Problem set from Y11 3U finals

You know who

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List of problems

1. For $P(x) = (4x + \frac{1}{k})^n$, the following is true:

   The coefficient of x^4 is 8 times that of x^3 and the coefficient of x^2 is 24 times that of x.

   Find n and k.

   (3)

2. For $0 \leq x < \pi$, evaluate x so that $\sin 4x = \sin 2x$.

   (2)

Soln. for problem 1

Understand that $P(x)$ may be expanded as:

$$\sum_{i=0}^{n} \binom{n}{i} (4x)^{n-i} \times k^{-n}$$

And that $\binom{n}{i} = \binom{n}{n-i}$ by the symmetric distribution of Pascal's triangle.
Thus taking the binomial coefficients of x^4, x^3, x^2, x the following is obtained:

- For x^4:

$$C = \binom{n}{4} \times 4^4 \times k^{4-n}$$

$$= \frac{n!}{(4!)(n-4)!} \times 256 \times k^{4-n}$$

$$= \frac{n(n-1)(n-2)(n-3)}{4!} \times 256 \times k^{4-n}$$

$$= \frac{32(n)(n-1)(n-2)(n-3)}{3} \times k^{4-n}$$

- For x^3:

$$C = \binom{n}{3} \times 4^3 \times k^{3-n}$$

$$= \frac{n!}{(3!)(n-3)!} \times 64 \times k^{3-n}$$

$$= \frac{(n)(n-1)(n-2)}{3!} \times 64 \times k^{3-n}$$

$$= \frac{32(n)(n-1)(n-2)}{3} \times k^{3-n}$$

- For x^2:

$$C = \binom{n}{2} \times 4^2 \times k^{2-n}$$

$$= \frac{n!}{(2!)(n-2)!} \times 16 \times k^{2-n}$$

$$= \frac{(n)(n-1)}{2!} \times 16 \times k^{2-n}$$

$$= 8(n)(n-1) \times k^{2-n}$$

- For x:

$$C = \binom{n}{1} \times 4^1 \times k^{1-n}$$

$$= 4nk^{1-n}$$

Soln. for problem 2

Considering the domain $0 \leq x < \pi$ as $0 \leq 2x < 2\pi$,
First rewrite $\sin 4x$ as $2 \sin(2x) \cos(2x)$, and thus:

- First define LHS and RHS:

$$LHS = \sin 4x = 2 \sin(2x) \cos(2x)$$
$$RHS = \sin 2x$$

- Thus the following cases are trivially obtained:

$$\sin 2x = 0$$
$$\cos 2x = \frac{1}{2}$$

- $\therefore$ For each of the following cases:

    1. $\sin 2x = 0$

$$\sin^{-1} 0 = 2x$$

$$\therefore 2x = 0, \pi$$

$$x = 0, \frac{\pi}{2}$$

    2. $\cos 2x = \frac{1}{2}$

$$\cos^{-1} \frac{1}{2} = 2x$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3},$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

- Thus:

$$x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$

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