Problem set from Y11 3U finals

You know who

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List of problems

1. For $P(x) = (4x + \frac{1}{k})^n$, the following is true:
   The coefficient of x^4 is 8 times that of x^3 and the coefficient of x^2 is 24 times that of x.
   Find n and k.

   (3 Marks)

2. For $0 \leq x < \pi$, evaluate x so that $\sin 4x = \sin 2x$.

   (2 Marks)

Soln. for problem 1

Understand that $P(x)$ may be expanded as:

$$\sum_{i=0}^{n} \binom{n}{i} (4x)^{n-i} \times k^{-n}$$

And that $\binom{n}{i} = \binom{n}{n-i}$ by the symmetric distribution of Pascal's triangle.
Thus taking the binomial coefficients of x^4, x^3, x^2, x the following is obtained:

- For x^4:

$$\begin{aligned} C &= \binom{n}{4} \times 4^4 \times k^{4-n} \\ &= \frac{n!}{(4!)(n-4)!} \times 256 \times k^{4-n} \\ &= \frac{n(n-1)(n-2)(n-3)}{4!} \times 256 \times k^{4-n} \\ &= \frac{32(n)(n-1)(n-2)(n-3)}{3} \times k^{4-n} \end{aligned} \tag{1}$$

- For x^3:

$$\begin{aligned} C &= \binom{n}{3} \times 4^3 \times k^{3-n} \\ &= \frac{n!}{(3!)(n-3)!} \times 64 \times k^{3-n} \\ &= \frac{(n)(n-1)(n-2)}{3!} \times 64 \times k^{3-n} \\ &= \frac{32(n)(n-1)(n-2)}{3} \times k^{3-n} \end{aligned} \tag{2}$$

- For x^2:

$$\begin{aligned} C &= \binom{n}{2} \times 4^2 \times k^{2-n} \\ &= \frac{n!}{(2!)(n-2)!} \times 16 \times k^{2-n} \\ &= \frac{(n)(n-1)}{2!} \times 16 \times k^{2-n} \\ &= 8(n)(n-1) \times k^{2-n} \end{aligned} \tag{3}$$

- For x:

$$\begin{aligned} C &= \binom{n}{1} \times 4^1 \times k^{1-n} \\ &= 4nk^{1-n} \end{aligned} \tag{4}$$

And now consider that:

- **Equation 1 = 8 × Equation 2**
- **Equation 3 = 24 × Equation 4**

Now forming two new equations:

- Equation 5

$$
\begin{aligned}
&\frac{32(n)(n-1)(n-2)(n-3)}{3} \times k^{4-n} = \frac{256(n)(n-1)(n-2)}{3} \times k^{3-n} \\
&(n-3)k^{4-n} = 8k^{3-n} \\
&\frac{n-3}{8} = k^{-1} \\
&\frac{8}{n-3} = k
\end{aligned}
\tag{5}
$$

- Equation 6

$$
\begin{aligned}
&8(n)(n-1) \times k^{2-n} = 96nk^{1-n} \\
&(n-1)k^{2-n} = 12k^{1-n} \\
&\frac{n-1}{12} = k^{-1} \\
&\frac{12}{n-1} = k
\end{aligned}
\tag{6}
$$

- And thus:

$$\frac{12}{n-1} = \frac{8}{n-3} = k$$

$$\frac{3}{2} = \frac{n-1}{n-3} \rightarrow 2(n-1) = 3(n-3) \rightarrow 2n-2 = 3n-9$$

$$\therefore n = 7$$

$$\therefore k = 2$$

Soln. for problem 2

Considering the domain $0 \leq x < \pi$ as $0 \leq 2x < 2\pi$,
First rewrite $\sin 4x$ as $2\sin(2x)\cos(2x)$, and thus:

- First define LHS and RHS:

$$LHS = \sin 4x = 2\sin(2x)\cos(2x)$$
$$RHS = \sin 2x$$

- Thus the following cases are trivially obtained:

$$\sin 2x = 0$$
$$\cos 2x = \frac{1}{2}$$

- $\therefore$ For each of the following cases:

1. $\sin 2x = 0$

$$\sin^{-1} 0 = 2x$$

$$\therefore 2x = 0, \pi$$

$$x = 0, \frac{\pi}{2}$$

2. $\cos 2x = \frac{1}{2}$

$$\cos^{-1} \frac{1}{2} = 2x$$

$$\therefore 2x = \frac{\pi}{3}, \frac{5\pi}{3},$$

$$x = \frac{\pi}{6}, \frac{5\pi}{6}$$

- Thus:

$$x = 0, \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$$