1 Q2. Inventory Pooling

Primitives
D = demands
Q = quantity ordered
P = price
h = inventory costs = c-s
b = backholding costs = p-c

First we will show that $Q_p^* = \sqrt{n}Q^* + \mu(n - \sqrt{n})$

$$\begin{aligned}
P(\sum_{L=1}^{n} D_i \leq Q_p^*) =& \frac{b}{b+R} \\
&= P(\sqrt{n}D_i + \mu(n - \sqrt{n}) \leq Q_p^*) = \frac{b}{b+R} \\
&= P(D_i \leq \frac{1}{\sqrt{n}}(Q_p^* - \mu(n - \sqrt{n}))) = \frac{b}{b+h}
\end{aligned}$$

which implies that this is equal to Q^*. Thus, we can use the hint to find the following:

$$\begin{aligned}
G(Q) &= CQ - PD + h(Q - D)^+ + b(D - Q)^+ \\
G(Q) &= CQ - PE[D] + hE[Q - D]^+ bE[D - Q]^+ \\
G'(Q^*) &= C + hP(D \leq Q^*) - b(1 - P[D \leq Q^*]) = 0 \\
P(D \leq Q^*) &= \frac{-c + b}{h + b}
\end{aligned}$$

By using the hint, we can prove the desired result:

$$\begin{aligned}
nG(Q^*) &= n[CQ^* - PE[D] + hE[(Q^* - D)^+] + bE[(D - Q^*)^+]] \\
G(Q_p^*) &= [CQ_p^* - PE[\sum_{i=1}^{n} D_i] + hE[(Q_p^* - \sum_{i=1}^{n} D_i)^+] + bE[(\sum_{i=1}^{n} D_i - Q_p^*)^+]]
\end{aligned}$$

Since $\sum_{i=1}^{n} D_i = \sqrt{n}D_i + \mu(n - \sqrt{n})$ and $Q_P^* = \sqrt{n}Q^* + \mu(n - \sqrt{n})$, we can show that:

$$\begin{aligned}
G(Q_P^*) =& [C\sqrt{n}Q^* + \mu(n - \sqrt{n}) - PE[\sqrt{n}D_i + \mu(n - \sqrt{n})^+] \\
&+ hE[(\sqrt{n}Q^* + \mu(n - \sqrt{n}) - \sqrt{n}D_i - \mu(n - \sqrt{n}))^+] \\
&+ bE[\sqrt{n}D_i + \mu(n - \sqrt{n}) - \sqrt{n}Q^* + \mu(n - \sqrt{n})^+]]
\end{aligned}$$

$$
\begin{aligned}
G(Q_P^*) =&[C\sqrt{n}Q^* + \mu(n-\sqrt{n}) - PE[\sqrt{n}D_i + \mu(n-\sqrt{n})^+] \\
&+ hE[(\sqrt{n}Q^* - \sqrt{n}D_i] \\
&+bE[\sqrt{n}D_i - \sqrt{n}Q^*]]
\end{aligned}
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