

# Problem set from Y11 3U finals

You know who

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## List of problems

1. For  $P(x) = (4x + \frac{1}{k})^n$ , the following is true:

The coefficient of  $x^4$  is 8 times that of  $x^3$  and the coefficient of  $x^2$  is 24 times that of  $x$ .

Find  $n$  and  $k$ .

(3 Marks)

2. For  $0 \leq x < \pi$ , evaluate  $x$  so that  $\sin 4x = \sin 2x$ .

(2 Marks)

## Soln. for problem 1

Understand that  $P(x)$  may be expanded as:

$$\sum_{i=0}^n \binom{n}{i} (4x)^{n-i} \times k^{-i}$$

And that  $\binom{n}{i} = \binom{n}{n-i}$  by the symmetric distribution of Pascal's triangle.

Thus taking the binomial coefficients of  $x^4, x^3, x^2, x$  the following is obtained:

- For  $x^4$ :

$$\begin{aligned} C &= \binom{n}{4} \times 4^4 \times k^{4-n} \\ &= \frac{n!}{(4!)(n-4)!} \times 256 \times k^{4-n} \\ &= \frac{n(n-1)(n-2)(n-3)}{4!} \times 256 \times k^{4-n} \\ &= \frac{32(n)(n-1)(n-2)(n-3)}{3} \times k^{4-n} \end{aligned} \tag{1}$$

- For  $x^3$ :

$$\begin{aligned} C &= \binom{n}{3} \times 4^3 \times k^{3-n} \\ &= \frac{n!}{(3!)(n-3)!} \times 64 \times k^{3-n} \\ &= \frac{(n)(n-1)(n-2)}{3!} \times 64 \times k^{3-n} \\ &= \frac{32(n)(n-1)(n-2)}{3} \times k^{3-n} \end{aligned} \tag{2}$$

- For  $x^2$ :

$$\begin{aligned} C &= \binom{n}{2} \times 4^2 \times k^{2-n} \\ &= \frac{n!}{(2!)(n-2)!} \times 16 \times k^{2-n} \\ &= \frac{(n)(n-1)}{2!} \times 16 \times k^{2-n} \\ &= 8(n)(n-1) \times k^{2-n} \end{aligned} \tag{3}$$

- For  $x$ :

$$\begin{aligned} C &= \binom{n}{1} \times 4^1 \times k^{1-n} \\ &= 4nk^{1-n} \end{aligned} \tag{4}$$

And now consider that:

- **Equation 1 = 8 × Equation 2**
- **Equation 3 = 24 × Equation 4**

Now forming two new equations:

- Equation 5

$$\frac{32(n)(n-1)(n-2)(n-3)}{3} \times k^{4-n} = \frac{256(n)(n-1)(n-2)}{3} \times k^{3-n}$$

$$(n-3)k^{4-n} = 8k^{3-n}$$

(5)

$$\frac{n-3}{8} = k^{-1}$$

$$\frac{8}{n-3} = k$$

- Equation 6

$$8(n)(n-1) \times k^{2-n} = 96nk^{1-n}$$

$$(n-1)k^{2-n} = 12k^{1-n}$$

(6)

$$\frac{n-1}{12} = k^{-1}$$

$$\frac{12}{n-1} = k$$

- And thus:

$$\frac{12}{n-1} = \frac{8}{n-3} = k$$

$$\frac{3}{2} = \frac{n-1}{n-3} \Rightarrow 2(n-1) = 3(n-3) \Rightarrow 2n-2 = 3n-9$$

$$\therefore n = 7$$

$$\therefore k = 2$$

## Soln. for problem 2

Considering the domain  $0 \leq x < \pi$  as  $0 \leq 2x < 2\pi$ ,

First rewrite  $\sin 4x$  as  $2 \sin(2x) \cos(2x)$ , and thus:

- First define LHS and RHS:

$$LHS = \sin 4x = 2 \sin(2x) \cos(2x)$$

$$RHS = \sin 2x$$

- Thus the following cases are trivially obtained:

$$\sin 2x = 0$$

$$\cos 2x = \frac{1}{2}$$

- $\therefore$  For each of the following cases:

1.  $\sin 2x = 0$

$$\sin^{-1} 0 = 2x$$

$$\therefore 2x = 0, \pi$$

$$x = 0, \frac{\pi}{2}$$

2.  $\cos 2x = \frac{1}{2}$

$$\cos^{-1} \frac{1}{2} = 2x$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

- Thus:

$$x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$