1 Q2. Inventory Pooling

Primitives
D = demands
Q = quantity ordered
P = price
h = inventory costs = c-s
b = backholding costs = p-c

First we will show that $Q_p^* = \sqrt{n}Q^* + \mu(n - \sqrt{n})$

$$\begin{aligned}
P(\sum_{L=1}^{n} D_i \leqslant Q_p^*) =& \frac{b}{b+R} \\
&= P(\sqrt{n}D_i + \mu(n-\sqrt{n}) \leqslant Q_p^*) = \frac{b}{b+R} \\
&= P(D_i \leqslant \frac{1}{\sqrt{n}}(Q_p^* - \mu(n-\sqrt{n}))) = \frac{b}{b+h}
\end{aligned}$$

which implies that this is equal to Q^*. Thus, we can use the hint to find the following:

$$\begin{aligned}
G(Q) &= h(Q-D)^+ + b(D-Q)^+ \\
G(Q) &= hE[Q-D]^+ bE[D-Q]^+ \\
G'(Q^*) &= hP(D \leqslant Q^*) - b(1 - P[D \leqslant Q^*]) = 0 \\
P(D \leqslant Q^*) &= \frac{b}{h+b}
\end{aligned}$$

Next, we will apply the hint to prove the desired result:

$$\begin{aligned}
nG(Q^*) &= n[hE[(Q^* - D)^+] + bE[(D - Q^*)^+]] \\
G(Q_p^*) &= [hE[(Q_p^* - \sum_{i=1}^{n} D_i)^+] + bE[(\sum_{i=1}^{n} D_i - Q_p^*)^+]]
\end{aligned}$$

Since $\sum_{i=1}^{n} D_i = \sqrt{n}D_i + \mu(n-\sqrt{n})$ and $Q_P^* = \sqrt{n}Q^* + \mu(n-\sqrt{n})$, we can show that:

$$\begin{aligned}
G(Q_P^*) = [hE[(\sqrt{n}Q^* + \mu(n-\sqrt{n}) - \sqrt{n}D_i - \mu(n-\sqrt{n}))^+] \\
+bE[\sqrt{n}D_i + \mu(n-\sqrt{n}) - \sqrt{n}Q^* + \mu(n-\sqrt{n})^+]]
\end{aligned}$$

$$= [\sqrt{n}hE[(Q^* - D_i] + \sqrt{n}\ bE[D_i - Q^*]] = \frac{nG(Q^*)}{\sqrt{n}}$$

Problem 4 (An Investment Problem)