_n}, \ i = 2, \dots, n. \tag{6.3}$$

Therefore, for a known distribution of footballers by sectors, it is possible to compute the quantities

$$c_i = \prod_{j=1}^{i} \frac{1 - g(x_j/y_j)}{g(x_j/y_j)}, i = 1, \dots n, \tag{6.4}$$

and the corresponding probabilities π_i of teams in each situation i. For convenience, suppose that players utilize unit amounts of infinitely divisible resources. The strategy of player I consists in a resource allocation vector $x = (x_1, \dots, x_n), x_i \geq 0, i = 1, \dots, n$ by different sectors, where $\sum_{i=1}^{n} x_i = 1$. Similarly, as his strategy, player II chooses a vector $y = (y_1, \dots, y_n), y_j \geq 0, j = 1, \dots, n$, where $\sum_{j=1}^{n} y_j = 1$.

Player I strives for maximizing π_m, whereas player II seeks to minimize this probability.

To evaluate an equilibrium in this antagonistic game, construct the Lagrange function

$$L(x, y) = \pi_m + \lambda_1(x_1 + \cdots + x_n - 1) + \lambda_2(y_1 + \cdots + y_n - 1),$$

and find the optimal strategies (x^*, y^*) from the condition $\partial L/\partial x = 0, \partial L/\partial y = 0$.

Notably,

$$\frac{\partial L}{\partial x_k} = \sum_{j=1}^{n} \frac{\partial \pi_m}{\partial c_j} \frac{\partial c_j}{\partial x_k} + \lambda_1.$$

According to (6.4),

$$\frac{\partial c_j}{\partial x_k} = 0, \text{ for } k > j.$$

Hence,

$$\frac{\partial L}{\partial x_k} = \sum_{j=k}^{n} \frac{\partial \pi_m}{\partial c_j} \frac{\partial c_j}{\partial x_k} + \lambda_1. \tag{6.5}$$

If $j \geq k$, we have

$$\frac{\partial c_j}{\partial x_k} = -\frac{g'\left(\frac{x_k}{y_k}\right)}{y_k g\left(\frac{x_k}{y_k}\right)\left(1 - g(\frac{x_k}{y_k})\right)} c_j = -\alpha_k c_j, j = k, \dots, n. \tag{6.6}$$

It appears from (6.5) and (6.6) that, under $k \geq m$,

$$\begin{aligned}\frac{\partial L}{\partial x_k} &= \sum_{j=k}^{n} \frac{1 + c_1 + \cdots + c_{m-1}}{(1 + c_1 + \cdots + c_n)^2}(-\alpha_k c_j) + \lambda_1 \\ &= -\frac{\alpha_k}{(1 + c_1 + \cdots + c_n)^2}(1 + c_1 + \cdots + c_{m-1})(c_k + \cdots + c_n) + \lambda_1. \end{aligned} \tag{6.7}$$

In the case of $k < m$,

$$\frac{\partial L}{\partial x_k} = \sum_{j=k}^{m-1} \frac{-(c_m + \cdots + c_n)}{(1 + c_1 + \cdots + c_n)^2}(-\alpha_k c_j) + \sum_{j=m}^{n} \frac{1 + c_1 + \cdots + c_{m-1})}{(1 + c_1 + \cdots + c_n)^2}(-\alpha_k c_j) + \lambda_1$$
$$= -\frac{\alpha_k}{(1 + c_1 + \cdots + c_n)^2}(1 + c_1 + \cdots + c_{k-1})(c_m + \cdots + c_n) + \lambda_1. \tag{6.8}$$

The expressions (6.7) and (6.8) can be united:

$$\frac{\partial L}{\partial x_k} = -\frac{\alpha_k}{(1 + c_1 + \cdots + c_n)^2}(1 + c_1 + \cdots + c_{m\wedge k-1})(c_{m\vee k} + \cdots + c_n) + \lambda_1, \tag{6.9}$$

where $i \wedge j = \min\{i,j\}$ and $i \vee j = \max\{i,j\}$.

Similar formulas take place for $\partial L/\partial y_k, k = 1, \ldots, n$. First, we find

$$\frac{\partial c_j}{\partial y_k} = \begin{cases} 0 & \text{if } j < k \\ \frac{x_k g'(\frac{x_k}{y_k})}{y_k^2 g(\frac{x_k}{y_k})(1-g(\frac{x_k}{y_k}))} c_j = \frac{x_k}{y_k}\alpha_k c_j, & \text{if } j \geq k, \end{cases} \tag{6.10}$$

and then

$$\frac{\partial L}{\partial y_k} = \frac{x_k}{y_k}\frac{\alpha_k}{(1 + c_1 + \cdots + c_n)^2}(1 + c_1 + \cdots + c_{m\vee k-1})(c_{m\vee k} + \cdots + c_n) + \lambda_2. \tag{6.11}$$

Now, evaluate the stationary point of the function $L(x, y)$. The condition $\partial L/\partial x_1 = 0$ and the expression (6.9) with $k = 1$ lead to the equation

$$\lambda_1 = \alpha_1 \frac{c_m + \cdots + c_n}{(1 + c_1 + \cdots + c_n)^2}.$$

Accordingly, the condition $\partial L/\partial y_1 = 0$ and (6.11) yield the equation

$$\lambda_2 = \frac{x_1}{y_1}\lambda_1.$$

In the case of $k \geq 2$, the conditions $\partial L/\partial x_k = 0, \partial L/\partial y_k = 0$ imply the following. Under $k > m$, we have the equalities

$$\begin{aligned} \alpha_k(1 + c_1 + \cdots + c_{m-1})(c_k + \cdots + c_n) &= \alpha_1(c_m + \cdots + c_n), \\ \frac{x_k}{y_k}\alpha_k(1 + c_1 + \cdots + c_{m-1})(c_k + \cdots + c_n) &= \frac{x_1}{y_1}\alpha_1(c_m + \cdots + c_n). \end{aligned} \tag{6.12}$$

If $k \leq m$, we obtain

$$\begin{aligned} \alpha_k(1 + c_1 + \cdots + c_{k-1}) &= \alpha_1, \\ \frac{x_k}{y_k}\alpha_k(1 + c_1 + \cdots + c_{m-1}) &= \frac{x_1}{y_1}\alpha_1. \end{aligned} \tag{6.13}$$

It appears from (6.12)–(6.13) that

$$\frac{x_k}{x_1} = \frac{y_k}{y_1}, \text{for all } k = 1, \ldots, n.$$

Since $\sum_{k=1}^{n} x_k = \sum_{k=1}^{n} y_k = 1$, summing up these expressions brings to $x_1 = y_1$ and, hence,

$$x_k = y_k, k = 1, \ldots, n.$$

In an equilibrium, both players have an identical allocation of their resources.

According to the condition $g(1) = 1/2$ and the definition (6.4), the fact of identical allocations requires that

$$c_1 = \ldots = c_n = 1.$$

It follows from (6.12)–(6.13) that

$$\alpha_k = \begin{cases} \frac{1}{k}\alpha_1 & \text{for } k \leq m, \\ \frac{n-m+1}{m(n-k+1)}\alpha_1 & \text{for } k > m. \end{cases}$$

Formula (6.6) shows that $\alpha_k = 4g'(1)/y_k$ in an equilibrium. Therefore, $y_k = y_1\alpha_1/\alpha_k$, which means that

$$x_k = y_k = \begin{cases} ky_1 & \text{for } k \leq m \\ \frac{m(n-k+1)}{n-m+1)}y_1 & \text{for } k > m. \end{cases}$$

Sum up all y_k over $k = 1, \ldots, n$:

$$\sum_{k=1}^{m} ky_1 + \sum_{k=m+1}^{n} \frac{m(n-k+1)}{n-m+1}y_1 = y_1\left(\sum_{k=1}^{m} k + \frac{m}{n-m+1}\sum_{k=1}^{n-m} k\right)$$
$$= y_1\left(\frac{m(m+1)}{2} + \frac{m}{n-m+1}\frac{(n-m)(n-m+1)}{2}\right) = y_1\frac{m(n+1)}{2}.$$

Having in mind that $\sum_{k=1}^{n} y_k = 1$, we get $y_1 = \frac{2}{m(n+1)}$.

Thus,

$$x_k = y_k = \begin{cases} \frac{2k}{m(n+1)} & \text{for } k \leq m \\ \frac{2(n-k+1)}{(n+1)(n-m+1)} & \text{for } k > m. \end{cases}$$

A match begins in the center of the ground: $m = (n + 1)/2$. Consequently, the optimal distribution of players on the ground is given by

$$x_k = y_k = \begin{cases} \frac{4k}{(n+1)^2} & \text{for } k \leq (n+1)/2 \\ \frac{4(n-k+1)}{(n+1)^2)} & \text{for } k > (n+1)/2. \end{cases} \tag{6.14}$$

Remark 5.3 Obviously, the optimal distribution of players has a triangular form. Moreover, players must be located such that the ball demonstrates symmetrical random walks. Suppose that a football ground comprises three sectors, namely, defense, midfield, and attack. According to (6.14), the optimal distribution must be $x^* = y^* = (1/4, 1/2, 1/4)$. In other words, the center contains 50% of resources, and the rest resources are equally shared by defense and attack. In real soccer with 11 footballers in the starting line-up, an equilibrium distribution corresponds to the formations $(3, 6, 2)$ or $(3, 5, 3)$.

Exercises

1. Poker with two players.

In the beginning of the game, players I and II contribute the buy-ins of 1. Subsequently, they are dealt cards of ranks x and y, respectively. Each player chooses between two strategies—passing or playing. In the second case, a player has to contribute the buy-in of $c > 0$. The extended form of this game is illustrated by Figure 5.16.

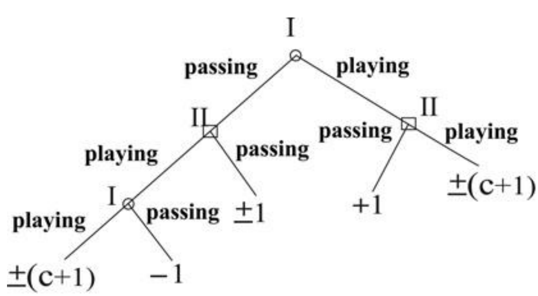


Figure 5.16 Poker with two players.

The player whose card has a higher denomination wins. Find optimal behavioral strategies and the value of this game.

2. Poker with bet raising.

In the beginning of the game, players I and II contribute the buy-ins of 1. Subsequently, they are dealt cards of ranks x and y, respectively. Each player chooses among three strategies—passing, playing, or raising. In the second and third cases, players have to contribute the buy-in of $c > 0$ and $d > 0$, respectively. The extended form of this game is illustrated by Figure 5.17.

Find optimal behavioral strategies and the value of this game.

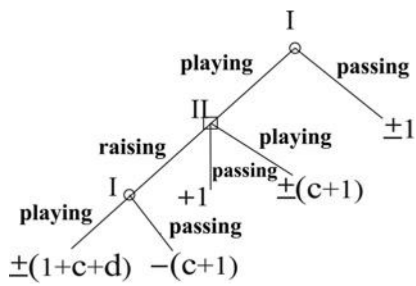


Figure 5.17 Poker with two players.

3. Poker with double bet raising.

In the beginning of the game, players *I* and *II* contribute the buy-ins of 1. Subsequently, they are dealt cards of ranks x and y, respectively. Each player chooses among three strategies—passing, playing, or raising. In the second (third) case, players have to contribute the buy-in of 2 (6, respectively). The extended form of this game is illustrated by Figure 5.18.

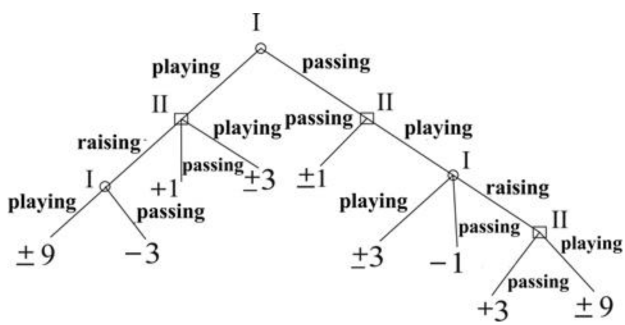


Figure 5.18 Poker with two players.

Find optimal behavioral strategies and the value of this game.

4. Construct the poker model for three or more players.

5. Suggest the two-player preference model with three cards and cards play.

6. The exchange game.

Players *I* and *II* are dealt cards of ranks x and y; these quantities represent independents random variables with the uniform distribution on [0,1]. Having looked at his card, each player may suggest exchange to the opponent. If both players agree, they exchange the cards. Otherwise, exchange occurs with the probability of p (when one of players agrees) or takes no place (when both disagree). The payoff of a player is the value of his card. Find the optimal strategies of players.

7. The exchange game with dependent cards.

Here regulations are the same as in game no. 6. However, the random variables x and y turn out dependent—they possess the joint distribution

$$f(x,y) = 1 - \gamma(1-2x)(1-2y), \quad 0 \le x, y \le 1 .$$

Find optimal strategies in the game.

8. Construct the preference model for three or more players.

9. Twenty-one.

Two players observe the sums of independent identically distributed random variables $S_n^{(1)} = \sum_{i=1}^{n} x_i^{(1)}$ and $S_n^{(2)} = \sum_{i=1}^{n} x_i^{(2)}$, where $x_i^{(1)}$ and $x_i^{(2)}$ have the exponential distribution with the parameters λ_1 and λ_2. The threshold to-be-not-exceeded equals $K = 21$. The winner is the player terminating observations with a higher sum than the opponent (but not exceeding K). Find the optimal strategies and the value of the game.

10. Twenty-one with Gaussian distribution.

Consider game no. 9 with the observations defined by $S_n^{(1)} = \sum_{i=1}^{n}(x_i^{(1)})^+$ and $S_n^{(2)} = \sum_{i=1}^{n}(x_i^{(2)})^+$, $(a^+ = \max(0, a))$, where $x_i^{(1)}$ and $x_i^{(2)}$ have the normal distributions with different mean values ($a_1 = -1$ and $a_2 = 1$) and the identical variance $\sigma = 1$. Find the optimal strategies and the value of the game.

6

Negotiation models

Introduction

Negotiation models represent a traditional field of research in game theory. Different negotiations run in everyday life. The basic requirements to negotiations are the following:

1. a well-defined list of negotiators;
2. a well-defined sequence of proposals;
3. well-defined payoffs of players;
4. negotiations finish at some instant;
5. equal negotiators have equal payoffs.

6.1 Models of resource allocation

A classical problem in negotiation theory lies in the so-called cake cutting (see Figure 6.1). The word "cake" represents a visual metaphor, actually indicating any (homogeneous or inhomogeneous) resource to-be-divided among parties with proper consideration of their interests.

6.1.1 Cake cutting

Imagine a cake and two players striving to divide it into two equal pieces. How could this be done to satisfy both players? The solution seems easy: one participant cuts the cake, whereas the other chooses an appropriate piece. Both players are satisfied—one believes that he has

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

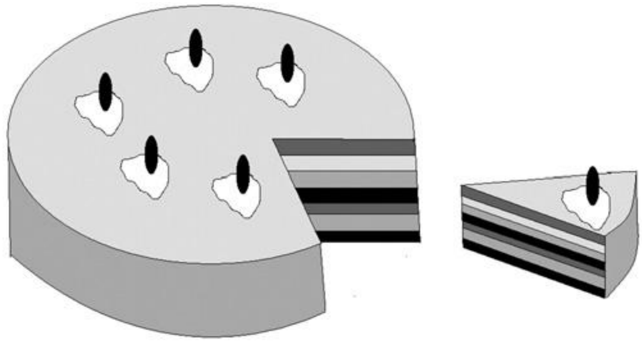


Figure 6.1 Cake cutting.

cut the cake into equal pieces, whereas the other has selected the “best” piece. We call the described procedure by the cutting-choosing procedure.

Now, suppose that cake cutting engages three players. Here the solution is the following (see Figure 6.2). Two players divide the cake by the cutting-choosing procedure. Subsequently, each of them divides his piece into three equal portions and invites player 3 to choose the best portion. All players are satisfied—they believe in having obtained, at least, $1/3$ of the cake.

In the case of n players, we can argue by induction. Let the problem be solved for player $n-1$: the cake is divided into $n-1$ pieces and all $n-1$ players feel satisfied. Subsequently, each player cuts his piece into n portions and invites player n. The latter chooses the best portion and his total portion of the cake makes up $(n-1) \times \frac{1}{(n-1)n} = \frac{1}{n}$. All n players are satisfied.

Even at the current stage, such cake cutting procedure is subjected to criticism. For instance, consider the case of $n \geq 3$ players; a participant that has already shared his piece with player n can be displeased with that another participant does the same (shares his piece with player n). On the one hand, he is sure of that his piece is not smaller than $1/n$. At the same time, he can be displeased with that his piece appears smaller than the opponent’s one.

Therefore, the notion of “fairness” may have different interpretations. Let us discuss this notion in a greater detail. In this model, players are identical but estimate the size of the

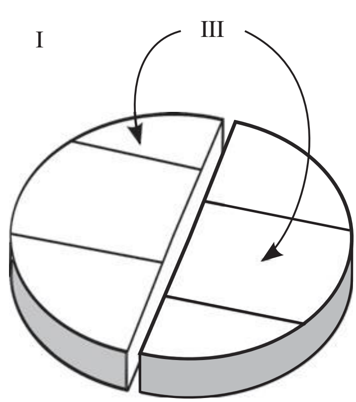


Figure 6.2 Cake cutting for three players.

Figure 6.3 Cake cutting by the moving-knife procedure.

cake in various ways (the cake is homogeneous). In the general case, the problem seems appreciably more sophisticated.

Analysis of the cake cutting problem can proceed from another negotiation model. Suppose that there exists an arbitrator moving a long knife along the cake (thus, gradually increasing the size of a cut portion). It is comfortable to reexpress the cake as the unit segment (see Figure 6.3), where the arbitrator moves the knife left-to-right. As soon as a participant utters "Stop!", he receives the given piece and gets eliminated from further allocation. If several participants ask to stop simultaneously, choose one of them in a random way. Subsequently, the arbitrator continues this procedure with the rest of the participants. Such cake cutting procedure will be called the moving-knife procedure.

If participants are identical and the cake appears homogeneous, the optimal strategy consists in the following. Stop the arbitrator when his knife passes the boundary of $1/n$. Before or after this instant, it is non-beneficial to stop the arbitrator (some participant gets a smaller piece).

6.1.2 Principles of fair cake cutting

The above allocation procedures enjoy fairness, if all participants identically estimate different pieces of the cake.

We identify the basic principles of fair cake cutting. Regardless of players' estimates, the cutting-choosing procedure with two participants guarantees that they surely receive the maximal piece. Player *I* achieves this during cutting, whereas his opponent—during choosing. In other words, this procedure agrees with the following principles.

P1. *The absence of discrimination*. Each of n participants feels sure that he receives, at least, $1/n$ of the case.

P2. *The absence of envy*. Each participant feels sure that he receives not less than other players (does not envy them).

P3. (*Pareto optimality*). Such procedure gives no opportunity to increase the piece of any participant such that the remaining players keep satisfied with their pieces.

The moving-knife procedure also meets these principles for two participants. Imagine that the knife passes the point, where the left and right portions are adequate for some participant. Then he stops the arbitrator, since further cutting reduces his piece. For another player, the right portion seems more beneficial.

Nevertheless, principle P2 (the absence of envy) is violated for three players in both procedures. Consider the cutting-choosing procedure; a certain player can think that cake cutting without his participation is incorrect. In the case of the moving-knife procedure, a player stopping the arbitrator first (he chooses 1/3 of the cake) can think the following. According to his opinion, further allocation is incorrect, as one of the remaining players receives more than 1/3 of the cake.

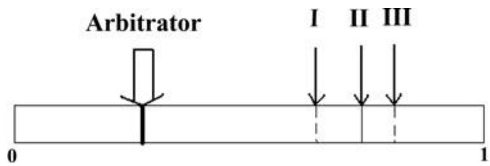


Figure 6.4 Cake cutting with an arbitrator: the case of three players.

Still, there exists the moving-knife procedure that matches the principle of the absence of envy for three players. Actually, it was pioneered by W. Stromquist [1980]. Here, an arbitrator moves the knife left-to-right, whereas three players hold their knives over the right piece (at the points corresponding to the middle of this piece by their viewpoints—see Figure 6.4). As some player utters "Stop!", he receives the left piece and the remaining players allocate the right piece as follows. The cake is cut at the point corresponding to the middle-position knife of these players. Moreover, the player holding his knife to the left from the opponent (closer to the arbitrator), receives the greater portion.

The stated procedure satisfies principle P2. Indeed, the player stopping the arbitrator estimates the left piece not less than 2/3 of the right piece (not smaller than the portions received by other players). His opponents estimate the cut (left) piece not smaller than their portions (in the right piece). Furthermore, each player believes that he receives the greatest piece. Unfortunately, this procedure could not be generalized to the case of $n \geq 3$ players.

6.1.3 Cake cutting with subjective estimates by players

Let us formalize the problem. Again, apply the negotiation model with a moving knife. For this, reexpress a cake as the unit segment (see Figure 6.3). An arbitrator moves a knife left-to-right. As soon as a participant utters "Stop!", he receives the given piece and gets eliminated from further allocation. Subsequently, the arbitrator continues this procedure with the rest participants.

Imagine that players estimate different portions of the cake in different ways. For instance, somebody prefers a rose on a portion, another player loves chocolate filler or biscuit, etc. Describe such estimation subjectivity by a certain measure. Accordingly, we represent the subjective measure of the interval $[0, x]$ for player i by the distribution function $M_i(x)$ with the density function $\mu_i(x)$, $x \in [0, 1]$. Moreover, set $M_i(1) = 1, i = 1, \dots, n$, which means the following. When the arbitrator's knife passes the point x, player i is sure that the size of the corresponding piece makes $M_i(x)$, $i = 1, \dots, n$. Then, for player i, the half of the cake is either the left or right half of the segment with the boundary $x : M_i(x) = 1/2$. Assume that the functions $M_i(x), i = 1, \dots, n$ appear continuous.

Consider the posed problem for two players. Players I and II strive for obtaining the half of the cake in their subjective measure. Allocate the cake according to the procedure below.

Denote by x and y the medians of the distributions M_1 and M_2, i.e., $M_1(x) = M_2(y) = 1/2$. If $x \leq y$, then player I receives the portion $[0, x]$, and his opponent gets $[y, 1]$. In the case of $x < y$, give the players the additional pieces $[x, z]$ and $(z, y]$, respectively, where $z : M_1(z) = 1 - M_2(z)$. Both players feel satisfied, since they believe in having obtained more than the half of the cake.

Now, suppose that $x > y$. Then the arbitrator gives the right piece $[y, 1]$ to player I and the left piece $[0, x]$ to player II. Furthermore, the arbitrator grants additional portions to the participants, $(z, x]$ (player I) and $[y, z]$ (player II), where $z : M_2(z) = 1 - M_1(z)$. Again, both players are pleased—they believe in having obtained more than the half of the cake.

Example 6.1 Take a biscuit cake with 50% chocolate coverage. For instance, player I estimates the chocolate portion of the cake two times higher than the biscuit portion. Consequently, his subjective measure can be defined by the density function $\mu_1(x) = 4/3, x \in [0, 1/2]$ and $\mu_1(x) = 2/3, x \in [1/2, 1]$. Player II has the uniformly distributed measure on $[0, 1]$. The condition $M_1(z) = 1 - M_2(z)$ brings to the equation

$$4/3z = 1 - z,$$

yielding the cutting point $z = 3/7$.

Example 6.2 Imagine that the subjective measure of player I is determined by $M_1(x) = 2x - x^2, x \in [0, 1]$, whereas player II possesses the uniformly distributed measure on $[0, 1]$. Similarly, the condition $M_1(z) = 1 - M_2(z)$ leads to the equation

$$2z - z^2 = 1 - z.$$

Its solution yields the cutting point $z = (3 - \sqrt{5})/2 \approx 0.382$. As a matter of fact, this represents the well-known golden section.

Now, consider the problem for n players. Demonstrate the feasibility of cake cutting such that each player receives a piece exceeding $1/n$. Let the subjective measures $M_i(x), x \in [0, 1]$ be defined for all players $i = 1, \dots, n$. Choose a point x_1 meeting the condition

$$\max_{i=1,\dots,n} \{M_i(x_1)\} = 1/n.$$

Player i_1 corresponding to the above maximum is called player 1. Cut the portion $[0, x_1]$ for him. This player feels satisfied, since he receives $1/n$. For the rest players, the residual piece $[x_1, 1]$ has a higher subjective measure than $1 - M_i(x_1) \geq 1 - 1/n, i = 2, \dots, n$. Then we choose the next portion $(x_1, x_2]$ such that

$$\max_{i=2,\dots,n} \{M_i(x_2)\} = 2/n.$$

By analogy, the player corresponding to this maximum is said to be player 2. He receives the portion $(x_1, x_2]$. Again, this player feels satisfied—his piece in the subjective measure is greater or equal to $1/n$. The remaining players are also pleased, as the residual piece $(x_2, 1]$ possesses the subjective measure not smaller than $1 - 2/n$ for them. We continue the described procedure by induction and arrive at the following result. The last player obtains the residual piece whose subjective measure is not less than $1 - (n - 1)/n = 1/n$.

Therefore, we have constructed a procedure guaranteeing everybody's satisfaction (each player receives a piece with the subjective measure not smaller than $1/n$).

Nevertheless, this procedure disagrees with an important principle as follows.

The principle of equality (P4) is a procedure, where all players get identical pieces in their subjective measures.

6.1.4 Fair equal negotiations

We endeavor to improve the cake cutting procedure suggested in the previous subsection. The ultimate goal lies in making it fair in the sense of principle P4.

Let us start similarly to the original statement. Introduce the parameter z and, for the time being, believe that $0 \le z \le 1/n$. Choose a point x_1 such that

$$\max_{i=1,\dots,n} \{M_i(x_1)\} = z.$$

Player i_1 corresponding to this maximum is called player 1; cut the portion $[0, x_1]$ for him. According to the subjective measure of this player, the portion has the value z.

Choose the next portion $(x_1, x_2]$ by the condition

$$\max_{i=2,\dots,n} \{M_i(x_2) - M_i(x_1)\} = z.$$

The player corresponding to the above maximum is called player 2; cut the portion $(x_1, x_2]$ for him. In the subjective measure of player 2, this portion is estimated by z precisely. Next, cut the portion for player 3, and so on. The procedure repeats until the portion $(x_{n-2}, x_{n-1}]$ is cut for player $n-1$. Recall that $x_{n-1} \le 1$. And finally, player n receives the piece $(x_{n-1}, 1]$.

As the result of this procedure, each of players $\{1, 2, \dots, n-1\}$ possesses a piece estimated by z in his subjective measure. Player n has the piece $(x_{n-1}, 1]$. While z is smaller than $1/n$, this piece appears greater or equal to z (in his subjective measure).

Now, gradually increase z by making it greater than $1/n$. Owing to continuity and monotonicity of the functions $M_i(x)$, the cutting points $x_i (i = 1, \dots, n-1)$ also represent continuous and monotonous functions of argument z such that $x_1 \le x_2 \le \dots \le x_{n-1}$. Moreover, the subjective measure of player n, $1 - M_n(x_{n-1})$, decreases in z (from 1 down to 0). And so, there exists z^* meeting the equality

$$M_1(x_1) = M_2(x_2) - M_2(x_1) = \dots = M_n(1) - M_n(x_{n-1}) = z^*.$$

The modified procedure above leads to the following. Each player gets the piece of value $1/n + z^*$ in his subjective measure. All players obtain equal pieces, negotiations are fair.

Example 6.3 Consider the cake cutting problem for three players whose subjective measures are defined by the following density functions. The subjective measure of player 1 takes the form $M_1(x) = 2x - x^2, x \in [0, 1]$ (he prefers the left boundary of the cake). Player 2 has the uniformly distributed subjective measure. And the density function for player 3 is given by $\mu(x) = |2 - 4x|, x \in [0, 1]$ (he prefers boundaries of the cake). Then the distribution functions acquire the form $M_1(x) = 2x - x^2, M_2(x) = x, x \in [0, 1]$, and $M_3(x) = 2x - 2x^2$, for $x \in [0, 1/2]$, and $M_3(x) = 1 - 2x + 2x^2$, for $x \in [1/2, 1]$ (see Figure 6.5).

The condition $M_1(x_1) = M_2(x_2) - M_2(x_1) = 1 - M_3(x_2)$ brings to the system of equations

$$2x_1 - x_1^2 = x_2 - x_1 = 1 - (1 - 2x_2 + 2x_2^2).$$

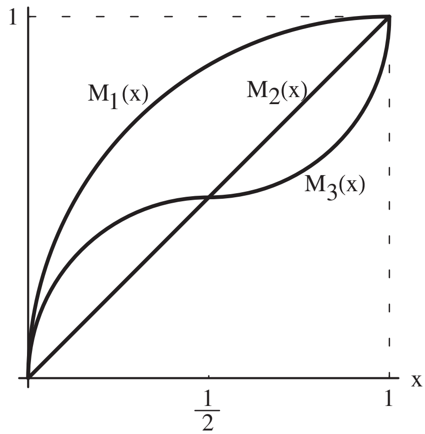


Figure 6.5 The subjective measures of three players.

Its solution gives the cutting points $x_1 \approx 0.2476, x_2 \approx 0.6816$. Each player receives a piece estimated (in his subjective measure) by $z^* \approx 0.4339$, which exceeds $1/3$.

6.1.5 Strategy-proofness

Interestingly, the negotiation model studied in subsection 6.1.4 compels players to be fair. We focus on the model with two players. Suppose that, e.g., player 2 acts according to his subjective measure M_2, whereas player 1 reports another measure M_1' to the arbitrator. And each player knows nothing about the subjective preferences of the opponent.

In this case, it may happen that the median m_1 (m_1') of the distribution M_1 (M_1') lies to the left (to the right, respectively) from that of the distribution M_2. By virtue of the procedure, player 1 gets the piece from the right boundary $(m_1', 1]$, which makes his payoff less than $1/2$.

6.1.6 Solution with the absence of envy

The procedure proposed in the previous subsections ensures cake cutting into identical portions in the subjective measures of players. Nevertheless, this does not guarantee the absence of envy (see principle P2). Generally speaking, the subjective measure of some player i may estimate the piece of another player j higher than the subjective measure of the latter. To establish the existence of such solution without envy, introduce new notions.

Still, we treat the cake as the unit segment $[0, 1]$ to-be-allocated among n players.

Definition 6.1 *An allocation $x = (x_1, \dots, x_n)$ is a vector of cake portions of appropriate players (in their order left-to-right). Consequently, $x_j \geq 0, j = 1, \dots, n$ and $\sum_{j=1}^n x_j = 1$.*

The set of allocations forms a simplex S in R^n. This simplex possesses nodes $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 occupies position i (see Figure 6.6). Node e_i corresponds to cake cutting, where portion i makes the whole cake. Denote by $S_i = \{x \in S : x_i = 0\}$ the set of allocations such that player i receives nothing.

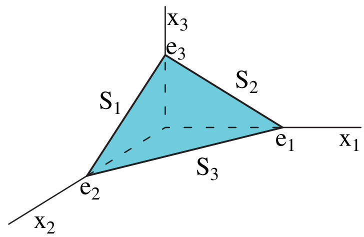


Figure 6.6 The set of allocations under $n = 3$.

Assume that, for each player i and allocation x, there is a given estimation function $f_i^j(x),\ i,j = 1, \dots, n$, *for piece j. We believe that this function takes values in R and is continuous in x. For instance, in terms of the previous subsection,*

$$f_i^j(x) = M_i(x_1 + \cdots + x_j) - M_i(x_1 + \cdots + x_{j-1}),\ i,j = 1, \dots, n.$$

Definition 6.2 *For a given allocation* $x = (x_1, \dots, x_n)$, *we say that player i prefers piece j if*

$$f_i^j(x) \geq f_i^k(x), \forall k = 1, \dots, n.$$

Note that a player may prefer one or more pieces. In addition, suppose that none of the players prefer an empty piece.

Then an allocation, where each player receives the piece he actually prefers, matches the principle of the absence of envy.

Theorem 6.1 *There exists an allocation with the absence of envy.*

Proof: Denote by A_{ij} the set of $x \in S$ such that player i prefers piece j. Since the functions $f_i^j(x)$ enjoy continuity, these sets turn out closed. For any player i, the sets A_{ij} cover S (see Figure 6.7). Moreover, due to the assumption, none of players prefers an empty piece. Hence, it follows that the sets A_{ij} and S_j do not intersect for any i, j.

Consider the sets

$$B_{ij} = \cap_{k \neq j}(S - A_{ik}),\ i,j = 1, \dots, n.$$

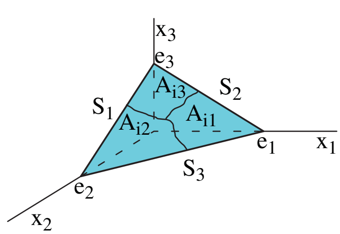


Figure 6.7 The preference set of player i.

For given i and j, B_{ij} represents the set of all allocations, where player i prefers only piece j. This is an open set. The sets B_{ij} do not cover S under a fixed i. The set $S - \cup_j B_{ij}$ consists of boundaries, where a player may prefer two or more pieces.

Now, introduce the structure $U_j = \cup_i B_{ij}$. In fact, this is a set of allocations such that a certain player prefers piece j exclusively. Each set U_j is open and does not intersect S_j. Consider $U = \cap_j U_j$. To prove the theorem, it suffices to argue that the set U is non-empty. Indeed, if $x \in U$, then x belongs to each U_j. In other words, piece j is preferred by some player only. Recall that the number of players and the number of pieces coincide. Therefore, each player prefers just one piece. And we have to demonstrate non-emptiness of U.

Lemma 6.1 *The intersection of sets $U_1, \dots, U_n$ is non-empty.*

Proof: We will consider two cases. First, suppose that the sets $U_j, j = 1, \dots, n$ cover S.

Let $d_j(x)$ be the distance from the point x to the set $S - U_j$ and denote $D(x) = \sum_j d_j(x)$. Since $S = \cup_j U_j$, then x belongs to some U_j and $d_j(x) > 0$. Hence, $D(x) > 0$ for all $x \in S$.

Define the mapping $f : S \to S$ by

$$f(x) = \sum_{j=1}^{n} \frac{d_j(x)}{D(x)} e_j.$$

This is a continuous self-mapping of the simplex S, where each face S_i corresponds to the interior of the simplex S. Really, if $x \in S_i$ (i.e., $x_i = 0$), then $x \notin U_i$, since the sets U_i and S_i do not intersect. In this case, $d_i(x) > 0$, which means that component i of the point $f(x)$ is greater than 0.

By Brouwer's fixed-point theorem, there exists a fixed point of the mapping $f(x)$, which lies in the interior of the simplex S. It immediately follows that there is an allocation x such that $d_j(x) > 0$ for all j. Consequently, $x \in U_j$ for all $j = 1, \dots, n$, *ergo* $x \in U$.

The case when the sets $U_j, j = 1, \dots, n$ do not cover S can be reduced to case 1. This is possible if the preferences of some players coincide (for such allocations, all players prefer two or more pieces). Then we modify the preferences of players and approximate the sets A_{ij} by the sets A'_{ij}, where all preferences differ. Afterwards, we pass to the limit.

This theorem claims that there exists an allocation without envy; each player receives a piece estimated in his subjective measure at least the same as the pieces of other players. However, such allocation is not necessarily fair.

6.1.7 Sequential negotiations

We study another cake cutting model with two players pioneered by A [1982]. Rubinstein. Suppose that players make sequential offers for cake cutting and the process finishes, as one of them accepts the offer of another. Assume that the payoff gets discounted with the course of time. At shot 1, the size of cake is 1; at shot 2, it makes $\delta < 1$, at shot 3, δ^2, and so on. For definiteness, we believe that, at shot 1 and all subsequent odd shots, player *I* makes his offer (and even shots correspond to the offers of player *II*). An offer can be represented as a pair (x_1, x_2), where x_1 indicates the share of cake for player *I*, and x_2 means the share for player *II*. We will seek for a subgame-perfect equilibrium, i.e., an equilibrium in all subgames of this game. Apply the backward induction technique.

We begin with the case of three shots. The scheme of negotiations is as follows.

1. Player I makes the offer $(x_1, 1 - x_1)$, where $x_1 \leq 1$. If player II agrees, the game finishes—players I and II receive the payoffs x_1 and $1 - x_1$, respectively. Otherwise, the game continues to the next shot.

2. Player II makes the new offer $(x_2, 1 - x_2)$, where $x_2 \leq 1$. If player I accepts it, the game finishes. Players I and II gain the payoffs x_2 and $1 - x_2$, respectively. If player I rejects the offer, the game continues to shot 3.

3. The game finishes such that players I and II get the payoffs y and $1 - y$, respectively ($y \leq 1$ is a given value). In the sequel, we will establish the following fact. This value has no impact on the optimal solution under a sufficiently large duration of negotiations.

   To find a subgame-perfect equilibrium, apply the backward induction method. Suppose that negotiations run at shot 2 and player II makes an offer. He should make a certain offer x_2 to player I such that his payoff is higher than at shot 3. Due to the discounting effect, the payoff of player I at the last shot constitutes δy. Therefore, player I agrees with the offer x_2, iff

$$x_2 \geq \delta y.$$

   On the other hand, if player II offers $x_2 = \delta y$ to the opponent, his payoff becomes $1 - \delta y$. However, if his offer appears non-beneficial to player I, the game continues to shot 3 and player II gains $\delta(1 - y)$ (recall the discounting effect). Note that $\delta(1 - y) < 1 - \delta y$. Hence, the optimal offer of player II is $x_2^* = \delta y$.

Now, imagine that negotiations run at shot 1 and the offer belongs to player I. He knows the opponent's offer at the next shot. And so, player I should make an offer $1 - x_1$ to the opponent such that the latter's payoff is at least the same as at shot 2: $\delta(1 - x_2^*) = \delta(1 - \delta y)$. Player II feels satisfied if $1 - x_1 \geq \delta(1 - \delta y)$ or

$$x_1 \leq 1 - \delta(1 - \delta y).$$

Thus, the following offer of player I is surely accepted by his opponent: $x_1 = 1 - \delta(1 - \delta y)$. If player I offers less, he receives the discounted payoff at shot 2: $\delta x_2^* = \delta^2 y$. Still, this quantity turns out smaller than $1 - \delta(1 - \delta y)$. Therefore, the optimal offer of player I forms $x_1^* = 1 - \delta(1 - \delta y)$, and it will be accepted by player II. The sequence $\{x_1^*, x_2^*\}$ represents a subgame-perfect equilibrium in this negotiation game with three shots.

Arguing by induction, assume that a subgame-perfect equilibrium in the negotiation game with n shots is such that

$$x_1^n = 1 - \delta + \delta^2 - \ldots + (-\delta)^{n-2} + (-\delta)^{n-1} y. \tag{1.1}$$

Now, consider shot 1 in the negotiation game consisting of $n + 1$ shots. Player I should offer to the opponent the share $1 - x_1^{(n+1)}$, which is not smaller than the discounted income of player II at the next shot. By the induction hypothesis, this income takes the form $\delta(1 - x_1^{(n)})$. And so, the offer is accepted by player II if

$$1 - x_1^{(n+1)} \geq \delta(1 - \delta + \delta^2 - \ldots + (-\delta)^{n-2} + (-\delta)^{n-1} y)$$

or

$$x_1^{(n+1)} = 1 - \delta(1 - \delta + \delta^2 - \ldots + (-\delta)^{n-2} + (-\delta)^{n-1} y). \tag{1.2}$$

The expression (1.2) coincides with (1.1) for the negotiation game with $n + 1$ shots.

For large n, the last summand in (1.1), containing y, becomes infinitesimal, whereas the optimal offer of player I equals $x_1^* = 1/(1 + \delta)$.

Theorem 6.2 *The sequential negotiation game of two players admits the subgame-perfect equilibrium*

$$\left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta} \right).$$

Again, these results can be generalized by induction to the case of sequential negotiations among n players. However, we evaluate a subgame-perfect equilibrium differently.

First, we describe the scheme of negotiations with n players.

1. Player 1 makes an offer to each player $(x_1, x_2, \ldots, x_n)$, where $x_1 + \cdots + x_n = 1$. If all players agree, the game finishes and player i gets the payoff x_i, $i = 1, \ldots, n$. If somebody disagrees, the game continues at the next shot, and player 1 becomes the last one.

2. Player 2 acts as the leader and makes a new offer $(x'_2, x'_3, \ldots, x'_n, x'_1)$, where $x'_1 + \cdots + x'_n = 1$. The game finishes if all players accept it; then player i gains x'_i, $i = 1, \ldots, n$. Otherwise (some player rejects the offer), the game continues at the next shot. And player 2 becomes the last one, accordingly.

3. Player 3 acts as the leader and makes his offer. And so on for all players. Actually, the game may have infinite duration.

By analogy, suppose that the payoff gets discounted by the quantity δ at each new shot. Due to this effect, players may benefit nothing by long-term negotiations. We evaluate a subgame-perfect equilibrium via the following considerations.

Player 1 makes an offer at shot 1. He should offer to player 2 a quantity x_2, being not smaller than the quantity player 2 would choose at the next shot. This quantity equals δx_1 by virtue of the discounting effect. Therefore, player 1 should make the following offer to player 2:

$$x_2 \geq \delta x_1.$$

Similarly, player 1 should offer to player 3 some quantity x_3, being not smaller than the quantity player 3 would choose at shot 3: $\delta^2 x_1$. This immediately leads to the inequality

$$x_3 \geq \delta^2 x_1.$$

Adhering to the same line of reasoning, we arrive at an important conclusion. The offer of player 1 satisfies the rest players under the conditions

$$x_i \geq \delta^{i-1} x_1, i = 2, \ldots, n.$$

By offering $x_i = \delta^{i-1}x_1, i = 2, \dots, n$, player 1 pleases the rest of the players. And he receives the residual share of

$$1 - \left(\delta x_1 + \delta^2 x_1 + \cdots + \delta^{n-1} x_1\right).$$

Anyway, this share must coincide with his own offer x_1. Such requirement

$$1 - \left(\delta x_1 + \delta^2 x_1 + \cdots + \delta^{n-1} x_1\right) = x_1$$

yields

$$x_1^* = \left(1 + \delta + \cdots + \delta^{n-1}\right)^{-1} = \frac{1-\delta}{1-\delta^n}.$$

Theorem 6.3 *The sequential negotiation game of n players admits the subgame-perfect equilibrium*

$$\left(\frac{1-\delta}{1-\delta^n}, \frac{\delta(1-\delta)}{1-\delta^n}, \dots, \frac{\delta^{n-1}(1-\delta)}{1-\delta^n}\right). \tag{1.3}$$

Formula (1.3) implies that player 1 has an advantage over the others—his payoff increases as the rate of discounting goes down. This is natural, since the cake rapidly vanishes as time evolves; the role of players with large order numbers becomes inappreciable.

6.2 Negotiations of time and place of a meeting

An important problem in negotiation theory concerns time and place of a meeting. As a matter of fact, time and place of a meeting represent key factors for participants of a business talk and a conference. These factors may predetermine a certain result of an event. For instance, a suggested time or place can be inconvenient for some negotiators. The "convenience" or "inconvenience" admits a rigorous formulation via a utility function. In this case, each participant strives for maximizing his utility. And the problem is to suggest negotiations design and find a solution accepted by negotiators. Let us apply the formal scheme proposed by C. Ponsati [2007, 2011] in [24–25]. For definiteness, we study the negotiation problem for time of meeting.

6.2.1 Sequential negotiations of two players

Imagine two players negotiating time of their meeting. Suppose that their utilities are described by continuous unimodal functions $u_1(x)$ and $u_2(x)$, $x \in [0, 1]$ with maximum points c_1 and c_2, respectively. If $c_1 = c_2$, this value makes the solution. And so, we believe that $c_1 > c_2$.

Assume that players sequentially announce feasible alternatives, and decision making requires the consent of both participants. Players may infinitely insist on alternatives beneficial for them. To avoid such situations, introduce the discounting factor $\delta < 1$. After each session of negotiations, the utility functions of both players get decreased proportionally to δ. Therefore, if players have not agreed about some alternative till instant t, their utilities at this instant acquire the form $\delta^{t-1}u_i(x), i = 1, 2$.

For definiteness, suppose that $u_1(x) = x$ and $u_2(x) = 1 - x$. In this case, the problem becomes equivalent to the cake cutting problem with two players (see Section 6.1.7). Indeed,

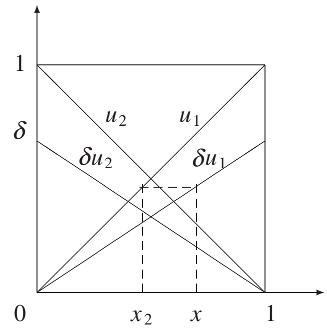


Figure 6.8a The best response of player *I*.

treat x as a portion of the cake. Then player *II* receives the residual piece $1 - x$. It seems interesting to study geometrical interpretations of the solution. Figure 6.8 shows the curves of the utilities $u_1(x)$ and $u_2(x)$, as well as their curves at the next shot, i.e., $\delta u_1(x)$ and $\delta u_2(x)$.

Imagine that player *II* knows the alternative x chosen by the opponent at the next shot. The alternative is accepted, if he offers to player *I* an alternative y such that his utility $u_1(y)$ is not smaller than the utility at the next shot—$\delta u_1(x)$ (see Figure 6.8a). This brings to the inequality $y \geq \delta x$. Furthermore, the maximal utility of player *II* is achieved under $y = \delta x$. Therefore, his optimal response to the opponent's strategy x makes up $x_2 = \delta x$.

Now, suppose that player *I* knows the strategy x_2 selected by player *II* at the next shot. Then his offer y is accepted by player *II* at the current shot, if the corresponding utility $u_2(y)$ of player *II* appears not smaller than at the next shot (the quantity $\delta u_2(x_2)$). This condition is equivalent to the inequality $1 - y \geq \delta(1 - \delta x)$, or

$$y \leq 1 - \delta(1 - \delta x).$$

Hence, the best response of player *I* at the current shot makes up $x_1 = 1 - \delta(1 - \delta x)$. The solution x gives a subgame-perfect equilibrium in negotiations if $x_1 = x$, or $x = 1 - \delta(1 - \delta x)$. It follows that

$$x = \frac{1}{1 + \delta}.$$

This result coincides with the solution obtained in Section 6.1.7.

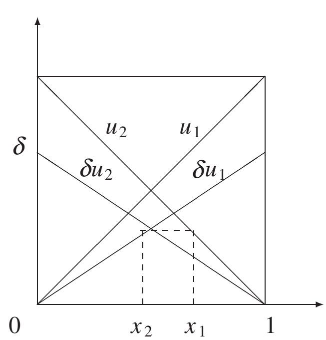


Figure 6.8b The best response of player *I*.

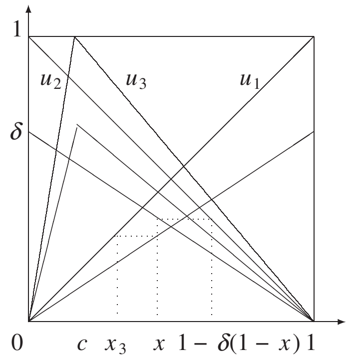


Figure 6.9 The best response of player *III*.

6.2.2 Three players

Now, add player *III* to the negotiations process. Suppose that his utility function possesses a unique maximum on the interval [0, 1]; denote it by $c : 0 < c < 1$. For simplicity, let $u_3(x)$ be a piecewise linear function (see Figure 6.9):

$$u_3(x) = \begin{cases} \frac{x}{c}, & \text{if } 0 \le x < c, \\ \frac{1-x}{1-c}, & \text{if } x \le c \le 1. \end{cases}$$

Players sequentially offer different alternatives; accepting an offer requires the consent of all players. We demonstrate the following aspect. An equilibrium may have different forms depending on the correlation between c and δ. First, consider the case of $c \le 1/3$. Figure 6.9 provides the corresponding illustrations.

The sequence of moves is $I \to II \to III \to I \to \dots$. Suppose that player *I* announces his strategy $x : x \le 1/3$. Being informed of that, player *III* can find his best response. And his offer y will be accepted by player *I*, if $u_1(y)$ is not smaller than $\delta u_1(x)$, i.e., $y \ge \delta x$. The offer y will be accepted by player *II*, if $u_2(y) \ge \delta u_2(x)$, i.e., $1 - y \ge \delta(1 - x)$. Therefore, any offer from the interval $I_3 = [\delta x, 1 - \delta(1 - x)]$ will be accepted. Finally, player *III* strives for maximizing his utility $u_3(y)$ (within the interval I_3, as well). Under the condition $c < \delta x$, the best response consists in $x_3 = \delta x$; if $c \ge \delta x$, the best response becomes $x_3 = c$ (see Figure 6.9).

We begin with the case of $c < \delta x$. The best response of player *III* makes $x_3 = \delta x$. Now, find the best response of player *II* to this strategy. His offer y is surely accepted by player *I*, if $u_1(y) \ge \delta u_1(x_3)$, or $y \ge \delta^2 x$. The offer y is accepted by player *III*, if $u_3(y) \ge \delta u_3(x_3)$, which is equivalent to

$$\delta(1 - \delta x)\frac{c}{1 - c} \le y \le 1 - \delta(1 - \delta x).$$

Clearly, the condition $c < \delta x$ implies that

$$\delta(1 - \delta x)\frac{c}{1 - c} \le \delta^2 x.$$

And so, the offer y is accepted by players *I* and *III*, if it belongs to the interval $I_2 = [\delta^2 x, 1 - \delta(1 - \delta x)]$. Consequently, the best response of player *II* lies in $x_2 = \delta^2 x$ (see Figure 6.10a).

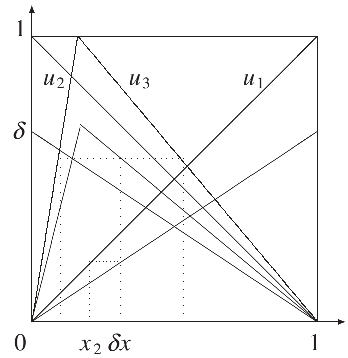


Figure 6.10a The best response of player *II*.

Evaluate the best response of player *I* to this strategy adopted by player *II*.

His offer y is accepted by player *II*, if $u_2(y) \geq \delta u_2(x_2)$, or $y \leq 1 - \delta(1 - \delta^2 x)$, and by player *III*, if $u_3(y) \geq \delta u_3(x_2)$, which is equivalent to the condition

$$\delta^3 x \leq y \leq 1 - \delta^3 x \frac{1-c}{c}.$$

Hence, any offer from the interval $I_1 = [\delta^3 x, 1 - \delta(1 - \delta^2 x)]$ is accepted by players *I* and *III*. The best response of player *I* lies in $x_1 = 1 - \delta(1 - \delta^2 x)$ (see Figure 6.10b).

The subgame-perfect equilibrium corresponds to a strategy x^*, where $x_1 = x$. This yields the equation

$$x = 1 - \delta(1 - \delta^2 x),$$

whence it follows that

$$x^* = \frac{1}{1 + \delta + \delta^2}.$$

Recall that this formula takes place under the condition $c < \delta x^*$, which appears equivalent to

$$c < \frac{\delta}{1 + \delta + \delta^2}.$$

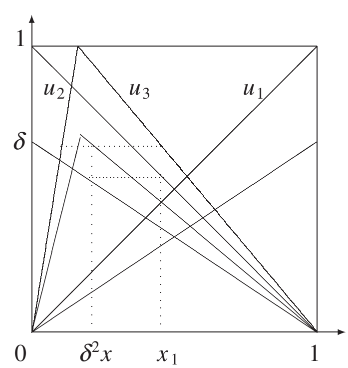


Figure 6.10b The best response of player *I*.

Now, suppose that

$$c \geq \frac{\delta}{1+\delta+\delta^2}.$$

In this case, the best response of player *III* becomes $x_3 = c$. As earlier, we find the best responses of player *II* and then of player *I*. These are the quantities $x_2 = c\delta$ and $x_1 = 1 - \delta + \delta^2 c$, respectively. Such result holds true under the condition $c \leq 1 - \delta(1 - x)$. Finally, we arrive at the following assertion.

Theorem 6.4 *For $n = 3$, the subgame-perfect equilibrium is defined by*

$$x^* = \begin{cases} \frac{1}{1+\delta+\delta^2}, & \text{if } c < \frac{\delta}{1+\delta+\delta^2} \\ 1 - \delta + \delta^2 c, & \text{if } \frac{\delta}{1+\delta+\delta^2} \leq c \leq \frac{1+\delta}{1+\delta+\delta^2} \\ \frac{1+\delta^2}{1+\delta+\delta^2}, & \text{if } c > \frac{1+\delta}{1+\delta+\delta^2}. \end{cases}$$

Theorem 6.4 implies that, in the subgame-perfect equilibrium, the offer of player *I* is not less than $1/3$; under small values of δ, the offer appears arbitrary close to its maximum. In this sense, player *I* dominates the opponents.

6.2.3 Sequential negotiations. The general case

Let us scrutinize the general case of n players. Their utilities are described by continuous quasiconcave unimodal functions $u_i(x), i = 1, 2, \ldots, n$. Recall that a function $u(x)$ is said to be quasiconcave, if the set $\{x : u(x) \leq a\}$ enjoys convexity for any a. Denote by $c_1, c_2, \ldots, c_n$ the maximum points of the utility functions.

Players sequentially offer different alternatives; accepting an alternative requires the consent of all participants. The sequence of moves is $1 \to 2 \to \ldots \to n \to 1 \to 2 \to \ldots$. We involve the same idea as in the case of three players.

Assume that player 1 announces his strategy x. Knowing this strategy, player n can compute his best response. And his offer y will be accepted by player j, if $u_j(y)$ appears not less than $\delta u_j(x)$; denote this set by $I_j(x)$. Note that, for any j, the set $I_j(x)$ is non-empty, so long as $x \in I_j(x)$. Since $u_j(x)$ is quasiconcave, $I_j(x)$ represents a closed interval. Consequently, there exists a closed interval $\Cap_{j=1}^{n-1} I_j(x)$, we designate it by $[a_n, b_n](x)$. Maximize the function $u_n(y)$ on the interval $[a_n, b_n](x)$. Actually, this is the best response of player n, and, by virtue of the above assumptions, it takes the form

$$x_n(x) = \begin{cases} a_n, & \text{if } a_n > c_n, \\ b_n, & \text{if } b_n < c_n, \\ c_n, & \text{if } a_n \leq c_n \leq b_n. \end{cases}$$

Now, imagine that player $n - 1$ is informed of the strategy x_n to-be-selected by player n at the next shot. Similarly, he will make such offer y to player j, and this offer is accepted if $u_j(y) \geq \delta u_j(x_n), j \neq n - 1$. For each j, the set of such offers forms a closed interval; moreover, the intersection of all such intervals is non-empty and turns out to be a closed interval

$[a_{n-1}, b_{n-1}](x_n)$. Again, maximize the function $u_{n-1}(y)$ on this interval. Actually, this maximum gets attained at the point

$$x_{n-1}(x_n) = \begin{cases} a_{n-1}, & \text{if } a_{n-1} > c_{n-1}, \\ b_{n-1}, & \text{if } b_{n-1} < c_{n-1}, \\ c_{n-1}, & \text{if } a_{n-1} \le c_{n-1} \le b_{n-1}. \end{cases}$$

Here x_{n-1} indicates the best response of player $n-1$ to the strategy x_n chosen by player n.

Following this line of reasoning, we finally arrive at the best response of player 1, *viz.*, the function $x_1(x_2)$.

By virtue of the assumptions, all constructed functions $x_i(x), i = 1, \dots, n$ appear continuous. And the superposition of the mappings

$$x_1(\dots(x_{n-1}(x_n)\dots)(x)$$

is a continuous self-mapping of the closed interval $[0, 1]$. Brouwer's fixed-point theorem claims that there exists a fixed point x^* such that

$$x_1(\dots(x_{n-1}(x_n)\dots)(x^*) = x^*.$$

Consequently, we have established the following result.

Theorem 6.5 *Negotiations of meeting time with continuous quasiconcave unimodal utility functions admit a subgame-perfect equilibrium.*

In fact, Theorem 6.5 seems non-constructive—it merely points to the existence of an optimal behavior in negotiations. It is possible to evaluate an equilibrium, e.g., by progressive approximation (start with some offer of player 1 and compute the best responses of the rest players).

Naturally enough, the utility functions of players may possess several equilibria. In this case, the issue regarding the existence of a subgame-perfect equilibrium remains far from settled.

6.3 Stochastic design in the cake cutting problem

Revert to the cake cutting problem with unit cake and n players. Modify the design of negotiations by introducing another independent participant (an arbitrator). The latter submits offers, whereas players decide to agree or disagree with them. The ultimate decision is either by majority or complete consent.

Assume that the arbitrator represents a random generator. Negotiations run on a given time interval K. At each shot, the arbitrator makes random offers. Players observe their offers and support or reject them. Next, it is necessary to calculate the number of negotiators satisfied by their offer; if this number turns out not less than a given threshold p, the offer is accepted. Otherwise, the offered alternative is rejected and players proceed to the next shot for considering another alternative. The size of the cake is discounted by some quantity δ, where $\delta < 1$. If negotiations result in no decision, each player receives a certain portion b, where $b << 1/n$.

Let the random generator be described by the Dirichlet distribution with the density function

$$f(x_1, \dots, x_n) = \frac{1}{B(k)} \prod_{i=1}^{n} x_i^{k_i - 1},$$

where $x_i \geq 0$, $\sum_{i=1}^{n} x_i = 1$ and $k_i \geq 1$. The constant $B(k)$ in this formula,

$$B(k) = B(k_1, \dots, k_n) = \frac{\prod_{i=1}^{n} \Gamma(k_i)}{\Gamma(k_1 + \cdots + k_n)},$$

depends on a set of parameters $(k_1, \dots, k_n)$. They serve for adjusting the weights of appropriate negotiators.

6.3.1 The cake cutting problem with three players

We begin with the case of three players. Negotiations cover the horizon of K shots. Let us count down—suppose that k shots remain. Players receive offers that form a vector (x_1^k, x_2^k, x_3^k). At each shot, offers represent random variables distributed according to the Dirichlet law. In other words, the joint density function takes the form

$$f(x_1, x_2, x_3) = \frac{\Gamma(k_1 + k_2 + k_3)}{\Gamma(k_1)\Gamma(k_2)\Gamma(k_3)} x_1^{k_1 - 1} x_2^{k_2 - 1} x_3^{k_3 - 1},$$

where $x_1 + x_2 + x_3 = 1$.

For a given offer vector (x_1, x_2, x_3), each player has to choose between two alternatives: (a) accepting a current offer (b) rejecting a current offer (waiting for a better offer at next shots). Below we analyze two possible scenarios of negotiations scheme, namely, complete consent and majority. In the former case, an allocation (x_1, x_2, x_3) takes place if all players agree at some shot of negotiations. In the latter case, an allocation occurs when most of players accept the offer (otherwise, players move to nest shot $k - 1$). And the discounting effect reduces the size of the cake to $\delta \leq 1$.

The described process continues until all players (complete consent) or, at least, two of them (majority) support an offer or shot $k = 0$ comes. If negotiations fail, all players receive small portions $b << 1/3$.

Complete consent.

Consider negotiations, where ultimate decision requires the complete consent of players. Denote by H_k the value of this game when k shots remain to the end of negotiations. Suppose that each player is informed of his personal offer only. Let (x_1, x_2, x_3) specify the offers for players I, II, III, respectively. Since $x_1 + x_2 + x_3 = 1$, it suffices to handle the variables x_1, x_2.

First, study the symmetrical case of the Dirichlet distribution, where $k_1 = k_2 = k_3 = 1$:

$$f(x_1, x_2) = 2, \quad x_1 + x_2 \leq 1, \quad x_1, x_2 \geq 0.$$

Introduce the strategies $\mu_i(x_i)$, where $i = 1, 2, 3$. These are the probabilities that player i accepts a current offer x_i. By virtue of problem's symmetry, an equilibrium (if any) belongs to the class of identical strategies of players.

Theorem 6.6 *The optimal strategies of players at shot k have the form*

$$\mu_i(x_i) = I_{\{x_i \geq \delta H_{k-1}\}}, \qquad i = 1, 2, 3,$$

where I_A means the indicator of event A.

The value of this game satisfies the recurrent formulas

$$H_k = \delta H_{k-1} + \frac{1}{3}(1 - 3\delta H_{k-1})^3, \qquad H_0 = b.$$

Proof: The optimality equation for player I payoff at shot k is defined by

$$H_k = \sup_{\mu_1} 2 \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left\{ \mu_1\mu_2\mu_3 x_1 + (1 - \mu_1\mu_2\mu_3)\delta H_{k-1} \right\}, \tag{3.1}$$

$k = 1, 2, \ldots, H_0 = b$. Here $\mu_1 = \mu_1(x_1),\ \ \mu_2 = \mu_2(x_2),\ \ \mu_3 = \mu_3(1 - x_1 - x_2)$. Rewrite (3.1) as

$$H_k = \sup_{\mu_1} 2 \int_0^1 \mu_1(x_1) dx_1 \int_0^{1-x_1} (x_1 - \delta H_{k-1})\mu_2\mu_3 dx_2 + \delta H_{k-1}. \tag{3.2}$$

Player I aims at maximizing his payoff. In the expression (3.2), a player can influence the value of the first integral only. Denote

$$G_k(x_1) = (x_1 - \delta H_{k-1}) \int_0^{1-x_1} \mu_2\mu_3 dx_2.$$

Clearly, the optimal strategy of player I becomes

$$\mu_1(x_1) = \begin{cases} 1, & \text{if} \quad G_k(x_1) \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Owing to problem's symmetry, the optimal behavior of players II and III must be identical: $\mu_2 = \mu_3$. Note that $G_k(0) \leq 0$ and $G_k(1) \geq 0$, since $0 \leq \delta H_{k-1} \leq 1$. And so, $\exists a$ such that $G_k(a) = 0$.

We seek for an equilibrium among threshold strategies. Let $\mu_2 = I_{\{x_2 \geq a\}}$ and $\mu_3 = I_{\{x_3 \geq a\}}$. Clearly, $G_k(x_1)$ has the form

$$\begin{aligned} G_k(x_1) &= (x_1 - \delta H_{k-1}) \int_0^{1-x_1} I_{\{x_2 \geq a, 1 - x_1 - x_2 \geq a\}} dx_2 \\ &= (x_1 - \delta H_{k-1})(1 - x_1 - 2a) I\{a \leq x_1 \leq 1 - 2a\} + 0 I\{x_1 > 1 - 2a\}. \end{aligned}$$

So far as $G_k(a) = 0$, one obtains $a = \delta H_{k-1}$.

Therefore, if players *II* and *III* adopt the threshold strategies $\mu_2 = I_{\{x_2 \geq \delta H_{k-1}\}}$ and $\mu_3 = I_{\{x_3 \geq \delta H_{k-1}\}}$, then the best response of player *I* must be $\mu_1 = I_{\{x_1 \geq \delta H_{k-1}\}}$.

Substitution of $G_k(x_1)$ into (3.2) yields the following equation in H_k:

$$
\begin{aligned}
H_k &= 2 \int\limits_{\delta H_{k-1}}^{1-2\delta H_{k-1}} (x_1 - \delta H_{k-1})(1 - x_1 - 2\delta H_{k-1}) dx_1 + \delta H_{k-1} \\
&= \delta H_{k-1} + \frac{1}{3}(1 - 3\delta H_{k-1})^3.
\end{aligned}
$$

Remark 6.1 If $\delta = 1$, then $\lim_{k\to\infty} H_k = \frac{1}{3}$. This is natural—in the case of no discounting and infinite horizon of negotiations, players wait for a shot when the arbitrator suggests 1/3 of the cake to everybody.

Majority rule.

Now, suppose that negotiations in the cake cutting problem obey the majority rule. An offer is accepted if, at least, two of three players support it. Again, we believe that (a) the horizon of negotiations is K shots and (b) offers at shot k, i.e., the components of the vector (x_1^k, x_2^k, x_3^k), have the Dirichlet distribution with the parameters $k_1 = k_2 = k_3 = 1$.

Denote by H_k the value of this game when k shots remain till the end. Let (x_1, x_2, x_3) specify the offers for players I, II, and III, respectively.

By analogy, introduce the vector $\mu_i(x_i)$, where $i = 1, 2, 3$. It defines the probability that player i accepts a current offer x_i. Set $\bar{\mu}(x) = 1 - \mu(x)$. Owing to symmetry, we search for an equilibrium among identical strategies.

Theorem 6.7 *The optimal strategies of players at shot k possess the form*

$$\mu_i(x_i) = I_{\{x_i \geq \delta H_{k-1}\}}, \qquad i = 1, 2, 3.$$

The value of this game meets the recurrent formulas

$$H_k = \frac{1}{3} - 2\delta^2 H_{k-1}^2 \left(1 - 3\delta H_{k-1}\right), \quad H_0 = b.$$

Proof: The optimality equation for player *I* payoff at shot k is given by

$$
\begin{aligned}
H_k = \sup_{\mu_1} 2 \int\limits_0^1 dx_1 \int\limits_0^{1-x_1} dx_2 \, &\big\{(\mu_1\mu_2\mu_3 + \bar{\mu}_1\mu_2\mu_3 + \mu_1\bar{\mu}_2\mu_3 + \mu_1\mu_2\bar{\mu}_3)x \\
&+(\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3 + \mu_1\bar{\mu}_2\bar{\mu}_3 + \bar{\mu}_1\mu_2\bar{\mu}_3 + \bar{\mu}_1\bar{\mu}_2\mu_3)\delta H_{k-1}\big\}, \quad k = 1, 2, \ldots \qquad (3.4)
\end{aligned}
$$

$H_0 = b$. Here $\mu_1 = \mu_1(x_1), \mu_2 = \mu_2(x_2), \mu_3 = \mu_3(1 - x_1 - x_2)$. Rewrite (3.4) as

$$H_k = \sup_{\mu_1} 2 \int_0^1 \mu_1(x_1)dx_1 \left[\int_0^{1-x_1} \left\{ (x_1 - \delta H_{k-1})(\mu_2 + \mu_3 - 2\mu_2\mu_3) \right\} dx_2 \right]$$
$$+2 \int_0^1 dx_1 \int_0^{1-x_1} \left\{ (x_1 - \delta H_{k-1})\mu_2\mu_3 + H_{k-1} \right\} dx_2. \qquad (3.5)$$

Take the bracketed expression in the first integral and denote it by

$$G_k(x_1) = \int_0^{1-x_1} \left\{ (x_1 - \delta H_{k-1})(\mu_2 + \mu_3 - 2\mu_2\mu_3) \right\} dx_2$$

Evidently, the optimal strategy of player I is

$$\mu_1(x_1) = I_{\{G_k(x_1) \geq 0\}}.$$

We have mentioned that, due to problem's symmetry, the optimal behavior of players II and III is identical: $\mu_2 = \mu_3$. Since $G_k(0) \leq 0$ and $G_k(1) \geq 0$, then $\exists a$ such that $G_k(a) = 0$.

Seek for an equilibrium among threshold strategies. Let $\mu_2 = I_{\{x_2 \geq a\}}$ and $\mu_3 = I_{\{x_3 \geq a\}}$. Clearly, $G_k(x_1)$ has different shape on three intervals:

$$G_k(x_1) = (x_1 - \delta H_{k-1}) \big(2aI\{x_1 \leq 1 - 2a\}$$
$$+2(1 - a - x_1)I\{1 - 2a < x_1 \leq 1 - a\} + 0I\{1 - a < x_1 \leq 1\} \big).$$

It follows from $G_k(a) = 0$ that $a = \delta H_{k-1}$. Thus, $G_k(x_1)$ can be expressed by

$$G_k(x_1) = (x_1 - \delta H_{k-1}) \big(2\delta H_{k-1} I\{x_1 \leq 1 - 2\delta H_{k-1}\}$$
$$+2(1 - \delta H_{k-1} - x_1)I\{1 - 2\delta H_{k-1} < x_1 \leq 1 - \delta H_{k-1}\}$$
$$+0I\{1 - \delta H_{k-1} < x_1 \leq 1\} \big).$$

And so, if players II and III select the threshold strategies $\mu_2 = I_{\{x_2 \geq \delta H_{k-1}\}}$ and $\mu_3 = I_{\{x_3 \geq \delta H_{k-1}\}}$, then the best response of player I must be $\mu_1 = I_{\{x_1 \geq \delta H_{k-1}\}}$.

Substitute $G_k(x_1)$ into (3.5) to derive

$$H_k = 2 \int_0^1 \mu_1(x_1) G_k(x_1) dx_1 + 2 \int_0^1 \int_0^{1-x_1} \left\{ (x_1 - \delta H_{k-1})\mu_2\mu_3 + \delta H_{k-1} \right\} dx_1 dx_2$$
$$= 4\delta H_{k-1} \int_{H_{k-1}}^{1-2\delta H_{k-1}} (x_1 - \delta H_{k-1}) dx_1 + 4 \int_{1-2\delta H_{k-1}}^{1-\delta H_{k-1}} (x_1 - \delta H_{k-1})(1 - \delta H_{k-1} - x_1) dx_1$$
$$+2 \int_0^{1-2\delta H_{k-1}} (x_1 - \delta H_{k-1})(1 - 2\delta H_{k-1} - x_1) dx_1 + \delta H_{k-1}.$$

This result brings to the recurrent formula

$$H_k = \delta H_{k-1} + \frac{1}{3}(1 - 3\delta H_{k-1})(1 - 6\delta^2 H_{k-1}^2).$$

The proof of Theorem 6.7 is completed.

6.3.2 Negotiations of three players with non-uniform distribution

We endeavor to change the parameters of the Dirichlet distribution and analyze the properties of the corresponding optimal solution. For instance, set $k_1 = k_2 = k_3 = 2$. Then the joint density function takes the form

$$f(x_1, x_2) = 120x_1x_2(1 - x_1 - x_2),$$

where $x_1, x_2 > 0$ and $x_1 + x_2 \leq 1$.

Solve this problem under the majority rule. As previously, $\mu_i(x_i)$ $(i = 1, 2, 3)$ indicates the probability that player i accepts a current offer x_i.

Theorem 6.8 *The optimal strategies of players at shot k are defined by*

$$\mu_i(x_i) = I_{\{x_i \geq \delta H_{k-1}\}}, \qquad i = 1, 2, 3.$$

The value of this game satisfies the recurrent formulas

$$H_k = \frac{1}{3} - 10\delta^4 H_{k-1}^4(1 - 3\delta H_{k-1})(3 - 4\delta H_{k-1}), \qquad H_0 = b.$$

Proof: For the payoff at shot k, the optimality equation is given by

$$\begin{aligned} H_k = 120 \int_0^1 x_1 dx_1 \int_0^{1-x_1} x_2(1 - x_1 - x_2)dx_2 \{&(\mu_1\mu_2\mu_3 + \bar{\mu}_1\mu_2\mu_3 \\ &+\mu_1\bar{\mu}_2\mu_3 + \mu_1\mu_2\bar{\mu}_3)x_1 + (\bar{\mu}_1\bar{\mu}_2\bar{\mu}_3 + \mu_1\bar{\mu}_2\bar{\mu}_3 \\ &+\bar{\mu}_1\mu_2\bar{\mu}_3 + \bar{\mu}_1\bar{\mu}_2\mu_3)\delta H_{k-1}\}, \qquad k = 1, 2, \dots, \end{aligned} \tag{3.6}$$

where $H_0 = b$, $\mu_1 = \mu_1(x_1)$, $\mu_2 = \mu_2(x_2)$, and $\mu_3 = \mu_3(1 - x_1 - x_2)$.

Some transformations of (3.6) yield

$$\begin{aligned} H_k = 120 \int_0^1 x_1 \cdot \mu_1(x_1)dx_1 \left[\int_0^{1-x_1} \left\{ \left(x_1 - \delta H_{k-1}\right)\left(\mu_2 + \mu_3 - 2\mu_2\mu_3\right) \right\} x_2(1 - x_1 - x_2)dx_2 \right] \\ +120 \int_0^1 x_1 dx_1 \int_0^{1-x_1} \left\{ \left(x_1 - \delta H_{k-1}\right)\mu_2\mu_3 + \delta H_{k-1} \right\} x_2(1 - x_1 - x_2)dx_2. \end{aligned} \tag{3.7}$$

Take the bracketed expression in the first integral and denote it by

$$G_k(x_1) = x_1 \int_0^{1-x_1} \left\{ \left(x_1 - \delta H_{k-1}\right) \left(\mu_2 + \mu_3 - 2\mu_2\mu_3\right) \right\} x_2(1 - x_1 - x_2)dx_2.$$

The optimal strategy of player I acquires the form (3.3). Find an equilibrium in the class of threshold strategies. Let $\mu_2 = I_{\{x_2 \geq a\}}$, $\mu_3 = I_{\{x_3 \geq a\}}$ and study three cases as follows:

1. If $0 \leq x_1 \leq 1 - 2a$, we have

$$\int_0^{1-x_1} \left(\mu_2 + \mu_3 - 2\mu_2\mu_3\right) x_2(1 - x_1 - x_2)dx_2$$
$$= \int_0^{a} x_2(1 - x_1 - x_2)dx_2 + \int_{1-x_1-a}^{1-x_1} x_2(1 - x_1 - x_2)dx_2 = \frac{1}{3}a^2 \left(3 - 3x_1 - 2a\right).$$

2. If $1 - 2a < x_1 \leq 1 - a$, the value of this integral obeys the formula

$$\int_0^{1-x_1-a} x_2(1 - x_1 - x_2)dx_2 + \int_a^{1-x_1} x_2(1 - x_1 - x_2)dx_2$$
$$= \frac{1}{3}(1 - x_1 + 2a)(1 - x_1 - a)^2.$$

3. If $1 - a < x_1 \leq 1$, the integral under consideration vanishes.

Obtain the corresponding expression for the second integral in (3.7):

$$\int_0^{1-x_1} \mu_2\mu_3 \cdot x_2(1 - x_1 - x_2)dx_2 = \int_a^{1-a-x_1} x_2(1 - x_1 - x_2)dx_2$$
$$= \frac{1}{6}(1 - x_1 - 2a)(1 + 2a - 2a^2 - 2x_1 - 2ax_1 + x_1^2).$$

By virtue of the above relationships, it is possible to write down

$$G_k(x_1) = x_1(x_1 - \delta H_{k-1})\Bigg(\frac{1}{3}a^2 \left(3 - 3x_1 - 2a\right) \cdot I\{x_1 \leq 1 - 2a\}$$
$$+\frac{1}{3}(1 - x_1 + 2a)(1 - x_1 - a)^2 \cdot I\{1 - 2a < x_1 \leq 1 - a\}$$
$$+0 \cdot I\{1 - a < x_1 \leq 1\}\Bigg).$$

So far as $G_k(a) = 0$, we have $a = \delta H_{k-1}$. Consequently,

$$G_k(x_1) = x_1(x_1 - \delta H_{k-1}) \Big(\frac{1}{3} \delta^2 H_{k-1}^2 \left(3 - 3x_1 - 2\delta H_{k-1}\right) \cdot I\{x_1 \le 1 - 2\delta H_{k-1}\}$$
$$+ \frac{1}{3}(1 - x_1 + 2\delta H_{k-1})(1 - x_1 - \delta H_{k-1})^2 \cdot I\{1 - \delta H_{k-1} < x_1 \le 1 - \delta H_{k-1}\}$$
$$+ 0 \cdot I\{1 - \delta H_{k-1} < x_1 \le 1\} \Big).$$

Therefore, if players *II* and *III* adopt the threshold strategies $\mu_2 = I_{\{x_2 \ge \delta H_{k-1}\}}$ and $\mu_3 = I_{\{x_3 \ge \delta H_{k-1}\}}$, then the best response of player *I* consists in $\mu_1 = I_{\{x_1 \ge \delta H_{k-1}\}}$, as well. Thus and so,

$$H_k = 120 \int_0^1 \mu_1(x_1) \cdot G_k(x_1) dx_1$$
$$+ 120 \int_0^1 x_1 dx_1 \int_0^{1-x_1} \left\{ \left(x_1 - \delta H_{k-1}\right) \mu_2 \mu_3 + \delta H_{k-1} \right\} x_2 (1 - x_1 - x_2) dx_2$$
$$= 40 \delta^2 H_{k-1}^2 \int_{\delta H_{k-1}}^{1 - 2\delta H_{k-1}} x_1 \left(x_1 - \delta H_{k-1}\right) \left(3 - 3x_1 - 2\delta H_{k-1}\right) dx_1$$
$$+ 40 \int_{1-2H_{k-1}}^{1-H_{k-1}} x_1 \left(x_1 - \delta H_{k-1}\right) (1 - x_1 + 2\delta H_{k-1})(1 - x_1 - \delta H_{k-1})^2 dx_1$$
$$+ 20 \int_0^{1-2H_{k-1}} x_1 (x_1 - \delta H_{k-1})(1 - x_1 - 2\delta H_{k-1}) \cdot$$
$$\cdot (1 + 2\delta H_{k-1} - 2\delta^2 H_{k-1}^2 - 2x_1 - 2\delta H_{k-1} x_1 + x_1^2) dx_1 + \delta H_{k-1}.$$

And finally, we derive the recurrent formula
$H_k = \delta H_{k-1} + \frac{1}{3} \left(1 - 3\delta H_{k-1}\right) \left(1 - 90 \delta^4 H_{k-1}^4 + 120 \delta^5 H_{k-1}^5\right).$

6.3.3 Negotiations of n players

This subsection deals with the general case of negotiations engaging n participants. Decision making requires, at least, $p \ge 1$ votes. Assume that $k_i = 1$, $i = 1, \dots, n$. The joint density function of the Dirichlet distribution is described by

$$f(x_1, \dots, x_n) = (n-1)!,$$

where $x_i > 0$, $i = 1, \dots, n$ and $\sum_{i=1}^{n} x_i = 1$.

Let H_k^n indicate the value of this game at shot k. Introduce the symbols $\mu^1 = \mu$ and $\mu^0 = 1 - \mu$. In the sequel, we use the notation μ^σ, where $\sigma = \{0, 1\}$. Accordingly,

$$H_k^n = (n-1)! \sup_{\mu_1} \left\{ \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-x_1-\dots-x_{n-2}} \sum_{(\sigma_1\sigma_2\dots\sigma_n)} \left\{ \mu_1^{\sigma_1} \mu_2^{\sigma_2} \dots \mu_n^{\sigma_n} \cdot \right. \right.$$

$$\left. \left. \cdot \begin{bmatrix} x_1, & \text{if } \sum_{i=1}^n \sigma_i \ge p \\ \delta H_{k-1}^n, & \text{if } \sum_{i=1}^n \sigma_i < p \end{bmatrix} \right\} dx_1 dx_2 \dots dx_{n-1} \right\}$$

$$= (n-1)! \sup_{\mu_1} \left\{ \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-x_1-\dots-x_{n-2}} \mu_1 \cdot \right.$$

$$\cdot \sum_{(\sigma_2\dots\sigma_n)} \left\{ \mu_2^{\sigma_2} \dots \mu_n^{\sigma_n} \cdot F_{1,k} \right\} dx_1 \dots dx_{n-1}$$

$$\left. + \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-x_1-\dots-x_{n-2}} (1-\mu_1) \cdot \sum_{(\sigma_2\dots\sigma_n)} \left\{ \mu_2^{\sigma_2} \dots \mu_n^{\sigma_n} \cdot F_{2,k} \right\} dx_1 \dots dx_{n-1} \right\},$$

where

$$F_{1,k} = \begin{bmatrix} x_1, & \text{if } \sum_{i=2}^n \sigma_i \ge p-1 \\ \delta H_{k-1}^n, & \text{if } \sum_{i=2}^n \sigma_i < p-1 \end{bmatrix}$$

$$F_{2,k} = \begin{bmatrix} x_1, & \text{if } \sum_{i=2}^n \sigma_i \ge p \\ \delta H_{k-1}^n, & \text{if } \sum_{i=2}^n \sigma_i < p. \end{bmatrix}$$

Certain transformations and grouping by μ_1 bring to

$$H_k^n = \sup_{\mu_1} (n-1)! \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-x_1-\dots-x_{n-2}} \mu_1 \cdot$$

$$\cdot \sum_{(\sigma_2\dots\sigma_n)} \left\{ \mu_2^{\sigma_2} \dots \mu_n^{\sigma_n} \cdot \left(F_{1,k} - F_{2,k} \right) \right\} dx_1 \dots dx_{n-1} + (n-1)! \cdot$$

$$\cdot \int_0^1 \int_0^{1-x_1} \dots \int_0^{1-x_1-\dots-x_{n-2}} \sum_{(\sigma_2\dots\sigma_n)} \left\{ \mu_2^{\sigma_2} \dots \mu_n^{\sigma_n} \cdot F_{2,k} \right\} dx_1 \dots dx_{n-1}. \quad (3.8)$$

Then

$$F_k = F_{1,k} - F_{2,k} = \begin{bmatrix} x_1 - \delta H^n_{k-1}, & \text{if} \quad p - 1 \le \sum_{i=1}^{n} \sigma_i < p \\ 0, & \text{otherwise.} \end{bmatrix}$$

The inequality $p - 1 \le \sum_{i=1}^{n} \sigma_i < p$ appears equivalent to $\sum_{i=1}^{n} \sigma_i = p - 1$; under this condition, we have $F_k \neq 0$.

As a result, the first integral in (3.8) admits the representation

$$\int_0^1 \mu_1(x_1)dx_1 \int_0^{1-x_1} \dots \int_0^{1-x_1-\ldots-x_{n-2}} \sum_{(\sigma_2\ldots\sigma_n)} \left\{\mu_2^{\sigma_2} \ldots \mu_n^{\sigma_n} \cdot F_k\right\} dx_2 \ldots dx_{n-1}$$
$$= \int_0^1 \mu_1(x_1) G^n_k(x_1) dx_1.$$

Select $\mu_i = I_{\{x_i \ge a\}}$, $i = 2, \ldots, n$ and find the best response of player I. Actually, it becomes

$$\mu_1(x_1) = \begin{cases} 1, & \text{if } G^n_k(x_1) \ge 0 \\ 0, & \text{otherwise.} \end{cases}$$

To evaluate $G^n_k(x_1)$, we introduce the notation

$$S_i(1) = \{x_i : x_i \ge a\} \cap [0, 1 - x_1 - \ldots - x_{i-1}]$$

and

$$S_i(0) = \{x_i : x_i < a\} \cap [0, 1 - x_1 - \ldots - x_{i-1}]$$

for $i = \overline{2, n-1}$. Then $G^n_k(x_1)$ can be reexpressed by

$$G^n_k(x_1) = \sum_{(\sigma'_2\ldots\sigma'_{n-1})} \int_{S_2(\sigma'_2)} \dots \int_{S_{n-1}(\sigma'_{n-1})} \sum_{(\sigma_2\ldots\sigma_n)} \left\{\mu_2^{\sigma_2} \ldots \mu_n^{\sigma_n} \cdot F_k\right\} dx_2 \ldots dx_{n-1}.$$

The following equality holds true:

$$\int_{S_2(\sigma'_2)} \dots \int_{S_{n-1}(\sigma'_{n-1})} \sum_{(\sigma_2\ldots\sigma_n)} \left\{\mu_2^{\sigma_2} \ldots \mu_n^{\sigma_n} \cdot F_k(x_1, \sigma_2, \ldots, \sigma_n)\right\} dx_2 \ldots dx_{n-1}$$
$$= \int_{S_2(\sigma'_2)} \dots \int_{S_{n-1}(\sigma'_{n-1})} \left\{1 \cdot F_k(x_1, \sigma'_2, \ldots, \sigma'_{n-1}, \sigma_n)\right\} dx_2 \ldots dx_{n-1}.$$

Note that $F_k(x_1, \sigma_2, \dots, \sigma_n) \neq 0$, if $\sum_{i=2}^{n} \sigma_i = p - 1$. The number of sets $(\sigma_2, \dots, \sigma_n)$ such that $F_k \neq 0$ equals C_{n-1}^{p-1}. Hence,

$$G_k^n(x_1) = C_{n-1}^{p-1}\left(x_1 - \delta H_{k-1}^n\right) \cdot$$
$$\cdot \int_0^{1-x_1} \dots \int_0^{1-x_1-\dots-x_{n-2}} I\left\{\cap_{i=2}^{p}\{x_i \geq a\} \cap_{i=p+1}^{n} \{x_i < a\}\right\} dx_2 \dots dx_{n-1}. \quad (3.9)$$

Thus, the optimal strategy of player I belongs to the class of threshold strategies; its threshold makes up $a = \delta H_{k-1}^n$.

Theorem 6.9 *Consider the cake cutting game with n players. The optimal strategies of players at shot k take the form*

$$\mu_i(x_i) = I_{\{x_i \geq \delta H_{k-1}\}}, \qquad i = 1, \dots, n.$$

The value of this game meets the recurrent expressions

$$H_k^n = (n-1)! \left\{ \int_{\delta H_{k-1}^n}^{1} G_k^n(x_1) dx_1 \right.$$
$$\left. + \int_0^1 dx_1 \sum_{(\sigma_2 \dots \sigma_{n-1})} \int_{S_2(\sigma_2)} \dots \int_{S_{n-1}(\sigma_{n-1})} F_{2,k} dx_2 \dots dx_{n-1} \right\}. \quad (3.10)$$

6.3.4 Negotiations of *n* players. Complete consent

Consider the cake cutting problem with n participants, where decision making requires complete consent: $p = n$.

In this case, the optimality equation is defined by

$$H_k^n = (n-1)! \int_{\delta H_{k-1}^n}^{1} G_k^n(x_1) dx_1 + \delta H_{k-1}^n. \quad (3.11)$$

According to (3.9), the function $G_k^n(x)$ acquires the form

$$G_k^n(x_1) = \left(x_1 - \delta H_{k-1}^n\right) \int_{\delta H_{k-1}^n}^{1-x_1} \dots \int_{\delta H_{k-1}^n}^{1-x_1-\dots-x_{n-2}} dx_2 \dots dx_{n-1}$$
$$= \begin{cases} \dfrac{\left(x_1 - \delta H_{k-1}^n\right)\left(1 - x_1 - (n-1)\delta H_{k-1}^n\right)^{n-2}}{(n-2)!}, & x_1 \leq 1 - (n-1)\delta H_{k-1}^n \\ 0, & x_1 > 1 - (n-1)\delta H_{k-1}^n. \end{cases}$$

Substitute this result into (3.11) and apply certain simplifications to get the recurrent equation

$$H_k^n = \delta H_{k-1}^n + \frac{(1 - n\delta H_{k-1}^n)^n}{n}. \tag{3.12}$$

Therefore, we have arrived at the following statement.

Theorem 6.10 *Consider the cake cutting problem with n players and decision making by complete consent. The optimal strategies of players at shot k are determined by*

$$\mu_i(x_i) = I_{\{x_i \geq \delta H_{k-1}\}}, i = 1, \ldots, n.$$

The value of this game H_k^n satisfies the recurrent formulas (3.12).

Remark 6.2 The described stochastic procedure of cake allocation can be adapted to different real situations. If participants possess identical weights, the parameters of the Dirichlet distribution should coincide. In this case, the cake allocation procedure guarantees equal opportunities for all players. However, if a certain participant has a higher weight, increase its parameter in the Dirichlet distribution. Moreover, the solution depends on the length of negotiations horizon.

6.4 Models of tournaments

Generally, negotiations cover not just a single point (e.g., price, the volume of supplies or the period of a contract), but a package of interconnected points. Any changes in a point of a package may affect the rest points and terms. Compiling a project, one should maximize the expected profits and take into account the behavior of opponents.

Suppose that players $i \in N = \{1, 2, \ldots, n\}$ submit their projects for a tournament. Projects are characterized by a set of parameters $x^i = (x_1^i, \ldots, x_m^i)$. An arbitrator or arbitration committee considers the incoming projects and chooses a certain project by a stochastic procedure with a known probability distribution. The winner (player k) obtains the payoff $h_k(x^k)$, which depends on the parameters of his project. Assume that project selection employs a multidimensional arbitration procedure choosing the project closest to the arbitrator's opinion.

6.4.1 A game-theoretic model of tournament organization

Study the following non-cooperative n player game with zero sum. Players $\{1, 2, \ldots, n\}$ submit projects for a tournament. Projects are characterized by the vectors $\{x^1, \ldots, x^n\}$ from some feasible set S in the space R^m. For instance, the description of a project can include required costs, implementation period, the number of employees, etc. An arbitrator analyzes the incoming proposals and choose a project by the following stochastic procedure. In the space R^m, it is necessary to model a random vector a with a certain probability distribution $\mu(x_1, \ldots, x_m)$ (all tournament participants are aware of this distribution). The vector a is called arbitrator's decision. A project x^k located at the shortest distance to a represents the winner of this tournament. The corresponding player k receives the payoff $h_k(x^k)$, which depends on the project parameters. Another interpretation of the vector a consists in the following. This

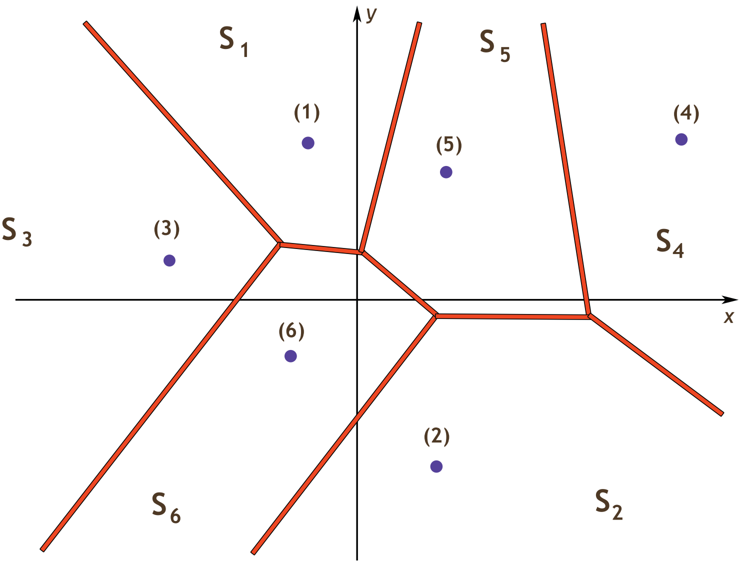


Figure 6.11 The Voronoi diagram on the set of projects.

is a set of expert's opinions, where each component specifies the decision of an appropriate expert. Moreover, experts can be independent or make correlated decisions.

Note that the decision of the arbitrator appears random. For a given set of projects $\{x^1, \dots, x^n\}$, the set $S \subset R^m$ gets partitioned into n subsets $S_1, \dots, S_n$ such that, if $a \in S_k$, then the arbitrator selects project k (see Figure 6.11). The described partition is known as the Voronoi diagram.

Therefore, the payoff of player k in this game can be defined through the mean value of his payoff as the arbitrator's decision hits the set S_k, i.e.,

$$H_k(x^1, \dots, x^n) = \int_{S_k} h_k(x^k)\mu(dx_1, \dots, dx_n) = h_k(x_k)\mu(S_k), k = 1, \dots, n.$$

And so, we seek for a Nash equilibrium in this game—a strategy profile $x^* = (x^1, \dots, x^n)$ such that

$$H_k(x^*||y^k) \le H_k(x^*), \quad \forall y^k, \quad k = 1, \dots, n.$$

For the sake of simplicity, consider the two-dimensional case when a project is described by a couple of parameters. Suppose that players have submitted their projects $x^i = (x_i, y_i), i = 1, \dots, n$ for a tournament, and two independent arbitrators assess them. The decision of the arbitrators is modeled by a random vector in the 2D space, whose density function takes the form $f(x, y) = g(x)g(y)$.

For definiteness, focus on player 1. The set S_1 (corresponding to the approval of his project) represents a polygon with sides $l_{i_1}, \dots, l_{i_k}$. Here l_j designates a straight-line segment passing through the bisecting point of the segment $[x^1, x^j]$ perpendicular to the latter (see Figure 6.11).

Clearly, the boundary l_j satisfies the equation

$$x(x_1 - x_j) + y(y_1 - y_j) = \frac{x_1^2 + y_1^2 - x_j^2 - y_j^2}{2},$$

or

$$y = l_j(x) = -\frac{x_1 - x_j}{y_1 - y_j}x + \frac{x_1^2 + y_1^2 - x_j^2 - y_j^2}{2(y_1 - y_j)}.$$

Let $x_{i_j}, j = 1, \dots, k$ denote the abscissas of all vertices of the polygon S_1. For convenience, we renumber them such that

$$x_{i_0} \le x_{i_1} \le x_{i_2} \le \dots \le x_{i_k} \le x_{i_{k+1}},$$

where $x_{i_0} = -\infty, x_{i_{k+1}} = \infty$.

All interior points $(x, y) \in S_1$ meet the following condition. The function $l_{i_j}(x)$ possesses the same sign as $l_{i_j}(x_1)$, or $l_{i_j}(x)l_{i_j}(x_1) > 0, j = 1, \dots, k$.

In this case, the measure $\mu(S_1)$ admits the representation

$$\mu(S_1) = \sum_{j=0}^{k+1} \int_{x_{i_j}}^{x_{i_{j+1}}} g(x)dx \int_{l_{i_j}(x)l_{i_j}(x_1)>0, j=1,\dots,k} g(y)dy.$$

A similar formula can be derived for any domain $S_i, i = 1, \dots, n$.

6.4.2 Tournament for two projects with the Gaussian distribution

Consider the model of a tournament with two participants and zero sum, where projects are characterized by two parameters. For instance, imagine a dispute on the partition of property (movable estate x and real estate y). Player I strives for maximizing the sum $x + y$, whereas the opponent (player II) seeks to minimize it. Suppose that, settling such a dispute, an arbitrator applies a procedure with the Gaussian distribution $f(x, y) = \frac{1}{2\pi} \exp\{-(x^2 + y^2)/2\}$.

Players submit their offers (x_1, y_1) and (x_2, y_2). The 2D space of arbitrator's decisions is divided into two subsets, S_1 and S_2. Their boundary represents a straight line passing through the bisecting point of the segment connecting the points (x_1, y_1) and (x_2, y_2) (see Figure 6.12). The equation of this line is defined by

$$y = -\frac{x_1 - x_2}{y_1 - y_2}x + \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2(y_1 - y_2)}.$$

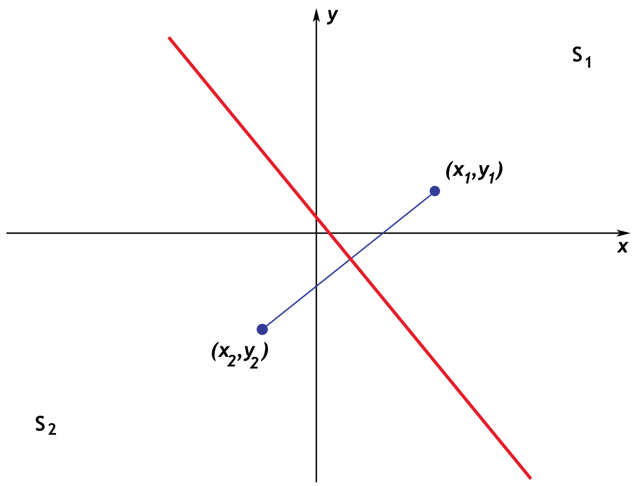


Figure 6.12 A tournament with two projects in the 2D space.

Therefore, the payoff of player I in this game acquires the form

$$\begin{aligned} H(x_1, y_1; x_2, y_2) &= (x_1 + y_1)\mu(S_1) \\ &= (x_1 + y_1) \int_R \int_R f(x,y) I \left\{ y \geq -\frac{x_1 - x_2}{y_1 - y_2} x + \frac{(x_1^2 - x_2^2 + y_1^2 - y_2^2)}{2(y_1 - y_2)} \right\} dx\, dy, \quad (4.1) \end{aligned}$$

where $I\{A\}$ means the indicator of the set A.

Problem's symmetry dictates that, in the optimal strategies, all players have identical values of the parameters. Let $x^2 = y^2 = -a$. Then it appears from (4.1) that

$$H(x_1, y_1) = (x_1 + y_1) \int_R \int_R f(x,y) I \left\{ y \geq -\frac{x_1 + a}{y_1 + a} x + \frac{(x_1^2 + y_1^2 - 2a^2)}{2(y_1 + a)} \right\} dxdy.$$

The best response of player I satisfies the condition

$$\frac{\partial H}{\partial x_1} = 0, \frac{\partial H}{\partial y_1} = 0.$$

And so,

$$\begin{aligned} \frac{\partial H}{\partial x_1} &= \mu(S_1) + (x_1 + y_1) \frac{\partial \mu(S_1)}{\partial x_1} \\ &= \mu(S_1) + (x_1 + y_1) \int_R \frac{1}{2\pi} \frac{x - x_1}{y_1 + a} \exp\left\{ -\frac{1}{2} \left[x^2 + \left(-\frac{x_1 + a}{y_1 + a} x + \frac{x_1^2 + y_1^2 - 2a^2}{2(y_1 + a)} \right)^2 \right] \right\} dx. \end{aligned} \quad (4.2)$$

Equate the expression (4.2) to zero and require that the solution is achieved at the point $x^1 = y^1 = a$. This leads to the optimal value of the parameter a. Note that, owing to symmetry, $\mu(S_1) = 1/2$. Consequently,

$$\frac{1}{2} - 2a \int_R \frac{1}{2\pi} \exp\left\{-\frac{1}{2}(x^2 + x^2)\right\} \frac{-x+a}{2a} dx = 0,$$

whence it follows that

$$\int_{-\infty}^{\infty} (-x + a)\frac{1}{2\pi} e^{-x^2} dx = \frac{1}{2}.$$

Finally, we obtain the optimal value

$$a = \sqrt{\pi}.$$

Readers can easily verify the sufficient maximum conditions of the function $H(x, y)$ at the point (a, a).

Therefore, the optimal strategies of players in this game consist in the offers $(-\sqrt{\pi}, -\sqrt{\pi})$ and $(\sqrt{\pi}, \sqrt{\pi})$, respectively.

6.4.3 The correlation effect

We have studied the model of a tournament, where projects are assessed by two criteria and arbitrator's decisions represent Gaussian random variables. Consider the same problem under the assumption that arbitrator's decisions are dependent. This corresponds to the case when each criterion belongs to a separate expert, and the decisions of experts are correlated.

Suppose that the winner is selected by a procedure with the Gaussian distribution $f(x, y) = \frac{1}{2\pi\sqrt{1-r^2}} \exp\{-\frac{1}{2(1-r^2)}(x^2 + y^2 - 2rxy)\}$. Here $r : r \le 1$ means the correlation factor.

Again, we take advantage of the problem symmetry. Imagine that player II adheres to the strategy $(-a, -a)$ and find the best response of player I in the form $(x_1 = y_1 = a)$. Perform differentiation of the payoff function (4.1) with the new distribution, and substitute the values $x_1 = y_1 = a$ to derive the equation

$$\int_{-\infty}^{\infty} (-x + a)\frac{1}{2\pi\sqrt{1 - r^2}} e^{-\frac{x^2}{1-r}} dx = \frac{1}{2}.$$

Its solution yields

$$a = \sqrt{\pi(1 + r)}.$$

Obviously, the relationship between arbitrator's decisions allows to increase the optimal offers of the players.

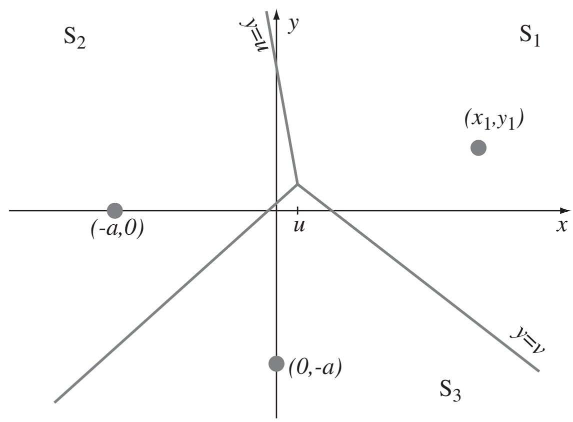


Figure 6.13 A tournament with three projects in the 2D space.

6.4.4 The model of a tournament with three players and non-zero sum

Now, analyze a tournament of projects submitted by three players. Here player I aims at maximizing the sum $x + y$, whereas players II and III strive for minimization of x and y, respectively. Suppose that an arbitrator is described by the Gaussian distribution in the 2D space: $f(x, y) = g(x)g(y)$, where $g(x) = \frac{1}{\sqrt{2\pi}} \exp\{-x^2/2\}$.

As usual, we utilize the problem symmetry. The optimal strategies must have the following form:

for player I: (c, c),
for player II: $(-a, 0)$,
for player III: $(0, -a)$.

To evaluate a and c, we proceed as follows. Assume that players II and III submit to the tournament the projects $(-a, 0)$ and $(0, -a)$, respectively. On the other hand, player I submits the project (x_1, y_1), where $x_1, y_1 \geq 0$. In this case, the space of projects gets decomposed into three sets (see Figure 6.13) delimited by the lines $y = x$ and

$$l_2 : y = -\frac{x_1 + a}{y_1}x + \frac{x_1^2 + y_1^2 - a^2}{2y_1},$$
$$l_3 : y = -\frac{x_1}{y_1 + a}x + \frac{x_1^2 + y_1^2 - a^2}{2(y_1 + a)}.$$

The three lines intersect at the same point $x = y = x_0$, where

$$x_0 = \frac{x_1^2 + y_1^2 - a^2}{2(x_1 + y_1 + a)}.$$

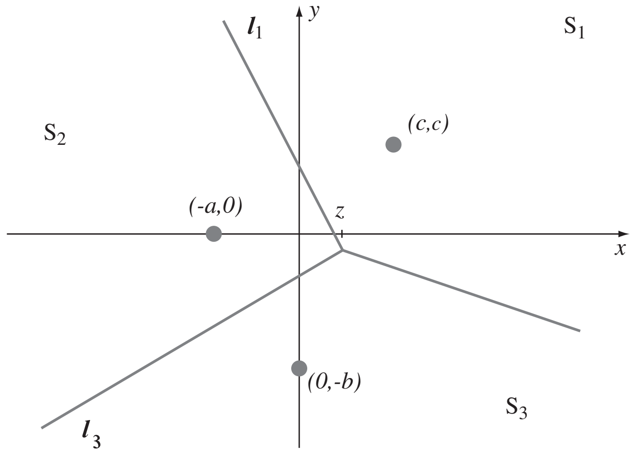


Figure 6.14 A tournament with three projects in the 2D space.

We are mostly concerned with the domain S_1 having the boundaries l_2 and l_3. Reexpress the payoff of player I as

$$H_1(x_1, y_1) = (x_1 + y_1)\left[\int_{-\infty}^{x_0} g(x)dx \int_{u}^{\infty} g(y)dy + \int_{x_0}^{\infty} g(x)dx \int_{v}^{\infty} g(y)dy\right], \quad (4.3)$$

where

$$u = -\frac{x_1 + a}{y_1}x + \frac{x_1^2 + y_1^2 - a^2}{2y_1},$$
$$v = -\frac{x_1}{y_1 + a}x + \frac{x_1^2 + y_1^2 - a^2}{2(y_1 + a)}.$$

Further simplifications of (4.3) yield

$$H_1(x_1, y_1) = (x_1 + y_1)\left[1 - \int_{-\infty}^{x_0} g(x)G(u)dx - \int_{x_0}^{\infty} g(x)G(v)dx\right], \quad (4.4)$$

where $G(x)$ is the Gaussian distribution function. The maximum of (4.4) is attained under $x_1 = y_1 = c$; actually, it appears a certain function of a.

Now, fix a strategy (c, c) of player I such that $c > 0$. Suppose that player III chooses the strategy $(0, -b)$ and seek for the best response $(-a, 0)$ of player II to the strategies adopted by the opponents. The space of projects is divided into three domains (see Figure 6.14). The boundaries of the domain S_2 are defined by

$$l_1 : y = -\frac{c + a}{c}x + \frac{2c^2 - a^2}{2c}$$

and

$$l_3 : y = \frac{a}{b}x - \frac{b^2 - a^2}{2b}.$$

The intersection point of these domains possesses the abscissa

$$z = \left(\frac{2c^2 - a^2}{2c} - \frac{a^2 - b^2}{2b}\right)\frac{1}{a/b + 1 + a/c}.$$

And the payoff of player *II* constitutes

$$\begin{aligned} H_2(a) &= a\left[\int_{-\infty}^{z} g(x)dx \int_{v_1}^{v_2} g(y)dy\right] \\ &= a[\int_{-\infty}^{z} \big(G(v_2) - G(v_1)\big) f(x)dx, \end{aligned} \tag{4.5}$$

where

$$\begin{aligned} v_1 &= \frac{a}{b}x - \frac{b^2 - a^2}{2b}, \\ v_2 &= -\frac{c + a}{c}x + \frac{2c^2 - a^2}{2c}. \end{aligned}$$

Due to the considerations of symmetry, the minimum of (4.5) must be attained at $a = b$. These optimization problems yield the optimal values of the parameters a and c. Numerical simulation leads to the following approximate values of the optimal parameters:

$$a = b \approx 1.7148, \quad c \approx 1.3736.$$

The equilibrium payoffs of the players make up

$$H_1 \approx 0.920, \quad H_2 = H_3 \approx 0.570,$$

and the probabilities of entering the appropriate domains equal

$$\mu(S_1) \approx 0.335, \quad \mu(S_2) = \mu(S_3) \approx 0.332.$$

Remark 6.3 The game-theoretical model of tournaments with arbitration procedures admits a simple implementation in a software environment.

To solve a practical task (e.g., house making), one organizes a tournament and creates a corresponding commission. Experts (arbitrators) assess this task in terms of each parameter. Subsequently, it is necessary to construct a probability distribution which agrees with the opinions of experts.

Then players submit their offers for the tournament. The commission may immediately reject the projects whose parameter values are dominated by other projects. And the phase of winner selection follows. The decisions of an arbitrator (or several arbitrators) are modeled by random variables in the space of projects. The winner is the project lying closer to the arbitrator's decision. Voting takes place in the case of the arbitration committee.

6.5 Bargaining models with incomplete information

Negotiations accompany any transactions on a market. Here participants are sellers and buyers. In recent years, such transactions employ the system of electronic tenders. There exist different mechanisms of negotiations. We begin with the double auction model proposed by K. Chatterjee and W.F. Samuelson [1983].

6.5.1 Transactions with incomplete information

Consider a two-player game with incomplete information. It engages player *I* (a seller) and player *II* (a buyer). Each player possesses private information unavailable to the opponent. Notably, the seller knows the manufacturing costs of a product (denote them by s), whereas the buyer assigns some value b to this product. These quantities are often called reservation prices. Assume that reservation prices (both for sellers and buyers) have the uniform distribution on a market. In other words, if we select randomly a seller and buyer on the market, their reservation prices s and b represent independent random variables with the uniform distribution within the interval $[0, 1]$.

Players appear on the market and announce their prices for a product, S and B, respectively. Note that these quantities may differ from the reservation prices. Actually, we believe that $S = S(s)$ and $B = B(b)$—they are some functions of the reservation prices. The transaction occurs if $B \geq S$. A natural supposition claims that $S(s) \geq s$ and $B(b) \leq b$, i.e., a seller overprices a product and a buyer underprices it. If the transaction takes place, we suppose that the negotiated price is $(S(s) + B(b))/2$.

In fact, players gain the difference between the reservation prices and the negotiated price: $(S(s) + B(b))/2 - s$ (the seller) and $b - (S(s) + B(b))/2$ (the buyer). Recall that b and s are random variables, and we define the payoff functions as the mean values

$$H_s(B, S) = E_{b,s} \left(\frac{S(s) + B(b)}{2} - s \right) I_{\{B(b) \geq S(s)\}} \tag{5.1}$$

and

$$H_b(B, S) = E_{b,s} \left(b - \frac{S(s) + B(b)}{2} \right) I_{\{B(b) \geq S(s)\}}. \tag{5.2}$$

The stated Bayesian game includes the functions $B(b)$ and $S(s)$ as the strategies of players. It seems logical that these are non-decreasing functions (the higher the seller's costs or the buyer's price, the greater are the offers of the players). Find a Bayesian equilibrium in the game with the payoff functions (5.1)–(5.2).

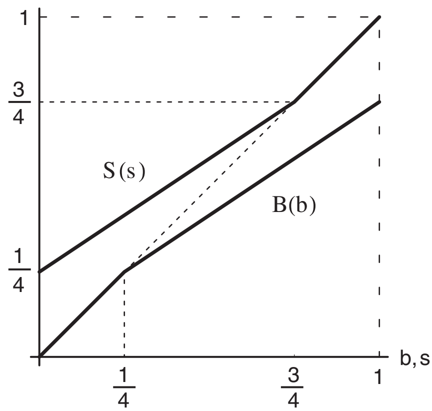


Figure 6.15 Optimal strategies.

Theorem 6.11 *The optimal strategies in the transaction problem have the following form:*

$$B(b) = \begin{cases} b & \text{if } b \le \frac{1}{4}, \\ \frac{2}{3}b + \frac{1}{12} & \text{if } \frac{1}{4} \le b \le 1, \end{cases} \tag{5.3}$$

$$S(s) = \begin{cases} \frac{2}{3}s + \frac{1}{4} & \text{if } 0 \le s \le \frac{3}{4}, \\ s & \text{if } \frac{3}{4} \le s \le 1. \end{cases} \tag{5.4}$$

Moreover, the probability of transaction constitutes $9/32$, *and each player gains* $9/64$.

Proof: The strategies (5.3)–(5.4) are illustrated in Figure 6.15. Assume that the buyer selects the strategy (5.3) and establishes the best response of the seller under different values of the parameter s.

Let $s \ge 1/4$. Then the transaction occurs under the condition $B(b) \ge S$. By virtue of (5.3), this is equivalent to

$$\frac{2}{3}b + \frac{1}{12} \ge S.$$

The last inequality can be reduced to

$$b \ge \frac{3}{2}S - \frac{1}{8},$$

where b denotes a random variable with the uniform distribution on $[0, 1]$. The seller's payoff acquires the form

$$H_s(B, S) = E_b\left(\frac{S + B(b)}{2} - s\right) I_{\{B(b)\ge S\}} = \int_{\frac{3}{2}S-\frac{1}{8}}^{1} \left(\frac{\frac{2}{3}b + \frac{1}{12} + S}{2} - s\right) db$$

$$= \frac{3}{128}(-3 + 16s - 12S)(-3 + 4S). \tag{5.5}$$

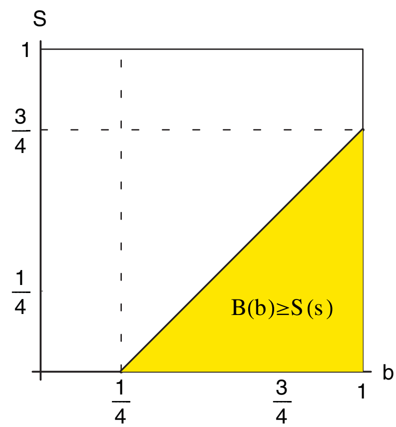


Figure 6.16 The domain of successful negotiations (the transaction domain).

As a matter of fact, this curve draws a parabola with the roots of $3/4$ and $4/3s - 1/4$. The maximum is achieved under

$$S = \frac{1}{2}\left(\frac{4}{3}s - \frac{1}{4} + \frac{3}{4}\right) = \frac{2}{3}s + \frac{1}{4}. \tag{5.6}$$

Interestingly, if $s > 3/4$, the quantity (5.6) appears smaller than s. Therefore, the best response of the seller becomes $S(s) = s$. And so, in the case of $s \geq 1/4$, the best response of the seller to the strategy (5.3) is given by

$$S = \max\left\{\frac{2}{3}s + \frac{1}{4}, s\right\}.$$

Now, suppose that $s < 1/4$. We demonstrate that the inequality $S(s) \geq 1/4$ holds true then.

Indeed, if $S(s) < 1/4$, the transaction occurs under the condition $B(b) = b \geq S$. Consequently, the seller's payoff makes up

$$H_s(B, S) = E_b\left(\frac{S(s) + B(b)}{2} - s\right) = \int_S^{\frac{1}{4}}\left(\frac{b + S}{2} - s\right)db + \int_{\frac{1}{4}}^1\left(\frac{\frac{2}{3}b + \frac{1}{12} + S}{2} - s\right)db$$
$$= -\frac{3}{4}S^2 + \left(\frac{1}{2} + s\right)S - s + \frac{13}{64}. \tag{5.7}$$

The function (5.7) increases within the interval $S \in [0, 1/4]$. Hence, the optimal response lies in $S(s) \geq 1/4$. The payoff function $H_s(B, S)$ acquires the form (5.5), and the optimal strategy of the seller is defined by (5.4).

Similarly, readers can show that the best response of the buyer to the strategy (5.4) coincides with the strategy (5.3).

The optimal strategies enjoy the property $B(b) \leq 3/4$ and $S(s) \geq 1/4$. Thus, the transaction fails if $b < 1/4$ or $s > 3/4$ (see Figure 6.16). However, if $b \geq 1/4$ and $s \leq 3/4$, the transaction takes place under $B(b) \geq S(s)$, which is equivalent to

$$b \geq s + 1/4.$$

Now, we compute the probability that the transaction occurs with the optimal behavior of the players:

$$P\{B(b) > S(s)\} = \int_{\frac{1}{4}}^{1} \int_{0}^{b-\frac{1}{4}} dsdb = \frac{9}{32} \approx 0.281.$$

In this case, the payoffs of the players equal

$$H_s = H_b = \int_{\frac{1}{4}}^{1} \int_{0}^{b-\frac{1}{4}} \left(\frac{2/3b + 1/12 + 2/3s + 1/4}{2} - s \right) dsdb = \frac{9}{128} \approx 0.070.$$

Remark 6.4 Compare the payoffs ensured by optimal behavior and honest negotiations (when players announce true reservation prices). Evidently, in the equilibrium the probability of transaction $P\{B(b) > S(s)\} = 0.281$ turns out appreciably smaller than in honest negotiations ($P\{b \geq s\} = 0.5$). Furthermore, the corresponding mean payoff of 0.070 is also considerably lower than in the case of truth-telling: $\int_0^1 ds \int_0^b \left(\frac{b+s}{2} - s\right) db = 1/12 \approx 0.0833$.

In this sense, the transaction problem is equivalent to prisoners' dilemma, where a good solution becomes unstable. The equilibrium solution yields slightly smaller payoffs to the players than their truth-telling.

6.5.2 Honest negotiations in conclusion of transactions

The notion of "honesty" plays a key role in the transaction problem. A transaction is called *honest*, if its equilibrium belongs to the class of pure strategies and the optimal strategies have the form $B(b) = b$ and $S(s) = s$. In other words, players benefit by announcing the reservation prices in honest transactions.

To make the game honest, we redefine it. There exist two approaches as follows.

The honest transaction model with a bonus.

Assume that, having concluded a transaction, players receive some bonus. Let $t_s(B, S)$ and $t_b(B, S)$ designate the seller's bonus and the buyer's bonus, respectively. Then the payoff functions in this game acquire the form

$$H_s(B, S) = E_{b,s} \left(\frac{S(s) + B(b)}{2} - s + t_s(B(b), S(s)) \right) I_{\{B(b) \geq S(s)\}}$$

and

$$H_b(B, S) = E_{b,s} \left(b - \frac{S(s) + B(b)}{2} + t_b(B(b), S(s)) \right) I_{\{B(b) \geq S(s)\}}.$$

It appears that, if the functions $t_s(B,S)$ and $t_b(B,S)$ are selected as $\frac{(B-S)^+}{2}$, the game becomes honest. Indeed, if

$$t_s(B,S) = t_b(B,S) = \frac{(B-S)^+}{2},$$

then, for an arbitrary strategy $B(b)$ of the buyer, the seller's payoff constitutes

$$\begin{aligned} H_s(B,S) &= E_b\left(\frac{S+B(b)}{2} - s + \frac{B(b)-S}{2}\right) I_{\{B(b)\ge S\}} \\ &= E_b\,(B(b)-s)\,I_{\{B(b)\ge S\}}. \end{aligned} \tag{5.8}$$

The integrand in (5.8) is non-negative, so long as $B(b) \ge S(s) \ge s$. As S goes down, the payoff (5.8) increases, since the domain in the integral grows. Hence, for a given s, the maximum of (5.8) corresponds to the minimal value of $S(s)$. Consequently, $S(s) = s$.

Similarly, we can argue that, for an arbitrary strategy of the seller, the buyer's optimal strategy acquires the form $B(b) = b$.

The honest transaction model with a penalty.

A shortcoming of the previous model concerns the following. Honest negotiations require that somebody pays the players for concluding a transaction. Moreover, players may act in collusion to receive the maximal bonus from the third party. For instance, they can announce extreme values and share the bonus equally. Another approach to honest negotiations dictates that players pay for participation in the transaction.

Denote by $q_s(B,S)$ ($q_b(B,S)$) the residual payoff of the seller (buyer, respectively) after transaction; of course, these quantities make sense if $B(b) \ge S(s)$. And the payoffs of the players are defined by

$$H_s(B,S) = E_{b,s}\left(\frac{S(s)+B(b)}{2} - s\right) q_s(B(b),S(s)) I_{\{B(b)\ge S(s)\}}$$

and

$$H_b(B,S) = E_{b,s}\left(b - \frac{S(s)+B(b)}{2}\right) q_b(B(b),S(s)) I_{\{B(b)\ge S(s)\}}.$$

Choose the functions q_s, q_b as

$$q_s = (B(b)-S(s))c_s, \quad q_b = (B(b)-S(s))c_b,$$

where c_s, c_b stand for positive constants. Such choice of penalties has the following grounds. It stimulates players to increase the difference between their offers and compel their truth-telling. Now, we establish this fact rigorously. For convenience, let $c_s = c_b = 1$.

Suppose that the buyer's strategy represents some non-decreasing function $B(b)$. Then the seller's payoff acquires the form

$$H_s(B,S) = E_b\left(\frac{S+B(b)}{2} - s\right)(B(b)-S)I_{\{B(b)\geq S\}}$$
$$= \int_{B^{-1}(S)}^{1}\left(\frac{B^2(b)-S^2}{2} - s(B(b)-S)\right)db \tag{5.9}$$

The function (5.9) decreases with respect to S. Really,

$$\frac{\partial H_s}{\partial S} = \int_{B^{-1}(S)}^{1}(-S+s)db \leq 0.$$

It follows that the maximal value of the seller's payoff is attained under the minimal admissible value of $S(s)$. So long as $S(s) \geq s$, the optimal strategy is honest: $S(s) = s$. By analogy, one can show that the above choice of the payoff functions brings to the honest optimal strategy of the buyer: $B(b) = b$.

6.5.3 Transactions with unequal forces of players

In the transaction model studied above, players are "in the same box." This fails in a series of applications. Assume that, if players reach an agreement, the transaction is concluded at the price of

$$kS(s) + (1-k)B(b).$$

Here $k \in (0,1)$ indicates a parameter characterizing "the alignment of forces" for sellers and buyers. In the symmetrical case, we have $k = 1/2$. If $k = 0$ ($k = 1$), the interests of the buyer (seller, respectively) are considered only.

Accordingly, a game with incomplete information arises naturally, where the payoff functions are defined by

$$H_s(B,S) = E_{b,s}\left(kS(s) + (1-k)B(b) - s\right)I_{\{B(b)\geq S(s)\}}$$

and

$$H_b(B,S) = E_{b,s}\left(b - kS(s) - (1-k)B(b)\right)I_{\{B(b)\geq S(s)\}}.$$

Similar reasoning yields the following result.

Theorem 6.12 *Consider the transaction problem with a force level k. The optimal strategies of the players have the form*

$$B(b) = \begin{cases} b & \text{if } b \leq \frac{1-k}{2}, \\ \frac{1}{1+k}b + \frac{(1-k)k}{2(1+k)} & \text{if } \frac{1-k}{2} \leq b \leq 1, \end{cases}$$

$$S(s) = \begin{cases} \frac{1}{2-k}s + \frac{1-k}{2} & \text{if } 0 \leq s \leq \frac{2-k}{2}, \\ s & \text{if } \frac{2-k}{2} \leq s \leq 1. \end{cases}$$

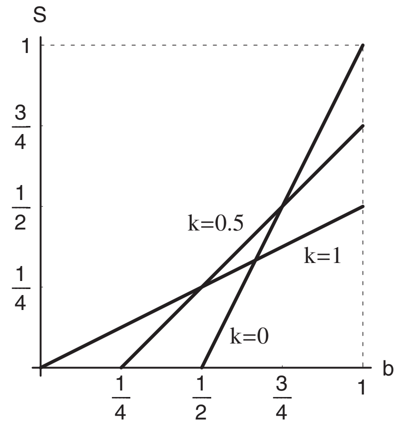


Figure 6.17 The transaction domain.

And the domain of successful negotiations $B(b) \geq S(s)$ (the transaction domain) is given by

$$b \geq \frac{1+k}{2-k}s + \frac{1-k}{2}.$$

This domain varies depending on k. In the case of $k = 0$, we obtain $b \geq s + 1/4$; if $k = 1$, then $b \geq 2s$ (see Figure 6.17). Recall that, in the symmetrical case, the probability of transaction equals $9/32 \approx 0.281$ (which is higher than the corresponding probability under $k = 0$ or $k = 1$, i.e., $1/4 = 0.25$).

6.5.4 The "offer-counteroffer" transaction model

Transactions with non-equal forces of players make payoffs essentially dependent on the force level k (see Section 6.5.3). Moreover, if $k = 0$ or $k = 1$, the payoffs of players *de facto* depend on the behavior of one player (known as a strong player). This player moves by announcing a certain price. If the latter exceeds the reservation price of another player, the transaction occurs. Otherwise, the second side makes its offer. The described model of transactions is called a sealed-bid auction (see Perry [1986]).

Suppose that the reservation prices of sellers and buyers have a non-uniform distribution on the interval $[0, 1]$ with distribution functions $F(s)$ and $G(b)$, respectively, where $b \in [0, 1]$. The corresponding density functions are $f(s), g(b), s, b \in [0, 1]$.

Imagine that the first offer is made by the seller. Another case, when the buyer moves first, can be treated by analogy. Under a reservation price s, he may submit an offer $S(s) \geq s$. A random buyer purchases this product with the probability of $1 - G(S)$. Therefore, the seller's payoff becomes

$$H_s(S) = (S - s)(1 - G(S)).$$

The maximum of this function follows from the equation

$$1 - G(S) - g(S)(S - s) = 0.$$

For instance, in the case of the uniform distribution of buyers, the optimal strategy of the seller satisfies the equation

$$1 - S - (S - s) = 0,$$

whence it appears that

$$S = \frac{1 + s}{2}.$$

Figure 6.17 demonstrates the transaction domain $b \geq S(s)$ which corresponds to $k = 1$. The probability of transaction equals $1/4 = 0.25$. The seller's payoff makes up

$$H_s = \int_{1/2}^{1} \int_{0}^{2b-1} \left(\frac{1 + s}{2} - s \right) dsdb = \frac{1}{12},$$

whereas the buyer's payoff constitutes

$$H_b = \int_{1/2}^{1} \int_{0}^{2b-1} \left(b - \frac{1 + s}{2} \right) dsdb = \frac{1}{24}.$$

In the mean, the payoff of the buyers is two times smaller than that of the sellers.

Now, analyze the non-uniform distribution of reservation prices on the market. For instance, suppose that the density function possesses the form $g(b) = 2(1 - b)$. This agrees with the case when many buyers value the product at a sufficiently low price. The optimal strategy of the seller meets the equation

$$1 - (2S - S^2) - 2(1 - S)(S - s) = 0.$$

Therefore, $S = (1 + 2s)/3$. In comparison with the uniform case, the seller should reduce the announced price.

6.5.5 The correlation effect

Up to this point, we have discussed the case of independent random variables representing the reservation prices of sellers and buyers. On a real market, reservation prices may be interdependent. In this context, it seems important to discover the impact of reservation prices correlation on the optimal strategies and payoffs of the players. Here we consider the case when the reservation prices (b, s) have the joint density function

$$f(b, s) = 1 + \gamma(1 - 2b)(1 - 2s), b, s \in [0, 1].$$

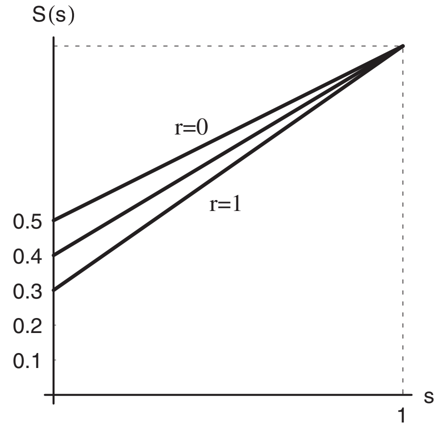


Figure 6.18 The optimal strategies for different values of γ.

The marginal distribution with respect to each parameter is uniform, and the correlation factor equals $\gamma/3$.

Assume that the seller makes the first offer. Under a reservation price s, the seller can submit an offer $S(s) \geq s$. And the seller's payoff becomes

$$H_s(S) = \int_S^1 (S-s)(1+\gamma(1-2s)(1-2b))db$$
$$= (S-s)(1-S)(1+\gamma(2s-1)S).$$

The maximum of this function lies at the point

$$S = \frac{-1+\gamma(1+s)(2s-1)+\sqrt{1+\gamma^2(2s-1)^2(1-s+s^2)+\gamma(1+s)(2s-1)}}{3\gamma(2s-1)}.$$

Figure 6.18 shows the strategy $S(s)$ for different values of γ. Evidently, as the correlation factor grows, the optimal behavior of the seller requires further reduction in his offers. To proceed, evaluate the payoffs of the players. The seller's payoff constitutes

$$H_s = \int_0^1 ds \int_{S(s)}^1 (S(s)-s)f(b,s)db = \int_0^1 (S(s)-s)(1-S(s))(1+\gamma(2s-1)S(s))ds,$$

whereas the buyer receives the payoff

$$H_b = \int_0^1 ds \int_{S(s)}^1 (b-S(s))f(b,s)db$$
$$= \frac{1}{6}\int_0^1 (1-S(s))^2\,(3-\gamma(1-2s)(1+2S(s)))\,ds.$$

The payoffs of sellers and buyers go down as the correlation of their reservation prices becomes stronger. This phenomenon admits an obvious explanation. Such tendency decreases the existing uncertainty in the price suggested by the partner. And so, a player must have moderate behavior.

6.5.6 Transactions with non-uniform distribution of reservation prices

Now, suppose that the reservation prices of sellers and buyers are distributed non-uniformly on the interval $[0, 1]$. For instance, let the reservation prices s and b represent independent random variables with the density functions

$$f(s) = 2s, s \in [0,1] \quad g(b) = 2(1-b), b \in [0,1]. \tag{5.10}$$

This corresponds to the following situation on a market. There are many sellers with high manufacturing costs of a product, and there are many buyers assessing the product at a low price.

Find the optimal strategies of the players. As usual, we believe that such strategies are some functions of the reservation prices, $S = S(s)$ and $B = B(b)$. The transaction occurs provided that $B \geq S$. If this is the case, assume that the transaction runs at the price of $(S(s) + B(b))/2$. The payoff functions of the players have the form (5.1) and (5.2); in the cited expressions, the expectation is evaluated with respect to appropriate distributions. To establish an equilibrium, involve the same considerations as in subsection 6.5.1.

Suppose that the buyer selects the strategy

$$B(b) = \begin{cases} b & \text{if } b \leq \frac{1}{6}, \\ \frac{4}{5}b + \frac{1}{30} & \text{if } \frac{1}{6} \leq b \leq 1. \end{cases} \tag{5.11}$$

Find the best response of the seller under different values of the parameter s.

Let $s \geq 1/6$. Then the transaction occurs under the condition $B(b) \geq S$, or

$$\frac{4}{5}b + \frac{1}{30} \geq S.$$

The last inequality is equivalent to

$$b \geq \frac{5}{4}S - \frac{1}{24}\ ,$$

where b designates a random variable with the distribution $g(b), b \in [0,1]$. Calculate the payoff of the seller:

$$\begin{aligned} H_s(B,S) &= E_b\left(\frac{S + B(b)}{2} - s\right) I_{\{B(b) \geq S\}} = \int_{\frac{5}{4}S - \frac{1}{24}}^{1} \left(\frac{\frac{4}{5}b + \frac{1}{30} + S}{2} - s\right) 2(1-b)db \\ &= -\frac{25}{12^4}(-5 + 36s - 30S)(5 - 6S)^2. \end{aligned} \tag{5.12}$$

The derivative of this function acquires the form

$$\frac{\partial H_s}{\partial S} = \frac{25}{1152}(5 + 24s - 30S)(5 - 6S).$$

It appears that the maximum of the payoff (5.12) is achieved at

$$S = \frac{4}{5}s + \frac{1}{6}.$$

If $s > 5/6$, the value of $S(s)$ becomes smaller than s. Therefore, in the case of $s \geq 1/4$, the seller's best response to the strategy (5.11) is defined by

$$S = \max\left\{\frac{4}{5}s + \frac{1}{6}, s\right\}. \tag{5.13}$$

Using the same technique as before, one can demonstrate optimality of this strategy in the case of $s < 1/6$, either.

Now, suppose that the seller adopts the strategy (5.13). Evaluate the buyer's best response under different values of the parameter b.

Let $b \leq 5/6$. The transaction occurs provided that $\frac{4}{5}s + \frac{1}{6} \leq B$, which is equivalent to

$$s \leq \frac{5}{4}B - \frac{5}{24}.$$

Here s represents a random variable with the distribution function $f(s), s \in [0, 1]$. Find the buyer's payoff:

$$H_b(B, S) = E_s\left(b - \frac{S(s) + B}{2}\right) I_{\{B \geq S(s)\}} = \int_0^{\frac{5}{4}B - \frac{5}{24}} \left(b - \frac{\frac{4}{5}s + \frac{1}{6} + B}{2}\right) 2sds$$
$$= -\frac{25}{12^4}(1 - 36b + 30B)(1 - 6B)^2.$$

The derivative of this function takes the form

$$\frac{\partial H_b}{\partial B} = \frac{25}{1152}(1 + 24b - 30B)(-1 + 6B).$$

It follows that the maximal payoff is achieved at

$$B = \frac{4}{5}b + \frac{1}{30}.$$

If $b < 1/6$, the function $B(b)$ has values higher than b. Therefore, the best response of the buyer to the strategy (5.13) becomes

$$B = \min\left\{\frac{4}{5}b + \frac{1}{30}, b\right\}.$$

Actually, we have established the following fact.

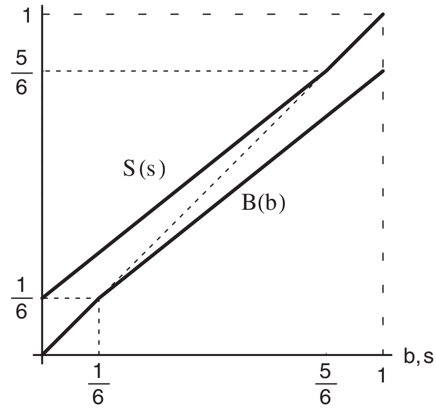


Figure 6.19 The optimal strategies of the players.

Theorem 6.13 *Consider the transaction problem with the reservation prices distribution (5.10). The optimal strategies of the players possess the form*

$$S = \max\left\{\frac{4}{5}s + \frac{1}{6}, s\right\}, \quad B = \min\left\{\frac{4}{5}b + \frac{1}{30}, b\right\}.$$

These optimal strategies are illustrated in Figure 6.19. In this situation, the transaction takes place if $B(b) \geq S(s)$, i.e.,

$$b \geq s + 1/6.$$

Figure 6.20 demonstrates the domain of successful negotiations.

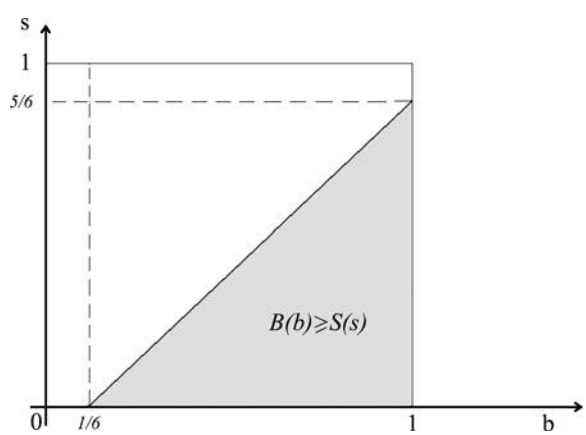


Figure 6.20 The transaction domain.

The optimal behavior results in the transaction with the probability

$$P\{B(b) > S(s)\} = \int_{\frac{1}{6}}^{1} \int_{0}^{b-\frac{1}{6}} 2s \cdot 2(1-b)ds \cdot db = \frac{1}{6}\left(\frac{5}{6}\right)^4 \approx 0.080.$$

This quantity is smaller than the probability of honest transaction $P\{b > s\} = \int_0^1 \int_0^b 2s \cdot 2(1-b)ds \cdot db = \frac{1}{6} \approx 0.166$. Moreover, the players receive the payoffs

$$H_s = H_b = \int_{\frac{1}{6}}^{1} \int_{0}^{b-\frac{1}{6}} \left(\frac{4/5b + 1/30 + 4/5s + 1/6}{2} - s\right) 2s \cdot 2(1-b)ds \cdot db = \frac{1}{36}\left(\frac{5}{6}\right)^4 \approx 0.0133,$$

being less than in the case of honest transaction:

$$\bar{H}_s = \bar{H}_b = \int_0^1 \int_0^b \left(\frac{b+s}{2} - s\right) 2s \cdot 2(1-b)ds \cdot db = \frac{1}{60} \approx 0.0166.$$

Interestingly, the honest game yields higher payoffs to the players, yet the corresponding strategy profile appears unstable. Similarly to prisoners' dilemma, in the honest game a player feels temptation to modify his strategy. We show this rigorously. For instance, the sellers adhere to truth-telling: $S(s) = s$. Obtain the optimal response of the buyers. The payoff of the buyer makes up

$$H_b(B, S) = \int_0^B \left(b - \frac{B+s}{2}\right) 2sds = B^2(b - 5/6B).$$

To define the optimal strategy, write down the derivative

$$\frac{\partial H_b}{\partial B} = B(2b - \frac{5}{2}B).$$

Hence, the optimal strategy of the buyer lies in $B(b) = 4/5b$. And the seller's payoff decreases twice

$$H_s(4/5b, s) = \int_0^1 \int_0^{4/5b} \left(\frac{\frac{4}{5}b + s}{2} - s\right) 2s \cdot 2(1-b)ds \cdot db = \frac{1}{3}\left(\frac{2}{5}\right)^4 \approx 0.008.$$

6.5.7 Transactions with non-linear strategies

We study another class of non-uniformly distributed reservation prices of buyers and sellers within the interval $[0, 1]$. Notably, consider linear distributions as above, but toggle their roles. In other words, suppose that the reservation prices s and b represent independent random variables with the density functions

$$f(s) = 2(1-s), s \in [0,1] \quad g(b) = 2b, b \in [0,1]. \tag{5.14}$$

This corresponds to the following situation on a market. There are many sellers supplying a product at a low prime cost and many rich buyers.

Find the optimal strategies of the players. As before we believe these are some functions of the reservation prices, $S = S(s)$ and $B = B(b)$, respectively (naturally enough, monotonically increasing functions). Then there exist the inverse functions $U = B^{-1}$ and $V = S^{-1}$, where $s = V(S)$ and $b = U(B)$.

Let us state optimality conditions for the distributions (5.14) of the reservation prices. The transaction occurs provided that $B \geq S$. If the transaction takes place, we assume that the corresponding price is $(S(s) + B(b))/2$. The payoff functions of the players have the form (5.1) and (5.2), where expectation engages appropriate distributions. It appears that an equilibrium is now achieved in the class of non-linear functions. For its evaluation, fix a buyer's strategy $B(b)$ and find the best response of the seller under different values of the parameter s.

The condition $B(b) \geq S$ is equivalent to $b \geq U(S)$. The seller's payoff equals

$$H_s(B, S) = E_b\left(\frac{S + B(b)}{2} - s\right) I_{\{B(b)\geq S\}} = \int_{U(S)}^{1}\left(\frac{B(b) + S}{2} - s\right) 2bdb. \tag{5.15}$$

Perform differentiation with respect to S in formula (5.15). The best response of the buyer meets the condition

$$\frac{\partial H_s}{\partial S} = -2(S - s)U(S)U'(S) + \frac{1 - U^2(S)}{2} = 0.$$

It yields the differential equation for the optimal strategies (i.e., the inverse functions) $U(B), V(S)$:

$$U'(S)(S - V(S))U(S) = \frac{1 - U^2(S)}{4} = 0. \tag{5.16}$$

By analogy, let $S(s)$ be a seller's strategy. We find the best response of the buyer under different values of the parameter b. Evaluate his payoff:

$$\begin{aligned} H_b(B, S) &= E_s\left(b - \frac{S(s) + B}{2}\right) I_{\{B\geq S(s)\}} = E_s\left(b - \frac{S(s) + B}{2}\right) I_{\{s\leq V(B)\}} \\ &= \int_0^{V(B)}\left(b - \frac{S(s) + B}{2}\right) 2(1 - s)ds. \end{aligned} \tag{5.17}$$

By differentiating (5.17) with respect to B, evaluate the best response of the buyer. Notably,

$$\frac{\partial H_b}{\partial B} = 2(b - B)V(B)V'(B) - \frac{V^2(B)}{2} = 0,$$

which gives the second differential equation for the optimal strategies $U(B), V(S)$:

$$V'(B)(U(B) - B)(1 - V(B)) = \frac{2V(B) - V^2(B)}{4}. \tag{5.18}$$

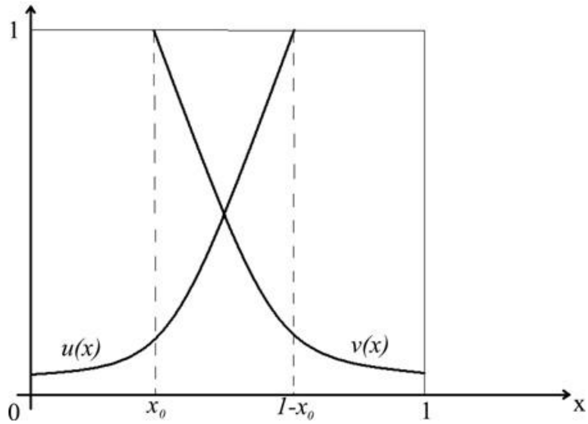


Figure 6.21 The curves of $u(x), v(x)$.

Introduce the change of variables $u(x) = U^2(x), v(x) = (1 - V(x))^2$ into (5.16) and (5.18) to derive the system of equations

$$u'(x)(x - 1 + \sqrt{v(x)}) = \frac{1 - u(x)}{2}, \quad v'(x)(x - \sqrt{u(x)}) = \frac{1 - v(x)}{2}. \tag{5.19}$$

Due to (5.19), the functions $u(x)$ and $v(x)$ are related by the expressions

$$u(x) = v(1 - x), v(x) = u(1 - x), u'(x) = -v'(1 - x), v'(x) = -u'(1 - x). \tag{5.20}$$

Rewrite the system (5.19) as

$$x - 1 + \sqrt{v(x)} = \frac{1 - u(x)}{2u'(x)}, \quad x - \sqrt{u(x)} = \frac{1 - v(x)}{2v'(x)}.$$

By taking into account the expressions (5.20), we arrive at the following equation in $v(x)$:

$$\left(\sqrt{v(x)} + \frac{1 - v(x)}{2v'(x)}\right) + \left(\sqrt{v(1 - x)} + \frac{1 - v((1 - x)}{2v'(1 - x)}\right) = 1. \tag{5.21}$$

Suppose that the function $v(x)$ decreases, i.e., $v'(x) < 0, x \in [0, 1]$. Formula (5.19) claims that the function $u(x)$ lies above the parabola x^2. By virtue of symmetry, $v(x)$ is above the parabola $1 - x^2$ and $u'(x) > 0$. Figure 6.21 demonstrates the curves of the functions $u(x)$ and $v(x)$; x_0 and $1 - x_0$ are the points where these functions equal 1.

We have $v(x_0) = 1$ at the point x_0. Then the second equation in (5.19) requires that $u(x_0) = x_0^2$ and, subsequently, $v(1 - x_0) = x_0^2$. By setting $x = x_0$ in (5.21), we obtain

$$1 + x_0 + \frac{1 - x_0^2}{2v'(1 - x_0)} = 1.$$

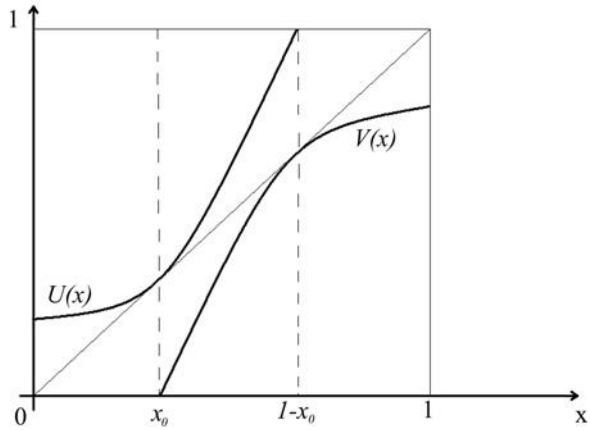


Figure 6.22 The curves of $U(x)$, $V(x)$.

And it follows that

$$v'(1-x_0) = -u'(x_0) = -\frac{1-x_0^2}{2x_0}. \tag{5.22}$$

Figure 6.22 shows the functions $U(x)$ and $V(x)$. Finally, it is possible to present the optimal strategies $B(b)$ and $S(s)$ (see Figure 6.23). The remaining uncertainty consists in the value of x_0. Actually, this is the marginal threshold for successful negotiations—the seller would not agree to a lower price, whereas the buyer would not suggest a higher price than $1 - x_0$.

Assume that the derivative $v'(x_0)$ is finite and non-zero. Apply L'Hospital's rule in (5.19) to get

$$v'(x_0) = \lim_{x \to x_0+0} \frac{1 - v(x)}{2(x - \sqrt{u(x)})} = \frac{-v'(x_0)}{2\left(1 - \frac{u'(x_0)}{2\sqrt{u(x_0)}}\right)}.$$

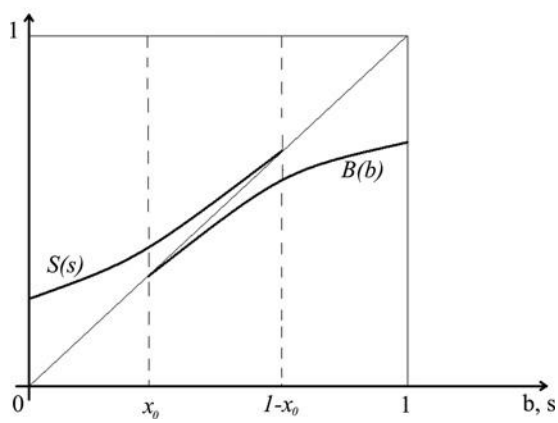


Figure 6.23 The optimal strategies.

And so,

$$1 = -\frac{\sqrt{u(x_0)}}{2\sqrt{u(x_0)} - u'(x_0)},$$

or

$$u'(x_0) = 3\sqrt{u(x_0)} = 3x_0.$$

In combination with (5.22), we obtain that

$$\frac{1 - x_0^2}{2x_0} = 3x_0.$$

Hence, $x_0 = 1/\sqrt{7} \approx 0.3779$.

Therefore, the following assertion has been argued.

Theorem 6.14 *Consider the transaction problem with the reservation prices distribution (5.14). The optimal strategies of the players possess the form*

$$S = V^{-1}(s), \quad B = U^{-1}(b),$$

where the functions $u = U^2, v = (1 - V)^2$ *satisfy the system of differential equations (5.19). The corresponding transaction takes place if the prices belong to the interval* $[1/\sqrt{7}, 1 - 1/\sqrt{7}] \approx [0.3779, 0.6221]$.

Figure 6.24 illustrates the domain of successful negotiations. It has a curved boundary.

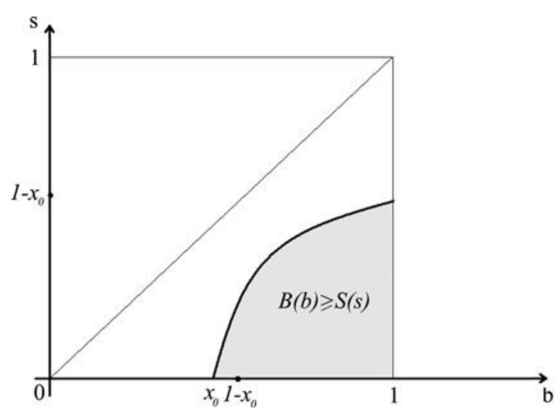


Figure 6.24 The transaction domain.

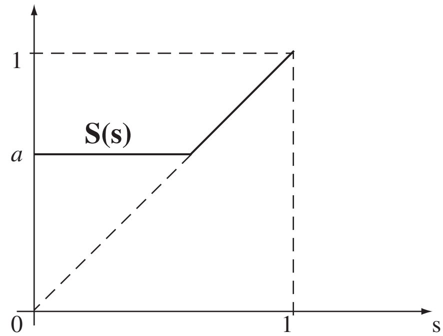


Figure 6.25 The seller's strategy.

6.5.8 Transactions with fixed prices

As before, we focus on the transaction models with non-uniformly distributed reservation prices. Assume that the reservation prices of the sellers and buyers (s and b) represent independent random variables. Denote the corresponding distribution functions and density functions by $F(s), f(s), s \in [0,1]$ and $G(b), g(b), b \in [0,1]$.

Suppose that the seller adopts the threshold strategy (see Figure 6.25)

$$S(s) = \begin{cases} a & \text{if } s \leq a, \\ s & \text{if } a \leq s \leq 1. \end{cases}$$

For small reservation prices, the seller quotes a fixed price a; only if s exceeds a, he announces the actual price s.

Find the best response of the buyer under different values of the parameter b. Note that the transaction occurs only if the buyer's reservation price b appears not less than a. In the case of $b \geq a$, the transaction may take place provided that $B \geq S(s)$.

Evaluate the buyer's payoff:

$$\begin{aligned} H_b(B,S) &= E_s\left(b - \frac{S(s)+B}{2}\right) I_{\{B \geq S(s)\}} \\ &= \int_0^a \left(b - \frac{a+B}{2}\right) f(s)ds + \int_a^B \left(b - \frac{s+B}{2}\right) f(s)ds. \end{aligned}$$

The derivative of this function acquires the form

$$\frac{\partial H_b}{\partial B} = (b-B)f(B) - \frac{F(B)}{2}. \tag{5.23}$$

If the expression $(1-B)f(B) - \frac{F(B)}{2}$ turns out non-positive within the interval $B \in [a,1]$, the derivative (5.23) takes non-positive values, either. Consequently, the maximal payoff is achieved under $B(b) = a, b \in [a,1]$.

We naturally arrive at

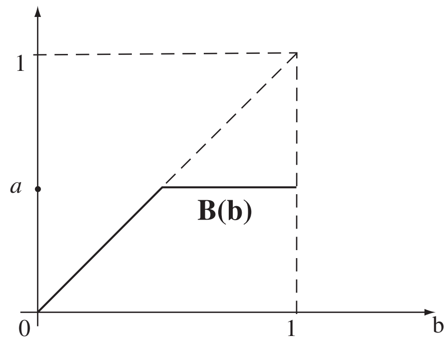


Figure 6.26 The buyer's strategy.

Lemma 6.2 *Let the seller's strategy be defined by* $S(s) = \max\{a, s\}$. *If the condition*

$$(1 - x)f(x) - \frac{F(x)}{2} \leq 0$$

holds true for all $x \in [a, 1]$, *then the best response of the buyer lies in the strategy* $B(b) = \min\{a, b\}$.

Similar reasoning applies to the buyer. Imagine that the latter selects the strategy $B(b) = \min\{a, b\}$ *(see Figure 6.26). In other words, he establishes a fixed price* a *for high values of* b *and prefers truth-telling for small ones* $b \leq a$.

The corresponding payoff of the seller makes up

$$H_s(B, S) = E_b \left(\frac{S + B(b)}{2} - s \right) I_{\{B(b) \geq S)\}}$$
$$= \int_S^a \left(\frac{b + S}{2} - s \right) g(b)db + \int_a^1 \left(\frac{a + S}{2} - s \right) g(b)\, db. \qquad (5.24)$$

Again, we perform differentiation:

$$\frac{\partial H_s}{\partial S} = (s - S)g(S) + \frac{1 - G(S)}{2}.$$

If for all $x \in [0, a]$:

$$-xg(x) + \frac{1 - G(x)}{2} \geq 0,$$

the derivative (5.24) is non-negative.

Lemma 6.3 *Let the buyer's strategy be given by* $B(b) = \min\{a, b\}$. *If the condition*

$$xg(x) - \frac{1 - G(x)}{2} \leq 0$$

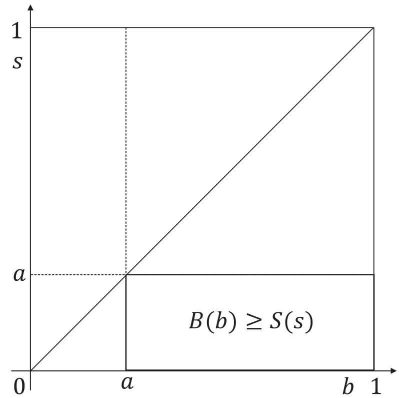


Figure 6.27 The transaction domain.

holds true for all $x \in [0, a]$*, then the best response of the seller consists in the strategy* $S(s) = \max\{a, s\}$.

Lemmas 6.2 and 6.3 lead to

Theorem 6.15 *Assume that the following inequalities are valid for some* $a \in [0, 1]$*:*

$$(1 - x)f(x) - \frac{F(x)}{2} \leq 0, \quad x \in [a, 1];$$
$$xg(x) - \frac{1 - G(x)}{2} \leq 0, \quad x \in [0, a].$$

Then the optimal strategies in the transaction problem have the form

$$S(s) = \max\{a, s\}, \quad B(b) = \min\{a, b\}.$$

The domain of successful negotiations is demonstrated by Figure 6.27.

Therefore, the transaction always runs at a fixed price a under the conditions of Theorem 6.15. Obviously, this is the case for the distributions

$$F(s) = 1 - (1 - s)^n, \quad G(b) = b^n, n \geq 3,$$

with $a = 1/2$. And the equilibrium becomes

$$S(s) = \max\left\{\frac{1}{2}, s\right\}, \quad B(b) = \min\left\{\frac{1}{2}, b\right\}.$$

The expected payoffs of the players constitute

$$H_b = H_s = \int_{\frac{1}{2}}^{1} nb^{n-1}db \int_{0}^{\frac{1}{2}} (1/2 - s)n(1 - s)^{n-1}ds = \frac{(2^n - 1)\,(2^n(n - 1) + 1)}{2(n + 1)4^n}.$$

As n grows, they converge to $1/2$ (for $n = 3$, we have $H_b = H_s = 0.232$).

Interestingly, the conditions of Theorem 6.15 fail when the reservation prices possess the uniform distribution ($n = 1$) and the linear distribution ($n = 2$). To find an equilibrium, introduce two-threshold strategies.

6.5.9 Equilibrium among *n*-threshold strategies

First, suppose that the seller chooses a strategy with two thresholds σ_1, σ_2 as follows:

$$S(s) = \begin{cases} a_1 & \text{if } 0 \le s < \sigma_1, \\ a_2 & \text{if } \sigma_1 < s \le \sigma_2, \\ s & \text{if } \sigma_2 \le s \le 1. \end{cases}$$

Here $a_1 \le a_2$ and $\sigma_2 = a_2$. For small reservation prices, the seller quotes a fixed price a_1; for medium reservation prices, he sets a fixed price a_2. And finally, if s exceeds a_2, the seller prefers truth-telling—announces the actual price s.

Find the best response of the buyer under different values of the parameter b. Note an important aspect. The transaction occurs only if the buyer's reservation price b is not less than a_1. In the case of $b \ge a_1$, the transaction may take place provided that $B \ge S(s)$.

To proceed, compute the buyer's payoff, whose reservation price equals b. For $B : a_1 \le B < a_2$, the payoff is defined by

$$H_b(B, S) = E_s\left(b - \frac{S(s) + B}{2}\right) I_{\{B \ge S(s)\}} = \int_0^{\sigma_1} \left(b - \frac{a_1 + B}{2}\right) f(s)ds. \tag{5.25}$$

If $B : a_2 \le B \le b$, we accordingly obtain

$$H_b(B, S) = \int_0^{\sigma_1} \left(b - \frac{a_1 + B}{2}\right) f(s)ds + \int_{\sigma_1}^{\sigma_2} \left(b - \frac{a_2 + B}{2}\right) f(s)ds$$
$$+ \int_{\sigma_2}^{B} \left(b - \frac{s + B}{2}\right) f(s)\, ds. \tag{5.26}$$

Recall that the transaction fails under $b < a_1$. Therefore, $B(b)$ may have arbitrary values. Assume that $b : a_1 \le b < a_2$. Then the relationship $H_b(B, S)$ acquires the form (5.25) (since $B \le b$) and represents a decreasing function of B. The maximal value of (5.25) is attained at $B = a_1$. And the corresponding payoff becomes

$$H_b(a_1, S) = (b - a_1)F(\sigma_1). \tag{5.27}$$

In the case of $b \ge a_2$, the payoff $H_b(B, S)$ may have the form (5.26), either. Its derivative is determined by

$$\frac{\partial H_b}{\partial B} = -\frac{1}{2}F(B) + (b - B)f(B). \tag{5.28}$$

Suppose that the inequality

$$(1-x)f(x) - \frac{F(x)}{2} \leq 0$$

holds true on the interval $[a_2, 1]$. Then the expression (5.28) appears non-positive for all $B \in [a_2, 1]$. Hence, the function $H_b(B, S)$ does not increase in B. Its maximal value at the point $B = a_2$ equals

$$H_b(a_2, S) = \left(b - \frac{a_1 + a_2}{2}\right) F(\sigma_1) + (b - a_2)(F(\sigma_2) - F(\sigma_1)). \tag{5.29}$$

For $b = a_2$, the expression (5.29) takes the value of $\frac{a_2 - a_1}{2} F(\sigma_1)$, which is two times smaller than the payoff (5.27). Thus, the buyer's best response to the strategy $S(s)$ lies in the strategy

$$B(b) = \begin{cases} b & \text{if } 0 \leq b \leq a_1, \\ a_1 & \text{if } a_1 \leq b < \beta_2, \\ a_2 & \text{if } \beta_2 \leq b \leq 1. \end{cases}$$

Here β_2 follows from equality of (5.27) and (5.29):

$$(b - a_1)F(\sigma_1) = \left(b - \frac{a_1 + a_2}{2}\right) F(\sigma_1) + (b - a_2)(F(\sigma_2) - F(\sigma_1)).$$

Readers can easily verify that

$$\beta_2 = a_2 + \frac{(a_2 - a_1)F(\sigma_1)}{2(F(\sigma_2) - F(\sigma_1))}. \tag{5.30}$$

Now, suppose that the buyer employs the strategy

$$B(b) = \begin{cases} b & \text{if } 0 \leq b \leq \beta_1, \\ a_1 & \text{if } \beta_1 \leq b < \beta_2, \\ a_2 & \text{if } \beta_2 \leq b \leq 1, \end{cases}$$

where $\beta_1 = a_1 \leq a_2 \leq \beta_2$. What is the best response of the seller having the reservation price s? Skipping the intermediate arguments (they are almost the same as in the case of the buyer), we provide the ultimate result. Under the condition

$$xg(x) - \frac{1 - G(x)}{2} \leq 0,$$

the best strategy of the seller on the interval $[0, a_1]$ acquires the form

$$S(s) = \begin{cases} a_1 & \text{if } 0 \leq s < \sigma_1, \\ a_2 & \text{if } \sigma_1 < s \leq \sigma_2, \\ s & \text{if } \sigma_2 \leq s \leq 1, \end{cases}$$

where

$$\sigma_1 = a_1 - \frac{(a_2 - a_1)(1 - G(\beta_2))}{2(G(\beta_2) - G(\beta_1))}. \tag{5.31}$$

Furthermore, $\sigma_1 \leq a_1 \leq a_2 = \sigma_2$.

Theorem 6.16 *Assume that, for some constants $a_1, a_2 \in [0, 1]$ such that $a_1 \leq a_2$, the following inequalities are valid:*

$$(1 - x)f(x) - \frac{F(x)}{2} \leq 0, \qquad x \in [a_2, 1];$$
$$xg(x) - \frac{1 - G(x)}{2} \leq 0, \qquad x \in [0, a_1].$$

Then the players engaged in the transaction problem have the optimal strategies

$$S(s) = \begin{cases} a_1 & \text{if } 0 \leq s < \sigma_1, \\ a_2 & \text{if } \sigma_1 < s \leq \sigma_2, \\ s & \text{if } \sigma_2 \leq s \leq 1, \end{cases}$$

$$B(b) = \begin{cases} b & \text{if } 0 \leq b \leq \beta_1, \\ a_1 & \text{if } \beta_1 \leq b < \beta_2, \\ a_2 & \text{if } \beta_2 \leq b \leq 1. \end{cases}$$

In the previous formulas, the quantities σ_1 and β_2 meet the expressions (5.30) and (5.31). In addition,

$$\sigma_1 \leq \beta_1 = a_1 \leq a_2 = \sigma_2 \leq \beta_2.$$

Let us generalize this scheme to the case of n-threshold strategies. Suppose that the seller chooses a strategy with n thresholds $\sigma_i, i = 1, \ldots, n$:

$$S(s) = \begin{cases} a_i & \text{if } \sigma_{i-1} \leq s < \sigma_i, \qquad i = 1, \ldots, n \\ s & \text{if } \sigma_n \leq s \leq 1, \end{cases}$$

where $\{a_i\}, i = 1, \ldots, n$ and $\{\sigma_i\}, i = 1, \ldots, n$ form a non-decreasing sequence such that $\sigma_i \leq a_i, i = 1, \ldots, n$. For convenience, we believe that $\sigma_0 = 0$.

Therefore, all sellers are divided into $n + 1$ groups depending on the values of their reservation prices. If the reservation price s belongs to group i i.e., $s \in [\sigma_{i-1}, \sigma_i)$, the seller announces the price $a_i, i = 1, \dots, n$. If the reservation price appears sufficiently high ($s \geq a_n$), the seller quotes the actual price s.

Find the best response of the buyer under different values of the parameter b. Note that the transaction occurs only if the buyer's reservation price b is not less than a_1. In the case of $b \geq a_1$, the transaction takes place provided that $B \geq S(s)$.

Evaluate the payoff of the buyer whose reservation price equals b. For $a_1 \leq B < a_2$, the payoff is defined by

$$\begin{aligned} H_b(B, S) &= E_s\left(b - \frac{S(s) + B}{2}\right) I_{\{B \geq S(s)\}} \\ &= \int_0^{\sigma_1} \left(b - \frac{a_1 + B}{2}\right) f(s)ds = \left(b - \frac{a_1 + B}{2}\right) F(\sigma_1). \end{aligned} \tag{5.32}$$

Next, if $a_{i-1} \leq B < a_i$, we obtain

$$\begin{aligned} H_b(B, S) &= \sum_{j=1}^{i-1} \int_{\sigma_{j-1}}^{\sigma_j} \left(b - \frac{a_j + B}{2}\right) f(s)ds \\ &= \sum_{j=1}^{i-1} \left(b - \frac{a_j + B}{2}\right) \left(F(\sigma_j) - F(\sigma_{j-1})\right), \quad i = 1, \dots, n. \end{aligned} \tag{5.33}$$

And finally, for $a_n \leq B \leq b$, the payoff makes up

$$\begin{aligned} H_b(B, S) &= \sum_{j=1}^{n} \int_{\sigma_{j-1}}^{\sigma_j} \left(b - \frac{a_j + B}{2}\right) f(s)ds + \int_{\sigma_n}^{B} \left(b - \frac{a_j + B}{2}\right) f(s)ds \\ &= \sum_{j=1}^{n} \left(b - \frac{a_j + B}{2}\right) \left(F(\sigma_j) - F(\sigma_{j-1})\right) + \int_{\sigma_n}^{B} \left(b - \frac{a_j + B}{2}\right) f(s)\, ds. \end{aligned} \tag{5.34}$$

If $b < a_1$, the transaction fails; hence, $B(b)$ may possess arbitrary values. Set $\beta_1 = a_1$.

Suppose that $a_1 \leq b < a_2$. So far as $B \leq b < a_2$, the function $H_b(B, S)$ has the form (5.32) and decreases in B. The maximal payoff is attained under $B = a_1$, i.e.,

$$H_b(a_1, S) = (b - a_1)F(\sigma_1).$$

This function increases in b; in the point $b = a_2$, its value becomes

$$(a_2 - a_1)F(\sigma_1). \tag{5.35}$$

Now, assume that $a_2 \leq b < a_3$. If $B < a_2$, then H_b acquires the form (5.32). However, $B(b)$ is greater or equal to a_2. In this case, formula (5.33) yields

$$H_b(B, S) = \left(b - \frac{a_2 + B}{2}\right) F(\sigma_2) + \frac{a_2 - a_1}{2} F(\sigma_1).$$

The above function is maximized in B at the point $B = a_2$:

$$H_b(a_2, S) = (b - a_2)F(\sigma_2) + \frac{1}{2}(a_2 - a_1)F(\sigma_1).$$

Interestingly, the payoff in the point $b = a_2$ is two times smaller than the one gained by the strategy $B = a_1$ (see (5.35)). And so, switching from the strategy $B = a_1$ to the strategy $B = a_2$ occurs in the point $b = \beta_2 \geq a_2$, where β_2 satisfies the equation

$$(b - a_1)F(\sigma_1) = (b - a_2)F(\sigma_2) + \frac{1}{2}(a_2 - a_1)F(\sigma_1).$$

It follows immediately that

$$\beta_2 = a_2 + \frac{(a_2 - a_1)F(\sigma_1)}{2(F(\sigma_2) - F(\sigma_1))}.$$

Let a_3 be chosen such that $a_3 \geq \beta_2$.

Further reasoning employs induction. Suppose that the inequality $H_b(a_{k-1}, S) \leq H_b(a_k, S)$ holds true for some k and any $b \in (\beta_{k-1}, a_k]$; here $a_{k-1} \leq \beta_{k-1}$, $k = 1, \dots, n$.

Consider the interval $a_k \leq b < a_{k+1}$. Due to $B \leq b < a_{k+1}$, the function $H_b(B, S)$ has the form (5.33), where $i \leq k + 1$. Moreover, this is a decreasing function of B. The maximal value of (5.33) exists for $B = a_{i-1}$. And the corresponding payoff constitutes

$$H_b(a_{i-1}, S) = \sum_{j=1}^{i-1} \left(b - \frac{a_j + a_{i-1}}{2} \right) \left(F(\sigma_j) - F(\sigma_{j-1}) \right)$$
$$= (b - a_{i-1})F(\sigma_{i-1}) + \frac{1}{2} \sum_{j=1}^{i-2} (a_{j+1} - a_j)F(\sigma_j).$$

Note that, if $i = k + 1$, i.e., $B \in [a_k, a_{k+1})$, the maximal payoff equals

$$H_b(a_k, S) = (b - a_k)F(\sigma_k) + \frac{1}{2} \sum_{j=1}^{k-1} (a_{j+1} - a_j)F(\sigma_j). \tag{5.37}$$

In the case of $i = k$, the maximal value is

$$H_b(a_{k-1}, S) = (b - a_{k-1})F(\sigma_{k-1}) + \frac{1}{2} \sum_{j=1}^{k-2} (a_{j+1} - a_j)F(\sigma_j). \tag{5.38}$$

At the point $b = a_k$, the expression (5.38) possesses, at least, the same value as the expression (5.37). By equating (5.37) and (5.38), we find the value of β_k, which corresponds to switching from the strategy $B = a_k$ to the strategy $B = a_{k+1}$:

$$\beta_k = a_k + \frac{(a_k - a_{k-1})F(\sigma_{k-1})}{2(F(\sigma_k) - F(\sigma_{k-1}))}, \qquad k = 1, \dots, n. \tag{5.39}$$

The value β_k lies within the interval $[a_k, a_{k+1})$ under the following condition:

$$a_k + \frac{(a_k - a_{k-1})F(\sigma_{k-1})}{2(F(\sigma_k) - F(\sigma_{k-1}))} \leq a_{k+1}, \qquad k = 1, \dots, n.$$

If $b \geq a_n$, the payoff $H_b(B, S)$ can also acquire the form (5.34). The derivative of this function is given by

$$\frac{\partial H_b}{\partial B} = -\frac{1}{2}F(B) + (b - B)f(B).$$

Assume that the inequality

$$(1 - x)f(x) - \frac{F(x)}{2} \leq 0, \quad x \in [a_2, 1]$$

takes place on the interval $[a_n, 1]$. Subsequently, the derivative turns out non-positive for all $B \in [a_n, 1]$, i.e., the function $H_b(B, S)$ does not increase in B. Its maximal value in the point $B = a_n$ makes up

$$H_b(a_n, S) = (b - a_n)F(\sigma_n) + \frac{1}{2}\sum_{j=1}^{n-1}(a_{j+1} - a_j)F(\sigma_j).$$

Switching from the strategy $B = a_{n-1}$ to the strategy $B = a_n$ occurs under $b = \beta_n$, where the quantity β_n solves the equation

$$(b - a_n)F(\sigma_n) + \frac{1}{2}\sum_{j=1}^{n-1}(a_{j+1} - a_j)F(\sigma_j)$$
$$= (b - a_{n-1})F(\sigma_{n-1}) + \frac{1}{2}\sum_{j=1}^{n-2}(a_{j+1} - a_j)F(\sigma_j).$$

Uncomplicated manipulations bring to the formula

$$\beta_n = a_n + \frac{(a_n - a_{n-1})F(\sigma_{n-1})}{2(F(\sigma_n) - F(\sigma_{n-1}))}.$$

Therefore, we have demonstrated that the best response of the buyer to the strategy $S(s)$ consists in the strategy

$$B(b) = \begin{cases} b & \text{if } 0 \leq b \leq \beta_1 = a_1, \\ a_i & \text{if } \beta_i \leq b < \beta_{i+1}, \qquad i = 1, \dots, n \end{cases} \tag{5.40}$$

where $\beta_k, k = 1, \dots, n$ is determined by (5.39), $\beta_{n+1} = 1$.

Similar arguments serve for calculating the seller's optimal response to a threshold strategy adopted by the buyer. By supposing that the buyer chooses the strategy (40), one can show that the seller's optimal response is

$$S(s) = \begin{cases} a_i & \text{if } \sigma_{i-1} \leq s < \sigma_i, \qquad i = 1, \dots, n \\ s & \text{if } \sigma_n \leq s \leq 1. \end{cases}$$

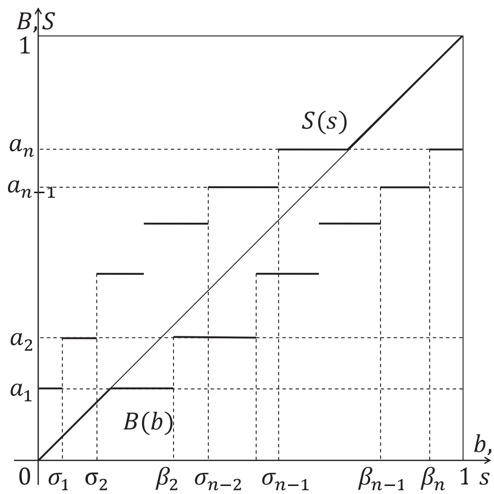


Figure 6.28 The equilibrium with n thresholds.

Here $\sigma_k, k = 1, \dots, n$ obey the formulas

$$\sigma_k = a_k - \frac{(a_{k+1} - a_k)(1 - G(\beta_{k+1}))}{2(G(\beta_{k+1}) - G(\beta_k))}, \qquad k = 1, \dots, n. \tag{5.41}$$

and, in addition, $\sigma_n = a_n$.

Thus, the following assertion remains in force in the general case with n thresholds.

Theorem 6.17 *Let a non-decreasing sequence $\{a_i\}, i = 1, \dots, n$ on the interval $[0, 1]$ meet the conditions*

$$xg(x) - \frac{1 - G(x)}{2} \leq 0, \quad x \in [0, a_1],$$
$$(1 - x)f(x) - \frac{F(x)}{2} \leq 0, \quad x \in [a_n, 1],$$

and

$$\beta_{k-1} \leq a_k \leq \sigma_{k+1}, \qquad k = 1, \dots, n.$$

Then the optimal strategies in the transaction problem have the form (see Figure 6.28)

$$S(s) = \begin{cases} a_i & \text{if } \sigma_{i-1} \leq s < \sigma_i, \qquad i = 1, \dots, n; \\ s & \text{if } \sigma_n \leq s \leq 1, \end{cases}$$

$$B(b) = \begin{cases} b & \text{if } 0 \leq b \leq \beta_1 = a_1, \\ a_i & \text{if } \beta_i \leq b < \beta_{i+1}, \qquad i = 1, \dots, n. \end{cases}$$

Here the quantities $\{\sigma_i\}$ and $\{\beta_i\}$ are defined by (5.39) and (5.41). The transaction domain is illustrated in Figure 6.29.

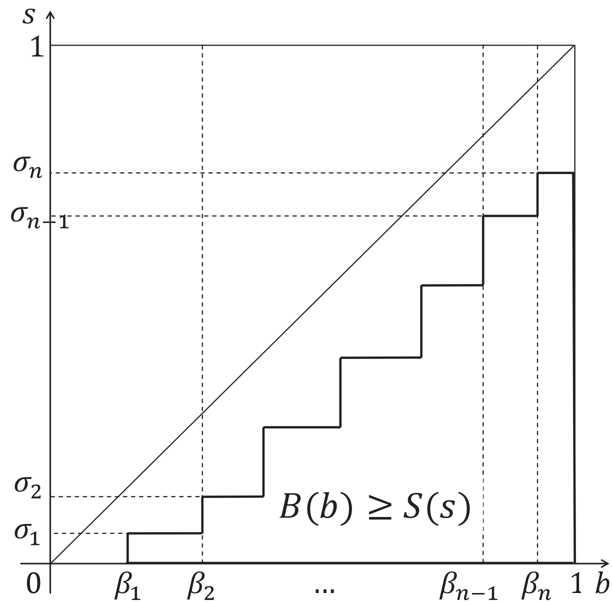


Figure 6.29 The transaction domain with n thresholds.

The uniform distribution. Two-threshold strategies. Assume that the reservation prices of the sellers and buyers have the uniform distribution on the market, i.e., $F(s) = s, s \in [0, 1]$ and $G(b) = b, b \in [0, 1]$. Let the players apply two-threshold strategies.

The conditions of Theorem 6.16 hold true for $a_1 \leq 1/3$ and $a_2 \geq 2/3$. It follows from (5.30)–(5.31) that

$$\sigma_1 = a_1 - \frac{(a_2 - a_1)(1 - \beta_2)}{2(\beta_2 - a_1)}, \quad \beta_2 = a_2 + \frac{(a_2 - a_1)\sigma_1}{2(a_2 - \sigma_1)}.$$

Under $a_1 = 1/3$ and $a_2 = 2/3$, we obtain

$$\sigma_1 = \frac{1}{12}(7 - \sqrt{17}) \approx 0.239, \quad \beta_2 = 1 - \sigma_1 = \frac{1}{12}(5 + \sqrt{17}) \approx 0.761.$$

Note that the equilibrium exists for arbitrary values of the thresholds a_1, a_2 such that $a_1 \leq 1/3$ and $a_2 \geq 2/3$. Therefore, it seems important to establish a_1, a_2 that maximize the total payoff of the sellers and buyers. This problem was posed by R. Myerson [1984]. The appropriate set of the parameters a_1, a_2 is called efficient.

The total payoff of the sellers and buyers selecting two-threshold strategies takes the form

$$H_b(B, S) + H_s(B, S) = E_{b,s}(b - s)I_{\{B(b)\geq S(s)\}}$$
$$= \int_{\beta_1}^{\beta_2} db \int_0^{\sigma_1} (b - s)ds + \int_{\beta_2}^{1} db \int_0^{\sigma_2} (b - s)ds.$$

It is possible to show that the maximal total payoff corresponds to $a_1 = 1/3, a_2 = 2/3$ and equals $H_b(B, S) + H_s(B, S) = (23 - \sqrt{17})/144 \approx 0.1311$.

Recall that the class of one-threshold strategies admits no equilibrium in the case of the uniform distribution. However, the continuum of equilibria exists in the class of two-threshold strategies.

The uniform distribution. Strategies with *n* thresholds.

Consider the uniform distribution of the reservation prices in the class of n-threshold strategies, where $n \geq 4$. For the optimal thresholds of the buyer and seller, the conditions (5.39) and (5.41) become

$$\beta_k = a_k + \frac{(a_k - a_{k-1})\sigma_{k-1}}{2(\sigma_k - \sigma_{k-1})}, \qquad k = 1, \dots, n,$$

$$\sigma_k = a_k - \frac{(a_{k+1} - a_k)(1 - \beta_{k+1})}{2(\beta_{k+1} - \beta_k)}, \qquad k = 1, \dots, n,$$

where we set $\sigma_0 = 0$ and $\beta_{n+1} = 1$.

As $n \to \infty$, these threshold strategies converge uniformly to the continuous strategies found in Theorem 6.13. Furthermore, the total payoff (the probability of transaction) tends to $9/64$ ($9/32$, respectively).

6.5.10 Two-stage transactions with arbitrator

Consider another design of negotiations between a seller and buyer involving an arbitrator. It was pioneered by D.M. Kilgour [1994]. Again, suppose that the reservation prices of the sellers and buyers (the quantities s and b, respectively) represent independent random variables on the interval $[0, 1]$. Their distribution functions and density functions are defined by $F(s)$, $G(b)$ and $f(s), s \in [0, 1]$, $g(b), b \in [0, 1]$, respectively.

The transaction comprises two stages. At stage 1, the seller and buyer inform the arbitrator of their reservation prices. Note that both players may report some prices $\bar{s}$ and $\bar{b}$, differing from their actual values s and b. If $\bar{b} < \bar{s}$, the arbitrator announces transaction fail. In the case of $\bar{s} \leq \bar{b}$, he announces that the transaction is possible; and the players continue to the next stage. At stage 2, the players make their offers for the transaction, S and B. Imagine that both transactions S, B enter the interval $[\bar{s}, \bar{b}]$; then the transaction occurs at the mean price $(S + B)/2$. If just one offer hits the interval $[\bar{s}, \bar{b}]$, the transaction runs at the corresponding price, but with the probability of $1/2$. And the transaction takes no place in the rest situations.

Therefore, the strategies of the seller and the buyer lie in pairs $(\bar{s}(s), S(s))$ and $(\bar{b}(b), B(b))$, respectively. These are some functions of the reservation prices. Naturally, the functions $(\bar{s}(s), S(s))$ and $(\bar{b}(b), B(b))$ do not decrease, $S \geq \bar{s}$ and $B \leq \bar{b}$.

For further analysis, it seems convenient to modify the rules as follows. One of players is chosen equiprobably at shot 2. For instance, assume that we have randomly selected the seller. If $S \leq \bar{b}$, the transaction occurs at the price S. Otherwise ($S > \bar{b}$), the transaction fails. Similar rules apply to the buyer. In other words, if $B \geq \bar{s}$, the transaction runs at the price B.

Such modifications do not affect negotiations and represent an equivalent transformation. Really, if the offers B and S belong to the interval $[\bar{s}, \bar{b}]$, the second design of negotiations requires that the seller or buyer is chosen equiprobably. Hence, the seller's payoff makes up

$$\frac{1}{2}(S-s)+\frac{1}{2}(B-s)=\frac{S+B}{2}-s,$$

whereas the buyer's payoff becomes

$$\frac{1}{2}(b-B)+\frac{1}{2}(b-S)=b-\frac{S+B}{2}.$$

Both quantities coincide with their counterparts in the original design of negotiations. And so, we proceed from the modified rules of shot 2.

Theorem 6.18 *Any seller's strategy $(\bar{s}, S)$ is dominated by the honest strategy (s, S) and any buyer's strategy $(\bar{b}, B)$ is dominated by the honest strategy (b, B).*

Proof: Let us demonstrate the first part of the theorem (the second one is argued by analogy). Assume that the buyer adopts the strategy $(\bar{b}(b), B(b))$. Find the best response of the seller, whose reservation price constitutes s, $s \geq \bar{s}$. Either the buyer or seller is chosen equiprobably. In the former case, the transaction occurs at the price B, if $B(b) \geq \bar{s}$ (equivalently, $b \geq B^{-1}(\bar{s})$). In the latter case, the transaction runs at the price S, if $S \leq \bar{b}(b)$ (equivalently, $b \geq \bar{b}^{-1}(S)$). Note that the inverse functions $B^{-1}(\bar{s}), \bar{b}^{-1}(S)$ exist owing to the monotonous property of the functions $B(b), \bar{b}(b)$.

Therefore, the expected payoff of the seller acquires the form

$$H_s(\bar{s}, S)=\frac{1}{2}\int_{B^{-1}(\bar{s})}^{1}(B(b)-s)dG(b)+\frac{1}{2}\int_{\bar{b}^{-1}(S)}^{1}(S-s)dG(b). \tag{5.42}$$

Evaluate the seller's best response in $\bar{s}$. The second summand in (5.42) appears independent from $\bar{s}$. Perform differentiation of (5.42) with respect to $\bar{s}$:

$$\frac{\partial H_s}{\partial \bar{s}}=-\frac{1}{2}(\bar{s}-s)g(B^{-1}(\bar{s}))\frac{dB^{-1}(\bar{s})}{d\bar{s}}. \tag{5.43}$$

Clearly, the expression (5.43) is non-negative (non-positive) under $\bar{s} < s$ (under $\bar{s} > s$, respectively). Thus, the function H_s gets maximized by $\bar{s} = s$ for any S. We have shown that the players should select truth-telling at stage 1 (i.e., report the actual reservation prices to the arbitrator).

Now, the payoff (5.42) can be rewritten as

$$\begin{aligned} H_s(s, S) &= \frac{1}{2}\int_{B^{-1}(s)}^{1}(B(b)-s)dG(b)+\frac{1}{2}\int_{S}^{1}(S-s)dG(b) \\ &= \frac{1}{2}\int_{B^{-1}(s)}^{1}(B(b)-s)dG(b)+\frac{1}{2}(S-s)(1-G(S)). \end{aligned} \tag{5.44}$$

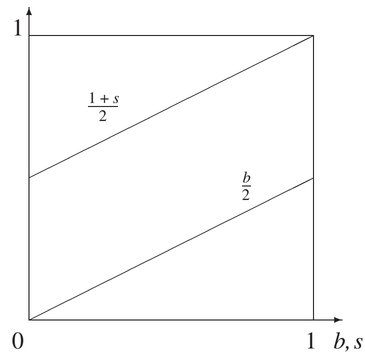


Figure 6.30 The optimal strategies of the players.

Find the seller's best response in S. The first expression in (5.44) does not depend on S. Consequently,

$$\frac{\partial H_s}{\partial S} = \frac{1}{2}(1 - G(S)) - \frac{1}{2}(S - s)g(S) = 0. \tag{5.45}$$

The optimal strategy S is defined implicitly by equation (5.45). There exists a solution to equation (5.45), since the expression (5.45) is non-negative (non-positive) under $S = s$ (under $S = 1$, respectively).

Theorem 6.19 *The equilibrium strategies* (S^*, B^*) *follow from the system of equations*

$$1 - G(S) = (S - s)g(S), \quad F(B) = (b - B)f(B). \tag{5.46}$$

For the uniform distributions $F(s) = s$ and $G(b) = b$, the optimal strategies of the seller and buyer are

$$S^* = \frac{s+1}{2}, B^* = \frac{b}{2}.$$

See Figure 6.30 for the curves of the optimal strategies. The seller's offer being chosen, the transaction takes place if $\frac{s+1}{2} \leq b$. In the case of buyer's offer selection, the transaction occurs provided that $\frac{b}{2} \geq s$. Figure 6.31 illustrates the transaction domain.

Finally, we compute the negotiators' payoffs in this equilibrium:

$$H_s^* = H_b^* = \frac{1}{2}\int_0^1 ds \int_{\frac{s+1}{2}}^1 \left(\frac{s+1}{2} - s\right) db + \frac{1}{2}\int_0^1 db \int_0^{\frac{b}{2}} \left(\frac{b}{2} - s\right) ds \approx 0.062.$$

In comparison with the one-stage negotiations and the uniform distribution of the reservation prices, we observe reduction of the transaction value.

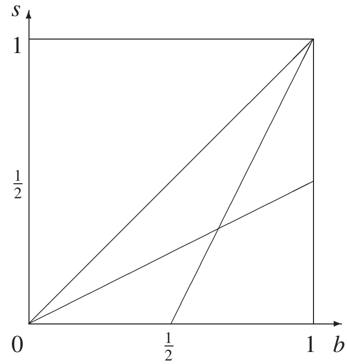


Figure 6.31 The transaction domain.

For the square distributions $F(s) = s^2$ and $G(b) = 1 - (1 - b)^2$, the optimal strategies of the seller and buyer make up

$$S^* = \frac{2s + 1}{3}, B^* = \frac{2b}{3}.$$

6.6 Reputation in negotiations

An important aspect of negotiations consists in the reputation of players. This characteristic forms depending on the behavior of negotiators during decision making. Therefore, at each stage of negotiations, players should maximize their payoff at this stage, but also think of their reputation (it predetermines their future payoffs).

In the sequel, we provide some possible approaches to formalization of the concept of reputation in negotiations.

6.6.1 The notion of consensus in negotiations

Let $N = \{1, 2, \dots, n\}$ be a set of players participating in negotiations to solve some problem. Each player possesses a specific opinion on the correct solution of this problem. Denote by $x = (x_1, x_2, \dots, x_n)$ the set of opinions of all players, $x_i \in S, i = 1, \dots, n$, where $S \subset R^k$ indicates the admissible set in the solution space.

Designate by $x(0)$ the opinions of players at the initial instant. Subsequently, players meet to discuss the problem, share their opinions and (possibly) change their opinions. In the general case, such discussion can be described by the dynamic system

$$x(t + 1) = f_t(x(t)), t = 0, 1, \dots$$

If the above sequence admits the limit $x = \lim_{t\to\infty} x(t)$ such that all components of the vector x coincide, this value is called a **consensus** in negotiations. However, are there such dynamic systems?

The next subsection presents the matrix model of opinions dynamics, where a consensus does exist. This model was first proposed by M.H. de Groot [1974] and extended by many authors.

6.6.2 The matrix form of dynamics in the reputation model

Suppose that R is the solution space. The key role belongs to the so-called confidence matrix $A \in [0,1]^{n\times n}$, whose elements a_{ij} specify the confidence level of player i in player j. By assumption, the matrix A is stochastic, i.e., $a_{ij} \geq 0$ and $\sum_{j=1}^{n} a_{ij} = 1, \forall i$. In other words, the confidence of player i is distributed among all players (including player i).

After a successive stage of negotiations, the opinions of players coincide with the weighted opinion of all negotiators (taking into account their confidence levels):

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij} x_j(t), \forall i.$$

This formula has the matrix representation:

$$x(t+1) = Ax(t), t = 0, 1, \dots \quad x(0) = x_0. \tag{6.1}$$

Perform integration of (6.1) t times to obtain

$$x(t) = A^t x(0). \tag{6.2}$$

The main issue concerns the existence of the limit matrix $A^\infty = \lim_{t\to\infty} A^t$. The behavior of stochastic matrices has been intensively studied within the framework of the theory of Markov chains. By an appropriate renumbering of players, any stochastic matrix can be rewritten as

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 & 0 \\ 0 & A_2 & \dots & 0 & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & \dots & A_m & 0 \\ & & A_{m+1} & & \end{pmatrix}.$$

Here the matrices $A_i, i = 1, \dots, m$ appear stochastic and correspond to different classes of communicating states. All states from the class A_{m+1} are non-essential; zeros correspond to these states in the limit matrix.

In terms of the theory of reputation, this fact means the following. A player entering an appropriate class A_i appreciates the reputation of players belonging to this class only. Players from the class A_{m+1} have no influence during negotiations.

A consensus exists iff the Markov chain is non-periodic and there is exactly one class of communicating states ($m = 1$). Then there exists $\lim_{t\to\infty} A^t$ known as the limit matrix A^∞. It comprises identical rows $(a_1, a_2, \dots, a_n)$. Therefore,

$$\lim_{t\to\infty} A^t x(0) = A^\infty x(0) = x(\infty) = (x, x, \dots, x).$$

The quantity a_i is the influence level of player i.

Example 6.4 The reputation matrix takes the form

$$A = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}.$$

Player I equally trusts himself and player II. However, player II trusts himself three times higher than player I. Obviously, the influence levels of the players become $a_1 = 1/3$ and $a_2 = 2/3$.

Example 6.5 The reputation matrix takes the form

$$A = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/6 & 1/3 & 1/2 \\ 1/6 & 1/2 & 1/3 \end{pmatrix}.$$

Here player I equally trusts all players. Player II feels higher confidence in player III, whereas the latter more trusts player II. The influence levels of the players make up $a_1 = 1/5$, $a_2 = a_3 = 2/5$.

All rows of the matrix A^∞ are identical. And so, all components of the limit vector $x(\infty)$ do coincide, representing the consensus x. Interestingly,

$$x = \sum_{i=1}^{n} a_i x_i(0),$$

where a_i and $x_i(0)$ denote the influence level and initial opinion of player i.

In the context of negotiations, we have the following interpretation. As the result of lengthy negotiations, the players arrive at the final common opinion owing to their mutual confidence.

6.6.3 Information warfare

According to the stated concept, negotiations bring to some consensus x. A consensus may represent, e.g., budgetary funds allocation to a certain construction project, assignment of fishing quotas or settlement of territorial problems. The resulting solution depends on the reputation of negotiators and their opinions. And so, the final solution can be affected by modifying the initial opinion of some participant. Of course, such manipulation guarantees greater efficiency, if the participant possesses higher reputation. However, this inevitably incurs costs.

Therefore, we arrive at the following optimization problem. Allocate a given amount of financial resources c among negotiators to maximize some utility function

$$H(y_1, \dots, y_n) = F\left(\sum_{j=1}^{n} a_j(x_j(0) + k_j y_j)\right) - G(y_1, \dots, y_n), \quad \sum_{j=1}^{n} y_j \le c.$$

Here the first summand answers for the payoff gained by variations of the initial opinion of negotiator i (k_i takes positive or negative values). And the second summand specifies the corresponding costs.

Imagine that several negotiators strive to affect the final solution. A game-theoretic problem known as information warfare arises immediately. This game engages m players operating certain amounts c_i of financial resources ($i = 1, \dots, m$). They allocate the above amounts among negotiators in order to maximize their own payoffs.

Definition 6.3 *Information warfare is the game*

$$\Gamma =< M, \{Y^i\}_{i=1}^m, \{H_i\}_{i=1}^m >,$$

where $M = \{1, 2, \dots, m\}$ *denotes the set of players,* $Y^i \subset R^n$ *indicates the strategy set of player i, representing the simplex* $\sum_{j=1}^n y_j^i \leq c_i,\ y_j^i \geq 0, j = 1, \dots, n$, *and the payoff of player i takes the form*

$$H_i(y^1, \dots, y^m) = F_i\left(\sum_{j=1}^n a_j(x_j(0) + \sum_{l=1}^m k_j^l y_j^l)\right) - G_i(y^1, \dots, y^m),\ \ \sum_{j=1}^n y_j^i \leq c_i,$$

$i = 1, \dots, m.$

6.6.4 The influence of reputation in arbitration committee. Conventional arbitration

The described influence approach can be adopted in negotiation models involving arbitration committees. Consider the conventional arbitration model in a salary conflict of two sides, the Labor Union (L) and the Manager (M). They address an arbitration court to resolve the conflict.

Suppose that the arbitrators have some initial opinions of the appropriate salary; designate these quantities by $x_i(0), i = 1, \dots, n$. In addition, the arbitrators enjoy certain reputation and corresponding influence levels $a_i, i = 1, \dots, n$. Their influence levels play an important role during conflict arrangement. Both sides of the conflict (players) submit their offers to the arbitration committee. The arbitrators meet to discuss the offers and make the final decision.

In the course of discussions, an arbitrator may correct his opinion according to the reputation model above. After lengthy negotiations, the arbitrators reach the consensus $x = \sum_{i=1}^n a_i x_i(0)$, which resolves the conflict.

Assume that player M has some amount c_M of financial resources to influence the original opinion of the arbitrators. This is an optimization problem defined by

$$H_M(y_1, \dots, y_n) = \sum_{j=1}^n a_j(x_j(0) - k_j y_j) + \sum_{j=1}^n y_j \to \min$$

subject to the constraints

$$\sum_{j=1}^n y_j \leq c_M,\ \ y_j \geq 0, j = 1, \dots, n.$$

Here the quantities $k_j, j = 1, \dots, n$ are non-negative. Interestingly, the initial opinions do not depend on y. Hence, the posed problem appears equivalent to the optimization problem

$$H_M(y_1, \dots, y_n) = \sum_{j=1}^n (a_j k_j - 1) y_j \to \max$$

subject to the constraints

$$\sum_{j=1}^{n} y_j \le c_M,\ y_j \ge 0, j = 1, \dots, n.$$

Its solution seems obvious. Consider only arbitrators j such that $a_j k_j > 1$ and choose the one with the maximal value of $a_j k_j$. Subsequently, invest all financial resources c_M in this arbitrator.

Now, suppose that player 2 (the Trade Union) disposes of some amount c_L of financial resources; it can be allocated to the arbitrators to “tilt the balance” in Trade Union’s favor. We believe that an arbitrator supports the player that has offered a greater amount of financial resources to him. Such statement leads to a two-player game with the payoff functions

$$H_M(y^M, y^L) = \sum_{j=1}^{n} (a_j k_j - 1) y_j^M I\{y_j^M > y_j^L\},$$

$$H_L(y^M, y^L) = \sum_{j=1}^{n} (a_j k_j - 1) y_j^L I\{y_j^L > y_j^M\}.$$

The strategies of both players meet the constraints

$$\sum_{j=1}^{n} y_j^M \le c_M,\ \sum_{j=1}^{n} y_j^L \le c_L,\ y_j^M, y_j^L \ge 0, j = 1, \dots, n.$$

This is a modification of the well-known Colonel Blotto game. Generally, this game is considered under $n = 2$. Both players allocate some resource between two objects; the winner becomes the player having allocated the greatest amount of the resource on a given position.

In our case, readers can easily obtain the following result. If there are two arbitrators and $c_M = c_L$, in the equilibrium each player should allocate his resource to arbitrator 1 with the probability of $a_1 k_1/(a_1 k_1 + a_2 k_2)$ and to arbitrator 2 with the probability of $a_2 k_2/(a_1 k_1 + a_2 k_2)$.

6.6.5 The influence of reputation in arbitration committee. Final-offer arbitration

Now, we analyze the reputation model in the final-offer arbitration procedure with several arbitrators. Their opinions obey some probability distributions.

Assume that arbitrators have certain reputations being significant in conflict settlement. The sides of a conflict submit their offers to an arbitration committee. The arbitrators meet to discuss the offers and make the final decision.

In the course of discussions, an arbitrator may correct his opinion according to the reputation model above. After lengthy negotiations, the arbitrators reach the consensus described by the common probability distribution. For instance, let the committee include n arbitrators whose opinions are expressed by the distribution functions $F_1, \dots, F_n$. Then the consensus is expressed by the distribution function $F_a = a_1 F_1 + \cdots + a_n F_n$, where a_i means the influence level of arbitrator i in the committee. This quantity depends on his reputation.

Clearly, all theorems from subsection 6.2.6 are applicable to this case, and F_a acts as the distribution function of one arbitrator. As an example, study the salary conflict involving an arbitration committee with two members. The opinions of arbitrators 1 and 2 are defined by the Gaussian distribution functions $N(1, 1)$ and $N(2, 1)$, respectively.

Take the reputation matrix from Example 1 and suppose that the influence levels of the players make up $a_1 = 1/3$ and $a_2 = 2/3$.

As the result of negotiations, the arbitrators reach a common opinion expressed by the common distribution $1/3N(1, 1) + 2/3N(2, 1)$. Following Theorem 2.11, find the median of the common distribution ($m_F \approx 1.679$) and the optimal strategies of players I and II:

$$x^* = m_F + \frac{1}{2f_a(m_F)} \approx 3.075, \ \ y^* = m_F - \frac{1}{2f_a(m_F)} \approx 0.283.$$

6.6.6 The influence of reputation on tournament results

We explore the impact of reputation on the optimal behavior in the tournament problem. Consider the tournament model in the form of a two-player zero-sum game. Projects are characterized by two parameters. As a matter of fact, this problem has been solved in Section 6.4. Player I strives for maximizing the sum $x + y$, whereas his opponent (player II) seeks to minimize it.

Assume that two invited arbitrators choose the winner. Their reputation is expressed by a certain matrix A. For simplicity, we focus on the symmetrical case—the opinions of the arbitrators are modeled by the two-dimensional Gaussian distributions $f_1(x, y) = \frac{1}{2\pi} \exp\{-((x + c)^2 + (y - c)^2)/2\}$ and $f_2(x, y) = \frac{1}{2\pi} \exp\{-((x - c)^2 + (y + c)^2)/2\}$, respectively. Here c stands for model's parameter. Recall that lengthy negotiations of the arbitrators yield the final distribution

$$f_a(x, y) = a_1 f_1(x, y) + a_2 f_2(x, y),$$

where a_1 and a_2 specify the influence levels of the arbitrators, $a_2 = 1 - a_1$.

Repeat the line of reasoning used in Section 6.4. The players submit their offers (x_1, y_1) and (x_2, y_2). The solution plane is divided into two sets S_1 and S_2. Their boundary represents the line passing through the bisecting point of the segment which connects the points (x_1, y_1) and (x_2, y_2). This line has the following equation:

$$y = -\frac{x_1 - x_2}{y_1 - y_2} x + \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2}{2(y_1 - y_2)}.$$

Thus, the payoff of player I takes the form

$$\begin{aligned} H(x_1, y_1; x_2, y_2) &= (x_1 + y_1)\mu(S_1) \\ &= (x_1 + y_1) \int_R \int_R f_a(x, y) I \left\{ y \geq -\frac{x_1 - x_2}{y_1 - y_2} x + \frac{(x_1^2 - x_2^2 + y_1^2 - y_2^2)}{2(y_1 - y_2)} \right\} dxdy. \end{aligned}$$

Fix the strategy (x_2, y_2) of player II; find the best response of his opponent from the conditions

$$\frac{\partial H}{\partial x_1} = 0, \frac{\partial H}{\partial y_1} = 0.$$

First, we obtain the derivatives:

$$\frac{\partial H}{\partial x_1} = \mu(S_1) + (x_1 + y_1)\frac{\partial \mu(S_1)}{\partial x_1}$$

$$= \mu(S_1) + (x_1 + y_1)\int_R \frac{x - x_1}{y_1 - y_2} f_a\left(x, -\frac{x_1 - x_2}{y_1 - y_2}x + \frac{(x_1^2 - x_2^2 + y_1^2 - y_2^2)}{2(y_1 - y_2)}\right)dx, \quad (6.3)$$

$$\frac{\partial H}{\partial y_1} = \mu(S_1) + (x_1 + y_1)\frac{\partial \mu(S_1)}{\partial y_1} = \mu(S_1)$$

$$+(x_1 + y_1)\int_R \left(-\frac{x_1 - x_2}{(y_1 - y_2)^2}x + \frac{x_1^2 - x_2^2}{2(y_1 - y_2)^2} - \frac{1}{2}\right) f_a\left(x, -\frac{x_1 - x_2}{y_1 - y_2}x + \frac{(x_1^2 - x_2^2 + y_1^2 - y_2^2)}{2(y_1 - y_2)}\right)dx. \quad (6.4)$$

Set the functions (6.3) and (6.4) equal to zero. Require that the solution is achieved at the point $x_1 = -y_2, y_1 = -x_2$. This follows from problem's symmetry with respect to the line $y = -x$. Note that, in this case, $\mu(S_1) = 1/2$. Consequently, we arrive at the system of equations

$$\frac{1}{2} + \int_R (x + y_2) f_a(x, -x)dx = 0,$$
$$\frac{1}{2} + \int_R (x_2 - x) f_a(x, -x)dx = 0.$$

The first equation

$$\int_{-\infty}^{\infty} (y_2 + x)\left(a_1 e^{-(x+c)^2} + a_2 e^{-(x-c)^2}\right)dx = -\pi$$

gives the optimal value of y_2:

$$y_2 = -\sqrt{\pi} + c(a_1 - a_2).$$

Similarly, the second equation yields

$$x_2 = -\sqrt{\pi} + c(a_2 - a_1).$$

And the optimal offer of player I makes $x_1 = \sqrt{\pi} + c(a_2 - a_1), y_1 = \sqrt{\pi} + c(a_1 - a_2)$.

Therefore, the optimal strategies of players in this game depend on the reputation of arbitrators. If the latter enjoy identical reputation, the equilibrium coincides with the above-mentioned one (both components of the offer are same). If the reputation of the players differs, the components of the offer are shifted to the arbitrator having a higher weight.

Exercises

1. Cake cutting.

Suppose that three players cut a cake using the scheme of random offers with the Dirichlet distribution, where $k_1 = 2, k_2 = 1$, and $k_3 = 1$. Negotiations have the maximal duration of $K = 3$ shots. Find the optimal strategies of the players when the final decision is made by (a) the majority of votes, (b) the consent of all players.

2. Meeting schedule for two players.

Imagine that two players negotiate the day of their meeting on a week. Player *I* prefers a day closer to Tuesday, whereas player *II* would like to meet closer to Thursday. The strategy set takes the form $X = Y = 1, 2, \dots, 7$. To choose the day of their meeting, the players adopt the random offer scheme as follows. The probabilistic mechanism equiprobably generates offers from 1 to 7. The players agree or disagree. The number of shots is $k = 5$. Find the optimal strategies of the players.

3. Meeting schedule for two players: the case of an arbitration procedure.

Two players negotiate the day of their meeting on a week $(1, 2, \dots, 7)$. Player *I* prefers the beginning of the week, whereas player *II* would like to meet at the end of the week. To choose the day of their meeting, the players make offers; subsequently, a random arbitrator generates some number a (3 or 4 equiprobably) and compares the offers with this number. As the day of meeting, he chooses the offer closest to a. Find the optimal strategies of the players.

4. Meeting schedule for three or more players.

Players $1, 2, \dots, n$ negotiate the date of a conference. For simplicity, we consider the months of a year: $T = 1, 2, \dots, 12$. The players make their offers $a_1, \dots, a_n \in T$. For player i, the inconvenience of other months is assessed by

$$f_i(t) = \min\{|t - a_i|, |t + 12 - a_i|\},\ t \in T,\ i = 1, \dots, n\,.$$

Using the random offer scheme, find the optimal behavioral strategies of the players on the horizon $K = 5$.

Study the case of three players and the majority rule. Analyze the case of four players and compare the payoffs under the thresholds of 2 and 3.

5. Equilibrium in the transaction model.

Consider the following transaction model. The reservation prices of the sellers have the uniform distribution on the market, whereas the reservation prices of the buyers possess the linear form $g(b) = 2b, b \in [0, 1]$. Evaluate a Bayesian equilibrium.

6. Networked auction.

The seller exhibits some product in an information network. The buyers do not know the quoted price c and bid for the product, gradually raising their offers. They act simultaneously, increasing their offers either by the same quantity or by a specific quantity $\alpha > 1$ (i.e., the players may have different values of α). The player who first announces a price higher or equal to c receives the product. If there are several such players, the seller prefers the highest offer. Each player wants to purchase the product. Find the optimal strategies of the players.

7. Traffic quotas.

Users $1, \ldots, n$ order from a provider the monthly traffic sizes of $x_1, \ldots, x_n$. Assume that $\sum_{i=1}^{n} x_i$ exceeds the channel's capacity c. Evaluate the quotas using the random offer scheme.

8. The 2D arbitration procedure.

Consider the tournament model in the form of a two-player zero-sum game in the 2D space. Player I strives for maximizing the sum $x + y$, whereas player II seeks to minimize it. The arbitrator is defined by the density function of the Cauchy distribution:

$$f(x, y) = \frac{1}{\pi^2(1 + x^2)(1 + y^2)}.$$

Find the optimal strategies of the players.

9. Tournament of construction projects.

Two companies seek to receive an order for house construction. Their offers represent a couple of numbers (t, c), where t indicates the period of construction and c specifies the costs of construction. The customer is interested in the reduction of the construction costs and period, whereas the companies have opposite goals. A random arbitrator is distributed in a unit circle with the density function $f(r, \theta) = \frac{3(1-r)}{\Pi}$ (in polar coordinates). Evaluate the optimal offers of the players under the following condition. Player I (player II) strives to maximize the construction period (the construction costs, respectively).

10. Reputation models.

Suppose that the reputation matrix in three-player negotiations has the form

$$A = \begin{pmatrix} 1/4 & 1/4 & 1/2 \\ 2/3 & 1/6 & 1/6 \\ 1/6 & 1/6 & 2/3 \end{pmatrix}.$$

Find the influence levels of the players.

11. The salary problem.

Consider the salary problem with final-offer arbitration. The arbitration committee consists of three arbitrators; the reputation matrix is given in exercise 10. The opinions of the arbitrators obey the Gaussian distributions $N(100, 10), N(120, 10)$, and $N(130, 10)$. Evaluate the optimal strategies of the players.

12. The filibuster problem.

A group of 10 filibusters captures a ship with 100 bars of gold. They have to divide the plunder. The allocation procedure is as follows. First, the captain suggests a possible allocation, and the players vote. If, at least, the half of the crew supports his offer, the allocation takes place. Otherwise, filibusters kill the captain, and the second player makes his offer. The procedure repeats until the players reach a consensus. Find the subgame-perfect equilibrium in this game.

7

Optimal stopping games

Introduction

In this chapter, we consider games with the following specifics. As their strategies, players choose stopping times for some observation processes. This class of problems is close to the negotiation models discussed in Chapter 6. Players independently observe the values of some random process; at any time moment, they can terminate such observations, saving the current value observed. Subsequently, the values of players are compared and their payoffs are calculated. Similar problems arise in choice models (the best object in a group of objects), behavioral models of agents on a stock exchange, dynamic games, etc.

Our analysis begins with a two-player game, where players *I* and *II* sequentially observe the values of some independent random processes x_n and y_n ($n = 0, 1, \dots, N$). At any moment, players can stop (the time moments τ and σ, respectively) with the current values x_τ and y_σ. Then these values are compared, and the winner is the player who has selected the greatest value. Therefore, the payoff function in the above non-cooperative game acquires the form

$$H(\tau, \sigma) = P\{x_\tau > y_\sigma\} - P\{x_\tau < y_\sigma\}.$$

Since observations represent random variables, the same applies to the strategies of the players. They take random integer values from the set $\{0, 1, \dots, N\}$. The decision on stopping time n must be made only using the observed values $x_1, \dots, x_n$, $n = 0, 1, \dots, N$. In other words, the events $\{\tau \leq n\}$ must be measurable in a non-decreasing sequence of σ-algebras $\mathcal{F}_n = \sigma\{x_1, \dots, x_n\}, n = 0, 1, \dots, N$.

To solve this class of games, we describe the general scheme of equilibrium construction via the backward induction method. Imagine that the strategy of a player (e.g., player *II*)

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

is fixed. Then the maximization problem $\sup_\tau H(\tau, \sigma)$ gets reduced to the optimal stopping problem for one player with a certain payoff function under stopping: $\sup_\tau Ef(x_\tau)$, where

$$f(x) = P\{y_\sigma < x\} - P\{y_\sigma > x\}.$$

In order to find

$$\sup_\tau Ef(x_\tau),$$

we employ the backward induction method. Denote by $v(x, n) = \sup_\tau E\{f(x_\tau)/x_n = x\}$ the optimal expected payoff of a player in the state when all $n-1$ observations are missed and the current observation at shot n equals x. In this state, the player obtains the payoff $f(x_n) = f(x)$ by terminating the observations. If he continues observations and follows the optimal strategy further, his expected payoff constitutes $E\{v(x_{n+1}, n+1)/x_n = x\}$. By comparing these quantities, one can suggest the optimal strategy. And the optimality equation takes the recurrent form

$$v(x, n) = \max\{f(x), E\{v(x_{n+1}, n+1)/x_n = x\}\}, n = 1, \ldots, N-1, \ v(x, N) = f(x).$$

For instance, if $\{x_1, x_2, \ldots, x_N\}$ is a sequence of independent identically distributed random variables with some distribution function $G(x)$, the expected payoff under continuation makes up

$$Ev(x_{n+1}, n+1) = \int_R v(x, n+1)dG(x).$$

If this is a Markov process on a finite set $E = \{1, 2, \ldots, k\}$ with a certain transition matrix $p_{ij}(i, j = 1, \ldots, k)$, the expected payoff under continuation becomes

$$E\{v(x_{n+1}, n+1)/x_n = x\}\} = \sum_{y=1}^{k} v(y, n+1)p_{xy}.$$

First, we establish an equilibrium for a simple game using standard techniques from game theory. Second, we construct a solution in a wider class of games by the backward induction method.

7.1 Optimal stopping game: The case of two observations

This game consists of two shots. At Shot 1, players I and II are offered the values of some random variables, x_1 and y_1, respectively. They can select or reject these values. In the latter case, both players are offered new values, x_2 to player I and y_2 to player II. The game ends, the players show their values to each other. The winner is the player with the highest value. A player possesses no information on the opponent's behavior. All random variables appear independent. For convenience, we believe they have the uniform distribution on the unit interval [0, 1].

It seems comfortable to define the strategies of the players by thresholds u and v ($0 \le u, v \le 1$) such that, if $x_1 \ge u$ ($y_1 \ge v$), player I (player II) stops on x_1 (on y_1, respectively). Otherwise, player I (player II) chooses the second value x_2 (y_2, respectively). Therefore, for given strategies the selected observations have the form

$$x_\tau = \begin{cases} x_1, & \text{if } x_1 \ge u, \\ x_2, & \text{if } x_1 < u, \end{cases} \quad y_\sigma = \begin{cases} y_1, & \text{if } y_1 \ge v, \\ y_2, & \text{if } y_1 < v. \end{cases}$$

To find the payoff function in this game, we need the distribution function of the random variables x_τ and y_σ. For $x < u$, the event $\{x_\tau \le x\}$ occurs only if the first observation x_1 is smaller than u, whereas the second observation turns out less than x. For $x \ge u$, the event $\{x_\tau \le x\}$ happens either if the first observation x_1 enters the interval $[u, x]$, or the first observation is less than u, whereas the second observation is smaller or equal to x. Therefore,

$$F_1(x) = P\{x_\tau \le x\} = ux + I_{\{x \ge u\}}(x - u).$$

Similarly,

$$F_2(y) = P\{y_\sigma \le y\} = vy + I_{\{y \ge v\}}(y - v).$$

For chosen strategies (u, v), the payoff function can be reexpressed by

$$H(u, v) = P\{x_\tau > y_\sigma\} - P\{x_\tau < y_\sigma\} = \int_0^1 [P\{y_\sigma < x\} - P\{y_\sigma > x\}]dF_1(x)$$

$$= \int_0^1 [2F_2(y) - 1]dF_1(x). \tag{1.1}$$

By making successive simplifications in (1.1) for $v \le u$, i.e.,

$$H(u, v) = 2\int_0^1 F_2(y)dF_1(x) - 1 = 2\left[\int_0^v udx \int_0^x vdy + \int_v^u udx\left(\int_0^v vdy + \int_v^x (1 + v)dy\right)\right.$$

$$\left. + \int_u^1 (1 + u)dx\left(\int_0^v vdy + \int_v^x (1 + v)dy\right)\right] - 1,$$

we arrive at the formula

$$H(u, v) = (u - v)(1 - u - uv). \tag{1.2}$$

In the case of $v > u$, the problem symmetry implies that

$$H(u, v) = -H(v, u) = (u - v)(1 - v - uv).$$

Imagine that the strategy v of player II is fixed. Then the best response of player I, which maximizes the function (1.2), meets the condition

$$\frac{\partial H(u,v)}{\partial u} = 1 - u - uv - (v+1)(u-v) = 0.$$

Hence, it follows that

$$u(v) = \frac{v^2 + v + 1}{2(v+1)}.$$

Again, the problem symmetry demands that the optimal strategies of the players do coincide. By setting $u(v) = v$, we obtain the equation

$$\frac{v^2 + v + 1}{2(v+1)} = v,$$

which is equivalent to

$$v^2 + v - 1 = 0.$$

Its solution represents the well-known "golden section"

$$u^* = v^* = \frac{\sqrt{5} - 1}{2}.$$

By analogy, we evaluate the best response of player I from the expression (1.3) under $v > 0$. This yields the equation

$$\frac{\partial H(u,v)}{\partial u} = 1 - v - uv - v(u-v) = 0,$$

whence it appears that

$$u(v) = \frac{v^2 - v + 1}{2v}.$$

By setting $u = v$, readers again arrive at the golden section $u^* = v^* = \frac{\sqrt{5}-1}{2}$.

Therefore, if some player adopts the golden section strategy, the opponent's best response lies in the same strategy. In other words, the strategy profile (u^*, v^*) forms a Nash equilibrium in this game.

Interestingly, the stated approach is applicable to games with an arbitrary number of observations; however, it becomes extremely cumbersome. There exists an alternative solution method for optimal stopping problems, and we will address it below. Actually, it utilizes backward induction adapted to the class of problems under consideration. This method allows to find solutions to the optimal stopping problem in the case of an arbitrary number of observations, as well as to many other optimal stopping games.

7.2 Optimal stopping game: The case of independent observations

Let $\{x_n, n = 1, \ldots, N\}$ and $\{y_n, n = 1, \ldots, N\}$ be two sets of independent identically distributed random variables with a continuous distribution function $G(x), x \in R$ and the density function $g(x), x \in R$. Consider the following game $\Gamma_N(G)$. At each time moment $n = 0, \ldots, N$, players I and II receive some value of the corresponding random variable. They can either stop on the current values x_n and y_n, respectively, or continue observations. At the last shot N, the observations are terminated (if the players have not still made their choice); a player receives the value of the last random variable. The strategies in this game are the stopping times τ and σ, representing random variables with integer values from the set $\{1, 2, \ldots, N\}$. Each player seeks to stop observations with a higher value than the opponent.

Find optimal stopping rules in the class of threshold strategies $u = (u_1, \ldots, u_{N-1})$ and $v = (v_1, \ldots, v_{N-1})$ of the form

$$\tau(u) = \begin{cases} 1, \text{ if } x_1 \geq u_1, \\ n, \text{ if } x_1 < u_1, \ldots, x_{n-1} < u_{n-1}, x_n \geq u_n, \\ N, \text{ if } x_1 < u_1, \ldots, x_{N-1} < u_{N-1}, \end{cases}$$

and

$$\sigma(v) = \begin{cases} 1, \text{ if } y_1 \geq v_1, \\ n, \text{ if } y_1 < v_1, \ldots, y_{n-1} < v_{n-1}, y_n \geq v_n, \\ N, \text{ if } y_1 < v_1, \ldots, y_{N-1} < v_{N-1}. \end{cases}$$

Here we believe that

$$u_{N-1} < u_{N-2} < \ldots < u_1, \text{ and } v_{N-1} < v_{N-2} < \ldots < v_1. \tag{2.1}$$

Similarly to the previous section, for the chosen class of strategies (u, v) the payoff function can be rewritten as

$$\begin{aligned} H(u, v) &= P\{x_\tau > y_\sigma\} - P\{x_\tau < y_\sigma\} \\ &= \int_0^1 [P\{y_\sigma < x\} - P\{y_\sigma > x\}]dF_1(x) = E\{2F_2(x_{\tau(u)}) - 1\}, \end{aligned} \tag{2.2}$$

where $F_1(x)$ and $F_2(y)$ mean the distribution functions of the random variables $x_{\tau(u)}$ and $y_{\sigma(v)}$, respectively.

Lemma 7.1 *For strategies $\tau(u)$ and $\sigma(v)$, the distribution functions $F_1(x)$ and $F_2(y)$ have the density functions*

$$f_1(u,x) = \left[\prod_{i=1}^{N-1} G(u_i) + \sum_{i=1}^{N-1}\prod_{j=1}^{i-1} G(u_j) I_{\{x \ge u_i\}}\right] g(x), \tag{2.3}$$

and

$$f_2(v,y) = \left[\prod_{i=1}^{N-1} G(v_i) + \sum_{i=1}^{N-1}\prod_{j=1}^{i-1} G(v_j) I_{\{y \ge v_i\}}\right] g(y). \tag{2.4}$$

Proof: We employ induction on N. For instance, we demonstrate (2.3). Actually, equality (2.4) is argued by analogy. The base case: for $N = 1$, we have $f_1(x) = g(x)$. The inductive step: assume that equality (2.3) holds true for some $N = n$, and show its validity for $N = n + 1$. By the definition of the threshold strategy $u = (u_1, \dots, u_n)$, one obtains

$$f_1(u_1, \dots, u_n, x) = G(u_1) f_1(u_2, \dots, u_n) + I_{x_1 \ge u_1} g(y).$$

In combination with the inductive hypothesis, this yields

$$f_1(u_1, \dots, u_n, x) = G(u_1)\left[\prod_{i=2}^{n} G(u_i) + \sum_{i=2}^{n}\prod_{j=2}^{i-1} G(u_j) I_{\{x \ge u_i\}}\right] g(x)$$
$$+ I_{\{x_1 \ge u_1\}} g(y) = \left[\prod_{i=1}^{n} G(u_i) + \sum_{i=1}^{n}\prod_{j=1}^{i-1} G(u_j) I_{\{x \ge u_i\}}\right] g(x).$$

The proof of Lemma 7.1 is concluded.

Fix the strategy $\sigma(v)$ of player *II* and find the best response of player *I*. In fact, player *I* has to maximize the expression $\sup_u E\{2F_2(x_{\tau(u)}) - 1\}$ with respect to u.

For simpler exposition, suppose that all observations have the uniform distribution on the interval $[0, 1]$. This causes no loss of generality; indeed, it is always possible to pass to observations $G(x_n), G(y_n), n = 1, \dots, N$ having the uniform distribution. To evaluate $\sup_u E\{2F_2(x_{\tau(u)}) - 1\}$, we apply the backward induction method (see the beginning of Chapter 7). Write down the optimality equation

$$v(x,n) = \max\left\{2\int_0^x f_2(v,t)dt - 1, Ev(x_{n+1}, n+1)\right\}, n = 1, \dots, N-1.$$

Illustrate its application in the case of $N = 2$, when player I observes the random variables x_1, x_2, whereas player II observes the random variables y_1, y_2. Lemma 1 claims that the density function $f_2(v, y)$ has the form

$$f_2(v, y) = \begin{cases} v, & \text{if } 0 \le y < v, \\ 1 + v, & \text{if } v \le y \le 1. \end{cases}$$

Then the payoff function $f(x) = 2\int_0^x f_2(v, t)dt - 1$ under stopping in the state x can be expressed by

$$f(x) = \begin{cases} 2vx - 1, & \text{if } 0 \le x < v, \\ 1 - 2(1 - x)(1 + v), & \text{if } v \le x \le 1. \end{cases} \tag{2.5}$$

Imagine that player I has received the observation x. If he decides to continue to the next shot, his payoff makes up

$$Ev(x_2, 2) = \int_0^1 f(t)dt = \int_0^v (2vt - 1)dt + \int_v^1 (1 - 2(1 - t)(1 + v))dt = v^2 - v. \tag{2.6}$$

According to the optimality equation, player I stops at shot 1 (having received the observation x) if $f(x) \ge Ev(x_2, 2)$, and passes to the next observation if $f(x) < Ev(x_2, 2)$. The function $f(x)$ defined by (2.5) increases monotonically, while the function $Ev(x_2, 2)$ of the form (2.6) turns out independent from x (see Figure 7.1). Hence, there exists a unique intersection point of these functions; denote it by $u' \in [0, 1]$. In the case of $u' \le v$, the quantity u' meets the condition

$$2vu' - 1 = v^2 - v. \tag{2.7}$$

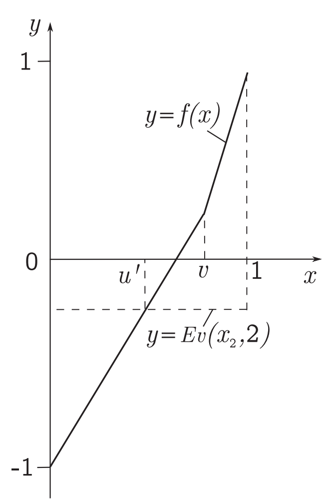


Figure 7.1 The payoff function $f(x)$.

If $u' > v$, it satisfies

$$1 - 2(1 - u')(1 + v) = v^2 - v. \tag{2.8}$$

Thus, if player II adopts the strategy with the threshold v, the optimal strategy of player I is determined by the stopping set $S = [u', 1]$ and the continuation set $C = [0, u')$.

Choose v such that it coincides with u'. Equations (2.7) and (2.8) bring to the same equation

$$v^2 + v - 1 = 0,$$

whose solution is the golden section $v^* = \frac{\sqrt{5}-1}{2}$. Therefore, if player II adheres to the threshold strategy v^*, the best response of player I is the threshold strategy with the same threshold $u^* = v^*$. The converse statement also holds true. This means optimality of the threshold strategies based on the golden section.

We have derived the same solution as in the previous section. However, the suggested evaluation scheme of the optimal stopping time possesses higher performance in the given class of problems. Furthermore, there is no a priori need to conjecture that the optimal strategies belong to the class of threshold ones. This circumstance follows directly from the optimality equation.

Section 7.3 shows the applicability of this scheme in stopping games with arbitrary numbers of observations.

7.3 The game $\Gamma_N(G)$ under $N \geq 3$

Consider the general case of the game $\Gamma_N(G)$, where the players receive independent uniformly distributed random variables $\{x_n, n = 1, \ldots, N\}$ and $\{y_n, n = 1, \ldots, N\}$, and $N \geq 3$. Suppose that player II uses a threshold strategy $\sigma(v)$ with thresholds meeting the condition

$$v_{N-1} < v_{N-2} < \ldots < v_1. \tag{3.1}$$

Due to Lemma 7.1, the density function $f_2(v, y)$ has the form

$$f_2(v, y) = \sum_{i=k}^{N} \prod_{j=0}^{i-1} v_j, \ \text{ if } v_k \leq y \leq v_{k-1},$$

where $k = 1, \ldots, N$ and $v_0 = 1, v_N = 0$ for convenience. Therefore, the function $f_2(v, y)$ jumps by the quantity $\prod_{j=0}^{k-1} v_j$ in the point $y = v_k$.

To construct the best response of player I to this strategy, we involve the optimality equation

$$v(x, n) = \max\left\{2\int_0^x f_2(v, t)dt - 1, Ev(x_{n+1}, n + 1)\right\}, n = 1, \ldots, N - 1, \tag{3.2}$$

with the boundary condition

$$v(x, N) = 2 \int_0^x f_2(v, t)dt - 1.$$

The payoff function $f(x) = 2 \int_0^x f_2(v, t)dt - 1$ under stopping in the state x can be rewritten as

$$f(x) = f(v_k) + 2(x - v_k) \sum_{i=k}^{N} \prod_{j=0}^{i-1} v_j, v_k \leq x \leq v_{k-1}, \tag{3.3}$$

or

$$f(x) = f(v_{k-1}) + 2(x - v_{k-1}) \sum_{i=k}^{N} \prod_{j=0}^{i-1} v_j, v_k \leq x \leq v_{k-1}.$$

The curve of $y = f(x)$ represents a jogged line ascending from point $(0, -1)$ to the point $(1, 1)$ (see Figure 7.2).

For any n, the maximands in equation (3.2) are the monotonically increasing function $f(x)$ and the function $Ev(x_{n+1}, n + 1)$ independent from x. This feature allows to simplify the optimality equation:

$$v(x, n) = \begin{cases} Ev(x_{n+1}, n + 1), & 0 \leq x \leq u_n, \\ f(x), & u_n \leq x \leq 1, \end{cases}$$

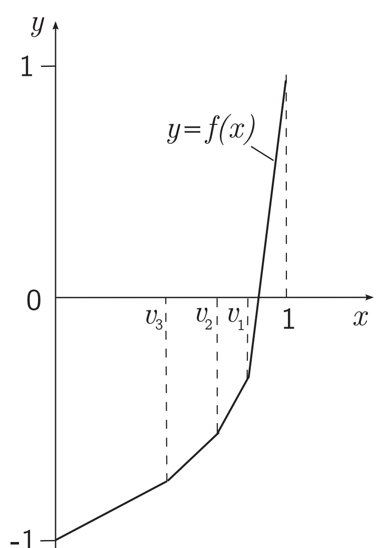


Figure 7.2 The payoff function $f(x)$.

where u_n designates the intersection point of the functions $y = f(x)$ and $y = Ev(x_{n+1}, n+1)$. Therefore, if at shot n the observation exceeds u_n, the player should stop observations (and continue them, otherwise). Clearly,

$$u_{N-1} < \dots < u_2 < u_1.$$

Which requires validity of the equality

$$u_n = v_n,\ n = 1, \dots, N-1.$$

This brings to the system of conditions

$$f(u_n) = Ev(u_{n+1}, n+1), n = 1, \dots, N-1. \tag{3.4}$$

Under $n = N-1$, formula (3.4) implies that

$$f(u_{N-1}) = Ev(u_N, N) = \int_0^1 f(t)dt. \tag{3.5}$$

According to (3.3), we obtain

$$\sum_{i=1}^{N-1} u_j(1-u_i) + 2\prod_{i=1}^{N-1} u_i u_{N-1} = 1. \tag{3.6}$$

Next, note that

$$f(u_{N-2}) = Ev(u_{N-1}, N-1) = \int_0^{u_{N-1}} v(t, N-1)dt + \int_{u_{N-1}}^1 f(t)dt$$

$$= \int_0^{u_{N-1}} f(u_{N-1})dt + \int_{u_{N-1}}^1 f(t)dt = \int_0^1 f(t)dt + \frac{u_{N-1}}{2}[1 + f(u_{N-1})].$$

Hence, by virtue of (3.5) and the notation $u_N = 0, f(u_N) = -1$, we have

$$f(u_{N-2}) - f(u_{N-1}) = \frac{u_{N-1} + u_N}{2}(f(u_{N-1}) - f(u_N)).$$

Readers can easily demonstrate the following equality by induction:

$$f(u_{n-1}) - f(u_n) = \frac{u_n + u_{n+1}}{2}(f(u_n) - f(u_{n+1})), n = 1, \dots, N-1. \tag{3.7}$$

It appears from (3.3) that

$$f(u_n) - f(u_{n+1}) = 2 \sum_{k=n+1}^{N-1} \prod_{j=1}^{k-1} u_j (u_n - u_{n+1}).$$

In combination with (3.7), this yields

$$2 \sum_{k=n}^{N-1} \prod_{j=1}^{k-1} u_j (u_{n-1} - u_n) = 2 \sum_{k=n+1}^{N-1} \prod_{j=1}^{k-1} u_j \frac{u_n^2 - u_{n+1}^2}{2}.$$

By canceling by $2 \prod_{j=1}^{n-1} u_j$, we obtain

$$(u_{n-1} - u_n) \left[1 + \sum_{k=n+1}^{N-1} \prod_{j=n}^{k-1} u_j \right] = \sum_{k=n+1}^{N-1} \prod_{j=n}^{k-1} u_j \frac{u_n^2 - u_{n+1}^2}{2}.$$

Now, reexpress u_{n-1} from the above relationship:

$$u_{n-1} = u_n + \frac{u_n - u_{n+1}}{2} \frac{\sum\limits_{k=n+1}^{N-1} \prod\limits_{j=n}^{k-1} u_j}{1 + \sum\limits_{k=n+1}^{N-1} \prod\limits_{j=n}^{k-1} u_j}, n = 2, \dots, N-1. \tag{3.8}$$

Standard analysis of the system of equations (3.6), (3.8) shows that there exists a solution of this system, which satisfies the condition $u_{N-1} < ... < u_2 < u_1$. Furthermore, as N increases, the value of u_1 gets arbitrarily close to 1, whereas u_{N-1} tends to some threshold value $u_{N-1}^* \approx 0.715$. Table 7.1 provides the numerical values of the optimal thresholds under different values of N.

Therefore, if player *II* chooses the strategy $\sigma(u^*)$ defined by the thresholds $u_n^*, n = 1, \dots, N-1$ that meet the system (3.6), (3.8), we have the following result. According to the backward induction method, the best response of player *I* has the same threshold form as that of player *II*. This means optimality of the corresponding strategy in the game under consideration. We summarize the above reasoning in

Table 7.1 The optimal thresholds for different N.

N	u_1^*	u_2^*	u_3^*	u_4^*	u_5^*
2	0.618				
3	0.742	0.657			
4	0.805	0.768	0.676		
5	0.842	0.821	0.781	0.686	
6	0.869	0.855	0.833	0.791	0.693

Theorem 7.1 *The game $\Gamma_N(G)$ admits an equilibrium in the class of threshold strategies of the form*

$$\tau(u^*) = \begin{cases} 1, & \text{if } G(x_1) \geq u_1^*, \\ n, & \text{if } G(x_1) < u_1^*, \dots, G(x_{n-1}) < u_{n-1}^*, G(x_n) \geq u_n^*, \\ N, & \text{if } G(x_1) < u_1^*, \dots, G(x_{N-1}) < u_{N-1}^*, \end{cases}$$

and

$$\sigma(u^*) = \begin{cases} 1, & \text{if } G(y_1) \geq u_1^*, \\ n, & \text{if } G(y_1) < u_1^*, \dots, G(y_{n-1}) < u_{n-1}^*, G(y_n) \geq u_n^*, \\ N, & \text{if } G(y_1) < u_1^*, \dots, G(y_{N-1}) < u_{N-1}^*, \end{cases}$$

where $u_n^*, n = 1, \dots, N-1$ satisfies the system (3.6), (3.8).

For instance, if $n = 3$, the system (3.6), (3.8) implies that the optimal thresholds make up $u_1^* = 0.742$, $u_2^* = 0.657$. In the case of $n = 4$, we obtain the optimal thresholds $u_1^* = 0.805$, $u_2^* = 0.768$, and $u_3^* = 0.676$.

Therefore, the above game with optimal stopping of a sequence of independent identically distributed random variables has a pure strategy Nash equilibrium. In the beginning of the game, the players evaluate their thresholds, and then compare the incoming observations with the thresholds. If the former exceed the latter, a player terminates observations. However, the equilibrium does not necessarily comprise pure strategies. In what follows, we find an equilibrium in the game with random walks, and it will exist among mixed strategies.

7.4 Optimal stopping game with random walks

Consider a two-player game $\Gamma(a, b)$ defined on random walks as follows. Let x_n and y_n be symmetrical random walks on the set of integers $E = \{0, 1, \dots, k\}$, starting in states $a \in E$ and $b \in E$, respectively. For definiteness, we believe that $a \leq b$. In any inner state $i \in E$, a walk equiprobably moves left or right and gets absorbed in the end points (0 and k). Players *I* and *II* observe the walks x_n and y_n and can stop them at certain time moments τ and σ. These random time moments represent the strategies of the players. Consequently, if $x_\tau > y_\sigma$, player *I* wins. In the case of $x_\tau < y_\sigma$, player *II* wins accordingly. Finally, the game is drawn provided that $x_\tau = y_\sigma$. A player has no information on the opponent's behavior.

As usual, this game is antagonistic with the payoff function

$$H(\tau, \sigma) = E\{I_{\{x_\tau > y_\sigma\}} - I_{\{x_\tau < y_\sigma\}}\}.$$

To solve the posed game, recall the technique adopted at the beginning of Chapter 7. Notably, reduce the game to two optimal stopping problems. At the outset, note an important feature. To establish an equilibrium, it suffices to find a pair of strategies (τ^*, σ^*) of players *I* and *II* such that

$$\sup_\tau H(\tau, \sigma^*) = \inf_\sigma H(\tau^*, \sigma) = H^*.$$

Then (τ^*, σ^*) forms an equilibrium, H^* gives the value of the game $\Gamma(a, b)$.

The first problem $\sup_\tau H(\tau, \sigma^*)$ is the optimal stopping problem $\sup_\tau Ef(x_\tau)$ for player I with the payoff function under stopping defined by

$$f_1(x) = P\{y_{\sigma^*} < x\} - P\{y_{\sigma^*} > x\}, x = 0, 1, \dots, k.$$

Here the solution is based on the backward induction method. Similarly, the second problem $\inf_\sigma H(\tau^*, \sigma)$ is the optimal stopping problem $\sup_\sigma Ef(y_\sigma)$ for player II with the payoff function under stopping determined by

$$f_2(y) = P\{x_{\tau^*} < y\} - P\{x_{\tau^*} > y\}, y = 0, \dots, k.$$

To simplify the form of the payoff functions under stopping, introduce vectors $s = (s_0, s_1, \dots, s_k)$ and $t = (t_0, t_1, \dots, t_k)$, where

$$s_i = P\{x_\tau = i\}, \quad t_i = P\{y_\sigma = i\}, \quad i = 0, 1, \dots, k.$$

These vectors are called *the spectra of the strategies* τ and σ. Now, if the strategy σ is fixed, the problem $\sup_\tau H(\tau, \sigma)$ gets reduced to the optimal stopping problem for the random walk x_n with the payoff function under stopping

$$f_1(x) = 2 \sum_{i=0}^{x} t_i - t_x - 1, \quad x = 0, 1, ...k. \tag{4.1}$$

By analogy, the problem $\inf_\sigma H(\tau, \sigma)$ with fixed τ represents the optimal stopping problem of the random walk y_n with the payoff function under stopping

$$f_2(y) = 2 \sum_{i=0}^{y} s_i - s_y - 1, \quad y = 0, 1, \dots, k. \tag{4.2}$$

We take advantage of the backward induction method to solve the derived problems. Write down the optimality equation (for definiteness, select player I). The optimal expected payoff of player I provided that the walk is in the state $x_n = x \in E$ will be denoted by

$$v_1(x) = \sup_\tau E\{f_1(x_\tau)/x_n = x\}.$$

By terminating observations in this state, the player gains the payoff $f(x_n) = f(x)$. If he continues observations and acts in the optimal way, his expected payoff constitutes

$$E\{v_1(x_{n+1})/x_n = x\} = \frac{1}{2} v_1(x-1) + \frac{1}{2} v_1(x+1).$$

By comparing these payoffs, one can construct the optimal strategy. And the optimality equation takes the recurrent form

$$v_1(x) = \max\{f_1(x), \frac{1}{2} v_1(x-1) + \frac{1}{2} v_1(x+1)\}, x = 1, \dots, k-1. \tag{4.3}$$

In absorbing states, we have

$$v_1(0) = f_1(0),\ \ v_1(k) = f_1(k).$$

Equation (4.3) can be solved geometrically. It appears from (4.3) that the solution meets the conditions $v_1(x) \ge f_1(x), x = 0, \dots, k$. In other words, the curve of $y = v_1(x)$ lies above the curve of $y = f_1(x)$. In addition,

$$v_1(x) \ge \frac{1}{2}v_1(x-1) + \frac{1}{2}v_1(x+1), x = 1, \dots, k-1.$$

The last equation implies that the function $v_1(x)$ is concave. Therefore, to solve (4.3), it suffices to draw the curve $y = f_1(x), x = 0, \dots, k$ and strain a thread above it. The position of the thread yields the solution $v_1(x)$, *eo ipso* defines the optimal strategy of player I. Notably, this player should stop in the states $S = \{x : v_1(x) = f_1(x)\}$, and continue observations in the states $C = \{x : v_1(x) > f_1(x)\}$. The same reasoning applies to the optimality equation for player II.

Consequently, the functions $v_1(x)$ and $v_2(y)$ being available, we have

$$\sup_\tau H(\tau, \sigma^*) = \sup_\tau Ef_1(x_\tau) = v_1(a),$$
$$\inf_\sigma H(\tau^*, \sigma) = -\sup_\sigma Ef_2(y_\sigma) = -v_2(b).$$

If some strategies (τ^*, σ^*) meet the equality

$$v_1(a) = -v_2(b) = H^*, \tag{4.4}$$

they actually represent optimal strategies.

Prior to optimal strategies design, let us discuss several properties of the spectra of strategies.

7.4.1 Spectra of strategies: Some properties

We demonstrate that the spectra of strategies form a certain polyhedron in the space R^{k+1}. This allows to reexpress the solution to the corresponding optimal stopping problem via a linear programming problem.

Theorem 7.2 *A vector $s = (s_0, \dots, s_k)$ represents the spectrum of some strategy τ iff the following conditions hold true:*

$$\sum_{i=0}^{k} is_i = a, \tag{4.5}$$

$$\sum_{i=0}^{k} s_i = 1, \tag{4.6}$$

$$s_i \ge 0,\ \ i = 0, \dots, k. \tag{4.7}$$

Proof: **Necessity.** Suppose that τ is the stopping time with the spectrum s. The definition of s directly leads to validity of the conditions (4.6)–(4.7). Next, the condition (4.8) results from considerations below. A symmetrical random walk enjoys the remarkable relationship

$$E\{x_{n+1}/x_n = x\} = \frac{1}{2}(x-1) + \frac{1}{2}(x+1) = x,$$

or

$$E\{x_{n+1}/x_n\} = x_n. \tag{4.8}$$

In this case, we say that the sequence $x_n, n \geq 0$ makes a *martingale*. It appears from the condition (4.8) that the mean value of the martingale is time-invariant: $Ex_n = Ex_0,\ n = 1, 2, ...$ Particularly, this takes place for the stopping time τ. Hence,

$$Ex_\tau = Ex_0 = a.$$

By noticing that

$$Ex_\tau = \sum_{i=0}^{k} iP\{x_\tau = i\} = \sum_{i=0}^{k} is_i,$$

readers naturally arrive at (4.5).

Among other things, this also implies the following. The stopping time τ_{ij} defined by two thresholds i and j $(0 \leq i < a < j \leq k)$, which dictates to continue observations until the random walk reaches one of states i or j, possesses the spectrum (see (4.5)–(4.6)) agreeing with the conditions

$$s_i + s_j = 1, \quad s_i \cdot i + s_j \cdot j = a.$$

And so, $s_i = \frac{j-a}{j-i}$ and $s_j = \frac{a-i}{j-i}$.

Consequently, the spectrum of the strategy τ_{ij} makes

$$s_{ij} = \left(0, \dots, 0, \frac{j-a}{j-i}, 0, \dots, 0, \frac{a-i}{j-i}, 0, \dots, 0\right). \tag{4.9}$$

Finally, note that the spectrum of the strategy $\tau_0 \equiv 0$ equals

$$s_0 = (0, \dots, 0, 1, 0, \dots, 0), \tag{4.10}$$

where all components except a are zeros.

Sufficiency. The set S of all vectors $s = (s_0, \dots, s_k)$ satisfying the conditions (4.5)–(4.7) represents a convex polyhedral, *ergo* coincides with the convex envelope of the finite set of extreme points.

A point s of the convex set S is called *an extreme point*, if it does not admit the representation

$$s = (s^{(1)} + s^{(2)})/2, \quad \text{where } s^{(1)} \neq s(2) \text{ and } s^{(1)},\ s^{(2)} \in S.$$

Show that the extreme points of the set S have the form (4.9) and (4.10). Indeed, any vector s described by (4.5)–(4.7), having at least three non-zero components $(s_l, s_i, s_j$, where $0 \leq l < i < j \leq k)$, can be reexpressed as the half-sum of the vectors

$$s^{(1)} = (..., s_l - e_1, \dots, s_i + e, \dots, s_j - e_2, ...),$$
$$s^{(2)} = (..., s_l + e_1, \dots, s_i - e, \dots, s_j + e_2, ...).$$

Here $e_1 + e_2 = e,\ e_1 = \frac{j-i}{i-l} \cdot e_2,\ 0 < e \leq \min\{s_l, s_i, s_j\}$ belong to the set S.

Therefore, all extreme points of the polyhedron S may have the form (4.9) or (4.10). Hence, any vector s enjoying the properties (4.5)–(4.7) can be rewritten as the convex envelope of the vectors s_0 and s_{ij}:

$$s = v_0 s_0 + \sum v_{ij} s_{ij},$$

where $v_0 \geq 0$, $v_{ij} \geq 0$,

$$v_0 + \sum v_{ij} = 1.$$

Here summation runs over all (i,j) such that $0 \leq i < a < j \leq k$. By choosing the strategy τ_0 with the probability v_0 and the strategies τ_{ij} with the probabilities v_{ij}, we build the mixed strategy v, which possesses the spectrum s. This substantiates the sufficiency of the conditions (4.5)–(4.7). The proof of Theorem 7.2 is finished.

7.4.2 Equilibrium construction

To proceed, we evaluate the equilibrium (τ^*, σ^*). According to Theorem 7.2, it suffices to construct the spectra of the optimal strategies (s^*, t^*). These are the vectors meeting the conditions

$$s_i \geq 0,\ i = 0, \dots, k;\ \ \sum_{i=0}^{k} s_i = 1;\ \ \sum_{i=0}^{k} s_i i = a; \tag{4.11}$$

$$t_i \geq 0,\ i = 0, \dots, k;\ \ \sum_{i=0}^{k} t_i = 1;\ \ \sum_{i=0}^{k} t_i i = b. \tag{4.12}$$

Lemma 7.2 *For player II, there exists a strategy σ^* such that*

$$\sup_{\tau} H(\tau, \sigma^*) = (a - b)/b. \tag{4.13}$$

Proof: Let us employ Theorem 7.2 and, instead of the strategy σ^*, construct its spectrum t^*. First, assume that

$$k < 2b - 1. \tag{4.14}$$

Define the vector $t^* = (t_0, t_1, \dots, t_k)$ by the expressions

$$t_i = \begin{cases} 1/b, \;\; i = 1, 3, 5, \dots, 2(k-b) - 1, \\ 1 - (k-b)/b, \;\; i = k, \\ 0, \quad \text{for the rest } i \in E. \end{cases} \tag{4.15}$$

Verify the conditions (4.12) for the vector (4.15). For all $i \in E$, the inequality $t_i \geq 0$ takes place (for $i = k$, this follows from (4.14)). Next, we find

$$\sum_{i=0}^{k} t_i = \frac{1}{b}(k-b) + 1 - \frac{k-b}{b} = 1,$$

$$\sum_{i=0}^{k} t_i i = \frac{1}{b}\,[1 + 3 + \dots + 2(k-b) - 1] + k\left(1 - \frac{k-b}{b}\right) = b.$$

And so, the conditions of Theorem 7.2 hold true, and the vector t^* represents the spectrum of some strategy σ^*.

Second, evaluate $\sup_\tau H(\tau, \sigma^*)$. For this (see the discussion above), it is necessary to solve the optimal stopping problem with the payoff function under stopping $f_1(x)$ defined by (4.1). Substitute (4.15) into (4.1) to get

$$f_1(i) = \begin{cases} i/b - 1, \quad i = 0, \dots, 2(k-b), \\ 2(k-b)/b - 1, \;\; i = 2(k-b) + 1, \dots, k-1, \\ k/b - 1, \quad i = k. \end{cases}$$

To solve the optimality equation, we use the same geometrical considerations as before. Figure 7.3 provides the curve of $y = f_2(x)$, which represents a straight line connecting the points $(x = 0, y = -1)$ and $(x = k, y = k/b - 1)$. The equation of this line takes the form $y = x/b - 1$. On the other hand, the optimality equation (4.3) is solved by the function

$$v_1(i) = i/b - 1.$$

In state $i = a$, we accordingly obtain

$$v_1(a) = (a - b)/b.$$

This proves (4.13).

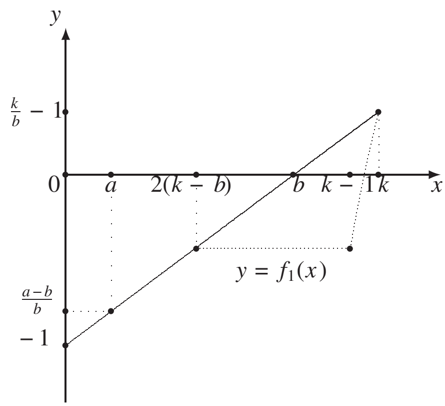


Figure 7.3 The payoff function under stopping $f_1(x)$.

If $k \geq 2b - 1$, define t^* by

$$t_i = \begin{cases} 1/b, & \text{if } i = 1, 3, \ldots, 2b-1, \\ 0, & \text{for the rest } i \in E. \end{cases}$$

The vector t^* of this form meets the conditions (4.12) and, hence, is the spectrum of a certain strategy. The function $f_1(x)$ acquires the form

$$f_1(i) = \begin{cases} i/b - 1, & i = \overline{0, \ldots, 2b-1}, \\ 1, & i = \overline{2b, \ldots, k}. \end{cases}$$

Its curve is illustrated in Figure 7.4. Clearly, the function $v_1(x)$ coincides with the function $f_1(x)$. And it follows that, within the interval $[0, 2b]$, the function $v_1(x)$ has the form

$$v_1(i) = i/b - 1,$$

which argues validity of (4.13). The proof of Lemma 7.2 is concluded.

Lemma 7.3 *For player I, there exists a strategy τ^* such that*

$$\inf_{\sigma} H(\tau^*, \sigma) = (a - b)/b. \tag{4.16}$$

Proof: It resembles that of Lemma 7.1. First, consider the case of

$$k \leq 2b.$$

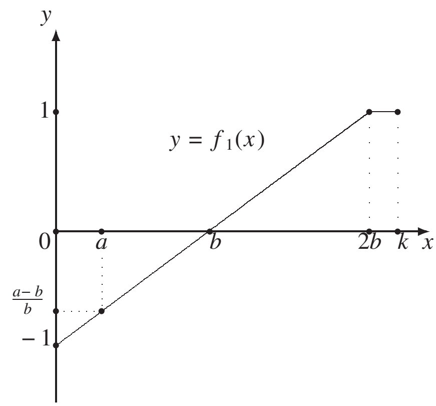


Figure 7.4 The payoff function under stopping $f_1(x)$.

Determine the vector $s^* = (s_0, s_1, \dots, s_k)$ by the expressions

$$s_i = \begin{cases} 1 - \frac{a}{b+1}, & \text{if } i = 0, \\ \frac{a}{b(b+1)}, & \text{if } i = 2, 4, \dots, 2(k-b-1), \\ \frac{a(2b-k+1)}{b(b+1)}, & \text{if } i = k, \\ 0, & \text{for the rest } i \in E. \end{cases} \tag{4.17}$$

Evidently, the vector (4.17) agrees with the conditions (4.11). According to Theorem 7.2, it makes the spectrum of some strategy τ^* of player I. Substitution of (4.17) into (4.2) gives

$$f_2(i) = \begin{cases} -\frac{a}{b+1}, i = 0, \\ \frac{a}{b(b+1)}(i-b) + \frac{b-a}{b}, \ i = 1, \dots, 2(k-b)-1, \\ f_2(2(k-b)-1), \ i = 2(k-b), \dots, k-1, \\ \frac{a}{b(b+1)}(k-b) + \frac{b-a}{b}, \ i = k. \end{cases} \tag{4.18}$$

Figure 7.5 demonstrates the curve of the function (4.18).

The straight line obeying the equation

$$y = \frac{a}{b(b+1)}(x-b) + \frac{b-a}{b},$$

coincides with the curve of $y = f_2(x)$ in the points $x = 1, 2, \dots, 2(k-b)$, $2(k-b)+1, \dots, k-1$ and $x = 0$ (see (4.18)). Moreover, the former lies above the latter in the points $x = 2(k-b), 2(k-b)+1, \dots, k-1$ and $x = 0$. Thus, we obtain the expressions

$$y(0) = -\frac{a}{b+1} + \frac{b-a}{b} \geq -\frac{a}{b+1} = f_2(0).$$

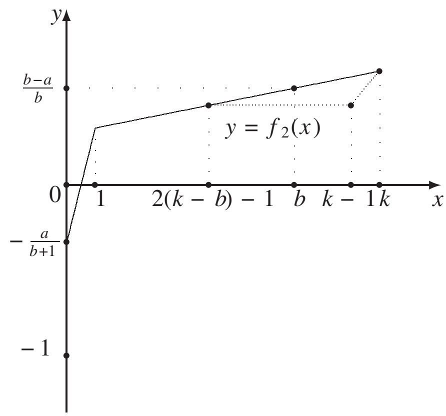


Figure 7.5 The payoff function under stopping $f_2(x)$.

But this implies that, within the interval $[1, k]$, the function $v_2(x)$ has the form

$$v_2(i) = \frac{a}{b(b+1)}(i-b) + \frac{b-a}{b}.$$

As a result,

$$v_2(b) = \frac{b-a}{b},$$

which proves formula (4.16).

Now, suppose that $k \geq 2b+1$. Define the spectrum s^* by

$$s_i = \begin{cases} 1 - \frac{a}{b+1}, & \text{if } i = 0, \\ \frac{a}{b(b+1)}, & \text{if } i = 2, 4, \dots, 2b, \\ 0, & \text{for the rest } i \in E. \end{cases}$$

In this case,

$$f_2(i) = \begin{cases} -\frac{a}{b+1}, \ i = 0, \\ \frac{a}{b(b+1)}(i-b) + \frac{b-a}{b}, \ i = \overline{1, 2b}, \\ 1, \ i = \overline{2b+1, k}. \end{cases}$$

Figure 7.6 shows that the function $v_2(x)$ coincides with $f_2(x)$, whence it appears that

$$v_2(b) = f_2(b) = (b-a)/b.$$

The proof of Lemma 7.3 is finished.

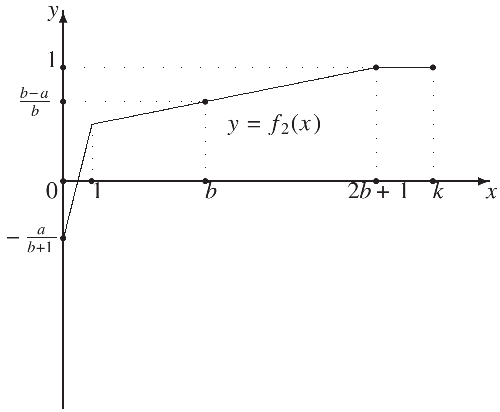


Figure 7.6 The payoff function under stopping $f_2(x)$.

Therefore, if we choose the strategies τ^* and σ^* according to Lemmas 7.1 and 7.2, the expressions (4.13), (4.16) lead to

$$\sup_{\tau} H(\tau, \sigma^*) = \inf_{\sigma} H(\tau^*, \sigma) = (a - b)/b.$$

By turn, this yields the following assertion.

Theorem 7.3 *Let $x_n(w)$ and $y_n(w)$ be symmetrical random walks on the set E. Then the value of the game $\Gamma(a, b)$ equals*

$$H^* = (a - b)/b.$$

Apparently, the solution of the game problem belongs to the class of mixed strategies being random distributions on the set of two-threshold strategies. With some probabilities, each player selects left and right thresholds i, j with respect to the starting point of the walk; he continues observations until the walk leaves these limits. The value of the game is the probability that the walk starting in point a reaches zero earlier than the point $-b$. Interestingly, the value does not depend on the right limit of the walking interval.

7.5 Best choice games

Best choice games are an alternative setting of optimal stopping games. Imagine N objects sorted by their quality; the best object has number 1. The objects are supplied randomly to a player, one by one. He can compare them, but is unable to get back to the viewed objects. The player's aim consists in choosing the best object. As a matter of fact, this problem is also known as the secretary problem, the bride problem, the parking problem, etc.

Here we deal with a special process of observations. Let us endeavor to provide a formal mathematical description. Suppose that all objects are assigned numbers $\{1, 2, \ldots, N\}$, where object 1 possesses the highest quality. The number of an object is called its rank. The objects arrive in a random order. All permutations $N!$ are equiprobable. Denote by a_n the absolute rank of the object that appears at time moment $n, n = 1, \dots, N$. The whole difficulty lies in the following. The player receives an object at time moment n, but does not know the absolute rank of this object. Being able to compare the objects with each other, the player knows merely the relative rank y_n of this object among the viewed objects. Such rank makes up

$$y_n = \sum_{i=1}^{n} I(a_i \le a_n),$$

where $I(A)$ acts as the indicator of the event A. If all objects which arrived before time moment n have ranks higher than the given object, the relative rank of the latter constitutes 1. Therefore, by observing the relative ranks y_n, the player has to make some conclusions regarding the absolute rank a_n. The relative ranks y_n represent random variables; owing to the equal probability of all permutations, we obtain $P\{y_n = i\} = 1/n$, $i = 1, \dots, n$. In other words, the relative rank of object n can be any value from 1 to n equiprobably. The best choice problem suggests two possible goals (criteria) for the player: (1) find the stopping rule τ, which maximizes the probability of best object finding, i.e., $P\{a_\tau = 1\}$, or (2) minimize the expected object's rank $E\{a_\tau\}$. We begin with the best choice problem, where the player maximizes the probability of best object finding.

Introduce the random sequence $x_n = P\{a_n = 1/y_n\}, n = 1, 2, \dots, N$. Note that, for any stopping time τ,

$$Ex_\tau = \sum_{n=1}^{N} E\{x_n I_{\{\tau=n\}}\} = \sum_{n=1}^{N} E\{P\{a_n = 1/y_n\} I_{\{\tau=n\}}\}.$$

The decision on stopping $\tau = n$ is made depending on the value of y_n. By the properties of conditional expectations,

$$Ex_\tau = \sum_{n=1}^{N} E\{I_{\{a_n=1\}} I_{\{\tau=n\}}\} = E\{I_{\{a_\tau=1\}}\} = P\{a_\tau = 1\}.$$

Therefore, the problem of stopping rule which maximizes the probability of best object finding is the optimal stopping problem for the random process $x_n, n = 1, \dots, N$. This sequence forms a Markov chain on the extended set of states $E = \{0, 1, \dots, N\}$ (we have added the stopping state 0).

Lemma 7.4 *The following formula holds true:*

$$x_n = P\{a_n = 1/y_n\} = \begin{cases} \frac{n}{N}, & \text{if } y_n = 1, \\ 0, & \text{if } y_n > 1. \end{cases}$$

Proof: Obviously, if $y_n > 1$, this object is not the best one. And so, $x_n = P\{a_n = 1/y_n\} = 0$. On the other part, if $y_n = 1$ (object n is the best among the viewed objects), we have

$$x_n = P\{a_n = 1/y_n = 1\} = \frac{P\{a_n = 1\}}{P\{a_n < \min\{a_1, \dots, a_{n-1}\}\}}.$$

Due to the equiprobability of all permutations, $P\{a_n = 1\} = 1/N$. The probability $P\{a_n < \min\{a_1, \dots, a_{n-1}\}\}$ that the minimal element in a permutation of n values holds position n also makes up $1/n$. And it appears that

$$x_n = P\{a_n = 1/y_n = 1\} = \frac{1/N}{1/n} = \frac{n}{N}.$$

This concludes the proof of Lemma 7.4.

According to Lemma 7.4, the optimal behavior prescribes stopping only on objects with the relative rank of 1. Such objects are called **candidates**. If a candidate comes at time moment n and we choose it, the probability that this is the best object is n/N. By comparing the payoffs in the cases of stopping and continuation of observations, one can find the optimal stopping rule. Revert to the backward induction method to establish the optimal rule. We define it by the optimal expected payoff function $v_n, n = 1, \dots., N$.

Consider the end time moment $n = N$. The player's payoff makes up x_N. This is either 0 or 1, depending on the status of the given object (a candidate or not, respectively). Let us set $v_N = 1$.

At shot $n = N - 1$, the player's payoff under stopping equals x_{N-1} (or 0, or $(N-1)/N$). If the player continues observations, his expected payoff is

$$Ex_N = \frac{1}{N} \cdot 1 + \left(1 - \frac{1}{N}\right) \cdot 0 = \frac{1}{N}.$$

By comparing these payoffs, we get the optimal stopping rule

$$v_{N-1} = \max\{x_{N-1}, Ex_N\} = \max\left\{\frac{N-1}{N}, \frac{1}{N}\right\}.$$

At shot $n = N - 2$, the player's payoff under stopping equals x_{N-2} (or 0, or $(N-2)/N$). If the player continues observations, a candidate appears at shot $N - 1$ with the probability $1/(N-1)$ and at shot N with the probability

$$\left(1 - \frac{1}{N-1}\right)\frac{1}{N} = \frac{N-2}{(N-1)N}.$$

And his expected payoff becomes

$$Ex_{N-1} = \frac{1}{N-1}\frac{N-1}{N} + \frac{N-2}{(N-1)N} = \frac{N-2}{N}\left(\frac{1}{N-2} + \frac{1}{N-1}\right).$$

Consequently,

$$v_{N-2} = \max\{x_{N-2}, Ex_{N-1}\} = \max\left\{\frac{N-2}{N}, \frac{N-2}{N}\left(\frac{1}{N-2} + \frac{1}{N-1}\right)\right\}.$$

Repeat these arguments for subsequent shots. At shot n, we arrive at the equation

$$v_n = \max\{x_n, Ex_{n+1}\} = \max\left\{\frac{n}{N}, \frac{n}{N}\left(\sum_{i=n+1}^{N}\frac{1}{i-1}\right)\right\}, n = N, \ldots 1.$$

If the player continues at shot n, a next candidate can appear at time moment $n+1$ with the probability $1/(n+1)$ and at time moment $i, i > n$ with the probability

$$\left(1-\frac{1}{n+1}\right)\left(1-\frac{1}{n+2}\right)\cdot \ldots \cdot\left(1-\frac{1}{i-1}\right)\frac{1}{i} = \frac{n}{(i-1)i}. \tag{5.1}$$

The stopping rule is defined as follows. The player should stop on candidates such that the payoff under stopping becomes greater or equal to the payoff under continuation. The stopping set is given by the inequalities

$$S = \left\{n : \frac{n}{N} \geq \frac{n}{N}\left(\sum_{i=n+1}^{N}\frac{1}{i-1}\right)\right\}.$$

Therefore, the set S has the form $[r, r+1, \ldots, N]$, where r meets the inequalities

$$\sum_{i=r}^{N-1}\frac{1}{i} \leq 1 < \sum_{i=r-1}^{N-1}\frac{1}{i}. \tag{5.2}$$

Theorem 7.4 *Consider the best choice game, where the player seeks to maximize the probability of best object finding. His optimal strategy is to stop on the object after time moment r defined by (5.2). Moreover, this object turns out the best among all viewed objects.*

Under large N, inequalities (5.2) can be rewritten in the integral form:

$$\int_{i=r}^{N-1}\frac{1}{t}dt \leq 1 < \int_{i=r-1}^{N-1}\frac{1}{t}dt.$$

This immediately yields $\lim_{N\to\infty}\frac{r}{N} = 1/e \approx 0.368$. Therefore, under large N, the player should stop on the first candidate after N/e viewed objects.

We have argued that the optimal behavior in this game is described by threshold strategies $\tau(r)$. The player chooses some threshold r, till this time moment he merely observes the incoming objects and finds the best one among them. After the stated time moment, the player stops on the first object being better than the previous ones. Undoubtedly, he may skip the best object (if it appears among the first $r-1$ objects) or not reach the best object (by terminating observations on the relatively best object). To compute the probability of best object finding under the optimal strategy, we find the spectrum of the threshold strategy $\tau(r)$.

Lemma 7.5 *Consider the best choice problem, where the player uses the threshold strategy $\tau(r)$. The spectrum of such strategy (the probability of choosing object i) takes the form*

$$P\{\tau(r) = i\} = \begin{cases} 0, \ \textit{if } i = 1, \dots, r-1, \\ \frac{r-1}{i(i-1)}, \ \textit{if } i = r, \dots, N, \\ \frac{r-1}{N}, \ \textit{if } i = 0. \end{cases}$$

Proof: The strategy $\tau(r)$ requires to stop only in states $r, r+1, \dots, N$; therefore, we have $P\{\tau(r) = i\} = 0$ for $i = 1, \dots, r-1$. The event $\{\tau = r\}$ is equivalent to that the last element in the sequence $a_1, \dots, a_r$ appears minimal. The corresponding probability makes up $P\{\tau(r) = r\} = 1/r$. On the other hand, the event $\tau(r) = i$, where $i = r+1, \dots, N$, is equivalent to the following event. The minimal element in the sequence $a_1, \dots, a_{r-1}, a_r, \dots, a_i$ holds position i, whereas the second smallest element is located on position j $(1 \le j \le r-1)$. The probability of such a complex event constitutes

$$P\{\tau = i\} = \sum_{j=1}^{r-1} \frac{(i-2)!}{i!} = \frac{(i-2)!}{i!}(r-1) = \frac{r-1}{i(i-1)}.$$

Finally, we find the probability of break $P\{\tau(r) = 0\}$ from the equality

$$P\{\tau(r) = 0\} = 1 - \sum_{i=r}^{N} \frac{r-1}{i(i-1)} = \frac{r-1}{N}.$$

The proof of Lemma 7.5 is completed.

As a matter of fact, the quantity $P\{\tau(r) = 0\} = (r-1)/N$ represents exactly the probability that the player skips the best object under the given threshold rule $\tau(r)$. Evidently, the optimal behavior ensures best object finding with the approximate probability of 0.368.

7.6 Best choice game with stopping before opponent

In the previous section, we have studied players acting independently (their payoff functions depend only on their individual behavior). Now, consider a nonzero-sum two-player game, where each player strives to find the best object earlier than the opponent. A possible interpretation of this game lies in the following. Two companies competing on a market wait for a favorable order or conduct research work to improve their products; the earlier a company succeeds, the more profits it makes.

And so, let us analyze the following game. Players I and II randomly receive objects ordered from 1 to N. Each player has a specific set of objects, and all $N!$ permutations appear equiprobable. Players make their choice at some time moments, and the chosen objects are compared. The payoff of a player equals 1, if he stops on the best object earlier than the opponent. We adopt the same notation as in Section 7.6. Designate by a_n, a'_n and y_n, y'_n the absolute and relative ranks for players I and II, respectively. Consequently, the payoff functions in this game acquire the form

$$\begin{aligned} H_1(\tau,\sigma) &= E\{I_{\{a_\tau=1,a'_\sigma\neq 1\}} + I_{\{a_\tau=1,a'_\sigma=1,\tau<\sigma\}}\} \\ &= P\{a_\tau = 1, a'_\sigma \neq 1\} + P\{a_\tau = 1, a'_\sigma = 1, \tau < \sigma\}, \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} H_2(\tau,\sigma) &= E\{I_{\{a_\tau\neq 1,a'_\sigma=1\}} + I_{\{a_\tau=1,a'_\sigma=1,\tau>\sigma\}}\} \\ &= P\{a_\tau \neq 1, a'_\sigma = 1\} + P\{a_\tau = 1, a'_\sigma = 1, \tau < \sigma\}. \end{aligned} \tag{6.2}$$

So long as the game enjoys symmetry, it suffices to find the optimal strategy of one player. For instance, we select player *I*.

Fix a certain strategy σ of player *II* and evaluate the best response of the opponent. Recall the scheme involved in the preceding section. Notably, take a random sequence

$$x_n = E\{I_{\{a_n=1,a'_\sigma\neq 1\}} + I_{\{a_n=1,a'_\sigma=1,n<\sigma\}}/y_n\},\ n = 1, 2, \dots, N.$$

Using similar arguments, one can show that

$$E\{x_\tau\} = H_1(\tau,\sigma).$$

Therefore, the optimal response problem of player *I* represents the optimal stopping problem of the random sequence x_n, $n = 1, \dots, N$. The independence of the random variables a_n^1, a_n^2 implies that x_n can be reexpressed by

$$\begin{aligned} x_n &= P\{a_n = 1/y_n\}\left[P\{a'_\sigma \neq 1\} + P\{a'_\sigma = 1, n < \sigma\}\right] \\ &= P\{a_n = 1/y_n\}\left[1 - P\{a'_\sigma = 1\} + P\{a'_\sigma = 1, n < \sigma\}\right]. \end{aligned}$$

Hence,

$$x_n = P\{a_n = 1/y_n\}\left[1 - P\{a'_\sigma = 1, \sigma \le n\}\right],\ n = 1, \dots, N.$$

By virtue of Lemma 7.4, we arrive at the representation

$$x_n = \frac{n}{N} I_{\{y_n=1\}}\left[1 - P\{a'_\sigma = 1, \sigma \le n\}\right],\ n = 1, \dots, N. \tag{6.3}$$

Formula (6.3) specifies the payoff of player *I* under stopping on object n. Clearly, to establish the best response of player *I*, one should follow up only the appearance of candidates in the sequence. Again, we address the backward induction method.

Seek for the optimal strategies among threshold strategies $\tau(r)$ dictating to terminate observations as soon as the sequence x_n enters the set $\{r, r+1, \dots, N\}$. Suppose that player *II* chooses the threshold strategy $\sigma(r)$. Then formula (6.3) immediately implies that, for $n \le r - 1$, the payoff under stopping equals

$$x_n = \frac{n}{N} I_{\{y_n=1\}},\ n = 1, \dots, r - 1.$$

In the case of $n \geq r$, we compute the probability $P\{a'_\sigma = 1, \sigma \leq n\}$:

$$P\{a'_\sigma = 1, \sigma \leq n\} = \sum_{i=r}^{n} P\{a'_j = 1, \sigma = j\} = \sum_{i=r}^{n} P\{\sigma = j\} P\{a'_j = 1/y'_j = 1\}.$$

According to Lemma 7.5, we have $P\{\sigma = j\} = \frac{r-1}{j(j-1)}$; and Lemma 7.4 yields $P\{a'_j = 1/y'_j = 1\} = j/N$. Therefore,

$$P\{a'_\sigma = 1, \sigma \leq n\} = \sum_{i=r}^{n} \frac{r-1}{j(j-1)} \frac{j}{N} = \frac{r-1}{N} \sum_{j=r}^{n} \frac{1}{j-1}.$$

Using (6.3), readers easily find that

$$x_n = \frac{n}{N} I_{\{y_n = 1\}} \left[1 - \frac{r-1}{N} \sum_{j=r}^{n} \frac{1}{j-1} \right], \ n = r, \ldots, N. \tag{6.4}$$

The payoff under stopping in state n has been successfully established. We can proceed and define the optimal stopping rule. This will employ the optimal expected payoff function $v_n, n = 1, \ldots., N$ and the backward induction method. Assume that incoming object n is the best among all previous ones.

Consider the end time moment $n = N$. Here the player's payoff equals x_N. Due to (6.4), this is the quantity

$$x_N = \left[1 - \frac{r-1}{N} \sum_{j=r}^{N} \frac{1}{j-1} \right].$$

At the last shot, a player must stop, since his payoff under continuation makes up 0. Set $v_N = x_N$.

At shot $n = N - 1$, the player's payoff under stopping is given by

$$x_{N-1} = \frac{N-1}{N} \left[1 - \frac{r-1}{N} \sum_{j=r}^{N-1} \frac{1}{j-1} \right].$$

If he continues, the expected payoff becomes

$$Ex_N = \frac{1}{N} \left[1 - \frac{r-1}{N} \sum_{j=r}^{N} \frac{1}{j-1} \right].$$

By comparing these expressions, we find the optimal stopping rule:

$$\begin{aligned} v_{N-1} &= \max\{x_{N-1}, Ex_N\} \\ &= \max\left\{ \frac{N-1}{N} \left[1 - \frac{r-1}{N} \sum_{j=r}^{N-1} \frac{1}{j-1} \right], \frac{1}{N} \left[1 - \frac{r-1}{N} \sum_{j=r}^{N} \frac{1}{j-1} \right] \right\}. \end{aligned}$$

Repeat these considerations accordingly; using the transition rate formulas (5.1), at shot n we get the equation

$$v_n = \max\{x_n, Ex_{n+1}\} = \max\left\{x_n, \sum_{i=n+1}^{N} \frac{n}{i(i-1)} x_i\right\}, n = N, \ldots 1.$$

Calculate the expected payoff under continuation by one shot.

For $r \leq n \leq N-1$, we have the representation

$$\begin{aligned} Ex_{n+1} &= \sum_{i=n+1}^{N} \frac{n}{i(i-1)} \frac{i}{N} \left[1 - \frac{r-1}{N} \sum_{j=r}^{i} \frac{1}{j-1}\right] \\ &= \frac{n}{N} \sum_{i=n}^{N-1} \frac{1}{i} - \frac{n(r-1)}{N^2} \sum_{i=n}^{N-1} \sum_{j=r-1}^{i} \frac{1}{j}. \end{aligned} \tag{6.5}$$

In the case of $1 \leq n \leq r-1$, the following relationship holds true:

$$\begin{aligned} Ex_{n+1} &= \sum_{i=n+1}^{r-1} \frac{n}{i(i-1)} \frac{i}{N} + \sum_{i=r}^{N} \frac{n}{i(i-1)} \frac{i}{N} \left[1 - \frac{r-1}{N} \sum_{j=r}^{i} \frac{1}{j-1}\right] \\ &= \frac{n}{N} \sum_{i=n}^{N-1} \frac{1}{i} - \frac{n(r-1)}{N^2} \sum_{i=r-1}^{N-1} \sum_{j=r-1}^{i} \frac{1}{j}. \end{aligned} \tag{6.6}$$

Figure 7.7 demonstrates the curves of the functions $y = x_n$ and $y = Ex_{n+1}$, $n = 0, \ldots, N$ under $N = 10$ and $r = 4$.

We have mentioned that, owing to the problem's symmetry, the optimal strategies of the players do coincide. Therefore, choose r such that

$$x_{r-1} < Ex_r, \;\; x_r \geq Ex_{r+1}.$$

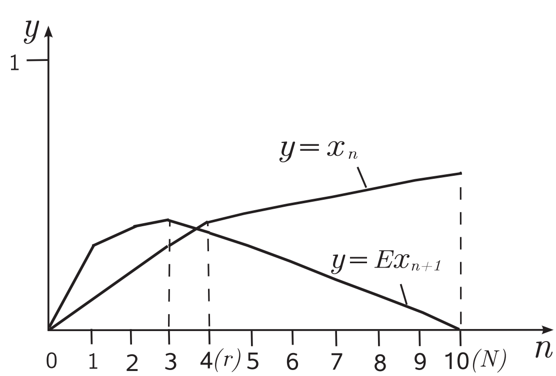


Figure 7.7 The functions x_n and Ex_{n+1}.

After simplifications, formulas (6.4)–(6.6) imply that r satisfies the inequalities

$$1 < \sum_{i=r-1}^{N-1} \frac{1}{i}\left(1 - \frac{r-1}{N}\sum_{j=r-1}^{i}\frac{1}{j}\right) \leq 1 + \frac{N-r}{N(r-1)}. \tag{6.7}$$

Theorem 7.5 *Consider the fastest best choice game. The equilibrium strategy profile is achieved among threshold strategies* $(\tau(r), \sigma(r))$*, where r meets the conditions (6.7).*

Proof: Suppose that player *II* adheres to the strategy $\sigma(r)$, where r agrees with (6.7). Below we demonstrate that the best response of player *I* represents the strategy $\tau(r)$ with the same threshold r. In fact, it suffices to show that

$$x_n < Ex_{n+1},\ n = 1, \dots, r-1,$$
$$x_n \geq Ex_{n+1}\ n = r, \dots, N.$$

According to (6.6), for $n = 1, \dots, r-1$ we have

$$Ex_{n+1} - x_n = \frac{n}{N}\left[\sum_{i=n}^{r-2}\frac{1}{i} + \sum_{i=r-1}^{N-1}\frac{1}{i}\left(1 - \frac{r-1}{N}\sum_{j=r-1}^{i}\frac{1}{j}\right) - 1\right].$$

This expression is strictly positive by virtue of the condition (6.7).

In the case of $n = r, \dots, N$, formulas (6.4), (6.5) lead to

$$Ex_{n+1} - x_n = \frac{n}{N}\left[\sum_{i=n}^{N-1}\frac{1}{i}\left(1 - \frac{r-1}{N}\sum_{j=r-1}^{i}\frac{1}{j}\right)\right.$$
$$\left. -1 + \frac{r-1}{N}\sum_{i=r-1}^{n-1}\frac{i}{i}\right] = \frac{n}{N}G(n).$$

Due to the second condition in (6.7), the bracketed expression $G(n)$ appears non-positive in the point $n = r$. This property remains in force in the rest points $n = r+1, \dots, N-1$, since the function $G(n)$ is non-increasing in n. Really, this follows from

$$G(n+1) - G(n) = -\frac{1}{n}\left[1 - \frac{r-1}{N}\sum_{i=r-1}^{n}\frac{1}{i} - \frac{r-1}{N}\right]$$

and non-negativity of the expression

$$1 - \frac{r-1}{N}\sum_{i=r-1}^{n}\frac{1}{i} - \frac{r-1}{N}, n = r, \dots, N-1.$$

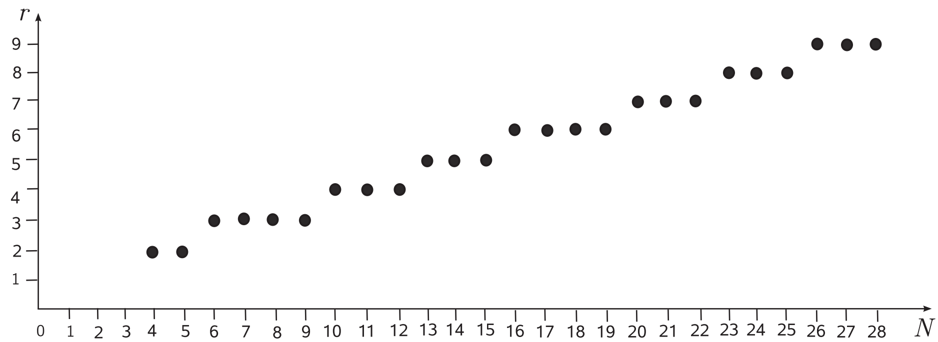


Figure 7.8 The optimal thresholds.

The last fact is immediate from the inequalities

$$1 - \frac{r-1}{N}\sum_{i=r-1}^{n}\frac{1}{i} - \frac{r-1}{N} \geq 1 - \frac{r-1}{N}\sum_{i=r-1}^{N-1}\frac{1}{i} - \frac{r-1}{N}$$
$$\geq \frac{r-1}{N}\left(\frac{N-r+1}{r-1} - \sum_{i=r-1}^{N-1}\frac{1}{i}\right) \geq 0.$$

The proof of Theorem 7.5 is finished.

Figure 7.8 shows the optimal thresholds in the fastest best choice problem under different values of N.

Finally, we explore the asymptotical setting of this game as $N \to \infty$. Imagine that the ratio r/N tends to some limit $z \in [0, 1]$. Under large N, the conditions (6.7) get reduced to the equation

$$-\ln z - z\ln^2\left(\frac{z}{2}\right) = 1.$$

Its solution $z^* \approx 0.295$ yields the asymptotically optimal value of r/N. In contrast to the solution of the previous problem (0.368), a player should stop earlier. As a result, errors grow appreciably—this is the cost of taking the lead over the opponent.

7.7 Best choice game with rank criterion. Lottery

Lemma 7.6 *Assume that y is the relative rank of the candidate at shot n. Then the expected value of its absolute rank makes up*

$$E\{a_n/y_n = y\} = Q(n, y) = \frac{N+1}{n+1}y.$$

Proof: Let the relative rank of candidate n be equal to y. Find the probability $P\{a_n = r|\, y_n = y\}$ (the absolute rank of this candidate is r, where $r = y, y+1, \ldots, N-n+y$). Consider the event that, after choice of n objects, the last object with the relative rank of y possesses the absolute rank of r; actually, this event is equivalent to the following. While choosing

n objects $k_1, \dots, k_{y-1}, k_y, \dots, k_n$ from N objects $1, 2, \dots, r-1, r, \dots, N$, one chooses objects $k_1, \dots, k_{y-1}$ from objects $1, \dots, r-1$ and objects $k_y, \dots, k_n$ from objects $r, \dots, N$. And the desired probability is defined by

$$P\{a_n = r | \, y_n = y\} = \frac{\binom{r-1}{y-1}\binom{N-r}{n-y}}{\binom{N}{n}}, \quad r = y, y+1, \dots, N-n+y. \tag{7.1}$$

Formula (7.1) specifies the negative hypergeometric distribution. Now, we evaluate the expected absolute rank of candidate n provided that its relative rank constitutes y. Notably,

$$Q(n,y) \equiv \sum_{r=y}^{N-(n-y)} r \binom{r-1}{y-1} \binom{N-r}{n-y} \Big/ \binom{N}{n}$$
$$= \frac{N+1}{n+1} y \sum_{r=y}^{N-(n-y)} \binom{r}{y} \binom{N-r}{n-y} \Big/ \binom{N+1}{n+1} = \frac{N+1}{n+1} y.$$

This concludes the proof of Lemma 7.6.

And so, as candidate n appears, the players observe the relative ranks $(y_n, z_n) = (y, z)$. If both players choose $(R-R)$, candidate n is rejected. Subsequently, the players interview candidate $(n+1)$ and pass to state y_{n+1}, z_{n+1}. However, if the players choose $(A-A)$, the game ends with the payoffs $\frac{N+1}{n+1} y$ (player I) and $\frac{N+1}{n+1} z$ (player II). In the case of different choices, lottery selects the decision of player I (or player II) with the probability p (the probability $1-p$, respectively). At the last shot, the last candidate is accepted anyway.

Define state (n, y, z), where

1. first $n-1$ candidates are rejected and players are shown candidate n,
2. the relative ranks of the current candidates equal $y_n = y$ and $z_n = z$.

Denote by u_n, v_n the optimal expected payoffs of the players at shot n, when first n candidates are rejected. Apply the backward induction method and write down the optimality equation:

$$(u_{n-1}, v_{n-1}) = n^{-2} \sum_{y,z=1}^{n} Val\, M_n(y,z). \tag{7.2}$$

Here $Val\, M_n(y,z)$ represents the value of the game with the matrix $M_n(y,z)$ defined by

$$\begin{array}{c|c|c|} & R & A \\ \hline R & u_n, v_n & \bar{p}Q(n,y) + pu_n, \bar{p}Q(n,z) + pv_n \\ \hline A & pQ(n,y) + \bar{p}u_n, pQ(n,z) + \bar{p}v_n & Q(n,y), Q(n,z) \\ \hline \end{array} \tag{7.3}$$

$$\left(n = 1, 2, \dots, N-1; u_{N-1} = v_{N-1} = \frac{1}{N} \sum_{y=1}^{N} y = \frac{N+1}{2} \right).$$

Without loss of generality, suppose that $1/2 \le p \le 1$.

Theorem 7.6 *The optimal strategies of the players in the game with the matrix (7.3) have the following form:*

player I chooses $A(R)$, if $Q(n, y) \le (>) u_n$ (regardless of z),
player II chooses $A(R)$, if $Q(n, z) \le (>) v_n$ (regardless of y).

The quantities u_n, v_n satisfy the recurrent equations

$$u_{n-1} = pE[Q(n, y_n) \wedge u_n] + \bar{p}E\left[\frac{N+1}{2} I\{Q(n, z_n) \le v_n\} + u_n I\{Q(n, z_n) > v_n\}\right], \quad (7.4)$$

$$v_{n-1} = \bar{p}E[Q(n, z_n) \wedge v_n] + pE\left[\frac{N+1}{2} I\{Q(n, y_n) \le u_n\} + v_n I\{Q(n, y_n) > u_n\}\right] \quad (7.5)$$

$$\left(n = 1, 2, \ldots, N-1; \quad u_{N-1} = v_{N-1} = \frac{N+1}{2}\right),$$

where $I\{C\}$ means the indicator of the event C. The optimal payoffs in the game $\Gamma_N(p)$ are $U^n = u_0$ and $V^n = v_0$.

Proof: Obviously, for any $(y, z) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$, the bimatrix game (7.3) admits the pure strategy equilibrium defined by

	$Q(n, z) > v_n$	$Q(n, z) \le v_n$
$Q(n, y) > u_n$	R-R u, v	R-A $\bar{p}Q(n, y) + pu, \bar{p}Q(n, z) + pv$
$Q(n, y) \le u_n$	A-R $pQ(n, y) + \bar{p}u, pQ(n, z) + \bar{p}v$	A-A $Q(n, y), Q(n, z)$

(7.6)

In each cell, we have the payoffs of players I and II, where indexes u_n, v_n are omitted for simplicity. Consider component 1 only and sum up all payoffs multiplied by n^{-2}:

$$n^{-2} \sum_{y,z=1}^{n} Q(n, y)[I\{Q(n, y) \le u, Q(n, z) \le v\} + pI\{Q(n, y) \le u, Q(n, z) > v\}$$
$$+\bar{p}I\{Q(n, y) > u, Q(n, z) \le v\}] + n^{-2}u \sum_{y,z=1}^{n} [\bar{p}I\{Q(n, y) \le u, Q(n, z) > v\}$$
$$+pI\{Q(n, y) > u, Q(n, z) \le v\} + I\{Q(n, y) > u, Q(n, z) > v\}]. \quad (7.7)$$

The first sum in (7.7) equals

$$n^{-2} \sum_{y,z=1}^{n} Q(n, y)[pI\{Q(n, y) \le u\} + \bar{p}I\{Q(n, z) \le v\}]$$
$$= pn^{-1} \sum_{y=1}^{n} Q(n, y)I\{Q(n, y) \le u\} + \bar{p}n^{-1} \sum_{z=1}^{n} \frac{N+1}{2} I\{Q(n, z) \le v\}, \quad (7.8)$$

so far as

$$n^{-1}\sum_{y=1}^{n} Q(n,y) = \frac{1}{n}\sum_{y=1}^{n}\frac{N+1}{n+1}y = \frac{N+1}{2}.$$

Consider the second sum in (7.7):

$$\begin{aligned} &n^{-2}u\sum_{y,z=1}^{n}[pI\{Q(n,y)>u\}+\bar{p}I\{Q(n,z)>v\}] \\ &\quad = n^{-1}u\left[p\sum_{y=1}^{n} I\{Q(n,y)>u\}+\bar{p}\sum_{z=1}^{n} I\{Q(n,z)>v\}\right]. \end{aligned} \tag{7.9}$$

By substituting (7.8) and (7.9) into (7.7), we obtain (7.4). Similarly, readers can establish the representation (7.5). The proof of Theorem 7.6 is finished.

Introduce the designation $\bar{y}_n = u_n\frac{n+1}{N+1}$ and $\bar{z}_n = v_n\frac{n+1}{N+1}$ for $n = 0, 1, \dots, N-1$ to reexpress the system (7.4)–(7.5) as

$$\bar{y}_{n-1} = \frac{p}{n+1}\left[\frac{1}{2}[\bar{y}_n]([\bar{y}_n]+1) + \bar{y}_n(n-[\bar{y}_n])\right] + \frac{\bar{p}}{n+1}\left[\frac{1}{2}(n+1)[\bar{z}_n] + \bar{y}_n(n-[\bar{z}_n])\right]$$

$$\bar{z}_{n-1} = \frac{\bar{p}}{n+1}\left[\frac{1}{2}[\bar{z}_n]([\bar{z}_n]+1) + \bar{z}_n(n-[\bar{z}_n])\right] + \frac{p}{n+1}\left[\frac{1}{2}(n+1)[\bar{y}_n] + \bar{z}_n(n-[\bar{y}_n])\right].$$

Here $[y]$ indicates the integer part of y, and $\bar{y}_{N-1} = \bar{z}_{N-1} = N/2$. Note that states with candidate acceptance, i.e.,

$$Q(n,y) = \frac{N+1}{n+1}y \le u_n,\ \ Q(n,z) = \frac{N+1}{n+1}z \le v_n$$

acquire the following form:

$$y \le \bar{y}_n,\ \ z \le \bar{z}_n.$$

In other words, $\bar{y}_n, \bar{z}_n$ represent the optimal thresholds for accepting candidates with the given relative ranks.

Under $p = 1/2$, the values of $\bar{y}_n$ and $\bar{z}_n$ do coincide. Denote these values by x_n; they meet the recurrent expressions

$$x_{n-1} = x_n + \frac{[x_n]}{4} - \frac{1}{n+1}([x_n]+1)(x_n - [x_n]/4), \tag{7.10}$$

$n = 1, \dots, N-1$. We investigate their behavior for large N.

Theorem 7.7 *Under $N \geq 10$, the following inequalities hold true:*

$$\frac{n+1}{3} \leq x_n \leq \frac{n}{2} \quad (n = 5, \dots, N-2). \tag{7.11}$$

Proof: It follows from (7.10) that

$$x_{N-2} = N/2 + \frac{[N/2]}{4} - \frac{1}{N}([N/2]+1)(N/2 - [N/2]/4).$$

Clearly, $x_{N-2} \geq (N-1)/3, \forall N$ and $x_{N-2} \leq (N-2)/2$ provided that $N \geq 10$. Therefore, formula (7.11) remains in force for $n = N-2$. Suppose that these conditions take place for $6 \leq n \leq N-2$. Below we demonstrate their validity for $n-1$.

Introduce the operator

$$T_x(s) = x + \frac{s}{4} - \frac{1}{n+1}(s+1)(x - s/4).$$

Then for $x - 1 \leq s \leq x$ we have

$$\begin{aligned} T'_x(s) &= \frac{1}{4(n+1)}(2s - 4x + 2 + n) \\ &\geq \frac{1}{4(n+1)}(2(x-1) - 4x + 2 + n) = \frac{1}{4(n+1)}(-2x + n). \end{aligned}$$

So long as $x_n \leq n/2$, the inequality $T'_{x_n}(s) \geq 0$ is the case for $x_n - 1 \leq s \leq x_n$. Hence,

$$\begin{aligned} x_{n-1} &= T_{x_n}([x_n]) \leq T_{x_n}(x_n) = \frac{1}{4}x_n\left(5 - \frac{3}{n+1}(x_n+1)\right) \\ &\leq \frac{n}{8}\left(5 - \frac{3(n+2)}{2(n+1)}\right) \leq \frac{n-1}{2}, \quad \text{for} \quad n \geq 6. \end{aligned}$$

Moreover, owing to $x_n \geq (n+1)/3$, we obtain

$$\begin{aligned} x_{n-1} &= T_{x_n}([x_n]) \geq T_{x_n}(x_n - 1) = \frac{5x_n - 1}{4} - \frac{x_n(3x_n+1)}{4(n+1)} \\ &\geq \frac{5(n+1) - 3}{12} - \frac{n+2}{12} \geq n/3. \end{aligned}$$

Now, it is possible to estimate x_0.

Corollary 7.1 $0.387 \leq x_0 \leq 0.404$.

Theorem 7.2 implies that $2 \leq x_5 \leq 2.5$. Using (7.10), we find successively $1.75 \leq x_4 \leq 2$, $1.4 \leq x_3 \leq 1.6$, $1.075 \leq x_2 \leq 1.175$, $0.775 \leq x_1 \leq 0.808$, and, finally, $0.387 \leq x_0 \leq 0.404$.

The following fact is well-known in the theory of optimal stopping. In the non-game setting of the problem ($p = 1$), there exists the limit value of $U^{(n)}$ as $n \to \infty$. It equals

Table 7.2 The limit values $y_0 = U^{(N)}/N, z_0 = V^{(N)}/N$ under different values of p.

p	0.5	0.6	0.7	0.8	0.9	1
y_0	0.390	0.368	0.337	0.305	0.249	0
z_0	0.390	0.407	0.397	0.422	0.438	0.5

$\prod_{j=1}^{\infty}(1+\frac{2}{j})^{\frac{1}{j+1}} \approx 3.8695$. Therefore, a priority player guarantees a secretary whose mean rank is smaller than 4. Moreover, for player 2 this secretary possesses the rank of 50% among all candidates.

In the case of $p < 1$, such limit value does not exist. For instance, if $p = 1/2$, the corollary of Theorem 7.7 claims that $x_0 \geq 0.387$. And it appears that $U^{(N)} = u_0 = x_0(N+1) \geq 0.387(N+1) \to \infty$ as $N \to \infty$. Table 7.2 combines the limit values $y_0 = \lim_{n\to\infty}\{U^{(n)}/n\}$ and $z_0 = \lim_{n\to\infty}\{V^{(n)}/n\}$ under different values of p.

Concluding this section, we compare the optimal payoffs with those ensured by other simple rules. If both players accept the first candidate, their payoffs (expected ranks) do coincide: $U^N = V^N = \sum_{y=1}^{N} y = \frac{N+1}{2}$. However, random strategies (i.e., choosing between (A) and (R) with the same probability of $1/2$) lead to

$$
\begin{aligned}
u_{n-1} &= E\left[\frac{1}{4}u_n + \frac{1}{4}(\bar{p}Q(n,y) + pu_n) + \frac{1}{4}(pQ(n,y) + \bar{p}u_n) + \frac{1}{4}Q(n,y)\right] \\
&= \frac{1}{2}\left[u_n + EQ(n,y)\right] = \frac{1}{2}\left[u_n + \frac{N+1}{2}\right] \quad (n = 1, 2, \dots, N-1),
\end{aligned}
$$

see Theorem 7.6. Then the condition $u_{N-1} = \frac{N+1}{2}$ implies that $u_0 = \dots = u_{N-1} = \frac{N+1}{2}$. Similarly, $v_0 = \dots = v_{N-1} = \frac{N+1}{2}$. Consequently, the first candidate strategy and the random strategy are equivalent. They both result in the candidate whose rank is the mean rank of all candidates (regardless of the players' priority p). Still, the optimal strategies found above yield appreciably higher payoffs to the players.

7.8 Best choice game with rank criterion. Voting

Consider the best choice game with m participants and final decision by voting. Assume that a commission of m players has to fill a vacancy. There exist N pretenders for this vacancy. For each player, the pretenders are sorted by their absolute ranks (e.g., communicability, language qualifications, PC skills, etc.). A pretender possessing the lowest rank is actually the best one. Pretenders appear in the commission one-by-one randomly such that all $N!$ permutations are equiprobable. During an interview, each player observes the relative rank of a current pretender against preceding pretenders. The relative ranks are independent for different players. A pretender is accepted, if at least k members of the commission agree to accept him (and the game ends). Otherwise, the pretender is rejected and the commission proceeds to the next pretender (the rejected one gets eliminated from further consideration). At shot N, the players have to accept the last pretender. Each player seeks to minimize the absolute rank of the selected pretender. We will find the optimal rule of decision making depending on the voting threshold k.

7.8.1 Solution in the case of three players

To begin, we take the case of $m = 3$ players. Imagine that a commission of three members has to fill a vacancy. There are N pretenders sorted by three absolute ranks. During an interview, each player observes the relative rank of a current pretender against preceding pretenders. Based on this information, he decides whether to accept or reject the pretender. A pretender is accepted, if the majority of the members (here, 2) agree to accept him (and the game ends). The pretender is rejected provided that, at least, two players disagree. And the commission proceeds to the next pretender (the rejected one gets eliminated from further consideration). At shot N, the players have to accept the last pretender.

Denote by x_n, y_n and z_n the relative ranks of the pretender at shot n for player 1, player 2, and player 3, respectively. The sequence $\{(x_n, y_n, z_n)\}_{n=1}^{N}$ composed of independent random variables obeys the distribution law $P\{x_n = x, y_n = y, z_n = z\} = \frac{1}{n^3}$, where x, y, and z take values from 1 to n.

After interviewing a current pretender, the players have to accept or reject him. Pretender n being rejected, the players pass to pretender $n + 1$. If pretender n is accepted, the game ends. In this case, the expected absolute rank for players 1–3 makes up $Q(n, x)$, $Q(n, y)$, and $Q(n, z)$, respectively. We have noticed that

$$Q(n, x) = \frac{N + 1}{n + 1} x.$$

When all pretenders except the last one are rejected, the players must accept the last pretender. Each player strives for minimization of his expected payoff.

Let u_n, v_n, w_n designate the expected payoffs of players 1–3, respectively, provided that n pretenders are skipped.

At shot n, this game can be described by the following matrix; as their strategies, players choose between A ("accept") and R ("reject").

R

	R	A
R	u_n, v_n, w_n	u_n, v_n, w_n
A	u_n, v_n, w_n	$Q(n, x), Q(n, y), Q(n, z)$

A

	R	A
R	u_n, v_n, w_n	$Q(n, x), Q(n, y), Q(n, z)$
A	$Q(n, x), Q(n, y), Q(n, z)$	$Q(n, x), Q(n, y), Q(n, z)$

According to the form of this matrix, strategy A dominates strategy R for players 1, 2, and 3 under $Q(n, x) \le u_n$, $Q(n, y) \le v_n$, and $Q(n, z) \le w_n$, respectively. Therefore, the optimal behavior of player 1 lies in accepting pretender n, if $Q(n, x) \leq u_n$; by analogy, player 2 accepts pretender n if $Q(n, y) \leq v_n$, and player 3 accepts pretender n if $Q(n, z) \leq w_n$. Then

$$\begin{aligned}
u_{n-1} = \frac{1}{n^3} \sum_{x,y,z=1}^{n} & Q(n,x)[I\{Q(n,x) \leq u_n, Q(n,y) \leq v_n, Q(n,z) \leq w_n\} \\
& + I\{Q(n,x) \leq u_n, Q(n,y) \leq v_n, Q(n,z) > w_n\} \\
& + I\{Q(n,x) \leq u_n, Q(n,y) > v_n, Q(n,z) \leq w_n\} \\
& + I\{Q(n,x) > u_n, Q(n,y) \leq v_n, Q(n,z) \leq w_n\}] \\
& + \frac{1}{n^3} u_n \sum_{x,y,z=1}^{n} [I\{Q(n,x) > u_n, Q(n,y) > v_n, Q(n,z) > w_n\} \\
& + I\{Q(n,x) > u_n, Q(n,y) \leq v_n, Q(n,z) > w_n\} \\
& + I\{Q(n,x) \leq u_n, Q(n,y) > v_n, Q(n,z) > w_n\} \\
& + I\{Q(n,x) > u_n, Q(n,y) > v_n, Q(n,z) \leq w_n\}]
\end{aligned}$$

or

$$\begin{aligned}
u_{n-1} = \frac{1}{n^2} \Bigg[& \sum_{x,y=1}^{n} Q(n,x) I\{Q(n,x) \leq u_n, Q(n,y) \leq v_n\} \\
& + \sum_{x,z=1}^{n} Q(n,x) I\{Q(n,x) \leq u_n, Q(n,z) \leq w_n\} \\
& + \sum_{y,z=1}^{n} \frac{N+1}{2} I\{Q(n,y) \leq v_n, Q(n,z) \leq w_n\} \\
& - \frac{2}{n} \sum_{x,y,z=1}^{n} Q(n,x) I\{Q(n,x) \leq u_n, Q(n,y) \leq v_n, Q(n,z) \leq w_n\} \Bigg] \\
& + \frac{1}{n^2} u_n \Bigg[\sum_{x,y=1}^{n} I\{Q(n,x) > u_n, Q(n,y) > v_n\} + \sum_{x,z=1}^{n} I\{Q(n,x) > u_n, Q(n,z) > w_n\} \\
& + \sum_{y,z=1}^{n} I\{Q(n,y) > v_n, Q(n,z) > w_n\} \\
& - \frac{2}{n} \sum_{x,y,z=1}^{n} I\{Q(n,x) > u_n, Q(n,y) > v_n, Q(n,z) > w_n\} \Bigg],
\end{aligned}$$

where $n = 1, 2, \ldots, N-1$ and $u_{N-1} = \frac{1}{N} \sum_{x=1}^{N} x = \frac{N+1}{2}$.

Here $I\{A\}$ is the indicator of the event A.

Owing to the problem's symmetry, $u_n = v_n = w_n$. And so, the optimal thresholds make up $\bar{x}_n = u_n \frac{n+1}{N+1}$.

Consequently,

$$\begin{aligned}
\bar{x}_{n-1} &= u_{n-1}\frac{n}{N+1}\\
&= \frac{1}{n(N+1)}\left[\frac{N+1}{(n+1)}[\bar{x}_n]^2([\bar{x}_n]+1)+\frac{N+1}{2}[\bar{x}_n]^2-\frac{N+1}{n(n+1)}[\bar{x}_n]^3([\bar{x}_n]+1)\right]\\
&\quad+\frac{\bar{x}_n}{n(n+1)}\left[3(n-[\bar{x}_n])^2-\frac{2}{n}(n-[\bar{x}_n])^3\right],
\end{aligned}$$

where $\bar{x}_{N-1}=\frac{N}{2}$, $[x]$ means the integer part of x.

Certain transformations lead to

$$\bar{x}_{n-1}=\frac{1}{2n^2(n+1)}[[\bar{x}_n]^2(2([\bar{x}_n]+1)(n-[\bar{x}_n])+n(n+1))+2\bar{x}_n(n+2[\bar{x}_n])(n-[\bar{x}_n])^2].$$

By substituting $N=100$ into this formula, we obtain the optimal expected rank of 33. Compare this quantity with the expected rank of 39.425 in the problem with two players ($p=1/2$) and with the optimal rank of 3.869 in the non-game problem. Obviously, the voting procedure ensures a better result than the equiprobable scheme involving two players.

Theorem 7.8 *Under $N \geq 19$, the optimal payoff in the best choice game with voting is higher than in the game with an arbitrator.*

Proof: It is required to demonstrate that, for $N \geq 19$, the inequality $\frac{n+2}{4} < \bar{x}_n < \frac{n-1}{2}$ holds true for $14 \leq n \leq N-2$. Apply the backward induction method.

In the case of $N \geq 19$, we have $\frac{N}{4} < \bar{x}_{N-2} < \frac{N-3}{2}$. Suppose that the inequality takes place for $15 \leq n \leq N-2$. Prove its validity under $n-1$, i.e., $\frac{n+1}{4} < \bar{x}_{n-1} < \frac{n-2}{2}$ for $15 \leq n \leq N-2$. Introduce the operator

$$T(x,y)=\frac{1}{2n^2(n+1)}[y^2(2(y+1)(n-y)+n(n+1))+2x(n+2y)(n-y)^2],$$

where $x-1<y\leq x$.

Find the first derivative:

$$\begin{aligned}
T'_y(x,y) &= \frac{1}{n^2(n+1)}(-4y^3+3y^2(n-1+2x)+y(n^2+3n-6xn))\\
&= \frac{y(n-y)(3+n-6x+4y)}{n^2(n+1)}.
\end{aligned}$$

So far as $x-1<y\le x$ and $\frac{n+2}{4}<x<\frac{n-1}{2}$, we obtain $T'_y(x,y)>0$. Then the function $T(x,y)$ increases. Hence, owing to $\bar{x}_n<\frac{n-1}{2}$,

$$\begin{aligned}\bar{x}_{n-1} &= T(\bar{x}_n,[\bar{x}_n])<T(\bar{x}_n,\bar{x}_n)\\ &= \frac{1}{2n^2(n+1)}\left(-2\bar{x}_n^4+2\bar{x}_n^3(n-1+2\bar{x}_n)+\bar{x}_n^2(n^2+3n-6\bar{x}_n n)+2\bar{x}_n n^3\right)\\ &\quad<\frac{(7n^2-3)(n-1)}{16n^2}<\frac{n-2}{2}\ \text{for } n\ge 9.\end{aligned}$$

Similarly, so long as $\bar{x}_n>\frac{n+2}{4}$, the following inequality holds true:

$$\begin{aligned}\bar{x}_{n-1} &= T(\bar{x}_n,[\bar{x}_n])>T(\bar{x}_n,\bar{x}_n-1)\\ &= \frac{1}{2n^2(n+1)}\left(2\bar{x}_n^4-2\bar{x}_n^3(2n+3)+\bar{x}_n^2(n^2+9n+6)+2\bar{x}_n(n^3-n^2-3n-1)+n^2+n\right)\\ &> \frac{65n^4+116n^3+32n^2-16n-16)}{256n^2(n+1)}>\frac{n+1}{4}\ \text{for } n\ge 19.\end{aligned}$$

By taking into account the inequality $\frac{n+2}{4}<\bar{x}_n<\frac{n-1}{2}$ for $14\le n\le N-2$ and $N\ge 19$, we get

$$\begin{gathered}4<\bar{x}_{14}<6.5,\ 3.837<\bar{x}_{13}<5.650,\ 3.580<\bar{x}_{12}<4.984,\dots,\\ 1.838<\bar{x}_5<2.029,\ 1.526<\bar{x}_4<1.736,\ 1.230<\bar{x}_3<1.372,\\ 0.961<\bar{x}_2<1.040,\ 0.641<\bar{x}_1<0.763,\ 0.320<\bar{x}_0<0.382.\end{gathered}$$

Recall that $0.387\le x_0\le 0.404$ in the problem with two players. Thus, the derived thresholds in the case of three players are smaller than in the case of two players. Voting with three players guarantees a better result than fair lottery.

7.8.2 Solution in the case of *m* players

This subsection concentrates on the scenario with m players. Designate by x_n^j $(j=1,\dots,m)$ the relative rank of pretender n for player j. Then the vector $\{(x_n^1,\dots,x_n^m)\}_{i=1}^N$ possesses the distribution $P\{x_n^1=x^1,\dots,x_n^m=x^m\}=\frac{1}{n^m}$ for $x^l=1,\dots,n$, where $l=1,\dots,m$.

A current pretender is accepted, if at least k members of the commission agree, $k=1,\dots,m$. If after the interview pretender n is accepted, the game ends. In this case, the expected value of the absolute rank for player j makes up the quantity

$$Q(n,x^j)=\frac{N+1}{n+1}x^j, j=1,\dots,m.$$

Table 7.3 The optimal expected absolute ranks.

k	1	2	3	4	5	k^*
$m = 1$ u_0	3.603					1
$m = 3$ u_0	47.815	33.002	19.912			3
$m = 4$ u_0	49.275	44.967	26.335	27.317		3
$m = 5$ u_0	49.919	47.478	40.868	26.076	33.429	4

Let u_n^j, $j = 1, \ldots, m$ indicate the expected payoff of player j provided that n pretenders are skipped. As above, the optimal strategy of player j consists in accepting pretender n if $Q(n, x^j) \le u_n^j$. Then

$$u_{n-1}^j = \frac{1}{n^m}\left[\sum_{x^1,x^2,\ldots,x^m=1}^{n} Q(n,x^j)[J_m + J_{m-1} + \ldots + J_{k+1} + J_k] + u_n^j \sum_{x^1,x^2,\ldots,x^m=1}^{n} [J_{k-1} + J_{k-2} + \ldots + J_0]\right],$$

where J_l gives the number of all events when the pretender has been accepted by l players exactly, $l = 0, 1, \ldots, m$.

Problem's symmetry dictates that $u_n^1 = u_n^2 = \ldots = u_n^m = u_n$. We set $x_n = \frac{n+1}{N+1} u_n$.

The optimal strategies acquire the form

$$x_{n-1} = \frac{1}{2n^{m-1}(n+1)} \sum_{j=1}^{m-k} \left[\left(\binom{m}{j}([x_n] + 1 + n) - \binom{m-1}{j} n\right)[x_n]^{m-j}(n - [x_n])^j\right]$$
$$+ [x_n]^m([x_n] + 1) + \frac{x_n}{n^{m-1}(n+1)} \sum_{j=1}^{k-1} \left[\binom{m}{j} [x_n]^j \left(n - [x_n]\right)^{m-j}\right] + (n - [x_n])^m;$$
$$u_n = x_n \frac{N+1}{n+1};$$
$$x_{N-1} = \frac{N}{2};$$

where $n = 1, \ldots, N-1$, and $[x]$ corresponds to the integer part of x.

Table 7.3 provides some numerical results for different m and k under $N = 100$.

Clearly, the best result k^* is achieved by the commission of three members. Interestingly, decision making by simple majority appears **insufficient in small commissions**.

7.9 Best mutual choice game

The previous sections have studied best choice games with decision making by just one side. However, a series of problems include mutual choice. For instance, such problems arise in biology and sociology (mate choice problems), in economics (modeling of market relations between buyers and sellers) and other fields.

Let us imagine the following situation. There is a certain population of male and female individuals. Individuals choose each other depending on some quality index. Each individual seeks to maximize the mate's quality. It may happen that one mate accepts another, whereas the latter disagrees. Thus, the choice rule must concern both mates.

Suppose that the populations of both genders have identical sizes and their quality levels are uniformly distributed on the interval $[0, 1]$. Denote by x and y the quality level of females and males, respectively; accordingly, their populations are designated by X and Y. Choose randomly two individuals of non-identical genders. This pair (x, y) is called the state of the game. Each player has some threshold for the mate's quality level (he/she does not agree to pair with a mate whose quality level is below the threshold). If, at least, one mate disagrees, the pair takes no place and both mates return to their populations. If they both agree, the pair is formed and the mates leave their populations.

Consider a multi-shot game, which models random meetings of all individuals from these populations. After each shot, the number of individuals with high quality levels decreases, since they form pairs and leave their populations. Players have less opportunities to find mates with sufficiently high quality levels. Hence, the demands of the remaining players (their thresholds) must be reduced with each shot. Our analysis begins with the game of two shots.

7.9.1 The two-shot model of mutual choice

Consider the following situation. At shot 1, all players from the populations meet each other; if there are happy pairs, they leave the game. At shot 2, the remaining players again meet each other randomly and form the pair regardless of the quality levels. Establish the optimal behavior of the players.

Assume that each player can receive the observations x_1 and x_2 (y_1 and y_2, respectively) and adopts the threshold rule $z : 0 \le z \le 1$. Due to the problem's symmetry, we apply the same rule to both genders. If a current mate possesses a smaller quality level than z, the mate is rejected, and the players proceed to the next shot. If the quality levels of both appears greater or equal to z, the pair is formed, and the players leave the game. Imagine that at shot 1 same-gender players have the uniform distribution on the segment $[0, 1]$; after shot 1, this distribution changes, since some players with quality levels higher than z leave the population (see Figure 7.9). For instance, we find the above distribution for x.

In the beginning of the game, the power of the player set X makes up 1. After shot 1, the remaining players are the ones whose quality levels belong to $[0, z)$ and the share $(1 - z)z$ of the players whose quality levels are between z and 1. Those mates whose quality levels exceed z (they are z^2 totally) leave the game. Therefore, just $z + (1 - z)z$ players of the same gender continue the game at shot 2. And the density function of players' distribution by their quality levels acquires the following form (see Figure 7.10)

$$f(x) = \begin{cases} \dfrac{1}{z + (1 - z)z}, & x \in [0, z), \\ \dfrac{z}{z + (1 - z)z}, & x \in [z, 1]. \end{cases}$$

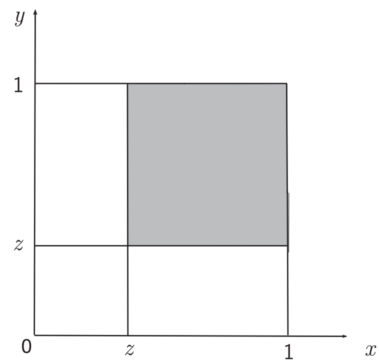


Figure 7.9 The two-short model.

Hence, if some player fails to find an appropriate pair at shot 1, he/she obtains the mean quality level of all opposite-gender mates at shot 2, i.e.,

$$Ex_2 = \int_0^1 xf(x)dx = \int_0^z \frac{x}{z + (1-z)z} dx + \int_z^1 \frac{zx}{z + (1-z)z} dx.$$

By performing integration, we arrive at the formula

$$Ex_2 = \frac{1 + z - z^2}{2(2-z)}.$$

Get back to shot 1. A player with a quality level y decides to choose a mate with a quality level x (and vice versa), if the quality level x appears greater or equal to the mean quality level Ex_2 at the next shot. Therefore, the optimal threshold for mate choice at shot 1 obeys the equation

$$z = \frac{1 + z - z^2}{2(2-z)}.$$

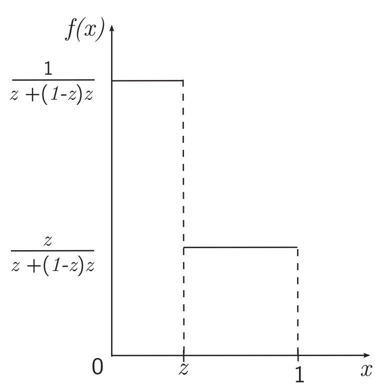


Figure 7.10 The density function.

Again, its solution $z = (3 - \sqrt{5})/2 \approx 0.382$ has close connection with the golden section ($z = 1 - z^*$, where z^* represents the golden section).

7.9.2 The multi-shot model of mutual choice

Now, suppose that players have $n + 1$ shots for making pairs. Let player *II* adhere to a threshold strategy with the thresholds $z_1, \dots, z_n$, where $0 < z_n \le z_{n-1} \le \dots \le z_1 \le z_0 = 1$. Evaluate the best response of player *I* and require that it coincides with the above threshold strategy. For this, we analyze possible variations of the players' distribution by their quality levels after each shot.

Initially, this distribution is uniform. Assume that the power of the player set equals $N_0 = 1$. After shot 1, players with quality levels higher than z_1 can create pairs and leave the game. Therefore, as shot 1 is finished, the mean power of the player set becomes $N_1 = z_1 + (1 - z_1)z_1$. This number can be rewritten as $N_1 = 2z_1 - z_1^2/N_0$. After shot 2, players whose quality levels exceed z_2 can find pairs and leave the game. And the mean power of the player set after shot 2 is given by $N_2 = z_2 + (z_1 - z_2)z_2/N_1 + (1 - z_1)z_1 z_2/(N_1 N_0)$. This quantity can be reexpressed as $N_2 = 2z_2 - z_2^2/N_1$. Apply such reasoning further to find that, after shot i, the number of remaining players is

$$N_i = z_i + \sum_{j=1}^{i-1}(z_j - z_{j+1}) \prod_{k=j}^{i-1} \frac{z_{k+1}}{N_k}, i = 1, \dots, n.$$

For convenience, rewrite the above formula in the recurrent form:

$$N_i = 2z_i - \frac{z_i^2}{N_{i-1}}, \ i = 1, \dots, n. \tag{9.1}$$

After each shot, the distribution of players by the quality levels has the following density function:

$$f_1(x) = \begin{cases} 1/N_1, & 0 \le x < z_1, \\ z_1/N_1, & z_1 \le x \le 1 \end{cases}$$

(after shot 1),

$$f_2(x) = \begin{cases} \dfrac{1}{N_2}, & 0 \le x < z_2, \\[2mm] \dfrac{z_2}{N_1 N_2}, & z_2 \le x < z_1, \\[2mm] \dfrac{z_1 z_2}{N_0 N_1 N_2}, & z_1 \le x \le 1 \end{cases}$$

(after shot 2), and finally,

$$f_i(x) = \begin{cases} \dfrac{1}{N_i}, & 0 \le x < z_i, \\ \prod_{j=k}^{i-1} \dfrac{z_{j+1}}{N_j} \dfrac{1}{N_i}, & z_{k+1} \le x < z_k, \ k = i-1, \dots, 1 \end{cases}$$

(after shot i, $i = 1, \dots, n$).

Now, address the backward induction method and consider the optimality equation. Denote by $v_i(x)$, $i = 1, \dots, n$, the optimal expected payoff of the player after shot i provided that he/she deals with a mate of the quality level x.

Suppose that, at shot n, the player observes a mate of the quality level x. If the player continues, he/she expects the quality level Ex_{n+1}, where x_{n+1} obeys the distribution $f_n(x)$. Hence, it appears that

$$v_n(x) = \max \left\{ x, \int_0^1 y f_n(y) dy \right\},$$

or

$$v_n(x) = \max \left\{ x, \int_0^{z_n} \frac{y}{N_n} dy + \int_{z_n}^{z_{n-1}} \frac{z_n y}{N_n N_{n-1}} dy + \dots + \int_{z_1}^1 \frac{z_n \dots z_1 y}{N_n \dots N_1} dy \right\}. \tag{9.2}$$

The maximand in equation (9.2) comprises an increasing function and a constant function. They intersect in one point representing the optimal threshold for accepting the pretender at shot n. Let us require that this value coincides with z_n. Such condition brings to the equation

$$z_n = \int_0^{z_n} \frac{y}{N_n} dy + \int_{z_n}^{z_{n-1}} \frac{z_n y}{N_n N_{n-1}} dy + \dots + \int_{z_1}^1 \frac{z_n \dots z_1 y}{N_n \dots N_1} dy,$$

whence it follows that

$$z_n = \frac{z_n^2}{2N_n} + \frac{z_n \left(z_{n-1}^2 - z_n^2 \right)}{2N_n N_{n-1}} + \dots + \frac{z_n \dots z_1 \left(1 - z_1^2 \right)}{2N_n \dots N_1}. \tag{9.3}$$

Then the function $v_n(x)$ acquires the form

$$v_n(x) = \begin{cases} z_n, & 0 \le x < z_n, \\ x, & z_n \le x \le 1. \end{cases}$$

Pass to shot $n-1$. Assume that the player meets a mate with the quality level x. Under continuation, his/her expected payoff makes up $Ev_n(x_n)$, where the function $v_n(x)$ has been obtained earlier and the expectation operator engages the distribution $f_{n-1}(x)$. The optimality equation at shot $n-1$ is defined by

$$v_{n-1}(x) = \max\left\{x, \int_0^{z_n} \frac{z_n}{N_{n-1}} dy + \int_{z_n}^{z_{n-1}} \frac{y}{N_{n-1}} dy + \dots + \int_{z_1}^{1} \frac{z_{n-1}\dots z_1 y}{N_{n-1}\dots N_1} dy\right\}.$$

Require that the threshold determining the optimal choice at shot $n-1$ coincides with z_{n-1}. This yields

$$z_{n-1} = \frac{z_n^2}{N_{n-1}} + \frac{z_{n-1}^2 - z_n^2}{2N_{n-1}} + \frac{z_{n-1}\left(z_{n-2}^2 - z_{n-1}^2\right)}{2N_{n-1}N_{n-2}} + \dots + \frac{z_{n-1}\dots z_1\left(1 - z_1^2\right)}{2N_{n-1}\dots N_1}. \tag{9.4}$$

Repeat such arguments to arrive at the following conclusion. The optimality equation at shot i, i.e.,

$$v_i(x) = \max\{x, Ev_{i+1}(x_{i+1})\}$$

gives the expression

$$z_i = \frac{1}{2N_i}\left[z_i^2 + z_{i+1}^2 + \sum_{k=0}^{i-1}\left(z_k^2 - z_{k+1}^2\right)\prod_{j=k}^{i-1}\frac{z_{j+1}}{N_j}\right], i = 1, \dots, n-1. \tag{9.5}$$

Next, compare the two equations for z_{i+1} and z_i. According to (9.5),

$$z_{i+1} = \frac{1}{2N_{i+1}}\left[z_{i+1}^2 + z_{i+2}^2 + \sum_{k=0}^{i}\left(z_k^2 - z_{k+1}^2\right)\prod_{j=k}^{i}\frac{z_{j+1}}{N_j}\right].$$

Rewrite this equation as

$$z_{i+1} = \frac{1}{2N_{i+1}}\left[z_{i+1}^2 + z_{i+2}^2 + \left(z_i^2 - z_{i+1}^2\right)\frac{z_{i+1}}{N_i} + \sum_{k=0}^{i-1}\left(z_k^2 - z_{k+1}^2\right)\frac{z_{i+1}}{N_i}\prod_{j=k}^{i-1}\frac{z_{j+1}}{N_j}\right].$$

Multiplication by $2\prod_{j=1}^{i+1} N_j$ leads to

$$2\prod_{j=1}^{i+1} N_j z_{i+1} = \left(z_{i+1}^2 + z_{i+2}^2\right)\prod_{j=1}^{i} N_j + \left(z_i^2 - z_{i+1}^2\right) z_{i+1}\prod_{j=1}^{i-1} N_j + \sum_{k=0}^{i-1}\left(z_k^2 - z_{k+1}^2\right)\prod_{j=k}^{i} z_{j+1}\prod_{j=1}^{k-1} N_j. \tag{9.6}$$

On the other hand, formula (9.1) implies that

$$2\prod_{j=1}^{i+1} N_j = 2\prod_{j=1}^{i-1} N_j\left(2z_{i+1}N_i - z_{i+1}^2\right).$$

By substituting this result into (9.6), we have

$$4\prod_{j=1}^{i} N_j z_{i+1}^2 = \left(z_{i+1}^2 + z_{i+2}^2\right)\prod_{j=1}^{i} N_j + \left(z_i^2 + z_{i+1}^2\right) z_{i+1}\prod_{j=1}^{i-1} N_j + \sum_{k=0}^{i-1}\left(z_k^2 - z_{k+1}^2\right)\prod_{j=k}^{i} z_{j+1}\prod_{j=1}^{k-1} N_j.$$

Comparison with equation (9.5) yields the expression

$$4z_{i+1}^2 = z_{i+1}^2 + z_{i+2}^2 + 2z_i z_{i+1}.$$

And so,

$$z_i = \frac{3}{2} z_{i+1} - \frac{1}{2}\frac{z_{i+2}^2}{z_{i+1}}, i = n-2, \dots, 1. \tag{9.7}$$

Taking into account (9.1), we compare (9.3) with (9.4) to get

$$z_n = \frac{2}{3} z_{n-1}.$$

Then formula (9.7) brings to

$$z_{n-2} = \frac{3}{2} z_{n-1} - \frac{1}{2}\left(\frac{z_n}{z_{n-1}}\right)^2 z_{n-1} = \frac{1}{2}\left(3 - \frac{4}{9}\right) z_{n-1},$$

or

$$z_{n-1} = \frac{2}{3 - 4/9} \cdot z_{n-2}.$$

The following recurrent expressions hold true:

$$z_i = a_i z_{i-1}\; i = 2, \dots, n,$$

where the coefficients a_i satisfy

$$a_i = \frac{2}{3 - a_{i+1}^2}, \; i = 1, \dots, n-1, \tag{9.8}$$

and $a_n = 2/3$.

Formulas (9.8) uniquely define the coefficients $a_i, i = 1, \dots, n$. Unique determination of $z_i, i = 1, \dots, n$ calls for specifying one of these quantities. We define z_1 using equation (9.5). Notably,

$$z_1 = \frac{1}{2N_1}\left[z_1^2 + z_2^2 + \left(1 - z_1^2\right) z_1\right].$$

Table 7.4 The optimal thresholds in the mutual best choice problem.

i	1	2	3	4	5	6	7	8	9	10
a_i	0.940	0.934	0.927	0.918	0.907	0.891	0.870	0.837	0.782	0.666
z_i	0.702	0.656	0.608	0.559	0.507	0.452	0.398	0.329	0.308	0.205

So long as $z_2 = a_2 z_1$, it appears that

$$2\left(2z_1 - z_1^2\right)z_1 = z_1^2 + a_2^2 z_1^2 + \left(1 - z_1^2\right)z_1.$$

We naturally arrive at a quadratic equation in z_1:

$$z_1^2 + z_1\left(a_2^2 - 3\right) + 1 = 0.$$

Since formulas (9.8) claim that $a_2^2 - 3 = -2/a_1$, the following equation arises immediately:

$$z_1^2 - 2\frac{z_1}{a_1} + 1 = 0.$$

Hence,

$$z_1 = \frac{1}{a_1}\left(1 - \sqrt{1 - a_1^2}\right).$$

Let us summarize the procedure. First, find the coefficients $a_i, i = n, n-1, \dots, 1$. Next, evaluate z_1, and compute recurrently the optimal thresholds $z_2, \dots, z_n$. For instance, calculations in the case of $n = 10$ are presented in Table 7.4.

Clearly, the optimal thresholds decrease monotonically. This is natural—the requirements to mate's quality level must go down as the game evolves.

Exercises

1. Two players observe random walks of a particle. It starts in position 0 and moves to the right by unity with some probability p or gets absorbed in state 0 with the probability $q = 1 - p$. The player who stops random walks in the rightmost position becomes the winner. Find the optimal strategies of the players.
2. Within the framework of exercise no. 1, suppose that each player observes his personal random Bernoulli sequence. These sequences are independent. Establish the optimal strategies of the players.
3. Evaluate an equilibrium in exercise no. 2 in the case of dependent observations.
4. Best choice game with incomplete information.

   Two players observe a sequence of pretenders for the position of a secretary. Pretenders come in a random order. The sequence of moves is first, player *I* and then *II*.

A pretender may reject from the position with the probability p. Find the optimal strategies of the players.

5. Two players receive observations representing independent random walks on the set $\{0, 1, \dots, k\}$ with absorption in the extreme states. In each state, a random walk moves by unity to the right (to the left) with the probability p (with the probability $1 - p$, respectively). The winner is the player who terminates walks in the state lying to the right from the corresponding state of the opponent's random walks. Find the optimal strategies of the players.

6. Evaluate an equilibrium in exercise no. 5 in the following case. Random walks in extreme states are absorbed with a given probability $\beta < 1$.

7. Best choice game with complete information.

Two players observe a sequence of independent random variables $x_1, x_2, \dots, x_\theta, x_{\theta+1}, \dots, x_n$, where at random time moment θ the distribution of the random variables switches from $p_0(x)$ to $p_1(x)$. First, the decision to stop is made by player I and then by player II. The players strive to select the observation with the maximal value. Find the optimal strategies of the players.

8. Consider the game described in exercise no. 7, but with random priority of the players. An observation is shown to player I with the probability p and to player II with the probability $1 - p$. Find the optimal strategies of the players.

9. Best choice game with partial information.

Two players receive observations representing independent identically distributed random variables. The players are unaware of the exact values of these observations. The only available knowledge is whether an observation exceeds a given threshold or not. The first move belongs to player I. Both players employ one-threshold strategies. The winner is the player who terminates observations on a higher value than the opponent. Find the optimal strategies and the value of this game.

10. Within the framework of exercise no. 9, assume that the priority of the players is defined by a random mechanism. Each observation is shown to player I with the probability p and to player II with the probability $1 - p$. Find the optimal strategies of the players.

8

Cooperative games

Introduction

In the previous chapters, we have considered games, where each player pursues individual interests. In other words, players do not cooperate to increase their payoffs. Chapter 8 concentrates on games, where players may form coalitions. The major problem here lies in distribution of the gained payoff among the members of a coalition. The set $N = \{1, 2, \dots, n\}$ will be called the grand coalition. Denote by 2^N the set of all its subsets, and let $|S|$ designate the number of elements in a set S.

8.1 Equivalence of cooperative games

Definition 8.1 *A cooperative game of n players is a pair* $\Gamma = < N, v >$, *where* $N = \{1, 2, \dots, n\}$ *indicates the set of players and* $v : 2^N \to R$ *is a mapping which assigns to each coalition* $S \in 2^N$ *a certain number such that* $v(\emptyset) = 0$. *The function v is said to be the characteristic function of the cooperative game.*

Generally, characteristic functions of cooperative games are assumed superadditive, i.e., for any coalitions S and T such that $S \cap T = \emptyset$ the following condition holds true:

$$v(S \cup T) \geq v(S) + v(T). \tag{1.1}$$

This appears a natural requirement stimulating players to build coalitions. Suppose that inequality (1.1) becomes an equality for all non-intersecting coalitions S and T; then the

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

corresponding characteristic function is called additive. Note that additive characteristic functions satisfy the formula

$$v(N) = \sum_{i \in N} v(i).$$

In this case, distribution runs in a natural way—each player receives his payoff $v(i)$. Such games are said to be **inessential**. In the sequel, we analyze only essential games that meet the inequality

$$v(N) > \sum_{i \in N} v(i). \tag{1.2}$$

Let us provide some examples.

Example 8.1 (A jazz band) A restaurateur invites a jazz band to perform on an evening and offers 100 USD. The jazz band consists of three musicians, namely, a pianist (player 1), a vocalist (player 2), and a drummer (player 3). They should distribute the fee. An argument during such negotiations is the characteristic function v defined by individual honoraria the players may receive by performing singly (e.g., $v(1) = 40, v(2) = 30, v(3) = 0$) or in pairs (e.g., $v(1, 2) = 80$, $v(1, 3) = 60$, $v(2, 3) = 50$).

Example 8.2 (The glove market) Consider the glove set $N = \{1, 2, \dots, n\}$, which includes left gloves (the subset L) and right gloves (the subset R). A glove costs nothing, whereas the price of a pair is 1 USD. Here the cooperative game $< N, v >$ can be determined by the characteristic function

$$v(S) = \min\{|S \cap L|, |S \cap R|\}, \ S \in 2^N.$$

Actually, it represents the number of glove pairs that can be formed from the set S.

Example 8.3 (Scheduling) Consider the set of players $N = \{1, 2, \dots, n\}$. Suppose that player i has a machine M_i and some production order J_i. It can be executed (a) by player i on his machine during the period t_{ii} or (b) by the coalition of players i and j on the machine M_j during the period t_{ij}. The cost matrix $T = \{t_{ij}\}, i, j = 1, \dots, n$ is given. For any coalition $S \in 2^N$, it is then possible to calculate the total costs representing the minimal costs over all permutations of players entering the coalition S, i.e.,

$$t(S) = \min_{\sigma} \sum_{i \in S} t_{i\sigma(i)}.$$

The characteristic function $v(S)$ in this game can be specified by the total time saved by the coalition (against the case when each player executes his order on his machine).

Example 8.4 (Road construction) Farmers agree to construct a road communicating all farms with a city. Construction of each segment of the road incurs definite costs; therefore, it seems beneficial to construct roads by cooperation. Each farm has a specific income from selling its agricultural products in the city. What is the appropriate cost sharing by farmers?

Prior to obtaining solutions, we partition the set of cooperative games into equivalence classes.

Definition 8.2 *Two cooperative games* $\Gamma_1 = < N, v_1 >$ *and* $\Gamma_2 = < N, v_2 >$ *are called equivalent, if there exist constants* $\alpha > 0$ *and* $c_i, i = 1, \dots, n$ *such that*

$$v_1(S) = \alpha v_2(S) + \sum_{i \in S} c_i$$

for any coalition $S \in 2^N$. *In this case, we write* $\Gamma_1 \sim \Gamma_2$.

Clearly, the relation $\sim$ represents an equivalence relation.

1. $v \sim v$ (reflexivity). This takes place under $\alpha = 1$ and $c_i = 0, i = 1, \dots, n$.
2. $v \sim v' \Rightarrow v' \sim v$ (symmetry). By setting $\alpha' = 1/\alpha$ and $c'_i = -c_i/\alpha$, we have $v'(S) = \alpha' + \sum_{i \in S} c'_i,\ S \in 2^N$, i.e., $v' \sim v$.
3. $v \sim v_1, v_1 \sim v_2 \Rightarrow v \sim v_2$ (transitivity). Indeed, $v(S) = \alpha v_1(S) + \sum_{i \in S} c_i$ and $v_1(S) = \alpha_1 v_2(S) + \sum_{i \in S} c'_i$. Hence, $v(S) = \alpha\alpha_1 v_2(S) + \sum_{i \in S} \left(\alpha c'_i + c_i\right)$.

Consequently, $\sim$ makes an equivalence relation. All cooperative games get decomposed into equivalence classes, and it suffices to solve one game from a given class. Clearly, all inessential games appear equivalent to games with zero characteristic function.

It seems comfortable to find solutions for cooperative games in the 0-1 form.

Definition 8.3 *A cooperative game in the 0-1 form is a game* $\Gamma = < N, v >$, *where* $v(i) = 0, i = 1, \dots, n$ *and* $v(N) = 1$.

Theorem 8.1 *Any essential cooperative game is equivalent to a certain game in the 0-1 form.*

Proof: It suffices to demonstrate that there exist constants $\alpha > 0$ and $c_i, i = 1, \dots, n$ such that

$$\alpha v(i) + c_i = 0, i = 1, \dots, n, \alpha v(N) + \sum_{i \in N} c_i = 1. \tag{1.3}$$

The system (1.3) uniquely determines these quantities:

$$\alpha = [v(N) - \sum_{i \in N} v(i)]^{-1},$$

$$c_i = -v(i)[v(N) - \sum_{i \in N} v(i)]^{-1}.$$

Note that, by virtue of (1.2), we have $\alpha > 0$.

8.2 Imputations and core

Now, we define the solution of a cooperative game.

The solution of a cooperative game is comprehended as some distribution of the total payoff gained by the grand coalition $v(N)$.

Definition 8.4 *An imputation in the cooperative game* $\Gamma = < N, v >$ *is a vector* $x = (x_1, \ldots, x_n)$ *such that*

$$x_i \geq v(i), \qquad i = 1, \ldots, n \tag{2.1}$$

$$\textstyle\sum_{i \in N} x_i = v(N). \tag{2.2}$$

According to the condition (2.1) (the property of individual rationality), each player gives not less than he can actually receive. The condition (2.2) is called the property of efficiency. The latter presumes that (a) it is unreasonable to distribute less than the grand coalition can receive and (b) it is impossible to distribute more than $v(N)$. We will designate the set of all imputations by $D(v)$. For equivalent characteristic functions v and v' such that $v(S) = \alpha v'(S) + \sum\limits_{i \in S} c_i$, $S \in 2^N$, imputations are naturally interconnected: $x_i = \alpha x_i' + c_i, i \in N$. Interestingly, the set of imputations for cooperative games in the 0-1 form represents the simplex $D(v) = \{x : \sum\limits_{i \in N} x_i = 1, x_i \geq 0, i = 1, \ldots, n\}$ in R^n.

There exist several optimal principles of choosing a point or a set of points on the set $D(v)$ that guarantee an acceptable solution of the payoff distribution problem in the grand coalition. We begin with the definition of core. First, introduce the notion of dominated imputations.

Definition 8.5 *An imputation x dominates an imputation y in a coalition S (which is denoted by $x \succ_S y$), if*

$$x_i > y_i, \quad \forall i \in S, \tag{2.3}$$

and

$$\sum_{i \in S} x_i \leq v(S). \tag{2.4}$$

The condition (2.3) implies that the imputation x appears more preferable than the imputation y for all members of the coalition S. On the other hand, the condition (2.4) means that the imputation x is implementable by the coalition S.

Definition 8.6 *We say that an imputation x dominates an imputation y, if there exists a coalition $S \in 2^N$ such that $x \succ_S y$.*

Here the dominance $x \succ y$ indicates the following. There exists a coalition supporting the given imputation x. Below we introduce the definition of core.

Definition 8.7 *The set of non-dominated imputations is called the core of a cooperative game.*

Theorem 8.2 *An imputation x belongs to the core of a cooperative game $< N, v >$ iff*

$$\sum_{i \in S} x_i \geq v(S), \quad \forall S \in 2^N. \tag{2.5}$$

Proof: Let us demonstrate the necessity of the condition (2.5) by contradiction. Suppose that $x \in C(v)$, but for some coalition S: $\sum_{i \in S} x_i < v(S)$. Note that $1 < |S| < n$ (otherwise, we violate the conditions of individual rationality and efficiency, see (2.1) and (2.2)). Suggest to the coalition S a new imputation y, where

$$y_i = x_i + \frac{v(S) - \sum_{i \in S} x_i}{|S|}, \quad i \in S,$$

and distribute the residual quantity $v(N) - v(S)$ among the members of the coalition $N \backslash S$:

$$y_i = \frac{v(N) - v(S)}{|N \backslash S|}, \quad i \in N \backslash S.$$

Obviously, y is an imputation and $y \succ x$. The resulting contradiction proves (2.5).

Finally, we argue the sufficiency part. Assume that x meets (2.5), but is dominated by another imputation y for some coalition S. Due to (2.3)–(2.4), we have

$$\sum_{i \in S} x_i < \sum_{i \in S} y_i \leq v(S),$$

which contradicts the condition (2.5).

8.2.1 The core of the jazz band game

Construct the core of the jazz band game. Recall that musicians have to distribute their honorarium of 100 USD. The characteristic function takes the following form:

$$v(1) = 40,\ v(2) = 30,\ v(3) = 0,\ v(1,2) = 80,\ v(1,3) = 60,\ v(2,3) = 50,\ v(1,2,3) = 100.$$

First, rewrite this function in the 0-1 form. We evaluate $\alpha = 1/[v(N) - v(1) - v(2) - v(3)] = 1/30$ and $c_1 = -4/3, c_2 = -1, c_3 = 0$. Then the new characteristic function is defined by

$$v'(1) = 0,\ v'(2) = 0,\ v'(3) = 0,\ v'(1,2,3) = 1,$$

$$v'(1,2) = \frac{8}{3} - \frac{4}{3} - 1 = \frac{1}{3},\ v'(1,3) = \frac{6}{3} - \frac{4}{3} = \frac{2}{3},\ v'(2,3) = \frac{5}{3} - 1 = \frac{2}{3}.$$

The core of this game lies on the simplex

$$E = \{x = (x_1, x_2, x_3) : x_1 + x_2 + x_3 = 1\},\ x_i \geq 0, i = 1, 2, 3.$$

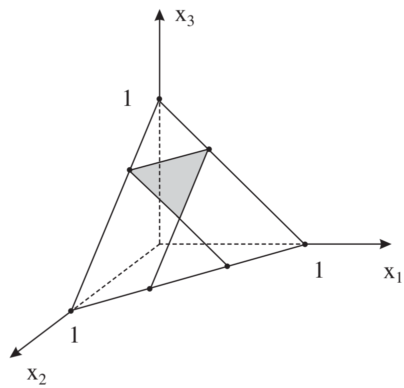


Figure 8.1 The core of the jazz band game.

According to (2.5), it obeys the system of inequalities

$$x_1 + x_2 \geq \frac{1}{3},\ x_1 + x_3 \geq \frac{2}{3},\ x_2 + x_3 \geq \frac{2}{3}.$$

So long as $x_1 + x_2 + x_3 = 1$, the inequalities can be reformulated as

$$x_3 \leq \frac{2}{3},\ x_2 \leq \frac{1}{3},\ x_1 \leq \frac{1}{3}.$$

In Figure 8.1, the core is illustrated by the shaded domain. Any element of the core is not dominated by another imputation. As a feasible solution, we can choose the center of gravity of the core: $x = (2/9, 2/9, 5/9)$.

Getting back to the initial game, we obtain the following imputation: $(140/3, 110/3, 50/3)$. And so, the pianist, the vocalist, and the drummer receive 46.6 USD, 36.6 USD, and 16.6 USD, respectively.

8.2.2 The core of the glove market game

Construct the core of the glove market game. Reexpress the glove set as $N = \{L, R\}$, where $L = \{l_1, \ldots, l_k\}$ is the set of left gloves and $R = \{r_1, \ldots, r_m\}$ gives the set of right gloves. For definiteness, we believe that $k \leq m$. It is possible to compile k pairs, therefore $v(N) = k$. The characteristic function acquires the form

$$v(l_{i_1}, \ldots, l_{i_s}, r_{j_1}, \ldots, r_{j_t}) = \min\{s, t\}, s = 1, \ldots, k;\ t = 1, \ldots, m.$$

Theorem 8.2 claims that the core of this game on the set of imputations

$$D = \{(x_1, \ldots, x_k, y_1, \ldots, y_m) : \sum_{i=1}^{k} x_i + \sum_{j=1}^{m} y_j = k, x \geq 0, y \geq 0\}$$

is described by the inequalities

$$x_{i_1} + \cdots + x_{i_s} + y_{j_1} + \cdots + y_{j_t} \geq \min\{s, t\}, s = 1, \ldots, k;\ t = 1, \ldots, m.$$

If $k < m$, these inequalities imply that

$$x_1 + \cdots + x_k + y_{j_1} + \cdots + y_{j_k} \geq k,$$

for any set k of right gloves $\{j_1, ..., j_k\}$. Since $\sum_{i=1}^{k} x_i + \sum_{j=1}^{m} y_j = k$, it appears that

$$\sum_{j \neq j_1, \ldots, j_k} y_j = 0.$$

Hence, $y_j = 0$ for all $j \neq j_1, \ldots, j_k$. However, the set of right gloves is arbitrary; and so, all y_j equal zero. Consequently, in the case of $k < m$, the core of the game consists of the point $(x_1 = \cdots = x_k = 1, y_1, \ldots, y_m = 0)$ only.

If $k = m$, readers can see that the core also comprises a single imputation of the form $x_1 = \cdots = x_k = y_1 = \cdots = y_k = \frac{1}{2k}$.

8.2.3 The core of the scheduling game

Construct the core of the scheduling game with $N = \{1, 2, 3\}$. In other words, we have three production orders and three machines for their execution. Suppose that the time cost matrix is determined by

$$T = \begin{pmatrix} 1 & 2 & 4 \\ 3 & 5 & 8 \\ 5 & 7 & 11 \end{pmatrix}.$$

The corresponding cooperative game $< N, v >$ is presented in Table 8.1. Time cost evaluation lies in minimization over all possible schemes of order execution using different coalitions. For instance, there exist two options for $S = \{1, 2\}$: each production order is executed on the corresponding machine, or the players exchange their production orders: $t(1, 2) = \min\{1 + 5, 2 + 3\} = 5$. The characteristic function $v(S)$ results from the difference $\sum_{i \in S} t_i - t(S)$.

Table 8.1 The characteristic function in the scheduling game.

S	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$t(S)$	0	1	5	11	5	9	15	14
$v(S)$	0	0	0	0	1	3	1	3

The core of such characteristic function is defined by the inequalities

$$x_1 + x_2 \geq 1,\ x_1 + x_3 \geq 3,\ x_2 + x_3 \geq 1,\ x_1 + x_2 + x_3 = 3,$$

or $C(v) = \{x : x_1 + x_3 = 3, x_2 = 0\}$. Therefore, the optimal solution prescribes to execute the second production order on machine 2, whereas machines 1 and 3 should exchange their orders.

8.3 Balanced games

We emphasize that the core of a game can be empty. Then this criterion of payoff distribution fails. The existence of core relates to the notion of balanced games suggested by O. Bondareva (1963) and L. Shapley (1967).

Definition 8.8 *Let $N = \{1, 2, \dots, n\}$ and 2^N denote the set of all subsets of N. A mapping $\lambda(S) : 2^N \to R^+$, defined for all coalitions $S \in 2^N$ such that $\lambda(\emptyset) = 0$, is called balanced if*

$$\sum_{S \in 2^N} \lambda(S) I(S) = I(N). \tag{3.1}$$

Here $I(S)$ means the indicator of the set S (i.e., $I_i(S) = 1$, if $i \in S$ and $I_i(S) = 0$, otherwise). Equality (3.1) holds true for each player $i \in N$.

For instance, if $N = \{1, 2, 3\}$, the following mappings are balanced:

$$\lambda(1) = \lambda(2) = \lambda(3) = \lambda(1, 2, 3) = 0,\ \lambda(1, 2) = \lambda(1, 3) = \lambda(2, 3) = \frac{1}{2},$$

or

$$\lambda(1) = \lambda(2) = \lambda(3) = \lambda(1, 2) = \lambda(1, 3) = \lambda(2, 3) = \frac{1}{3}, \lambda(1, 2, 3) = 0.$$

Definition 8.9 *A cooperative game $< N, v >$ is called a balanced game, if for each balanced mapping $\lambda(S)$ we have the condition*

$$\sum_{S \in 2^N} \lambda(S) v(S) \leq v(N). \tag{3.2}$$

Theorem 8.3 *Consider a cooperative game $< N, v >$. Its core appears non-empty iff the game is balanced.*

Proof: Based on the duality theorem of linear programming. Take the linear programming problem

$$\min \sum_{i=1}^{n} x_i,$$

$$\sum_{i \in S} x_i \geq v(S),\ \ \forall S \in 2^N. \tag{3.3}$$

The core being non-empty, Theorem 8.2 (see Section 8.1) claims that this problem admits a solution coinciding with $v(N)$. The converse proposition takes place instead. If there exists a solution of the problem (3.3), which equals $v(N)$, then $C(v) \neq \emptyset$.

Let us analyze the dual problem for (3.3):

$$\max \sum_{S \in 2^N} \lambda(S) v(S),$$

$$\sum_{S \in 2^N} \lambda(S) I(S) = I(N), \quad \lambda \geq 0. \tag{3.4}$$

The constraints in the dual problem (3.4) define the balanced mapping $\lambda(S)$. Therefore, the problem (3.4) consists in seeking for the maximal value of the functional $\sum_{S \in 2^N} \lambda(S) v(S)$ among all balanced mappings. For the balanced mapping $\lambda(N) = 1, \lambda(S) = 0, \forall S \subset N$, its value makes up $v(N)$. Hence, the value of the problem (3.4) is greater or equal to $v(N)$.

The duality theory of linear programming states the following. If there exist admissible solutions to the direct and dual problems, then these problems admit optimal solutions and their values coincide. And so, a necessary and sufficient condition of the non-empty core consists in $\sum_{S \in 2^N} \lambda(S) v(S) \leq v(N)$ for any balanced mapping $\lambda(S)$.

8.3.1 The balance condition for three-player games

Consider a three-player game in the 0-1 form with the characteristic function

$$v(1) = v(2) = v(3) = 0, \; v(1,2) = a, \; v(1,3) = b, \; v(2,3) = c, \; v(1,2,3) = 1.$$

For a balanced mapping $\lambda(S)$, the condition (3.2) acquires the form

$$\sum_{S \in 2^N} \lambda(S) v(S) = \lambda(1,2) a + \lambda(1,3) b + \lambda(2,3) c + \lambda(1,2,3) \leq 1,$$

which is equivalent to the inequality $a + b + c \leq 2$. Therefore, the core of a three-player cooperative game appears non-empty iff $a + b + c \leq 2$.

8.4 The τ-value of a cooperative game

In Section 8.2, we have defined a possible solution criterion for cooperative games (core). It has been shown that core may not exist. Even if core is non-empty, there arises an uncertainty in choosing a specific imputation from this set. One feasible principle of such choice was proposed by S. Tijs (1981). This is the so-called τ-value.

Consider a cooperative game $< N, v >$. Define the maximum and minimum possible payoffs of each player.

Definition 8.10 *An utopia imputation (upper vector) $M(v)$ is a vector*

$$M(v) = (M_1(v), \ldots, M_n(v)),$$

where the payoff of player i takes the form

$$M_i(v) = v(N) - v(N\backslash i),\ \ i = 1, \dots, n.$$

As a matter of fact, $M_i(v)$ specifies the maximum possible payoff of player i. If a player wants a higher payoff, the grand coalition benefits from eliminating this player from its staff.

Definition 8.11 *The minimum rights vector (lower vector)* $m(v) = (m_1(v), \dots, m_n(v))$ *is the vector with the components*

$$m_i(v) = \max_{S: i \in S} v(S) - \sum_{j \in S\backslash i} M_j(v),\ \ i = 1, \dots, n.$$

The minimum rights vector enables each player i to join a coalition, where all other players are satisfied with the membership of player i. Indeed, they guarantee the maximum possible (utopia) payoffs.

Theorem 8.4 *Let* $< N, v >$ *be a cooperative game with non-empty core. Then for any* $x \in C(v)$ *we have*

$$m(v) \le x \le M(v), \tag{4.1}$$

or

$$m_i(v) \le x_i \le M_i(v),\ \ \forall i \in N.$$

Proof: Really, the property of efficiency implies that for any player $i \in N$:

$$x_i = \sum_{j \in N} x_j - \sum_{j \in N\backslash i} x_j = v(N) - \sum_{j \in N\backslash i} x_j.$$

As far as x lies in the core,

$$\sum_{j \in N\backslash i} x_j \ge v(N\backslash i).$$

Hence,

$$x_i = v(N) - \sum_{j \in N\backslash i} x_j \le v(N) - v(N\backslash i) = M_i(v).$$

This argues the right-hand side of inequalities (4.1).

Since $x \in C(v)$, we obtain $\sum_{j \in S} x_j \ge v(S)$. Furthermore, the results established earlier lead to the following. Any coalition S containing player i meets the inequality

$$\sum_{j \in S\backslash i} x_j \le \sum_{j \in S\backslash i} M_j(v).$$

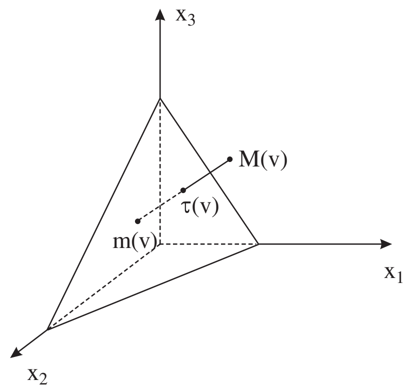


Figure 8.2 The τ-value of a quasibalanced game.

It appears that

$$x_i = \sum_{j\in S} x_j - \sum_{j\in S\setminus i} x_j \geq v(S) - \sum_{j\in S\setminus i} M_j(v)$$

for any coalition S containing player i. Therefore,

$$x_i \geq \max_{S:i\in S}\{v(S) - \sum_{j\in S\setminus i} M_j(v)\} = m_i(v).$$

The proof of Theorem 8.4 is finished.

Let us summarize the outcomes. If core is non-empty and we connect the vectors $m(v)$ and $M(v)$ by a segment, then there exists a point x (lying on this segment and belonging to a hyperplane in R^n), which contains the core. Moreover, this point is uniquely defined.

We also make an interesting observation. If the core does not exist, but the inequalities

$$m(v) \leq M(v), \ \sum_{i\in N} m_i(v) \leq v(N) \leq \sum_{i\in N} M_i(v) \tag{4.2}$$

hold true, the segment $[m(v), M(v)]$ necessarily has a unique point intersecting the hyperplane $\sum_{i\in N} x_i = v(N)$ (see Figure 8.2).

Definition 8.12 *A cooperative game $< N, v >$ obeying the conditions (4.2) is called quasi-balanced.*

Definition 8.13 *In a quasibalanced game, the vector $\tau(v)$ representing the intersection of the segment $[m(v), M(v)]$ and the hyperplane $\sum_{i\in N} x_i = v(N)$ is said to be the τ-value of this cooperative game.*

8.4.1 The τ-value of the jazz band game

Find the τ-value of the jazz band game. The characteristic function has the form

$$v(1) = 40, v(2) = 30, v(3) = 0, v(1,2) = 80, v(1,3) = 60, v(2,3) = 50, v(1,2,3) = 100.$$

Evaluate the utopia imputation:

$$M_1(v) = v(1,2,3) - v(2,3) = 50,\ M_2(v) = v(1,2,3) - v(1,3) = 40,$$
$$M_3(v) = v(1,2,3) - v(1,2) = 20,$$

and the equal rights vector:

$$m_1(v) = \max\{v(1), v(1,2) - M_2(v), v(1,3) - M_3(v), v(1,2,3) - M_2(v) - M_3(v)\} = 40.$$

By analogy, we obtain $m_2(v) = 30, m_3(v) = 10$.

The τ-value lies on the intersection of the segment $\lambda M(v) + (1-\lambda)m(v)$ and the hyperplane $x_1 + x_2 + x_3 = 100$. Hence, the following equality is valid:

$$\lambda(M_1(v) + M_2(v) + M_3(v)) + (1-\lambda)(m_1(v) + m_2(v) + m_3(v)) = 100.$$

This leads to $\lambda = 2/3$. Thus, $\tau(v) = (140/3, 110/3, 50/3)$ makes the center of gravity for the core of the jazz band game.

8.5 Nucleolus

The concept of nucleolus was suggested by D. Schmeidler (1969) as a solution principle for cooperative games. Here a major role belongs to the notion of lexicographic order and excess.

Definition 8.14 *The excess of a coalition S is the quantity*

$$e(x,S) = v(S) - \sum_{i\in S} x_i,\ x \in D(v), S \in 2^N.$$

Actually, excess represents the measure of dissatisfaction with an offered imputation x in a coalition S. For instance, core (if any) rules out unsatisfied players, all excesses are negative.

We form the vector of excesses for all $2^n - 1$ non-empty coalitions by placing them in the descending order:

$$e(x) = (e_1(x), e_2(x), \dots, e_m(x)),\ \text{where } e_i(x) = e(x, S_i), i = 1, 2, \dots, m = 2^n - 1,$$

and $e_1(x) \geq e_2(x) \geq \dots \geq e_m(x)$.

A natural endeavor is to find an imputation minimizing the maximal measure of dissatisfaction. To succeed, we introduce the concept of a lexicographic order.

Definition 8.15 *Let $x, y \in R^m$. We say that a vector x is lexicographically smaller than a vector y (and denote this fact by $x \leq_e y$), if $e(x) = e(y)y$ or there exists $k : 1 \leq k \leq m$ such that $e_i(x) = e_i(y)$ for all $i = 1, \dots, k-1$ and $e_k(x) < e_k(y)$.*

For instance, a vector with the excess $(3, 2, 0)$ appears lexicographically smaller than a vector with the excess $(3, 3, -10)$.

Definition 8.16 *The lexicographic minimum with respect to the preference $<_e$ is called the nucleolus of a cooperative game.*

Therefore, nucleolus minimizes the maximum dissatisfaction of all coalitions. In what follows, we establish its existence and uniqueness.

Theorem 8.5 *There exists a unique nucleolus for each cooperative game $< N, v >$.*

Proof: It is necessary to demonstrate the existence of the lexicographic minimum. Note that the components of the excess vector can be rewritten as

$$
\begin{aligned}
e_1(x) &= \max_{i=1,\dots,m} \{e(x, S_i)\}, \\
e_2(x) &= \min_{j=1,\dots,m} \left\{ \max_{i \neq j} \{e(x, S_i)\} \right\}, \\
e_3(x) &= \min_{j,k=1,\dots,m; j \neq k} \left\{ \max_{i \neq j,k} \{e(x, S_i\} \right\} \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
e_m(x) &= \min_{i=1,\dots,m} \{e(x, S_i)\}.
\end{aligned}
\tag{5.1}
$$

For each i, the functions $e(x, S_i)$ enjoy continuity. The maxima and minima of the continuous functions in (5.1) are also continuous. Thus, all functions $e_i(x), i = 1, \dots, m$ turn out continuous.

The imputation set $D(v)$ is compact. The continuous function $e_1(x)$ attains its minimal value m_1 on this set. If the above value is achieved in the single point x_1, this gives the minimal element (which proves the theorem).

Suppose that the minimal value is achieved on the set $X_1 = \{x \in D(v) : e_1(x) = m_1\}$. Since the function $e_1(x)$ enjoys continuity, X_1 represents a compact set. We look to achieve the minimum of the continuous function $e_2(x)$ on the compact set X_1. It does exist; let the minimal value equal $m_2 \leq m_1$. If this value is achieved in a single point, we obtain the lexicographic minimum; otherwise, this is the compact set $X_2 = \{x \in X_1 : e_2(x) = m_2\}$. By repeating the described process, we arrive at the following result. There exists a point or set yielding the lexicographic minimum.

Now, we prove its uniqueness. Suppose the contrary, i.e., there exist two imputations x and y such that $e(x) = e(y)$. Note that, despite the equal excesses of these imputations, they can be expressed for different coalitions. Consider the vector $e(x)$ and let $e_1(x) = ...e_k(x)$ be the maximal components in it, $e_k(x) > e_{k+1}(x)$. Moreover, imagine that the above components represent the excesses for the coalitions $S_1, \dots, S_k$, respectively: $e_1(x) = e(x, S_1), \dots, e_k(x) = e(x, S_k)$. Then for these coalitions, yet another imputation y, we obtain the conditions

$$
e(y, S_i) \leq e(x, S_i), \text{ or } v(S_i) - \sum_{j \in S_i} y_j \leq v(S_i) - \sum_{j \in S_i} x_j, \; i = 1, \dots, k. \tag{5.2}
$$

Suppose that, for $i = 1, \dots, \hat{\imath} - 1$, formulas (5.2) become equalities, while strict inequalities here take place for a certain coalition $S_{\hat{\imath}}$, i.e.,

$$v(S_{\hat{\imath}}) - \sum_{j \in S_{\hat{\imath}}} y_j < v(S_{\hat{\imath}}) - \sum_{j \in S_{\hat{\imath}}} x_j. \tag{5.3}$$

Inequality (5.3) remains in force for the new imputation $z = \epsilon y + (1 - \epsilon)x$ under any $\epsilon > 0$:

$$v(S_{\hat{\imath}}) - \sum_{j \in S_{\hat{\imath}}} z_j < v(S_{\hat{\imath}}) - \sum_{j \in S_{\hat{\imath}}} x_j.$$

As far as

$$e(x, S_{\hat{\imath}}) > e_j(x), \ j = k + 1, \dots, m, \tag{5.4}$$

the continuous property of the functions $e_j(x)$ leads to inequality (5.4) for the imputation z under sufficiently small ϵ:

$$e(z, S_{\hat{\imath}}) > e_j(z), \ j = k + 1, \dots, m.$$

Hence, for such ϵ, the imputation z appears lexicographically smaller than x. This contradiction proves that inequalities (5.2) are actually equalities.

Further application of such considerations by induction for smaller components brings us to the following conclusion. All coalitions $S_i, i = 1, \dots, m$ satisfy the equality $e(y, S_i) = e(x, S_i)$ or

$$\sum_{j \in S_i} y_j = \sum_{j \in S_i} x_j, \ i = 1, \dots, m,$$

whence it follows that $x = y$. The proof of Theorem 8.5 is concluded.

Theorem 8.6 *Suppose that the core of a cooperative game $< N, v >$ is non-empty. Then the nucleolus belongs to $C(v)$.*

Proof: Denote the nucleolus by x^*. Take an arbitrary imputation x belonging to the core. In the case of $x \in C(v)$, the excesses of all coalitions are non-positive, i.e., $e(x, S_j) \le 0, j = 1, \dots, m$. However, $x^* <_L x$; hence, this condition holds true for x^*: $e(x^*, S_j) \le 0, j = 1, \dots, m$. And so,

$$\sum_{i \in S_j} x_i^* \ge v(S_j), \ j = 1, \dots, m,$$

which means that x^* belongs to the core.

8.5.1 The nucleolus of the road construction game

Three farmers agree to construct a road communicating all farms with a city. Construction of each segment of the road incurs definite costs. Each farm has a specific income from selling

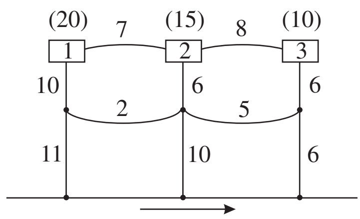


Figure 8.3 Road construction.

its agricultural products in the city. Road infrastructure, the construction cost of each road segment and the incomes of the farmers are illustrated in Figure 8.3.

Compute the characteristic function for each coalition. Obviously, each player would benefit nothing by independent road construction:

$$v(1) = 20 - 21 = -1,\ v(2) = 15 - 16 = -1,\ v(3) = 10 - 12 = -2.$$

Owing to cooperation, the players receive the following incomes:

$$v(1,2) = 35 - 23 = 12,\ v(2,3) = 25 - 20 = 5,\ v(1,3) = 30 - 27 = 3,$$
$$v(1,2,3) = 45 - 27 = 18.$$

The stages of nucleolus evaluation are demonstrated in Table 8.2. We begin with an arbitrary imputation, e.g., $x = (8, 6, 4)$. Compute the excesses $e(x, S)$ for all coalitions. Actually, the maximal excess corresponds to the coalition $S = \{1, 2\}$. Since $e(x, S) = x_3 - 6$, it can be reduced by decreasing x_3. Let us decrease x_3 and simultaneously increase x_2, until this excess coincides with $e(x, \{3\}) = -2 - x_3$ (until $x_3 = 2$). Such procedure leads to the imputation $(8, 8, 2)$. It is impossible to vary x_3 with increasing $e(x, \{3\})$ or $e(x, \{1, 2\})$. In other words, these excesses have reached their minimal values. The next excess (in descending order) is $e(x, \{2, 3\}) = x_1 - 13$; we will reduce it by decreasing x_1 and simultaneously increasing x_2 (until this excess equals the next excess $e(x, \{1, 3\}) = x_2 - 15$). This happens under $x_1 = 7, x_2 = 9$. And none of the excesses allows further reduction. Therefore, the nucleolus of the road construction game lies in the imputation $(7, 9, 2)$.

Now, we easily distribute the road construction cost among the farmers. The road proper is marked by thin line in Figure 8.3. The total cost of road construction makes up 27, and the shares of the players are $c_1 = 20 - 7 = 13,\ c_2 = 15 - 9 = 6,\ c_3 = 10 - 2 = 8$.

Table 8.2 Nucleolus evaluation.

S	v	$e(x, S)$	$(8, 6, 4)$	$(8, 8, 2)$	$(7, 9, 2)$
$\{1\}$	-1	$-1 - x_1$	-9	-9	-8
$\{2\}$	-1	$-1 - x_2$	-7	-9	-10
$\{3\}$	-2	$-2 - x_3$	-6	-4	-4
$\{1, 2\}$	12	$12 - x_1 - x_2 = x_3 - 6$	-2	-4	-4
$\{1, 3\}$	3	$3 - x_1 - x_3 = x_2 - 15$	-9	-7	-6
$\{2, 3\}$	5	$5 - x_2 - x_3 = x_1 - 13$	-5	-5	-6

8.6 The bankruptcy game

In 1985, R.J. Aumann and M.B. Maschler studied the following game based on a text from the Talmud. Three lenders call in the debts of 300, 200, and 100 units, respectively. For convenience, denote them by d_1, d_2, and d_3. Depending on the state E of a bankrupt, the lenders offer different payment schemes, see Table 8.3.

Clearly, under the small state ($E = 100$), it is recommended to pay all lenders equally. If $E = 300$, the debts should be distributed proportionally. In the intermediate case ($E = 200$), the offer seems even more difficult to explain. To define the distribution principle, we address some methods from the theory of cooperative games.

Construct a cooperative game related to the bankruptcy problem. For each coalition S, specify the characteristic function in this game by

$$v(S) = (E - \sum_{i \in N \setminus S} d_i)^+, \tag{6.1}$$

where $a^+ = max(a, 0)$. Let us analyze the case of three players only.

Theorem 8.7 *Consider a three-player game* $< N = \{1, 2, 3\}, v >$ *in the 0-1 form, where* $v(1, 2) = c_3$, $v(1, 3) = c_2$, $v(2, 3) = c_1$ *and* $c_1 \le c_2 \le c_3 \le 1$. *The nucleolus has the form*

$$NC = \begin{cases} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) & \text{if } c_1 \le c_2 \le c_3 \le \frac{1}{3} \\ \left(\frac{1+c_3}{4}, \frac{1+c_3}{4}, \frac{1-c_3}{2}\right) & \text{if } c_3 > \frac{1}{3},\ c_2 \le \frac{1-c_3}{2} \\ \left(\frac{c_2+c_3}{2}, \frac{1-c_2}{2}, \frac{1-c_3}{2}\right) & \text{if } c_3 > \frac{1}{3},\ c_2 > \frac{1-c_3}{2},\ c_1 \le \frac{1-c_3}{2} \\ \left(\frac{1-2c_1+2c_2+c_3}{4}, \frac{1+2c_1-2c_2+c_3}{4}, \frac{1-c_3}{2}\right) & \text{if } c_3 > \frac{1}{3},\ c_1 > \frac{1-c_3}{2},\ c_1+c_2 \le \frac{1+c_3}{2} \\ \left(\frac{1-2c_1+c_2+c_3}{3}, \frac{1+c_1-2c_2+c_3}{3}, \frac{1+c_1+c_2-2c_3}{3}\right) & \text{if } c_3 > \frac{1}{3},\ c_1 > \frac{1-c_3}{2},\ c_1+c_2 > \frac{1+c_3}{2}. \end{cases}$$

Proof: Without loss of generality, we believe that $c_1 \le c_2 \le c_3$; otherwise, just renumber the players. For convenience, describe different stages of the proof as the diagram in Figure 8.4.

Table 8.3 Imputations in the bankruptcy game.

		Debts Player 1 $d_1 = 300$	Player 2 $d_2 = 200$	Player 3 $d_3 = 100$
	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
E	200	75	75	50
	300	150	100	50

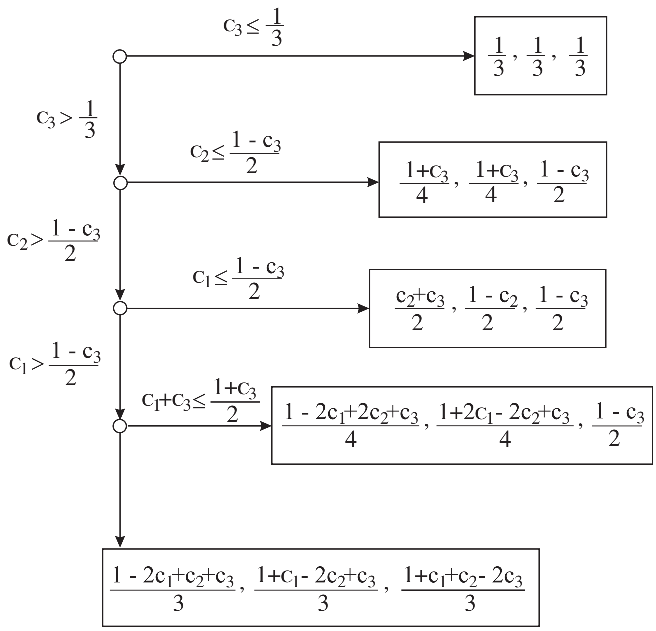


Figure 8.4 The nucleolus of the three-player game.

We begin with the case of $c_3 \leq 1/3$. Evaluate the excesses for all coalitions $S_1 = \{1\}, S_2 = \{2\}, S_3 = \{3\}, S_4 = \{1, 2\}, S_5 = \{1, 3\}, S_6 = \{2, 3\}$, and the imputation $x = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. They are combined in Table 8.2. Obviously, for the coalitions S_1, S_2, and S_3, the excesses equal $e(x, S_1) = e(x, S_2) = e(x, S_3) = 1/3$. If $c_3 \leq 1/3$, then $-1/3 \geq c_3 - 2/3$, and it follows that $-1/3 \geq e(x, S_4)$. Therefore, in the case of $c_3 \leq 1/3$, we obtain the following order of the excesses:

$$e(x, S_1) = e(x, S_2) = e(x, S_3) \geq e(x, S_4) \geq e(x, S_5) \geq e(x, S_6).$$

Any variations in a component of the imputation surely increase the maximal excess. Consequently, the vector $(1/3, 1/3, 1/3)$ forms the nucleolus.

Now, suppose that $c_3 > 1/3$. Furthermore, let

$$c_2 \leq \frac{1 - c_3}{2} \tag{6.2}$$

and consider the imputation $x_1 = x_2 = \frac{1+c_3}{4}, x_3 = \frac{1-c_3}{2}$. Table 8.4 provides the corresponding excesses. Notably, the maximal excesses belong to the coalitions S_3 and S_4:

$$e(x, S_3) = e(x, S_4) = -\frac{1 - c_3}{2}.$$

Table 8.4 The nucleolus of the three-player game.

S	v	$e(x,S)$	$(1/3,1/3,1/3)$	$\left(\frac{1+c_3}{4},\frac{1+c_3}{4},\frac{1-c_3}{2}\right)$	$\left(\frac{c_2+c_3}{2},\frac{1-c_2}{2},\frac{1-c_3}{2}\right)$
$\{1\}$	0	$-x_1$	$-1/3$	$-\frac{1+c_3}{4}$	$-\frac{c_2+c_3}{2}$
$\{2\}$	0	$-x_2$	$-1/3$	$-\frac{1+c_3}{4}$	$-\frac{1-c_2}{2}$
$\{3\}$	0	$-x_3$	$-1/3$	$-\frac{1-c_3}{2}$	$-\frac{1-c_3}{2}$
$\{1,2\}$	c_3	c_3-1+x_3	$c_3-2/3$	$-\frac{1-c_3}{2}$	$-\frac{1-c_3}{2}$
$\{1,3\}$	c_2	c_2-1+x_2	$c_2-2/3$	$\frac{1+c_3}{4}-1+c_2$	$-\frac{1-c_2}{2}$
$\{2,3\}$	c_1	c_1-1+x_1	$c_1-2/3$	$\frac{1+c_3}{4}-1+c_1$	$\frac{c_2+c_3}{2}-1+c_1$
S	$\left(\frac{1-2c_1+2c_2+c_3}{4},\frac{1+2c_1-2c_2+c_3}{4},\frac{1-c_3}{2}\right)$	$\left(\frac{1-2c_1+c_2+c_3}{3},\frac{1+c_1-2c_2+c_3}{3},\frac{1+c_1+c_2-2c_3}{3}\right)$			
$\{1\}$	$-\frac{1-2c_1+2c_2+c_3}{4}$	$-\frac{1-2c_1+c_2+c_3}{3}$			
$\{2\}$	$-\frac{1+2c_1-2c_2+c_3}{4}$	$-\frac{1+c_1-2c_2+c_3}{3}$			
$\{3\}$	$-\frac{1-c_3}{2}$	$-\frac{1+c_1+c_2-2c_3}{3}$			
$\{1,2\}$	$-\frac{1-c_3}{2}$	$\frac{-2+c_1+c_2+c_3}{3}$			
$\{1,3\}$	$\frac{-3+2c_1+2c_2+c_3}{4}$	$\frac{-2+c_1+c_2+c_3}{3}$			
$\{2,3\}$	$\frac{-3+2c_1+2c_2+c_3}{4}$	$\frac{-2+c_1+c_2+c_3}{3}$			

Indeed,

$$e(x, S_3) = -\frac{1 - c_3}{2} > -\frac{1 + c_3}{4} = e(x, S_1) = e(x, S_2),$$

since $c_3 > 1/3$, and the assumption (6.2) brings to

$$e(x, S_1) = -\frac{1 + c_3}{4} \geq \frac{1 + c_3}{4} - 1 + c_2 = e(x, S_5).$$

And so, the excesses in this case possess the following order:

$$e(x, S_3) = e(x, S_4) > e(x, S_1) = e(x, S_2) \geq e(x, S_5) \geq e(x, S_6).$$

The maximal excesses $e(x, S_3), e(x, S_4)$ comprise x_3 with reverse signs. Hence, variations of x_3 increase the maximal excess. Fix the quantity x_3. The second largest excesses $e(x, S_1), e(x, S_2)$ do coincide. Any variations of x_1, x_2 would increase the second largest excess. Thus, the imputation $x_1 = x_2 = \frac{1+c_3}{4}, x_3 = \frac{1-c_3}{2}$ makes the nucleolus.

To proceed, assume that $c_2 > \frac{1-c_3}{2}$ and let

$$c_1 \leq \frac{1 - c_3}{2}. \tag{6.3}$$

Consider the imputation $x_1 = \frac{c_2+c_3}{2}, x_2 = \frac{1-c_2}{2}, x_3 = \frac{1-c_3}{2}$. Again, Table 8.4 presents the corresponding excesses. Here the maximal excesses are $e(x, S_3) = e(x, S_4) = -\frac{1-c_3}{2}$, since

$$-\frac{1 - c_3}{2} \geq -\frac{1 - c_2}{2} = e(x, S_2) = e(x, S_5),$$

and the condition $c_2 > \frac{1-c_3}{2}$ implies that

$$e(x, S_2) = -\frac{1 - c_2}{2} > -\frac{c_2 + c_3}{2} = e(x, S_1).$$

On the other hand, due to (6.3), we have

$$e(x, S_2) = -\frac{1 - c_2}{2} \geq \frac{c_1 + c_2}{2} - 1 + c_1 = e(x, S_6).$$

Therefore,

$$e(x, S_3) = e(x, S_4) \geq e(x, S_2) = e(x, S_5) \geq \max\{e(x, S_1), e(x, S_6)\}.$$

Recall that the maximal excesses $e(x, S_3), e(x, S_4)$ coincide and incorporate x_3 with reverse signs. Hence, it is not allowed to change x_3. Any variations of x_2 cause further growth of the second largest excess due to the equality of the second largest excesses $e(x, S_2)$ and $e(x, S_5)$. This means that $x_1 = \frac{c_2+c_3}{2}, x_2 = \frac{1-c_2}{2}, x_3 = \frac{1-c_3}{2}$ form the nucleolus.

Next, take the case of $c_1 > \frac{1-c_3}{2}$; accordingly, we have $c_2 > \frac{1-c_3}{2}$. Suppose validity of the following inequality:

$$c_1 + c_2 \leq \frac{1 + c_3}{2}. \tag{6.4}$$

Demonstrate that the imputation $x_1 = \frac{1-2c_1+2c_2+c_3}{4}, x_2 = \frac{1+2c_1-2c_2+c_3}{4}, x_3 = \frac{1-c_3}{2}$ represents the nucleolus. Clearly, all $x_i \in [0, 1]$, $i = 1, 2, 3$. The corresponding excesses can be found in Table 8.4. As previously, it is necessary to define the lexicographic order of the excesses. In this case, we have the inequality

$$e(x, S_3) = e(x, S_4) \geq e(x, S_5) = e(x, S_6) > e(x, S_2) \geq e(x, S_1).$$

The first inequality

$$-\frac{1 - c_3}{2} \geq \frac{-3 + 2c_1 + 2c_2 + c_3}{4}$$

appears equivalent to (6.4), whereas the second one is equivalent to the condition $c_2 > \frac{1-c_3}{2}$. The first equality $e(x, S_3) = e(x, S_4)$ claims that any variations of x_3 would increase the maximal excess. And the second equality $e(x, S_5) = e(x, S_6)$ states that any variations of x_2 and x_3 cause an increase in the second largest excess.

Finally, suppose that $c_2 > \frac{1-c_3}{2}$ and

$$c_1 + c_2 > \frac{1 + c_3}{2}. \tag{6.5}$$

We endeavor to show that the imputation $x_1 = \frac{1-2c_1+c_2+c_3}{3}, x_2 = \frac{1+c_1-2c_2+c_3}{3}, x_3 = \frac{1+c_1+c_2-2c_3}{3}$ is the nucleolus. Table 8.4 gives the corresponding excesses. They are in the following lexicographic order:

$$e(x, S_4) = e(x, S_5) = e(x, S_6) > e(x, S_3) \geq e(x, S_2) \geq e(x, S_1).$$

The first inequality appears equivalent to (6.5); the rest are clear. The equalities $e(x, S_4) = e(x, S_5) = e(x, S_6)$ imply that the maximal excess increases by any variations in the imputation. This concludes the proof of Theorem 8.7.

Revert to the bankruptcy problem, see the beginning of this section. The characteristic function (6.1) takes the form $v(1) = v(2) = v(3) = 0$, $v(1, 2, 3) = E$ and

$$v(1, 2) = (E - d_3)^+ = (E - 100)^+,\ v(1, 3) = (E - d_2)^+ = (E - 200)^+,$$
$$v(2, 3) = (E - d_1)^+ = (E - 300)^+.$$

If $E = 100$, we have

$$v(1, 2) = 0,\ v(1, 3) = 0,\ v(2, 3) = 0.$$

This agrees with the case when all values of the characteristic function do not exceed 1/3 of the payoff. According to the theorem, the nucleolus dictates equal sharing.

Next, if $E = 200$, the characteristic function becomes

$$v(1,2) = 100, \ v(1,3) = 0, \ v(2,3) = 0.$$

This matches the second condition of the theorem when $v(1,2)$ is greater than 1/3 of the payoff E, but $v(1,3)$ does not exceed 1/2 of the residual $E - v(1,2)$. Then the nucleolus is require to give this quantity $(E - v(1,2))/2 = (200 - 100)/2 = 50$ to player 3, and to distribute the remaining shares between players 1 and 2 equally (by 75 units).

If $E = 300$, we get

$$v(1,2) = 200, \ v(1,3) = 100, \ v(2,3) = 0.$$

This corresponds to the third case when $v(1,3)$ is greater than 1/2 of $E - v(1,2)$, whereas $v(2,3)$ does not exceed this quantity. And the nucleolus distributes the debt proportionally, i.e., 150 units to player 1, 100 units to player 2, and 50 units to player 3.

Clearly, the variant described in Table 8.1 coincides with the nucleolus of the cooperative game with the characteristic function (6.1).

If $E = 400$, the characteristic function acquires the form $v(1,2) = 300$, $v(1,3) = 200$, $v(2,3) = 100$. This relates to the fourth case of the theorem. By evaluating the nucleolus, we obtain the imputation $(225, 125, 50)$. And finally, if $E = 500$, the fifth case of the theorem arises naturally; the nucleolus equals (266.66, 166.66, 66.66).

8.7 The Shapley vector

A popular solution in the theory of cooperative games consists in the Shapley vector [1953]. Consider a cooperative game $< N, v >$. Denote by $\sigma = (\sigma(1), \dots, \sigma(n))$ an arbitrary permutation of players $1, \dots, n$. Imagine the following situation. Players get together randomly in some room to form a coalition. By assumption, all permutations σ are equiprobable. And the probability each permutation makes up $1/n!$.

Consider a certain player i. We believe that the coalition is finally formed with his arrival. Designate by $P_\sigma(i) = \{j \in N : \sigma^{-1}(j) < \sigma^{-1}(i)\}$ the set of his forerunners in the permutation σ. Evaluate the contribution of player i to this coalition as

$$m_i(\sigma) = v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i)).$$

Definition 8.17 *The Shapley vector is the mean value of contributions for each player over all possible permutations, i.e.,*

$$\phi_i(v) = \frac{1}{n!}\sum_\sigma m_i(\sigma) = \frac{1}{n!}\sum_\sigma [v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))], i = 1, \dots, n. \tag{7.1}$$

8.7.1 The Shapley vector in the road construction game

Construct the Shapley vector in the road construction game stated before. Recall that the characteristic function takes the form

$$v(1) = -1,\ v(2) = -1,\ v(3) = -2,\ v(1,2) = 12,\ v(2,3) = 5,\ v(1,3) = 3,\ v(1,2,3) = 18.$$

Computations of the Shapley vector can be found in Table 8.5 below. The left column presents all possible permutations, as well as the contributions of all players (for each permutation) according to the given characteristic function. For the first permutation $(1,2,3)$, the contribution of player 1 constitutes $v(1) - v(\emptyset) = -1$, the contribution of player 2 makes up $v(1,2) - v(1) = 12 - (-1) = 13$, and the contribution of player 3 equals $v(1,2,3) - v(1,2) = 18 - 12 = 6$. Calculations yield the Shapley vector $\phi = (7,8,3)$ in this problem. Note that this solution differs from the nucleolus $x^* = (9,7,2)$. Accordingly, we observe variations in the cost of each player in road construction. Now, the shares of players' costs have the form $c_1 = 20 - 7 = 13,\ c_2 = 15 - 8 = 7,\ c_3 = 10 - 3 = 7$.

Find a more convenient representation for Shapley vector evaluation. The bracketed expression in (7.1) is $v(S) - v(S\backslash\{i\})$, where player i belongs to the coalition S. Therefore, summation in formula (7.1) can run over all coalitions S containing player i. In each coalition S, player i is on the last place, whereas forerunners can enter the coalition in $(|S| - 1)!$ ways. On the other hand, players from the coalition $N\backslash S$ can come after player i in $(n - |S|)!$ ways. Thus, the number of permutations in the sum (7.1), which correspond to the same coalition S containing player i, equals $(|S| - 1)!(n - |S|)!$. Hence, formula (7.1) can be rewritten as

$$\phi_i(v) = \sum_{S:i\in S} \frac{(|S| - 1)!(n - |S|)!}{n!}[v(S) - v(S\backslash\{i\})], i = 1, \dots, n. \tag{7.2}$$

The quantities $\frac{(|S|-1)!(n-|S|)!}{n!}$ stand for the probabilities that player i forms the coalition S. And so,

$$\sum_{S:i\in S} \frac{(|S| - 1)!(n - |S|)!}{n!} = 1,\ \forall i. \tag{7.3}$$

We demonstrate that the introduced vector is an imputation.

Table 8.5 Evaluation of the Shapley vector.

σ	Player 1	Player 2	Player 3	The total contribution
$(1,2,3)$	-1	13	6	18
$(1,3,2)$	-1	15	4	18
$(2,1,3)$	13	-1	6	18
$(2,3,1)$	13	-1	6	18
$(3,1,2)$	5	15	-2	18
$(3,2,1)$	13	7	-2	18
The mean value	7	8	3	18

Lemma 8.1 *The vector ϕ satisfies the properties of individual rationality and efficiency, i.e., $\phi_i(v) \geq v(i)$, $\forall i$ and $\sum_{i \in N} \phi_i(v) = v(N)$.*

Proof: Due to the superadditive property of the function v, we have the inequality $v(S) - v(S \setminus \{i\}) \geq v(i)$ for any coalition S. Then it follows from (7.2) and (7.3) that

$$\phi_i(v) \geq v(i) \sum_{S: i \in S} \frac{(|S|-1)!(n-|S|)!}{n!} = v(i), \quad i = 1, \dots, n.$$

Now, let us show the equality $\sum_{i \in N} \phi_i(v) = v(N)$. Address the definition (7.1) and consider the sum

$$\sum_{i \in N} \phi_i(v) = \frac{1}{n!} \sum_{\sigma} \sum_{i \in N} [v(P_\sigma(i) \cup \{i\}) - v(P_\sigma(i))]. \tag{7.4}$$

For each permutation σ, the sum (7.4) comprises the contributions of all players in this permutation, i.e.,

$$v(\sigma(1)) + [v(\sigma(1), \sigma(2)) - v(\sigma(1))] + [v(\sigma(1), \sigma(2), \sigma(3)) - v(\sigma(1), \sigma(2))]$$
$$+ \cdots + [v(\sigma(1), \dots, \sigma(n)) - v(\sigma(1), \dots, \sigma(n-1))] = v(\sigma(1), \dots, \sigma(n)) = v(N).$$

Hence,

$$\sum_{i \in N} \phi_i(v) = \frac{1}{n!} \sum_{\sigma} v(N) = v(N).$$

The proof of Lemma 8.1 is completed.

We have already formulated some criteria for the solution of a cooperative game. For core, the matter concerns an undominated offer. In the case of nucleolus, a solution minimizes the maximal dissatisfaction from other solutions. L.S. Shapley stated several desired properties to-be-enjoyed by an imputation in a cooperative game.

8.7.2 Shapley's axioms for the vector $\varphi_i(v)$

1. **Efficiency**. $\sum_{i \in N} \varphi_i(v) = v(N)$.

2. **Symmetry**. If players i and j are such that $v(S \cup \{i\}) = v(S \cup \{j\}$ for any coalition S without players i and j, then $\varphi_i(v) = \varphi_j(v)$.

3. **Dummy player property**. Player i such that $v(S \cup \{i\}) = v(S)$ for any coalition S without player i meets the condition $\varphi_i(v) = 0$.

4. **Linearity**. If v_1 and v_2 are two characteristic functions, then $\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2)$.

Axiom 1 declares that the whole payoff must be completely distributed among participants. The symmetry property consists in that, if the characteristic function is symmetrical for players

i and j, they must receive equal shares. A player giving no additional utility to any coalition is called a **dummy player**. Of course, his share must equal zero. The last axiom reflects the following fact. If a series of games are played, the share of each player in the series must coincide with the sum of shares in each game.

Theorem 8.8 *There exists a unique vector $\varphi(v)$ satisfying Axioms 1–4.*

Proof: Consider the elementary characteristic functions.

Definition 8.18 *Let $S \subset N$. The elementary characteristic function is the function*

$$v_S(T) = \begin{cases} 1 \text{ if } S \subset T \\ 0 \text{ otherwise.} \end{cases}$$

Therefore, in a cooperative game with such characteristic function, the coalition T wins if it contains some minimal winning coalition S. We endeavor to find the vector $\varphi(v_S)$ agreeing with Shapley's axioms. Interestingly, any player outside the minimal winning coalition represents a zero player. According to Axiom 2, $\varphi_i(v_S) = 0$ if $i \notin S$. The symmetry axiom implies that $\varphi_i(v_S) = \varphi_j(v_S)$ for all players i, j entering the coalition S. In combination with the efficiency axiom, this yields $\sum\limits_{i \in N} \varphi_i(v_S) = v_S(N) = 1$. Hence, $\varphi_i(v_S) = 1/|S|$ for all players in the coalition S. Analogous reasoning applies to the characteristic function cv_S, where c indicates a constant factor. Then

$$\varphi_i(cv_S) = \begin{cases} \dfrac{c}{|S|} & \text{if } i \in S \\ 0 & \text{if } i \notin S. \end{cases}$$

Lemma 8.2 *The elementary characteristic functions form a basis on the set of all characteristic functions.*

Proof: We establish that any characteristic function can be rewritten as a linear combination of the elementary functions, i.e., there exist constants c_S such that

$$v = \sum_{S \in 2^N} c_S v_S. \tag{7.5}$$

Choose $c_\emptyset = 0$. Argue the existence of the constants c_S by induction over the number of elements in the set S. Select

$$c_T = v(T) - \sum_{S \subset T, S \neq T} c_S.$$

In other words, the value of c_T is determined via the quantities c_S, where the number of elements in the set S appears smaller than in the set T.

Since $v_S(T)$ is non-zero only for coalitions $S \subset T$, the above-defined constants c_S obey the equality

$$\sum_{S \in 2^N} c_S v_S(T) = \sum_{S \subset T} c_S = c_T + \sum_{S \subset T, S \neq T} c_S = v(T).$$

This proves formula (7.5).

Consequently, each characteristic function is uniquely represented as the sum of the elementary characteristic functions $c_S v_S$. By virtue of linearity, the vector $\varphi(v)$ turns out uniquely defined, either:

$$\varphi_i(v) = \sum_{i \in S \in 2^N} \frac{c_S}{|S|}.$$

The proof of Theorem 8.8 is finished.

Now, we note that the Shapley vector—see (7.2)—meets Axioms 1–4. Its efficiency has been rigorously shown in Lemma 8.1.

Symmetry comes from the following fact. If players i and j satisfy the condition $v(S \cup \{i\}) = v(S \cup \{j\}$ for any coalition S without players i and j, the contributions of the players to the sum in (7.2) do coincide, and hence $\phi_i(v) = \phi_j(v)$.

Imagine that player i represents a zero player. Then all his contributions (see the bracketed expressions in (7.2)) vanish and $\phi_i(v) = 0$, i.e., the property of zero player holds true.

Finally, linearity follows from the additive form of the expression in (7.2).

Satisfaction of Shapley's axioms and uniqueness of an imputation meeting these axioms (see Theorem 8.8) lead to an important result.

Theorem 8.9 *A unique imputation agreeing with Axioms 1–4 is the Shapley vector $\phi(v) = (\phi_1(v), \ldots, \phi_n(v))$, where*

$$\phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{(|S|-1)!(n-|S|)!}{n!}[v(S) - v(S \backslash \{i\})], i = 1, \ldots, n.$$

Remark 8.1 If summation in (7.2) runs over coalitions excluding player i, the Shapley vector formula becomes

$$\phi_i(v) = \sum_{S \subseteq N : i \notin S} \frac{(|S|)!(n-|S|-1)!}{n!}[v(S \cup i) - v(S)], i = 1, \ldots, n. \tag{7.6}$$

8.8 Voting games. The Shapley–Shubik power index and the Banzhaf power index

We should mention political science (especially, political decision-making) among important applications of cooperative games. Generally, a political decision is made by voting in some public authority, e.g., a parliament. In these conditions, a major role belongs to the power of factions within such an authority. The matter concerns political parties possessing a certain set of votes, or other political unions. Political power definition was studied by L. Shapley and M. Shubik [1954], as well as by J.F. Banzhaf [1965]. In their works, the researchers employed certain methods from the theory of cooperative games to define the power or influence level of voting sides.

Definition 8.19 *A voting game is a cooperative game $< N, v >$, where the characteristic function takes only two values,* 0 *and* 1*,* $v(N) = 1$*. A coalition S such that $v(S) = 1$ is called a winning coalition. Denote by W the set of winning coalitions.*

Within the framework of voting games, the contribution of each player in any coalition equals 0 or 1. Therefore, the Shapley vector concept can be modified for such games.

Definition 8.20 *The Shapley–Shubik vector in a voting game $< N, v >$ is the vector $\phi(v) = (\phi_1(v), \dots, \phi_n(v))$, where the index of player i has the form*

$$\phi_i(v) = \sum_{S \notin W, S \cup \in W} \frac{(|S|)!(n - |S| - 1)!}{n!}, i = 1, \dots, n.$$

According to Definition 8.20, the influence of player i is defined as the mean number of coalitions, where his participation guarantees win and non-participation leads to loss.

In fact, there exists another definition of player's power, *viz.*, the so-called Banzhaf index. For player i, we say that a pair of coalitions $(S \cup i, S)$ is a switching, when $(S \cup i)$ appears as a winning coalition and the coalition S does not. In this case, player i is referred to as the **key player** in the coalition S. For each player $i \in N$, evaluate the number of all switchings in the game $< N, v >$ and designate it by $\eta_i(v)$. The total number of switchings makes up $\eta(v) = \sum_{i \in N} \eta_i(v)$.

Definition 8.21 *The Banzhaf vector in a voting game $< N, v >$ is the vector $\beta(v) = (\beta_1(v), \dots, \beta_n(v))$, where the index of player i obeys the formula*

$$\beta_i(v) = \frac{\eta_i(v)}{\sum_{j \in N} \eta_j(v)}, i = 1, \dots, n.$$

Now, let us concentrate on voting games proper. We believe that each player i in a voting game is described by some number of votes w_i, $i = 1, \dots, n$. Furthermore, the affirmative decision requires a given threshold q of votes.

Definition 8.22 *A weighted voting game is a cooperative game $< q; w_1, \dots, w_n >$ with the characteristic function*

$$v(S) = \begin{cases} 1, & \text{if } w(S) \geq q \\ 0, & \text{if } w(S) < q. \end{cases}$$

Here $w(S) = \sum_{i \in S} w_i$ specifies the sum of votes of players in a coalition S.

To compute power indices, one can involve generating functions. Recall that the **generating function** of a sequence $\{a_n, n \geq 0\}$ is the function

$$G(x) = \sum_{n \geq 0} a_n x^n.$$

For a sequence $\{a(n_1, \dots, n_k), n_i \geq 0, i = 1, \dots, k\}$, this function acquires the form

$$G(x_1, \dots, x_k) = \sum_{n_1 \geq 0} \dots \sum_{n_k \geq 0} a(n_1, \dots, n_k) x_1^{n_1} \dots x_k^{n_k},$$

The generating function for Shapley–Shubik power index evaluation was found by D.G. Cantor [1962]. In the case of the Banzhaf power index, the generating function was obtained by S.J. Brams and P.J. Affuso [1976].

Theorem 8.10 *Suppose that* $< q; w_1, \dots, w_n >$ *represents a weighted voting game. Then the Shapley–Shubik power index is defined by*

$$\phi_i(v) = \sum_{s=0}^{n-1} \frac{s!(n-s-1)!}{n!} \left(\sum_{k=q-w_i}^{q-1} A^i(k, s) \right), i = 1, \dots, n. \tag{8.1}$$

Here $A^i(k, s)$ *means the number of coalitions* S *comprising exactly* s *players and* $i \notin S$, *whose power equals* $w(S) = k$. *In addition, the generating function takes the form*

$$G_i(x, z) = \prod_{j \neq i} (1 + zx^{w_j}). \tag{8.2}$$

Proof: Consider the product $(1 + zx^{w_1})...(1 + zx^{w_n})$. By removing all brackets, we get the following expression. As their coefficients, identical degrees z^k hold the quantity x raised to the power $w_{i_1} + \dots + w_{i_k}$ for different combinations of $(i_1, \dots, i_k)$, i.e.,

$$(1 + zx^{w_1})...(1 + zx^{w_n}) = \sum_{S \subset N} z^{|S|} x^{\sum_{i \in S} w_i}. \tag{8.3}$$

Now, take this sum and extract terms having identical degrees of x (they correspond to coalitions with the same power $w(S)$). Such manipulations yield

$$(1 + zx^{w_1})...(1 + zx^{w_n}) = \sum_{k \geq 0} \sum_{s \geq 0} A(k, s) x^k z^s,$$

where the factor $A(k, s)$ equals the number of all coalitions with s participants, whose power constitutes k. By eliminating the factor $(1 + zx^{w_i})$ in the product (8.3), we construct the generating function for $A_i(k, s)$.

Formula (8.1) is immediate from the following fact. Coalitions, where player i appears the key one, are coalitions S with the power levels $w(S) \in \{q - w_i, q - w_i + 1, \dots, q - 1\}$. Really, in this case, we have $w(S \cup i) = w(S) + w_i \geq q$. The proof of Theorem 8.10 is concluded.

Theorem 8.11 *Let* $< q; w_1, \dots, w_n >$ *be a weighted voting game. The number of switchings* $\eta_i(v)$ *in the Banzhaf power index can be rewritten as*

$$\eta_i(v) = \sum_{k=q-w_i}^{q-1} b^i(k), i = 1, \dots, n, \tag{8.4}$$

where $b^i(k)$ *stands for the number of coalitions* $S : i \notin S$, *whose power makes up* $w(S) = k$. *Furthermore, the generating function has the form*

$$G_i(x) = \prod_{j \neq i} (1 + x^{w_j}). \tag{8.5}$$

Proof: Similarly to Theorem 8.10, consider the product $(1 + x^{w_1})...(1 + x^{w_n})$ and remove the brackets:

$$G(x) = (1 + x^{w_1})...(1 + x^{w_n}) = \sum_{S \subset N} \prod_{i \in S} x^{w_i} = \sum_{S \subset N} x^{\sum_{i \in S} w_i}. \tag{8.6}$$

Again, extract summands with coinciding degrees of x, that correspond to coalitions having the identical power $w(S)$. This procedure brings to

$$G(x) = \sum_{k \geq 0} b(k) x^k,$$

where the factor $b(k)$ is the number of all coalitions with power k. By eliminating the factor $(1 + x^{w_i})$ from the product (8.6), we derive the generating function (8.5) for $b_i(k)$. Formula (8.4) follows from when coalitions, where player i is the key one, are actually coalitions S with the power $w(S) \in \{q - w_i, q - w_i + 1, \dots, q - 1\}$. This completes the proof of Theorem 8.11.

Theorems 8.10 and 8.11 provide a simple computation technique for the above power indices. As examples, we select the 14th Bundestag (the national parliament of the Federal Republic of Germany, 1998–2002) and the 3rd State Duma (the lower chamber of the Russian parliament, 2000–2003).

8.8.1 The Shapley–Shubik power index for influence evaluation in the 14th Bundestag

The 14th Bundestag consisted of 669 members from five political parties:

- The Social Democratic Party of Germany (*Sozialdemokratische Partei Deutschlands*, SPD), 298 seats
- The Christian Democratic Union of Germany (*Christlich Demokratische Union Deutschlands*, CDU), 245 seats
- The Greens (*Die Grünen*), 47 seats
- The Free Democratic Party (*Freie Demokratische Partei*, FDP), 43 seats
- The Party of Democratic Socialism (*Partei des Demokratischen Sozialismus*, PDS), 36 seats.

For a draft law, the enactment threshold q is the simple majority of votes, i.e., 335 votes. To find the influence levels of each party, we apply the Shapley–Shubik power indices. It is necessary to calculate the generating function (8.2).

For the SDP, the generating function acquires the form

$$\begin{aligned} G_1(x,z) &= (1+zx^{245})(1+zx^{47})(1+zx^{43})(1+zx^{36}) \\ &= 1+z(x^{36}+x^{43}+x^{47}+x^{245})+z^2(x^{79}+x^{83}+x^{90}+x^{281}+x^{288}+x^{292}) \\ &\quad +z^3(x^{126}+x^{324}+x^{328}+x^{335})+z^4x^{371}. \end{aligned} \tag{8.7}$$

Now, we can define the number of coalitions, where the SDP appears to be the key coalition. This requires that the influence level of a coalition before SDP entrance lies between $q-w_1 = 335-298 = 37$ and $q-1 = 334$. The first bracketed expression shows that, for $s=1$, the number of such coalitions makes up 3. In the cases of $s=2$ and $s=3$, we use the second bracketed expression to find that the number of such coalitions is 6 and 3, respectively. In the rest cases, the SDP does not represent the key coalition. Therefore,

$$\sum_{k=q-w_1}^{q-1} A^1(k,0)=0, \quad \sum_{k=q-w_1}^{q-1} A^1(k,1)=3, \quad \sum_{k=q-w_1}^{q-1} A^1(k,2)=6,$$

$$\sum_{k=q-w_1}^{q-1} A^1(k,3)=3, \quad \sum_{k=q-w_1}^{q-1} A^1(k,4)=0.$$

Hence, the Shapley–Shubik index equals

$$\phi_1(v) = \frac{1!3!}{5!}3 + \frac{2!2!}{5!}6 + \frac{3!1!}{5!}3 = 0.5.$$

We can perform similar computations for other parties in the 14th Bundestag. The CDU becomes the key coalition, if the influence level of a coalition before its entrance is within the limits of $q-w_2 = 90$ and $q-1 = 334$. For the CDU, the generating function is described by

$$\begin{aligned} G_2(x,z) &= (1+zx^{298})(1+zx^{47})(1+zx^{43})(1+zx^{36}) \\ &= 1+z(x^{36}+x^{43}+x^{47}+x^{298})+z^2(x^{79}+x^{83}+x^{90}+x^{334}+x^{341}+x^{345}) \\ &\quad +z^3(x^{126}+x^{377}+x^{381}+x^{388})+z^4x^{424}, \end{aligned}$$

whence it follows that $\sum_{k=q-w_2}^{q-1} A^2(k,0)=0$, $\sum_{k=q-w_2}^{q-1} A^2(k,1)=1$, $\sum_{k=q-w_2}^{q-1} A^2(k,2)=2$, $\sum_{k=q-w_2}^{q-1} A^2(k,3)=1$, $\sum_{k=q-w_2}^{q-1} A^2(k,4)=0$. The corresponding Shapley–Shubik power index constitutes

$$\phi_2(v) = \frac{1!3!}{5!}1 + \frac{2!2!}{5!}2 + \frac{3!1!}{5!}1 = \frac{1}{6} \approx 0.1667.$$

In the case of the Greens, we obtain

$$
\begin{aligned}
G_3(x,z) &= (1+zx^{298})(1+zx^{245})(1+zx^{47})(1+zx^{36}) \\
&= 1 + z(x^{36}+x^{47}+x^{245}+x^{298}) + z^2(x^{83}+x^{281}+x^{292}+x^{334}+x^{345}+x^{543}) \\
&\quad + z^3(x^{328}+x^{381}+x^{579}+x^{590}) + z^4x^{626}.
\end{aligned}
$$

This party is the key coalition for coalitions whose influence levels lie between $q - w_3 = 292$ and $q - 1 = 334$. The number of coalitions, where the Greens form the key coalition, coincides with the appropriate number for the CDU; thus, their power indices are identical. The same applies to the FDP.

The PDS possesses the generating function

$$
\begin{aligned}
G_5(x,z) &= (1+zx^{298})(1+zx^{245})(1+zx^{47})(1+zx^{43}) \\
&= 1 + z(x^{43}+x^{47}+x^{245}+x^{298}) + z^2(x^{90}+x^{288}+x^{292}+x^{341}+x^{345}+x^{543}) \\
&\quad + z^3(x^{335}+x^{388}+x^{586}+x^{590}) + z^4x^{633}.
\end{aligned}
$$

The PDS is the key coalition for coalitions with the influence levels in the range $[q - w_5 = 299, 334]$. However, the shape of the generating function implies the non-existence of such coalitions. Hence, $\phi_5(v) = 0$.

And finally,

$$
\phi_1(v) = \frac{1}{2},\ \phi_2(v) = \phi_3(v) = \phi_4(v) = \frac{1}{6},\ \phi_5(v) = 0.
$$

Therefore, the greatest power in the 14th Bundestag belonged to the Social Democrats. Yet the difference in the number of seats of this party and the CDU was not so significant (298 and 245, respectively). Meanwhile, the power indices of the CDU, the Greens, and the Free Democrats turned out the same despite considerable differences in their seats (245, 47, and 43, respectively). The PDS possessed no influence at all, as it was not the key one in any coalition.

8.8.2 The Banzhaf power index for influence evaluation in the 3rd State Duma

The 3rd State Duma of the Russian Federation was represented by the following political parties and factions:

The Agro-industrial faction (AIF), 39 seats
The Unity Party (UP), 82 seats
The Communist Party of the Russian Federation (CPRF), 92 seats
The Liberal Democratic Party (LDPR), 17 seats
The People's Deputy faction (PDF), 57 seats
Fartherland-All Russia (FAR), 46 seats
Russian Regions (RR), 41 seats
The Union of Rightist Forces (URF), 32 seats
Yabloko (YAB), 21
Independent deputies (IND), 23 seats.

Table 8.6 Banzhaf index evaluation for the State Duma of the Russian Federation.

Parties	AIF	UP	CPRF	LDPR	PDF
Switching thresholds	187-225	144-225	134-225	209-225	169-225
The number of switchings	96	210	254	42	144
The Banzhaf index	0.084	0.185	0.224	0.037	0.127
Parties	FAR	RR	URF	YAB	IND
Switching thresholds	180-225	185-225	194-225	205-225	203-225
The number of switchings	110	100	78	50	52
The Banzhaf index	0.097	0.088	0.068	0.044	0.046

The 3rd State Duma included 450 seats totally. For a draft law, the enactment threshold q is the simple majority of votes, i.e., 226 votes. To evaluate the influence level of each party and faction, we employ the Banzhaf power index. To compute the generating function $G_i(x) = \prod_{j \neq i}(1 + x^{w_j}), i = 1, \dots, 10$ for each player, it is necessary to remove brackets in the product (8.5). For instance, we can use any software for symbolic computations, e.g., *Mathematica* (procedure *Expand*). For each player $i = 1, \dots, 10$, remove brackets and count the number of terms of the form x^k, where k varies from $q - w_i$ to $q - 1$. This corresponds to the number of coalitions, where players are key ones. For the AIF, these thresholds make up $q - w_1 = 225 - 39 = 187$ and 225. Table 8.6 combines the thresholds for calculating the number of switchings for each player. Moreover, it presents the resulting Banzhaf indices of the parties.

In addition to the power indices suggested by Shapley–Shubik and Banzhaf, researchers sometimes employ the Deegan–Packel power index [1978] and the Holler index [1982]. They are defined through minimal winning coalitions.

Definition 8.23 *A minimal winning coalition is a coalition where each player appears the key one.*

Definition 8.24 *The Deegan–Packel vector in a weighted voting game $< N, v >$ is the vector $dp(v) = (dp_1(v), \dots, dp_n(v))$, where the index of player i has the form*

$$dp_i(v) = \frac{1}{m} \sum_{S \in M : i \in S} \frac{1}{s}, \ i = 1, \dots, n.$$

Here M denotes the set of all minimal winning coalitions, m means the total number of minimal winning coalitions and s is the number of members in a coalition S.

Definition 8.25 *The Holler vector in a weighted voting game $< N, v >$ is the vector $h(v) = (h_1(v), \dots, h_n(v))$, where the index of player i has the form*

$$h_i(v) = \frac{m_i(v)}{\sum_{i \in N} m_i(v)}, \ i = 1, \dots, n.$$

Here m_i specifies the number of minimal winning coalitions containing player i $(i = 1, \dots, n)$.

Evaluate these indices for defining the influence levels of political parties in the parliament of Japan.

8.8.3 The Holler power index and the Deegan–Packel power index for influence evaluation in the National Diet (1998)

The National Diet is Japan's bicameral legislature. It consists of a lower house (the House of Representatives) and an upper house (the House of Councillors). We will analyze the House of Councillors only, which comprises 252 seats. After the 1998 elections, the majority of seats were accumulated by six parties:

The Liberal Democratic Party (LDP), 105 seats
The Democratic Party of Japan (DPJ), 47 seats
The Japanese Communist Party (JCP), 23 seats
The Komeito Party (KP), 22 seats
The Social Democratic Party (SDP), 13 seats
The Liberal Party (LP), 12 seats
The rest parties (RP), 30 seats.

For a draft law, the enactment threshold q is the simple majority of votes, i.e., 127 votes. Obviously, the minimal winning coalitions are (LDP, DPJ), (LDP, JCP), (LDP, KP), (LDP, RP), (LDP, SDP, LP), (DPJ, JCP, KP, SDP, RP), and (DPJ, JCP, KP, LP, RP). Therefore, the LDP appears in five minimal winning coalitions, the DPJ, the JCP, the KP, the RP belong to three minimal winning coalitions, the SDP, and the LP enter two minimal winning coalitions. And the Holler power indices make up $h_1(v) = 5/21 \approx 0.238$, $h_2(v) = h_3(v) = h_4(v) = h_7(v) = 3/21 \approx 0.144$, $h_5(v) = h_6(v) = 2/21 \approx 0.095$.

Now, evaluate the Deegan–Packel power indices:

$$dp_1(v) = \frac{1}{7}\left(4\frac{1}{2} + \frac{1}{3}\right) = 1/3 \approx 0.333,$$

$$dp_2(v) = dp_3(v) = dp_4(v) = dp_7(v) = \frac{1}{7}\left(\frac{1}{2} + 2\frac{1}{5}\right) \approx 0.128,$$

$$dp_5(v) = dp_6(v) = \frac{1}{7}\left(\frac{1}{3} + \frac{1}{5}\right) \approx 0.076.$$

Readers can see that the influence of the Liberal Democrats is more than two times higher than any party in the House of Councillors. The rest parties get decomposed into two groups of almost identical players.

8.9 The mutual influence of players. The Hoede–Bakker index

The above models of voting ignore the mutual influence of players. However, real decision-making includes situations when a certain player or some players modify an original decision under the impact of others.

Consider a voting game $< N, v >$, where the characteristic function possesses two values, 0 and 1, as follows.

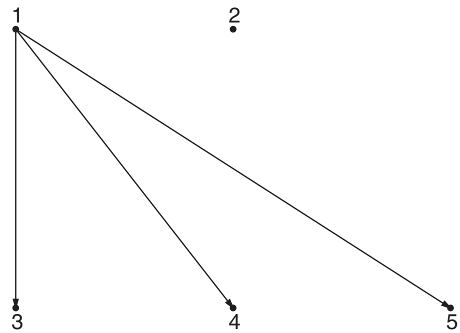


Figure 8.5 Influence graph. Player 1 influences players 3, 4, and 5. Player 2 is fully independent.

Imagine that players $N = \{1, 2, \dots, n\}$ have to select a certain draft law. The weights of the players are represented by the vector $w = (w_1, \dots, w_n)$. Next, their initial preferences form a binary vector $\pi = (\pi_1, \dots, \pi_n)$, where π_i equals 1, if player i supports the draft law, and equals 0 otherwise. At stage 1, players hold a consultation, and the vector π gets transformed into a new decision vector $b = B\pi$ (which is also a binary vector). The operator B can be defined, e.g., using the mutual influence graph of players (see Figure 8.5). Stage 2 lies in calculating the affirmative votes for the draft law; the affirmative decision follows, if their number is not smaller than a given threshold q, $1 \le q \le n$. Therefore, the collective decision is defined by the characteristic function

$$v(b) = v(B\pi) = I\left\{\sum_{i=1}^{n} b_i w_i \ge q\right\}. \tag{9.1}$$

Here $I\{A\}$ means the indicator of the set A. Suppose that the function v matches a couple of axioms below. As a matter of fact, this requirement applies to the operator B.

A1. Denote by $\bar{\pi}$ the complement vector, where $\bar{\pi}_i = 1 - \pi_1$. Then any preference vector π meets the equality

$$v(B\bar{\pi}) = 1 - v(B\pi).$$

A2. Consider vectors π, π' and define the order $\pi \le \pi'$, if $\{i \in N : \pi_i = 1\} \subset \{i \in N : \pi'_i = 1\}$. Then any preference vectors π, π' such that $\pi \le \pi$ satisfy the condition $v(B\pi) \le v(B\pi')$.

According to Axiom A1, the decision-making rule must be such that, if all players reverse their initial opinions, the collective decision is also contrary. Axiom A2 claims that, if players with initial affirmative decision are supplemented by other players, the final collective decision either remains the same or may turn out affirmative (if it was negative).

Definition 8.26 *The Hoede–Bakker index of player i is the quantity*

$$HB_i(v(B)) = \frac{1}{2^{n-1}} \sum_{\pi:\pi_i=1} v(B\pi), i = 1, \dots, n. \tag{9.2}$$

Figure 8.6 Influence graph. Player A influences player B. Players C and D are fully independent.

In the expression (9.2), summation runs over all binary preference vectors π, where the opinion of player i is affirmative. The Hoede–Bakker index reflects the influence level of player i on draft law adoption under equiprobable preferences of other players (the latter may have mutual impacts).

Concluding this section, we study the following example. A parliament comprises four parties A, B, C, and D that have 10, 20, 30, and 40 seats, respectively. Assume that (a) draft law enactment requires 50 votes and (b) party A exerts an impact on party B (see Figure 8.6).

First, compute the Banzhaf indices:

$$\beta_1(v) = \frac{1}{12},\ \beta_2(v) = \beta_3(v) = \frac{3}{12},\ \beta_4(v) = \frac{5}{12}.$$

Evidently, the influence level of party A is minimal; however, we neglect the impact of party A on party B.

Second, calculate the Hoede–Bakker indices, see Table 8.7.

We find that

$$HB_1(v) = HB_3(v) = HB_4(v) = \frac{3}{4},\ HB_2(v) = \frac{1}{2}.$$

Now, the influence level of party A goes up, reaching the levels of parties C and D.

Table 8.7 Hoede–Bakker index evaluation.

π	0,0,0,0	0,0,0,1	0,0,1,0	0,1,0,0	1,0,0,0	0,0,1,1	0,1,0,1	0,1,1,0
$B\pi$	0,0,0,0	0,0,0,1	0,0,1,0	0,0,0,0	1,1,0,0	0,0,1,1	0,0,0,1	0,0,1,0
$v(B\pi)$	0	0	0	0	0	1	0	0
π	1,0,0,1	1,0,1,0	1,1,0,0	0,1,1,1	1,0,1,1	1,1,0,1	1,1,1,0	1,1,1,1
$B\pi$	1,1,0,1	1,1,1,0	1,1,0,0	0,0,1,1	1,1,1,1	1,1,0,1	1,1,1,0	1,1,1,1
$v(B\pi)$	1	1	0	1	1	1	1	1

Exercises

1. The jazz band game with four players.

   A restaurateur invites a jazz band to perform one evening and offers 100 USD. The jazz band consists of four musicians, namely, a pianist (player 1), a vocalist (player 2), a drummer (player 3), and a guitarist (player 4). They should distribute the fee. An argument during such negotiations is the characteristic function v defined by individual honoraria the players may receive by performing singly, in pairs or triplets. The characteristic function has the form

$$v(1) = 40,\ v(2) = 30,\ v(3) = 20,\ v(4) = 0,$$
$$v(1,2) = 80,\ v(1,3) = 70,\ v(1,4) = 50,\ v(2,3) = 60,\ v(2,4) = 35,\ v(3,4) = 25,$$
$$v(1,2,3) = 95,\ v(1,2,4) = 85,\ v(1,3,4) = 75,\ v(2,3,4) = 65.$$

   Construct the core and the τ-equilibrium of this game.
2. Find the Shapley vector in exercise no. 1.
3. The road construction game with four players.

   Four farms agree to construct a road communicating all farms with a city. Construction of each segment of the road incurs definite costs. Each farm has a specific income from selling its agricultural products in the city. Road infrastructure, the construction cost of each road segment and the incomes of the farmers are illustrated in Figure 8.7.

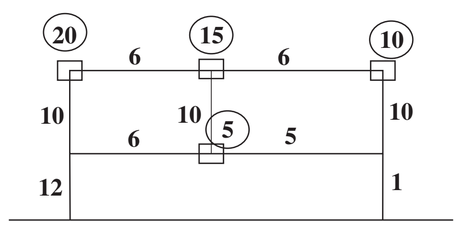


Figure 8.7 Road construction.

   Build the nucleolus of this game.
4. Construct the core and the τ-equilibrium of the game in exercise no. 2.
5. The shoes game.

   This game involves four sellers. Sellers 1 and 2 have right shoes, whereas sellers 3 and 4 have left shoes. The price of a single shoe is 0 USD, the price of a couple makes up 10 USD. Sellers strive to obtain some income from shoes. Build the core and the τ-core of this game.
6. Take the game from exercise no. 5 and construct the Shapley vector.
7. Give an example of a cooperative game which is not quasibalanced.
8. Give an example of a cooperative game, which is quasibalanced, but fails to be balanced.

9. The parliament of a small country has 40 seats distributed as follows:

Party 1, 20 seats

Party 2, 15 seats

Party 3, 5 seats

For a draft law, the enactment threshold q is the simple majority of votes, i.e., 21 votes. Evaluate the Shapley–Shubik power index of the parties.

10. Consider the parliament from exercise no. 9 and find the Banzhaf power index of the parties.

9

Network games

Introduction

Games in information networks form a modern branch of game theory. Their development was connected with expansion of the global information network (Internet), as well as with organization of parallel computations on supercomputers. Here the key paradigm concerns the non-cooperative behavior of a large number of players acting independently (still, their payoffs depend on the behavior of the rest participants). Each player strives for transmitting or acquiring maximum information over minimum possible time. Therefore, the payoff function of players is determined either as the task time or as the packet transmission time over a network (to-be-minimized). Another definition of the payoff function lies in the transmitted volume of information or channel capacity (to-be-maximized).

An important aspect is comparing the payoffs of players with centralized (cooperative) behavior and their equilibrium payoffs under non-cooperative behavior. Such comparison provides an answer to the following question. Should one organize management in a system (thus, incurring some costs)? If this sounds inefficient, the system has to be self-organized.

Interesting effects arise naturally in the context of equilibration. Generally speaking, in an equilibrium players may obtain non-maximal payoffs. Perhaps, the most striking result covers Braess's (1968) paradox (network expansion reduces the equilibrium payoffs of different players).

There exist two approaches to network games analysis. According to the first one, a player chooses a route for packet transmission; a packet is treated as an indivisible quantity. Here we mention the works by Papadmitrious and Koutsoupias (1999). Accordingly, such models will be called the KP-models. The second approach presumes that a packet can be divided into segments and transmitted by different routes. It utilizes the equilibrium concept suggested by J.G. Wardrop (1952).

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

9.1 The KP-model of optimal routing with indivisible traffic. The price of anarchy

We begin with an elementary information network representing m parallel channels (see Figure 9.1).

Consider a system of n users (players). Player i ($i = 1, \dots, n$) intends to send traffic of some volume w_i through a channel. Each channel $l = 1, \dots, m$ has a given capacity c_l. When traffic of a volume w is transmitted by a channel with a capacity c, the channel delay equals w/c.

Each user pursues individual interests, endeavoring to occupy the minimal delay channel. The pure strategy of player i is the choice of channel l for his traffic. Consequently, the vector $L = (l_1, \dots, l_n)$ makes the pure strategy profile of all users; here l_i means the number of the channel selected by user i. His mixed strategy represents the probabilistic distribution $p_i = (p_i^1, \dots, p_i^m)$, where p_i^l stands for the probability of choosing channel l by user i. The matrix P composed of the vectors p_i is the mixed strategy profile of the users.

In the case of pure strategies for user i, the traffic delay in the channel l_i is determined by $\lambda_i = \frac{\sum\limits_{k:l_k=l_i} w_k}{c_{l_i}}$.

Definition 9.1 *A pure strategy profile $(l_1, \dots, l_n)$ is called a Nash equilibrium, if for each user i we have $\lambda_i = \min\limits_{j=1,\dots,m} \frac{w_i + \sum\limits_{k \neq i: l_k = j} w_k}{c_j}$.*

In the case of mixed strategies, it is necessary to introduce the expected traffic delay for user i employing channel l. This characteristic makes up $\lambda_i^l = \frac{w_i + \sum\limits_{k=1, k\neq i}^{n} p_k^l w_k}{c_l}$. The minimal expected delay of user i equals $\lambda_i = \min\limits_{l=1,\dots,m} \lambda_i^l$.

Definition 9.2 *A strategy profile P is called a Nash equilibrium, if for each user i and any channel adopted by him the following condition holds true: $\lambda_i^l = \lambda_i$, if $p_i^l > 0$, and $\lambda_i^l > \lambda_i$, if $p_i^l = 0$.*

Definition 9.3 *A mixed strategy equilibrium P is said to be a completely mixed strategy equilibrium, if each user selects each channel with a positive probability, i.e., for any $i = 1, \dots, n$ and any $l = 1, \dots, m$: $p_i^l > 0$.*

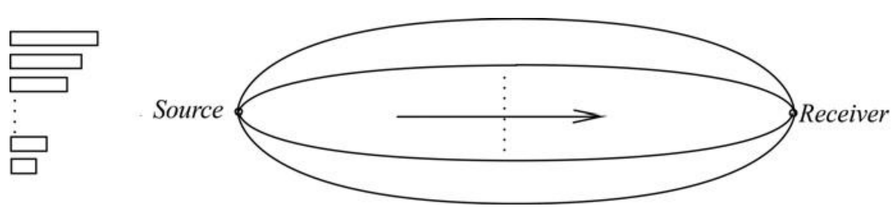


Figure 9.1 A network of parallel channels.

The quantity λ_i describes the minimum possible individual costs of user i to send his traffic. Pursuing personal goals, each user chooses strategies ensuring this value of the expected delay. The so-called *social costs* characterize the general costs of the system due to channels operation. One can involve the following social costs functions $SC(w, L)$ for a pure strategy profile:

1. the linear costs $LSC(w, L) = \sum_{l=1}^{m} \frac{\sum_{k:l_k=l} w_k}{c_l}$;

2. the quadratic costs $QSC(w, L) = \sum_{l=1}^{m} \frac{\left(\sum_{k:l_k=l} w_k\right)^2}{c_l}$;

3. the maximal costs $MSC(w, L) = \max_{l=1,\dots,m} \frac{\sum_{k:l_k=l} w_k}{c_l}$.

Definition 9.4 *The social costs for a mixed strategy profile P are the expected social costs $SC(w, L)$ for a random pure strategy profile L:*

$$SC(w, P) = E\left(SC(w, L)\right) = \sum_{L=(l_1,\dots,l_n)} \left(\prod_{k=1}^{n} p_k^{l_k} \cdot SC(w, L)\right).$$

Denote by opt $= \min_P SC(w, P)$ the optimal social costs. The global optimum in the model considered follows from social costs minimization. Generally, the global optimum is found by enumeration of all admissible pure strategy profiles. However, in a series of cases, it results from solving the continuous conditional minimization problem for social costs, where the mixed strategies of users (the vector P) act as variables.

Definition 9.5 *The price of anarchy is the ratio of the social costs in the worst-case Nash equilibrium and the optimal social costs:*

$$PA = \sup_{P-equilibrium} \frac{SC(w, P)}{opt}.$$

Moreover, if sup affects equilibrium profiles composed of pure strategies only, we mean the pure price of anarchy. Similarly, readers can state the notion of the mixed price of anarchy. The price of anarchy defines how much the social costs under centralized control differ from the social costs when each player acts according to his individual interests. Obviously, $PA \geq 1$ and the actual deviation from 1 reflects the efficiency of centralized control.

9.2 Pure strategy equilibrium. Braess's paradox

Study several examples of systems, where the behavior of users represents pure strategy profiles only. As social costs, we select the maximal social costs function. Introduce the notation $(w_{i_1}, \dots, w_{i_k}) \to c_l$ for a situation when traffic segments $w_{i_1}, \dots, w_{i_k}$ belonging to users $i_1, \dots, i_k \in \{1, \dots, n\}$ are transmitted through the channel with the capacity c_l.

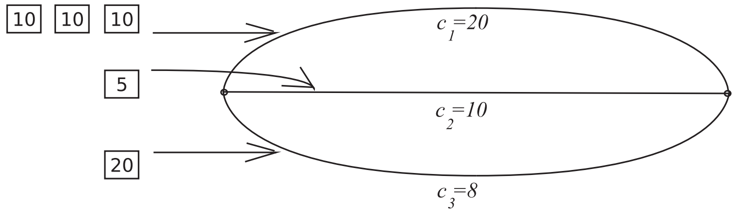


Figure 9.2 The worst-case Nash equilibrium with the delay of 2.5.

Example 9.1 Actually, it illustrates Braess's paradox under elimination of one channel. Consider the following set of users and channels: $n = 5$, $m = 3$, $w = (20, 10, 10, 10, 5)$, $c = (20, 10, 8)$ (see Figure 9.2). In this case, there exist several Nash equilibria. One of them consists in the strategy profile

$$\{(10, 10, 10) \to 20, 5 \to 10, 20 \to 8)\}.$$

Readers can easily verify that any deviation of a player from this profile increases his delay. However, such equilibrium maximizes the social costs:

$$MSC(w; c; (10, 10, 10) \to 20, 5 \to 10, 20 \to 8) = 2.5.$$

We call this equilibrium the **worst-case equilibrium**.

Interestingly, the global optimum of the social costs is achieved in the strategy profile $(20, 10) \to 20, (10, 5) \to 10, 10 \to 8$; it makes up 1.5. Exactly this value represents the best-case pure strategy Nash equilibrium. If we remove channel 8 (see Figure 9.3), the worst-case social costs become

$$MSC(w; c; (20, 10, 10) \to 20, (10, 5) \to 10) = 2.$$

This strategy profile forms the best-case pure strategy equilibrium and the global optimum.

Example 9.2 Set $n = 4$, $m = 3$, $w = (15, 5, 4, 3)$, and $c = (15, 10, 8)$. The social costs in the worst-case equilibrium constitute

$$MSC(w; c; (5, 4) \to 15, 15 \to 10, 3 \to 8) = 1.5.$$

Under the best-case equilibrium, the global optimum of 1 gets attained in the strategy profile $15 \to 15, (5, 3) \to 10, 4 \to 8$. The non-equilibrium strategy profile $15 \to 15, (5, 4) \to 10, 3$

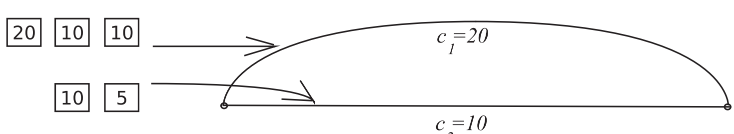


Figure 9.3 Delay reduction owing to channel elimination.

$\to 8$ is globally optimal, either. As the result of channel 10 removal, the worst-case equilibrium becomes $(15, 5) \to 15, (4, 3) \to 8$ (the corresponding social costs equal 1.333). The global optimum and the best-case equilibrium are achieved in $(15, 3) \to 15, (5, 4) \to 8$, and the social costs make up 1.2.

Example 9.3 Set $n = 4$, $m = 3$, $w = (15, 8, 4, 3)$, and $c = (15, 8, 3)$. The social costs in the worst-case equilibrium constitute

$$MSC(w; c; (8, 4, 3) \to 15, 15 \to 8) = 1.875.$$

Under the best-case equilibrium, the global optimum of 1.2666 gets attained in the strategy profile $(15, 4) \to 15, 8 \to 8, 4 \to 3$. By eliminating channel 8, we obtain the worst-case equilibrium $(15, 8, 4) \to 15, 3 \to 3$ with the social costs of 1.8. Finally, the global optimum and the best-case equilibrium are observed in $(15, 8, 3) \to 15, 4 \to 3$, and the corresponding social costs equal 1.733.

Example 9.4 (Braess's paradox.) This model was proposed by D. Braess in 1968. Consider a road network shown in Figure 9.4. Suppose that 60 automobiles move from point A to point B. The delay on the segments (C, B) and (A, D) does not depend on the number of automobiles (it equals 1 h). On the segments (A, C) and (D, B), the delay is proportional to the number of moving automobiles (measured in mins). Obviously, here an equilibrium lies in the equal distribution of automobiles between the routes (A, C, B) and (A, D, B), i.e., 30 automobiles per route. In this case, for each automobile the trip consumes 1.5 h.

Now, imagine that we have connected points C and D by a speedway, where each automobile has zero delay (see Figure 9.5). Then automobiles that have previously selected the route (A, D, B) benefit from moving along the route (A, C, D, B). This applies to automobiles that have previously chosen the route (A, C, B)—they should move along the route (A, C, D, B) as well. Hence, the Nash equilibrium (worst case) is the strategy profile, where all automobiles move along the route (A, C, D, B). However, each automobile spends 2 h for the trip.

Therefore, we observe a self-contradictory situation—the costs of each participant have increased as the result of highway construction. This makes Braess's paradox.

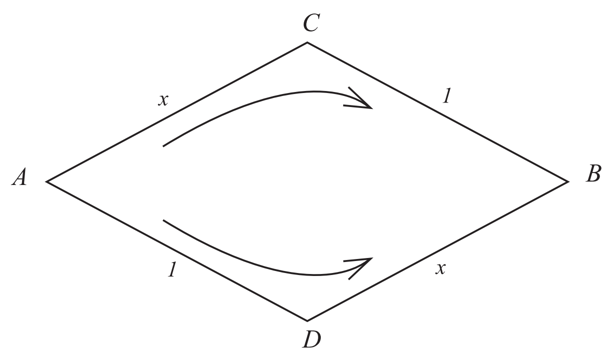


Figure 9.4 In the equilibrium, players are equally distributed between the routes.

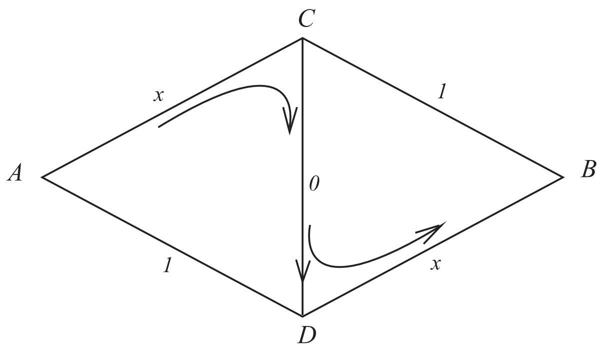


Figure 9.5 In the equilibrium, all players choose the route *ACDB*.

9.3 Completely mixed equilibrium in the optimal routing problem with inhomogeneous users and homogeneous channels

In the current section, we study a system with identical capacity channels. Suppose that the capacity of each channel l equals $c_l = 1$. Let us select linear social costs.

Lemma 9.1 *Consider a system with n users and m parallel channels having identical capacities. There exists a unique completely mixed Nash equilibrium such that for any user i and channel l the equilibrium probabilities make up $p_i^l = 1/m$.*

Proof: By the definition of an equilibrium, each player i has the same delay on all channels, i.e.,

$$\sum_{k \neq i} p_k^j w_k = \lambda_i, \ \ i = 1, \dots, n; \ \ j = 1, \dots, m.$$

First, sum up these equations over $j = 1, \dots, m$:

$$\sum_{j=1}^{m} \sum_{k \neq i} p_k^j w_k = \sum_{k \neq i} w_k = m\lambda_i.$$

Hence it follows that

$$\lambda_i = \frac{1}{m} \sum_{k \neq i} w_k, \ \ i = 1, \dots, n.$$

Second, sum up these equations over $i = 1, \dots, n$:

$$\sum_{i=1}^{n} \sum_{k \neq i} p_k^j w_k = (n-1) \sum_{k=1}^{n} p_k^j w_k = \sum_{i=1}^{n} \lambda_i.$$

This yields

$$\sum_{k=1}^{n} p_k^j w_k = \frac{1}{n-1} \sum_{i=1}^{n} \lambda_i = \frac{1}{n-1} \cdot \frac{(n-1)W}{m} = \frac{W}{m},$$

where $W = w_1 + \cdots + w_n$. The equilibrium equation system leads to

$$p_i^j w_i = \sum_{k=1}^{n} p_k^j w_k - \sum_{k \neq i} p_k^j w_k = \frac{W}{m} - \frac{1}{m} \sum_{k \neq i} w_k = \frac{w_i}{m},$$

whence it appears that

$$p_i^j = \frac{1}{m}, \ i = 1, \ldots, n; \ j = 1, \ldots, m.$$

Denote by F the completely mixed equilibrium in this model and find the corresponding social costs:

$$LSC(w, F) = E\left(\sum_{l=1}^{m} \sum_{k:l_k=l} w_k\right) = \sum_{l=1}^{m} \sum_{k=1}^{n} E(w_k \cdot I_{l_k=l})$$
$$= \sum_{l=1}^{m} \sum_{k=1}^{n} w_k p_l^k = \sum_{k=1}^{n} w_k.$$

9.4 Completely mixed equilibrium in the optimal routing problem with homogeneous users and inhomogeneous channels

This section deals with a system, where users send the same volumes of traffic. Suppose that the traffic volume of any user i makes up $w_i = 1$. Define the total capacity of all channels: $C = \sum_{l=1}^{m} c_l$. We select linear and quadratic social costs. Without loss of generality, sort the channels in the ascending order of their capacities: $c_1 \leq c_2 \leq \ldots \leq c_m$.

Lemma 9.2 *Consider the model with n homogeneous users and m parallel channels. A unique completely mixed Nash equilibrium exists iff $c_1(m + n - 1) > C$. Furthermore, for each channel $l = 1, \ldots, m$ and any user $i = 1, \ldots, n$ the equilibrium probabilities take the form $p_i^l = p^l = \frac{c_l(m+n-1)-C}{C(n-1)}$ and the individual equilibrium delays do coincide, being equal to $\frac{m+n-1}{C}$.*

Proof: Suppose that a completely mixed equilibrium exists. Then the expected traffic delay of each user i on any channel must be the same:

$$\frac{1 + \sum_{k \neq i} p_k^l}{c_l} = \frac{1 + \sum_{k \neq i} p_k^j}{c_j} \text{ for } i = 1, \ldots, n \text{ and } l, j = 1, \ldots, m.$$

Multiply both sides of each identity by c_l. Next, perform summation over l for each group of identities with the same indexes i and j. Bearing in mind that $\sum_{l=1}^{m} p_k^l = 1$ for $k = 1, \dots, n$, we obtain

$$m + (n-1) = C\frac{1 + \sum_{k\neq i} p_k^j}{c_j} \text{ for } i = 1, \dots, n, j = 1, \dots, m$$

$$\frac{m+n-1}{C} = \frac{1 + \sum_{k\neq i} p_k^j}{c_j} = \lambda_i^j \text{ for } i = 1, \dots, n, j = 1, \dots, m.$$

Since the left-hand side of the identity takes the same value for any missed term $\frac{p_i^j}{c_j}$, all quantities $p_i^j = p^j$ for any i. The identity can be transformed to

$$\frac{m+n-1}{C} = \frac{1+(n-1)p^j}{c_j} \text{ for } j = 1, \dots, m,$$

$$\text{whence it follows that } p^j = \frac{c_j(m+n-1) - C}{C(n-1)} \text{ for } j = 1, \dots, m.$$

Clearly, the sum of the equilibrium probabilities over all channels constitutes 1. Thus, a necessary and sufficient admissibility condition of the derived solution lies in the inequality $p^l > 0$ valid for all $l = 1, \dots, m$. In other words, the condition $c_1(m+n-1) > C$ must hold true.

Social costs evaluation in a completely mixed equilibrium involves a series of identities below.

Lemma 9.3 *For any $x \in [0, 1]$ and integer n, we have the expressions*

$$\sum_{k=1}^{n} C_n^k k x^k (1-x)^{n-k} = nx.$$

Proof: Address some properties of the binomial distribution. Let each independent random variable ξ_i, where $i = 1, \dots, n$, possess values 0 or 1 and $E\xi_i = x$. In this case,

$$\sum_{k=1}^{n} C_n^k k x^k (1-x)^{n-k} = E\left(\sum_{i=1}^{n} \xi_i\right) = \sum_{i=1}^{n} E\xi_i = nx.$$

Now, find the social costs for the completely mixed equilibrium.

$$\begin{aligned}
LSC(c, F) &= E\left(\sum_{l=1}^{m} \frac{\text{the number of users } l}{c_l}\right) \\
&= \sum_{l=1}^{m} \frac{1}{c_l} \sum_{k=1}^{n} C_n^k k (1-p^l)^{n-k} (p^l)^k = n \sum_{l=1}^{m} \frac{p^l}{c_l} \\
&= \frac{mn(m+n-1)}{C(n-1)} - \frac{n}{n-1} \sum_{l=1}^{m} \frac{1}{c_l}.
\end{aligned}$$

Analyze the possible appearance of Braess's paradox in this model, i.e., when adding a new channel worsens the completely mixed equilibrium (increases the social costs for the completely mixed equilibrium). We believe that the completely mixed equilibrium exists in the original system, *viz.*, the condition $c_1(m+n-1) > C$ takes place. Adding a new channel must not violate the existence of a completely mixed equilibrium. Notably, we add a channel c_0 such that $c_0(m+n) > C + c_0$ and $c_1(m+n) > C + c_0$.

Theorem 9.1 *Consider the model with n homogeneous users and m inhomogeneous parallel channels. The social costs in a completely mixed equilibrium increase as the result of adding a new channel with the capacity* $\frac{C}{m+n-1} < c_0 < \frac{C}{m}$ *such that* $c_0(m+n) > C + c_0$ *and* $c_1(m+n) > C + c_0$.

Proof: Let F be the completely mixed equilibrium strategy profile in the model with n homogeneous users and m inhomogeneous parallel channels. Assume that we have added a channel with some capacity c_0, and F_0 indicates the completely mixed equilibrium in the resulting system.

Then the variation of the linear social costs becomes

$$\begin{aligned}
&LSC(w, F_0) - LSC(w, F)\\
&= -\frac{n}{(n-1)c_0} + \frac{(m+1)n(m+n)}{(C+c_0)(n-1)} - \frac{mn(m+n-1)}{C(n-1)}\\
&= \frac{n}{(n-1)Cc_0(C+c_0)}\left(Cc_0(2m+n-1) - C^2 - mc_0^2(m+n-1)\right).
\end{aligned}$$

The above difference appears negative, if $Cc_0(2m+n-1) - C^2 - mc_0^2(m+n-1) > 0$. The left-hand side of this inequality represents a parabolic function in c_0 with a non-negative coefficient held by c_0^2. Therefore, all positive values of the function lie between its roots $\frac{C}{m+n-1}$ and $\frac{C}{m}$. Adding a new channel with some capacity $\frac{C}{m+n-1} < c_0 < \frac{C}{m}$ increase the linear social costs.

Example 9.5 Choose a system with four users and two parallel channels of capacity 1. Here the completely mixed equilibrium is the strategy profile, where all equilibrium strategies equal 0.5. The linear social costs in the equilibrium make up 4. If we add a new channel with any capacity $\frac{2}{5} < c_0 < 1$, the completely mixed equilibrium exists; the equilibrium probabilities are $\frac{5c_0-2}{3(c_0+2)}$ (the new channel) and $\frac{4-c_0}{3(c_0+2)}$ (the two former channels). The linear social costs in the equilibrium equal $\frac{52c_0-8c_0^2-8}{3c_0(c_0+2)}$, which exceeds 4.

9.5 Completely mixed equilibrium: The general case

To proceed, we concentrate on the general case model, where users send traffic of different volumes through channels with different capacities. Again, select the linear social costs and let $W = \sum_{i=1}^{n} w_i$ be the total volume of user traffic, $C = \sum_{l=1}^{m} c_l$ represent the total capacity of all channels.

The following theorem provides the existence condition of a completely mixed equilibrium and the corresponding values of equilibrium probabilities.

Theorem 9.2 *A unique completely mixed equilibrium exists iff the condition*

$$\left(1-\frac{mc_l}{C}\right)\left(1-\frac{W}{(n-1)w_i}\right)+\frac{c_l}{C}\in(0,1)$$

holds true for all users $i=1,\dots,n$ and all channels $l=1,\dots,m$. The corresponding equilibrium probabilities make up

$$p_i^l=\left(1-\frac{mc_l}{C}\right)\left(1-\frac{W}{(n-1)w_i}\right)+\frac{c_l}{C}.$$

Obviously, for any user the sum of the equilibrium probabilities over all channels equals 1. Therefore, we should verify the above inequality only in one side, i.e., for all users $i=1,\dots,n$ and all channels $l=1,\dots,m$:

$$\left(1-\frac{mc_l}{C}\right)\left(1-\frac{W}{(n-1)w_i}\right)+\frac{c_l}{C}>0. \tag{5.1}$$

Evaluate the linear social costs for the completely mixed equilibrium F:

$$LSC(w,c,F)=E\left(\sum_{l=1}^{m}\frac{\sum_{k:l_k=l}w_k}{c_l}\right)=\sum_{l=1}^{m}\frac{\sum_{k=1}^{n}E(w_k\cdot I_{l_k=l})}{c_l}$$
$$=\sum_{l=1}^{m}\frac{\sum_{k=1}^{n}w_kp_l^k}{c_l}=\frac{mW(n+m-1)}{C(n-1)}-\frac{W}{n-1}\sum_{l=1}^{m}\frac{1}{c_l}.$$

Study the possible revelation of Braess's paradox in this model. Suppose that the completely mixed equilibrium exists in the original system (the condition (5.1) takes place). Adding a new channel must preserve the existence of a completely mixed equilibrium. In other words, we add a certain channel c_0 meeting the analog of the condition (5.1) in the new system of $m+1$ channels.

Theorem 9.3 *Consider the model with n inhomogeneous users and m inhomogeneous parallel channels. The linear social costs in the completely mixed equilibrium increase as the result of adding a new channel with some capacity $\frac{C}{m+n-1}<c_0<\frac{C}{m}$ such that for any users $i=1,\dots,n$ and all channels $l=0,\dots,m$:*

$$\left(1-\frac{(m+1)c_l}{C+c_0}\right)\left(1-\frac{W}{(n-1)w_i}\right)+\frac{c_l}{C+c_0}>0.$$

Proof: Let F denote the completely mixed equilibrium strategy profile in the model with n inhomogeneous users and m inhomogeneous parallel channels. Suppose that we add a channel with a capacity c_0, and F_0 is the completely mixed equilibrium in the resulting system.

Then the linear social costs become

$$\begin{aligned}&LSC(w,c,F_0) - LSC(w,c,F)\\&= -\frac{W}{(n-1)c_0} + \frac{(m+1)W(m+n)}{(C+c_0)(n-1)} - \frac{mW(m+n-1)}{C(n-1)}\\&= \frac{W}{(n-1)Cc_0(C+c_0)}\left(Cc_0(2m+n-1) - C^2 - mc_0^2(m+n-1)\right).\end{aligned}$$

The remaining part of the proof coincides with that of Theorem 9.2.

Example 9.6 Choose a system with two users sending their traffic of the volumes $w_1 = 1$ and $w_2 = 3$, respectively, through two parallel channels of the capacities $c_1 = c_2 = 1$. The completely mixed equilibrium is the strategy profile, where all equilibrium probabilities equal 0.5. The linear social costs in the equilibrium constitute 4. If we add a new channel with some capacity $\frac{6}{7} < c_0 < 1$, a completely mixed equilibrium exists and the equilibrium prices make up

$$p_1^0 = \tfrac{7c_0-6}{2+c_0}; \quad p_2^0 = \tfrac{5c_0-2}{3(2+c_0)}; \quad p_1^1 = p_1^2 = \tfrac{4-3c_0}{2+c_0}; \quad p_2^1 = p_2^2 = \tfrac{4-c_0}{3(2+c_0)}.$$

In the new system, the linear social costs in the equilibrium become $\frac{28c_0-8c_0^2-8}{c_0(c_0+2)} > 4$.

9.6 The price of anarchy in the model with parallel channels and indivisible traffic

Revert to the system with m homogeneous parallel channels and n players. Take the maximal social costs $MSC(w,c,L)$. Without loss of generality, we believe that the capacity c of all channels is 1 and $w_1 \geq w_2 \geq \ldots \geq w_n$. Let P be some Nash equilibrium. Designate by p_i^j the probability that player i selects channel j. The quantity M^j specifies the expected traffic in channel j, $j = 1,\ldots,m$. Then

$$M_j = \sum_{i=1}^{n} p_i^j w_i. \tag{6.1}$$

In the Nash equilibrium P, the optimal strategy of player i is employing only channels j, where his delay $\lambda_i^j = w_i + \sum\limits_{k=1,k\neq i}^{n} p_k^j w_k$ attains the minimal value ($\lambda_i^j = \lambda_i$, if $p_i^j > 0$, and $\lambda_i^j > \lambda_i$, if $p_i^j = 0$). Reexpress the quantity λ_i^j as

$$\lambda_i^j = w_i + \sum_{k=1,k\neq i}^{n} p_k^j w_k = M_j + (1-p_i^j)w_i. \tag{6.2}$$

Denote by S_i the support of player i strategy, i.e., $S_i = \{j : p_i^j > 0\}$. In the sequel, we write $S_i^j = 1$, if $p_i^j > 0$, and $S_i^j = 0$, otherwise. Suppose that we know the supports $S_1, \dots, S_n$ of the strategies of all players. In this case, the strategies proper are defined by

$$M^j + (1 - p_i^j)w_i = \lambda_i, \quad S_i^j > 0, \quad i = 1, \dots, n; \quad j = 1, \dots, m.$$

Hence it appears that

$$p_i^j = \frac{M^j + w_i - \lambda_i}{w_i}. \tag{6.3}$$

According to (6.1), for all $j = 1, \dots, m$ we have

$$M^j = \sum_{i=1}^{n} S_i^j (M^j + w_i - \lambda_i).$$

Moreover, since the equality $\sum_{j=1}^{m} p_i^j = 1$ takes place for all players i, we also obtain

$$\sum_{j=1}^{m} S_i^j (M^j + w_i - \lambda_i) = w_i, \quad i = 1, \dots, n.$$

Reexpress the social costs as the expected maximal traffic over all channels:

$$SC(w, P) = \sum_{j_1=1}^{m} \dots \sum_{j_n=1}^{m} \prod_{i=1}^{n} p_i^{j_i} \max_{l=1,\dots,m} \sum_{k:l_k=l} w_k. \tag{6.4}$$

Denote by $\text{opt} = \min_P SC(w, P)$ the optimal social costs.

Now, calculate the price of anarchy in this model. Recall that it represents the ratio of the social costs in the worst-case Nash equilibrium and the optimal social costs:

$$PA = \sup_{P-\text{equilibrium}} \frac{SC(w, P)}{\text{opt}}.$$

Let P indicate some mixed strategy profile and q_i be the probability that player i chooses the maximal delay channel. Then

$$SC(w, P) = \sum_{i=1}^{m} w_i q_i.$$

In addition, introduce the probability that players i and k choose the same channel—the quantity t_{ik}. Consequently, the inequality $P(A \cup B) = P(A) + P(B) - P(A \cap B) \le 1$ implies that

$$q_i + q_k \le 1 + t_{ik}.$$

Lemma 9.4 *The following condition holds true in the Nash equilibrium P:*

$$\sum_{k\neq i} t_{ik} w_k = \lambda_i - w_i, \quad i = 1, \dots, n.$$

Proof: First, note that $t_{ik} = \sum_{j=1}^{m} p_i^j p_k^j$. In combination with (6.1), this yields

$$\sum_{k\neq i} t_{ik} w_k = \sum_{j=1}^{m} p_i^j \sum_{k\neq i} p_k^j w_k = \sum_{j=1}^{m} p_i^j \left(M^j - p_i^j w_i \right).$$

According to (6.3), if $p_i^j > 0$, then $M^j - p_i^j w_i = \lambda_i - w_i$. Thus, we can rewrite the last expression as

$$\sum_{k\neq i} t_{ik} w_k = \sum_{j=1}^{m} p_i^j (\lambda_i - w_i) = \lambda_i - w_i.$$

Lemma 9.5 *The following estimate takes place:*

$$\lambda_i \leq \frac{1}{m} \sum_{i=1}^{n} w_i + \frac{m-1}{m} w_i, \quad i = 1, \dots, n.$$

Proof: *Proof* is immediate from the expressions

$$\lambda_i = \min_j \left\{ M^j + \left(1 - p_i^j w_i\right) \right\} \leq \frac{1}{m} \sum_{j=1}^{m} \left\{ M^j + \left(1 - p_i^j w_i\right) \right\}$$
$$= \frac{1}{m} \sum_{j=1}^{m} M^j + \frac{m-1}{m} w_i = \frac{1}{m} \sum_{i=1}^{n} w_i + \frac{m-1}{m} w_i.$$

Now, we can evaluate the price of anarchy in a two-channel network (see Figure 9.6).

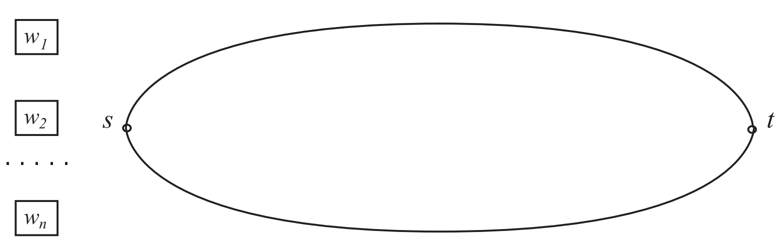


Figure 9.6 A two-channel network.

Theorem 9.4 *Consider the model with n inhomogeneous users and two homogeneous parallel channels. The corresponding price of anarchy constitutes 3/2.*

Proof: Construct an upper estimate for the social costs $SC(w, P)$. Rewrite them as

$$SC(w, P) = \sum_{k=1}^{m} q_k w_k = \sum_{k \neq i} q_k w_k + q_i w_i = \sum_{k \neq i} (q_i + q_k) w_k - \sum_{k \neq i} q_i w_k + q_i w_i. \tag{6.5}$$

Since $q_i + q_k \leq 1 + t_{ik}$, we have

$$\sum_{k \neq i} (q_i + q_k) w_k \leq \sum_{k \neq i} (1 + t_{ik}) w_k.$$

In the case of $m = 2$, Lemmas 9.4 and 9.5 imply that

$$\sum_{k \neq i} t_{ik} w_k = c_i - w_i \leq \frac{1}{2} \sum_{k=1}^{n} w_k - \frac{1}{2} w_i = \frac{1}{2} \sum_{k \neq i}^{n} w_k.$$

Hence it appears that

$$\sum_{k \neq i} (q_i + q_k) w_k \leq \frac{3}{2} \sum_{k \neq i}^{n} w_k,$$

and the function (6.5) can be estimated by

$$SC(w, P) \leq \left(\frac{3}{2} - q_i \right) \sum_{k=1}^{m} w_k + \left(2q_i - \frac{3}{2} \right) w_i.$$

Note that

$$\text{opt} \geq \max \left\{ w_1, \frac{1}{2} \sum_k w_k \right\}.$$

Indeed, if $w_1 \geq \frac{1}{2} \sum_k w_k$, then $w_1 \geq w_2 + \cdots + w_n$. The optimal strategy lies in transmitting the packet w_1 through one channel, whereas the rest packets should be sent by another channel. Accordingly, the delay makes up w_1. If $w_1 < \frac{1}{2} \sum_k w_k$, the optimal strategy is distributing each packet between the channels equiprobably; the corresponding delay equals $\frac{1}{2} \sum_k w_k$.

Then, if some player i meets the inequality $q_i \geq 3/4$, one obtains

$$SC(w, P) \leq \left(\frac{3}{2} - q_i \right) 2\text{opt} + \left(2q_i - \frac{3}{2} \right) \text{opt} = \frac{3}{2} \text{opt}.$$

At the same time, if all players i are such that $q_i < 3/4$, we get

$$SC(w, P) = \sum_{k=1}^{m} q_k w_k \leq \frac{3}{4} \sum_k w_k \leq \frac{3}{2} \text{opt}.$$

Therefore, all Nash equilibria P satisfy the inequality $SC(w, P) \leq \frac{3}{2}\text{opt}$. And so,

$$PA = \sup_P \frac{SC(w, P)}{\text{opt}} \leq \frac{3}{2}.$$

To derive a lower estimate, consider a system with two homogeneous channels and two players, where $w_1 = w_2 = 1$. Obviously, the worst-case equilibrium is $p_i^j = 1/2$ for $i = 1, 2; j = 1, 2$. The expected maximal load of the network makes up $1 \cdot 1/2 + 2 \cdot 1/2 = 3/2$. The maximal value of opt $= 1$ is achieved when each channel transmits a single packet. Thus, we have found the precise estimate for the price of anarchy in a system with two homogeneous channels.

9.7 The price of anarchy in the optimal routing model with linear social costs and indivisible traffic for an arbitrary network

Up to this point, we have explored networks with parallel channels. Now, switch to network games with an arbitrary topology.

Interpret the optimal routing problem as a non-cooperative game $\Gamma = \langle N, G, Z, f \rangle$, where users (players) $N = (1, 2, \ldots, n)$ send traffic via some channels of a network $G = (V, E)$. The symbol G stands for an undirected graph with a node set V and an edge set E (see Figure 9.7). For each user i, there exists Z_i—a set of routes from s_i to t_i via channels G. We suppose that the volume of user traffic is 1. Further analysis covers two types of network games, *viz.*, symmetrical ones (all players have identical strategy sets Z_i), and asymmetrical ones (all players have different strategy sets).

Each channel $e \in E$ possesses a given capacity $c_e > 0$. Users pursue individual interests—they choose routes of traffic transmission to minimize the maximal traffic delay on the way from s to t. Each user selects a specific strategy $R_i \in Z_i$, which represents the route used by player i for his traffic. Consequently, the vector $R = (R_1, \ldots, R_n)$ forms the pure strategy profile of all users. For a strategy profile R, we again introduce the notation $(R_{-i}, R_i') = (R_1, \ldots, R_{i-1}, R_i', R_{i+1}, \ldots, R_n)$. It indicates that user i has modified his strategy from R_i to R_i', while the rest users keep their strategies invariable.

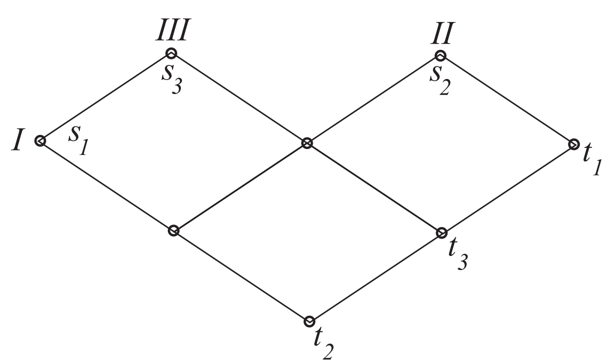


Figure 9.7 An asymmetrical network game with 10 channels.

For each channel, define its load $n_e(R)$ as the number of players involving channel e in the strategy profile R. The traffic delay on a given route depends on the loads of channels in this route. Consider the linear latency function $f_e(k) = a_e k + b_e$, where a_e and b_e specify non-negative constants. For the sake of simplicity, we take the case $f_e(k) = k$. All relevant results are easily extended to the general case.

Each user i strives to minimize the total traffic delay over all channels in his route:

$$c_i(R) = \sum_{e \in R_i} f_e(n_e(R)) = \sum_{e \in R_i} n_e(R).$$

This function represents the individual costs of user i.

A Nash equilibrium is defined as a strategy profile such that none of the players benefits by unilateral deviation from this strategy profile (provided that the rest players still follow their strategies).

Definition 9.6 *A strategy profile R is called a Nash equilibrium, if for each user $i \in N$ we have $c_i(R) \le c_i(R_{-i}, R_i')$.*

We emphasize that this game is a special case of the congestion game analyzed in Section 3.4. Recall that players choose some objects (channels) from their feasible sets $Z_i, i \in N$, and the payoff function of a player depends on the number of other players choosing the same object. Such observation guarantees that the game in question always admits a pure strategy equilibrium. Therefore, further consideration focuses on pure strategies only.

Take the linear (total) costs of all players as the social costs, i.e.,

$$SC(R) = \sum_{i=1}^{n} c_i(R) = \sum_{i=1}^{n} \sum_{e \in R_i} n_e(R) = \sum_{e \in E} n_e^2(R).$$

Designate by opting the minimal social costs. Evaluate the ratio of the social costs in the worst-case Nash equilibrium and the optimal costs. In other words, find the price of anarchy

$$PA = \sup_{R-\text{equilibrium}} \frac{SC(R)}{\text{opt}}.$$

Theorem 9.5 *In the asymmetrical model with indivisible traffic and linear delays, the price of anarchy constitutes 5/2.*

Proof: We begin with upper estimate derivation. Let R^* be a Nash equilibrium and R form an arbitrary strategy profile (possibly, the optimal one). To construct an upper estimate for the price of anarchy, compare the social costs in these strategy profiles. In the Nash equilibrium R^*, the costs of player i under switching to the strategy R_i do not decrease:

$$c_i(R^*) = \sum_{e \in R_i^*} n_e(R^*) \le \sum_{e \in R_i} n_e(R_{-i}^*, R_i).$$

In the case of switching by player i, the number of players on each channel may increase by unity only. Therefore,

$$c_i(R^*) \le \sum_{e \in R_i} (n_e(R^*) + 1).$$

Summing up these inequalities over all i yields

$$SC(R^*) = \sum_{i=1}^{n} c_i(R^*) \leq \sum_{i=1}^{n} \sum_{e \in R_i} (n_e(R^*) + 1) = \sum_{e \in E} n_e(R)(n_e(R^*) + 1).$$

We will need the following technical result.

Lemma 9.6 *Any non-negative integers α, β meet the inequality*

$$\beta(\alpha + 1) \leq \frac{1}{3}\alpha^2 + \frac{5}{3}\beta^2.$$

Proof: Fix β and consider the function $f(\alpha) = \alpha^2 + \beta^2 - 3\beta(\alpha + 1)$. This is a parabola, whose node lies in the point $\alpha = 3/2\beta$. The minimal value equals

$$f\left(\frac{3}{2}\beta\right) = \frac{1}{4}\beta(11\beta - 12).$$

If $\beta \geq 2$, the above value appears positive. Hence, the lemma holds true for $\beta \geq 2$. In the cases of $\beta = 0, 1$, the inequality can be verified directly.

Using Lemma 9.6, we obtain the upper estimate

$$SC(R^*) \leq \frac{1}{3} \sum_{e \in E} n_e^2(R^*) + \frac{5}{3} \sum_{e \in E} n_e^2(R) = \frac{1}{3} SC(R^*) + \frac{5}{3} SC(R),$$

whence it follows that

$$SC(R^*) \leq \frac{5}{2} SC(R)$$

for any strategy profiles R. This immediately implies that $PA \leq 5/2$.

To argue that $PA \geq 5/2$, we provide an example of a network, where the price of anarchy is $5/2$. Consider a network with the topology illustrated by Figure 9.8. Three players located in node 0 send their traffic through network channels, $\{h_1, h_2, h_3, g_1, g_2, g_3\}$. Each player chooses between just two pure strategies. For player 1, these are the routes (h_1, g_1) or (h_2, h_3, g_2). For player 2, these are the routes (h_2, g_2) or (h_1, h_3, g_3). And finally, for player 3, these are the routes (h_3, g_3) or (h_1, h_2, g_1). Evidently, the optimal distribution of players consists in choosing the first strategies, (h_1, g_1), (h_2, g_2), and (h_3, g_3). The corresponding social costs constitute 2. The worst-case Nash equilibrium results from selection of the second strategies: $(h_2, h_3, g_2), (h_1, h_3, g_3), (h_1, h_2, g_1)$. Really, imagine that, e.g., player 1 (the equilibrium costs are 5) switches to the first strategy (h_1, g_1). Then his costs still make up 5.

Therefore, the price of anarchy in the described network is $5/2$. This concludes the proof of Theorem 9.5.

The symmetrical model, where all players have the same strategy set, ensures a smaller price of anarchy.

Theorem 9.6 *Consider the n-player symmetrical model with indivisible traffic and linear delays. The price of anarchy equals $(5n - 2)/(2n + 1)$.*

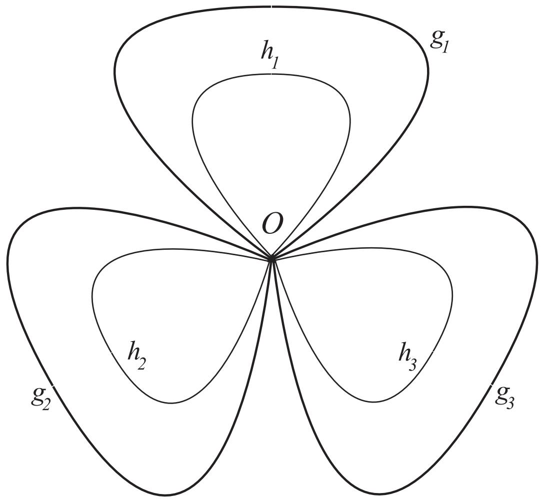


Figure 9.8 A network with three players and three channels $(h_1, h_2, h_3, g_1, g_2, g_3)$.

Proof: Let R^* be a Nash equilibrium and R represent the optimal strategy profile, which minimizes the social costs. We estimate the costs of player i in the equilibrium, i.e., the quantity $c_i(R^*)$. As he deviates from the equilibrium by choosing another strategy R_j (this is possible, since the strategy sets of all players coincide), the costs rise accordingly:

$$c_i(R^*) = \sum_{e\in R_i^*} n_e(R^*) \le \sum_{e\in R_j} n_e(R^*_{-i}, R_j).$$

Moreover, $n_e(R^*_{-i}, R_j)$ differs from $n_e(R^*)$ by 1 in the channels, where $e \in R_j - R_i^*$. Hence it appears that

$$c_i(R^*) \le \sum_{e\in R_j} n_e(R^*) + |R_j - R_i^*|,$$

where $|R|$ means the number of elements in R. As far as $A - B = A - A \cap B$, we have

$$c_i(R^*) \le \sum_{e\in R_j} n_e(R^*) + |R_j| - |R_j \cap R_i^*|.$$

Summation over all $j \in N$ brings to the inequalities

$$\begin{aligned} nc_i(R^*) &\le \sum_{j=1}^{n} \sum_{e\in R_j} n_e(R^*) + \sum_{j=1}^{n}(|R_j| - |R_j \cap R_i^*|) \\ &\le \sum_{e\in E} n_e(R) n_e(R^*) + \sum_{e\in E} n_e(R) - \sum_{e\in R_i^*} n_e(R). \end{aligned}$$

Now, by summing up over all $i \in N$, we get

$$\begin{aligned} nSC(R^*) &\leq \sum_{i=1}^{n} \sum_{e \in E} n_e(R) n_e(R^*) + \sum_{i=1}^{n} \sum_{e \in E} n_e(R) - \sum_{i=1}^{n} \sum_{e \in R_i^*} n_e(R) \\ &= n \sum_{e \in E} n_e(R) n_e(R^*) + n \sum_{e \in E} n_e(R) - \sum_{e \in E} n_e(R) n_e(R^*) \\ &= (n-1) \sum_{e \in E} n_e(R) n_e(R^*) + n \sum_{e \in E} n_e(R). \end{aligned}$$

Rewrite this inequality as

$$SC(R^*) \leq \frac{n-1}{n} \sum_{e \in E} \left(n_e(R) n_e(R^*) + n_e(R) \right) + \frac{1}{n} \sum_{e \in E} n_e(R),$$

and apply Lemma 9.6:

$$\begin{aligned} SC(R^*) &\leq \frac{n-1}{3n} \sum_{e \in E} n_e^2(R^*) + \frac{5(n-1)}{3n} \sum_{e \in E} n_e^2(R) + \frac{1}{n} \sum_{e \in E} n_e^2(R) \\ &= \frac{n-1}{3n} SC(R^*) + \frac{5n-2}{3n} SC(R). \end{aligned}$$

This immediately implies that

$$SC(R^*) \leq \frac{5n-2}{2n+1} SC(R),$$

and the price of anarchy enjoys the upper estimate $PA \leq (5n-2)/(2n+1)$.

To obtain a lower estimate, it suffices to give an example of a network with the price of anarchy $(5n-1)/(2n+1)$. We leave this exercise to an interested reader.

Theorem 9.6 claims that the price of anarchy is smaller in the symmetrical model than in its asymmetrical counterpart. However, as n increases, the price of anarchy reaches the level of $5/2$.

9.8 The mixed price of anarchy in the optimal routing model with linear social costs and indivisible traffic for an arbitrary network

In Section 9.6 we have estimated the price of anarchy by considering only pure strategy equilibria. To proceed, find the mixed price of anarchy for arbitrary networks with linear delays. Suppose that players can send traffic of different volumes through channels of a network $G = (V, E)$.

Well, consider an asymmetrical optimal routing game $\Gamma = \langle N, G, Z, w, f \rangle$, where players $N = (1, 2, \ldots, n)$ transmit traffic of corresponding volumes $\{w_1, w_2, \ldots, w_n\}$. For each user i, there is a given set Z_i of pure strategies, i.e., a set of routes from s_i to t_i via channels of the network G. The traffic delay on a route depends on the load of engaged channels. We understand the load of a channel as the total traffic volume transmitted through this channel. Assume that the latency function on channel e has the linear form $f_e(k) = a_e k + b_e$, where k indicates channel load, a_e and b_e are non-negative constants. Then the total traffic delay on the complete route makes the sum of traffic delays on all channels of a route.

Users pursue individual interests and choose routes for their traffic to minimize the delay during traffic transmission from s to t. Each user $i \in N$ adopts a mixed strategy P_i, i.e., player i sends his traffic w_i by the route $R_i \in Z_i$ with the probability $p_i(R_i)$, $i = 1, \dots, n$,

$$\sum_{R_i \in Z_i} p_i(R_i) = 1.$$

A set of mixed strategies forms a strategy profile $P = \{P_1, \dots, P_n\}$ in this game.

Each user i strives to minimize the expected delay of his traffic on all engaged routes:

$$c_i(P) = \sum_{R \in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e \in R_i} f_e(n_e(R)) = \sum_{R \in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e \in R_i} \left(a_e n_e(R) + b_e\right),$$

where $n_e(R)$ is the load of channel e under a given strategy profile R. The function $c_i(P)$ specifies the individual costs of user i. On the other hand, the function

$$SC(P) = \sum_{i=1}^{n} w_i c_i(P) = \sum_{R \in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e \in E} n_e(R) f_e(n_e(R))$$

gives the social costs.

Let P^* be a Nash equilibrium. We underline that a Nash equilibrium exists due to strategy set finiteness. Denote by R^* the optimal strategy profile ensuring the minimal social costs. Obviously, it consists of pure strategies of players, i.e., $R^* = (R_1^*, \dots, R_n^*)$. Then each user $i \in N$ obeys the inequality

$$c_i(P^*) \le c_i(P_{-i}^*, R_i^*).$$

Here (P_{-i}^*, R_i^*) means that in the strategy profile P^* player i chooses the pure strategy R_i^* instead of the mixed strategy P^*i. In the equilibrium, we have the condition

$$c_i(P^*) \le c_i(P_{-i}^*, R_i^*) = \sum_{R \in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e \in R_i^*} f_e(n_e(R_{-i}, R_i^*)), i = 1, \dots, n.$$

Note that, in any strategy profile (R_{-i}, R_i^*), only player i deviates actually. And so, the load of any channel in the route R_i^* may increase at most by w_i, i.e.,$f_e(n_e(R_{-i}, R_i^*)) \le f_e(n_e(R) + w_i)$. Hence it follows that

$$c_i(P^*) \le \sum_{R \in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e \in R_i^*} f_e(n_e(R) + w_i), i = 1, \dots, n.$$

Multiply these inequalities by w_i and perform summation from 1 to n. Such manipulations lead to

$$SC(P^*) = \sum_{i=1}^{n} w_i c_i(P^*) \le \sum_{R\in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{i=1}^{n} \sum_{e\in R_i^*} w_i f_e(n_e(R) + w_i).$$

Using the linear property of the latency functions, we arrive at the inequalities

$$\begin{aligned} SC(P^*) &\le \sum_{R\in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{i=1}^{n} \sum_{e\in R_i^*} w_i \left(a_e(n_e(R) + w_i) + b_e\right) \\ &= \sum_{R\in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{i=1}^{n} \left(\sum_{e\in R_i^*} a_e n_e(R) w_i + a_e w_i^2 \right) + \sum_{i=1}^{n} \sum_{e\in R_i^*} b_e w_i \\ &= \sum_{R\in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e\in E} a_e \left(n_e(R) n_e(R^*) + n_e(R^*)^2\right) + \sum_{e\in E} b_e n_e(R^*). \end{aligned} \tag{8.1}$$

Further exposition will employ the estimate from Lemma 9.7.

Lemma 9.7 *Any non-negative numbers α, β meet the inequality*

$$\alpha\beta + \beta^2 \le \frac{z}{2}\alpha^2 + \frac{z+3}{2}\beta^2, \tag{8.2}$$

where $z = (\sqrt{5} - 1)/2 \approx 0.618$ is the golden section of the interval $[0, 1]$.

Proof: Fix β and consider the function

$$f(\alpha) = \frac{z}{2}\alpha^2 + \frac{z+3}{2}\beta^2 - \alpha\beta - \beta^2 = \frac{z}{2}\alpha^2 + \frac{z+1}{2}\beta^2 - \alpha\beta.$$

This is a parabola with the vertex $\alpha = \beta/z$. The minimal value of the parabola equals

$$f\left(\frac{\beta}{z}\right) = \beta^2 \left(z + 1 - \frac{1}{z}\right).$$

The expression in brackets (see the value of z equal to the golden section) actually vanishes. This directly gives inequality (9.2).

In combination with inequality (9.2), the condition (9.1) implies that

$$\begin{aligned} SC(P^*) &\le \sum_{R\in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e\in E} a_e \left(\frac{z}{2} n_e(R)^2 + \frac{z+3}{2} n_e(R^*)^2\right) + \sum_{e\in E} b_e n_e(R^*) \\ &\le \frac{z}{2} \sum_{R\in Z} \prod_{j=1}^{n} p_j(R_j) \sum_{e\in E} \left(a_e n_e(R)^2 + b_e n_e(R)\right) + \frac{z+3}{2} \left(a_e n_e(R^*)^2 + b_e n_e(R^*)\right) \\ &= \frac{z}{2} SC(P^*) + \frac{z+3}{2} SC(R^*). \end{aligned} \tag{8.3}$$

Now, it is possible to estimate the price of anarchy for a pure strategy Nash equilibrium.

Theorem 9.7 *Consider the n-player asymmetrical model with indivisible traffic and linear delays. The mixed price of anarchy does not exceed* $z + 2 = (\sqrt{5} + 3)/2 \approx 2.618$.

Proof: It follows from (9.3) that

$$SC(P^*) \leq \frac{z+3}{2-z} SC(R^*).$$

By virtue of golden section properties, $\frac{z+3}{2-z} = z + 2$. Consequently, the ratio of the social costs in the Nash equilibrium and the optimal costs, i.e., the quantity

$$PA = \frac{SC(P^*)}{SC(R^*)},$$

does not exceed $z + 2 \approx 2.618$. The proof of Theorem 9.7 is finished.

Remark. The price of anarchy in pure strategies is $5/2 = 2.5$. Transition to mixed strategies slightly increases the price of anarchy (up to 2.618). This seems natural, since the worst-case Nash equilibrium can be achieved in mixed strategies.

9.9 The price of anarchy in the optimal routing model with maximal social costs and indivisible traffic for an arbitrary network

We have demonstrated that, in the case of linear social costs, the price of anarchy possesses finite values. However, if we select the maximal costs of a player as the social costs, the price of anarchy takes arbitrary large values. Illustrate this phenomenon by an example—consider a network in Figure 9.9.

The network comprises the basic nodes $\{v_0, v_1, \ldots, v_k\}$. The nodes v_i, v_{i+1} are connected through k routes; one of them (see abscissa axis) has the length of 1, whereas the rest routes possess the length of k. Player 1 (suffering from the maximal costs) sends his traffic from node v_0 to node v_k. Note that he can employ routes lying on abscissa axis only. Each node $v_i, i = 0, \ldots, k-1$ contains $k-1$ players transmitting their traffic from v_i to v_{i+1}. Evidently, the optimal social costs of k are achieved if player 1 sends his traffic via the route $v_0, v_1, \ldots, v_k$,

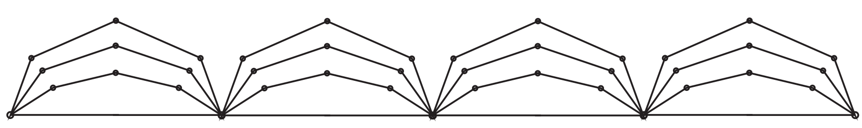


Figure 9.9 A network with $k^2 - k + 1$ players. Player 1 follows the route $(v_0, v_1, \ldots, v_k)$. Node v_i has $k - 1$ players following the route (v_i, v_{i+1}). The delay on the main channel equals 1, the delay on the rest channels is k. The price of anarchy makes up k.

and all $k-1$ players in the node v_i are distributed among the rest routes (a player per a specific route).

Readers can easily observe the following. Here, the worst-case Nash equilibrium is when all $n = (k-1)k + 1$ players send their traffic through routes lying on abscissa axis. However, then the costs of player 1 (*ergo*, the maximal social costs) constitute $(k-1)k + 1$. Hence, the price of anarchy in this model is defined by

$$PA = \frac{k^2 - k + 1}{k} = \sqrt{n} + O(1).$$

Now, construct an upper estimate for the price of anarchy in an arbitrary network with indivisible traffic. Suppose that R^* is a Nash equilibrium and R designates the optimal strategy profile guaranteeing the minimal social costs. In our case, the social costs represent the maximal costs of players. Without loss of generality, we believe that in the equilibrium the maximal costs are attained for player 1, i.e., $SC(R^*) = c_1(R^*)$. To estimate the price of anarchy, apply the same procedure as in the proof of Theorem 9.6. Compare the maximal costs $SC(R^*)$ in the equilibrium and the maximal costs for the strategy profile R—the quantity $SC(R) = \max_{i \in N} c_i(R)$.

So long as R^* forms a Nash equilibrium, we have

$$c_1(R^*) \le \sum_{e \in R_1} (n_e(R^*) + 1) \le \sum_{e \in R_1} n_e(R^*) + |R_1| \le \sum_{e \in R_1} n_e(R^*) + c_1(R).$$

The last inequality follows from clear considerations: if player 1 chooses channels from R_1, his delay is greater or equal to the number of channels in R_1.

Finally, let us estimate $\sum_{e \in R_1} n_e(R^*)$. Using the inequality $(\sum_{i=1}^n a_i)^2 \le n \sum_{i=1}^n a_i^2$, we have

$$\left(\sum_{e \in R_1} n_e(R^*) \right)^2 \le |R_1| \sum_{e \in R_1} n_e^2(R^*) \le |R_1| \sum_{e \in E} n_e^2(R^*) = \sum_{i=1}^n c_i(R^*).$$

According to Theorem 9.5,

$$\sum_{i=1}^n c_i(R^*) \le \frac{5}{2} \sum_{i=1}^n c_i(R).$$

And so,

$$\left(\sum_{e \in R_1} n_e(R^*) \right)^2 \le |R_1| \frac{5}{2} \sum_{i=1}^n c_i(R),$$

which means that

$$c_1(R^*) \le c_1(R) + \sqrt{|R_1| \frac{5}{2} \sum_{i=1}^n c_i(R)}.$$

Since $|R_1| \leq c_1(R)$ and $c_i(R) \leq SC(R)$, we get the inequality

$$c_1(R^*) \leq SC(R)\left(1 + \sqrt{\frac{5}{2}n}\right).$$

The last expression implies that the price of anarchy admits the upper estimate $1 + \sqrt{5/2n}$. As a matter of fact, we have established the following result.

Theorem 9.8 *Consider the n-player asymmetrical model with indivisible traffic and the maximal costs of players as the social costs. The price of anarchy constitutes $O(\sqrt{n})$.*

This theorem shows that the price of anarchy may possess arbitrary large values.

9.10 The Wardrop optimal routing model with divisible traffic

The routing model studied in this section is based on the Wardrop model with divisible traffic suggested in 1952. Here the optimality criterion lies in traffic delay minimization.

The optimal traffic routing problem is treated as a game $\Gamma = \langle n, G, w, Z, f \rangle$, where n users transmit their traffic by network channels; the network has the topology described by a graph $G = (V, E)$. For each user i, there exists a certain set Z_i of routes from s_i to t_i via channels G and a given volume of traffic w_i. Next, each channel $e \in E$ possesses some capacity $c_e > 0$. All users pursue individual interests and choose routes for their traffic to minimize the maximal delay during traffic transmission from s to t. Each user selects a specific strategy $x_i = \{x_{iR_i} \geq 0\}_{R_i \in Z_i}$. The quantity x_{iR_i} determines the volume of traffic sent by user i through route R_i, and $\sum_{R_i \in Z_i} x_{iR_i} = w_i$. Then $x = (x_1, \ldots, x_n)$ represents a strategy profile of all users. For a strategy profile x, we again introduce the notation $(x_{-i}, x_i') = (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$. It indicates that user i has modified his strategy from x_i to x_i', while the rest users keep their strategies invariable.

For each channel $e \in E$, define its load (the total traffic through this channel) by

$$\delta_e(x) = \sum_{i=1}^{n} \sum_{R_i \in Z_i : e \in R_i} x_{iR_i}.$$

The traffic delay on a given route depends on the loads of channels in this route. The continuous latency function $f_{iR_i}(x) = f_{iR_i}(\{\delta_e(x)\}_{e \in R_i})$ is specified for each user i and each route R_i engaged by him. Actually, it represents a non-decreasing function with respect to the loads of channels in a route (*ergo*, with respect to x_{iR_i}).

Each user i strives to minimize the maximal traffic delay over all channels in his route:

$$PC_i(x) = \max_{R_i \in Z_i : x_{iR_i} > 0} f_{iR_i}(x).$$

This function represents the individual costs of user i.

A Nash equilibrium is defined as a strategy profile such that none of the players benefit by unilateral deviation from this strategy profile (provided that the rest players still follow their strategies). In terms of the current model, the matter concerns a strategy profile such that none of the players can reduce his individual costs by modifying his strategy.

Definition 9.7 *A strategy profile x is called a Nash equilibrium, if for each user i and any strategy profile $x' = (x_{-i}, x'_i)$ we have $PC_i(x) \leq PC_i(x')$.*

Within the framework of network models, an important role belongs to the concept of a Wardrop equilibrium.

Definition 9.8 *A strategy profile x is called a Wardrop equilibrium, if for each i and any $R_i, \rho_i \in Z_i$ the condition $x_{iR_i} > 0$ leads to $f_{iR_i}(x) \leq f_{i\rho_i}(x)$.*

This definition can be restated similarly to the definition of a Nash equilibrium.

Definition 9.9 *A strategy profile x is a Wardrop equilibrium, if for each i the following condition holds true: the inequality $x_{iR_i} > 0$ leads to $f_{iR_i}(x) = \min_{\rho_i \in Z_i} f_{i\rho_i}(x) = \lambda_i$ and the equality $x_{iR_i} = 0$ yields $f_{iR_i}(x) \geq \lambda_i$.*

Such explicit definition provides a system of equations and inequalities for evaluating Wardrop equilibrium strategy profiles. Strictly speaking, the definitions of a Nash equilibrium and a Wardrop equilibrium are not equivalent. Their equivalence depends on the type of latency functions in channels.

Theorem 9.9 *If a strategy profile x represents a Wardrop equilibrium, then x is a Nash equilibrium.*

Proof: Let x be a strategy profile such that for all i we have the following condition: the inequality $x_{iR_i} > 0$ brings to $f_{iR_i}(x) = \min_{\rho_i \in Z_i} f_{i\rho_i}(x) = \lambda_i$ and the equality $x_{iR_i} = 0$ implies $f_{iR_i}(x) \geq \lambda_i$. Then for all i and R_i one obtains

$$\max_{\rho_i \in Z_i : x_{i\rho_i} > 0} f_{i\rho_i}(x) \leq f_{iR_i}(x).$$

Suppose that user i modifies his strategy from x_i to x'_i. In this case, denote by $x' = (x_{-i}, x'_i)$ a strategy profile such that, for user i, the strategies on all his routes $R_i \in Z_i$ change to $x'_{iR_i} = x_{iR_i} + \Delta_{R_i}$, where $\sum_{R_i \in Z_i} \Delta_{R_i} = 0$. The rest users $k \neq i$ adhere to the same strategies as before, i.e., $x'_k = x_k$.

If all $\Delta_{R_i} = 0$, then $PC_i(x) = PC_i(x')$. Assume that $x \neq x'$, *viz.*, there exists a route R_i such that $\Delta_{R_i} > 0$. This route meets the condition $f_{iR_i}(x) \leq f_{iR_i}(x')$, since $f_{iR_i}(x)$ is a non-decreasing function in x_{iR_i}. As far as $x'_{iR_i} > 0$, we get

$$f_{iR_i}(x') \leq \max_{\rho_i \in Z_i : x_{i\rho_i} > 0} f_{i\rho_i}(x').$$

Finally,

$$\max_{\rho_i \in Z_i : x_{i\rho_i} > 0} f_{i\rho_i}(x) \leq \max_{\rho_i \in Z_i : x_{i\rho_i} > 0} f_{i\rho_i}(x'),$$

or $PC_i(x) \leq PC(x')$. Hence, due to the arbitrary choice of i and x'_i, we conclude that the strategy profile x forms a Nash equilibrium.

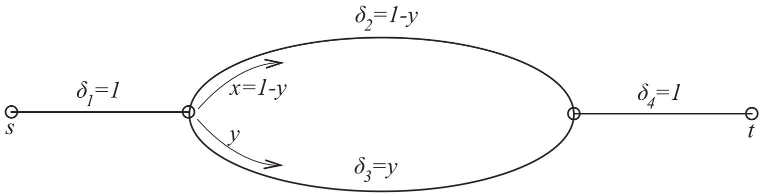


Figure 9.10 A Nash equilibrium mismatches a Wardrop equilibrium.

Any Nash equilibrium in the model considered represents a Wardrop equilibrium under the following sufficient condition imposed on all latency functions. For a given user, it is possible to redistribute a small volume of his traffic from any route to other (less loaded) routes for this user such that the traffic delay on this route becomes strictly smaller.

Example 9.7 Consider a simple example explaining the difference between the definitions of a Nash equilibrium and a Wardrop equilibrium. A system contains one user, who sends traffic of volume 1 from node s to node t via two routes (see Figure 9.10).

Suppose that the latency functions on route 1 (which includes channels (1,2,4)) and on route 2 (which includes channels (1,3,4)) have the form $f_1(x) = \max\{1, x, 1\} = 1$ and $f_2(y) = \min\{1, y, 1\} = y$, respectively; here $x = 1 - y$. Both functions are continuous and non-decreasing in x and y, respectively. The inequality $f_1(x) > f_2(y)$ takes place for all feasible strategy profiles (x, y) such that $x + y = 1$. However, any reduction in x (the volume of traffic through channel 1) does not affect $f_1(x)$. In the described model, a Nash equilibrium is any strategy profile $(x, 1 - x)$, where $0 \le x \le 1$. Still, the delays in both channels coincide only for the strategy profile $(0, 1)$.

Definition 9.10 *Let x indicate some strategy profile. The social costs are the total delay of all players under this strategy profile:*

$$SC(x) = \sum_{i=1}^{n} \sum_{R_i \in Z_i} x_{iR_i} f_{iR_i}(x).$$

Note that, if x represents a Wardrop equilibrium, then (by definition) for each player i the delays on all used routes R_i equal $\lambda_i(x)$. Therefore, the social costs in the equilibrium acquire the form

$$SC(x) = \sum_{i=1}^{n} w_i \lambda_i(x).$$

Designate by opt $= \min_x SC(x)$ the minimal social costs.

Definition 9.11 *We call the price of anarchy the maximal value of the ratio $SC(x)/opt$, where the social costs are evaluated only in Wardrop equilibria.*

9.11 The optimal routing model with parallel channels. The Pigou model. Braess's paradox

We analyze the Wardrop model for a network with parallel channels.

Example 9.8 The Pigou model (1920). Consider a simple network with two parallel channels (see Figure 9.11). One channel possesses the fixed capacity of 1, whereas the second channel has the capacity proportional to traffic. Imagine very many users transmitting their traffic from node s to node t such that the total load is 1. Each user seeks to minimize his costs. Then a Nash equilibrium lies in employing the lower channel for each user. Indeed, if the upper channel comprises a certain quantity of players, the lower channel always guarantees a smaller delay than the upper one. Therefore, the costs of each player in the equilibrium make up 1. Furthermore, the social costs constitute 1 too.

Now, assume that some share x of users utilize the upper channel, and the rest users (the share $1 - x$) employ the lower channel. Then the social costs become $x \cdot 1 + (1 - x) \cdot (1 - x) = x^2 - x + 1$. The minimal social costs of $3/4$ correspond to $x = 1/2$. Obviously, the price of anarchy in the Pigou model is $PA = 4/3$.

Example 9.9 Consider the same two-channel network, but set the delay in the lower channel equal to x^p, where p means a certain parameter. A Nash equilibrium also consists in sending the traffic of all users through the lower channel (the social costs make up 1). Next, send some volume ϵ of traffic by the upper channel. The corresponding social costs $\epsilon \cdot 1 + (1 - \epsilon)^{p+1}$ possess arbitrary small values as $\epsilon \to 0$ and $p \to \infty$. And so, the price of anarchy can have arbitrary large values.

Example 9.10 (Braess's paradox). Recall that we have explored this phenomenon in the case of indivisible traffic. Interestingly, Braess's paradox arises in models with divisible traffic. Select a network composed of four nodes, see Figure 9.4. There are two routes from node s to node t with the identical delays of $1 + x$. Suppose that the total traffic of all users equals 1. Owing to the symmetry of this network, all users get partitioned into two equal groups with the identical costs of $3/2$. This forms a Nash equilibrium.

To proceed, imagine that we have constructed a new superspeed channel (CD) with zero delay. Then, for each user, the route $A \to C \to D \to B$ is always not worse than the route $A \to C \to B$ or $A \to D \to B$. Nevertheless, the costs of all players increase up to 2 in the new

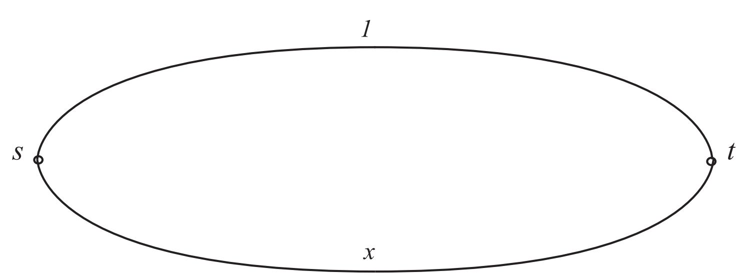


Figure 9.11 The Pigou model.

equilibrium. This example shows that adding a new channel may raise the costs of individual players and the social costs.

9.12 Potential in the optimal routing model with indivisible traffic for an arbitrary network

Let $\Gamma = \langle n, G, w, Z, f \rangle$ be the Wardrop model, where n users send traffic via channels of a network. Its topology is defined by a graph $G = (V, E)$. The quantity $W = \sum_{i=1}^{n} w_i$ specifies the total volume of data packets of all players. Denote by x_{iR_i} the strategy of player i; actually, this is the part of traffic transmitted through the channel R_i. Note that $\sum_{R_i \in Z_i} x_{iR_i} = w_i$, $x_{iR_i} \geq 0$. For each edge e, a given strictly increasing continuous function $f_e(\delta(x))$ taking non-negative values on $[0, W]$ characterizes the delay on this edge. We believe that the delay of player i on the route R_i has the additive form

$$f_{iR_i}(\delta(x)) = \sum_{e \in R_i} f_e(\delta_e(x)),$$

i.e., represents the sum of delays on all channels of this route.

Consider a game with the payoff functions

$$PC_i(x) = \max_{R_i \in Z_i : x_{iR_i} > 0} f_{iR_i}(x) = \max_{R_i \in Z_i : x_{iR_i} > 0} \sum_{e \in R_i} f_e(\delta_e(x)).$$

Introduce the potential

$$P(x) = \sum_{e \in E} \int_0^{\delta_e(x)} f_e(t) dt.$$

Since $\int_0^{\delta} f_e(t) dt$ is a differentiable function with non-decreasing derivative, the above function enjoys convexity.

Theorem 9.10 *A strategy profile x forms a Wardrop equilibrium* (ergo, *a Nash equilibrium) iff $P(x) = \min_y P(y)$.*

Proof: Let x be a Wardrop equilibrium and y mean an arbitrary strategy profile. The convexity of the function $P(x)$ implies that

$$P(y) - P(x) \geq \sum_{i=1}^{n} \sum_{R_i \in Z_i} \frac{\partial P(x)}{\partial x_{iR_i}} (y_{iR_i} - x_{iR_i}). \qquad (12.1)$$

Clearly,

$$\frac{\partial P(x)}{\partial x_{iR_i}} = \sum_{e \in R_i} f_e(\delta_e(x)).$$

According to the Wardrop equilibrium condition, for any player i we have

$$\lambda_i(x) = \sum_{e \in R_i} f_e(\delta_e(x)), \ x_{iR_i>0},$$
$$\lambda_i(x) \leq \sum_{e \in R_i} f_e(\delta_e(x)), \ x_{iR_i=0}.$$

Expand the second sum in (12.1) into two sums as follows. Where $y_{iR_i} - x_{iR_iI} \geq 0$, take advantage of the inequality $\frac{\partial P(x)}{\partial x_{iR_i}} \geq \lambda_i(x)$. In the second sum, we have $y_{iR_i} - x_{iR_i} < 0$. Hence, $x_{iR_i} > 0$, and the equilibrium condition brings to $\frac{\partial P(x)}{\partial x_{iR_i}} = \lambda_i(x)$.

As a result,

$$P(y) - P(x) \geq \sum_{i=1}^{n} \sum_{R_i \in Z_i} \lambda_i(x)(y_{iR_i} - x_{iR_i}) = \sum_{i=1}^{n} \lambda_i(x) \sum_{R_i \in Z_i} (y_{iR_i} - x_{iR_i}).$$

On the other hand, for any player i and any strategy profile we have $\sum_{R_i \in Z_i} y_{iR_i} = \sum_{R_i \in Z_i} x_{iR_i} = w_i$. Then it follows that

$$P(y) \geq P(x), \ \forall y.$$

And so, x minimizes the potential $P(y)$.

Now, let the strategy profile x be the minimum point of the function $P(y)$. Assume that x is not a Wardrop equilibrium. Then there exists player i and two routes $R_i, \rho_i \in Z_i$ such that $x_{R_i} > 0$ and

$$\sum_{e \in R_i} f_e(\delta_e(x)) > \sum_{e \in \rho_i} f_e(\delta_e(x)). \tag{12.2}$$

Next, take the strategy profile x and replace the traffic on the routes R_i and ρ_i such that $y_{R_i} = x_{R_i} - \epsilon$ and $y_{\rho_i} = x_{R_i} + \epsilon$. This is always possible for a sufficiently small ϵ, so long as $x_{R_i} > 0$. Then the inequality

$$\begin{aligned} P(x) - P(y) &\geq \sum_{i=1}^{n} \sum_{R_i \in Z_i} \frac{\partial P(y)}{\partial x_{iR_i}}(y_{iR_i} - x_{iR_i}) \\ &= \epsilon \left(\sum_{e \in R_i} f_e(\delta_e(y)) - \sum_{e \in \rho_i} f_e(\delta_e(y)) \right) > 0 \end{aligned}$$

holds true for a sufficiently small ϵ by virtue of inequality (12.2) and the continuity of the function $f_e(\delta_e(y))$. This contradicts the hypothesis that $P(x)$ is the minimal value of the potential. The proof of Theorem 9.10 is finished.

We emphasize that potential represents a continuous function defined on the compact set of all feasible strategy profiles x. Hence, this function admits a minimum, and a Nash equilibrium exists.

Generally, researchers employ linear latency functions $f_e(\delta) = a_e\delta + b_e$, as well as latency functions of the form $f_e(\delta) = 1/(c_e - \delta)$ or $f_e(\delta) = \delta/(c_e - \delta)$, where c_e indicates the capacity of channel e.

9.13 Social costs in the optimal routing model with divisible traffic for convex latency functions

Consider a network with an arbitrary topology, where the latency functions $f_e(\delta)$ are differentiable increasing convex functions. Then the social costs acquire the form

$$SC(x) = \sum_{i=1}^{n} \sum_{R_i \in Z_i} x_{iR_i} \sum_{e \in R_i} f_e(\delta_e(x)) = \sum_{e \in E} \delta_e(x) f_e(\delta_e(x)),$$

i.e., become a convex function. Note that

$$\frac{\partial SC(x)}{\partial x_{iR_i}} = \sum_{e \in R_i} \left(f_e(\delta_e(x)) + \delta_e(x) f_e'(\delta_e(x)) \right) = \sum_{e \in R_i} f_e^*(\delta_e(x)).$$

The expression $f_e^*(\delta_e(x))$ will be called the marginal costs on channel e.

By repeating argumentation of Theorem 9.10 for the function $SC(x)$ (instead of potential), we arrive at the following assertion.

Theorem 9.11 *A strategy profile x minimizes the social costs $SC(x) = \min_y SC(y)$ iff the inequality*

$$\sum_{e \in R_i} f_e^*(\delta_e(x)) \le \sum_{e \in \rho_i} f_e^*(\delta_e(x))$$

holds true for any i and any routes $R_i, \rho_i \in Z_i$, where $x_{iR_i > 0}$.

For instance, choose the linear latency functions $f_e(\delta) = a_e\delta + b_e$. The marginal costs are determined by $f_e^*(\delta) = 2a_e\delta + b_e$, and the minimum condition of the social costs in the strategy profile x takes the following form. For any player i and any routes $R_i, \rho_i \in Z_i$, where $x_{iR_i > 0}$, we have

$$\sum_{e \in R_i} \left(2a_e\delta_e(x) + b_e\right) \le \sum_{e \in \rho_i} \left(2a_e\delta(x) + b_e\right).$$

The last condition can be reexpressed as follows. For any player i, the inequality $x_{iR_i} > 0$ implies that $\sum_{e\in R_i}(2a_e\delta_e(x) + b_e) = \lambda_i^*(x)$, while the equality $x_{iR_i} = 0$ brings to $\sum_{e\in R_i}(2a_e\delta_e(x) + b_e) \geq \lambda_i^*(x)$

Compare this result with the conditions when a strategy profile x forms a Wardrop equilibrium.

Corollary. If a strategy profile x is a Wardrop equilibrium in the model $\langle n, G, w, Z, f\rangle$ with the linear latency function, then the strategy profile $x/2$ minimizes the social costs in the model $\langle n, G, w/2, Z, f\rangle$, where the traffic of all players is cut by half.

9.14 The price of anarchy in the optimal routing model with divisible traffic for linear latency functions

Consider the game $\langle n, G, w, Z, f\rangle$ with the linear latency functions $f_e(\delta) = a_e\delta + b_e$, where $a_e > 0, e \in E$. Let x^* be a strategy profile ensuring the optimal social costs $SC(x^*) = \min_y SC(y)$.

Lemma 9.8 *The social costs in the Wardrop model with doubled traffic*

$$\langle n, G, 2w, Z, f\rangle$$

grow, at least, to the quantity

$$SC(x^*) + \sum_{i=1}^{n} \lambda_i^*(x^*)w_i.$$

Proof: Take an arbitrary strategy profile x in the model with double traffic. The following inequality can be easily verified:

$$(a_e\delta_e(x) + b_e)\delta_e(x) \geq (a_e\delta_e(x^*) + b_e)\delta_e(x^*) + (\delta_e(x) - \delta_e(x^*)(2a_e\delta_e(x^*) + b_e).$$

It appears equivalent to the inequality $(\delta(x) - \delta(x^*))^2 \geq 0$. In the accepted system of symbols, this inequality takes the form

$$f_e(\delta_e(x))\delta_e(x) \geq f_e(\delta_e(x^*))\delta_e(x^*) + (\delta_e(x) - \delta_e(x^*)f_e^*(\delta_e(x^*)).$$

Summation over all $e \in E$ yields the expressions

$$SC(x) = \sum_{e\in E} f_e(\delta_e(x))\delta_e(x) \geq \sum_{e\in E} f_e(\delta_e(x^*))\delta_e(x^*) + \sum_{e\in E}(\delta_e(x) - \delta_e(x^*))f_e^*(\delta_e(x^*)).$$

And so,

$$SC(x) \geq SC(x^*) + \sum_{i=1}^{n} \sum_{R_i\in Z_i} \left(x_{iR_i} - x_{iR_i}^*\right) \sum_{e\in R_i} f_e^*(\delta_e(x^*)).$$

Since x^* specifies the minimum point of $SC(x)$, Theorem 9.10 implies that $\sum_{e\in R_i} f_e^*(\delta_e(x^*)) = \lambda_i^*(x^*)$ under $x^*_{iR_i} > 0$ and $\sum_{e\in R_i} f_e^*(\delta_e(x^*)) \geq \lambda_i^*(x^*)$ under $x^*_{iR_i} = 0$. Hence, it follows that

$$SC(x) \geq SC(x^*) + \sum_{i=1}^{n} \lambda_i^*(x^*) \sum_{R_i \in Z_i} \left(x_{iR_i} - x^*_{iR_i} \right).$$

By the assumption, $\sum_{R_i\in Z_i}(x_{iR_i} - x^*_{iR_i}) = 2w_i - w_i = w_i$. Therefore,

$$SC(x) \geq SC(x^*) + \sum_{i=1}^{n} \lambda_i^*(x^*) w_i.$$

This concludes the proof of Lemma 9.8.

Theorem 9.12 *The price of anarchy in the Wardrop model with linear latency functions constitutes* $PA = 4/3$.

Proof: Suppose that x represent a Wardrop equilibrium in the model

$$\langle n, G, w, Z, f \rangle.$$

Then, according to the corollary of Theorem 9.11, the strategy profile $x/2$ yields the minimal social costs in the model

$$\langle n, G, w/2, Z, f \rangle.$$

Lemma 9.8 claims that, if we double traffic in this model (i.e., getting back to the initial traffic w), for any strategy profile y the social costs can be estimated as follows:

$$SC(y) \geq SC(x/2) + \sum_{i=1}^{n} \lambda_i^*(x/2) \frac{w_i}{2} = SC(x/2) + \frac{1}{2} \sum_{i=1}^{n} \lambda_i(x) w_i.$$

Recall that x forms a Wardrop equilibrium. Then $\sum_{i=1}^{n} \lambda_i(x) w_i = SC(x)$, whence it appears that

$$SC(y) \geq SC(x/2) + \frac{1}{2} SC(x).$$

Furthermore,

$$SC(x/2) = \sum_{e\in E} \delta_e(x/2) f_e(\delta_e(x/2)) = \sum_{e\in E} \frac{1}{2}\delta_e(x) \left(\frac{1}{2} a_e \delta_e(x) + b_e \right)$$
$$\geq \frac{1}{4} \sum_{e\in E} \left(a_e \delta_e^2(x) + b_e \delta_e(x) \right) = \frac{1}{4} SC(x).$$

These inequalities lead to $SC(y) \geq \frac{3}{4}SC(x)$ for any strategy profile y (particularly, for the strategy profile guaranteeing the minimal social costs). Consequently, we obtain the upper estimate for the price of anarchy:

$$PA = \sup_{x\text{–equilibrium}} \frac{SC(x)}{\text{opt}} \leq \frac{4}{3}.$$

The corresponding lower estimate has been established in the Pigou model, see Section 9.10. The proof of Theorem 9.12 is completed.

9.15 Potential in the Wardrop model with parallel channels for player-specific linear latency functions

In the preceding sections, we have studied models with identical latency functions of all players on each channel (this latency function depends on channel load only). However, in real games channel delays may have different prices for different players. In this case, we speak about network games with player-specific delays. Consider the Wardrop model $\langle n, G, w, Z, f\rangle$ with parallel channels (Figure 9.12) and linear latency functions of the form $f_{ie}(\delta) = a_{ie}\delta$. Here the coefficients a_{ie} are different for different players $i \in N$ and channels $e \in E$.

Let $x = \{x_{ie}, i \in N, e \in E\}$ be some strategy profile, $\sum_{e\in E} x_{ie} = w_i, i = 1, \dots, n$. Introduce the function

$$P(x) = \sum_{i=1}^{n}\sum_{e\in E} x_{ie} \ln a_{ie} + \sum_{e\in E} \delta_e(x) \ln \delta_e(x).$$

Theorem 9.13 *A strategy profile x makes a Wardrop equilibrium iff $P(x) = \min_y P(y)$.*

Proof: We begin with the essentials. Assume that x is a Wardrop equilibrium. Find the derivative of the function P:

$$\frac{\partial P(x)}{\partial x_{ie}} = 1 + \ln a_{ie} + \ln\left(\sum_{k=1}^{n} x_{ke}\right) = 1 + \ln\left(a_{ie}\sum_{k=1}^{n} x_{ke}\right).$$

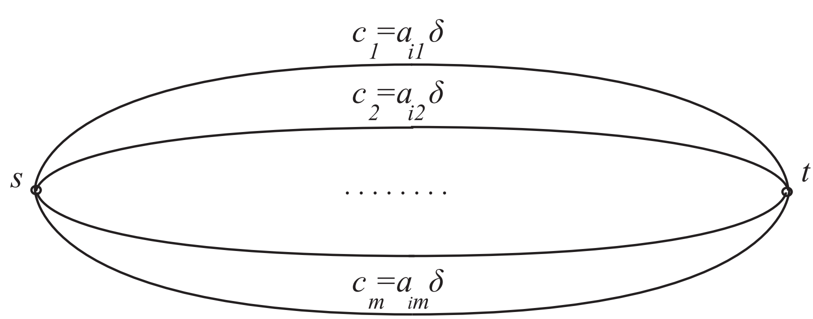


Figure 9.12 The Wardrop model with parallel channels and linear delays.

The equilibrium conditions require that for all $i \in N$ and $e, l \in E$:

$$x_{ie} > 0 \Rightarrow a_{ie} \sum_{k=1}^{n} x_{ke} \leq a_{il} \sum_{k=1}^{n} x_{kl}.$$

Due to the monotonicity of the function $\ln x$, this inequality leads to

$$x_{ie} > 0 \Rightarrow \frac{\partial P(x)}{\partial x_{ie}} \leq \frac{\partial P(x)}{\partial x_{il}}, \forall i, e, l.$$

Subsequent reasoning is similar to that of Theorem 9.10. The function $x \ln x$, as well as the linear function are convex. On the other hand, the sum of convex functions also represents a convex function. Thus and so, $P(x)$ becomes a convex function. This function is continuously differentiable. The convexity of $P(x)$ implies that

$$P(y) - P(x) \geq \sum_{i=1}^{n} \sum_{e \in E} \frac{\partial P(x)}{\partial x_{ie}}(x)(y_{ie} - x_{ie}).$$

By virtue of the equilibrium conditions, we have

$$x_{ie} > 0 \Rightarrow \frac{\partial P(x)}{\partial x_{ie}} = \lambda_i, \forall e \in E,$$
$$x_{ie} = 0 \Rightarrow \frac{\partial P(x)}{\partial x_{ie}} \geq \lambda_i, \forall e \in E.$$

Under the second condition $x_{ie} = 0$, we get $y_{ie} - x_{ie} \geq 0$, and then $\frac{\partial P(x)}{\partial x_{ie}}(x)(y_{ie} - x_{ie}) \geq \lambda_i(y_{ie} - x_{ie})$. This brings to the expressions

$$P(y) - P(x) \geq \sum_{i=1}^{n} \sum_{e \in E} \lambda_i(y_{ie} - x_{ie}) = \sum_{i=1}^{n} \lambda_i \sum_{e \in E}(y_{ie} - x_{ie}) = 0.$$

Consequently, $P(y) \geq P(x)$ for all y; hence, x is the minimum point of the function $P(x)$.

Now, argue the sufficiency part of Theorem 9.13. Imagine that x is the minimum point of the function $P(y)$. Proceed by *reductio ad absurdum*. Conjecture that x is not a Wardrop equilibrium. Then for some player k there are two channels p and q such that $x_{kp} > 0$ and $a_{kp}\delta_p(x) > a_{kq}\delta_q(x)$. In this case, there exists a number $z : 0 < z < x_{kp}$ meeting the condition

$$a_{kp}(\delta_p(x) - z) \geq a_{kq}(\delta_q(x) + z).$$

Define a new strategy profile y such that all strategies of players $i \neq k$ remain the same, whereas the strategy of player k acquires the form

$$y_{ke} = \begin{cases} x_{kp} - z, & \text{if } e = p \\ x_{kq} + z, & \text{if } e = q \\ x_{ke}, & \text{otherwise.} \end{cases}$$

Consider the difference

$$P(x) - P(y) = \sum_{i=1}^{n} \sum_{e \in E} (x_{ie} - y_{ie}) \ln a_{ie} + \sum_{e \in E} (\delta_e(x) \ln \delta_e(x) - \delta_e(y) \ln \delta_e(y)). \quad (15.1)$$

Both sums in (15.1) have non-zero terms corresponding to player k and channels p, q only:

$$\begin{aligned} P(x) - P(y) &= z(\ln a_{kp} - \ln a_{kq}) + \delta_p(x) \ln \delta_p(x) + \delta_q(x) \ln \delta_q(x) \\ &\quad -(\delta_p(x) - z) \ln(\delta_p(x) - z) - (\delta_q(y) + z) \ln(\delta_q(x) + z) \\ &= \ln \left(a_{kp}^z \cdot \delta_p(x)^{\delta_p(x)} \cdot \delta_q(x)^{\delta_q(x)} \right) - \ln \left(a_{kq}^z \cdot (\delta_p(x) - z)^{\delta_p(x) - z} \cdot (\delta_q(x) + z)^{\delta_q(x) + z} \right). \end{aligned}$$

Below we demonstrate Lemma 9.9, which claims that the last expression is strictly positive. But, in this case, one obtains $P(x) > P(y)$. This obviously contradicts the condition that x is the minimum point of the function $P(y)$. And the desired conclusion follows.

Lemma 9.9 *Let a, b, u, v, and z be non-negative, $u \geq z$. If $a(u - z) \geq b(v + z)$, then*

$$a^z \cdot u^u \cdot v^v > b^z \cdot (u - z)^{u - z} \cdot (v + z)^{v + z}.$$

Proof: First, show the inequality >

$$\left(\frac{\alpha}{\alpha - 1} \right)^{\alpha} > e > \left(1 + \frac{1}{\beta} \right)^{\beta}, \; \alpha > 1, \beta > 0. \quad (15.2)$$

It suffices to notice that the function

$$f(\alpha) = \left(1 + \frac{1}{\alpha - 1} \right)^{\alpha} = \exp \left(\alpha \ln \left(1 + \frac{1}{\alpha - 1} \right) \right),$$

being monotonically decreasing, tends to e as $\alpha \to \infty$. The monotonous property follows from the negativity of the derivative

$$f'(\alpha) = f(\alpha) \left(\ln \left(1 + \frac{1}{\alpha - 1} \right) - \frac{1}{\alpha - 1} \right) < 0 \;\text{ for all } \alpha > 1.$$

By analogy, readers can verify the right-hand inequality.

Now, set $\alpha = u/z$ and $\beta = v/z$. Then the condition $a(u - z) \geq b(v + z)$ implies that $a(\alpha z - z) \geq b(\beta z + z)$, whence it appears that $a(\alpha - 1) \geq b(\beta + 1)$. Due to inequality (15.2), we have

$$a \alpha^{\alpha} \beta^{\beta} > a(\alpha - 1)^{\alpha} (\beta + 1)^{\beta} \geq b(\alpha - 1)^{\alpha - 1} (\beta + 1)^{\beta + 1}.$$

Multiply the last inequality by $z^{\alpha + \beta}$,

$$a(z\alpha)^{\alpha} (z\beta)^{\beta} > b(z\alpha - z)^{\alpha - 1} (z\beta + z)^{\beta + 1},$$

and raise to the power of z to get

$$a^z(z\alpha)^{z\alpha}(z\beta)^{z\beta} > b^z(z\alpha - z)^{z\alpha - z}(z\beta + z)^{z\beta + z}.$$

This proves Lemma 9.9.

9.16 The price of anarchy in an arbitrary network for player-specific linear latency functions

consider the Wardrop model $\langle n, G, w, Z, f\rangle$ for an arbitrary network with divisible traffic and linear latency functions of the form $f_{ie}(\delta) = a_{ie}\delta$. Here the coefficients a_{ie} are different for different players $i \in N$ and channels $e \in E$. An important characteristic lies in

$$\Delta = \max_{i,k \in N, e \in E} \left\{ \frac{a_{ie}}{a_{ke}} \right\},$$

i.e., the maximal ratio of delays over all players and channels. We have demonstrated that the price of anarchy in an arbitrary network with linear delays (identical for all players) equals $4/3$. The price of anarchy may grow appreciably, if the latency functions become player-specific. Still, it is bounded by the quantity Δ.

The proof of this result employs the following inequality.

Lemma 9.10 *For any $u, v \geq 0$ and $\Delta > 0$, we have*

$$uv \leq \frac{1}{2\Delta}u^2 + \frac{\Delta}{2}v^2.$$

Proof is immediate from the representation

$$\frac{1}{2\Delta}u^2 + \frac{\Delta}{2}v^2 - uv = \frac{\Delta}{2}\left(\frac{u}{\Delta} - v\right)^2 \geq 0.$$

Theorem 9.14 *The price of anarchy in the Wardrop model with player-specific linear costs does not exceed Δ.*

Proof: Let x be a Wardrop equilibrium and x^* denote a strategy profile minimizing the social costs. Consider the social costs in the equilibrium:

$$SC(x) = \sum_{i=1}^{n} \sum_{R_i \in Z_i} x_{iR_i} \sum_{e \in R_i} a_{ie}\delta_e(x).$$

By the definition of a Wardrop equilibrium, the delays on all used channels coincide, i.e., $\sum_{e \in R_i} a_{ie}\delta_e(x) = \lambda_i$, if $x_{iR_i} > 0$. This implies that

$$SC(x) = \sum_{i=1}^{n} \sum_{R_i \in Z_i} x_{iR_i} \sum_{e \in R_i} a_{ie}\delta_e(x) \leq \sum_{i=1}^{n} \sum_{R_i \in Z_i} x^*_{iR_i} \sum_{e \in R_i} a_{ie}\delta_e(x).$$

Rewrite the last expression as

$$\sum_{i=1}^{n} \sum_{R_i \in Z_i} x^*_{iR_i} \sum_{e \in R_i} \frac{a_{ie}}{\delta_e(x^*)} \delta_e(x^*)\delta_e(x),$$

and take advantage of Lemma 9.10:

$$\begin{aligned}
SC(x) &\le \sum_{i=1}^{n} \sum_{R_i \in Z_i} x^*_{iR_i} \sum_{e \in R_i} \frac{a_{ie}}{\delta_e(x^*)} \left(\frac{\Delta}{2}\delta_e^2(x^*) + \frac{1}{2\Delta}\delta_e^2(x) \right) \\
&= \frac{\Delta}{2} \sum_{i=1}^{n} \sum_{R_i \in Z_i} x^*_{iR_i} \sum_{e \in R_i} a_{ie}\delta_e(x^*) + \frac{1}{2\Delta} \sum_{e \in E} \sum_{i=1}^{n} \sum_{R_i \in Z_i : e \in R_i} \frac{x^*_{iR_i}}{\delta_e(x^*)} a_{ie}\delta_e^2(x) \\
&= \frac{\Delta}{2} SC(x^*) + \frac{1}{2\Delta} \sum_{e \in E} \sum_{i=1}^{n} \sum_{R_i \in Z_i : e \in R_i} \frac{x^*_{iR_i}}{\delta_e(x^*)} \cdot a_{ie}\delta_e^2(x).
\end{aligned}$$

To estimate the second term in the last formula, make the following observation. If the equalities

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n = 1$$

hold true with non-negative summands, one obtains the estimate

$$\frac{a_1x_1 + a_2x_2 + \cdots + a_nx_n}{a_1y_1 + a_2y_2 + \cdots + a_ny_n} \le \frac{\max\{a_i\}}{\min\{a_i\}}, \ a_i > 0, i = 1, \dots, n.$$

On the other hand, in the last expression for any $e \in E$ we have

$$\sum_{i=1}^{n} \sum_{R_i \in Z_i : e \in R_i} \frac{x^*_{iR_i}}{\delta_e(x^*)} = \sum_{i=1}^{n} \sum_{R_i \in Z_i : e \in R_i} \frac{x_{iR_i}}{\delta_e(x)} = 1.$$

Hence,

$$SC(x) \le \frac{\Delta}{2} SC(x^*) + \frac{1}{2\Delta}\Delta \sum_{e \in E} \sum_{i=1}^{n} \sum_{R_i \in Z_i : e \in R_i} \frac{x_{iR_i}}{\delta_e(x)} \cdot a_{ie}\delta_e^2(x).$$

Certain simplifications yield

$$SC(x) \le \frac{\Delta}{2} SC(x^*) + \frac{1}{2} \sum_{i=1}^{n} \sum_{R_i \in Z_i} x_{iR_i} \sum_{e \in R_i} a_{ie}\delta_e(x) = \frac{\Delta}{2} SC(x^*) + \frac{1}{2} SC(x).$$

And so, the estimate

$$\frac{SC(x)}{SC(x^*)} \le \Delta$$

is valid for any equilibrium. This proves Theorem 9.14.

Therefore, in an arbitrary network with player-specific linear delays, the price of anarchy appears finite and depends on the ratio of latency function coefficients of different players.

Exercises

1. Three identical parallel channels are used to send four data packets $w_1 = 2$, $w_2 = 2$, $w_3 = 3$, and $w_4 = 4$. The latency function has the form $f(w) = \frac{w}{c}$. Find a pure strategy Nash equilibrium and a completely mixed equilibrium.

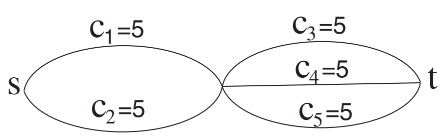
2. Three parallel channels with the capacities $c_1 = 1.5$, $c_2 = 2$ and $c_3 = 2.5$ transmit four identical data packets. Evaluate a pure strategy Nash equilibrium and a completely mixed equilibrium under linear delays.

3. Two parallel channels with the capacities $c_1 = 1$ and $c_2 = 2$ transmit five packets $w_1 = 2$, $w_2 = 2$, $w_3 = 2$, $w_4 = 4$, and $w_5 = 5$. Find a Nash equilibrium in the class of pure strategies and mixed strategies. Calculate the social costs in the linear, quadratic, and maximin forms.

4. Three parallel channels with the capacities $c_1 = 2$, $c_2 = 2.5$, and $c_3 = 4$ are used to send four data packets $w_1 = 5$, $w_2 = 7$, $w_3 = 10$, and $w_4 = 12$. The latency function possesses the linear form. Evaluate the worst-case Nash equilibrium. Compute the corresponding social costs and the price of anarchy.

5. Two channels with the capacities $c_1 = 1$ and $c_2 = 2$ transmit four data packets $w_1 = 3$, $w_2 = 4$, $w_3 = 6$, and $w_4 = 8$. The latency functions are linear. One channel is added to the network. Which capacity of the additional channel will increase the social costs?

6. Consider a network with parallel channels. The social costs have the quadratic form. One channel is added to the network. Is it possible that the resulting social costs go up?



7. Take a Wardrop network illustrated by the figure above.

Four data packets $w_1 = 1$, $w_2 = 1$, $w_3 = 2$, and $w_4 = 3$ are sent from node s to node t. The latency function is defined by $f(w) = \frac{1}{c-w}$. Find a Wardrop equilibrium. Calculate the linear and maximal social costs.

8. In a Wardrop network, three parallel channels with the capacities $c_1 = 3$, $c_2 = 3$, and $c_3 = 4$ transmit four data packets $w_1 = 1$, $w_2 = 1$, $w_3 = 2$, and $w_4 = 3$. The latency function is defined by $f(w) = \frac{1}{c-w}$. Find a Wardrop equilibrium, calculate the linear social costs and evaluate the price of anarchy.

9. Consider the Wardrop model. In a general form network, the delay on channel e possesses the form $f_e(w) = \frac{1}{c_e-w}$. Find the potential of this network.

10. Choose the player-specific Wardrop model. A network comprises parallel channels. For player i, the delay on channel e is given by $f_{ie}(w) = a_e w + b_{ie}$. Find the potential of such network.

10

Dynamic games

Introduction

Dynamic games are remarkable for their evolvement over the course of time. Here players control some object or system whose dynamics is described by a set of difference equations or differential equations. In the case of objects moving in certain space, pursuit games arise naturally. Players strive to approach an opponent's object at minimum time or maximize the probability of opponent's object detection. Another interpretation concerns economic or ecological systems, where players seek to gain the maximal income or cause the minimal environmental damage.

Definition 10.1 *A dynamic game is a game* $\Gamma = < N, x, \{U_i\}_{i=1}^n, \{H_i\}_{i=1}^n >$, *where* $N = \{1, 2, \dots, n\}$ *denotes the set of players,*

$$x'(t) = f(x, u_1, \dots, u_n, t),\ x(0) = x_0,\ x = (x_1, \dots, x_m),\ 0 \le t \le T,$$

indicates a controlled system in the space R^m, $U_1, \dots, U_n$ *are the strategy sets of players* $1, \dots, n$, *respectively, and a function* $H_i(u_1, \dots, u_n)$ *specifies the payoff of player* $i \in N$.

A controlled system is considered on a time interval $[0, T]$ (finite or infinite). Player strategies represent some functions $u_i = u_i(t), i = 1, \dots, n$. Depending on selected strategies, each player receives the payoff

$$H_i(u_1, \dots, u_n) = \int_0^T g_i(x(t), u_1(t), \dots, u_n(t), t)dt + G_i(x(T), T), i = 1, \dots, n.$$

Actually, it consists of **the integral component and the terminal component**; g_i and $G_i, i = 1, \dots, n$ are given functions.

Mathematical Game Theory and Applications, First Edition. Vladimir Mazalov.

Companion website: http://www.wiley.com/go/game_theory

There exist cooperative and non-cooperative dynamic games. Solutions of non-cooperative games are comprehended in the sense of Nash equilibria.

Definition 10.2 *A Nash equilibrium in the game* Γ *is a set of strategies* $(u_1^*, \dots, u_n^*)$ *such that*

$$H_i\left(u_{-i}^*, u_i\right) \le H_i(u^*)$$

for arbitrary strategies $u_i, i = 1, \dots, n$.

Our analysis begins with discrete-time dynamic games often called "fish wars."

10.1 Discrete-time dynamic games

Imagine a certain dynamic system governed by the system of difference equations

$$x_{t+1} = (x_t)^\alpha, \; t = 0, 1, \dots,$$

where $0 < \alpha \le 1$. For instance, this is the evolution of some fish population. The initial state x_0 of the system is given. Interestingly, the system admits the stationary state $x = 1$. If $x_0 > 1$, the population diminishes approaching infinitely the limit state $x = 1$. In the case of $x_0 < 1$, the population spreads out with the same asymptotic line.

Suppose that two countries (players) perform fishing; they aim at maximizing the income (the amount of fish) on some time interval. The utility function of each player depends on the amount of fish u caught by this player and takes the form $\log(u)$. The discounting coefficients β_1 (player 1) and β_2 (player 2) are given, $0 < \beta_i < 1, \; i = 1, 2$. Find a Nash equilibrium in this game.

10.1.1 Nash equilibrium in the dynamic game

First, we study this problem on a finite interval. Take the one-shot model. Assume that players decide to catch the amounts u_1 and u_2 at the initial instant (here $u_1 + u_2 \le x_0$). At the next instant $t = 1$, the population of fish has the size $x_1 = (x_0 - u_1 - u_2)^\alpha$. The game finishes and, by the agreement, players divide the remaining amount of fish equally. Consequently, the payoff of player 1 makes up

$$\begin{aligned} H_1(u_1, u_2) &= \log u_1 + \beta_1 \log\left(\tfrac{1}{2}(x - u_1 - u_2)^\alpha\right) = \\ &= \log u_1 + \alpha\beta_1 \log(x - u_1 - u_2) - \beta_1 \log 2, \; x = x_0. \end{aligned}$$

The factor β_1 corresponds to payoff reduction due to the discounting effect. Similarly, player 2 obtains the payoff

$$H_2(u_1, u_2) = \log u_2 + \alpha\beta_2 \log(x - u_1 - u_2) - \beta_2 \log 2, \; x = x_0.$$

The functions $H_1(u_1, u_2)$ and $H_2(u_1, u_2)$ are convex, and a Nash equilibrium exists. For its evaluation, solve the system of equations $\partial H_1/\partial u_1 = 0$, $\partial H_2/\partial u_2 = 0$, or

$$\frac{1}{u_1} - \frac{\alpha\beta_1}{x - u_1 - u_2} = 0, \ \frac{1}{u_2} - \frac{\alpha\beta_2}{x - u_1 - u_2} = 0.$$

And so, the equilibrium is defined by

$$u_1' = \frac{\alpha\beta_2}{(1 + \alpha\beta_1)(1 + \alpha\beta_2) - 1} \cdot x, \ u_2' = \frac{\alpha\beta_1}{(1 + \alpha\beta_1)(1 + \alpha\beta_2) - 1} \cdot x.$$

The size of the population after fishing constitutes

$$x - u_1' - u_2' = \frac{\alpha^2\beta_1\beta_2}{(1 + \alpha\beta_1)(1 + \alpha\beta_2) - 1} \cdot x.$$

The players' payoffs in the equilibrium become

$$H_1\left(u_1', u_2'\right) = (1 + \alpha\beta_1) \log x + a_1, \ H_2\left(u_1', u_2'\right) = (1 + \alpha\beta_2) \log x + a_2,$$

where the constants a_1, a_2 follow from the expressions

$$a_i = \log\left(\frac{\alpha\beta_j(\alpha^2\beta_1\beta_2)^{\alpha\beta_i}}{[(1 + \alpha\beta_1)(1 + \alpha\beta_2) - 1]^{1+\alpha\beta_i}}\right) - \beta_i \log 2, \ i, j = 1, 2, \ i \neq j.$$

Now, suppose that the game includes two shots, i.e., players can perform fishing twice. As a matter of fact, we have determined the optimal behavior and payoffs of both players at the last shot (though, under another initial condition). Hence, the equilibrium in the two-shot model results from maximization of the new payoff functions

$$H_1^2(u_1, u_2) = \log u_1 + \alpha\beta_1(1 + \alpha\beta_1) \log(x - u_1 - u_2) + \beta_1 a_1, \ x = x_0,$$
$$H_2^2(u_1, u_2) = \log u_1 + \alpha\beta_2(1 + \alpha\beta_2) \log(x - u_1 - u_2) + \beta_2 a_2, \ x = x_0.$$

Still, the payoff functions keep their convexity. The Nash equilibrium appears from the system of equations

$$\frac{1}{u_1} - \frac{\alpha\beta_1(1 + \alpha\beta_1)}{x - u_1 - u_2} = 0, \ \frac{1}{u_2} - \frac{\alpha\beta_2(1 + \alpha\beta_2)}{x - u_1 - u_2} = 0.$$

Again, it possesses the linear form:

$$u_1^2 = \frac{\alpha\beta_2(1 + \alpha\beta_2)}{(1 + \alpha\beta_1)(1 + \alpha\beta_2) - 1} \cdot x, \ u_2^2 = \frac{\alpha\beta_1(1 + \alpha\beta_1)}{(1 + \alpha\beta_1)(1 + \alpha\beta_2) - 1} \cdot x.$$

We continue such construction procedure to arrive at the following conclusion. In the n-shot fish war game, the optimal strategies of players are defined by

$$u_1^n = \frac{\alpha\beta_2 \sum_{j=0}^{n-1} (\alpha\beta_2)^j}{\sum_{j=0}^{n} (\alpha\beta_1)^j \sum_{j=1}^{n} (\alpha\beta_2)^j - 1} \cdot x, \ u_2^n = \frac{\alpha\beta_1 \sum_{j=0}^{n-1} (\alpha\beta_1)^j}{\sum_{j=0}^{n} (\alpha\beta_1)^j \sum_{j=1}^{n} (\alpha\beta_2)^j - 1} \cdot x. \tag{1.1}$$

After shot n, the population of fish has the size

$$x - u_1^n - u_2^n = \frac{\alpha^2\beta_1\beta_2 \sum_{j=0}^{n-1} (\alpha\beta_1)^j \sum_{j=0}^{n-1} (\alpha\beta_2)^j}{\sum_{j=0}^{n} (\alpha\beta_1)^j \sum_{j=1}^{n} (\alpha\beta_2)^j - 1} \cdot x. \tag{1.2}$$

As $n \to \infty$, the expressions (1.1), (1.2) admit the limits

$$u_1^* = \lim_{n\to\infty} u_1^n = \frac{\alpha\beta_2(1-\alpha\beta_1)x}{1-(1-\alpha\beta_1)(1-\alpha\beta_2)}, \ u_2^* = \lim_{n\to\infty} u_2^n = \frac{\alpha\beta_1(1-\alpha\beta_2)x}{1-(1-\alpha\beta_1)(1-\alpha\beta_2)}.$$

Therefore,

$$x - u_1^* - u_2^* = \lim_{n\to\infty} x - u_1^n - u_2^n = kx,$$

where

$$k = \frac{\alpha^2\beta_1\beta_2 x}{1-(1-\alpha\beta_1)(1-\alpha\beta_2)}.$$

To proceed, we revert to the problem in its finite horizon setting. Suppose that at each shot players adhere to the strategies u_1^*, u_2^*. Starting from the initial state x_0, the system evolves according to the law

$$x_{t+1} = (x_t - u_1^*(x_t) - u_2^*(x_t))^\alpha = k^\alpha x_{t-1}^\alpha = k^\alpha \left(kx_{t-1}^\alpha\right)^\alpha = k^{\alpha+\alpha^2} x_{t-1}^{\alpha^2} = \ldots$$
$$= k^{\sum_{j=1}^{t} \alpha^j} \cdot x_0^{\alpha^t}, \ t = 0, 1, 2, \ldots$$

Under large t, the system approaches the stationary state

$$\bar{x} = \left(\frac{1}{\frac{1}{\alpha\beta_1} + \frac{1}{\alpha\beta_2} - 1} \right)^{\frac{\alpha}{1-\alpha}}. \tag{1.3}$$

In the case of $\beta_1 = \beta_2 = \beta$, the stationary state has the form $\bar{x} = \left(\frac{\alpha\beta}{2-\alpha\beta}\right)^{\frac{\alpha}{1-\alpha}}$.

We focus on the special linear case. Here the population of fish demonstrates the following dynamics:

$$x_{t+1} = r(x_t - u_1 - u_2), \ \ r > 1.$$

Apply the same line of reasoning as before to get the optimal strategies of players in the Nash equilibrium for the multi-shot finite-horizon game:

$$u_1^n = \frac{\beta_2 \sum_{j=0}^{n-1} (\beta_2)^j}{\sum_{j=0}^{n} (\beta_1)^j \sum_{j=1}^{n} (\beta_2)^j - 1} \cdot x, \ \ u_2^n = \frac{\beta_1 \sum_{j=0}^{n-1} (\beta_1)^j}{\sum_{j=0}^{n} (\beta_1)^j \sum_{j=1}^{n} (\beta_2)^j - 1} \cdot x.$$

As $n \to \infty$, we obtain the limit strategies

$$u_1^* = \frac{\beta_2(1-\beta_1)x}{1-(1-\beta_1)(1-\beta_2)}, \ \ u_2^* = \frac{\beta_1(1-\beta_2)x}{1-(1-\beta_1)(1-\beta_2)}.$$

As far as

$$x - u_1^* - u_2^* = \frac{x}{\frac{1}{\beta_1} + \frac{1}{\beta_2} - 1},$$

the optimal strategies of the players lead to the population dynamics

$$x_t = \frac{r}{\frac{1}{\beta_1} + \frac{1}{\beta_2} - 1} \cdot x_{t-1} = \left(\frac{r}{\frac{1}{\beta_1} + \frac{1}{\beta_2} - 1} \right)^t x_0, \ \ t = 0, 1, \ldots$$

Obviously, the population dynamics in the equilibrium essentially depends on the coefficient $r/(\frac{1}{\beta_1} + \frac{1}{\beta_2} - 1)$. The latter being smaller than 1, the population degenerates; if this coefficient exceeds 1, the population grows infinitely. And finally, under strict equality to 1, the population possesses the stable size. In the case of identical discounting coefficients ($\beta_1 = \beta_2 = \beta$), further development or extinction of the population depends on the sign of $\beta(r + 1) - 2$.

10.1.2 Cooperative equilibrium in the dynamic game

Get back to the original model $x_t = x_{t-1}^{\alpha}$ with $\alpha < 1$. Assume that players agree about joint actions. We believe that $\beta_1 = \beta_2 = \beta$. Denote by $u = u_1 + u_2$ the general control. Arguing by analogy, readers can easily establish the optimal strategy in the n-shot game:

$$u^n = \frac{1 - \alpha\beta}{1 - (\alpha\beta)^{n+1}} \cdot x.$$

The corresponding limit strategy is $u^* = (1 - \alpha\beta)x$. And the population dynamics in the cooperative equilibrium acquires the form

$$x_t = (\alpha\beta x_{t-1})^{\alpha} = (\alpha\beta)^{\alpha+\alpha^2+\ldots+\alpha^t} \cdot x_0^{\alpha^t},\ t = 0, 1, \ldots$$

For large t, it tends to the stationary state

$$\hat{x} = (\alpha\beta)^{\frac{\alpha}{1-\alpha}}. \tag{1.4}$$

By comparing the stationary states (1.3) and (1.4) in the cooperative equilibrium and in the Nash equilibrium, we can observe that

$$\hat{x} = (\alpha\beta)^{\frac{\alpha}{1-\alpha}} \geq \bar{x} = \left(\frac{\alpha\beta}{2 - \alpha\beta}\right)^{\frac{\alpha}{1-\alpha}}.$$

In other words, cooperative actions guarantee a higher size of the population. Now, juxtapose the payoffs of players in these equilibria. In the cooperative equilibrium, at each shot players have the total payoff

$$u_c = (1 - \alpha\beta)\hat{x} = (1 - \alpha\beta)(\alpha\beta)^{\frac{\alpha}{1-\alpha}}. \tag{1.5}$$

Non-cooperative play brings to the following sum of their payoffs (under $\beta_1 = \beta_2$):

$$u_n = u_1^* + u_2^* = \frac{2\alpha\beta(1 - \alpha\beta)}{1 - (1 - \alpha\beta)^2} \cdot \bar{x} = \frac{2(1 - \alpha\beta)}{2 - \alpha\beta} \left(\frac{\alpha\beta}{2 - \alpha\beta}\right)^{\frac{\alpha}{1-\alpha}}. \tag{1.6}$$

Obviously,

$$2 < (2 - \alpha\beta)^{\frac{1}{1-\alpha}},\ 0 < \alpha, \beta < 1,$$

which means that $u_c > u_n$. Thus and so, cooperative behavior results in a benevolent scenario for the population and, furthermore, ensures higher payoffs to players (as against their independent actions).

This difference has the largest effect in the linear case $x_{t+1} = rx_t, t = 0, 1, \ldots$. Here the cooperative behavior $u = (1 - \beta)x$ leads to the population dynamics

$$x_t = r\beta x_{t-1} = \ldots = (r\beta)^t x_0\ t = 0, 1, \ldots$$

Its stationary state depends on the value of $r\beta$. If this expression is higher (smaller) than 1, the population grows infinitely (diminishes, respectively). The case $\beta = 1/r$ corresponds to stable population size. By virtue of $r\beta/(2 - \beta) \leq r\beta$, we may have a situation when $r\beta > 1$ and $r\beta/(2 - \beta) < 1$. This implies that, under the cooperative behavior of players, the population increases infinitely, whereas their egoistic behavior (each player pursues individual interests only) destroys the population.

10.2 Some solution methods for optimal control problems with one player

10.2.1 The Hamilton–Jacobi–Bellman equation

The principle of optimality was introduced by R. Bellman in 1958. Originally, the author suggested the following statement:

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

Consider the discrete-time control problem for one player. Let a controlled system evolve according to the law

$$x_{t+1} = f_t(x_t, u_t).$$

Admissible control actions satisfy certain constraints $u \in U$, where U forms some domain in the space R^m.

The player seeks to minimize the functional

$$J(u) = G(x_N) + \sum_{t=0}^{N-1} g_t(x_t, u_t),$$

where $u_t = u_t(x_t)$.

Introduce the Bellman function

$$B_k(x_k) = \min_{u_k,\cdots,u_{N-1}\in U} \sum_{i=k}^{N} g_i(x_i, u_i) + G(x_N).$$

We will address Bellman's technique. Suppose that we are at the point x_{N-1} and it suffices to make one shot, i.e., choose u_{N-1}. The payoff at this shot makes up

$$J_{N-1} = g_{N-1}(x_{N-1}, u_{N-1}) + G(x_N) = g_{N-1}(x_{N-1}, u_{N-1}) + G(f_{N-1}(x_{N-1}, u_{N-1})).$$

Therefore, on the last shot the functional represents a function of two variables, x_{N-1} and u_{N-1}. The minimum of this functional in u_{N-1} is the Bellman function

$$\min_{u_{N-1}\in U} J_{N-1} = B_{N-1}(x_{N-1}).$$

Take two last shots:

$$J_{N-2} = g_{N-2}(x_{N-2}, u_{N-2}) + g_{N-1}(x_{N-1}, u_{N-1}) + G(x_N).$$

We naturally have

$$\begin{aligned}
B_{N-2}(x_{N-2}) &= \min_{u_{N-2},u_{N-1}} J_{N-2} \\
&= \min_{u_{N-2}} g_{N-2}(x_{N-2}, u_{N-2}) + \min_{u_{N-1}} \{g_{N-1}(x_{N-1}, u_{N-1}) + G(x_N)\} \\
&= \min_{u_{N-2}} g_{N-2}(x_{N-2}, u_{N-2}) + B_{N-1}(x_{N-1}) \\
&= \min_{u_{N-2}} \{g_{N-2}(x_{N-2}, u_{N-2}) + B_{N-1}(f_{N-2}(x_{N-2}, u_{N-2}))\}.
\end{aligned}$$

Proceeding by analogy, readers can construct the recurrent expression known as the Bellman equation:

$$B_{N-k}(x_{N-k}) = \min_{u_{N-k}\in U} \{g_{N-k}(x_{N-k}, u_{N-k}) + B_{N-k+1}(f_{N-k}(x_{N-k}, u_{N-k}))\}. \tag{2.1}$$

Consequently, for optimal control evaluation, we search for B_k backwards. At any shot, implementing the minimum of the Bellman function gives the optimal control on this shot, $u_k^*(x_k)$.

Now, switch to the continuous-time setting of the optimal control problem:

$$\begin{aligned}
&x'(t) = f(t, x(t), u(t)), \;\; x(0) = x_0, \;\; u \in U, \\
&J(u) = \int_0^T g(t, x(t), u(t))dt + G(x(T)) \to \min.
\end{aligned}$$

Introduce the Bellman function

$$V(x, t) = \min_{u(s), t\le s\le T} \left[\int_t^T g(s, x(s), u(s))ds + G(x(T)) \right]$$

meeting the initial condition

$$V(x, T) = G(x(T)).$$

Using Bellman's principle of optimality, rewrite the function as

$$\begin{aligned}
V(x, t) &= \min_{u(s), t\le s\le T} \left[\int_t^{t+\Delta t} g(s, x(s), u(s))ds + \int_{t+\Delta t}^T g(s, x(s), u(s))ds + G(x(T)) \right] \\
&= \min_{u(s), t\le s\le t+\Delta t} \left\{ \int_t^{t+\Delta t} g(s, x(s), u(s))ds \right. \\
&\quad \left. + \min_{u(s), t+\Delta t\le s\le T} \left[\int_{t+\Delta t}^T g(s, x(s), u(s))ds + G(x(T)) \right] \right\} \\
&= \min_{u(s), t\le s\le t+\Delta t} \left\{ \int_t^{t+\Delta t} g(s, x(s), u(s))ds + V(x(t + \Delta t), t + \Delta t) \right\}.
\end{aligned}$$

By assuming that the function $V(x, t)$ is continuously differentiable, apply Taylor's expansion to the integral to obtain

$$V(x,t) = \min_{u(s), t \le s \le t+\Delta t} \{g(t, x(t), u(t))\Delta t + V(x(t), t) + \frac{\partial V(x,t)}{\partial t}\Delta t + \frac{\partial V(x,t)}{\partial x} f(t, x(t), u(t))\Delta t + o(\Delta t)\}.$$

As $\Delta t \to 0$, we derive the Hamilton–Jacobi–Bellman equation

$$-\frac{\partial V(x(t),t)}{\partial t} = \min_{u(t)\in U}\left[\frac{\partial V(x(t),t)}{\partial x} f(t, x(t), u(t)) + g(t, x(t), u(t))\right] \tag{2.2}$$

with the initial condition $V(x, T) = G(x(T))$.

Theorem 10.1 *Suppose that there exists a unique continuously differentiable solution $V_0(x, t)$ of the Hamilton–Jacobi–Bellman equation (2.2) and there exists an admissible control law $u_0(x, t)$ such that*

$$\min_{u\in U}\left[\frac{\partial V_0(x,t)}{\partial x} f(t, x, u) + g(t, x, u)\right] = \frac{\partial V_0(x,t)}{\partial x} f(t, x, u_0) + g(t, x, u_0).$$

Then $u_0(x, t)$ makes the optimal control law, and the corresponding Bellman function is $V_0(x, t)$.

Proof: Write down the total derivative of the function $V_0(x, t)$ by virtue of equation (2.2):

$$V_0'(x,t) = \frac{\partial V_0(x,t)}{\partial t} + \frac{\partial V_0(x,t)}{\partial x} f(t, x, u_0) = -g(t, x, u_0).$$

Substitute into this equality the state $x = x_0(t)$ corresponding to the control action $u_0(t)$ to get

$$V_0'(x_0(t), t) = -g(t, x_0(t), u_0(x_0(t), t)).$$

Integration over t from 0 to T yields

$$V_0(x_0, 0) = V(T, x(T)) + \int_0^T g(t, x_0(t), u_0(x_0(t), t)) = J(u_0).$$

Assume that $u(x, t)$ is any admissible control and $x(t)$ designates the corresponding trajectory of the process. Due to equation (2.2), we have

$$V_0'(x(t), t) \ge -g(t, x(t), u(x(t), t)).$$

Again, perform integration from 0 to T to obtain

$$J(u) \ge V_0(x_0, 0),$$

whence it follows that

$$J(u_0) = V_0(x_0, 0) \le J(u).$$

This proves optimality of u_0.

Let the function $V(t, x)$ be twice continuously differentiable. Suppose that the Hamilton–Jacobi–Bellman equation admits a unique continuous solution $V(x, t)$ and there exists admissible control $u_0(x, t)$, meeting the conditions of Theorem 10.1 with the trajectory $x_0(t)$.

Introduce the following functions:

$$\psi(t) = -\frac{\partial V(x(t), t)}{\partial x},$$
$$H(t, x, u, \psi) = \psi(t) f(t, x, u) - g(t, x, u).$$

Using Theorem 10.1, we have

$$\begin{aligned} H(t, x_0, u_0, \psi) &= -\frac{\partial V(x_0, t)}{\partial x} f(t, x_0, u_0) - g(t, x_0, u_0) \\ &= \max_{u \in U} \left[-\frac{\partial V(x_0, t)}{\partial x} f(t, x_0, u) - g(t, x_0, u) \right] = \max_{u \in U} H(t, x_0, u, \psi). \end{aligned}$$

According to Theorem 10.1,

$$\frac{\partial V(x, t)}{\partial t} = H(t, x, u_0, \psi).$$

By differentiating with respect to x and setting $x = x_0$, one obtains that

$$\frac{\partial^2 V(x_0, t)}{\partial t \partial x} = \frac{\partial H(t, x_0, u_0, \psi)}{\partial x} = -\psi'(t).$$

Similarly, differentiation of the initial conditions brings to

$$\frac{\partial V(x_0(T), T)}{\partial x} = -\psi(T) = G'_x(x_0(T)).$$

Therefore, we have derived the maximum principle for the fixed-time problem. Actually, it can be reformulated for more general cases.

10.2.2 Pontryagin's maximum principle

Consider the continuous-time optimal control problem

$$J(u) = \int_0^T f_0(x(t), u(t)) dt + G(x(T)) \to min,$$
$$x'(t) = f(x(t), u(t)), \; x(0) = x^0, \; u \in U,$$

where $x = (x_1, \cdots, x_n)$, $u = (u_1, \cdots, u_r)$, and $f(x, u) = (f_1(x, u), \cdots, f_n(x, u))$.

Introduce the Hamiltonian function

$$H(x, u, \psi) = \sum_{i=0}^{n} \psi_i f_i(x, u), \tag{2.3}$$

with $\psi = (\psi_0, \cdots, \psi_n)$ indicating the vector of conjugate variables.

Theorem 10.2 *(the maximum principle) Suppose that the functions $f_i(x, u)$ and $G(x)$ have partial derivatives and, along with these derivatives, are continuous in all their arguments for $x \in R^n$, $u \in U$, $t \in [0, T]$. A necessary condition for the control law $u^*(t)$ and the trajectory $x^*(t)$ to be optimal is that there exists a non-zero vector-function $\psi(t)$ satisfying*

1. *the maximum condition*

$$H(x^*(t), u^*(t), \psi(t)) = \max_{u \in U} H(x^*(t), u, \psi(t));$$

2. *the conjugate system in the conjugate variables*

$$\psi'(t) = -\frac{\partial H(x^*, u^*, \psi)}{\partial x}; \tag{2.4}$$

3. *the transversality condition*

$$\psi(T) = -G'_x(x^*(T)); \tag{2.5}$$

4. *the normalization condition*

$$\psi_0(t) = -1.$$

The proof of the maximum principle in the general statement appears rather complicated. Therefore, we are confined to the problem with fixed T and a free endpoint of the trajectory.

Take a controlled system described by the differential equations

$$\frac{dx}{dt} = f(x, u),\ \ x(0) = x^0, \tag{2.6}$$

where $x = (x_1, \cdots, x_n)$, $u = (u_1, \cdots, u_r)$, and $f(x, u) = (f_1(x, u), \cdots, f_n(x, u))$.

The problem consists in choosing an admissible control law $u(t)$ which minimizes the functional

$$Q = \int_0^T f_0(x(t), u(t))dt,$$

where T is a fixed quantity.

Introduce an auxiliary variable $x_0(t)$ defined by the equation

$$\frac{dx_0}{dt} = f_0(x, u),\ \ x_0(0) = 0. \tag{2.7}$$

This leads to the problem

$$Q = x_0(T) \to \min. \tag{2.8}$$

Theorem 10.3 *A necessary condition for an admissible control $u(t)$ and the corresponding trajectory $x(t)$ to solve the problem (2.6), (2.8) is that there exists a non-zero continuous vector-function $\psi(t)$ meeting the conjugate system (2.4) such that*

1. $H(x^*(t), u^*(t), \psi(t)) = \max_{u \in U} H(x^*(t), u, \psi(t))$;
2. $\psi(T) = (-1, 0, \cdots, 0)$.

Proof: Let $u^*(t)$ be an optimal control law and $x^*(t)$ mean the corresponding optimal trajectory of the system. We address the method of needle-shaped variations—this is a standard technique to prove the maximum principle in the general case. Notably, consider an infinitely small time interval $\tau - \varepsilon < t < \tau$, where $\tau \in (0, T)$ and ε denotes an infinitesimal quantity.

Now, vary the control law $u^*(t)$ through replacing it (on the infinitely small interval) by another control law u.

Find the corresponding variation in the system trajectory. According to (2.6) and (2.7), we have

$$\begin{aligned} x_j(\tau) - x_j^*(\tau) &= \left[\left(\frac{dx_j}{dt} \right)_{t=\tau} - \left(\frac{dx_j^*}{dt} \right)_{t=\tau} \right] \varepsilon + o(\varepsilon) \\ &= [f_j(x(\tau), u(\tau)) - f_j(x^*(\tau), u^*(\tau))]\varepsilon + o(\varepsilon), \; j = 0, \cdots, n. \end{aligned}$$

And so, the quantity $x_j(\tau) - x_j^*(\tau)$ possesses the same infinitesimal order as ε. Hence, this property also applies to the quantity $f_j(x(\tau), u(\tau)) - f_j(x^*(\tau), u(\tau))$. Consequently,

$$x_j(\tau) - x_j^*(\tau) = [f_j(x^*(\tau), u(\tau)) - f_j(x^*(\tau), u^*(\tau))]\varepsilon + o(\varepsilon), \; j = 0, \cdots, n. \tag{2.9}$$

Introduce the following variations of the coordinates:

$$x_j(t) = x_j^*(t) + \delta x_j(t), \; j = 0, \cdots, n.$$

Due to (2.9), their values at the moment $t = \tau$ are

$$\delta x_j(\tau) = [f_j(x^*(\tau), u(\tau)) - f_j(x^*(\tau), u^*(\tau))]\varepsilon.$$

Substitute the variations of the coordinates $x_j(t)$ into equations (2.6) and (2.7) to get

$$\frac{dx_j^*}{dt} + \frac{d(\delta x_j)}{dt} = f_j(x^* + \delta x, u^*), \; j = 0, \cdots, n.$$

Expansion in the Taylor series in a neighborhood of $x^*(t)$ yields

$$\frac{dx_j^*}{dt} + \frac{d(\delta x_j)}{dt} = f_j(x^*, u^*) + \sum_{i=0}^{n} \frac{\partial f_j(x^*, u^*)}{\partial x_i}\delta x_i + o(\delta x_0, \cdots, \delta x_n).$$

Since $x^*(t)$ obeys equation (2.6) under $u^*(t)$, further reduction brings to

$$\frac{d(\delta x_j)}{dt} = \sum_{i=0}^{n} \frac{\partial f_j(x^*, u^*)}{\partial x_i}\delta x_i, \tag{2.10}$$

known as the variation equations.

Consider conjugate variables $\psi = (\psi_0 \cdots, \psi_n)$ such that

$$\frac{d\psi_k}{dt} = -\sum_{i=0}^{n} \frac{\partial f_i(x^*, u^*)}{\partial x_k}\psi_i.$$

Find the time derivative of $\sum_{i=0}^{n} \psi_i \delta x_i$ using formula (2.10):

$$\begin{aligned}\frac{d}{dt}\sum_{i=0}^{n} \psi_i \delta x_i &= \sum_{i=0}^{n} \frac{d\psi_i}{dt}\delta x_i + \sum_{i=0}^{n} \frac{d(\delta x_i)}{dt}\psi_i \\ &= -\sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\partial f_j(x^*, u^*)}{\partial x_i}\psi_j \delta x_i + \sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\partial f_j(x^*, u^*)}{\partial x_i}\psi_j \delta x_i = 0.\end{aligned}$$

And so,

$$\sum_{i=0}^{n} \psi_i \delta x_i = const, \;\; \tau \le t \le T. \tag{2.11}$$

Let δQ be the increment of the functional under the variation u. According to (2.8) and due to the fact that u^* minimizes the functional Q, we naturally obtain

$$\delta Q = \delta x_0(T) \ge 0.$$

Introduce the initial conditions for the conjugate variables:

$$\psi_0(T) = -1, \; \psi_1(T) = 0, \cdots, \psi_n(T) = 0.$$

In this case,

$$\sum_{i=0}^{n} \psi_i(T)\delta x_i(T) = -\delta x_0(T) = -\delta Q \le 0.$$

Having in mind the expression (2.11), one can rewrite this inequality as

$$-\delta Q = \sum_{i=0}^{n} \psi_i(\tau)\delta x_i(\tau).$$

Next, substitute here the known quantity $\delta x_i(\tau)$ to get

$$\sum_{i=0}^{n} \psi_i(\tau)[f_j(x^*(\tau), u(\tau)) - f_j(x^*(\tau), u^*(\tau))]\varepsilon \leq 0.$$

So long as $\varepsilon > 0$ and τ can be any moment t, then

$$\sum_{i=0}^{n} \psi_i(\tau) f_j(x^*(\tau), u(\tau)) \leq \sum_{i=0}^{n} \psi_i(\tau) f_j(x^*(\tau), u^*(\tau)).$$

Finally, we have established that

$$H(x^*, u, \psi) \leq H(x^*, u^*, \psi),$$

which proves the validity of the maximum conditions.

Further exposition focuses on the discrete-time optimal control problem

$$\begin{aligned} I(u) &= \sum_{0}^{N} f^0(x_t, u_t)dt + G(x_N) \to min, \\ x_{t+1} &= f(x_t, u_t),\ x_0 = x^0,\ u \in U, \end{aligned} \tag{2.12}$$

where $x = (x^1, \cdots, x^n)$, $u = (u^1, \cdots, u^r)$, and $f(x, u) = (f^1(x, u), \cdots, f^n(x, u))$. Our intention is to formulate the discrete-time analog of Pontryagin's maximum principle.

Consider the Hamiltonian function

$$H(\psi_{t+1}, x_t, u_t) = \sum_{i=0}^{n} \psi_{t+1}^i f^i(x_t, u_t),\ t = 0, \cdots, N-1, \tag{2.13}$$

where $\psi = (\psi^0, \cdots, \psi^n)$ designates the vector of conjugate variables.

Theorem 10.4 (the maximum principle for the discrete-time optimal control problem) *A necessary condition for admissible control u_t^* and the corresponding trajectory x_t^* to be optimal is that there exists a set of non-zero continuous vector-functions $\psi_t^1, \cdots, \psi_t^n$ satisfying*

1. *the maximum condition*

$$H\left(\psi_{t+1}, x_t^*, u_t^*\right) = \max_{u_t \in U} H\left(\psi_{t+1}, x_t^*, u_t\right), \quad t = 0, \cdots, N-1;$$

2. *the conjugate system in the conjugate variables*

$$\psi_t = -\frac{\partial H\left(\psi_{t+1}, x_t^*, u_t^*\right)}{\partial x_t}; \tag{2.14}$$

3. *the transversality condition*

$$\psi_N = -\frac{\partial G(x_N)}{\partial x_N}; \tag{2.15}$$

4. *the normalization condition*

$$\psi_t^0 = -1.$$

We prove the discrete-time maximum principle in the terminal state optimization problem. In other words, the functional takes the form

$$I = G(x_N) \to max. \tag{2.16}$$

It has been demonstrated above that, in the continuous-time case, the optimization problem with the sum-type performance index, i.e., the functional

$$I = \sum_{t=0}^{N} f^0(x_t, u_t)dt,$$

can be easily reduced to the problem (2.16).

The Hamiltonian function in the problem (2.16) takes the form

$$H(\psi_{t+1}, x_t, u_t) = \sum_{i=1}^{n} \psi_{t+1}^i f^i(x_t, u_t), \ t = 0, \cdots, N-1.$$

For each $u \in U$, consider the cone of admissible variations

$$K(u) = \{\delta u \,|\, u + \varepsilon \delta u \in U\}, \ \varepsilon > 0.$$

Assume that the cone $K(u)$ is convex and contains inner points.

Denote by $\delta_u H(\psi, x, u)$ the admissible differential of the Hamiltonian function:

$$\delta_u H(\psi, x, u) = \left(\frac{\partial H(\psi, x, u)}{\partial u}, \delta u\right) = \sum_{i=0}^{r} \frac{\partial H(\psi, x, u)}{\partial u^i} \delta u^i,$$

where $\delta u \in K(u)$.

Below we argue

Lemma 10.1 *Let* $u^* = \{u_0^*, \cdots, u_{N-1}^*\}$ *be the optimal control under the initial state* $x_0 = x^0$ *in the problem (2.16). The inequality*

$$\delta_u H\left(\psi_{t+1}^*, x_t^*, u_t^*\right) \leq 0$$

takes place for any $\delta u_t^* \in K(u_t^*)$*, where the optimal values* x^* *follow from the system (2.12), whereas the optimal values* ψ^* *follow from the conjugate system (2.14) with the boundary condition (2.15).*

Moreover, if u_t^* *makes an inner point of the set* U*, then* $\delta H(u_t^*) = 0$ *for any admissible variations at this point. In the case of* $\delta H(u_t^*) < 0$*, the point* u_t^* *represents a boundary point of the set* U*.*

Proof: Fix the optimal process $\{u^*, x^*\}$ and consider the variation equation on this process:

$$\delta x_{t+1}^* = \frac{\partial f\left(x_t^*, u_t^*\right)}{\partial x_t}\delta x_t^* + \frac{\partial f\left(x_t^*, u_t^*\right)}{\partial u_t}\delta u_t^*, \quad t = 0, \cdots, N-1.$$

Suppose that the vectors ψ_t^* are evaluated from the conjugate system. Analyze the scalar product

$$\left(\psi_{t+1}^*, \delta x_{t+1}^*\right) = \left(\psi_t^*, \delta x_t^*\right) + \left(\psi_{t+1}^*, \frac{\partial f\left(x_t^*, u_t^*\right)}{\partial u_t}\delta u_t^*\right). \tag{2.17}$$

Perform summation over $t = 0, \cdots, N-1$ in formula (2.17) and use the equalities $\delta x^*(0) = 0$ and (2.15). Such manipulations lead to

$$\delta G\left(x_N^*\right) = \sum_{t=0}^{N-1} \delta_u H\left(\psi_{t+1}^*, x_t^*, u_t^*\right),$$

where

$$\delta G\left(x_N^*\right) = \left(\frac{\partial G\left(x_N^*\right)}{\partial x_N}, \delta x_N^*\right). \tag{2.18}$$

Since x_N^* is the optimal state, then $\delta G(x_N^*) \leq 0$ for any $\delta u_t^* \in K(u_t^*)$ (readers can easily verify this).

Assume that there exists a variation δx_N^* such that $(\frac{\partial G(x_N^*)}{\partial x_N}, \delta x_N^*) > 0$.

By the definition of the cone $K(x^*)$, there exists $\varepsilon_1 > 0$ such that $x_N^* + \varepsilon \delta x_N^* \in R$ for any $0 < \varepsilon < \varepsilon_1$.

Consider the expansion

$$G(x + \varepsilon\delta x) - G(x) = \varepsilon\delta G(x) + o(\varepsilon) = \varepsilon\left(\frac{\partial G(x)}{\partial x}, \delta x\right) + o(\varepsilon) > 0,$$

which is valid for admissible variations ensuring increase of the function $G(x)$.

The above expansion and our assumption imply that it is possible to choose ε such that $G(x^* + \varepsilon\delta x^*) > G(x^*)$. This contradicts optimality.

Thus, we have shown that $\delta G(x_N^*) \le 0$ for any $\delta u_t^* \in K(u_t^*)$.

Select $\delta u_j^* = 0, j \ne t$, $\delta u_t^* \ne 0$; then it follows from (2.18) that $\delta_u H(\psi_{t+1}^*, x_t^*, u_t^*) \le 0$ for any $\delta u_t^* \in K(u_t^*)$.

Now, imagine that u_t^* makes an inner point of the set U for some t. In this case, the cone $K(u_t^*)$ is the whole space of variations. Therefore, if $\delta u_t^* \in K(u_t^*)$, then necessarily $-\delta u_t^* \in K(u_t^*)$. Consequently, we arrive at

$$\frac{\partial H\left(\psi_{t+1}^*, x_t^*, u_t^*\right)}{\partial u_t} = 0.$$

On the other hand, if $\delta_u H(\psi_{t+1}^*, x_t^*, u_t^*) < 0$ at a certain point $u_t^* \in U$, the latter is not an inner point of the set U. The proof of Lemma 10.1 is completed.

Actually, we have demonstrated that the admissible differential of the Hamiltonian function possesses non-positive values on the optimal control.

In other words, the necessary maximum conditions of the function $H(u_t)$ on the set U hold true on the optimal control.

If u_t^* forms an inner point of the set U, then

$$\frac{\partial H\left(u_t^*\right)}{\partial u_k} = 0,$$

i.e., we obtain the standard necessary maximum conditions of a multivariable function.

Note that, if $\delta H(u_t^*) < 0$ (*viz.*, the gradient of the Hamiltonian function has non-zero components and is non-orthogonal to all admissible variations at the point u_t^*), then the point u_t^* provides a local maximum of the function $H(u_t)$ under some regularity assumption. This is clear from the expansion

$$H(u_t) - H\left(u_t^*\right) = \varepsilon \delta u H\left(u_t^*\right) + o(\varepsilon).$$

10.3 The maximum principle and the Bellman equation in discrete- and continuous-time games of N players

Consider a dynamic N player game in discrete time. Suppose that the dynamics obeys the equation

$$x_{t+1} = f_t\left(x_t, u_t^1, \cdots, u_t^N\right),\ t = 1, \cdots, n,$$

where x_1 is given. The payoff functions of players have the form

$$J^i(u^1, \cdots, u^N) = \sum_{j=1}^{n} g_j^i\left(u_j^1, \cdots, u_j^N, x_j\right) \to \min.$$

Theorem 10.5 *Let the functions f_t and g_t^i be continuously differentiable. If $(u^{1*}, \cdots, u^{N*})$ makes a Nash equilibrium in this game and x_t^* denotes the corresponding trajectory of the process, then for each $i \in N$ there exists a finite set of n-dimensional vectors $\psi_2^i, \cdots, \psi_{n+1}^i$ such that the following conditions hold true:*

$$
\begin{aligned}
x_{t+1}^* &= f_t\left(x_t^*, u_t^{1*}, \cdots, u_t^{N*}\right), \ x_1^* = x_1, \\
u_t^{i*} &= argmin_{u_t^i \in U_t^i} H_t^i\left(\psi_{t+1}^i, u_t^{1*}, \cdots, u_t^{i-1*}, u_t^i, u_t^{i+1*}, \cdots, u_t^{N*}, x_t^*\right), \\
\psi_t^i &= \frac{\partial f_t\left(x_t^*, u_t^{1*}, \cdots, u_t^{N*}\right)}{\partial x_t}\psi_{t+1}^i + \frac{\partial g_t^i\left(u_t^{1*}, \cdots, u_t^{N*}, x_t^*\right)}{\partial x_t}, \ \psi_{n+1}^i = 0,
\end{aligned}
$$

where

$$
H_t^i\left(\psi_{t+1}^i, u_t^1, \cdots, u_t^N, x_t\right) = g_t^i\left(u_t^1, \cdots, u_t^N, x_t\right) + \psi_{t+1}^i f_t\left(x_t, u_t^1, \cdots, u_t^N\right).
$$

Proof: For each player i, the Nash equilibrium condition acquires the form

$$
J^{i*}(u^{1*}, \cdots, u^{i-1*}, u^{i*}, u^{i+1*}, \cdots, u^{N*} \leq J^i(u^{1*}, \cdots, u^{i-1*}, u^i, u^{i+1*}, \cdots, u^{N*}).
$$

This inequality takes place when the minimum of J^i is attained on u^{i*} under the dynamics

$$
x_{t+1} = f_t(x_t, u^{1*}, \cdots, u^{i-1*}, u^{i*}, u^{i+1*}, \cdots, u^{N*}).
$$

Actually, we have obtained the optimal control problem for a single player. And the conclusion follows according to Theorem 10.4.

Theorem 10.6 *Consider the infinite discrete-time dynamic game of N players. Strategies $(u_t^{i*}(x_t))$ represent a Nash equilibrium iff there exist functions $V^i(t,x)$ meeting the conditions*

$$
\begin{aligned}
V^i(t,x) &= \min_{u_t^i \in U_t^i}\left[g_t^i\left(\bar{u}_t^i, x\right) + V^i\left(t+1, f_t(x, \bar{u}_t^i)\right)\right] \\
&= g_t^i\left(u_t^{1*}(x), \cdots, u_t^{N*}(x), x\right) \\
&\quad + V^i(t+1, f_t\left(x, u_t^{1*}(x), \cdots, u_t^{N*}(x)\right)), \quad V^i(n+1, x) = 0,
\end{aligned}
$$

where $\bar{u}_t^i = (u_t^{1*}(x), \cdots, u_t^{i-1*}(x), u_t^i, u_t^{i+1*}(x), \cdots, u_t^{N*}(x)$.

Proof: For each player i, the Nash equilibrium condition is defined by

$$
J^{i*}(u^{1*}, \cdots, u^{i-1*}, u^{i*}, u^{i+1*}, \cdots, u^{N*}) \leq J^i(u^{1*}, \cdots, u^{i-1*}, u^i, u^{i+1*}, \cdots, u^{N*}).
$$

Again, this inequality is the case when the maximum of J^i is reached on u^{i*} under the dynamics

$$
x_{t+1} = f_t(x_t, u^{1*}, \cdots, u^{i-1*}, u^{i*}, u^{i+1*}, \cdots, u^{N*}).
$$

This makes the optimal control problem for a single player. And the desired result of Theorem 10.6 follows from the Bellman equation (2.1).

To proceed, we analyze a two-player game, where one player is the leader. Assume that the dynamics satisfies the equation

$$x_{t+1} = f_t\left(x_t, u_t^1, u_t^2\right), \ t = 1, \cdots, n.$$

The payoff functions of players have the form

$$J^i\left(u^1, u^2\right) = \sum_{j=1}^{n} g_j^i(u_j^1, u_j^2, x_j) \to \min,$$

where $u^i \in U^i$ is a compact set in R^m.

Theorem 10.7 *Let the function g_t^1 be continuously differentiable, the functions f_t and g_t^2 be twice continuously differentiable. Moreover, suppose that the minimum of $H_t^2(\psi_{t+1}, u_t^1, u_t^2, x_k)$ in u_t^2 is achieved in an inner point for any $u_t^1 \in U^1$. Then, if (u^{1^*}, u^{2^*}) forms a Stackelberg equilibrium in this game (player 1 is the leader) and x_t^* indicates the corresponding trajectory of the process, there exist three finite sets of n-dimensional vectors $\lambda_1, \cdots, \lambda_n$, $\mu_1, \cdots, \mu_n$, $\nu_1, \cdots, \nu_n$ such that*

$$x_{t+1}^* = f_t\left(x_t^*, u_t^{1^*}, u_t^{2^*}\right), \ x_1^* = x_1,$$

$$\nabla_{u_t^1} H_t^1\left(\lambda_t, \mu_2, \nu_2, \psi_{t+1}^*, u_t^{1*}, u_t^{2*}, x_t^*\right) = 0,$$

$$\nabla_{u_t^2} H_t^1\left(\lambda_t, \mu_2, \nu_2, \psi_{t+1}^*, u_t^{1*}, u_t^{2*}, x_t^*\right) = 0,$$

$$\lambda_{t-1} = \frac{\partial H_t^1\left(\lambda_t, \mu_2, \nu_2, \psi_{t+1}^*, u_t^{1*}, u_t^{2*}, x_t^*\right)}{\partial x_t}, \ \lambda_n = 0,$$

$$\mu_{t+1} = \frac{\partial H_t^1\left(\lambda_t, \mu_2, \nu_2, \psi_{t+1}^*, u_t^{1*}, u_t^{2*}, x_t^*\right)}{\partial \psi_{t+1}}, \ \mu_1 = 0,$$

$$\nabla_{u_t^2} H_t^2\left(\psi_{t+1}^*, u_t^{1*}, u_t^{2*}, x_t^*\right) = 0,$$

$$\psi_t^* = F_t\left(x_t^*, \psi_{t+1}^*, u_t^{1*}, u_t^{2*}\right), \ \psi_{n+1} = 0,$$

where

$$H_t^1 = g_t^1\left(u_t^1, u_t^2, x_t\right) + \lambda_t f_t\left(x_t, u_t^1, u_t^2\right) + \mu_t F_t\left(x_t, u_t^1, u_t^2, \psi_{t+1}\right) + \nu_t \nabla_{u_t^2} H_t^2\left(\psi_{t+1}, u_t^1, u_t^2, x_t\right),$$

$$F_t = \frac{\partial f_t\left(x_t, u_t^1, u_t^2\right)}{\partial x_t} \psi_{t+1} + \frac{\partial g_t^2\left(u_t^1, u_t^2, x_t\right)}{\partial x_t},$$

$$H_t^2\left(\psi_{t+1}, u_t^1, u_t^2, x_t\right) = g_t^2\left(u_t^1, u_t^2, x_t\right) + \psi_{t+1} f_t\left(x_t, u_t^1, u_t^2\right).$$

Proof: First, assume that we know the leader's control u_1. In this case, the optimal response $\bar{u}_2$ of player 2 results from Theorem 10.5:

$$\bar{x}_{t+1} = f_t\left(\bar{x}_t, u_t^1, \bar{u}_t^2\right),\ \bar{x}_1 = x_1,$$
$$\bar{u}_t^2 = argmin_{u_t^2 \in U_t^2} \lim H_t^2\left(\psi_{t+1}, u_t^1, u_t^2, \bar{x}_t\right),$$
$$\psi_t = \frac{\partial f_t\left(\bar{x}_t, u_t^1, \bar{u}_t^2\right)}{\partial x_t}\psi_{t+1} + \frac{\partial g_t^2\left(u_t^1, \bar{u}_t^2, \bar{x}_t\right)}{\partial x_t},\ \psi_{n+1}^i = 0,$$

where

$$H_t^2\left(\psi_{t+1}^i, u_t^1, u_t^2, x_t\right) = g_t^2\left(u_t^1, u_t^2, x_t\right) + \psi_{t+1} f_t\left(x_t, u_t^1, u_t^2 N\right),$$

and $\psi_1, \cdots \psi_{n+1}$ is a sequence of n-dimensional conjugate vectors for this problem.

Recall that, by the premise, the Hamiltonian function admits its minimum in an inner point. Therefore, the maximum condition can be rewritten as

$$\nabla_{u_t^2} H_t^2\left(\psi_{t+1}, u_t^1, \bar{u}_t^2, \bar{x}_t\right) = 0.$$

To find the leader's control, we have to solve the problem

$$min_{u^1 \in U^1} J^1(u^1, u^2)$$

subject to the constraints

$$\bar{x}_{t+1} = f_t\left(\bar{x}_t, u_t^1, u_t^2\right),$$

$$\psi_t = F_t\left(x_t, \psi_{t+1}, u_t^1, u_t^2\right),\ \psi_{n+1} = 0,$$

$$\nabla_{u_t^2} H_t^2\left(\psi_{t+1}, u_t^1, u_t^2, x_t\right) = 0,$$

where

$$F_t = \frac{\partial f_t\left(x_t, u_t^1, u_t^2\right)}{\partial x_t}\psi_{t+1} + \frac{\partial g_t^2\left(u_t^1, u_t^2, x_t\right)}{\partial x_t}.$$

Apply Lagrange's method of multipliers. Construct the Lagrange function for this constrained optimization problem:

$$L = \sum_t g_t^1\left(u_t^1, u_t^2, x_t\right) + \lambda_t[f_t\left(x_t, u_t^1, u_t^2\right) - x_{t+1}]$$
$$+ \mu_t\left[f_t\left(x_t, \psi_{t+1}, u_t^1, u_t^2\right) - \psi_t\right] + \nu \frac{\partial H_t^2\ \left(\psi_{t+1}, u_t^1, u_t^2, x_t\right)}{\partial u_t^2},$$

where λ_t, μ_t, and ν_t stand for corresponding Lagrange multipliers. For $u_t^{1^*}$ to be a solution of the posed problem, it is necessary that

$$\nabla_{u_t^1} L = 0,\ \nabla_{u_t^2} L = 0,\ \nabla_{x_t} L = 0,\ \nabla_{\psi_{t+1}} L = 0.$$

These expressions directly lead to the conditions of Theorem 10.7.

Now, we switch to continuous-time games. Suppose that the dynamics is described by the equation

$$x'(t) = f(t, x(t), u^1(t), \cdots, u^N(t)),\ 0 \leq t \leq T,\ x(0) = x_0,\ u^i \in U^i.$$

The payoff functions of players have the form

$$J^i(u^1, \cdots, u^N) = \int_0^T g^i(t, x(t), u^1(t), \cdots, u^N(t))dt + G^i(x(T)) \to \min.$$

Theorem 10.8 *Let the functions f and g^i be continuously differentiable. If $(u^{1^*}(t), \cdots, u^{N^*}(t))$ represents a Nash equilibrium in this game and $x^*(t)$ specifies the corresponding trajectory of the process, then for each $i \in N$ there exist N functions $\psi^i(\cdot) : [0, T] \in R^n$ such that*

$$x^{*\prime}(t) = f(t, x^*(t), u^{1^*}(t), \cdots, u^{N^*}(t)),\ x^*(0) = x_0,$$

$$u^{i*}(t) = argmin_{u^i \in U^i} \lim H^i(t, \psi^i(t), x^*(t), u^{1*}(t), \cdots, u^{i-1*}(t), u^i, u^{i+1^*}(t), \cdots, u^{N*}(t)),$$

$$\psi^{i\prime}(t) = -\frac{\partial H^i(t, \psi^i(t), x^*, u^{1^*}(t), \cdots, u^{N^*}(t))}{\partial x},\ \psi(T) = \frac{\partial G^i(x^*(T))}{\partial x},$$

where

$$H^i(t, \psi^i, x, u^1, \cdots, u^N) = g^i(t, x, u^1, \cdots, u^N) + \psi^i f(t, x, u^1, \cdots, u^N).$$

Proof is immediate from the maximum principle for continuous-time games (see Theorem 10.2).

Theorem 10.9 *Consider a dynamic continuous-time game of N players. Strategies $(u^{i*}(t, x))$ form a Nash equilibrium iff there exist functions $V^i : [0, T]R^n \in R$ meeting the conditions*

$$\begin{aligned}
-\frac{\partial V^i(t, x)}{\partial t} &= \min_{u^i \in S^i} \lim \left[\frac{\partial V^i(t, x)}{\partial x} f(t, x, u^{1*}(t, x), \cdots, u^{i-1*}(t, x), u^i, u^{i+1*}(t, x), \cdots, u^{N*}(t, x))\right. \\
&\quad \left. + g^i(t, x, u^{1*}(t, x), \cdots, u^{i-1*}(t, x), u^i, u^{i+1*}(t, x), \cdots, u^{N*}(t, x))\right] \\
&= \frac{\partial V^i(t, x)}{\partial x} f(t, x, u^{1*}(t, x), \cdots, u^{N*}(t, x)) + g^i(t, x, u^{1*}(t, x), \cdots, u^{N*}(t, x)), \\
&\quad V^i(T, x) = G^i(x).
\end{aligned}$$

Proof follows from the Hamilton–Jacobi–Bellman equation (2.2).

In addition, we analyze a two-player game, where one player is a leader. Suppose that the dynamics has the equation

$$x'(t) = f(t, x(t), u^1(t), u^2(t)),\ x(0) = x_0.$$

The payoff functions of players take the form

$$J^i(u^1, u^2) = \int_0^T g^i(t, x(t), u^1(t), u^2(t))\, dt + G^i(x(T)) \to \min,$$

where $u^i \in U^i$ is a compact set in R^m.

Theorem 10.10 *Let the function g^1 be continuously differentiable in R^n, while the functions f, g^2, G^1 and G^2 be twice continuously differentiable in R^n. Moreover, assume that the function $H^2(t, \psi, u^1, u^2)$ appears continuously differentiable and strictly convex on U^2. If $(u^{1^*}(t), u^{2^*}(t))$ makes a Stackelberg equilibrium in this game and $x^*(t)$ means the corresponding strategy of the process, then there exist continuously differentiable functions $\psi(\cdot),\ \lambda_1(\cdot),\ \lambda_2(\cdot) : [0, T] \in R^n$ and a continuous function $\lambda_3(\cdot) : [0, T] \in R^m$ such that*

$$\begin{gathered}
x^{*\prime}(t) = f(t, x^*(t), u^{1^*}(t), u^{2^*}(t)),\ x^*(0) = x_0,\\
\psi'(t) = -\frac{\partial H^2(t, \psi, x^*, u^{1*}, u^{2*})}{\partial x},\ \psi(T) = \frac{\partial G^2(x^*(T))}{\partial x},\\
\lambda_1'(t) = -\frac{\partial H^1(t, \psi, \lambda_1, \lambda_2, \lambda_3, x^*, u^{1*}, u^{2*})}{\partial x},\ \lambda_1(T) = \frac{\partial G^1(x^*(T))}{\partial x} - \frac{\partial^2 G^2(x^*(T))}{\partial x^2}\lambda_2(T),\\
\lambda_2'(t) = -\frac{\partial H^1(t, \psi, \lambda_1, \lambda_2, \lambda_3, x^*, u^{1*}, u^{2*})}{\partial x},\ \lambda_2(0) = 0,\\
\nabla_{u^1} H^1(t, \psi, \lambda_1, \lambda_2, \lambda_3, x^*, u^{1*}, u^{2*}) = 0,\\
\nabla_{u^2} H^1(t, \psi, \lambda_1, \lambda_2, \lambda_3, x^*, u^{1*}, u^{2*}) = \nabla_{u^2} H^2(t, \psi, x^*, u^{1*}, u^{2*}) = 0,
\end{gathered}$$

where

$$\begin{aligned}
H^2(t, \psi, u^1, u^2) &= g^2(t, x, u^1, u^2) + \psi f(t, x, u^1, u^2),\\
H^1 &= g^1(t, x, u^1, u^2) + \lambda_1 f(t, x, u^1, u^2) - \lambda_2 \frac{\partial H^2}{\partial x} + \lambda_3 \nabla_{u^2} H^2.
\end{aligned}$$

Proof: We proceed by analogy to the discrete-time case. First, imagine that we know leader's control u_1. The optimal response $\bar{u}_2$ of player 2 follows from Theorem 10.7. Notably, we have

$$\begin{gathered}
x'(t) = f(t, x(t), u^1(t), \bar{u}^2(t)),\ x(0) = x_0,\\
\bar{u}^2(t) = argmin_{u^2 \in U^2} \lim H^2(t, \psi(t), x, u^1(t), u^2(t)),\\
\psi'(t) = -\frac{\partial H^2(t, \psi(t), x, u^1(t), \bar{u}^2(t))}{\partial x},\ \psi(T) = \frac{\partial G^2(x(T))}{\partial x},
\end{gathered}$$

where

$$H^2(t,\psi,x,u^1,u^2) = g^2(t,x,u^1,u^2) + \psi f(t,x,u^1,u^2),$$

and $\psi(t)$ is the conjugate variable for this problem.

By the premise, the Hamiltonian function possesses its minimum in an inner point. Therefore, the maximum condition can be reexpressed as

$$\nabla_{u^2} H^2(t,\psi,x,u^1,\bar{u}^2) = 0.$$

Now, to find leader's control, we have to solve the optimization problem

$$\min_{u^1\in U^1} \lim J^1(u^1,u^2)$$

subject to the constraints

$$x'(t) = f(t,x(t),u^1(t),u^2(t)),\ x(0) = x_0,$$

$$\psi'(t) = -\frac{\partial H^2(t,\psi,x,u^1,u^2)}{\partial x},\ \psi(T) = \frac{\partial G^2(x(T))}{\partial x},$$

$$\nabla_{u^2} H^2(t,\psi,x,u^1,\bar{u}^2) = 0.$$

Again, take advantage of Theorem 10.7. Compile the Hamiltonian function for this problem:

$$H^1 = g^1(t,x,u^1,u^2) + \lambda_1 f(t,x,u^1,u^2) - \lambda_2\frac{\partial H^2}{\partial x} + \lambda_3\nabla_{u^2}H^2.$$

The conjugate variables of the problem meet the equations

$$\lambda_1'(t) = -\frac{\partial H^1(t,\psi,\lambda_1,\lambda_2,\lambda_3,x^*,u^{1*},u^{2*})}{\partial x},\ \lambda_1(T) = \frac{\partial G^1(x^*(T))}{\partial x} - \frac{\partial^2 G^2(x^*(T))}{\partial x^2}\lambda_2(T),$$

$$\lambda_2'(t) = -\frac{\partial H^1(t,\psi,\lambda_1,\lambda_2,\lambda_3,x^*,u^{1*},u^{2*})}{\partial x},\ \lambda_2(0) = 0.$$

In addition, the maximum conditions are valid in inner points:

$$\nabla_{u^1} H^1 = 0,\ \nabla_{u^2} H^2 = 0,$$

which completes the proof of Theorem 10.10.

10.4 The linear-quadratic problem on finite and infinite horizons

Consider the linear-quadratic problem of bioresource management. Let the dynamics of a population have the form

$$x'(t) = \varepsilon x(t) - u_1(t) - u_2(t), \tag{4.1}$$

where $x(t) \geq 0$ is the population size at instant t, $u_1(t)$ and $u_2(t)$ indicate the control laws applied by player 1 and player 2, respectively.

Both players strive for maximizing their profits on the time interval [0,T]. We select the following payoff functionals of the players:

$$\begin{aligned} J_1 &= \int_0^T e^{-\rho t}\left[p_1 u_1(t) - c_1 u_1^2(t)\right]dt + G_1(x(T)), \\ J_2 &= \int_0^T e^{-\rho t}\left[p_2 u_2(t) - c_2 u_2^2(t)\right]dt + G_2(x(T)). \end{aligned} \tag{4.2}$$

Here c_1 and c_2 are the fishing costs of the players, p_1 and p_2 specify the unit price of caught fish.

Denote

$$c_{i\rho} = c_i e^{-\rho t}\,,\ p_{i\rho} = p_i e^{-\rho t},\ i = 1, 2.$$

Find a Nash equilibrium in this problem by Pontryagin's maximum principle.

Construct the Hamiltonian function for player 1:

$$H_1 = p_{1\rho}u_1 - c_{1\rho}u_1^2 + \lambda_1(\varepsilon x - u_1 - u_2).$$

Hence, it appears that

$$u_1(t) = \frac{p_{1\rho} - \lambda_1(t)}{2c_{1\rho}},$$

and the conjugate variable equation acquires the form

$$\lambda_1'(t) = -\frac{\partial H_1}{\partial x} = -\varepsilon\lambda_1(t),\ \ \lambda_1(T) = G_1'(x(T)).$$

By obtaining the solution of this equation and reverting to the original variables, we express the optimal control of player 1:

$$u_1^*(t) = \frac{p_1 - G_1'(x(T))e^{(\rho-\varepsilon)t}e^{\varepsilon T}}{2c_1}.$$

Similarly, the Hamiltonian function for player 2 is defined by

$$H_2 = p_{2\rho}u_2 - c_{2\rho}u_2^2 + \lambda_2(\varepsilon x - u_1 - u_2).$$

This leads to

$$u_2(t) = \frac{p_{2\rho} - \lambda_2(t)}{2c_{2\rho}},$$

and the conjugate variable equation becomes

$$\lambda_2'(t) = -\frac{\partial H_2}{\partial x} = -\varepsilon\lambda_2(t), \;\; \lambda_2(T) = G_2'(x(T)).$$

Finally, the optimal control of player 2 is given by

$$u_2^*(t) = \frac{p_2 - G_2'(x(T))e^{(\rho-\varepsilon)t}e^{\varepsilon T}}{2c_2}.$$

Actually, we have demonstrated the following result.

Theorem 10.11 *The control laws*

$$u_1^*(t) = \frac{p_1 - G_1'(x(T))e^{(\rho-\varepsilon)t}e^{\varepsilon T}}{2c_1}, \;\; u_2^*(t) = \frac{p_2 - G_2'(x(T))e^{(\rho-\varepsilon)t}e^{\varepsilon T}}{2c_2}$$

form the Nash-optimal solution of the problem (4.1)–(4.2).

Proof: Generally, the maximum principle states the necessary conditions of optimality. However, in the linear-quadratic case it appears sufficient. Let us show such sufficiency for the above model.

Fix $u_2^*(t)$ and study the problem for player 1. Designate by $x^*(t)$ the dynamics corresponding to the optimal behavior of both players. Consider the perturbed solution $x^*(t) + \Delta x$, $u_1^*(t) + \Delta u_1$. Here $x^*(t)$ and $u_1^*(t)$ satisfy equation (4.1), whereas Δx meets the equation $\Delta x' = \varepsilon\Delta x - \Delta u_1$ (since $(x^*)' + \Delta x' = \varepsilon x^* - u_1^* - u_2^* + \varepsilon\Delta x - \Delta u_1$).

Under the optimal behavior, the payoff constitutes

$$J_1^* = \int_0^T \left[p_{1\rho}u_1^*(t) - c_{1\rho}\left(u_1^*(t)\right)^2\right] dt + G_1(x^*(T)).$$

Its perturbed counterpart equals

$$J_1 = \int_0^T \left[p_{1\rho}u_1^*(t) + p_{1\rho}\Delta u_1(t) - c_{1\rho}\left(u_1^*(t) + \Delta u_1(t)\right)^2\right] dt + G_1(x^*(T) + \Delta x(T)).$$

Their difference is

$$\begin{aligned}
J_1^* - J_1 &= \int_0^T c_{1\rho}\Delta u_1^2 - \lambda_1(t)\Delta u_1 \, dt + G_1(x^*(T)) - G_1(x^*(T) + \Delta x(T)) \\
&= \int_0^T c_{1\rho}\Delta u_1^2 \, dt - G_1'(x^*(T))\Delta x(T) - \int_0^T \lambda_1 \Delta u_1 \, dt \\
&= \int_0^T c_{1\rho}\Delta u_1^2 \, dt - G_1'(x^*(T))\Delta x(T) - \int_0^T \lambda_1(\varepsilon\Delta x - (\Delta x)') \, dt \\
&= \int_0^T c_{1\rho}\Delta u_1^2 \, dt - G_1'(x^*(T))\Delta x(T) + \Delta x\lambda_1\big|_0^T = \int_0^T c_{1\rho}\Delta u_1^2 \, dt > 0.
\end{aligned}$$

This proves the optimality of $u_1^*(t)$ for player 1.

By analogy, readers can demonstrate that the control law $u_2^*(t)$ becomes optimal for player 2.

To proceed, we analyze the above problem with infinite horizon. Let the dynamics of a fish population possess the form (4.1). The players aim at minimization of their costs on infinite horizon. We adopt the following cost functionals of the players:

$$\begin{aligned} J_1 &= \int_0^\infty e^{-\rho t}\left[c_1 u_1^2(t) - p_1 u_1(t)\right]dt, \\ J_2 &= \int_0^\infty e^{-\rho t}\left[c_2 u_2^2(t) - p_2 u_2(t)\right]dt. \end{aligned} \tag{4.3}$$

In these formulas, c_1 and c_2 mean the fishing costs of the players, p_1 and p_2 are the unit prices of caught fish.

Apply the Bellman principle for Nash equilibrium evaluation.

Fix a control law of player 2 and consider the optimal control problem for his opponent. Define the function $V(x)$ by

$$V(x) = \min_{u_1}\left\{\int_0^\infty e^{-\rho t}\left[c_1 u_1^2(t) - p_1 u_1(t)\right]dt\right\}.$$

The Hamilton–Jacobi–Bellman equation acquires the form

$$\rho V(x) = \min_{u_1}\lim\left\{c_1 u_1^2 - p_1 u_1 + \frac{\partial V}{\partial x}(\varepsilon x - u_1 - u_2)\right\}.$$

The minimum with respect to u_1 is given by

$$u_1 = \left(\frac{\partial V}{\partial x} + p_1\right)\Big/ 2c_1.$$

Substitution of this quantity into the equation yields

$$\rho V(x) = -\frac{\left(\frac{\partial V}{\partial x} + p_1\right)^2}{4c_1} + \frac{\partial V}{\partial x}(\varepsilon x - u_2).$$

Interestingly, a quadratic form satisfies the derived equation. And so, set $V(x) = a_1 x^2 + b_1 x + d_1$.

In this case, the control law equals

$$u_1(x) = \frac{2a_1 x + b_1 + p_1}{2c_1},$$

where the coefficients follow from the system of equations

$$\begin{cases} \rho a_1 = 2a_1\varepsilon - \frac{a_1^2}{c_1}, \\ \rho b_1 = \varepsilon b_1 - 2a_1 u_2 - \frac{a_1(p_1+b_1)}{c_1}, \\ \rho d_1 = -b_1 u_2 - \frac{(p_1+b_1)^2}{4c_1}. \end{cases} \tag{4.4}$$

Similarly, for player 2, one can get the formula

$$u_2(x) = \frac{2a_2x + b_2 + p_2}{2c_2},$$

where the coefficients obey the system of equations

$$\begin{cases} \rho a_2 = 2a_2\varepsilon - \frac{a_2^2}{c_2}, \\ \rho b_2 = \varepsilon b_2 - 2a_2u_1 - \frac{a_2(p_2+b_2)}{c_2}, \\ \rho d_2 = -b_2u_1 - \frac{(p_2+b_2)^2}{4c_2}. \end{cases} \tag{4.5}$$

In fact, we have established

Theorem 10.12 *The control laws*

$$u_1^*(x) = \frac{2c_1c_2\varepsilon x(2\varepsilon - \rho) - c_1p_2(2\varepsilon - \rho) + \varepsilon c_2p_1}{2c_1c_2(3\varepsilon - \rho)},$$

$$u_2^*(x) = \frac{2c_1c_2\varepsilon x(2\varepsilon - \rho) - c_2p_1(2\varepsilon - \rho) + \varepsilon c_1p_2}{2c_1c_2(3\varepsilon - \rho)}$$

form a Nash equilibrium in the problem (4.1)–(4.3).

Proof: These control laws are expressed from the systems (4.4) and (4.5). Note that the Bellman principle again provides the necessary and sufficient conditions of optimality.

10.5 Dynamic games in bioresource management problems. The case of finite horizon

We divide the total area S of a basin into two domains S_1 and S_2, where fishing is forbidden and allowed, respectively. Let x_1 and x_2 be the fish resources in the domains S_1 and S_2. Fish migrates between these domains with the exchange coefficients γ_i. Two fishing artels catch fish in the domain S_2 during T time instants.

Within the framework of this model, the dynamics of a fish population is described by the following equations:

$$\begin{cases} x_1'(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t)), \\ x_2'(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - u(t) - v(t), \quad x_i(0) = x_i^0. \end{cases} \tag{5.1}$$

Here $x_1(t) \geq 0$ denotes the amount of fish at instant t in the forbidden domain, $x_2(t) \geq 0$ means the amount of fish at instant t in the allowed domain, ε is the natural growth coefficient of the population, γ_i corresponds to the migration coefficients, and $u(t)$, $v(t)$ are the control laws of player 1 and player 2, respectively.

We adopt the following payoff functionals of the players:

$$
\begin{aligned}
J_1 &= \int_0^T e^{-rt}[m_1((x_1(t)-\bar{x}_1)^2 + (x_2(t)-\bar{x}_2)^2) + c_1u^2(t) - p_1u(t)]dt, \\
J_2 &= \int_0^T e^{-rt}[m_2((x_1(t)-\bar{x}_1)^2 + (x_2(t)-\bar{x}_2)^2) + c_2v^2(t) - p_2v(t)]dt,
\end{aligned}
\tag{5.2}
$$

where $\bar{x}_i$, $i = 1,2$ indicates the optimal population size in the sense of reproduction, c_1, c_2 are the fishing costs of the players, and p_1, p_2 designate the unit prices of produced fish.

Introduce the notation

$$
c_{ir} = c_i e^{-rt}, \; m_{ir} = m_i e^{-rt}, \; p_{ir} = p_i e^{-rt}, \; i = 1,2.
$$

Analyze the problem (5.1)–(5.2) by different principles of optimality.

10.5.1 Nash-optimal solution

To find optimal controls, we use Pontryagin's maximum principle. Construct the Hamiltonian function for player 1:

$$
\begin{aligned}
H_1 &= m_{1r}((x_1-\bar{x}_1)^2 + (x_2-\bar{x}_2)^2) + c_{1r}u^2 - p_{1r}u \\
&\quad + \lambda_{11}(\varepsilon x_1 + \gamma_1(x_2 - x_1)) + \lambda_{12}(\varepsilon x_2 + \gamma_2(x_1 - x_2) - u - v).
\end{aligned}
$$

Hence, it appears that

$$
u(t) = \frac{\lambda_{12}(t) + p_{1r}}{2c_{1r}},
$$

and the conjugate variable equations take the form

$$
\begin{aligned}
\lambda'_{11}(t) &= -\frac{\partial H_1}{\partial x_1} = -2m_{1r}(x_1(t)-\bar{x}_1) - \lambda_{11}(t)(\varepsilon - \gamma_1) - \lambda_{12}(t)\gamma_2, \\
\lambda'_{12}(t) &= -\frac{\partial H_1}{\partial x_2} = -2m_{1r}(x_2(t)-\bar{x}_2) - \lambda_{12}(t)(\varepsilon - \gamma_2) - \lambda_{11}(t)\gamma_1,
\end{aligned}
$$

with the transversality conditions $\lambda_{1i}(T) = 0$, $i = 1,2$.

In the case of player 2, the same technique leads to

$$
\begin{aligned}
H_2 &= m_{2r}((x_1-\bar{x}_1)^2 + (x_2-\bar{x}_2)^2) + c_{2r}v^2 - p_{2r}v \\
&\quad + \lambda_{21}(\varepsilon x_1 + \gamma_1(x_2 - x_1)) + \lambda_{22}(\varepsilon x_2 + \gamma_2(x_1 - x_2) - u - v).
\end{aligned}
$$

Therefore,

$$
v(t) = \frac{\lambda_{22}(t) + p_{2r}}{2c_{2r}},
$$

and the conjugate variable equations become

$$\lambda'_{21}(t) = -\frac{\partial H_2}{\partial x_1} = -2m_{2r}(x_1(t) - \bar{x}_1) - \lambda_{21}(t)(\varepsilon - \gamma_1) - \lambda_{22}(t)\gamma_2,$$
$$\lambda'_{22}(t) = -\frac{\partial H_2}{\partial x_2} = -2m_{2r}(x_2(t) - \bar{x}_2) - \lambda_{22}(t)(\varepsilon - \gamma_2) - \lambda_{21}(t)\gamma_1,$$

with the transversality conditions $\lambda_{2i}(T) = 0\,,\ i = 1, 2.$

In terms of the new variables $\bar{\lambda}_{ij} = \lambda_{ij} e^{rt}$, the system of differential equations for optimal control laws acquires the form

$$\begin{cases} x'_1(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t))\,,\ x_1(0) = x_1^0, \\ x'_2(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - \frac{\bar{\lambda}_{12}(t)+p_1}{2c_1} - \frac{\bar{\lambda}_{22}(t)+p_2}{2c_2}\,,\ x_2(0) = x_2^0, \\ \bar{\lambda}'_{11}(t) = -2m_1(x_1(t) - \bar{x}_1) - \bar{\lambda}_{11}(t)(\varepsilon - \gamma_1 - r) - \bar{\lambda}_{12}(t)\gamma_2,\ \bar{\lambda}_{11}(T) = 0, \\ \bar{\lambda}'_{12}(t) = -2m_1(x_2(t) - \bar{x}_2) - \bar{\lambda}_{12}(t)(\varepsilon - \gamma_2 - r) - \bar{\lambda}_{11}(t)\gamma_1,\ \bar{\lambda}_{12}(T) = 0, \\ \bar{\lambda}'_{21}(t) = -2m_2(x_1(t) - \bar{x}_1) - \bar{\lambda}_{21}(t)(\varepsilon - \gamma_1 - r) - \bar{\lambda}_{22}(t)\gamma_2,\ \bar{\lambda}_{21}(T) = 0, \\ \bar{\lambda}'_{22}(t) = -2m_2(x_2(t) - \bar{x}_2) - \bar{\lambda}_{22}(t)(\varepsilon - \gamma_2 - r) - \bar{\lambda}_{21}(t)\gamma_1,\ \bar{\lambda}_{22}(T) = 0. \end{cases} \tag{5.3}$$

Theorem 10.13 *The control laws*

$$u^*(t) = \frac{\bar{\lambda}_{12}(t) + p_1}{2c_1}, \quad v^*(t) = \frac{\bar{\lambda}_{22}(t) + p_2}{2c_2},$$

where the conjugate variables result from (5.3), are the Nash-optimal solution of the problem (5.1)–(5.2).

Proof: We argue the optimality of such controls. Fix $v^*(t)$ and consider the optimal control problem for player 1. Suppose that $\{x_1^*(t), x_2^*(t), u(t)\}$ represents the solution of the system (5.3). Take the perturbed solution $x_1^*(t) + \Delta x_1$, $x_2^*(t) + \Delta x_2$, $u^*(t) + \Delta u$, where $x_1^*(t)$, Δx_1, and $x_2^*(t)$ meet the system (5.1), whereas Δx_2 satisfies the equation $\Delta x'_2 = \varepsilon \Delta x_2 + \gamma_2(\Delta x_1 - \Delta x_2) - \Delta u$ (as far as $(x_2^*)' + \Delta x'_2 = \varepsilon x_2^* + \gamma_2(x_1^* - x_2^*) - u^* - v^* + \varepsilon \Delta x_2 + \gamma_2(\Delta x_1 - \Delta x_2) - \Delta u$).

Under the optimal behavior, the payoff makes up

$$J_1^* = \int_0^T \left[m_{1r}\left(\left(x_1^*(t) - \bar{x}_1\right)^2 + \left(x_2^*(t) - \bar{x}_2\right)^2\right) + c_{1r}(u^*(t))^2 - p_{1r}u^*(t)\right] dt.$$

The corresponding perturbed payoff is

$$J_1 = \int_0^T [m_{1r}\left(\left(x_1^*(t) + \Delta x_1 - \bar{x}_1\right)^2 + \left(x_2^*(t) + \Delta x_2 - \bar{x}_2\right)^2\right)$$
$$+ c_1(u^*(t) + \Delta u)^2 - p_{1r}u^*(t) - p_{1r}\Delta u]\, dt.$$

Again, study their difference:

$$J_1 - J_1^* = \int_0^T m_{1r}\Delta x_1^2 + m_{1r}\Delta x_2^2 + \Delta x_1(-\lambda'_{11} - \lambda_{11}(\varepsilon - \gamma_1) - \lambda_{12}\gamma_2)$$
$$+ c_{1r}\Delta u^2 + \Delta x_2(-\lambda'_{12} - \lambda_{12}(\varepsilon - \gamma_2) - \lambda_{11}\gamma_1) + \lambda_{12}\Delta u\, dt$$
$$= \int_0^T m_{1r}\Delta x_1^2 + m_{1r}\Delta x_2^2 + c_{1r}\Delta u^2 - \lambda'_{11}\Delta x_1 - \lambda'_{12}\Delta x_2 - \lambda_{11}\Delta x'_1 - \lambda_{12}\Delta x'_2\, dt$$
$$= \int_0^T m_{1r}\Delta x_1^2 + m_{1r}\Delta x_2^2 + c_{1r}\Delta u^2\, dt > 0.$$

This substantiates the optimality of $u^*(t)$ for player 1.

Similarly, one can prove that the optimal control law of player 2 lies in $v^*(t)$.

10.5.2 Stackelberg-optimal solution

For optimal control evaluation, we employ the following modification of Pontryagin's maximum principle for two-shot games. Compile the Hamiltonian function for player 2:

$$H_2 = m_{2r}((x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2) + c_{2r}v^2 - p_{2r}v$$
$$+ \lambda_{21}(\varepsilon x_1 + \gamma_1(x_2 - x_1)) + \lambda_{22}(\varepsilon x_2 + \gamma_2(x_1 - x_2) - u - v).$$

Then it appears that

$$v(t) = \frac{\lambda_{22}(t) + p_{2r}}{2c_{2r}},$$

and the conjugate variable equations are defined by

$$\lambda'_{21}(t) = -\frac{\partial H_2}{\partial x_1} = -2m_{2r}(x_1(t) - \bar{x}_1) - \lambda_{21}(t)(\varepsilon - \gamma_1) - \lambda_{22}(t)\gamma_2,$$
$$\lambda'_{22}(t) = -\frac{\partial H_2}{\partial x_2} = -2m_{2r}(x_2(t) - \bar{x}_2) - \lambda_{22}(t)(\varepsilon - \gamma_2) - \lambda_{21}(t)\gamma_1,$$

with the transversality conditions $\lambda_{2i}(T) = 0,\ i = 1, 2$.

Substitute this control of player 2 to derive the system of differential equations

$$\begin{cases} x'_1(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t)),\ x_1(0) = x_1^0, \\ x'_2(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - u(t) - \frac{\lambda_{22}(t)+p_{2r}}{2c_{2r}},\ x_2(0) = x_2^0, \\ \lambda'_{21}(t) = -2m_{2r}(x_1(t) - \bar{x}_1) - \lambda_{21}(t)(\varepsilon - \gamma_1) - \lambda_{22}(t)\gamma_2,\ \lambda_{21}(T) = 0, \\ \lambda'_{22}(t) = -2m_{2r}(x_2(t) - \bar{x}_2) - \lambda_{22}(t)(\varepsilon - \gamma_2) - \lambda_{21}(t)\gamma_1,\ \lambda_{22}(T) = 0. \end{cases}$$

Apply Pontryagin's maximum principle to find the optimal control of player 1:

$$\begin{aligned}H_1 &= m_{1r}((x_1-\bar{x}_1)^2+(x_2-\bar{x}_2)^2)+c_{1r}u^2-p_{1r}u+\lambda_{11}(\varepsilon x_1+\gamma_1(x_2-x_1))\\ &+\lambda_{12}\left(\varepsilon x_2+\gamma_2(x_1-x_2)-u-\frac{\lambda_{22}+p_{2r}}{2c_{2r}}\right)+\mu_1(-2m_{2r}(x_1-\bar{x}_1)-\lambda_{21}(\varepsilon-\gamma_1))\\ &-\mu_1\lambda_{22}\gamma_2+\mu_2(-2m_{2r}(x_2-\bar{x}_2)-\lambda_{22}(\varepsilon-\gamma_2)-\lambda_{21}\gamma_1).\end{aligned}$$

This leads to

$$u(t)=\frac{\lambda_{12}(t)+p_{1r}}{2c_{1r}},$$

and the conjugate variable equations have the form

$$\begin{cases}\lambda'_{11}(t)=-2m_{1r}(x_1(t)-\bar{x}_1)-\lambda_{11}(t)(\varepsilon-\gamma_1)-\lambda_{12}(t)\gamma_2+2m_{2r}\mu_1(t),\\ \lambda'_{12}(t)=-2m_{1r}(x_2(t)-\bar{x}_2)-\lambda_{12}(t)(\varepsilon-\gamma_2)-\lambda_{11}(t)\gamma_1+2m_{2r}\mu_2(t),\\ \mu'_1(t)=-\frac{\partial H_1}{\partial\lambda_{21}}=\mu_1(t)(\varepsilon-\gamma_1)+\mu_2(t)\gamma_1,\\ \mu'_2(t)=-\frac{\partial H_1}{\partial\lambda_{22}}=\frac{\lambda_{12}(t)}{2c_{2r}}+\mu_2(t)(\varepsilon-\gamma_2)+\mu_1(t)\gamma_2,\end{cases}$$

with the transversality conditions $\lambda_{2i}(T)=0\,,\ \mu_i(0)=0$.

Finally, in terms of the new variables $\bar{\lambda}_{ij}=\lambda_{ij}e^{rt}$, the system of differential equations for optimal controls is determined by

$$\begin{cases}x'_1(t)=\varepsilon x_1(t)+\gamma_1(x_2(t)-x_1(t)),\\ x'_2(t)=\varepsilon x_2(t)+\gamma_2(x_1(t)-x_2(t))-u-\frac{\bar{\lambda}_{22}(t)+p_2}{2c_2},\\ \bar{\lambda}'_{11}(t)=-2m_1(x_1(t)-\bar{x}_1)-\bar{\lambda}_{11}(t)(\varepsilon-\gamma_1-r)-\bar{\lambda}_{12}(t)\gamma_2+2m_2\mu_1(t),\\ \bar{\lambda}'_{12}(t)=-2m_1(x_2(t)-\bar{x}_2)-\bar{\lambda}_{12}(t)(\varepsilon-\gamma_2-r)-\bar{\lambda}_{11}(t)\gamma_1+2m_2\mu_2(t),\\ \bar{\lambda}'_{21}(t)=-2m_2(x_1(t)-\bar{x}_1)-\bar{\lambda}_{21}(t)(\varepsilon-\gamma_1-r)-\bar{\lambda}_{22}(t)\gamma_2,\\ \bar{\lambda}'_{22}(t)=-2m_2(x_2(t)-\bar{x}_2)-\bar{\lambda}_{22}(t)(\varepsilon-\gamma_2-r)-\bar{\lambda}_{21}(t)\gamma_1,\\ \mu'_1(t)=\mu_1(t)(\varepsilon-\gamma_1)+\mu_2(t)\gamma_1,\\ \mu'_2(t)=\frac{\bar{\lambda}_{12}(t)}{2c_2}+\mu_2(t)(\varepsilon-\gamma_2)+\mu_1(t)\gamma_2,\\ \bar{\lambda}_{i1}(T)=\bar{\lambda}_{i2}(T)=0\,,\ x_i(0)=x_i^0\,,\ \mu_i(0)=0.\end{cases}\tag{5.4}$$

Consequently, we have proved

Theorem 10.14 *For the control laws*

$$u^*(t)=\frac{\bar{\lambda}_{12}(t)+p_1}{2c_1},\quad v^*(t)=\frac{\bar{\lambda}_{22}(t)+p_2}{2c_2}$$

to be the Stackelberg-optimal solution of the problem (5.1)–(5.2), it is necessary that the conjugate variables follow from (5.4).

10.6 Dynamic games in bioresource management problems. The case of infinite horizon

As before, the dynamics of a fish population is described by the equations

$$\begin{cases} x_1'(t) = \varepsilon x_1(t) + \gamma_1(x_2(t) - x_1(t)), \\ x_2'(t) = \varepsilon x_2(t) + \gamma_2(x_1(t) - x_2(t)) - u(t) - v(t)\,, \ x_i(0) = x_i^0. \end{cases} \tag{6.1}$$

All parameters have been defined in the preceding section.

We adopt the following payoff functionals of the players:

$$\begin{aligned} J_1 &= \int_0^\infty e^{-rt}[m_1((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_1u^2(t) - p_1u(t)]dt, \\ J_2 &= \int_0^\infty e^{-rt}[m_2((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_2v^2(t) - p_2v(t)]dt, \end{aligned} \tag{6.2}$$

where $\bar{x}_i$, $i = 1, 2$ means the optimal population size in the sense of reproduction, c_1, c_2 specify the fishing costs of the players, and p_1, p_2 are the unit prices of caught fish.

In the sequel, we study the problem (6.1)–(6.2) using different principles of optimality.

10.6.1 Nash-optimal solution

Fix the control law of player 2 and consider the optimal control problem for his opponent. Determine the function $V(x)$ by

$$V(x_1, x_2) = \min_u \left\{ \int_0^\infty e^{-rt}[m_1((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_1u^2(t) - p_1u(t)]\,dt \right\}.$$

The Hamilton–Jacobi–Bellman equation acquires the form

$$\begin{aligned} rV(x_1, x_2) = \min_u \lim\{ &m_1((x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2) + c_1u^2 - p_1u \\ &+ \frac{\partial V}{\partial x_1}(\varepsilon x_1 + \gamma_1(x_2 - x_1)) + \frac{\partial V}{\partial x_2}(\varepsilon x_2 + \gamma_2(x_1 - x_2) - u - v)\}. \end{aligned}$$

Find the minimum in u:

$$u = \left(\frac{\partial V}{\partial x_2} + p_1\right) \Big/ 2c_1.$$

Substitute this result into the above equation to get

$$rV(x_1, x_2) = m_1((x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2) - \frac{\left(\frac{\partial V}{\partial x_2} + p_1\right)^2}{4c_1}$$
$$+ \frac{\partial V}{\partial x_1}(\varepsilon x_1 + \gamma_1(x_2 - x_1)) + \frac{\partial V}{\partial x_2}(\varepsilon x_2 + \gamma_2(x_1 - x_2) - v).$$

Interestingly, a quadratic form satisfies this equation.
Set $V(x_1, x_2) = a_1 x_1^2 + b_1 x_1 + a_2 x_2^2 + b_2 x_2 + k x_1 x_2 + l$.
The corresponding control law makes

$$u(x) = \frac{2a_2 x_2 + b_2 + kx_1 + p_1}{2c_1},$$

where the coefficients meet the system of equations

$$\begin{cases} ra_1 &= m_1 - \frac{k^2}{4c_1} + 2a_1(\varepsilon - \gamma_1) + k\gamma_2, \\ rb_1 &= -2m_1\bar{x}_1 - \frac{kb_2}{2c_1} - \frac{kp_1}{2c_1} + b_1(\varepsilon - \gamma_1) + b_2\gamma_2 - kv, \\ ra_2 &= m_1 - \frac{a_2^2}{c_1} + 2a_2(\varepsilon - \gamma_2) + k\gamma_1, \\ rb_2 &= -2m_1\bar{x}_2 - \frac{a_2 b_2}{c_1} - \frac{a_2 p_1}{c_1} + b_2(\varepsilon - \gamma_2) + b_1\gamma_1 - 2a_2 v, \\ rk &= -\frac{a_2 k}{c_1} + k(\varepsilon - \gamma_1) + 2a_1\gamma_1 + 2a_2\gamma_2 + k(\varepsilon - \gamma_2), \\ rl &= m_1\bar{x}_1^2 + m_1\bar{x}_2^2 - \frac{b_2^2}{4c_1} - \frac{b_2 p_1}{2c_1} - \frac{p_1^2}{4c_1} - b_2 v. \end{cases} \tag{6.3}$$

Similar reasoning for player 2 yields

$$v(x) = \frac{2\alpha_2 x_2 + \beta_2 + k_2 x_1 + p_2}{2c_2},$$

where the coefficients follow from the system of equations

$$\begin{cases} r\alpha_1 &= m_2 - \frac{k_2^2}{4c_1} + 2\alpha_1(\varepsilon - \gamma_1) + k_2\gamma_2, \\ r\beta_1 &= -2m_2\bar{x}_1 - \frac{k_2\beta_2}{2c_1} - \frac{k_2 p_1}{2c_1} + \beta_1(\varepsilon - \gamma_1) + \beta_2\gamma_2 - k_2 u, \\ r\alpha_2 &= m_2 - \frac{\alpha_2^2}{c_1} + 2\alpha_2(\varepsilon - \gamma_2) + k_2\gamma_1, \\ r\beta_2 &= -2m_2\bar{x}_2 - \frac{\alpha_2\beta_2}{c_1} - \frac{\alpha_2 p_1}{c_1} + \beta_2(\varepsilon - \gamma_2) + \beta_1\gamma_1 - 2\alpha_2 u, \\ rk_2 &= -\frac{\alpha_2 k_2}{c_1} + k_2(\varepsilon - \gamma_1) + 2\alpha_1\gamma_1 + 2\alpha_2\gamma_2 + k_2(\varepsilon - \gamma_2), \\ rl_2 &= m_2\bar{x}_1^2 + m_2\bar{x}_2^2 - \frac{\beta_2^2}{4c_1} - \frac{\beta_2 p_1}{2c_1} - \frac{p_1^2}{4c_1} - \beta_2 u. \end{cases} \tag{6.4}$$

Consequently, we have established

Theorem 10.15 *The control laws*

$$u^*(x) = \frac{2a_2x_2 + b_2 + kx_1 + p_1}{2c_1}, \quad v^*(x) = \frac{2\alpha_2x_2 + \beta_2 + k_2x_1 + p_2}{2c_2},$$

where the coefficients result from (6.3) and (6.4), are the Nash-optimal solution of the problem (6.1)–(6.2).

10.6.2 Stackelberg-optimal solution

The Hamilton–Jacobi–Bellman equation for player 2 brings to

$$v(x) = \frac{2\alpha_2x_2 + \beta_2 + k_2x_1 + p_2 + \sigma u}{2c_2},$$

where the coefficients meet the system (6.4).

Define the function $V(x)$ in the optimal control problem for player 1 as

$$V(x_1, x_2) = \min_u \left\{ \int_0^\infty e^{-rt}[m_1((x_1(t) - \bar{x}_1)^2 + (x_2(t) - \bar{x}_2)^2) + c_1u^2(t) - p_1u(t)]\, dt \right\}.$$

The Hamilton–Jacobi–Bellman equation takes the form

$$rV(x_1, x_2) = \min_u \lim \left\{ m_1((x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2) + c_1u^2 - p_1u + \frac{\partial V}{\partial x_1}(\varepsilon x_1 + \gamma_1(x_2 - x_1)) \right.$$
$$\left. + \frac{\partial V}{\partial x_2}\left(\varepsilon x_2 + \gamma_2(x_1 - x_2) - u - \frac{2\alpha_2x_2 + \beta_2 + k_2x_1 + p_2 + \sigma u}{2c_2}\right)\right\}.$$

Again, find the minimum in u:

$$u = \left(\frac{\partial V}{\partial x_2}(2c_2 + \sigma) + 2p_1c_2\right) \Big/ 4c_1c_2.$$

Substitute this expression in the above equation to obtain

$$rV(x_1, x_2) = m_1((x_1 - \bar{x}_1)^2 + (x_2 - \bar{x}_2)^2) + \left(\frac{\partial V}{\partial x_2}\right)^2 \frac{(2c_2 + \sigma)^2}{8c_1c_2^2} + \frac{p_1^2}{2c_1}\frac{\partial V}{\partial x_1}(\varepsilon x_1 + \gamma_1(x_2 - x_1))$$
$$+ \frac{\partial V}{\partial x_2}\left(x_1\left(\gamma_2 - \frac{k_2}{2c_2}\right) + x_2\left(\varepsilon - \gamma_2 - \frac{\alpha_2}{c_2}\right) + \frac{2c_2 + \sigma}{2c_1c_2} - \frac{\beta_2 + p_2}{2c_2}\right).$$

Note that a quadratic form satisfies this equation.

Set $V(x_1, x_2) = a_1x_1^2 + b_1x_1 + a_2x_2^2 + b_2x_2 + gx_1x_2 + l$.

Then the control law becomes

$$u(x) = ((2a_2x_2 + b_2 + gx_1)(2c_2 + \sigma) + 2p_1c_2)/4c_1c_2,$$

where the coefficients follow from the system of equations

$$
\begin{cases}
ra_1 = m_1 + g^2 \frac{(2c_2+\sigma)^2}{8c_1c_2^2} + 2a_1(\varepsilon - \gamma_1) + g\left(\gamma_2 - \frac{k_2}{2c_2}\right), \\
rb_1 = -2m_1\bar{x}_1 + 2b_2 g \frac{(2c_2+\sigma)^2}{8c_1c_2^2} + g\frac{(2c_2+\sigma)}{2c_1c_2} b_1(\varepsilon - \gamma_1) + b_2\left(\gamma_2 - \frac{k_2}{2c_2}\right) - \frac{g(\beta_2+p_2)}{2c_2}, \\
ra_2 = m1 + a_2\frac{(2c_2+\sigma)^2}{2c_1c_2^2} + g\gamma_1 + 2a_2\left(\varepsilon - \gamma_2 - \frac{\alpha_2}{c_2}\right), \\
rb_2 = -2m_1\bar{x}_2 + a_2 b_2 \frac{(2c_2+\sigma)^2}{2c_1c_2^2} + a_2\frac{(2c_2+\sigma)}{c_1c_2} + b_2\left(\varepsilon - \gamma_2 - \frac{\alpha_2}{c_2}\right) \\
\qquad + b_1\gamma_1 - a_2\frac{\beta_2+p_2}{c_2}, \\
rg = a_2 g\frac{(2c_2+\sigma)^2}{2c_1c_2^2} + g(\varepsilon - \gamma_1) + 2a_1\gamma_1 + 2a_2(\gamma_2 - \frac{k_2}{2c_2}) + g\left(\varepsilon - \gamma_2 - \frac{\alpha_2}{c_2}\right), \\
rl = m_1\bar{x}_1^2 + m_1\bar{x}_2^2 + b_2^2\frac{(2c_2+\sigma)^2}{8c_1c_2^2} + b_2\frac{(2c_2+\sigma)}{2c_1c_2} + \frac{p_1^2}{2c_1} - \frac{b_2(\beta_2+p_2)}{2c_2}.
\end{cases}
\tag{6.5}
$$

Therefore, we have proved

Theorem 10.16 *The control laws*

$$
u^*(x) = \frac{(2a_2x_2 + b_2 + gx_1)(2c_2 + \sigma) + 2p_1c_2}{4c_1c_2},
$$

$$
v^*(x) = \frac{\sigma(2a_2x_2 + b_2 + gx_1)(2c_2 + \sigma)}{8c_1c_2^2} + \frac{2c_1(2\alpha_2x_2 + \beta_2 + k_2x_1) + (\sigma p_1 + 2c_1p_2)}{8c_1c_2},
$$

where the coefficients are defined from (6.4) and (6.5), represent the Stackelberg-optimal solution of the problem (6.1)–(6.2).

Finally, we provide numerical examples with the following parameters:
$q = 0.2, \gamma_1 = \gamma_2 = 2q, \varepsilon = 0.08, m_1 = m_2 = 0.09, c_1 = c_2 = 10, p_1 = 100, p_2 = 100, T = 200$, and $r = 0.1$.

Let the optimal population sizes in the sense of reproduction be equal to $\bar{x}_1 = 100$ and $\bar{x}_2 = 100$. The initial population sizes constitute $x_1(0) = 50$ and $x_2(0) = 50$.

In the case of Nash equilibrium, Figure 10.1 shows the curve of the population size dynamics in the forbidden domain (S_1). Figure 10.2 demonstrates the same curve in the allowed domain (S_2). And Figure 10.3 illustrates the control laws of both players (they coincide).

In the case of Stackelberg equilibrium, Figure 10.4 shows the curve of the population size dynamics in the forbidden domain (S_1). Figure 10.5 demonstrates the same curve in the allowed domain (S_2). And Figures 10.6 and 10.7 present the control laws of player 1 and player 2, respectively.

It suffices to compare the costs of both players under different equilibrium concepts.

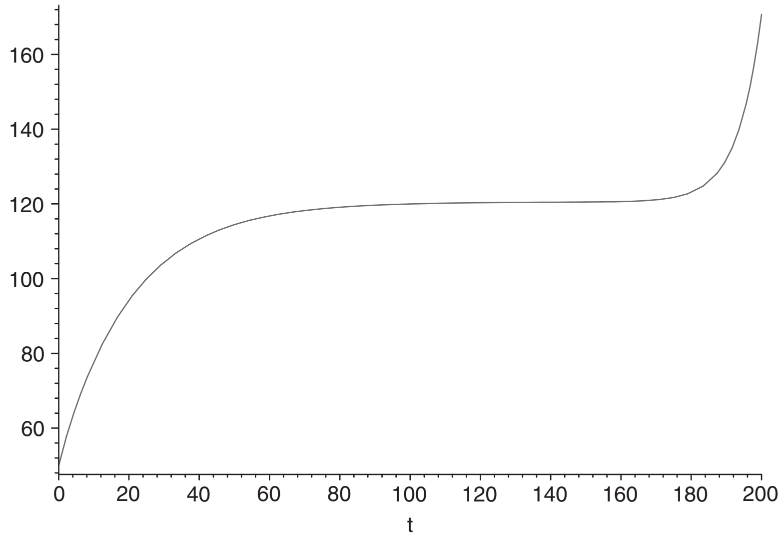


Figure 10.1 The values of $x_1^*(t)$.

In the case of Nash equilibrium, both players are "in the same boat." And so, their control laws and payoffs appear identical ($J_1 = J_2 = 93.99514185$).

In the case of Stackelberg equilibrium, player 1 represents a leader. According to the above example, this situation is more beneficial to player 1 ($J_1 = -62.73035267$) than to player 2 ($J_2 = 659.9387578$). In fact, such equilibrium even gains profits to player 1, whereas his opponent incurs all costs to maintain the admissible population size.

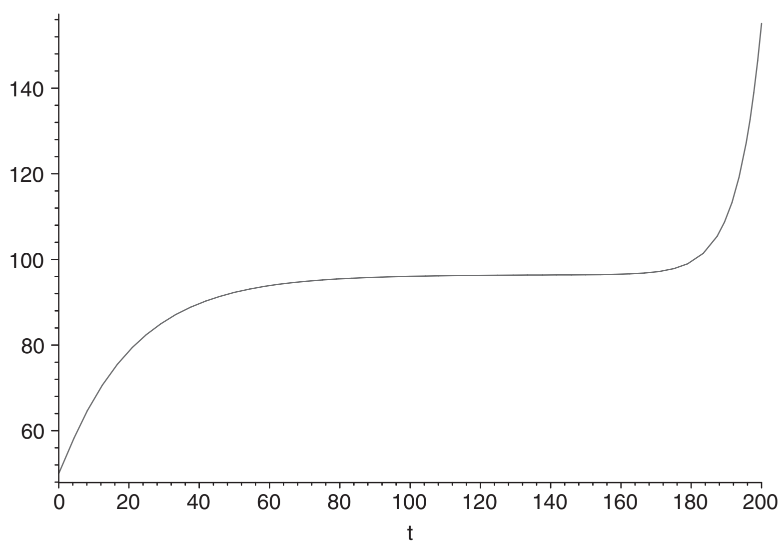


Figure 10.2 The values of $x_2^*(t)$.

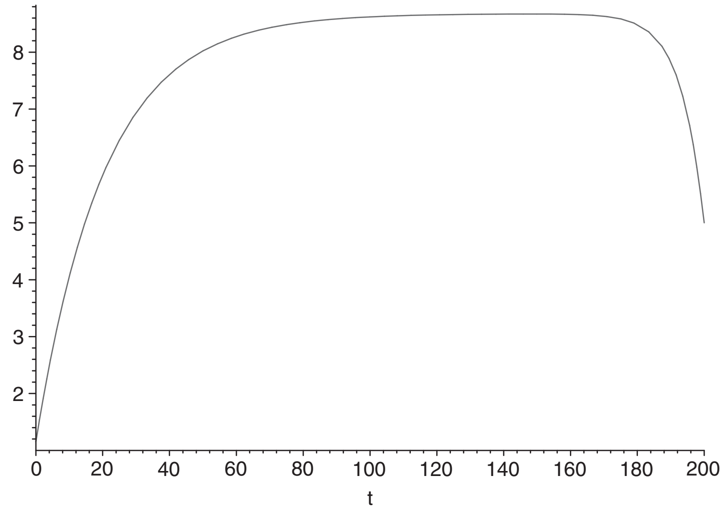


Figure 10.3 The values of $u^*(t)$.

10.7 Time-consistent imputation distribution procedure

Consider a dynamic game in the cooperative setting. The payoff of the grand coalition is the total payoff of all players. Total payoff evaluation makes an optimal control problem. The total payoff being found, players have to distribute it among all participants. For this, it is necessary to calculate the characteristic function, i.e., the payoff of each coalition. Below we discuss possible methods to construct characteristic functions.

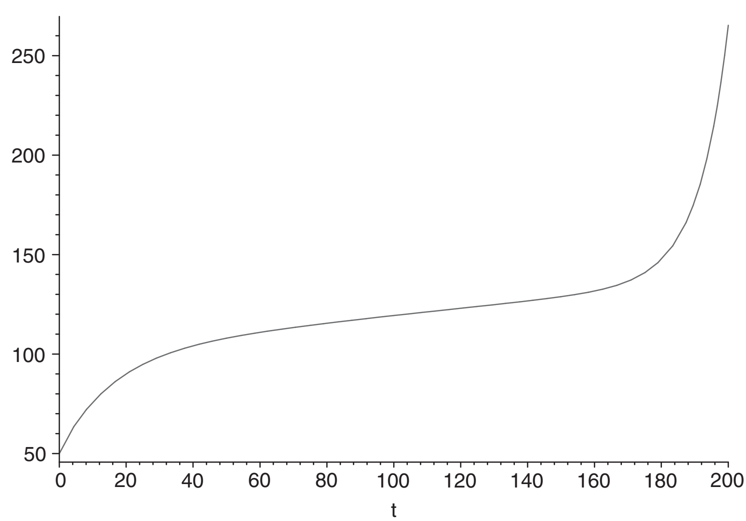


Figure 10.4 The values of $x_1^*(t)$.

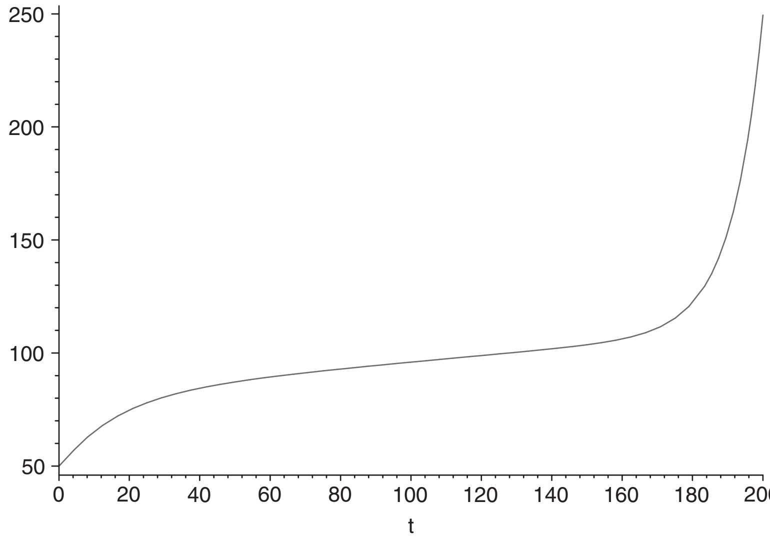


Figure 10.5 The values of $x_2^*(t)$.

A fundamental difference of dynamic cooperative games from common cooperative games is that their characteristic functions, *ergo* all imputations depend on time. To make an imputation distribution procedure time-consistent, we adopt the special distribution procedure suggested by L.A. Petrosjan (2003). To elucidate the basic ideas of this approach, let us take the "fish wars" model.

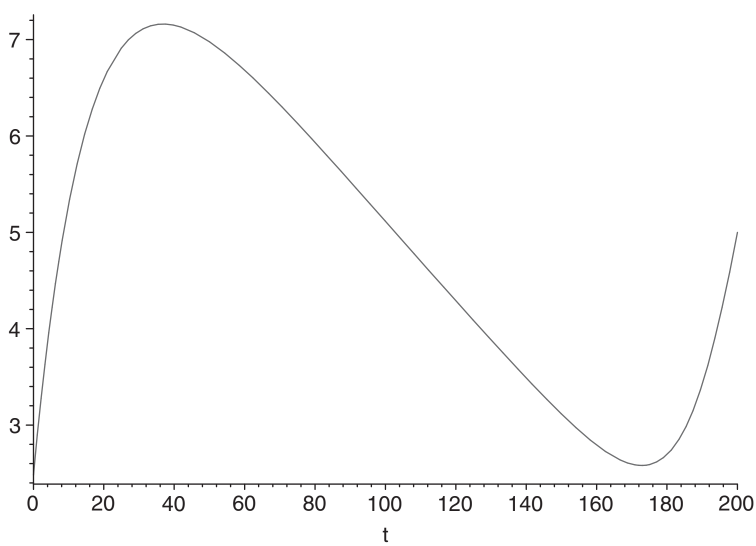


Figure 10.6 The values of $u^*(t)$.

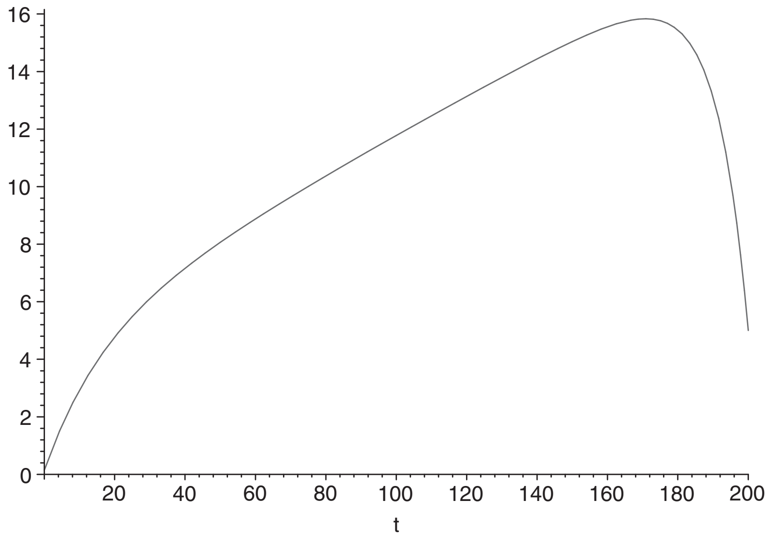


Figure 10.7 The values of $v^*(t)$.

10.7.1 Characteristic function construction and imputation distribution procedure

Imagine several countries (players) $I = \{1, 2, \dots, n\}$ that plan fishing in the ocean. The dynamics of fish resources evolves in discrete time:

$$x_{t+1} = f(x_t, u_t), \quad x_0 = x,$$

where $u_t = (u_{1t}, \dots, u_{nt})$, x_t denotes the amount of fish resources at instant t, u_{it} is the amount of fish catch by player i, $i = 1, \dots, n$.

Each player seeks to maximize his income—the sum of discounted incomes at each instant

$$J_i = \sum_{t=0}^{\infty} \lim \delta^t g_i(u_{it}).$$

The quantity $g_i(u_{it})$ designates the payoff of player i at instant t, and δ is the discounting parameter, $0 < \delta < 1$.

Before starting fishing, the countries have to negotiate the best way of fishing. Naturally, a cooperative behavior guarantees higher payoffs to each country. And so, it is necessary to solve the corresponding optimization problem, where the players aim at maximizing the total income of all participants, i.e., the function $\sum_{i=1}^{n} J_i$. Denote by $u_t^c = (u_{1t}^c, \dots, u_{nt}^c)$ the optimal strategies of the players under such cooperative behavior, and let x_t^c be the corresponding behavior of the ecological system under consideration.

If cooperation fails, each country strives to maximize its individual income. Let $u_t^N = (u_{1t}^N, \dots, u_{2t}^N)$ be a Nash equilibrium in this dynamic game. The total payoff

$$J^N = \sum_{t=0}^{\infty} \lim \delta^t \sum_{i=1}^{n} g_i(u_{it})$$

appears smaller in comparison with the cooperative case.

Some countries can form coalitions. We define by $J^S(u) = \sum_{t=0}^{\infty} \lim \delta^t \sum_{i\in S} \lim g_i(u_{it})$ the payoff of a coalition $S \in N$.

Suppose that the process evolves according to the cooperative scenario. Then it is necessary to distribute the income. For this, we apply certain methods from the theory of cooperative games. Define the characteristic function $V(S, 0)$ as the income of a coalition S in the equilibrium, when all players from S act as one player and all other players have individual strategies. In this case,

$$V(S,0) = \max_{u_i, i\in S} \lim J^S(u^N/u^S),$$

where $(u^N/u^S) = \{u_j^N, j \notin S, u_i, i \in S\}$.

Below we analyze two behavioral scenarios of independent players (lying outside the coalition). According to scenario 1, these players adhere to the same strategies as in the Nash equilibrium in the absence of the coalition S. This corresponds to the model, where players know nothing about coalition formation. In scenario 2, players outside the coalition S are informed about it and choose new strategies by making a Nash equilibrium in the game with $N\backslash K$ players. We call these scenarios by the model without information and the model with information, respectively.

As soon as the characteristic function is found, construct the imputation set

$$\xi = \{\xi(0) = (\xi_1(0), \dots, \xi_n(0)) :$$
$$\sum_{i=1}^{n} \lim \xi_i(0) = V(N,0), \xi_i(0) \geq V(i,0),\ i = 1, \dots, n\}.$$

Similarly, one can define the characteristic function $V(S, t)$ and the imputation set $\xi(t) = (\xi_1(t), \dots, \xi_n(t))$ at instant t for any subgame evolving from the state x_t^c. Subsequently, it is necessary to evaluate the optimal imputation by some principle from the theory of cooperative games (e.g., a Nash arbitration solution, the C–core, the Shapley vector, etc.). Note that, once selected, the principle of imputation choice remains invariant. We follow the time-consistent imputation distribution procedure proposed by Petrosjan [1996, 2003].

Definition 10.3 *A vector-function $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ forms an imputation distribution procedure (IDP) if*

$$\xi_i(0) = \sum_{t=0}^{\infty} \delta^t \beta_i(t),\ i = 1, \dots, n.$$

The major idea of this procedure lies in distributing the cooperative payoff along a game trajectory. Then β_i can be interpreted as the payment to player i at instant t.

Definition 10.4 *A vector-function $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ forms a time-consistent IDP, if for any $t \geq 0$:*

$$\xi_i(0) = \sum_{\tau=0}^{t} \delta^\tau \beta_i(\tau) + \delta^{t+1}\xi_i(t+1),\ i = 1, \dots, n.$$

This definition implies the following. Adhering to the cooperative trajectory, players can further receive payments in the form of IDP. In other words, they have no reason to leave the cooperative agreement.

Theorem 10.17 *The vector-function $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$, where*

$$\beta_i(t) = \xi_i(t) - \delta\xi_i(t+1), i = 1, 2, \dots, n,$$

forms a time-consistent IDP.

Proof: By definition,

$$\sum_{t=0}^{\infty} \delta^t \beta_i(t) = \sum_{t=0}^{\infty} \delta^t \xi_i(t) - \sum_{t=0}^{\infty} \delta^{t+1}\xi_i(t+1) = \xi_i(0).$$

Thus, $\beta(t)$ forms an IDP. Now, we demonstrate the time-consistency of this IDP.

Actually, this property is immediate from the following equalities:

$$\sum_{\tau=0}^{t} \delta^\tau \beta_i(\tau) + \delta^{t+1}\xi_i(t+1) = \sum_{\tau=0}^{t} \delta^\tau \xi_i(\tau) - \sum_{\tau=0}^{t} \delta^{\tau+1}\xi_i(\tau+1) + \delta^{t+1}\xi_i(t+1) = \xi_i(0).$$

Definition 10.5 *An imputation $\xi = (\xi_1, \dots, \xi_n)$ meets the irrational-behavior-proof condition if*

$$\sum_{\tau=0}^{t} \delta^\tau \beta_i(\tau) + \delta^{t+1} V(i, t+1) \geq V(i, 0)$$

for all $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ is a time-consistent IPD.

This condition introduced by D.W.K. Yeung [2006] guarantees that, even in the case of cooperative agreement cancelation, the participants obtain payoffs not smaller than under their initial non-cooperative behavior.

As applied to our model, the irrational-behavior-proof condition acquires the form

$$\xi_i(0) - \xi_i(t)\delta^t \geq V(i, 0) - \delta^t V(i, t),\ \ i = 1, \dots, n.$$

There exist another condition (see Mazalov and Rettieva [2009]), being stronger than Yeung's condition yet easily verifiable.

Definition 10.6 *An imputation $\xi = (\xi_1, \dots, \xi_n)$ satisfies the incentive condition for rational behavior at each shot if*

$$\beta_i(t) + \delta V(i, t+1) \geq V(i, t) \tag{7.1}$$

for $t \geq 0$, where $\beta(t) = (\beta_1(t), \dots, \beta_n(t))$ is a time-consistent IDP.

The suggested condition stimulates cooperation maintenance by a player, since at each shot the latter benefits more from cooperation than from independent behavior.

For our model, the condition (7.1) takes the form

$$\xi_i(t) - \delta\xi_i(t+1) \geq V(i, t) - \delta V(i, t+1),\ i = 1, \dots, n. \tag{7.2}$$

Clearly, the condition (7.2) directly leads to Yeung's condition. For evidence, just consider (7.2) at instant τ, multiply by δ^τ and perform summation over $\tau = 0, \dots, t$.

10.7.2 Fish wars. Model without information

We illustrate the application of the time-consistent imputation distribution procedure in the "fish war" model. In the case of two players, optimal control laws in fish wars have been established in Section 10.1. These results can be easily generalized to the case of n players.

And so, n countries participate in fishing within a fixed time period. The dynamics of this bioresource obeys the equation (see Levhari and Mirman [1980])

$$x_{t+1} = \left(\varepsilon x_t - \sum_{i=1}^{n} u_{it}\right)^\alpha, \quad x_0 = x,$$

where $x_t \geq 0$ is the population size at instant t, $\varepsilon \in (0, 1)$ stands for the mortality parameter, $\alpha \in (0, 1)$ corresponds to the fertility parameter, and $u_{it} \geq 0$ means the fish catch amount of player i, $i = 1, \dots, n$.

Consider the dynamic game with the logarithmic utility function of the countries. Then the incomes of the players on infinite horizon make up

$$J_i = \sum_{t=0}^{\infty} \lim \delta^t \log(u_{it}),$$

where $0 < \delta < 1$ is the discounting coefficient, $i = 1, \dots, n$.

Construct the characteristic function in the following case. Any players forming a coalition do not report of this fact to the rest players.

For Nash equilibrium evaluation, we address the dynamic programming approach. It is necessary to solve the Bellman equation

$$V_i(x) = \max_{u_i \geq 0} \lim \left\{ \log u_i + \delta V_i \left(\varepsilon x - \sum_{i=1}^{n} u_i \right)^\alpha \right\},\ i = 1, \dots, n,$$

We seek for its solution in the form

$$V_i(x) = A_i \log x + B_i,\ i = 1, \dots, n.$$

Accordingly, optimal control search runs in the class of $u_i = \gamma_i x, i = 1, \dots, n$. Recall that all players appear homogeneous. Hence, the Bellman equation yields the optimal amounts of fish catch

$$u_i^N = \frac{1 - \alpha\delta}{n - \alpha\delta(n-1)} \varepsilon x \tag{7.3}$$

and the payoffs

$$V_i(x) = \frac{1}{1 - \alpha\delta} \log x + \frac{1}{1 - \delta} B_i, \tag{7.4}$$

where

$$B_i = \frac{1}{1 - \alpha\delta} \log\left(\frac{\varepsilon}{n - \alpha\delta(n-1)}\right) + \log(1 - \alpha\delta) + \frac{\alpha\delta}{1 - \alpha\delta} \log(\alpha\delta).$$

Next, denote $a = \alpha\delta$.

The population dynamics in the non-cooperative case is given by

$$x_t = x_0^{\alpha^t} \hat{x}^{N \sum_{j=1}^{t} \lim \alpha^j}, \tag{7.5}$$

where

$$\hat{x}^N = \frac{\varepsilon a}{n - a(n-1)}.$$

Find the payoff of each coalition K engaging k players. By assumption, all players outside the coalition adopt their Nash equilibrium strategies defined by (7.3).

Take players from the coalition K; seek for the solution of the Bellman equation

$$V_K(x) = \max_{u_i \in K} \lim \left\{ \sum_{i \in K} \log u_i + \delta V_K \left(\varepsilon x - \sum_{i \in K} u_i - \sum_{i \in N \setminus K} u_i^N \right)^{\alpha} \right\} \tag{7.6}$$

among the functions

$$V_K(x) = A_K \log x + B_K.$$

The optimal control laws have the form $u_i = \gamma_i^K x,\ i \in K$. Again, all players in the coalition K are identical. It follows from equation (7.6) that the optimal amount of fish catch constitutes

$$u_i^K = \frac{(1-a)(k - a(k-1))}{k(n - a(n-1))} \varepsilon x \tag{7.7}$$

and the payoff of the coalition becomes

$$V_K(x) = \frac{k}{1-\alpha\delta}\log x + \frac{1}{1-\delta}B_K, \tag{7.8}$$

where

$$B_K = \frac{k}{1-a}\log\left(\frac{\epsilon(k-a(k-1))}{n-a(n-1)}\right) + k(\log(1-a) - \log(k)) + \frac{ka}{1-a}\log(a).$$

For further exposition, we need the equality

$$B_K = kB_i + k\left(\frac{1}{1-a}\log(k-a(k-1)) - \log(k)\right). \tag{7.9}$$

Under the existing coalition K, the population dynamics acquires the form

$$x_t = {x_0}^{\alpha^t}\hat{x}^{K\sum_{j=1}^{t}\lim\alpha^j}, \tag{7.10}$$

where

$$\hat{x}^K = \frac{\epsilon a(k-a(k-1))}{n-a(n-1)}.$$

Finally, find the payoff and optimal strategies in the case of complete cooperation. Formulas (7.7) and (7.8) bring to

$$u_i^I = \frac{(1-a)}{n}\epsilon x, \tag{7.11}$$

$$V_I(x) = \frac{n}{1-\alpha\delta}\log x + \frac{1}{1-\delta}B_I, \tag{7.12}$$

where

$$B_I = nB_i + n\left(\frac{1}{1-a}\log(n-a(n-1)) - \log(n)\right).$$

The dynamic control under complete cooperation is determined by

$$x_t = x_0^{\alpha^t}\hat{x}^{I\sum_{j=1}^{t}\lim\alpha^j},$$

where

$$\hat{x}^I = \epsilon a.$$

Theorem 10.18 *Cooperative behavior ensures a higher population size than non-cooperative one.*

Proof: Obviously,

$$\hat{x}^I = \varepsilon a > \frac{\varepsilon a}{n - a(n-1)} = \hat{x}^N.$$

However, the optimal amounts of fish catch meet the inverse inequality

$$\gamma_i^I = \frac{(1-a)\varepsilon}{n} < \frac{(1-a)\varepsilon}{n - a(n-1)} = \gamma_i^N.$$

Now, find the characteristic function for the game evolving from the state x at instant t:

$$V(L, x, t) = \begin{cases} 0, & L = 0, \\ V(\{i\}, x, t) = V_i(x), & L = \{i\}, \\ V(K, x, t) = V_K(x), & L = K, \\ V(I, x, t) = V_I(x), & L = I. \end{cases} \tag{7.13}$$

Here $V_i(x)$, $V_K(x)$, and $V_I(x)$ are defined by (7.4), (7.8) and (7.12), respectively.

Demonstrate that the constructed characteristic function enjoys superadditivity. For this, take advantage of

Lemma 10.2 *If $c > d$, the function $f(z) = \frac{1}{z}\log\left(\frac{1+zc}{1+zd}\right)$ decreases in z.*

Proof: Consider

$$f'(z) = \frac{1}{z^2}\left[z\left(\frac{c}{1+zc} - \frac{d}{1+zd}\right) - \log\left(\frac{1+zc}{1+zd}\right)\right] = \frac{1}{z^2}g(z).$$

The function $g(z)$ appears non-positive, as far as $g(0) = 0$ and

$$g'(z) = -z\left(\frac{c^2}{(1+zc)^2} - \frac{d^2}{(1+zd)^2}\right) \le 0$$

for $c > d$.

This implies that $f'(z) < 0$.

Theorem 10.19 *The characteristic function (7.13) is superadditive, i.e.,*

$$V(K \cup L, x, t) \ge V(K, x, t) + V(L, x, t), \quad \forall t.$$

Proof: It suffices to show that

$$\begin{aligned} &V(K \cup L, x, t) - V(K, x, t) - V(L, x, t) \\ &\quad = A_{K\cup L}\log(x^{K\cup L}) - A_K\log(x^K) - A_L\log(x^L) + \frac{1}{1-\delta}(B_{K\cup L} - B_K - B_L) \\ &\quad = A_K\log\left(\frac{x^{K\cup L}}{x^K}\right) + A_L\ln\left(\frac{x^{K\cup L}}{x^L}\right) + \frac{1}{1-\delta}(B_{K\cup L} - B_K - B_L) \ge 0. \end{aligned}$$

First of all, we notice that

$$x^K = x^{\alpha^t}\left(\frac{\varepsilon a(k-a(k-1))}{n-a(n-1)}\right)^{\sum_{j=1}^{t}\lim \alpha^j},$$

and so

$$\log\left(\frac{x^{K\cup L}}{x^K}\right) = \sum_{j=1}^{t} \lim \alpha^j \log\left(\frac{k+l-a(k+l-1)}{k-a(k-1)}\right) > 0,$$

since

$$\frac{k+l-a(k+l-1)}{k-a(k-1)} - 1 = \frac{l(1-a)}{k-a(k-1)} > 0.$$

Next, consider the second part and utilize the property (7.9):

$$\begin{aligned}
B_{K\cup L} - B_K - B_L = (k+l)B_i &+ (k+l)\left(\frac{1}{1-a}\log(k+l-a(k+l-1)) - \log(k+l)\right)\\
&- kB_i - k\left(\frac{1}{1-a}\log(k-a(k-1)) - \log(k)\right) - lB_i\\
&- l\left(\frac{1}{1-a}\log(l-a(l-1)) - \log(l)\right)\\
= k&\left(\frac{1}{1-a}\log\left(\frac{k+l-a(k+l-1)}{k-a(k-1)}\right) - \log\left(\frac{k+l}{k}\right)\right)\\
&+ l\left(\frac{1}{1-a}\log\left(\frac{k+l-a(k+l-1)}{l-a(l-1)}\right) - \log\left(\frac{k+l}{l}\right)\right).
\end{aligned}$$

Analyze the expression

$$f(a) = \frac{1}{1-a}\log\left(\frac{k+l-a(k+l-1)}{k-a(k-1)}\right) - \log\left(\frac{k+l}{k}\right).$$

Denote $z = 1-a$, then

$$f(z) = \frac{1}{z}\log\left(\frac{1+(k+l-1)z}{1+(k-1)z}\right) - \log\left(\frac{k+l}{k}\right).$$

It is possible to use the lemma with $k+l-1 = c > d = k-1$. The function $f(z)$ decreases in z. Therefore, $f(a)$ represents an increasing function in a and $f(0) = 0$. Hence, $f(a)$ possesses non-negative values.

Similarly, one can prove that

$$\frac{1}{1-a}\log\left(\frac{k+l-a(k+l-1)}{l-a(l-1)}\right) - \log\left(\frac{k+l}{l}\right) \geq 0.$$

Therefore, we have argued that

$$B_{K\cup L} - B_K - B_L \geq 0.$$

10.7.3 The Shapley vector and imputation distribution procedure

Subsection 10.7.3 selects the Shapley vector as the principle of imputation distribution. In this case, the cooperative income is allocated among the participants in the quantities

$$\xi_i = \sum_{K\subset N,\, i\in K} \frac{(n-k)!(k-1)!}{n!}\left[V_{\{K\}} - V_{\{K\setminus i\}}\right],\ i \in N = \{1, \dots, n\},$$

where k indicates the number of players in a coalition K, $V_{\{K\}}$ is the payoff of the coalition K and $V_{\{K\}} - V_{\{K\setminus i\}}$ gives the contribution of player i to the coalition K.

Theorem 10.20 *The Shapley vector in this game takes the form*

$$\xi_i(t) = \frac{1}{1-a}\log x_t + \frac{1}{1-\delta}(B_i + B_\xi), \tag{7.14}$$

with

$$B_\xi = \frac{1}{1-a}\log(1 + (n-1)(1-a)) - \log(n) \geq 0.$$

Proof: Evaluate the contribution of player i to the coalition K:

$$\begin{aligned} V_K(x_t) - V_{K\setminus i}(x_t) &= (A_K - A_{K\setminus i})\log(x_t) + \frac{1}{1-\delta}(B_K - B_{K\setminus i}) = \frac{1}{1-a}\log x_t \\ &\quad + \frac{1}{1-\delta}B_i + \frac{1}{1-\delta}\left(k\left(\frac{1}{1-a}\log(1+(k-1)(1-a)\right) - \log(k)\right) \\ &\quad - (k-1)\left(\frac{1}{1-a}\log(1+(k-2)(1-a)) - \log(k-1))\right). \end{aligned}$$

This expression turns out independent from i, which means that

$$\begin{aligned} \xi_i(t) &= \sum_{K\subset N,\, i\in K} \frac{(n-k)!(k-1)!}{n!}[V_K(x_t) - V_{K\setminus i}(x_t)] = \sum_{k=1}^{n} \frac{1}{n}[V_K(x_t) - V_{K\setminus i}(x_t)] \\ &= \frac{1}{1-a}\log x_t + \frac{1}{1-\delta}\left(B_i + \frac{1}{1-a}\log(1+(n-1)(1-a)) - \log(n)\right). \end{aligned}$$

Theorem 10.21 *The Shapley vector (7.14) forms a time-consistent imputation distribution procedure and the incentive condition for rational behavior (7.1) holds true.*

Proof: It follows from Theorem 10.17 that

$$\beta_i(t) = \frac{1}{1-a}(\log x_t - \delta \log x_{t+1}) + B_i + B_\xi.$$

For each shot, the incentive condition for rational behavior (7.2) becomes

$$\frac{1}{1-a}(\log x_t - \delta \log x_{t+1}) + B_i + B_\xi \geq \frac{1}{1-a}(\log x_t - \delta \log x_{t+1}) + B_i.$$

It is valid, so long as $B_\xi \geq 0$.

10.7.4 The model with informed players

Now, consider another scenario when players outside a coalition K are informed on its appearance. Subsequently, they modify their strategies to achieve a new Nash equilibrium in the game with $N \backslash K$ players.

In comparison with the previous case, the whole difference concerns evaluation of the characteristic function V_K. Let us proceed by analogy. Take players from the coalition K and solve the Bellman equation

$$\tilde{V}_K(x) = \max_{u_i \in K} \lim \left\{ \sum_{i \in K} \log u_i + \delta \tilde{V}_K \left(\epsilon x - \sum_{i \in K} u_i - \sum_{i \in N \backslash K} \tilde{u}_i^N \right)^\alpha \right\}, \tag{7.15}$$

where $\tilde{u}_i^N$ corresponds to the Bellman equation solution for players outside the coalition K:

$$\tilde{V}_i(x) = \max_{\tilde{u}_i \in N \backslash K} \lim \left\{ \log \tilde{u}_i + \delta \tilde{V}_i \left(\epsilon x - \sum_{i \in K} u_i - \sum_{i \in N \backslash K} \tilde{u}_i \right)^\alpha \right\}. \tag{7.16}$$

Seek for solutions of these equations in the form

$$\tilde{V}_K(x) = \tilde{A}_K \log x + \tilde{B}_K, \quad \tilde{V}_i(x) = \tilde{A}_i \log x + \tilde{B}_i,$$

and the optimal control laws defined by $u_i = \gamma_i^K x,\ i \in K$ and $\tilde{u}_i = \tilde{\gamma}_i^N x$. It follows from (7.15) that the optimal amounts of fish catch of the players belonging to the coalition K are

$$\tilde{u}_i^K = \frac{1-a}{k(1+(n-k)(1-a))} \epsilon x. \tag{7.17}$$

And so, their payoff makes up

$$\tilde{V}_K(x) = \frac{k}{1-a} \log x + \frac{1}{1-\delta} \tilde{B}_K, \tag{7.18}$$

where

$$\tilde{B}_K = k \left(\frac{1}{1-a} \log\left(\frac{\epsilon}{1+(n-k)(1-a)} \right) + \log(1-a) + \frac{a}{1-a} \log(a) - \log(k) \right).$$

We present a relevant inequality for further reasoning:

$$\tilde{B}_K = kB_i + k\left(\frac{1}{1-a}\log\Big(\frac{1+(n-1)(1-a)}{1+(n-k)(1-a)}\Big) - \log(k)\right). \qquad (7.19)$$

For players outside the coalition K, the optimal amounts of fish catch constitute

$$\tilde{u}_i^N = \frac{1-a}{1+(n-k)(1-a)}\varepsilon x$$

and the payoffs equal

$$\tilde{V}_i(x) = \frac{1}{1-a}\log x + \frac{1}{1-\delta}\tilde{B}_i,$$

where

$$\tilde{B}_i = \frac{1}{1-a}\log\Big(\frac{\varepsilon}{1+(n-k)(1-a)}\Big) + \log(1-a) + \frac{a}{1-a}\log(a).$$

The corresponding dynamics in the case of the coalition K acquires the form

$$x_t = x_0^{\alpha^t}\tilde{x}^{K\sum_{j=1}^{t}\lim\alpha^j},$$

where

$$\tilde{x}^K = \frac{\varepsilon a}{1+(n-k)(1-a)}.$$

In the grand-coalition I, the optimal amounts of fish catch and payoffs do coincide with the previous scenario. Therefore, Theorem 10.18 remains in force as well.

The characteristic function of the game evolving from the state x at instant t is determined by

$$V(L,x,t) = \begin{cases} 0, & L = 0, \\ V(\{i\},x,t) = V_i(x), & L = \{i\}, \\ V(K,x,t) = \tilde{V}_K(x), & L = K, \\ V(I,x,t) = V_I(x), & L = I, \end{cases}$$

where $V_i(x)$, $\tilde{V}_K(x)$, and $V_I(x)$ obey formulas (7.4), (7.18) and (7.12), respectively.

Similarly to the model without information, we find the Shapley vector and the time-consistent imputation distribution procedure.

It appears from (7.18) and (7.19) that

$$\xi_i(t) = \frac{1}{1-a}\log x_t + \frac{1}{1-\delta}(B_i + B_\xi),$$

where

$$\begin{aligned}
B_\xi &= \sum_{K \in N} \lim \frac{(n-k)!(k-1)!}{n!} \left[k \left(\frac{1}{1-a} \log \left(\frac{1+(n-1)(1-a)}{1+(n-k)(1-a)} \right) - \log(k) \right) \right. \\
&\quad \left. - (k-1) \left(\frac{1}{1-a} \log \left(\frac{1+(n-1)(1-a)}{1+(n-k+1)(1-a)} \right) - \log(k-1) \right) \right] \\
&= \sum_{k=1}^{n} \lim \frac{1}{n} \left[k \left(\frac{1}{1-a} \log \left(\frac{1+(n-1)(1-a)}{1+(n-k)(1-a)} \right) - \log(k) \right) \right. \\
&\quad \left. - (k-1) \left(\frac{1}{1-a} \log \left(\frac{1+(n-1)(1-a)}{1+(n-k+1)(1-a)} \right) - \log(k-1) \right) \right] \\
&= \frac{1}{1-a} \log(1+(n-1)(1-a)) - \log(n).
\end{aligned}$$

By analogy to Theorem 10.21, one can prove

Theorem 10.22 *The Shapley vector defines the time-consistent IDP and the incentive condition for rational behavior holds true.*

Finally, we compare these scenarios.

Theorem 10.23 *The payoffs of free players in the second model are higher than in the first one.*

Proof: Consider players outside the coalition K and calculate the difference in their payoffs:

$$\tilde{V}_i(x) - V_i(x) = \frac{1}{1-\delta}(\tilde{B}_i - B_i) = \frac{1}{(1-\delta)(1-a)} \log\left(\frac{1+(n-1)(1-a)}{1+(n-k)(1-a)} \right) > 0.$$

Theorem 10.24 *The payoff of the coalition K in the first model is higher than in the second one.*

Proof: Consider players from the coalition K and calculate the difference in their payoffs:

$$\begin{aligned}
V_K(x) - \tilde{V}_K(x) &= \frac{1}{1-\delta}(B_K - \tilde{B}_K) \\
&= \frac{k}{(1-\delta)(1-a)} \log\left(\frac{(1+(n-k)(1-a))(1+(k-1)(1-a))}{1+(n-1)(1-a)} \right) > 0,
\end{aligned}$$

so long as

$$\frac{(1+(n-k)(1-a))(1+(k-1)(1-a))}{1+(n-1)(1-a)} - 1 = \frac{(k-1)(1-a)^2(n-k)}{1+(n-1)(1-a)} > 0.$$

Theorem 10.25 *The population size under coalition formation in the first model is higher than in the second one.*

Proof: Reexpress the corresponding difference as

$$x^K - \tilde{x}^K = \frac{\epsilon a(k - a(k-1))}{1 + (n-1)(1-a)} - \frac{\epsilon a}{1 + (n-k)(1-a)}$$
$$= \frac{(1-a)^2(n-k)(k-1)}{(1 + (n-1)(1-a))(1 + (n-k)(1-a))}.$$

Actually, it possesses positive values, and the conclusion follows.

Exercises

1. Two companies exploit a natural resource with rates of usage $u_1(t)$ and $u_2(t)$. The resource dynamics meets the equation

$$x'(t) = \epsilon x(t) - u_1(t) - u_2(t), \ x(0) = x_0.$$

The payoff functionals of the players take the form

$$J_i(u_1, u_2) = \int_0^\infty \left[c_i u_i(t) - u_i^2(t)\right] dt, i = 1, 2.$$

Find a Nash equilibrium in this game.

2. Two companies manufacture some commodity with rates of production $u_1(t)$ and $u_2(t)$, but pollute the atmosphere with same rates. The pollution dynamics is described by

$$x_{t+1} = \alpha x_t + u_1(t) + u_2(t), \ t = 0, 1, ...$$

The initial value x_0 appears fixed, and the coefficient α is smaller than 1. The payoff functions of the players represent the difference between their incomes and the costs of purification procedures:

$$J_i(u_1, u_2) = \sum_{t=0}^{\infty} \beta^t \left[(a - u_1(t) - u_2(t))u_i(t) - cu_i(t)\right] dt, i = 1, 2.$$

Evaluate a Nash equilibrium in this game.

3. A two-player game satisfies the equation

$$x'(t) = u_1(t) + u_2(t), \ x(0) = 0, \ u_1, u_2 \in [0, 1].$$

The payoff functionals of the players have the form

$$J_i(u_1, u_2) = x(1) - \int_0^1 u_i^2(t)dt, \ \ i = 1, 2.$$

Under the assumption that player 1 makes a leader, find a Nash equilibrium and a Stackelberg equilibrium in this game.

4. Two companies exploit a natural resource with rates of usage $u_1(t)$ and $u_2(t)$. The resource dynamics meets the equation

$$x'(t) = rx(t)(1 - x(t)/K) - u_1(t) - u_2(t), \ \ x(0) = x_0.$$

The payoff functionals of the players take the form

$$J_i(u_1, u_2) = \int_0^T e^{-\beta t} \left(c_i u_i(t) - u_i^2(t)\right) dt, \ \ i = 1, 2.$$

Evaluate a Nash equilibrium in this game.

5. Find a Nash equilibrium in exercise no. 4 provided that both players utilize the resource on infinite time horizon.

6. Two players invest unit capital in two production processes evolving according to the equations

$$\begin{aligned} x_{t+1}^i &= a^i x_t^i + b^i u_t^i, \\ y_{t+1}^i &= c^i y_t^i + d^i \left(x_t^i - u_t^i\right), \ \ t = 1, 2, \dots, T-1. \end{aligned}$$

Their initial values x_0^i and y_0^i $(i = 1, 2)$ are fixed. The payoffs of the players have the form

$$J_i(u_1, u_2) = \delta^i \left(x_T^i\right)^2 - \sum_{t=0}^{T-1} \left[\left(y_t^i - y_t^j\right)^2 + \left(u_t^i\right)^2\right], \quad i \neq j, \quad i = 1, 2.$$

Find a Nash equilibrium in this game.

7. Consider a dynamic game of two players described by the equation

$$x'(t) = \varepsilon + u_1(t) + u_2(t), \ \ x(0) = x_0.$$

The payoff functionals of the players are defined by

$$\begin{aligned} J_1(u_1, u_2) &= a_1 x^2(T) - \int_0^T \left[b_1 u_1^2(t) - c_1 u_2^2(t)\right] dt, \\ J_2(u_1, u_2) &= a_2 x^2(T) - \int_0^T \left[b_2 u_2^2(t) - c_2 u_1^2(t)\right] dt. \end{aligned}$$

Evaluate a Nash equilibrium in this game.

8. Find the cooperative payoff in exercises no. 6 and 7. Construct the time-consistent imputation distribution procedure under the condition of equal payoff sharing by the players.
9. Consider the fish war model with three countries and the core as the imputation distribution criterion. Construct the time-consistent imputation distribution procedure.
10. Verify the incentive conditions for rational behavior at each shot in exercises no. 8 and 9.

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