1.1 Modulus Functions

=> The modulus of a number is the magnitude of that number without a sign -> absolute value

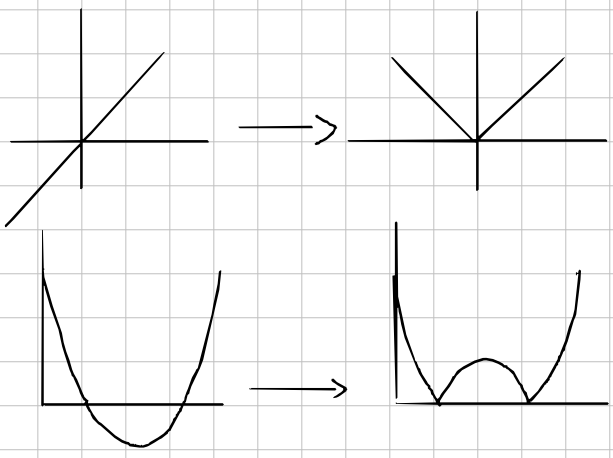
=> modulus is denoted as '| |'

example: $|3| = 3$ $\quad |-3| = 3$

Modulus as a Piece-Wise Function

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

*where 0 might be any point at which the original function crosses the x-axis



Modulus of Functions

$|x| = k \qquad k \geq 0$ ← can't be smaller

$x_1 = k \qquad x_2 = -k$

$|ax+b| = cx+d$

↳ $ax+b = -(cx+d)$

↳ $ax+b = cx+d$

! Always check if solutions satisfy the equation, <u>some don't work</u> !

Solve Equation of two Modulus Functions

note: '$\Longleftrightarrow$' means logically equivalent

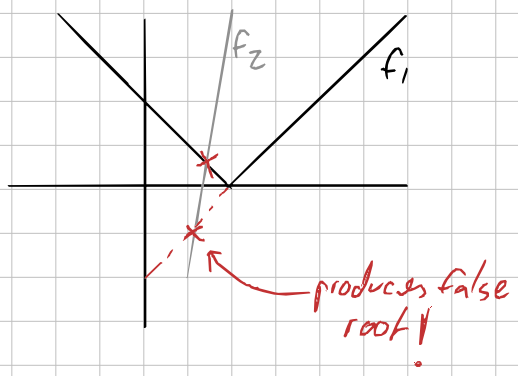
Rule: $\boxed{|a| = |b| \Longleftrightarrow a^2 = b^2}$

example: $|cx+d| = |ex+f|$

$(cx+d)^2 = (ex+f)^2$

False Solutions for Roots

emagine following functions:



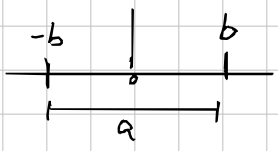
Solving Modulus Inequalities

Rules:

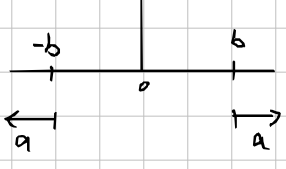
$|a| \leq b \Longleftrightarrow -b \leq a \leq b$

$|a| \geq b \Longleftrightarrow -b \geq a \leq b$

$|a| \leq b$



$|a| \geq b$



Practice

1A

9. $|3x-2y-11| + 2(31-8x+5y)^{\frac{1}{2}} = 0$

$|3x-2y-11| = 0$

$3x-2y-11=0$ and $-3x+2y+11=0$

and $2(31-8x+5y)^{\frac{1}{2}}=0$ $\quad 1\cdot 2(1)^{\frac{1}{2}}$

$31-8x+5y=0$

$31+5y=8x$

$x=\frac{31+5y}{8}$

$3x-2y-11=0$

$3\left(\frac{31+5y}{8}\right)-2y-11=0$

$\frac{31}{24}+\frac{5}{24}y-2y-11=0$

$\frac{43}{24}y-\frac{233}{24}=0$

$\frac{43}{24}y=\frac{233}{24}$

$y=$

8. $|2x+1|+|2x-1|=3$

$|2x+1|=3-|2x-1|$

$2x+1=3-|2x-1|$ and $2x+1=|2x-1|-3$

$2-2x=|2x-1|$

$2-2x=1-2x$

$2=1$

$2x+1=|2x-1|-3$

$2x+4=|2x-1|$

$\cancel{2x}+4=\cancel{2x}-1$ and $2x+4=1-2x$

$4\neq -1$ $\quad 4x=-3$

$x=-\frac{3}{4}$

$x_1=\frac{3}{4}$ $\quad x_2=-\frac{3}{4}$

test x_1:

$|2\cdot\frac{3}{4}+1|+|2\cdot\frac{3}{4}-1|=3$

$\frac{5}{2}+\frac{1}{2}=3$ ✓

test x_2:

$|2\cdot(-\frac{3}{4})+1|+|2\cdot(-\frac{3}{4})-1|=3$

$|-\frac{1}{2}|+|(-\frac{5}{2})|=3$

$\frac{1}{2}+\frac{5}{2}=3$ ✓

$x=\pm\frac{3}{4}$

7. $x^2-5|x|+6=0$

$x^2+6=5|x|$

$\frac{x^2+6}{5}=|x|$

$x=\frac{x^2+6}{5}$ $\qquad x=-\frac{x^2+6}{5}$

$5x=x^2+6$ $\qquad -5x=-x^2-6$

$0=x^2-5x+6$ $\qquad 0=-x^2+5x-6$ $\quad |\cdot(-1)$

$0=x^2-5x+6$ $\quad$ same, neglect

$A\cdot 5$ $\quad M\ 6$

-2 $\quad -3$

$0=(x-2)(x-3)$

$x_1=2$ $\quad x_2=3$

Test:

$2^2-5\cdot|2|+6=0$ $\quad 3^2-5\cdot|3|+6=0$

$4-10+6=0$ $\quad 9-15+6=0$

$0=0$ ✓ $\quad 9-9=0$

$0=0$ ✓

8) $x=0$

1. a) $|4x-3|=7$

$7=4x-3$ $\qquad -7=4x-3$

$10=4x$ $\qquad -4=4x$

$x=\frac{5}{2}$ $\qquad x=-1$

test $|4\cdot\frac{5}{2}-3|=7$ $\qquad |4\cdot(-1)-3|=7$

$|10-3|=7$ $\qquad |-4-3|=7$

$7=7$ ✓ $\qquad 7=7$ ✓

c) $\left|\frac{3x-2}{5}\right|=4$

$4=\frac{3x-2}{5}$ $\qquad 4=-\frac{3x-2}{5}$

$20=3x-2$ $\qquad 20=-3x+2$

$22=3x$ $\qquad 18=-3x$

$x_1=\frac{22}{3}$ $\qquad x_2=-\frac{18}{3}=-6$

test

$\left|\frac{3\cdot\frac{22}{3}-2}{5}\right|=4$ $\qquad \left|\frac{3\cdot(-6)-2}{5}\right|=4$

$\frac{20}{5}=4$ $\qquad \left|\frac{-20}{5}\right|=4$

$4=4$ ✓ $\qquad 4=4$ ✓

$x_1=\frac{22}{3}$ $\quad x_2=-6$

7. f) $8-|1-2x|=x$

$|1-2x|+x=8$

$|1-2x|=8-x$

$1-2x=8-x$ $\qquad 1-2x=x-8$

$-7=x_1$ $\qquad 9=3x$

$x_2=3$

test $8-|1-2(-7)|=-7$ $\qquad 8-|1-2(3)|=3$

$8-|15|=-7$ $\qquad 8-|-5|=3$

$-7=-7$ ✓ $\qquad 3=3$ ✓

30. $3|2x-1|=|\frac{1}{2}x-3|$

$3(2x-1)^2=(\frac{1}{2}x-3)^2$

$3(4x^2-4x+1)=\frac{1}{4}x^2-3x+9$

$36x^2-36x+18=\frac{1}{4}x^2-3x+9$

$35.75x^2-33x=0$

$x(35.75x-33)=0$

$x_1=0$ $\qquad 35.75x-33=0$

$35.75x=+33$

$x_2=\frac{33}{35.75}$

test $3|2\cdot 0-1|=|\frac{1}{2}\cdot 0-3|$ $\qquad 3|2\cdot\frac{33}{35.75}-1|=|\frac{1}{2}\cdot\frac{33}{35.75}-3|$

$3|-1|=|-3|$ $\qquad \left|\frac{33}{35.75}-1\right|=\left|-\frac{33}{35.75}\right|$

$3=3$ ✓ $\qquad \frac{33}{35.75}=\frac{33}{35.75}$ ✓

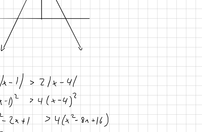
1. c) $y=|5-\frac{1}{2}x|$

$0=|5-\frac{1}{2}x|$

$5-\frac{1}{2}x=0$ $\qquad \frac{1}{2}x-5=0$

$5=\frac{1}{2}x$ $\qquad \frac{1}{2}x=5$

$x=10$ $\qquad x=10$ same



$y=\begin{cases}5-\frac{1}{2}x, & x\leq 10\\ \frac{1}{2}x-5, & x\geq 10\end{cases}$

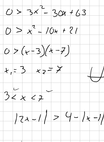
3. e) $y=-(x+2)^2+1$ $\qquad$ flip over x-axis

just to check $\qquad$ shift by $(-2,1)$

f) $y=5-2|x|$

flip over x-axis, stretch by factor, shift up 5

4. f



1c.

1. $|x-1|>2|x-4|$

$(x-1)^2>4(x-4)^2$

$x^2-2x+1>4(x^2-8x+16)$

$x^2-2x+1>4x^2-32x+64$

$0>3x^2-30x+63$

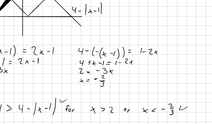
$0>x^2-10x+21$ $\qquad A\cdot 10 \quad M\ 21$

$0>(x-3)(x-7)$ $\qquad -3 \quad -7$

$x_1=3$ $\quad x_2=7$

$3<x<7$

2. $|2x-1|>4-|x-1|$



$4-(x-1)=2x-1$ $\qquad 4-(-(x-1))=1-2x$

$4-x+1=2x-1$ $\qquad 4+x-1=1-2x$

$6=3x$ $\qquad 2x=-3x$

$x=2$ $\qquad x=-\frac{3}{5}$

$2x-1>4-|x-1|$ for $x>2$ or $x<-\frac{3}{5}$

1.1 Devision of Polynomials

Long-Division

Divide $x^3+2x^2-11x+6$ by $x-2$

divisor → dividend

$$x-2\,\overline{)\,x^3+2x^2-11x+6}$$

Step 1: devide x^3 by the first place of the divisor (x) and write it above the second term

$$\begin{array}{r} x^2 \\ x-2\,\overline{)\,x^3+2x^2-11x+6} \end{array}$$

$x^3 : x = x^2$

Step 2: Multiply the result (x^2) by the dividend $(x-2)$ and write it bellow the term

$$\begin{array}{r} x^2 \\ x-2\,\overline{)\,x^3+2x^2-11x+6} \\ \underline{x^3-2x} \end{array}$$

$(x-2)x^2 = x^3-2x^2$

Step 3: Take the difference, pull down the next term, repeat.

$$\begin{array}{r} x^2 \quad 4x \quad -3 \\ x-2\,\overline{)\,x^3+2x^2-11x+6} \\ \underline{x^3-2x^2} \\ 4x^2-11x \\ \underline{4x^2-8x} \\ -3x+6 \\ \underline{-3x+6} \\ 0 \end{array}$$

$4x^2 : x = 4x$

$4x \cdot (x-2) = 4x^2-8x$

$-3x : x = -3$

$-3(x-2) = -3x+6$

no rest

Final: The result can be found at the top:

$$\overline{x^2 \quad 4x \quad -3}$$

$(x^3+2x^2-11x+6) : (x-2) = x^2+4x-3$

Polynomial Division with Remainder

$(2x^3-x+52) = (x+3)(2x^2-6x+17)+1$

Formula: $\boxed{\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}}$

↳ When there is a remainder, the result will include both the quotient and the divisor

Vocabulary help

$$\frac{3}{6} = 0.5$$

dividend, divisor, quotient

Factor Theorem

When a polynomial is divisable by $(x-c)$, then it can be written as follows:

$P(x) = Q(x)(x-c)$

$P(x) \hat{=}$ original polynomial
$Q(x) \hat{=}$ quotient
$(x-c) \hat{=}$ divisor
$c \hat{=}$ some constant

now it's garuanteed that $P(c)=0$:

$P(c) = Q(c) \cdot (c-c) = 0$

This means that when $P\left(\frac{a}{b}\right)=0$ $(ax-b)$ has to be a factor.

example: $P\left(\frac{2}{1}\right)=0 \rightarrow c=2 \rightarrow (x-c)=(x-2)$
↳ $(x-2)$ is a factor

$P\left(\frac{3}{2}\right)=0 \rightarrow c=\frac{3}{2} \rightarrow \left(x-\frac{3}{2}\right) \rightarrow (2x-3)$
↳ $(2x-b)$ is a factor

$$\boxed{\begin{array}{l} P\left(\frac{b}{a}\right)=0 \\ (ax-b) \text{ is a factor} \end{array}}$$

Remainder Theorem

Remember how $P(c)=0$ in the factor theorem?
↳ The remainder is zero

↳ $\boxed{P(c) = \text{remainder}}$ when $Q(x) = \frac{P(x)}{(x-c)}$

Remember how $P\left(\frac{b}{a}\right)=0$ when the remainder is zero?

when $Q(x) = \frac{P(x)}{(ax-b)}$ $\boxed{P\left(\frac{b}{a}\right) = \text{remainder}}$

Chapter Review

$|2x-3| = |5x+1|$

$(2x-3)^2 = (5x+1)^2$

$4x^2-12x+9 = 25x^2+10x+1$

$0 = 21x^2+22x-8$

$x_{1,2} = \frac{-22 \pm \sqrt{22^2-4(21)(-8)}}{2(21)}$ $\quad x_1 = \frac{2}{3}$ $\quad x_2 = -\frac{4}{7}$

$|2x\frac{1}{3}-3| = |5\frac{1}{3}+1|$ $\quad |-\frac{4}{3}-3| = |-\frac{4}{7}+1|$

$|-\frac{12}{3}| = |\frac{16}{3}|$ $\quad \frac{13}{3} = \frac{13}{7}$

$\frac{12}{3} = \frac{16}{3}$

2. $|4x-3| \geq 7$

$4x-3 \geq 7$ $\quad 3-4x \geq 7$

$4x \geq 10$ $\quad -4x \geq 4$

$x \geq 2$ $\quad x \leq -\frac{4}{4}$

$2 \leq x \leq -\frac{4}{4}$

3. $|2(x-3)| < |2-3x|$

$|2(x-3)| < |2-3x|$

$|2x-6| < |2-3x|$ $\quad 6-4x+9x > 2$

$3x^2-9x+9 < 4-12x+9x^2$ $\quad 5x > -4$

$x_{1,2} = \frac{9 \pm \sqrt{9^2-4(3)5}}{2 \cdot 3}$ $\quad x_1 = \frac{2}{3}$ $\quad x_2 = 1$

$|x| < \frac{2}{3}$

4. $|2^x-1| = 11$

$|2^x-1| = 11$ $\quad 1-2^x = 11$

$2^x+1 = 11$ $\quad 2^x+1 = -10$

$2^x = 10$ $\quad 2^x = -9$

$|2^x-1| = 11$

$11 = 11$

$x_1 = 3$ $\quad x_2 = -3$

$x_1 = 3$ $\quad x_2 = -\sqrt{3}$

5. a) $(x-4) < 2$

$x(x^2-7x^2-12x^2-4x-2x-4) < 2$

$6x^3-7x^2$

$11x^2-4x$

$11x^2-4x$

$3x-12$

$3x+12$

0

$(x-4)(6x^3+6x+3)$

$(x-4)(2x+1)(3x+1)$

x	4
$2x$	-1
$3x$	-1
6	3

$6x^3+3x+2x+3$

$3x(2x+1)+(2x+1)$

$(3x+1)(2x+1)$

b) $4 > -5$ $\quad c > 5$

$c(5x^2-2x(x^2-3x(5x)+15+0$

$5x^2+2x^2-3x$

$x-5$ $\quad 7x^2-2x-3x+15$

$7x^2-3x$

$-3x+15$

$-3x+15$

0

$(x-5)(7x+3)/(x+3)$

x	5
$7x$	-3
x	-3

3

$3x^2+2x-3$

$3x(2x+3)-(2x+3)$

$(3x-1)(2x+3)$

6. $(x+2) < 1$ $\quad c > 1$

$(-3)^2-(-2x-3)^2-(-3)+2 < 0$

$-27-2x-4x < 0$

$-27-2x-4x-2 < 0$

$-3x-2 < 0$

$x > -1/3$

1^2-5x^2+4x+4

$-4x+4x-6$

$-2x+2$

$6-2 < -1/3-2+2$

7. $(x+2)$ $\quad c < 3$ $\quad c > 1$

$(-3)^2-(-2x-3)^2-(-3)+2 < 0$

$-27-2x-4x < 0$

$-27-2x-4x-2 < 0$

$-3x-2 < 0$

$x > -1/3$

1^2-5x^2+4x+4

$-4x+4x-6$

$-2x+2$

$6-2 < -1/3-2+2$

8. x^2-5x^2+2x-3

x^2-2x-1 $\quad \frac{(x^2-5x^2+2x-3)}{x^2-2x-1}$

x^2-2x-1

$-3x^2+3x-3$

$-3x^2+6x+3$

0

Remainder: $3x+6$

Quot: $x-3$

9. $x-3$ $\quad c > 3$

$f(3)=0$

$(3)^4-(-7)^3+2(3)-3 = 0$

$81+343+6-3 = 0$

$424-336 = 0$

$88 = 0$

10. a) x^2-1 $\quad (x^2-1)(x^2+2x-3)$ set $= 6x^4+8x^3+2x^2-4x^2-2x-2$

$6x^4+8x^3-2x^2-4x-2 = x^4+2x^3-3x^2-x^2-2x-3$

$0 = x^4-1$ $\quad 6x^4+4x-3$

$1 = x^4$ $\quad 6x^4+4x-3$

$x^4 = 1$ $\quad 2x^2+2x-3$

x^4-1

$(x-1)(x^2+1)$

b) $(x^2-1)(x^2+2x-3)$ set $= 6x^4+8x^3+2x^2-4x^2-2x-2$

$6x^4+8x^3-2x^2-4x-2 = x^4+2x^3-3x^2-x^2-2x-3$

$0 = x^4-1$ $\quad 6x^4+4x-3$

$1 = x^4$ $\quad 6x^4+4x-3$

$x^4 = 1$ $\quad 2x^2+2x-3$

x^4-1

$(x-1)(x^2+1)$

11. a) $f(-2)=0$

$2(-2)^3+(-2)^2-3(-2)+2 = 0$

$-16+4+6+2 = 0$

$-6+6 = 0$

$0 = 0$

b) $(x+2)$ $\quad c > -2$

$f(-2)=0$

$2(-2)^3+(-2)^2-3(-2)+2 = 0$

$-16+4+6+2 = 0$

$-6+6 = 0$

$0 = 0$

12. a) $(x-2)$ $\quad c > 2$

$f(2) = 0$

$2(2)^3+2(2)^2-3(2)+2 = 0$

$16+8-6+2 = 0$

$16+2 = 0$

$18 = 0$

b) x^2+2x-3 $\quad \frac{2x^3+2x^2-3x+2}{x^2+2x-3}$

x^2+2x-3

$2x^3+2x^2-3x+2$

$2x^3+2x^2-3x$

0

$2x^3+2x^2-3x$

$3x(x^2+2x-1)$

$(3x-1)(x+3)$

x	2
$2x$	1
x	3
6	6

$(3x-1)(x+3)(x-2)$

13. a) $x-2$ $\quad c > 2$

$f(2) = \text{remainder}$

$2(2)^3+3(2)^2-11(2)+33$

$\frac{2x^3+3x^2-11x+33}{x-2}$

$16+12+22+33 = 83$

$x^3+3x^2-11x+33$

x^3-2x^2

$5x^2-11x+33$

$5x^2-10x$

$-1x+33$

$-1x+2$

31

Rem: $-1x+35$

Quot: $2x^2+5$

14. a) $x-3$ $\quad c > 3$ $\quad x \neq 1$ $\quad c > -1$

$x(5)^3+(6)(5)^2-3+12 = 0$

$x(-2)^3+6(-2)^2-(-6)+12 = 0$

$2x-6+3+5 = 0$ $\quad 2^3/(6+13)+6+5 = 16$

$-8+6+13 = 0$ $\quad 3x6+3+5 = 0$

$x = 6+13$ $\quad 3x6+3+5 = 0$

$x = -10/13 = 3$ $\quad 3x6+3+5 = 0$

$x = 3$

b) $(x-3)(x+1)$

x^2-2x-3 $\quad \frac{2x^3+6x^2-3x+12}{x^2-2x-3}$

x^2-2x-3

$-2x^2+5x+12$

$-2x^2+4x+6$

$9x+18$

$9x+18$

0

$(x-3)(2x+1)(x+3)=0$

$x=3$ $\quad 2x+1=0$ $\quad 2x=-1$

$x=3$ $\quad 2x=-1$

$x=3$ $\quad x=-\frac{1}{2}$ $\quad x=-\frac{3}{2}$

15. a) $2(-2)^3+(-2)^2+6(-2)+10 = 0$

$-16+4-12+10 = 0$

$-2x-2 = 0$

$x = -1$

b) $2x^2+2x+10$

$f(x)=0$ $\quad c < 1$ $\quad (x-2)$

$x=2$ $\quad \frac{2x^3+2x^2+6x+10}{x-2}$

$6x^2+12x+10$

$2x^3-4x$

$2x^2+6x+10$

$2x^2-4x$

$10x+10$

$10x-20$

30

$2x^2+2x+10$

$2(x^2+x+5)$

$2(x+2)(x+1)$

$x+2=0$ $\quad 2x+1=0$

$x=-2$ $\quad 2x=-1$

$x=-2$ $\quad x=-\frac{1}{2}$

16. a) $2(-2)^3+4(-2)^2+6(-2)+6=0$

$2(3)^3+4(3)^2+6(3)+6=0$

$-16+16-12+6=0$

$-10+6=-2+6$

$-4=-4$

$54+36+18+6=0$

$54+36=-18-6$

$90=-24$

b) $2x^2+4x-6$

$f(x)=0$

$x=2$ $\quad \frac{2x^3+4x^2-6x+6}{x-2}$

$6x^2+12x+6$

$2x^3-4x^2$

$6x^2-6x$

$6x^2-12x$

$6x+6$

$6x-12$

18

$2x^2+4x-6$

$2(x^2+2x-3)$

$2(x+3)(x-1)$

$x+3=0$ $\quad x-1=0$

$x=-3$ $\quad x=1$

17. a) $2x^2+3x-6=(2x^2-4x+6)$

$2x^2+3x-6=2x^2-4x+6$

$7x=12$

$x=12/7$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

18. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

19. a) $x-3$ $\quad c > 3$ $\quad x \neq 1$ $\quad c > -1$

$x(5)^3+(6)(5)^2-3+12 = 0$

$x(-2)^3+6(-2)^2-(-6)+12 = 0$

$2x-6+3+5 = 0$ $\quad 2^3/(6+13)+6+5 = 16$

$-8+6+13 = 0$ $\quad 3x6+3+5 = 0$

$x = 6+13$ $\quad 3x6+3+5 = 0$

$x = -10/13 = 3$ $\quad 3x6+3+5 = 0$

$x = 3$

b) $(x-3)(x+1)$

x^2-2x-3 $\quad \frac{2x^3+6x^2-3x+12}{x^2-2x-3}$

x^2-2x-3

$-2x^2+5x+12$

$-2x^2+4x+6$

$9x+18$

$9x+18$

0

$2x^2+4x-6$

$2(x^2+2x-3)$

$2(x+3)(x-1)$

$x+3=0$ $\quad x-1=0$

$x=-3$ $\quad x=1$

20. a) $2(-2)^3+4(-2)^2+6(-2)+6=0$

$-16+16-12+6=0$

$-10+6=-2+6$

$-4=-4$

$54+36+18+6=0$

$54+36=-18-6$

$90=-24$

b) $2x^2+4x-6$

$f(x)=0$

$x=2$ $\quad \frac{2x^3+4x^2-6x+6}{x-2}$

$6x^2+12x+6$

$2x^3-4x^2$

$6x^2-6x$

$6x^2-12x$

$6x+6$

$6x-12$

18

$2x^2+4x-6$

$2(x^2+2x-3)$

$2(x+3)(x-1)$

$x+3=0$ $\quad x-1=0$

$x=-3$ $\quad x=1$

21. a) $x^2-2x^2-4x-(x^2-4x+4)$

$x^2-2x^2-5=x^2-4x+4$

$2x=9$

$x=9/2$

b) $x-3$ $\quad c > 3$

$f(3)=0$

$(3)^4-(-7)^3+2(3)-3 = 0$

$81+343+6-3 = 0$

$424-336 = 0$

$88 = 0$

Rem: -3

Quot: $x^3+3x^2+9x+27$

22. a) x^2-1 $\quad (x^2-1)(x^2+2x-3)$ set $= 6x^4+8x^3+2x^2-4x^2-2x-2$

$6x^4+8x^3-2x^2-4x-2 = x^4+2x^3-3x^2-x^2-2x-3$

$0 = x^4-1$ $\quad 6x^4+4x-3$

$1 = x^4$ $\quad 6x^4+4x-3$

$x^4 = 1$ $\quad 2x^2+2x-3$

x^4-1

$(x-1)(x^2+1)$

b) $(x^2-1)(x^2+2x-3)$ set $= 6x^4+8x^3+2x^2-4x^2-2x-2$

$6x^4+8x^3-2x^2-4x-2 = x^4+2x^3-3x^2-x^2-2x-3$

$0 = x^4-1$ $\quad 6x^4+4x-3$

$1 = x^4$ $\quad 6x^4+4x-3$

$x^4 = 1$ $\quad 2x^2+2x-3$

x^4-1

$(x-1)(x^2+1)$

23. a) $x-2$ $\quad c > 2$

$W(-2)^3+(-2)^2-(-2)+13 = 0$

$-8+4+2+13 = 0$

$-4+15 = 0$

$11 = 0$

b) $\frac{x^3+x^2-x+13}{x-2}$

x^3+x^2-x+13

x^3-2x^2

$3x^2-x+13$

$3x^2-6x$

$5x+13$

$5x-10$

23

x^3+x^2-x+13

x^3+x^2-x+13

$0+13$

13

Rem: 13

Quot: x^2+3x+5

24. a) x^2-3 $\quad c > 3$ $\quad x \neq 1$ $\quad c > -1$

$x(5)^3+(6)(5)^2-3+12 = 0$

$x(-2)^3+6(-2)^2-(-6)+12 = 0$

$2x-6+3+5 = 0$ $\quad 2^3/(6+13)+6+5 = 16$

$-8+6+13 = 0$ $\quad 3x6+3+5 = 0$

$x = 6+13$ $\quad 3x6+3+5 = 0$

$x = -10/13 = 3$ $\quad 3x6+3+5 = 0$

$x = 3$

b) $(x-3)(x+1)$

x^2-2x-3 $\quad \frac{2x^3+6x^2-3x+12}{x^2-2x-3}$

x^2-2x-3

$-2x^2+5x+12$

$-2x^2+4x+6$

$9x+18$

$9x+18$

0

$2x^2+4x-6$

$2(x^2+2x-3)$

$2(x+3)(x-1)$

$x+3=0$ $\quad x-1=0$

$x=-3$ $\quad x=1$

25. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

26. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

27. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

28. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

29. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

30. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

31. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

32. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

33. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

34. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

35. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

36. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

37. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

38. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

39. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

40. a) $\frac{2x^3+2x^2+6x}{x+1}$ $\quad \frac{2x^3+2x^2+6x+17}{x+1}$

$2x^3+2x^2+6x$

$2x^3+2x^2+6x$

0

$2x^3+2x^2+6x+17$

$2x^3+2x^2+6x+17$

$0+17$

17

Rem: 17

Quot: $2x^2+6$

b) $x^2-2x+1-6=(x^2-4x+4)$

$x^2-2x-5=x^2-4x+4$

$2x=9$

$x=9/2$

1.2 Logarythms

=> a logarythmic function is the inverse of an exponential function

Basic Log Rules

$y = a^x \overset{*}{\Longleftrightarrow} x = \log_a y$

* '$\Longleftrightarrow$' means logically identical

$\log_a x = \log_a y \Longleftrightarrow x = y$ logs cancel

$y = a^x$ inverse of $y = \log_a x$

- usually log is base 10, but 'a' can be any of the following:

$a > 0$, $a \neq 1$
$x > 0$

Log of Zero Rule:

$\log_a 1 = 0$ always $\rightarrow a^0 = 1$

Negative Log Results:

$\log_4 16 = 2$ then $\log_4 \frac{1}{16} = -2 \rightarrow 4^{-2} = \frac{1}{4^2}$

↳ if $0 < \text{value} < 1$: result is negative

Fraction Log Results:

$\log_{216} \frac{1}{6} = -\frac{1}{3}$

↳ when the base (216) is larger then what's inside the log, the result is a fraction

↳ value inside the log is a fraction, so resut is negative

Change Log Base

$\log_a b = \frac{\log_c b}{\log_c a}$ where c can be any value

Log With same Base and Value

$\log_a a = 1$ allways $\rightarrow a^1 = a$

Log as Exponent

$\log_a a^x = x$ $\quad a^{\log_a x} = x \rightarrow \cancel{a}^{\log_{\cancel{a}} x}$

↳ bases cancel

Laws of Logarithms

Multiplication Law

$\log_a (xy) = \log_a x + \log_a y$

Division Law

$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

Power Law

$\log_a (x^m) = m \log_a x$ → it's x^2 not $(\log(\,))^2$

Law of Inverse (my name)

$\log_a \left(\frac{1}{x}\right) = -\log_a x$

Natural Logarithms

=> a type of logarithm where the base is e ($e \approx 2.718$)

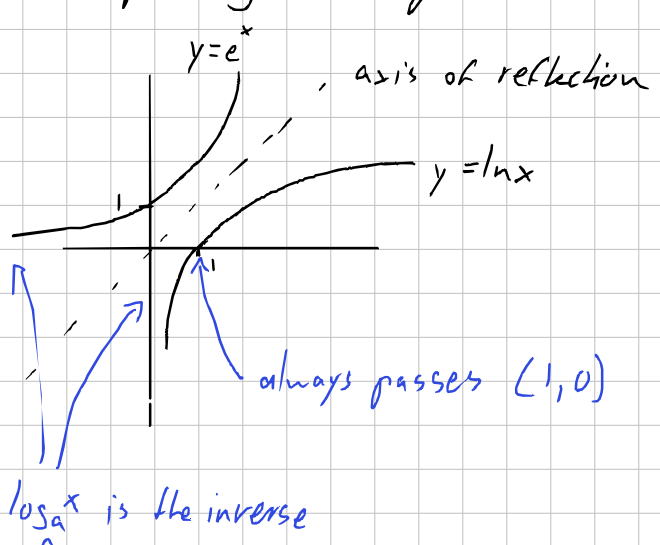
$y = e^x$ is a natural exponential function

$\log_e y = x \Leftrightarrow \ln y = x$

'ln' stands for natural logarithmic function

$y = e^x \Longleftrightarrow x = \ln y$

Graphing example



$\log_a x$ is the inverse of a^x

Transforming Exponential to Linear

=> convert relations of $y = ka^x$ or of $y = kx^n$ to a linear function using logarithms

- this works by changing both axis to functions, in this case to $\ln y$ and $\ln x$
- logs of any base can be used

Process:
- Transform the function to $Y = mX + b$ where X and Y are functions (e.g $\ln x$, $\ln y$)
- Then mark the axis X and Y and replace X and Y in the function with the actual x and y

$y = ka^x \quad | \ln$

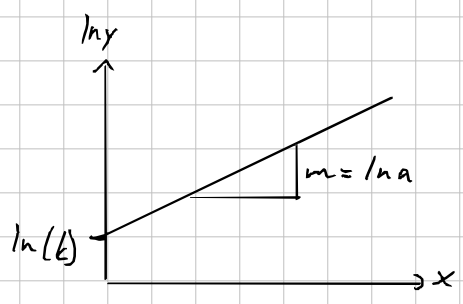
$\ln y = \ln(ka^x)$

$\ln y = \ln k + \ln a^x$

$\ln y = \ln(a)x + \ln k$

$Y = m \cdot X + b$

- X and Y must only contain x and y with no other constants
  - ↳ no a, k, n, etc.
  - ↳ logs are ok (e.g $\log_e(\,)$ ← constant



End Of Chapter Review

1. $\frac{1}{2}^{-2} \div 2^{-3}$ $\quad \log_2(\frac{1}{2})$
$\log_2 2^{-2} \div \log_2 2^{-3}$
$-2 \div -3 = \frac{2}{3}$

2. $\log_3 x + \log_3 y = \log_3(xy)$ $\quad$ $y = x^2$
$\log_3(xy) = \log_3(x^2)$
$\log_3(\frac{x}{y}) = \log_3(\frac{1}{x})$
$\log_3(xy) - \log_3(\frac{x}{y}) = 0$
$\log_3(\frac{x^2}{x}) = 1$
$\log_3(x) = 1$
$x = 3$

3. $\log_2(x) - \log_2(x+2) = 3$
$\log_2(\frac{x}{x+2}) = 3$
$\frac{x}{x+2} = 2^3$
$x = 8(x+2)$
$x = 8x + 16$
$-7x = 16$
$x = -\frac{16}{7}$

4. $x^{\log_3 x} = 3^{\log_3 x^2}$
$\log_3(x^{\log_3 x}) = \log_3(3^{\log_3 x^2})$
$\log_3 x \cdot \log_3 x = \log_3 x^2 \cdot \log_3 3$
$\log_3 x \cdot \log_3 x = \log_3 x^2 \cdot \log_3 3$
$\log_3 x \cdot \log_3 x = 2\log_3 x \cdot \log_3 3$
$\log_3 x = 2\log_3 3$
$x = 3^{2\log_3 3} = 3^{\log_3 9}$
$x = 9$

5. $\log_2(x) - \log_2(x-1) = \log_2 x$
$\log_2(\frac{x}{x-1}) = \log_2(x)$
$\frac{x}{x-1} = x$
$x = x(x-1)$
$x = x^2 - x$
$2x = x^2$
$x = 2$ or $x = 0$

6. $\ln(7x+5) - \ln(3x) = \ln 2$
$\ln(\frac{7x+5}{3x}) = \ln 2$
$\frac{7x+5}{3x} = 2$
$7x + 5 = 6x$
$x = -5$

7. $\log_2(x+5) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+5}{x-3}) = \log_2(x)$
$\frac{x+5}{x-3} = x$
$x + 5 = x^2 - 3x$
$0 = x^2 - 4x - 5$
$0 = (x-5)(x+1)$
$x = 5$ or $x = -1$
$x = 5$ is the only sol.
$\{x \mid x = 5\}$

8. $y = 8x^2$
$\log_2 y = \log_2(8x^2)$
$\log_2 y = \log_2 8 + \log_2 x^2$
$\log_2 y = \log_2 8 + 2\log_2 x$
$\log_2 y = 3 + 2\log_2 x$
$\log_2 y - 2\log_2 x = 3$
$\log_2(\frac{y}{x^2}) = 3$
$\frac{y}{x^2} = 2^3$
$\frac{y}{x^2} = 8$
$y = 8x^2$

9. $y = 2x^3$
$\log_2 y = \log_2(2x^3)$
$\log_2 y = \log_2 2 + \log_2 x^3$
$\log_2 y = \log_2 2 + 3\log_2 x$
$\log_2 y - 3\log_2 x = \log_2 2$
$\log_2(\frac{y}{x^3}) = \log_2 2$
$\frac{y}{x^3} = 2$
$y = 2x^3$

10. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$y = \frac{8}{x}$

11. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

12. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

13. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

14. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

15. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

16. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

17. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

18. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

19. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

20. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

21. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

22. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
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$x = \frac{8}{y}$

23. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

24. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

25. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

26. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

27. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

28. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
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$x = \frac{8}{y}$

29. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

30. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
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$x = \frac{8}{y}$

31. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
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32. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
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$\log_2(\frac{x}{y}) = 1$
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$\frac{x}{y} = 2$
$x = 2y$

34. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
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$xy = 8$
$x = \frac{8}{y}$

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$\log_2(\frac{x}{y}) = 1$
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$\log_2(xy) = 3$
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$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

38. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

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40. $\log_2 x + \log_2 y = 3$
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$x = \frac{8}{y}$

47. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

48. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

49. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
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$\frac{x}{y} = 2$
$x = 2y$

50. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

51. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

53. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
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$\frac{x}{y} = 2$
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$xy = 8$
$x = \frac{8}{y}$

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$\frac{x}{y} = 2$
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$x = \frac{8}{y}$

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$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
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$x = \frac{8}{y}$

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$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

61. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

62. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

63. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

64. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

65. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

66. $\log_2 x + \log_2 y = 3$
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$\log_2(\frac{x}{y}) = 1$
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$\frac{x}{y} = 2$
$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
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$x = \frac{8}{y}$

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$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
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$x = \frac{8}{y}$

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$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

73. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
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$x = \frac{8}{y}$

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$\log_2(\frac{x}{y}) = 1$
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$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

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$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
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$x = 2y$

78. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
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$\log_2(\frac{x}{y}) = 1$
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$x = 2y$

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$\log_2(xy) = 3$
$xy = 2^3$
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$x = \frac{8}{y}$

81. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

82. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

83. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

84. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

85. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

86. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

87. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

88. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

89. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

90. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

91. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

92. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

93. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

94. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

95. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

96. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

97. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

98. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

99. $\log_2 x - \log_2 y = 1$
$\log_2(\frac{x}{y}) = 1$
$\frac{x}{y} = 2^1$
$\frac{x}{y} = 2$
$x = 2y$

100. $\log_2 x + \log_2 y = 3$
$\log_2(xy) = 3$
$xy = 2^3$
$xy = 8$
$x = \frac{8}{y}$

Unit 1: Math Gen

9.0

1. $\ln(7x+4) + \ln(5x+4)$
$7x + 4 > 0$
$x > -\frac{4}{7}$
$5x + 4 > 0$
$x > -\frac{4}{5}$
$-\frac{4}{7} \le x \le 1$

2a. $\log_3(x+1) + \log_3(x-1) = 2$
$\log_3(x^2 - 1) = 2$
$x^2 - 1 = 3^2$
$x^2 - 1 = 9$
$x^2 = 10$
$x = \pm\sqrt{10}$
$x = \sqrt{10}$ $\quad$ $x = -\sqrt{10}$ ✗

2b. $\log_3(x+1) + \log_3(x-1) = 2$
$\log_3(x^2 - 1) = 2$
$x^2 - 1 = 3^2$
$x^2 - 1 = 9$
$x^2 = 10$
$x = \pm\sqrt{10}$
$x = \sqrt{10}$ $\quad$ $x = -\sqrt{10}$ ✗

3. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

4a. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

4b. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

5. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

6. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

7. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

8. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

9. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
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10. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

11. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

12. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

13. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
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14. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

15. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
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16. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

17. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
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18. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

19. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

20. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

21. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

22. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

23. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
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24. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

25. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

26. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

27. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
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28. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

29. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
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$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

31. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
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32. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

33. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
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34. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

35. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

36. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

37. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

38. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

39. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

40. $\log_2(x+3) + \log_2(x-3) = 3$
$\log_2(x^2 - 9) = 3$
$x^2 - 9 = 2^3$
$x^2 - 9 = 8$
$x^2 = 17$
$x = \pm\sqrt{17}$
$x = \sqrt{17}$ $\quad$ $x = -\sqrt{17}$ ✗

41. $\log_2(x+3) - \log_2(x-3) = \log_2 x$
$\log_2(\frac{x+3}{x-3}) = \log_2(x)$
$\frac{x+3}{x-3} = x$
$x + 3 = x^2 - 3x$
$0 = x^2 - 4x - 3$
$x = \frac{4 \pm \sqrt{16 + 12}}{2}$
$x = \frac{4 \pm \sqrt{28}}{2}$
$x = 2 \pm \sqrt{7}$
$x = 2 + \sqrt{7}$ $\quad$ $x = 2 - \sqrt{7}$ ✗

42. $\log_2(x+3) + \log_2(x-3) = 3$
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Differentiation

Exponential Functions

$$\frac{d}{dx} e^{x} = e^{x}$$

$$\frac{d}{dx}\left[e^{ax+b}\right] = ae^{ax+b}$$

$$\frac{d}{dx} e^{f(x)} = f'(x) \cdot e^{f(x)}$$

Natural logarithms:

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\frac{d}{dx}(\ln(ax+b)) = \frac{a}{ax+b}$$

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

Trigonometric functions

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$\frac{d}{dx}(\sin(f(x)))$ -> chain rule: $\cos(f(x)) \cdot f'(x)$

Teacher Teacher Teacher

$\sqrt{\cos x - \sin x} = \cos(2x) - \sin(2x)$

$\sqrt{\cos x - \sin x} = \cos^2 x - \sin^2 x - 2\sin x\cos x$

$\sqrt{\cos x - \sin x} = \cos^2 x - \sin^2 x$

$\sqrt{\cos x - \sin x} = \sqrt{\cos x} + \sqrt{\sin x}$

$\sqrt{\cos x} - \sin x = \sqrt{\cos x}$

$\sqrt{\cos x} - \sin x = \sqrt{\cos x} + \sqrt{\sin

Cometic Hurdles

$\sin(x+y) = \sin x\cos y + \cos x \sin y$
$\sin(x-y) = \sin x\cos y - \cos x \sin y$
$\cos(x+y) = \cos x\cos y - \sin x \sin y$
$\cos(x-y) = \cos x\cos y + \sin x \sin y$
$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$
$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

Compound angle formulas

Double Angle formulas

$\sin 2x = 2\sin x \cos x$
$\cos 2x = \cos^2 x - \sin^2 x$
$= 2\cos^2 x - 1$
$= 1 - 2\sin^2 x$
$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$

Half Angle formulas

Find all values of x for $0 \le x \le 2\pi$

1) $\frac{\sqrt{3}}{2} = \sin x$
$x = \frac{\pi}{3}, \frac{2\pi}{3}$

2) $\frac{\sqrt{3}}{2} = \cos x$
$x = \frac{\pi}{6}, \frac{11\pi}{6}$

3) $\sqrt{3} = \tan x$
$x = \frac{\pi}{3}, \frac{4\pi}{3}$

4) $\sin x = \frac{1}{2}$
$x = \frac{\pi}{6}, \frac{5\pi}{6}$

5) $\cos x = \frac{1}{2}$
$x = \frac{\pi}{3}, \frac{5\pi}{3}$

6) $\tan x = 1$
$x = \frac{\pi}{4}, \frac{5\pi}{4}$

7) $\sin(2x) = \frac{\sqrt{3}}{2}$
$2x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$
$x = \frac{\pi}{6}, \frac{\pi}{3}, \frac{7\pi}{6}, \frac{4\pi}{3}$

8) $\cos(2x) = \frac{1}{2}$
$2x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{11\pi}{3}$
$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

9) $\tan(2x) = 1$
$2x = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$
$x = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

10) $\sin x = \cos x$
$\tan x = 1$
$x = \frac{\pi}{4}, \frac{5\pi}{4}$

11) $\sin x = \cos x$
$\tan x = 1$
$x = \frac{\pi}{4}, \frac{5\pi}{4}$

12) $\sin x = \tan x$
$\sin x = \frac{\sin x}{\cos x}$
$\sin x \cos x = \sin x$
$\sin x \cos x - \sin x = 0$
$\sin x(\cos x - 1) = 0$
$\sin x = 0 \quad \cos x = 1$
$x = 0, \pi, 2\pi$

13) $\sin(2x) = \cos x$
$2\sin x \cos x = \cos x$
$2\sin x \cos x - \cos x = 0$
$\cos x(2\sin x - 1) = 0$
$\cos x = 0 \quad 2\sin x = 1$
$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \sin x = \frac{1}{2}$
$x = \frac{\pi}{6}, \frac{5\pi}{6}$

14) $\cos(2x) = \sin x$
$1 - 2\sin^2 x = \sin x$
$2\sin^2 x + \sin x - 1 = 0$
$(2\sin x - 1)(\sin x + 1) = 0$
$\sin x = \frac{1}{2} \quad \sin x = -1$
$x = \frac{\pi}{6}, \frac{5\pi}{6} \quad x = \frac{3\pi}{2}$

15) $\sin(2x) = \cos x$
$2\sin x \cos x = \cos x$
$2\sin x \cos x - \cos x = 0$
$\cos x(2\sin x - 1) = 0$
$\cos x = 0 \quad 2\sin x = 1$
$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \sin x = \frac{1}{2}$
$x = \frac{\pi}{6}, \frac{5\pi}{6}$

16) $\sin(2x) = \sin x$
$2\sin x \cos x = \sin x$
$2\sin x \cos x - \sin x = 0$
$\sin x(2\cos x - 1) = 0$
$\sin x = 0 \quad 2\cos x = 1$
$x = 0, \pi, 2\pi \quad \cos x = \frac{1}{2}$
$x = \frac{\pi}{3}, \frac{5\pi}{3}$

17) $\cos(2x) = \cos x$
$2\cos^2 x - 1 = \cos x$
$2\cos^2 x - \cos x - 1 = 0$
$(2\cos x + 1)(\cos x - 1) = 0$
$\cos x = -\frac{1}{2} \quad \cos x = 1$
$x = \frac{2\pi}{3}, \frac{4\pi}{3} \quad x = 0, 2\pi$

18) $\sin(2x) = \sin(x+\pi)$
$2\sin x \cos x = \sin x \cos \pi + \cos x \sin \pi$
$2\sin x \cos x = -\sin x$
$2\sin x \cos x + \sin x = 0$
$\sin x(2\cos x + 1) = 0$
$\sin x = 0 \quad 2\cos x = -1$
$x = 0, \pi, 2\pi \quad \cos x = -\frac{1}{2}$
$x = \frac{2\pi}{3}, \frac{4\pi}{3}$

19) $\sin(2x) = \cos(x+\pi)$
$2\sin x \cos x = \cos x \cos \pi - \sin x \sin \pi$
$2\sin x \cos x = -\cos x$
$2\sin x \cos x + \cos x = 0$
$\cos x(2\sin x + 1) = 0$
$\cos x = 0 \quad 2\sin x = -1$
$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \sin x = -\frac{1}{2}$
$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

20) $\sin(2x) = \cos(x+\pi)$
$2\sin x \cos x = \cos x \cos \pi - \sin x \sin \pi$
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$2\sin x \cos x = \cos x \cos \pi - \sin x \sin \pi$
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$2\sin x \cos x = \cos x \cos \pi - \sin x \sin \pi$
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$\cos x = 0 \quad 2\sin x = -1$
$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \sin x = -\frac{1}{2}$
$x = \frac{7\pi}{6}, \frac{11\pi}{6}$

26) $\sin(2x) = \cos(x+\pi)$
$2\sin x \cos x = \cos x \cos \pi - \sin x \sin \pi$
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$2\sin x \cos x = \cos x \cos \pi - \sin x \sin \pi$
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$\cos x = 0 \quad 2\sin x = -1$
$x = \frac{\pi}{2}, \frac{3\pi}{2} \quad \sin x = -\frac{1}{2}$
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$2\sin x \cos x = \cos x \cos \pi - \sin x \sin \pi$
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3.1 Further Integration

<u>Integration Formulae</u>: (Trig edition)

$$\int e^x \, dx = e^x + c \qquad \int e^{f(x)} \, dx = \frac{1}{f'(x)} e^{f(x)} + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$ → $\ln x$ is only defined for $x > 0$, therefore the absolute value is taken

$$\int \frac{1}{f(x)} \, dx = \frac{1}{f'(x)} \ln|f(x)| + c \qquad \int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)|$$

$$\int \cos x \, dx = \sin x + c \qquad \int \cos(f(x)) \, dx = \frac{1}{f'(x)} \sin(f(x)) + c$$

$$\int \sin x \, dx = -\cos x + c \qquad \int \sin(f(x)) \, dx = -\frac{1}{f'(x)} \cos(f(x)) + c$$

$$\int \sec^2 x \, dx = \tan x + c \qquad \int \sec^2(f(x)) \, dx = \frac{1}{f'(x)} \tan(f(x)) + c$$

<u>Trapezium Rule</u>

⇒ Used to estimate area under any curve, provided the function is known

$$\int_a^b f(x) \, dx = \frac{h}{2}\left[y_0 + y_n + 2(y_1 + y_2 \ldots + y_{n-1})\right]$$

Trapez Formula: $A = \frac{a+b}{2} \cdot h$

so $A_{total} = \frac{h}{2}(a + b + b + c + c + d \ldots + y + z)$

$A_{total} = \frac{h}{2}\left[a + z + 2(b + c + \ldots y)\right]$

Review

Q1

1.a) $\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

b) $\int_0^1 x^3\,dx = \left[\frac{1}{4}x^4\right]_0^1 = \frac{1}{4}$

c) $\int_0^1 x^4\,dx = \left[\frac{1}{5}x^5\right]_0^1 = \frac{1}{5}$

d) $\int_0^1 x^5\,dx = \left[\frac{1}{6}x^6\right]_0^1 = \frac{1}{6}$

2.) $\int_0^1 (x^2+x^3)\,dx = \int_0^1 x^2\,dx + \int_0^1 x^3\,dx = \frac{1}{3}+\frac{1}{4} = \frac{7}{12}$

3.) $\int_0^1 (x^2+x^3+x^4)\,dx = \int_0^1 x^2\,dx + \int_0^1 x^3\,dx + \int_0^1 x^4\,dx = \frac{1}{3}+\frac{1}{4}+\frac{1}{5} = \frac{47}{60}$

4.) $\int_0^1 (x^2+x^3+x^4+x^5)\,dx = \int_0^1 x^2\,dx + \int_0^1 x^3\,dx + \int_0^1 x^4\,dx + \int_0^1 x^5\,dx = \frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6} = \frac{37}{60}$

5.) $\int_0^1 \frac{(x^2+x^3+x^4+x^5)}{x^2}\,dx = \int_0^1 \frac{x^2+x^3+x^4+x^5}{x^2}\,dx = \int_0^1 (1+x+x^2+x^3)\,dx = \int_0^1 1\,dx + \int_0^1 x\,dx + \int_0^1 x^2\,dx + \int_0^1 x^3\,dx$

$= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$

Q2

a) $\int_0^1 \frac{x^2}{x^2+1}\,dx = \int_0^1 \left(1 - \frac{1}{x^2+1}\right)dx = \int_0^1 1\,dx - \int_0^1 \frac{1}{x^2+1}\,dx$

$= 1 - \left[\tan^{-1}x\right]_0^1 = 1 - \tan^{-1}(1) - \tan^{-1}(0) = 1 - \frac{\pi}{4} + 0 = 1 - \frac{\pi}{4}$

b) $\int_0^1 \frac{x^3}{x^2+1}\,dx = \int_0^1 \left(x - \frac{x}{x^2+1}\right)dx = \int_0^1 x\,dx - \int_0^1 \frac{x}{x^2+1}\,dx$

$= \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 = \frac{1}{2} - \frac{1}{2}\ln(2) + \frac{1}{2}\ln(1)$

$= \frac{1}{2} - \frac{1}{2}\ln(2)$

Q3

1. $\int_0^1 \frac{1}{(1+x)^2}\,dx = \int_0^1 (1+x)^{-2}\,dx = \left[\frac{(1+x)^{-1}}{-1}\right]_0^1 = \left[-\frac{1}{1+x}\right]_0^1 = -\frac{1}{2} + 1 = \frac{1}{2}$

2. $\int_0^1 \frac{1}{(1+x)^3}\,dx = \int_0^1 (1+x)^{-3}\,dx = \left[\frac{(1+x)^{-2}}{-2}\right]_0^1 = \left[-\frac{1}{2(1+x)^2}\right]_0^1 = -\frac{1}{8} + \frac{1}{2} = \frac{3}{8}$

3. $\int_0^1 \frac{1}{(1+x)^4}\,dx = \int_0^1 (1+x)^{-4}\,dx = \left[\frac{(1+x)^{-3}}{-3}\right]_0^1 = \left[-\frac{1}{3(1+x)^3}\right]_0^1 = -\frac{1}{24} + \frac{1}{3} = \frac{7}{24}$

4. $\int_0^1 \frac{1}{(1+x)^5}\,dx = \int_0^1 (1+x)^{-5}\,dx = \left[\frac{(1+x)^{-4}}{-4}\right]_0^1 = \left[-\frac{1}{4(1+x)^4}\right]_0^1 = -\frac{1}{64} + \frac{1}{4} = \frac{15}{64}$

5. $\int_0^1 \frac{1}{(1+x)^6}\,dx = \int_0^1 (1+x)^{-6}\,dx = \left[\frac{(1+x)^{-5}}{-5}\right]_0^1 = \left[-\frac{1}{5(1+x)^5}\right]_0^1 = -\frac{1}{160} + \frac{1}{5} = \frac{31}{160}$

Q4

a) $\int_0^1 \frac{1}{(1+x)^n}\,dx = \int_0^1 (1+x)^{-n}\,dx = \left[\frac{(1+x)^{-n+1}}{-n+1}\right]_0^1 = \left[-\frac{1}{(n-1)(1+x)^{n-1}}\right]_0^1 = -\frac{1}{(n-1)2^{n-1}} + \frac{1}{n-1} = \frac{1}{n-1}\left(1 - \frac{1}{2^{n-1}}\right)$

b) $\int_0^1 \frac{x^3}{(1+x)^n}\,dx = \int_0^1 \left(x - \frac{x}{(1+x)^n}\right)dx = \int_0^1 x\,dx - \int_0^1 \frac{x}{(1+x)^n}\,dx = \left[\frac{x^2}{2}\right]_0^1 - \left[\frac{1}{2}\ln(x^2+1)\right]_0^1 = \frac{1}{2} - \frac{1}{2}\ln(2) + \frac{1}{2}\ln(1) = \frac{1}{2} - \frac{1}{2}\ln(2)$

Q5

a) $\int_0^1 \frac{x^2}{(1+x)^2}\,dx = \int_0^1 \left(1 - \frac{1}{(1+x)^2}\right)dx = \int_0^1 1\,dx - \int_0^1 \frac{1}{(1+x)^2}\,dx = 1 - \left[-\frac{1}{1+x}\right]_0^1 = 1 - \left(-\frac{1}{2} + 1\right) = \frac{1}{2}$

b) $\int_0^1 \frac{x^3}{(1+x)^2}\,dx = \int_0^1 \left(x - \frac{x}{(1+x)^2}\right)dx = \int_0^1 x\,dx - \int_0^1 \frac{x}{(1+x)^2}\,dx = \left[\frac{x^2}{2}\right]_0^1 - \left[-\frac{1}{1+x}\right]_0^1 = \frac{1}{2} - \left(-\frac{1}{2} + 1\right) = \frac{1}{2} - \frac{1}{2} = 0$

Q6

$\int_0^1 \frac{x^2}{(1+x)^3}\,dx = \int_0^1 \left(1 - \frac{1}{(1+x)^3}\right)dx = \int_0^1 1\,dx - \int_0^1 \frac{1}{(1+x)^3}\,dx = 1 - \left[-\frac{1}{2(1+x)^2}\right]_0^1 = 1 - \left(-\frac{1}{8} + \frac{1}{2}\right) = \frac{1}{8}$

$\int_0^1 \frac{x^3}{(1+x)^3}\,dx = \int_0^1 \left(x - \frac{x}{(1+x)^3}\right)dx = \int_0^1 x\,dx - \int_0^1 \frac{x}{(1+x)^3}\,dx = \left[\frac{x^2}{2}\right]_0^1 - \left[-\frac{1}{2(1+x)^2}\right]_0^1 = \frac{1}{2} - \left(-\frac{1}{8} + \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{8} = \frac{3}{8}$

Q7

$\int_0^1 \frac{x^2}{(1+x)^4}\,dx = \int_0^1 \left(1 - \frac{1}{(1+x)^4}\right)dx = \int_0^1 1\,dx - \int_0^1 \frac{1}{(1+x)^4}\,dx = 1 - \left[-\frac{1}{3(1+x)^3}\right]_0^1 = 1 - \left(-\frac{1}{24} + \frac{1}{3}\right) = \frac{1}{24}$

$\int_0^1 \frac{x^3}{(1+x)^4}\,dx = \int_0^1 \left(x - \frac{x}{(1+x)^4}\right)dx = \int_0^1 x\,dx - \int_0^1 \frac{x}{(1+x)^4}\,dx = \left[\frac{x^2}{2}\right]_0^1 - \left[-\frac{1}{3(1+x)^3}\right]_0^1 = \frac{1}{2} - \left(-\frac{1}{24} + \frac{1}{3}\right) = \frac{1}{2} - \frac{1}{24} = \frac{11}{24}$

Q8

a) $\int_0^1 \frac{x^2}{(1+x)^n}\,dx = \int_0^1 \left(1 - \frac{1}{(1+x)^n}\right)dx = \int_0^1 1\,dx - \int_0^1 \frac{1}{(1+x)^n}\,dx = 1 - \left[-\frac{1}{(n-1)(1+x)^{n-1}}\right]_0^1 = 1 - \left(-\frac{1}{(n-1)2^{n-1}} + \frac{1}{n-1}\right) = 1 - \frac{1}{n-1}\left(1 - \frac{1}{2^{n-1}}\right)$

b) $\int_0^1 \frac{x^3}{(1+x)^n}\,dx = \int_0^1 \left(x - \frac{x}{(1+x)^n}\right)dx = \int_0^1 x\,dx - \int_0^1 \frac{x}{(1+x)^n}\,dx = \left[\frac{x^2}{2}\right]_0^1 - \left[-\frac{1}{(n-1)(1+x)^{n-1}}\right]_0^1 = \frac{1}{2} - \left(-\frac{1}{(n-1)2^{n-1}} + \frac{1}{n-1}\right) = \frac{1}{2} - \frac{1}{n-1}\left(1 - \frac{1}{2^{n-1}}\right)$

Exhibit Chapter 85

1. $\int \frac{1}{(1+x)\sqrt{1+x^2}}\,dx$ $\quad (\tan^{-1}(x)^2 - 1)$

$\int \frac{1}{(1+x)\sqrt{1+x^2}}\,dx = \int \frac{1}{(1+x)\sqrt{1+x^2}}\,dx

3.2 Approximate via Iteration

=> Sometimes it's not possible to solve an equation algebraically.

=> Roots / intersections of equations can be approximated through the <u>iterative process</u>

=> The approximated value is denoted 'α'

Iterative Process

1. Define x in terms of a function of x

$$f(x) = 0 \rightarrow x = F(x)$$

e.g. $f(x) = x^5 + x - 1 = 0 \rightarrow x = \sqrt[5]{1-x}$

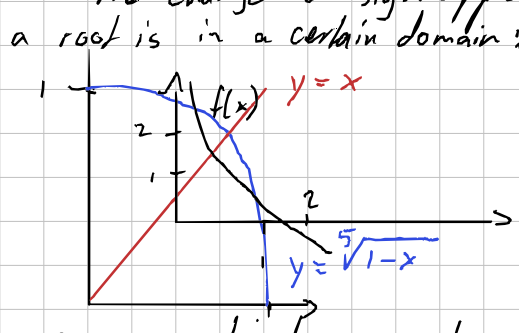
- It's required that $-1 < f'(\alpha) < 1$ where $f'(\alpha)$ is the slope at α
  - ↳ inside this domain the values for α will converge
  - ↳ outside they will diverge (no good solution)

↳ Keep this in mind when defining the function

e.g $x = \sqrt[5]{1-x}$ instead of $x = 1 - x^5$

2. Find a good starting point → x-value close to root

- The approximate value is called 'α'
  - ↳ The known range for the root is $a < \alpha < b$
- use the change of sign approach to determine if a root is in a certain domain:



$f(1) = 0.3$
$f(2) = -0.4$
root between $x = 1$ and $x = 2$

- use a graphical approach where you / your calculator plots the graph ... really?

3. Iterate over the function to get closer to the true value of α:

x_1 = first rough guess

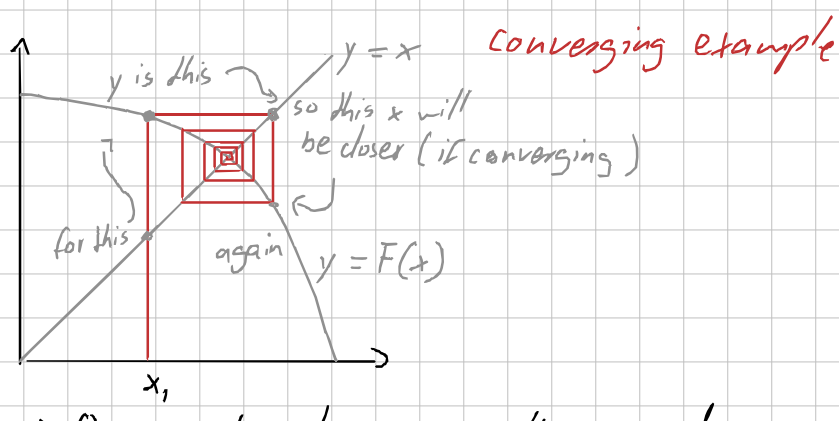
$x_2 = F(x_1)$ $\quad x_{n+1} = F(x_n)$
$x_3 = F(x_2)$ ..

- when the value for x_n shares a certain number of decimal places with x_{n+1} you can stop

e.g. $x_{10} = \underline{7.3215}$ $\quad x_{11} = \underline{7.3227}$
≈ 7.32 $\quad \approx 7.32$ same

Cobweb Diagram

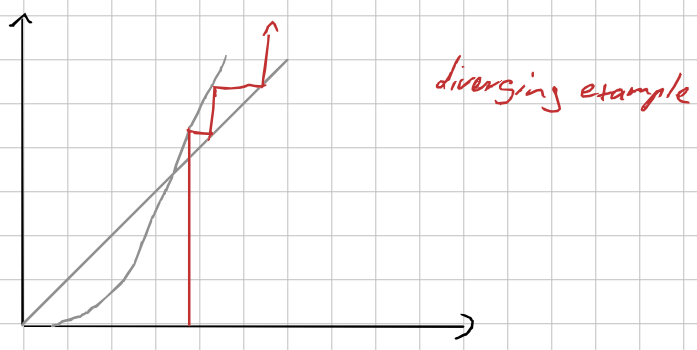
=> A diagram that shows spiral-like convergence / divergence



=> Occurs when slope is negative around α

Staircase Diagram

=> A diagram that converges / diverges in a zig-zag manner



=> Occurs when slope is positive around α

[a mixture of Cobweb & Staircase diagram is possible

Practice Questions

Trig and Graphs Review

1. $x_1 = 1.5 \quad f(x_1) = \frac{5}{2}\left(1 + \frac{1}{1.5}\right)$

$x_2 = 1.5 \quad f(x_2) = \frac{5}{2}\left(1 + \frac{1}{1.5}\right)$

$x_3 = 1.5 \quad f(x_3) = 1.5$

$x_4 = 1.5 \quad f(x_4) = 1.5$

$x_5 = 1.5 \quad f(x_5) = 1.5$

$x = 1.5 \quad \approx .666$

$x = 1.5 \quad \approx .666$

$x = 1.5 \quad \approx .666

Exam Topic 2

1. $y = \frac{x-2}{x+2}$ for $x \neq -2$

$\frac{dy}{dx} = \frac{(x+2)(1) - (1)(x-2)}{(x+2)^2}$ $= \frac{1x+2-1x+2}{(x+2)^2}$

$= \frac{4}{(x+2)^2}$

as $x \to -2$, $(x+2)^2 > 0$, $(x+2)^2 > 0$

and therefore $\frac{4}{(x+2)^2} < 0$ $\therefore$ y is decreasing

2. $y = 5\tan(2x)$

$\frac{dy}{dx} = 5\sec^2(2x) \times 2$ $\quad \sec^2(2x)$

$= 10\sec^2(2x)$

$\sec^2(2x) = 1$

$\cos^2(2x) = \frac{1}{2}$

$\cos(2x) = \pm\frac{1}{\sqrt{2}}$

$2x = \frac{\pi}{4}, \frac{7\pi}{4}$ $\quad 2x = \frac{3\pi}{4}, \frac{5\pi}{4}$

$x = \frac{\pi}{8}, \frac{7\pi}{8}$ $\quad x = \frac{3\pi}{8}, \frac{5\pi}{8}$

$m = 10 \times 1, \frac{1}{2}$ $\quad m = 10 \times -1, -\frac{1}{2}$

$m = \frac{5}{2}, \frac{7}{2}$ $\quad m = -5, -\frac{7}{2}$

3.a. $y = 2\sin(x) + \frac{1}{2}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ $\quad = \frac{0.5y - 2.5}{1} \times \frac{1}{\cos x}$

$= \frac{0.5y}{\cos x} - \frac{2.5}{\cos x}$

$2y' = 2y\sec x - 5\sec x$

$2y' - 2y\sec x = -5\sec x$

$y(2y' - 2\sec x) = -5\sec x$

$y(2y' - 2\sec x) = 0$

$y = 0$ or $2y' - 2\sec x = 0$

$y = 0$ $\quad y' = \sec x$

$y = 0 \quad 2 \times 0 + \frac{1}{2} = 0$

$0 = \frac{1}{2}$ $\quad$ ✗

$y' = \sec x$

$2y' = 2\sec x$

$y' = \sec x$

3.b. $y = 2$

$\frac{dy}{dx} = 0$ $\quad f(x) = \frac{1}{\cos x}$

$f(0) = 0.75$

$f(\frac{\pi}{2}) = 0.6282$

$f(\pi) = 0.6429$

$f(\frac{3\pi}{2}) = 0.7282 \times 0.0282 \times 0.0282$

$f(2\pi) = 0.7$ $\quad$ 3:20 / 3:00

4. $\int \frac{(x^2-1)^2}{x^2} dx = \int (x^2 - 1x^{-2} + 1)^2 x^{-2} dx$

$= \int (1 - 1x^{-2} + 1x^{-2})^2 dx = \int 1 + 1x^{-4} - 2x^{-2} dx$

5. a.

3:00 / 2:10

b. $f(x) = 2x - 6x$

$f(x) = 0.75x$

$f(\frac{1}{2}) = 0.0625$

$f(1) = 0.0625$

$f(\frac{3}{2}) = 0.0625 \times 0.0625 \times 0.0625$

$f(2) = 0.1$ $\quad$ 3:20 / 3:00

6. $y = \frac{1}{2}$

$\frac{dy}{dx} = \frac{1}{2}$ $\quad f(x) = \frac{1}{\cos x}$

$f(0) = 0.75$

$f(\frac{\pi}{2}) = 0.6282$

$f(\pi) = 0.6429$

$f(\frac{3\pi}{2}) = 0.7282 \times 0.0282 \times 0.0282$

$f(2\pi) = 0.7$ $\quad$ 3:20 / 3:00

7. $\frac{dy}{dx} = 3\sin^2(x)\cos(x)$

$\frac{dy}{dx} = \frac{\sin 3x + 3\sin 2x\cos x}{\cos^2 x}$ $\quad \frac{3\sin^2 x\cos x}{\cos^2 x}$

$\frac{dy}{dx} = \frac{\sin^3 x\cos x}{\cos^2 x}$ $\quad \frac{3\sin^2 x\cos x}{\cos^2 x}$ $\quad \frac{\sin^3 x\cos x}{\cos^2 x}$

$\frac{3\sin^2 x}{\cos^2 x}$ $\quad \frac{\sin^3 x}{\cos^2 x}$ $\quad \tan x$

$\tan x$

12:00 / 11:40

8. $m = 3\cos x$

$y - 1.5 = \tan x\left(x - \frac{\pi}{2}\right)$

$y - 1.5 = \tan x - \frac{\tan x}{2}$

$y + 3\tan x = \frac{3\tan x}{2}$

$y + 3\tan x = \frac{3\tan x}{2}$

$y = \tan x - \frac{3\tan x}{2}$

$y = \tan x - \frac{3\sin x}{2\cos x}$

$y = \tan x - \frac{3\sin x}{2\cos x}$

$y = \tan x - \tan x$

12:00 / 2:00

9. $\int \left(\frac{3x^2}{2} + 5\right) dx = 100$

$\left[\frac{3x^3}{2} + 5x\right] = 100$

$\frac{3x^3}{2} + 5x - \left(\frac{3\times 0^3}{2} + 5\times 0\right) = 100$

$\frac{3x^3}{2} + 5x = 100$ $\quad$ $\frac{3x^3}{2} + 5x - 100 = 0$

$x = \sqrt[3]{\frac{2}{3}(100 - 5x)}$ $\quad$ $x = \sqrt[3]{\frac{2}{3}(100 - 5x)}$

$x = \sqrt[3]{\frac{2}{3}(100 - 5\times 3.4)}$

$x = \sqrt[3]{\frac{2}{3}(100 - 17)}$

$x = \sqrt[3]{\frac{2}{3}(100 - 17)}$

$x = \sqrt[3]{\frac{2}{3}(83)}$

$x = \sqrt[3]{\frac{166}{3}}$

$x = 3.4$

3:00 / 2:00

10. $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{1 - \sin^2 x}\, dx$

$= \int_0^{\frac{\pi}{2}} \sqrt{1 - x^2

4.1 Further Algebra

Proper & Improper Fractions

Proper numerical fraction: $\frac{1}{3}$ where numerator < denominator

Improper numerical fraction: $\frac{2}{5}$ or $\frac{5}{5}$ num. ≥ denom.

Proper Algebraic fraction:

$\frac{P(x)}{Q(x)}$ where degree of $P(x) < Q(x)$

* $P(x)$ and $Q(x)$ are polynomials $(ax^b + cx^d + \dots)$

Improper Algebraic fraction:

$\frac{P(x)}{Q(x)}$ where degree of $P(x) \geq Q(x)$

Partial Fraction Splitting

Linear Factors

=> Breaking up proper algebraic fractions when the denominator contains two or more linear factors

$(ax+b)$ -> linear factor $\quad px+q$ -> doesn't have to be linear!

$$\frac{px+q}{(ax+b)(cx+d)} \equiv \frac{A}{ax+b} + \frac{B}{cx+d}$$

=> solve for A and B by equating coefficients. They're constants

$$\frac{px+q}{(ax+b)(cx+d)(ex+f)} = \frac{A}{ax+b} + \frac{B}{cx+d} + \frac{C}{ex+f}$$

=> any number of linear factors

Quadratic Factor

$$\frac{px+q}{(ax+b)^2} \equiv \frac{A}{ax+b} + \frac{B}{(ax+b)^2}$$

Not factorizable Quadratic factor

=> If cx^2+d can't be simplified to $(mx+b)^2$ then:

$$\frac{px+q}{(ax+b)(cx^2+d)} \equiv \frac{A}{ax+b} + \frac{Bx+C}{cx^2+d}$$

Binomial Expansion

Taylors Series

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

for $|x| < 1$

$$(1+x)^n = \sum_{k=0} \binom{n}{k} x^{n-k}$$ only works for positive integers

=> can be used for all real values of n (e.g. $-\frac{3}{4}$)

=> For the series to be converging $|x|$ has to be equal to 1

For $(a+x)^n$ take out the a

$(a+x)^n = a^n\left(1+\frac{x}{a}\right)^n$ then

$$(a+x)^n = a^n\left[1 + n\left(\frac{x}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \dots\right]$$

Practice

1. $(1-2x)^{-4}$

$= 1 + n(-2x) + \frac{n(n-1)}{2!}(-2x)^2 + \frac{n(n-1)(n-2)}{3!}(-2x)^3 \ldots$

$= 1 + 8x + \frac{(-4)(-5)}{2}4x^2 + \frac{(-4)(-5)(-6)}{6}(-8)x^3 \ldots$

$= 1 + 8x + 40x^2 + 160x^3 \ldots$ ✓

2. $(1-6x)^{\frac{1}{3}}$

$= 1 + \frac{1}{3}(-6x) + \frac{\frac{1}{3}(\frac{1}{3}-1)}{2!}(-6x)^2 + \frac{\frac{1}{3}(\frac{1}{3}-1)(\frac{1}{3}-2)}{3!}(-6x)^3$

$= 1 - 2x - 4x^2 - \frac{46}{3}x^3$ ✓

3. $(2-x)(1+2x)^{-\frac{3}{2}}$

$= (2-x)\left(1 + (-\frac{3}{2})(2x) + \frac{(-\frac{3}{2})(-\frac{3}{2}-1)}{2!}(2x)^2\right)$

$= (2-x)(1 - 3x + \frac{15}{2}x^2)$

$= 2 - 6x + 15x^2 - x + 3x^2 - \frac{15}{2}x^3$

$= \cancel{-\frac{15}{2}x^3} + 18x^2 - 7x + 2 \quad = 2 - 7x + 18x^2$ ✓

4. $(1+3x)(1+2x)^{-\frac{1}{2}}$

$= (1+3x)\left(1 + (-\frac{1}{2})(2x) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2!}(2x)^2 + \ldots\right)$

$= (1+3x)(1 - x + 1.5x^2)$

$= 1 - x + \frac{3}{2}x^2 + 3x - 3x^2 + \cancel{\frac{9}{2}x^3}$

$= 1 + 2x - \frac{3}{2}x^2$ ✓

10. $\frac{7x^2-3x+2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$

$7x^2 - 3x + 2 = A(x^2+1) + Bx^2 + Cx$

$7x^2 - 3x + 2 = Ax^2 + A + Bx^2 + Cx$

$7x^2 - 3x + 2 = (A+B)x^2 + Cx + A$

$A = 2 \quad C = -3 \quad 2 + B = 7$
$B = 5$

$= \frac{2}{x} + \frac{5x-3}{x^2+1}$ ✓

11. $9x^3 - 11x^2 + 8x - 4$

$= Ax^2(3x-2) + Bx(3x-2) + C(3x-2) + Dx^2$

$= 3Ax^3 - 2Ax^2 + B3x^2 - 2Bx + 3xC - 2C + Dx^2$

$= 3Ax^3 + (3B - 2A + D)x^2 + (3C - 2B)x - 2C$

$3A = 9 \quad -2C = -4 \quad 3(-1) - 2(3) + D = -11$
$A = 3$ ✓ $\quad C = 2$ ✓ $\quad -9 + D = -11$
$3C - 2B = 8 \quad D = -2$ ✓
$3 \cdot 2 - 2B = 8$
$-2B = 2$
$B = -1$ ✓

15. $\frac{2x^2-7x-1}{(x-2)(x^2+3)} = \frac{A}{x-2} + \frac{Bx+C}{x^2+3}$

$= A(x^2+3) + (Bx+C)(x-2)$

$= Ax^2 + 3A + Bx^2 - 2Bx + Cx - 2C$

$= (A+B)x^2 + (C-2B)x + 3A - 2C$

$A + B = 2 \quad A + 3 = 2$
$C - 2B = -7 \quad A = -1$
$3A - 2C = -1 \quad 3(-1) - 2C = -1$
$-3 - 2C = -1$
$-2C = 2$
$C = -1$ ✓

$C - 2B = -7$
$C = 2B - 7$
$3A - 2C = -1$
$3A - 2(2B - 7) = -1$
$3A - 4B + 14 = -1$
$3A = -15 + 4B$
$A = \frac{4}{3}B - 5$

$= \frac{-1}{x-2} + \frac{3x-1}{x^2+3}$

$\frac{4}{3}B - 5 + B = 2$

$\frac{7}{3}B - 5 = 2$

$\frac{7}{3}B = 7$

$B = \cancel{7} \cdot \frac{3}{\cancel{7}} = 3$ ✓

ii $-(x-2)^{-1} + (3x-1)(x^2+3)^{-1} -$

$= -(-2+x)^{-1} + (3x-1)(3+x^2)^{-1}$

$= -(-2)^{-1}(1 + \frac{x}{-2})^{-1} + 3^{-1}(1+x^2)^{-1}(3x-1)$

$= \frac{1}{2}(1 + \frac{x}{-2})^{-1} + \frac{1}{3}(1+x^2)^{-1} \ldots$

$= \frac{1}{2}\left(1 + (-1)(\frac{x}{-2}) + \frac{(-1)(-1-1)}{2!}(\frac{x}{-2})^2\right) \ldots$

$= \frac{1}{2}(1 + \frac{1}{2}x + \frac{1}{4}x^2) + (3x-1)(\quad)^{-1}$

$= \ldots$ sure I got it

4.2 Calculus Extended

Derivative of $\tan^{-1}(x)$

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{x^2+1}$$

$y = \tan^{-1}(x)$ $\quad \frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{\tan^2 y - 1}$

$\tan y = x$

$\sec^2 y \cdot \frac{dy}{dx} = 1$ $\quad \frac{dy}{dx} = \frac{1}{x^2+1}$

Integration of $\frac{1}{x^2+a^2}$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}(x) + c$$

extended:

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$$

Integration of $\frac{kf'(x)}{f(x)}$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

extended:

$$\int \frac{kf'(x)}{f(x)}\,dx = k\ln|f(x)| + c$$

Integration by Substitution

=> uses the concept of $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

=> use u to replace a part of the equation

e.g. $\int 4x(2x+1)^{\frac{1}{2}}\,dx \qquad u = 2x+1$

=> Take the derivative of u and solve for dx in terms of du and x in terms of u

e.g. $u = 2x+1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$

and: $u = 2x+1 \Rightarrow 2x = u-1 \Rightarrow x = \frac{u-1}{2}$

=> Replace x and dx with the equivalent terms of u and du

e.g. $\int \cancel{4}\left(\frac{u-1}{\cancel{2}}\right)(u)^{\frac{1}{2}}\,\frac{du}{\cancel{2}}$

=> Integrate and replace x in the result

e.g. $\int (u-1)\,u^{\frac{1}{2}}\,du = \int \left(u^{\frac{3}{2}} - u^{\frac{1}{2}}\right)du$

$= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{5}(2x+1)^{\frac{5}{2}} - \frac{3}{2}(2x+1)^{\frac{3}{2}} +$

Integration by Parts

=> Uses the concept of $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

↳ aka $\frac{dy}{dx}\left(f(x)\cdot g(x)\right) = g(x)f'(x) + g'(x)f(x)$

Formula:

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

5.2 Vectors

=> A vector consists of a direction and a magnitude

=> moving a point, vector, curve, etc. is called a translation or displacement

Displacement Notation For Points

=> Displacement from point A to point B: $\overrightarrow{AB}$

=> From B to A: $\overrightarrow{BA}$

=> A translation isn't specific to a given location. The displacement is relative to the line point:

=> A displacement vector is used to translate points, curves etc.

Vector notation

Vector: $\overrightarrow{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$

for three dimensions:

$$\overrightarrow{AB} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Vector in standard position

=> Vector starts at zero

Lower Case Vector Notation

Acceptable notations:

bold: $\mathbf{b}$ or line: $\underline{a}$ $\underline{b}$ $\underline{x}$ $\underline{z}$

=> all the same

Scalar

=> A value used to scale (re define the magnitude) of a vector

$$\vec{c} = \underbrace{-\frac{3}{4}}_{\text{scalar}} \underbrace{\vec{b}}_{\text{vector}}$$

$\vec{c}$ is defined as a scaled version of $\vec{b}$

Scalar multiplication: $2\begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 6 \\ 10 \\ 12 \end{pmatrix}$

Parallel Vectors

=> Vectors are parallel when only their magnitude (i.e. scalar) is different

=> λ (lambda) is probably for scalars

$\vec{a} = \lambda \vec{b}$ $\quad$ $\vec{a}$ is parallel to $\vec{b}$

Vector Addition

$\vec{a} + \vec{b} = \vec{c}$

$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a+c \\ b+d \end{pmatrix}$

=> the sum is called the resultant vector

Derived Formulas:

$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

$\overrightarrow{AB} = -\overrightarrow{BA}$

$\vec{a} + \vec{b} = \vec{b} + \vec{a}$

Magnitude Notation

=> magnitude is always positive!

=> The magnitude (abs modulus) of $\vec{p}$ is written as $|\vec{p}|$

=> Use the pythagorean theorem to find the magnitude (length of the vector)

Unit Vectors

=> A vector of the magnitude 1

=> notation: $\hat{p}$ where $|\hat{p}| = 1$

Find unit vector:

=> Solve for magnitude $|\vec{p}|$

=> divide the vector components by the magnitude

$$\vec{p} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \qquad \hat{p} = \begin{pmatrix} \frac{p_1}{|\vec{p}|} \\ \frac{p_2}{|\vec{p}|} \\ \frac{p_3}{|\vec{p}|} \end{pmatrix}$$

Position Vectors

=> Vectors that describe a specific position relative to a defined origin, usually O

=> They're not free vectors (aka displacement vectors)

=> O is the origin of the cartesian coordinate system

Position Vector Notation

=> $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors for the x, y and z axis

=> Position vectors are often given in the form: $x\vec{i} + y\vec{j} + z\vec{k}$ where (x,y,z) are coordinates of a point

$\overrightarrow{OP} = x\vec{i} + y\vec{j}$

or for three dimensions:

$\overrightarrow{OP} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $\vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

=> x and y are scalars and the coordinates for point P.

Vector Multiplication

=> Taking the cross product of two vectors results in a third vector:

$\vec{a} \times \vec{b} = \vec{c}$ (I won't have to learn this, this year)

↳ This is the scalar product

=> Taking the dot product results in a scalar

$\vec{a} \cdot \vec{b} = c$

↳ This is the scalar product

Scalar Product

$\Rightarrow \vec{a} \cdot \vec{b} = |\vec{a}| \times |\vec{b}| \cos\theta$

where θ is the angle between $\vec{a}$ and $\vec{b}$

=> $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

Desired Formula:

=> $\vec{a} \cdot \vec{a}$ when $\theta = 0$:
$= |\vec{a}||\vec{a}| \cos 0$

=> $\vec{a} \cdot \vec{a} = |\vec{a}| \cdot |\vec{a}| = |\vec{a}|^2$

=> $\vec{a} \cdot \vec{b}$ when $\theta = 90°$
$= |\vec{a}||\vec{b}| \cos 90° = 0$

Scalar Product With Component Form

$\Rightarrow \vec{a}, \vec{b}$ is in the form

$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ $\quad$ $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$

=> simplifies to

$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}| \cos\theta$

Vector Equation of a Line

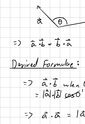
Required to find the equation:

=> point on line (A) and direction ($\vec{b}$) of line or any parallel vector works

$\vec{r} = \vec{a} + t\vec{b}$

=> where $\vec{r}$ can describe all points on the line as t changes

OR two points on line:



=> The direction vector of the line is found by subtracting the position vector $\vec{a}$ from $\vec{c}$

$\vec{r} = \vec{a} + t(\vec{c} - \vec{a})$

seems a bit wrong

Parametric Form

$\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ can be written as

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

where when $\vec{r} = \vec{a} + t\vec{b}$

$$\left.\begin{aligned} x &= a_1 + tb_1 \\ y &= a_2 + tb_2 \\ z &= a_3 + tb_3 \end{aligned}\right\} \text{ parametric equations}$$

Intersecting Lines

$r = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \lambda \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$ $\quad$ $r = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + \mu \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$

use parametric equations:

$x = a_1 + \lambda b_1 = c_1 + \mu d_1$

$y = a_2 + \lambda b_2 = c_2 + \mu d_2$

=> solve for λ and μ

=> Check in the third equation if no mistakes were made by plugging in values

$z = a_3 + \lambda b_3 = c_3 + \mu d_3$

=> If all worked correctly, solve for x, y, z

5.2 Differential Equations

=> any equation that have a derivative, e.g. $\frac{dy}{dx}$ or $\frac{d^2y}{dx^2}$ as one of their variables

=> <u>First order</u> differential equations has $\frac{dy}{dx}$ as one of its terms

General and Particular solution

=> $\frac{dy}{dx} = 3x^2$ is a differential solution

=> To solve the differential equation, the direct relationship between x and y has to be found

=> The solution to a differential equation will allways be a function

General Solution

=> Only describes the <u>direct relation</u> between x and y

e.g. $\frac{dy}{dx} = 3x^2 \Rightarrow y = x^3 + C$

= $y = x^3 + C$ is a <u>direct equation</u>

Particular Solution

=> Requires additional information but includes defined values for constants, e.g. C

Separating Variables

=> Variables x and y are seperated, so that
- side 1 is integrated with respect to x
- side 2 is integrated with respect to y

=> Rearrange to the form

$$f(y)\frac{dy}{dx} = g(x)$$

=> $\frac{dy}{dx}$ cancels with dx, so the left side will be an integral in terms of y and dy

$$\int f(y)\frac{dy}{dx}\,dx = \int f(y)\,dy$$

=> Integrate both sides, combine constants, and solve for y

=> When rearranging x and y, you might exclude values
e.g. $\frac{dy}{dx} = xy^3 \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = x$ where we assume $y \neq 0$

↳ Check if excluded values satisfy the original differential equations

e.g. $\frac{dy}{dx} = xy$ | seperate x and y

$$\frac{1}{y}\frac{dy}{dx} = x$$

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x\,dx$$ | Integrate both sides

$$\int \frac{1}{y}\,dy = \int x\,dx$$

$$\ln|y| + A = \frac{1}{2}x^2 + B$$ | constants combine

$$\ln|y| = \frac{1}{2}x^2 + D$$

$$y = e^{\frac{x^2}{2}} + e^D$$

$$y = e^{\frac{x^2}{2}} + C$$

5.3 Complex Numbers

$i = \sqrt{-1}$, $i^2 = -1$

=> i is called the unit imaginary number

↳ all other imaginary numbers are multiples of i
=> iy where y is a real value

Complex Numbers

=> $z = x + iy$ where $z \hat{=}$ "a complex number"
$x \hat{=} \operatorname{Re} z$ (real part of z)
$iy \hat{=} \operatorname{Im} z$ (imaginary part of z)

Equal Complex Numbers

=> Complex numbers are only ever equal when their real and imaginary parts are identical

$z_1 = a + bi$ $z_2 = c + di$

if $z_1 = z_2$ then $a = c$ and $b = d$

Complex Conjugate

=> A conjugate of two terms is created by flipping the sign between them

=> $z = x + iy$ then $z^* = x - iy$

$z^* \hat{=}$ conjugate of z

Calculations with Complex Numbers

$z_1 = x + yi$ $z_2 = a + bi$

Addition & Subtraction

=> Add together the real and the imaginary terms

$z_1 + z_2 = x + a + (y + b)i$

Multiplication

=> multiply the brackets

$z_1 z_2 = (x + yi)(a + bi) = ax + xbi + yai + yb(-1) \ldots$

Division

=> multiply the fraction $\frac{z_1}{z_2}$ by the conjugate z_2^*

=> $z \cdot z^* = (x + iy)(x - iy) = x^2 + y^2$ not $x^2 - y^2$ as $i^2 = -1$

$\frac{z_1}{z_2} = \frac{x + yi}{a + bi} = \frac{(x + yi)(a - bi)}{a^2 + b^2} \ldots$

Forms to Represent Complex Numbers

Cartesian Form

$z = x + yi$

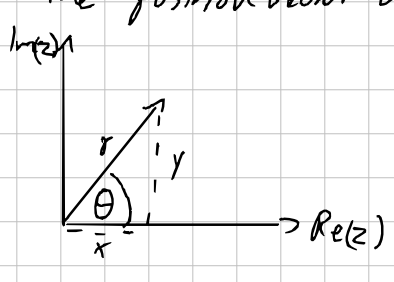
Modulus-Argument Form

=> a geometric representation, that interprets complex numbers as vectors in the complex plane

=> where $x + yb = \begin{pmatrix} x \\ y \end{pmatrix}$ ← this is the "argument" aka direction

=> modulus is the magnitude, i.e. if $z = x + iy$ then $|z|$ ← use $\sqrt{x^2 + y^2} = |z|$

=> Principal argument is the angle between the position vector and the positive x-axis, denoted 'θ'



- usually θ is given in the domain $-\pi < \theta \leq \pi$
- sometimes $0 < \theta < \pi$ is the case
- θ is given in radians

=> Putting all together:

$\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$x = r\cos\theta$ $y = r\sin\theta$

$z = x + iy$ => $z = r\cos\theta + i(r\sin\theta)$

=> $z = r(\cos\theta + i\sin\theta)$ where $r = |z|$ ← magnitude of z position vector

Exponential Form

=> $z = r(\cos\theta + i\sin\theta)$

Exponential Form: $z = re^{i\theta}$

Argument: $\operatorname{Arg}(z)$
↳ Describes angle of complex number vector z to positive x-axis

Important derived Rules:

Multiplication: $(z_1 z_2)$

1. $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ for magnitude
   ↑ magnitude of $z_1 z_2$
2. $\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2$ for direction
   ↑ angle of resulting vector from $z_1 z_2$ to positive x

Division: $\left(\frac{z_1}{z_2}\right)$

1. $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$ for magnitude
2. $\arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$ for direction

Numbers of Complex Roots of Polynomials

=> A polynomial of the degree n has n complex solutions (remember complex includes pure real answers)

=> Complex solutions for polynomials with real coefficients occur in complex conjugate pairs

Linear: always real
Quadratic: two real or two complex as $x \pm yi$
Cubic: three real or one real and two complex as $x \pm yi$
Quartic: four real or
two real and two complex as $x \pm yi$
four complex as $x \pm yi$

Loci

=> The path traced by an inequality or an equation

Circle with Centre (0,0):

$|z| = r$
=> circumfrance of a circle
=> center at (0,0)

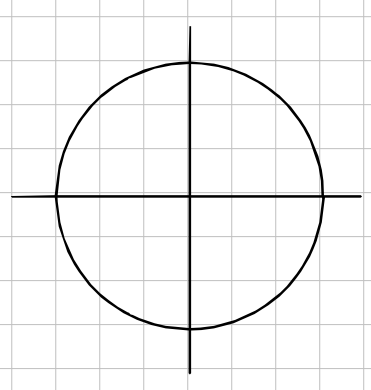
$|z| < r$
=> area inside a circle, excluding circumfrance
=> center at (0,0)

$|z| \leq r$
=> area inside a circle, including circumfrance
=> center at (0,0)

where $|z| \hat{=}$ magnitude of position vector of complex number in complex plane

$r \hat{=}$ some constant

e.g. $|z| = 3$



Circle with Centre (a,b):

$z = x + yi$ $z_1 = a + bi$

$|z - z_1| = r$
=> circumfrance of a circle
=> center at (a,b)

$|z - z_1| < r$
=> area inside a circle, excluding circumfrance
=> center at (a,b)

$|z - z_1| \leq r$
=> area inside a circle, including circumfrance
=> center at (a,b

6.1 Representation of Data

Types of Data

Qualitative Data: aka categorical data

=> non-numerical data

Quantitative Data:

=> either discrete or continuous values

Discrete Data: can only take exact values e.g. money
Continuous Data: can take any value, possibly within a limited range, to a certain degree of measured accuracy
e.g. time to run 100m

Stem-and-Leaf Diagrams

=> A way of presenting discrete data in an easy to compare way

Data: 58, 59, 62, 44, 69, 54, 73, 76, 92, 76

4	4
5	4 8 9
6	2 9
7	3 6 6
8	
9	2

Key: 4 | 4 represents a score of 44 points

} a row is called a "class" of values

stem — "leaves"

=> stem is ordered vertically, according

=> leaves are ordered horizontally, smallest value close to stem

=> A key has to be added that provides a unit, e.g. points, ms^{-1} and an interpretation of one datapoint

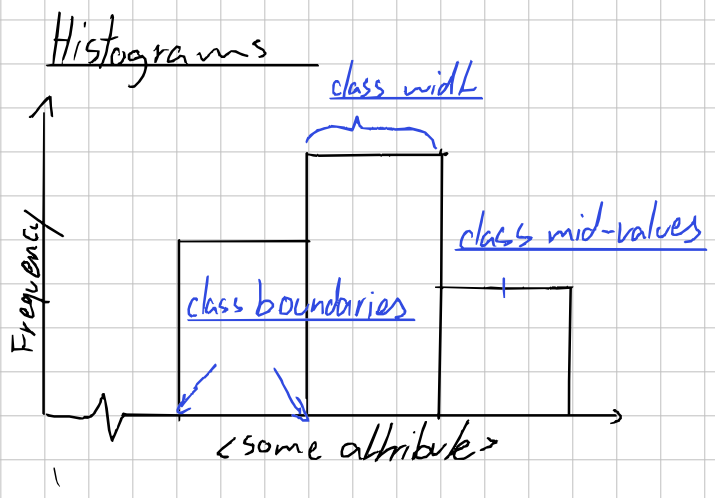
Back-to-Back Stem Diagrams:

=> smallest values always next to stem

e.g. class grades:

Girls		Boys
2	4	0 1 8
9 6	5	7
3 8 8 4	6	2 3 8
5 3	7	6 7
	8	1
9 9 3 2	9	9

Histograms



=> great for representing continuous data

Grouped Frequency Table

Age (A years)	$16 \le A < 18$	$18 \le A < \ldots$ Ranges
No. students (f)	42	36

Histograms with Different Class Ranges

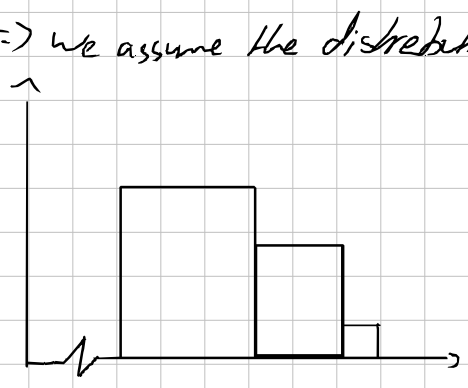
=> Classes don't have to have the same width

=> Always frequency $\propto$ area$_{class}$, meaning that with unequal class widths the heigh has to be adjusted

=> simple "frequency" is replaced by frequency density, where

$$\text{frequency density} = \frac{\text{class frequency}}{\text{class width}}$$

=> we assume the distribution is equal accross a class interval



Cumulative Frequency Graphs

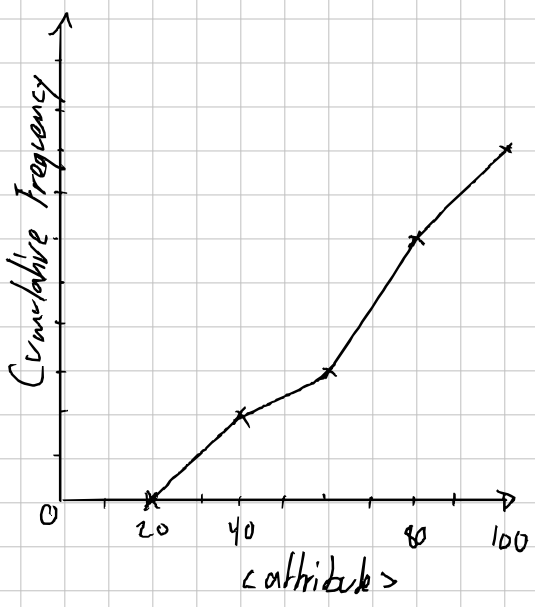
=> Used to represent continuous data

=> cumulative frequency = class frequency + all previous class frequencies

Class	1	2	3 ...
frequency	f_1	f_2	f_3 ...

Class	1	2	3 ...
cumulative frequency	f_1	f_1+f_2	$f_1+f_2+f_3$

=> include the lowest pound of the first class as zero



=> connect the dots with lines or a curve

=> allways plot the accumulative frequency at the class upper boundaries

6.2 Measures of Central Tendency

=> Three common types: Average, Mode, Mean

Mode and Modal Class

=> **Mode**: most frequently occuring value

=> **Modal Class**: Class with highest frequency density

Mean

=> aka arithmatic mean is the average number

$$\text{Mean} = \frac{\text{sum of values}}{\text{number of values}}$$, mean is denoted as $\bar{x}$

notation for ungrouped data:

$$\bar{x} = \frac{\sum x}{n}$$

$\bar{x} \hat{=}$ mean
$\sum x \hat{=}$ sum of all values
$n \hat{=}$ number of values

notation for grouped data:

$$\bar{x} = \frac{\sum xf}{\sum f}$$

$\bar{x} \hat{=}$ estimated mean
$\sum(x \cdot f) \hat{=}$ sum of estimated class total
$x \hat{=}$ class mid-values
$f \hat{=}$ class frequency

=> can only calculate an estimated version of the mean

Coded Data

=> "coding data" means shifting all data values by some value, e.g. +5

=> makes data easier to handle

Calculating Mean of Coded Data

=> the value b that was used to translate the data points is simply added back to the mean

$$\text{Mean}(x) = \text{mean}(x-a) + a$$

for ungrouped data: $\bar{x} = \frac{\sum(x-b)}{n} + b$

for grouped data: $\bar{x} = \frac{\sum(x-b)f}{\sum f} + b$

Median

=> The value in the middle of an ordered dataset

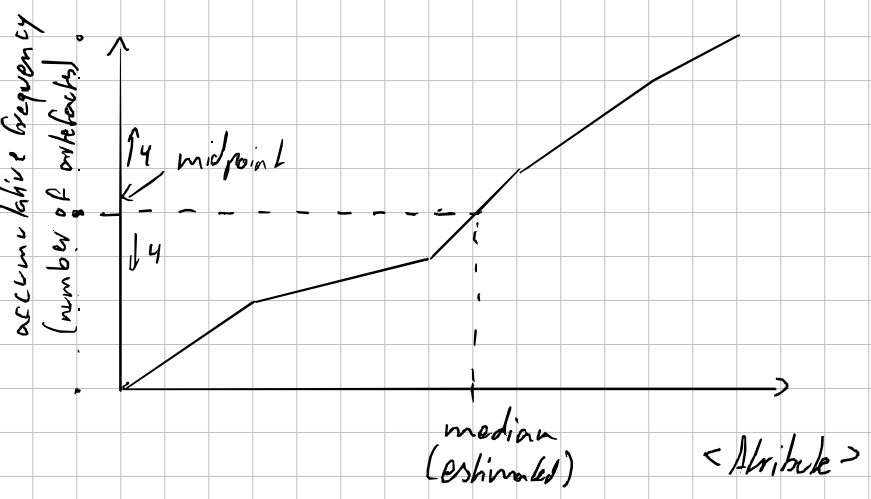
=> median = $\left(\frac{n+1}{2}\right)$th value $n \hat{=}$ number of values in dataset

=> if n is even, the mean equals the average of the two central values

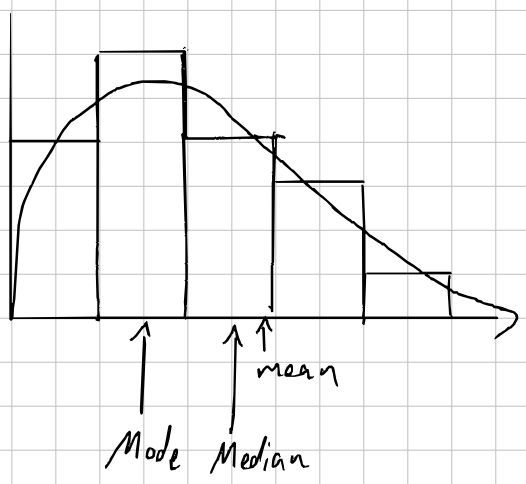
Estimate Median for Grouped Data

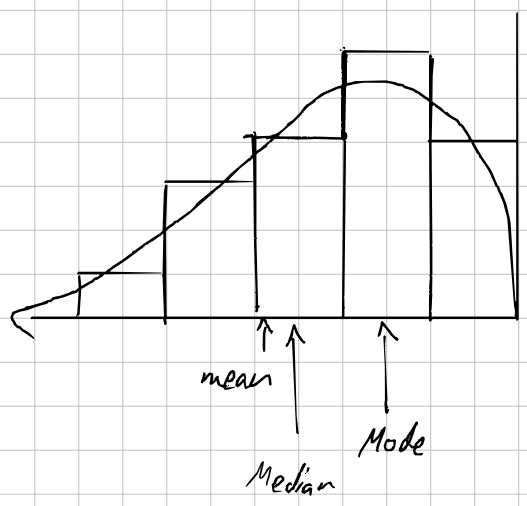
=> Plot the grouped data on a cumulative frequency graph

=> Half way of the cumulative frequency axis the central value of the dataset is located



Skew & Different Averages in Relation





Positive Skew

=> mode < median < mean

Negative Skew

=> mean < median < mode

7.1 Measure of Variation

=> A set is best described by a
1. measure of central tendency
2. measure of variation

-> Variation refers to the <u>spread</u>

=> Three common measures of variation:
1. <u>range</u>
2. <u>interquartile range</u>
3. <u>standard deviation</u>

Range

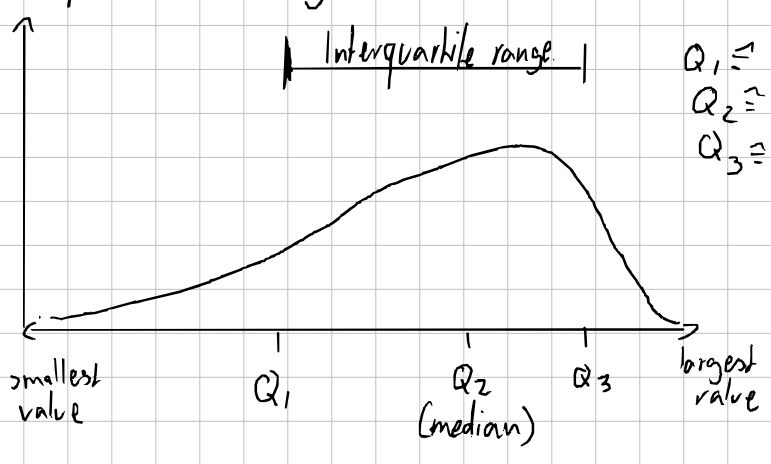
=> numerical difference between minimum and maximum

Advantages: Easy to calculate
Disadvantages: only takes extreme values into account, is relative

ungrouped data: range = max − min

grouped data: range = most upper boundary − most lower boundary

Interquartile Range and Percentiles



$Q_1 \hat{=}$ lower quartile
$Q_2 \hat{=}$ median / middle quartile
$Q_3 \hat{=}$ upper quartile

=> the interquartile includes 50% of the values

=> $IQR = Q_3 - Q_1$ $\quad$ IQR $\hat{=}$ interquartile range

Advantages:
- shows how varied middle values are
- not really affected by outliers (extreme values)

Positions Of Quarters

Ungrouped Data

Even number of data points:
1. split dataset in lower half and upper half
2. Q_1 and Q_2 are the medians of the lower and upper half

Odd number of data points:
1. split the dataset at the median, then discard the median
2. same as for even

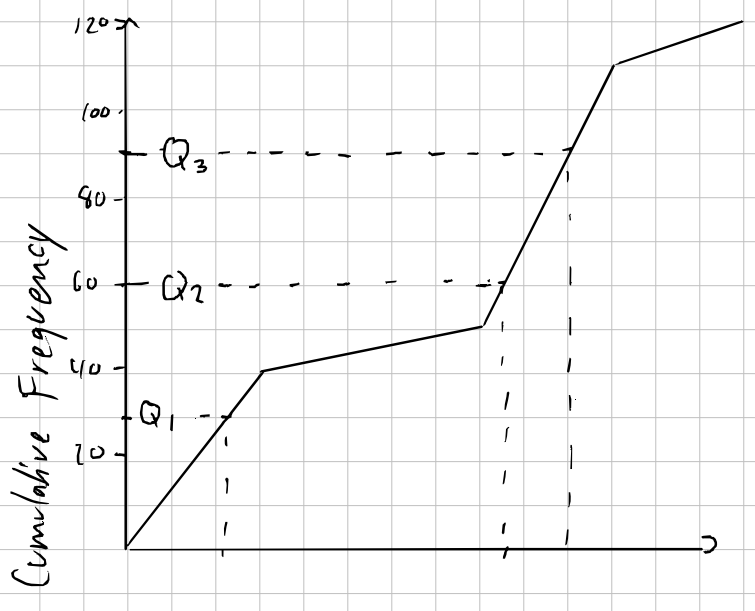
Grouped Data

-> Use a cumulative frequency graph

=> Place the quarters at

$Q_1 = \frac{1}{4}\sum f \quad Q_2 = \frac{1}{2}\sum f \quad Q_3 = \frac{3}{4}\sum f \quad Q_4 = \frac{3}{4}\sum f$

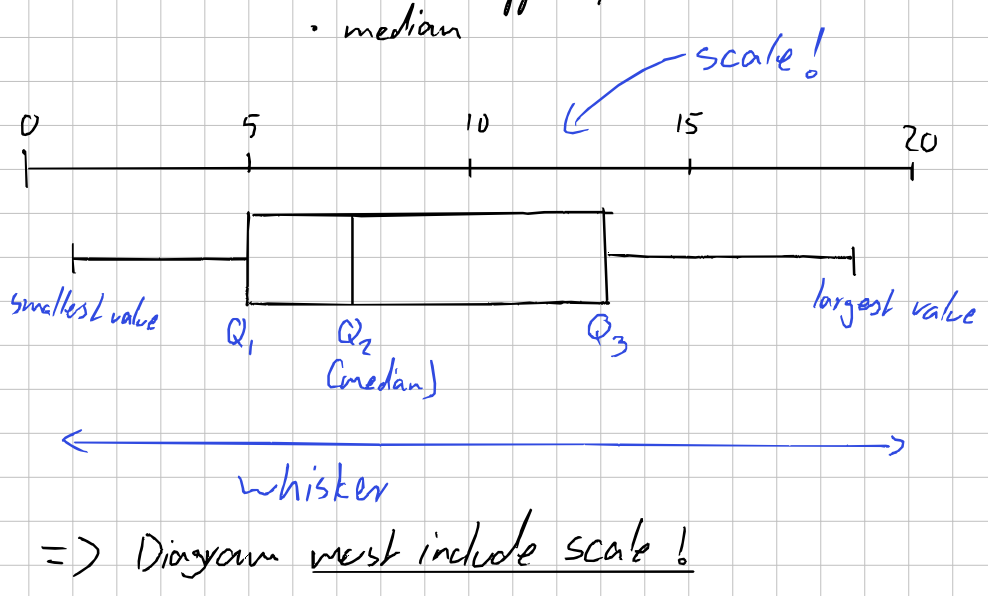
where $\sum f \hat{=}$ total frequency (sum of all frequences)



$\sum f = 120$

Box-and-Whisker Diagram

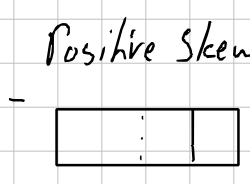
=> Shows:
- smallest & largest value
- lower & upper quartile
- median



=> Diagram <u>must include scale!</u>

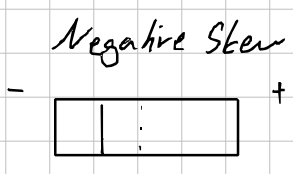
=> <u>Identify a skew</u>:

Positive Skew



=> median is to the right of the box's center

Negative Skew



=> median is to the left of the box's center

Standard Deviation

=> Quantifies how much all values in a dataset variate around the mean, denoted SD(x)

=> Standard deviation punishes outliers, as larger distances from the mean are more amplified by the square

=> Variance is defined as "mean square deviation from the mean" and denoted Var()

$$Var(x) \Rightarrow \frac{\sum (x-\bar{x})^2}{n}$$

Ungrouped Data:

standard deviation of $x = \sqrt{Var(x)} = \sqrt{\frac{\sum (x-\bar{x})^2}{n}} = \sqrt{\frac{\sum x^2}{n} - \bar{x}^2}$

where $\bar{x} = \frac{\sum x}{n}$ and x = data value

Grouped Data

standard deviation of $x = \sqrt{Var(x)} = \sqrt{\frac{\sum (x-\bar{x})f}{\sum f}} = \sqrt{\frac{\sum x^2 f}{\sum f} - \bar{x}^2}$

where $\bar{x} = \frac{\sum xf}{\sum f}$ and x = class interval midpoint

=> when calculating variance or standard deviation a small rounding error in the mean causes substantial errors in the result!

=> use $\bar{x} = \frac{\sum x}{n}$ or $\bar{x} = \frac{\sum xf}{\sum f}$ instead

Advantages of Standard Deviation:
- all data values are taken into account
- data doesn't have to be ordered

Disadvantages:
- strongly affected by outliers
- larger deviations have a disproportionally large impact

=> When given three of n, $\sum x$, $var(x)$, $\sum x^2$ you can determine the missing one

Combining Datasets

=> x and y are datasets that will be combined

$$\text{new mean} = \frac{\sum x + \sum y}{n_x + n_y}$$

$$\text{new variance} = \frac{\sum x^2 + \sum y^2}{n_x + n_y} - \left(\frac{\sum x + \sum y}{n_x + n_y}\right)^2$$

Coded Data & Standard Deviation

=> when a shift b was applied to the dataset, <u>variance isn't affected!</u>

=> As standard deviation = $\sqrt{\text{variance}}$, it's not affected either

=> $Var(x) = Var(x-b)$

8.1 Probability

Technical Terms

=> In a probability experiment,
- => one result ≙ elementary event e.g rolling a dice
- => multiple results ≙ events e.g. rolling dice twice
- => favourable outcomes: events that fullfill a condition

=> Random selection:
- => selecting at random ≙ same likelyhood to be selected
- => fair, unbiased, equiprobable ≙ selecting at random

Probability Notation

=> $P(A)$ or $P(B)$

probability event event

=> P(event) can be any value between 0 and 1, inclusive

=> $P(A \cap B)$ means "probability A and B occur

=> $P(A \cup B)$ means "probability A or B occur

Random Selection

=> when one object is randomly selected,

$P(\text{select any particular object}) = \frac{1}{n}$ n ≙ number of objects

=> Equally likelly events:

$P(\text{event}) \quad \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$

Exhaustive Events

=> A set of events that contain all possible outcomes

=> The complement of event A is "not A", i.e. all events except A
↳ The complement of A is denoted A'

=> $P(A) + P(A') = 1$

↳ probability A occurs + probability anything but A occurs = 1

Trials and Expectation

=> Trial ≙ each repetition of an experiment

=> Relative Frequency = $\frac{\text{number of trials where event occurs}}{\text{total number of trials}}$

↳ used to estimate probabilities

=> Expectation = The theoretically known probability of an event occuring.

Mutually Exclusive Events & Addition Law

=> Mutually exclusive events don't share any favourable outcomes
↳ never can both events occur

=> To find the combind probability that the mutually exclusive events A or B occur, their probabilities are simply added

$\emptyset$ ≙ empty set

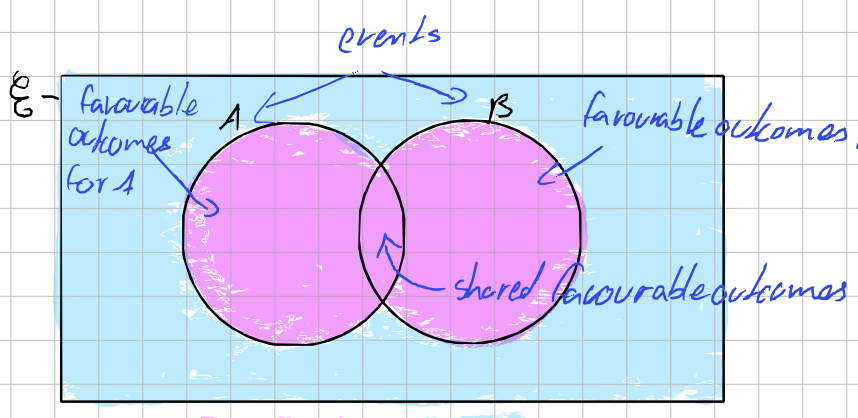
$P(A \cup B) = P(A) + P(B)$

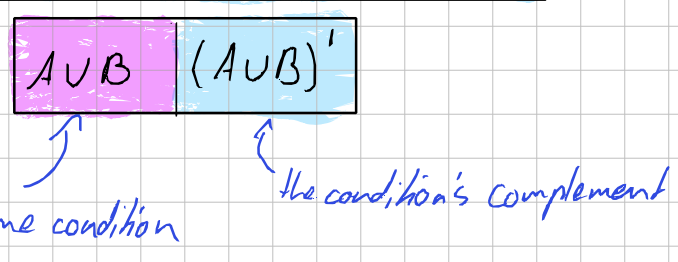
where A and B are mutually exclusive → $A \cap B = \emptyset$

=> This can be extended to any number of mutually exclusive events
e.g. $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

Venn Diagrams

Show:
- numbers of outcomes favourable to A → denoted $n(A)$
- set of outcomes not favourable to A → denoted A'
- additional sets that might share outcomes with A





ε ≙ complete set of outcomes, aka "possibility space"

Inclusion - Exclusion Law

=> To find the probability that either A or B will occur, both probabilities are added

=> We have to ensure the shared favourable outcomes aren't counted twice

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Independent Events & Multiplication Law

=> Independent events don't affect, and aren't affected by other events, e.g. rolling a dice

$P(A \cap B \cap C ..) = P(A) \times P(B) \times P(C) \times ...$

where A, B, C ... are independent events

=> Can be used to prove two sets are independent
↳ if $P(A \cap B) = P(A) \times P(B)$, they must be mutually exclusive

Conditional Probability

=> Probability that depends on an additional condition

e.g. ABC $P(\text{select } E) = \frac{1}{3}$

$P(\text{select } E, \text{ given that choice will be a vowel}) = \frac{1}{2}$

=> $P(A|B)$ uses "|" for "given that"

↳ i.e probability that A occurs, given that B occurs

Independence & Conditional Probability

=> extension to the definition of "independence"

=> X and Y are dependent only if

$P(A \cap B) = P(A) \times P(B)$ such that

$P(X|Y) = P(X|Y')$

↳ X and Y are only independent if neither is affected by the occurence of the other

Dependend Conditional Probability

=> find out the probability that $P(A|B)$ occurs

=> start with $P(A \cap B) = P(B) \times P(A|B)$

i.e. the probability that A and B occur equals the probability that B occurs times the probability that A occurs in a world where B already certainly occured.

=> rearange: $P(A|B) = \frac{P(A \cap B)}{P(B)}$

Practice Ch 8

1. $\int_0^1 e^{2x}\,dx$

$u = x \qquad \frac{du}{dx} = 1$

$\frac{dv}{dx} = e^{2x} \qquad v = \frac{1}{2}e^{2x}$

$= \left[\frac{1}{2}e^{2x}x\right]_0^1 - \int_0^1 1\cdot\frac{1}{2}e^{2x}\,dx$

$= \left[\frac{1}{2}e^{2x}x\right]_0^1 - \frac{1}{2}\int_0^1 e^{2x}\,dx$

$= \frac{1}{2}(e)e^{2(1)} - \frac{1}{2}(0)e^{2(0)} - \frac{1}{2}\left[\frac{e^{2x}}{2}\right]_0^1$

$= \frac{e}{2} - \frac{1}{2}\left(\frac{e^{2(1)}}{2} - \frac{e^{2(0)}}{2}\right)$

$= \frac{e}{2} - \frac{1}{2}\cdot\frac{e^2}{2} + \frac{1}{2}\cdot\frac{1}{2}$

$= \frac{e}{2} - \frac{e^2}{4} + \frac{1}{4} = \frac{1}{4}(2e - e^2 + 1)$

2. $\int \frac{\ln x}{x^2}\,dx = \int \ln x \cdot x^{-2}\,dx$

$u = \ln x \qquad \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^{-2} \qquad v = -x^{-1}$

$= \left[-x^{-1}\cdot\ln x\right]_1^2 - \int_1^2 -x^{-2}\cdot\frac{1}{x}\,dx$

$= \left[-x^{-1}\ln x\right]_1^2 - \int_1^2 -x^{-3}\,dx$

$= -2^{-1}\ln 2 - \left[\frac{-x^{-2}}{-2}\right]_1^2$

$= -\frac{1}{2}\ln 2 - \left[-\frac{1}{2}x^{-2}\right]_1^2$

$= -\frac{1}{2}\ln 2 - \left(-\frac{1}{2}\cdot 2^{-2} - -\frac{1}{2}\cdot 1^{-2}\right)$

$= -\frac{1}{2}\ln 2 - \frac{1}{2}\cdot\frac{1}{4} + \frac{1}{2}\cdot\frac{1}{2}$

$= -\frac{1}{2}\ln 2 - \frac{1}{8} + \frac{1}{2} = \frac{3}{8} - \frac{1}{2}\ln 2$

3. $\int \sin^{-1}\ln x\,dx$

$u = \sin^{-1}\ln x \qquad \frac{du}{dx} = \frac{1}{\sqrt{1-(\ln x)^2}}\cdot\frac{1}{x}$

$\frac{dv}{dx} = 1 \qquad v = x$

$= \left[\ln x\cdot\sin^{-1}\ln x\right]_1^e - \int_1^e \ln x\cdot\frac{1}{\sqrt{1-(\ln x)^2}}\cdot\frac{1}{x}\,dx$

$= \left[x\ln x\cdot\sin^{-1}\ln x\right]_1^e - \int_1^e \frac{\ln x}{x\sqrt{1-(\ln x)^2}}\,dx$

$= \left[e\ln e\cdot\sin^{-1}\ln e - 1\ln 1\cdot\sin^{-1}\ln 1\right] - \int_1^e \frac{\ln x}{x\sqrt{1-(\ln x)^2}}\,dx$

$= \left[e\cdot 1\cdot\sin^{-1}1 - 1\cdot 0\cdot\sin^{-1}0\right] - \int_1^e \frac{\ln x}{x\sqrt{1-(\ln x)^2}}\,dx$

$= \left[e\cdot\frac{\pi}{2} - 0\cdot 0\right] - \int_1^e \frac{\ln x}{x\sqrt{1-(\ln x)^2}}\,dx$

$= \frac{e\pi}{2} - \int_1^e \frac{\ln x}{x\sqrt{1-(\ln x)^2}}\,dx$

4. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \frac{1-\cos^2\theta}{\cos^2\theta}\,d\theta = \int \left(\frac{1}{\cos^2\theta} - \frac{\cos^2\theta}{\cos^2\theta}\right)d\theta$

$= \int \left(\sec^2\theta - 1\right)d\theta = \tan\theta - \theta + C$

5. $\int_0^{\frac{\pi}{2}} \sin^3\theta\,d\theta = \int_0^{\frac{\pi}{2}} \sin^2\theta\cdot\sin\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} (1-\cos^2\theta)\sin\theta\,d\theta$

$= \int_0^{\frac{\pi}{2}} \sin\theta - \cos^2\theta\sin\theta\,d\theta$

$= \left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\frac{\pi}{2}}$

$= \left[-\cos\frac{\pi}{2} + \frac{1}{3}\cos^3\frac{\pi}{2}\right] - \left[-\cos 0 + \frac{1}{3}\cos^3 0\right]$

$= \left[-0 + \frac{1}{3}\cdot 0\right] - \left[-1 + \frac{1}{3}\cdot 1\right]$

$= 0 - \left(-\frac{2}{3}\right) = \frac{2}{3}$

6. $\int \frac{\sec^2\theta}{\tan\theta}\,d\theta = \int \frac{\sec^2\theta}{\tan\theta\cdot\cos\theta}\,d\theta$

$= \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \frac{1}{\cos\theta}\cdot\frac{\cos\theta}{\sin\theta}\,d\theta$

$= \int \frac{1}{\sin\theta}\,d\theta = \int \csc\theta\,d\theta$

$= \ln|\csc\theta - \cot\theta| + C$

7. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

8. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

9. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

10. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

11. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

12. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

13. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

14. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

15. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

16. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

17. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

18. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

19. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

20. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

21. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

22. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

23. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

24. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

25. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

26. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

27. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

28. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

29. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

30. $\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta = \int \tan^2\theta\,d\theta$

$= \int \sec^2\theta - 1\,d\theta$

$= \tan\theta - \theta + C$

8.2 Combinations & Permutations

note that $0! = 1$

Selections & Arrangements

=> A selection is the act of choosing one or multiple objects from a set

=> <u>combination</u> : a selection for which the <u>order</u> of the objects <u>doesn't matter</u>

↳ how many ways to combine these objects

=> permutation : a selection where the <u>order</u> of objects <u>does matter</u>

↳ how many ways to arrange these objects

Example: A B C where a selection of two objects is made

Permutations: AB AC
BA BC
CA CB
$= 6$

Combinations: AB
BC
AC
$= 3$

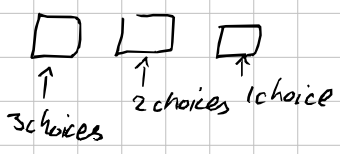
Permutations of n objects

=> how many ways are there to arrange n objects?

explained: A · B · C

=> If objects are distinct

$${}^nP_n = n!$$



=> choice decreases as letters can only be placed once

$3 \times 2 \times 1 = 6$ total possible choices

=> If objects aren't distinct:

$n_1 + n_2 + n_3 \ldots = n$

where e.g. n_1 = number of type 1 objects
or n_2 = number of type 2 objects

$$\frac{{}^nP_n}{n_1! \times n_2! \times n_3! \ldots} = \frac{n!}{n_1! \times n_2! \times \times n_3! \ldots}$$

=> why: SUCCESS

3x 'S', there are
3! ways of arranging 'S'
2! ways of arranging 'C'

↳ SUC'C²ESS and SUC²C'ESS
↳ as there are 2! ways to arrange 'C' for n! ways of arrangement, we devide n! by 2!

Permutation of r Objects from n Objects

=> How many ways are there to arange r number of objects, when there are n distinct objects

$${}^nP_r = \frac{n!}{(n-r)!}$$

explained: 6 People, four chairs

possibilities $6 \times 5 \times 4 \times 3$
chair nr. 1 2 3 4

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot \underbrace{2 \cdot 1}_{\text{remove } 2!}$

$\frac{6!}{2!} = 6 \times 5 \times 4 \cdot 3$

where $n - r = 6 - 4 = 2$

Combinations

=> A selection where the order of selection isn't important i.e. there is no order

${}^nC_r \hat{=}$ number of combinations of r objects from n objects

${}^rP_r \hat{=}$ number of ways to arrange r objects

$${}^nC_r \times {}^rP_r = {}^nP_r$$

$${}^nC_r \times r! = \frac{n!}{(n-r)!}$$

modern type of ${}^nC_r$ notation

$$\Rightarrow {}^nC_r = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

3. Probability Distributions

Discrete Random Variables

=> A discrete random value can only take on certain values, and does so in some random manner

=> notation: e.g. $S \in \{0, 1, 2, 3\}$

where 'S' is some random variable

Probability Distribution

=> Displays all discrete random values of a variable, showing how probable each of them is

=> The sum of all displayed probabilities is allways 1

$$\sum p = 1$$

↳ This should allways be checke

Expectation

=> expected value = mean of a random discrete variable

↳ $E(X)$ = expected value, ie. long-term average

where $X \hat{=}$ random discrete variable
$E() \hat{=}$ mean, ie. expected value

=> $E(X) = \sum x p$ where $x \hat{=}$ discrete random value(s)
$p \hat{=}$ probability of said value

or $E(X) = \dfrac{\sum x p}{\sum p} = \dfrac{\sum x f}{\sum f}$ where $f \hat{=}$ frequencies (ie. how often it happened)
$x \hat{=}$ discrete random value(s)

Variance

=> variance and standard deviation measure the spread of values around the mean

Taking the known variance formula:

$$Var(X) = \frac{\sum x^2 f}{\sum f} - \bar{x}^2 \Rightarrow Var(X) = \frac{\sum x^2 p}{\cancel{\sum p}\ 1} - [E(X)]^2$$

we get:

$$Var(X) = \sum x^2 p - [E(X)]^2$$

9.2 Binomial & Geometric Distribution

Binomial Distribution

=> Used to find the probability that in a number n independent trials with constant probabilities there will be r number of successes

e.g. How likely is it to roll three times the six in five trials?

=> Requirements:
- n repeated independent trials
- n is finite
- there are only two possible outcomes for a trial: success, failure
- the probability for success stays constant accross trials

$X \hat{=}$ random discrete variable
$B(n, p) \hat{=}$ Binomial distribution for n trials and constant trial success probability of p

If $X \sim B(n, p)$

then probability of r successes: $p_r = \binom{n}{r} p^r (1-p)^{n-r}$

Expectation & Variance

=> Using known formula for mean:

expectation = mean = $E(X) = \sum x p$, we get..

$E(X) = \mu = np$ where $\mu \hat{=}$ expected number of successes for n trials
$n \hat{=}$ number of trials
$p \hat{=}$ probability for a trial to be successful

=> $Var(X) = \sigma^2 = np(1-p)$ where $\sigma \hat{=}$ standard deviation
↳ σ^2 = variance
$n \hat{=}$ number of trials
$p \hat{=}$ probability of success for independent trial

Geometric Distribution

=> How probable is it that it will take r trials until a trial results in a success?

=> p = probability of success for a trial
$(1-p)$ = probability of failure

success on: trial 1 : p success
trial 2 : $(1-p)p$ fail, success
trial 3 : $(1-p)^2 p$ fail, fail, success

=> probability that success is on rth trial:

If $X \sim Geo(p)$ $p_r = p(1-p)^{r-1}$ for $r = 1, 2, 3 \ldots$

↳ Requirements
- repeated trials are independent
- repeated trials can be infinite in number
- there are only two outcomes for a trial: success, failure
- constant probability of success accros trial

=> This is a geometric sequence with $a_1 = p$ and common ratio of $(1-p)$

=> Sum of those probabilities must be 1

$$\sum [P(X=r)] = S_\infty = \frac{a_1}{1 - \text{common term}} = \frac{p}{1-(1-p)} = 1$$

Inequalities:

=> $P(X \leq r) = P(\text{success within the first r trials})$
$= 1 - P(\text{failure on first r trials})$
$= 1 - (1-p)^r$

=> $P(X > r) = P(\text{success after rth trial})$
$= P(\text{failure on first r trials})$
$= (1-p)^r$

Mode of Geometric Distribution

=> For any geometric distribution the mode (value that maximizes P) is allways 1

$P(X=1)$, as it's likelier that a success occurs emideately, rather then after some number of fails:

r = 1	success	success on 1st trial is allways most likely
r = 3	fail, fail, success	

Expectation of Geometric Distribution

=> If $X \sim Geo(p)$ then $\mu = \frac{1}{p}$

↳ The mean number of trials it takes until a trial is successful $= \frac{1}{p}$

10.1 Normal Distribution

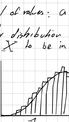
Continuous Random Variables

⇒ Can take on an infinite number of values

⇒ Are defined over an interval of values: $a \le X \le b$

⇒ for showing the probability distribution we calculate the probability for X to be in a small range many times:

$P(a \le X \le b)$



many small ranges

Representation of Probability Distribution

⇒ For data with class intervals of unequal width:

probability density = $\frac{\text{relative frequency}}{\text{class width}}$

relative frequency = observed probability for some class

⇒ For data of equal-width class intervals:

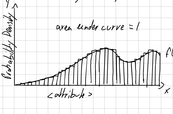
probability density = relative frequency

⇒ column means of frequencies

⇒ Area under the curve $= \Sigma p = 1$

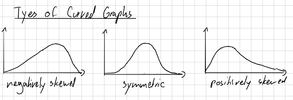
Probability Density Functions (PDFs)

for real for real



⇒ $f(x)$ is the probability density function

Types of Curved Graphs

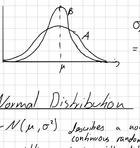


negatively skewed | symmetric | positively skewed

The Normal Curve

Attributes of a Normal Curve:

- symmetric bell-shaped curve
- mean = median = mode
- peak is at mean (μ) – line of symmetry
- probabilities decrease better from mean
- if standard deviation (σ) decreases
  - → width decreases
  - → height increases as Area = 1 always



$\sigma_A > \sigma_B$

⇒ The graph actually never touches the x-axis!

The Normal Distribution

⇒ $X \sim N(\mu, \sigma^2)$ describes a normally distributed continuous random variable

X: random variable

μ: mean

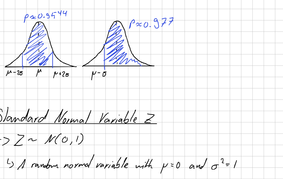
σ^2: variance (normal distribution)^2

has its variables distribution is a normal curve

⇒ Probability that X is between a and b equals the area under the curve between $x=a$ and $x=b$

$$P(a \le X \le b) = \int_a^b f(x)\,dx$$

Distribution Facts



Standard Normal Variable Z

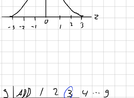
⇒ $Z \sim N(0,1)$

↳ A random normal variable with $\mu = 0$ and $\sigma^2 = 1$

⇒ normal random variables can be transformed to Z to simplify calculations

⇒ Attributes of Z's distribution:

- $\mu_z = 0$, $\sigma^2 = 1$
- $\pm 1, \pm 2, \pm 3$ on the graph actually represent standard deviations correctly
- function of area beneath $\phi(z)$
- area under $z = \phi(z) = 1$
- $\phi(z)$ is found using a table, not by integration
- $\phi(0) = 0.5$



Accumulation Z Table

z	0	1	2	3	4 … 9	0	1	2	3	4 … 9
0.0	0.5000	0.5040	…							
0.1	0.…	0.5398	0.5438			14	20	31		…
0.2	0.…	…	…	0.5793		19	26	15		
0.3	⋮	⋮	⋮	⋮		⋮	⋮	26		
⋮										

⇒ if $z = 0.1\boxed{3}$ then $\phi(z) = 0.5773$

*Za ← add to last two digits

$= 0.5773$

⇒ For negative values of z $\phi(z)$ produces exactly the same values but by symmetry

Standardizing a Normal Distribution

⇒ Any normal random variable can be normalized to Z

↳ this allows us to find probabilities using the table for Z

⇒ Normalize X: $\mu_z = 0$ so $X - \mu = 0$

$\sigma_z = 1$ so $(X - \mu) \cdot \frac{1}{\sigma_X}$

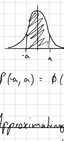
⇒ Recall:

when $X \sim N(\mu, \sigma^2)$ then $Z = \frac{X-\mu}{\sigma}$

and $z = \frac{x-\mu}{\sigma}$ ← how many standard deviations is x from μ

Calculating P From $\phi(z)$

For $0 < a < b$



$P(a < Z < b) = \phi(b) - \phi(a)$

For $a < 0 < b$

$P(a < Z < b) = \phi(b) + \phi(-a) - 1$

↳ $P(0, a) = \phi(a) - \phi(0)$

$= \phi(a) - 0.5$

$P(0, b) = \phi(b) - \phi(0)$

$= \phi(b) - 0.5$

$P(0, b) + P(0, a) = \phi(b) + \phi(a) - 1$

$P(-a, a) = \phi(a)^2 - 1$

Approximating Binomial Distributions

⇒ Given that the binomial distribution looks normal, it can be approximated with a normal distribution

↳ it's easier to calculate probabilities with the normal distribution

⇒ A binomial distribution is considered normal enough, when:

$np > 5$ and $nq > 5$ ← q = probability of failure $= (1-p)$

then $X \sim B(n, p)$ is approximated by

$N(\mu, \sigma^2)$ $\mu = np$ $\sigma^2 = (np)^2 = (np(1-p))^2$

⇒ Requires Continuity Correction

⇒ Discrete values from the binomial distribution will be interpreted by class of continuous values

e.g. $X = 13$ ⇒ $12.5 \le 13 < 13.5$

⇒ if we want to include 13, we use 13.5, the 12.5

$P(X < 13)$ ⇒ $X = 12.5$

$P(X \le 13)$ ⇒ $X = 13.5$

⇒ Basically $P(X < a) \approx P(X \le (a-1))$