Stochastic Models and Optimization: Problem Set 4

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March 18, 2017

Q1

We are given a linear-quadratic problem with perfect state information but with a forecast. We first set up the primitives of the system:
x_k: the state in period k
u_k: the decision variable in period k
w_k: the disturbances in period k
y_k: an accurate prediction that w_k will be selected according to a particular probability distribution $P_{k|y_k}$

We set up the dynamics of the problem with a linear system, as follows:

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

Where A and B are $n \times n$ matrices.
We also have a quadratic cost:

$$g_N(x_N) = x_N' Q_N x_N$$
$$g_k(x_k) = x_k' Q_k x_k + u_k' R_k u_k$$

Where Q_k and R_k are $n \times n$ positive definite matrices.

Our problem is thus to minimise:

$$E[\sum_{k=0}^{N-1}(x_k' Q_k x_k + u_k' R_k u_k) + x_N' Q_N x_N]$$

Subject to:

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

We can now set up our DP-algorithm:

$$J_N(x_N, y_N) = x_N' Q_N x_N$$
$$J_k(x_k, y_k) = \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + J_{k+1}(x_{k+1}, y_{k+1})]$$

We can see that $J_N(x_N)$ is of the form $J(x_k, y_k) = x_k' K_k x_k + x_k' b_k(y_k) + c(y_k)$, with$Q_n = K_n$, $x_N' b_N(y_N) = 0$ and $c(y_k) = 0$.
We assume that this is also true at stage k+1, so that:

$$J(x_{k+1}, y_{k+1}) = x_{k+1}' K_{k+1} x_{k+1} + x_{k+1}' b_{k+1}(y_{k+1}) + c(y_{k+1})$$

Where $b_{k+1}(y_{k+1})$ is an n-dimensional vector and $c(y_{k+1})$ is a scalar. Using this, we can compute $J_k(x_k, y_k)$:

$$
\begin{aligned}
J_k(x_k, y_k) &= \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + J_{k+1}(x_{k+1}, y_{k+1})] = \\
&= \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + x_{k+1}' K_{k+1} x_{k+1} + x_{k+1}' b_{k+1}(y_{k+1}) + c(y_{k+1}) \\
&= \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + (A_k x_k + B_k u_k + w_k)' K_{k+1}(A_k x_k + B_k u_k + w_k) + \\
&+ (A_k x_k + B_k u_k + w_k)' b_{k+1}(y_{k+1}) + c(y_{k+1})] = x_k'(Q_k + A_k' K_{k+1} A_k) x_k + E[w_k' K_{k+1} w_k] + \\
&+ \min_{u_k \in R^n} [u_k'(R_k + B_k' K_{k+1} B_k) u_k + 2x_k' A_k K_{k+1} B_k u_k + 2u_k B_k K_{k+1} E[w_k | y_k] + \\
&+ u_k' B_k' b_{k+1}(y_{k+1})] + x_k' A_k' K_{k+1} E[w_k | y_k] + x_k' A_k' b_{k+1}(y_{k+1}) + E[w_k | y_k]' b_{k+1}(y_{k+1})
\end{aligned}
$$

We then find our optimal decision by taking the derivative of $J_k(x_k, y_k)$ with respect to u_k and set it equal to zero in order to solve for our optimal solution $u*$:

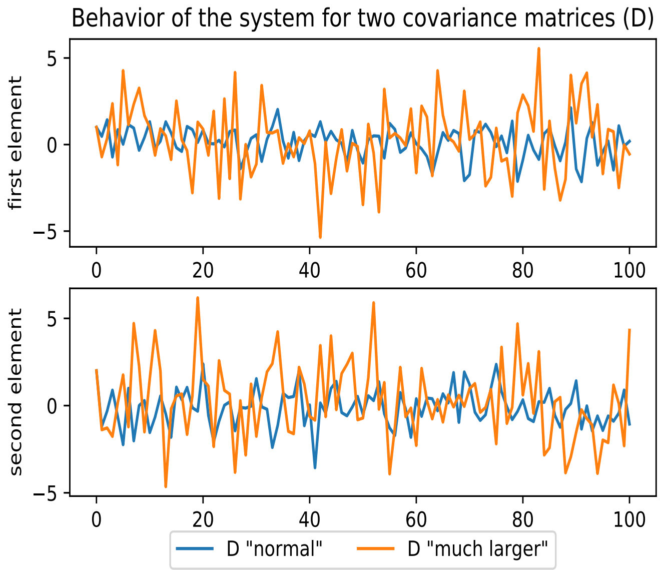
$$2(R_k + B_k' K_{k+1} B_k) u_k^* + 2B_k' K_{k+1} A_k x_k + 2B_k K_{k+1} B_k + 2B_k K_{k+1} E[w_k | y_k] + B_k' b_{k+1}(y_{k+1}) = 0$$

$$u_k^* = (R_k + B_k' K_{k+1} B_k)^{-1} B_k' K_{k+1}(A_k x_k + E[w_k | y_k]) + \alpha_k$$

Where $\alpha_k = B_k' b_{k+1}(y_{k+1})$

Q2

(ii)



Python Code:

```
# Import libraries
import numpy as np #Load numpy library for matrices operations
import matplotlib.pyplot as plt # library for graphics
from matplotlib.backends.backend_pdf import PdfPages # save graphics as pdf
import control #Load control for solving Riccati Equation
np.random.seed(1234)

# System components that are not going to be modified
N = 100 # horizon (time periods)
n = 2

A = np.array([[2, 0], [1, 0]])
B = np.array([[0, 2], [1, 1]])
C = np.array([[0, 3]])

Q = np.matmul(np.transpose(C), C)
#np.linalg.eigvals(Q)

# Behaviour of the system
# ii ---------------------------------------------------------------
# Fix R and x[0], D_1 = normal - D_2 = "much larger"
R = np.diag(np.repeat(1, [n], axis = 0))
x_1 = np.empty([N + 1, n])
x_1[0] = [1, 2]
```

```
D_1 = np.diag(np.repeat(1, [n], axis = 0))
w_1 = np.random.multivariate_normal(mean = np.repeat(0, [n], axis = 0), cov = D_1, size = N)
x_2 = np.empty([N + 1, n])
x_2[0] = x_1[0]
D_2 = np.diag(np.repeat(5, [n], axis = 0))
w_2 = np.random.multivariate_normal(mean = np.repeat(0, [n], axis = 0), cov = D_2, size = N)
#np.linalg.eigvals(K)
K = np.empty([N + 1, n, n])
K[N] = Q
for i in range(100,0,-1):
    K[i-1] = np.matmul(np.matmul(np.transpose(A), K[i] - np.matmul(np.matmul(K[i], B), np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K[i]), B) + R), np.matmul(np.transpose(B), K[i]))), A) + Q

L = np.empty([N, n, n])
for i in range(0, N):
    L[i] = - np.matmul(np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K[i+1]), B) + R), np.matmul(np.matmul(np.transpose(B), K[i+1]), A))

for i in range(0, N):
    # Values normal
    x_1[i+1] = np.matmul(A + np.matmul(B, L[i]), x_1[i]) + w_1[i]
    # Values much larger
    x_2[i+1] = np.matmul(A + np.matmul(B, L[i]), x_2[i]) + w_2[i]

# plot
plot_fig2 = plt.figure(2)
plt.subplot(211)
plt.title('Behavior of the system for two covariance matrices (D)')
plt.ylabel('first element')
plt.plot(x_1.T[0])
plt.plot(x_2.T[0])
plt.subplot(212)
plt.ylabel('second element')
plt.plot(x_1.T[1], label = 'D "normal"')
plt.plot(x_2.T[1], label = 'D "much larger"')
plt.legend(loc='lower center', bbox_to_anchor=(0.5, -0.4), ncol=2)

# iii ------------------------------------------------------------
# Fix R and D, x_1[0] = normal - x_2[0] = "much larger"
R = np.diag(np.repeat(1, [n], axis = 0))
x_1 = np.empty([N + 1, n])
x_1[0] = [1, 2]
D = np.diag(np.repeat(1, [n], axis = 0))
w = np.random.multivariate_normal(mean = np.repeat(0, [n], axis = 0), cov = D, size = N)
x_2 = np.empty([N + 1, n])
x_2[0] = [35, 41]
K = np.empty([N + 1, n, n])
K[N] = Q
for i in range(100,0,-1):
    K[i-1] = np.matmul(np.matmul(np.transpose(A), K[i] - np.matmul(np.matmul(K[i
```

```python
], B), np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K[i]), B) + R), np.matmul(np.transpose(B), K[i]))), A) + Q

L = np.empty([N, n, n])
for i in range(0, N):
    L[i] = - np.matmul(np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K[i+1]), B) + R), np.matmul(np.matmul(np.transpose(B), K[i+1]), A))

for i in range(0, N):
    # Values normal
    x_1[i+1] = np.matmul(A + np.matmul(B, L[i]), x_1[i]) + w[i]
    # Values much larger
    x_2[i+1] = np.matmul(A + np.matmul(B, L[i]), x_2[i]) + w[i]

# plot
plot_fig3 = plt.figure(3)
plt.subplot(211)
plt.title('Behavior of the system for two initial states (x0)')
plt.ylabel('first element')
plt.plot(x_1.T[0])
plt.plot(x_2.T[0])
plt.subplot(212)
plt.ylabel('second element')
plt.plot(x_1.T[1], label = 'x0 "normal"')
plt.plot(x_2.T[1], label = 'x0 "much larger"')
plt.legend(loc='lower center', bbox_to_anchor=(0.5, -0.4), ncol=2)

# iv ------------------------------------------------------------
# Fix x[0] and D, R_1[0] = normal - R_2[0] = "much larger"
R_1 = np.diag(np.repeat(1, [n], axis = 0))
R_2 = np.diag(np.repeat(100, [n], axis = 0))
x_1 = np.empty([N + 1, n])
x_1[0] = [1, 2]
D = np.diag(np.repeat(1, [n], axis = 0))
w = np.random.multivariate_normal(mean = np.repeat(0, [n], axis = 0), cov = D, size = N)
x_2 = np.empty([N + 1, n])
x_2[0] = x_1[0]

K_1 = np.empty([N + 1, n, n])
K_1[N] = Q
for i in range(100,0,-1):
    K_1[i-1] = np.matmul(np.matmul(np.transpose(A), K_1[i] - np.matmul(np.matmul(K_1[i], B), np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K_1[i]), B) + R_1), np.matmul(np.transpose(B), K_1[i])))), A) + Q

L_1 = np.empty([N, n, n])
for i in range(0, N):
    L_1[i] = - np.matmul(np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K_1[i+1]), B) + R_1), np.matmul(np.matmul(np.transpose(B), K_1[i+1]), A))

K_2 = np.empty([N + 1, n, n])
K_2[N] = Q
for i in range(100,0,-1):
```

```
    K_2[i-1] = np.matmul(np.matmul(np.transpose(A), K_2[i] - np.matmul(np.matmul
        (K_2[i], B), np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K_2[i]),
        B) + R_2), np.matmul(np.transpose(B), K_2[i])))), A) + Q

L_2 = np.empty([N, n, n])
for i in range(0, N):
    L_2[i] = - np.matmul(np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K_2[
        i+1]), B) + R_2), np.matmul(np.matmul(np.transpose(B), K_2[i+1]), A))

for i in range(0, N):
    # Values normal
    x_1[i+1] = np.matmul(A + np.matmul(B, L_1[i]), x_1[i]) + w[i]
    # Values much larger
    x_2[i+1] = np.matmul(A + np.matmul(B, L_2[i]), x_2[i]) + w[i]

# plot
plot_fig4 = plt.figure(4)
plt.subplot(211)
plt.title('Behavior of the system for two input-cost matrices (R)')
plt.ylabel('first element')
plt.plot(x_1.T[0])
plt.plot(x_2.T[0])
plt.subplot(212)
plt.ylabel('second element')
plt.plot(x_1.T[1], label = 'R "normal"')
plt.plot(x_2.T[1], label = 'R "much larger"')
plt.legend(loc='lower center', bbox_to_anchor=(0.5, -0.4), ncol=2)

# v ---------------------------------------------------------------
# Fix x[0] and D, R_1[0] = normal - R_2[0] = "much larger"
R = np.diag(np.repeat(1, [n], axis = 0))
x_1 = np.empty([N + 1, n])
x_1[0] = [1, 2]
D = np.diag(np.repeat(1, [n], axis = 0))
w = np.random.multivariate_normal(mean = np.repeat(0, [n], axis = 0), cov = D,
    size = N)
x_2 = np.empty([N + 1, n])
x_2[0] = x_1[0]

K_1 = np.empty([N + 1, n, n])
K_1[N] = Q
for i in range(100,0,-1):
    K_1[i-1] = np.matmul(np.matmul(np.transpose(A), K_1[i] - np.matmul(np.matmul
        (K_1[i], B), np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K_1[i]),
        B) + R), np.matmul(np.transpose(B), K_1[i])))), A) + Q

L_1 = np.empty([N, n, n])
for i in range(0, N):
    L_1[i] = - np.matmul(np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K_1[
        i+1]), B) + R), np.matmul(np.matmul(np.transpose(B), K_1[i+1]), A))

K_2, G, E = control.dare(A = A, B = B, Q = Q, R = R)

L_2 = - np.matmul(np.linalg.inv(np.matmul(np.matmul(np.transpose(B), K_2), B) +
```

```
    R), np.matmul(np.matmul(np.transpose(B), K_2), A))

for i in range(0, N):
    # Values normal
    x_1[i+1] = np.matmul(A + np.matmul(B, L_1[i]), x_1[i]) + w[i]
    # Values much larger
    x_2[i+1] = np.matmul(A + np.matmul(B, L_2), x_2[i]) + w[i]

# plot
plot_fig5 = plt.figure(5)
plt.subplot(211)
plt.title('Behavior of the system optimal control vs steady-state control')
plt.ylabel('first element')
plt.plot(x_1.T[0])
plt.plot(x_2.T[0])
plt.subplot(212)
plt.ylabel('second element')
plt.plot(x_1.T[1], label = 'Optimal control')
plt.plot(x_2.T[1], label = 'Steady-state control')
plt.legend(loc='lower center', bbox_to_anchor=(0.5, -0.4), ncol=2)

pp = PdfPages('/home/chpmoreno/Dropbox/Documents/BGSE/Second_Term/SMO/
    Problemsets/PS4/figures.pdf')
pp.savefig(plot_fig2)
pp.savefig(plot_fig3)
pp.savefig(plot_fig4)
pp.savefig(plot_fig5)
pp.close()
```

Q3

Asset selling w/offer estimation

Primitives

- x_k current offer.
- $\{w_k\}$ iid offers with an unknown underlying distribution F_1 or F_2.
- constraints: selling (u_1) or not selling (u_2)): $\begin{cases} \{u^1, u^2\} \text{ if } x_k \neq \text{T} \\ 0, \text{otherwise} \end{cases}$
- rewards: $g_n(x_N) = \begin{cases} x_N, \text{ if } x_N \neq \text{T} \\ 0, \text{otherwise} \end{cases}$
  $g_k(x_k, u_k, w_k) = \begin{cases} (1+r)^{N-k} x_k, \text{ if } x_k \neq \text{T and if } u_k = u^1 \\ 0, \text{otherwise} \end{cases}$
- q = prior belief that F_1 is true
- $q_{k+1} = \frac{\mathbb{P}\{y_k = y^1 | w_0, \ldots, w_k\}}{\mathbb{P}(w_1 = w_1)} = \frac{q_k F_1(w_k)}{q_k F_1(w_k) + (1-q) F_2(w_k)}$

Now, we can apply the DP algorithm to find an optimal asset selling policy

$$J_{N-1}(x_{N-1}) = \begin{cases} \max\{(1+r)x_k, (q_{N-1}\mathbb{E}_{F_1}[J_N(w_{N-1})] + (1-q_{N-1})\mathbb{E}_{F_2}[J_N(w_{N-1})]) \text{ if } x_{N-1} \neq T \\ 0, \text{otherwise} \end{cases}$$

$$J_k(x_k) = \begin{cases} \max(q_k \mathbb{E}_{F_1}[J_{k+1}(w_k)] + (1-q_k)\mathbb{E}_{F_2}[J_{k+1}(w_k)]), (1+r)^{N-k} x_k \text{ if } x_k \neq T \\ 0, \text{otherwise} \end{cases}$$

Thus, the threshold for selling an asset will be: $(1+r)^{N-k} x_k \geqslant q_k \mathbb{E}_{F_1}[J_{k+1}(w_k)] + (1-q_k)\mathbb{E}_{F_2}[J_{k+1}(w_k)]$

So, the optimal asset selling policy will be: $\mu^*(x_k) = \begin{cases} u^1, x_{k+1} \geqslant \frac{q_k \mathbb{E}_{F_1}[J_{k+1}(w_k)] + (1-q_k)\mathbb{E}_{F_2}[J_{k+1}(w_k)]}{(1+r)^{N-k}} \\ u^2, \text{otherwise} \end{cases}$

Q4

This problem is basically the same as the inventory management considering the demand as a random variable following an unknown distribution. It is a case with imperfect state information, in which the distribution of demand will be either F_1 or F_2. The probability that the demand follows F_1 is updated at each period k after observing the realization of the demand. That will effect the way the expectation of the demand is computed.

Primitives:
x_k: items in the inventory at period k.
u_k: quantity ordered at period k.
w_k: demand during period k. w_k are iid with probability distribution either F_1 or F_2.
q_k: probability that w_k follows distribution F_1.
$q_0 = q$: a priori probability that demand follows the distribution F_1.

Dynamics:
$x_{k+1} = x_k + u_k - w_k$
$q_{k+1} = \frac{q_k f_1(w_k)}{q_k f_1(w_k) + (1-q_k) f_2(w_k)}$, where $f_i(w)$ is the pdf of the distribution F_i.

Cost:
$g_N(x_N) = 0$.
$g_k(x_k, u_k, w_k) = cu_k + h\max\{0, w_k - x_k - u_k\} + p\max\{0, x_k + u_k - w_k\}$, where c, h, p are positive and $p > c$.

DP algorithm:

$J_N(x_N) = 0$
$J_k(x_k) = \min\limits_{u_k \geqslant 0} \mathbb{E}\left[cu_k + h\max\{0, w_k - x_k - u_k\} + p\max\{0, x_k + u_k - w_k\} + J_{k+1}(x_{k+1})\right]$

In order to solve it we can introduce the variable $y_k = x_k + u_k$, and the we have
$J_k(y_k) = \min\limits_{u_k \geqslant x_k} G_k(y_k) - cx_k$, where

$$G_k(y_k) = cy + h\mathbb{E}[\max\{0, w_k - y_k\}] + p\mathbb{E}[\max\{0, y_k - w_k\}] + \mathbb{E}[J_{k+1}(y_k - w_k)].$$

Now, since w_k is drawn from F_1 with probability q_k and from F_2 with probability F_2 we can apply the law of total probabilities, leading to

$$\begin{aligned} G(y_k) = cy_k &+ q_k(h\mathbb{E}_{w_k|w\sim F_1}[\max\{0, w_k - y_k\}] + p\mathbb{E}_{w_k|w\sim F_1}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w\sim F_1}[J_{k+1}(y_k - w_k)]) + \\ &+ (1-q_k)(h\mathbb{E}_{w_k|w\sim F_2}[\max\{0, w_k - y_k\}] + p\mathbb{E}_{w_k|w\sim F_2}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w\sim F_2}[J_{k+1}(y_k - w_k)]). \end{aligned}$$

We saw in class that $cy_k + h\mathbb{E}_{w_k|w\sim F_i}[\max\{0, w_k - y_k\}] + p\mathbb{E}_{w_k|w\sim F_i}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w\sim F_i}[J_{k+1}(y_k - w_k)]$ is convex, since we have a sum of convex, our $G(y_k)$ will also be convex. So, there exists a S_k that will represent the optimal stock we seek at period k. However, S_k could be smaller than x_k, so it would not be reachable (in which case we would not by stock). Then, the policy will be

$$\mu_k^*(x_k) = \begin{cases} S_k - x_k & \text{if } S_k > x_k \\ 0 & \text{otherwise.} \end{cases}$$

Q5

(a) To find the policy that will minimize the cost function in the worst case. We can write a DP-like algorithm to solve this problem as

$J_N(x_N) = g_N(x_N)$

$J_k(x_k) = \min_{\mu_k \in U_k} \max_{w_k \in W_k(x_k, \mu_k(x_k))} [g_k(x_k, \mu_k(x_k), w_k) + J_{k+1}(x_{k+1})]$, where $x_{k+1} = f_k(x_k, \mu_k(x_k), w_k)$ $\forall k = 0, \ldots, N-1$.

(b) In this problem the state x_k needs to belong to a certain set X_k at each state k, so we have to choose a policy that will assure that this will happen. We write the dynamics in a general form like $x_{k+1} = f(x_k, \mu_k(x_k)) + h_k(w_k)$.

We consider a cost of the policy $g(\mu_k(x_k))$ and the cost of being outside X_k infinite to ensure we stay whenever possible there.

DP algorithm:

$$J_N(x_N) = \begin{cases} 0 & \text{if } x_N \in X_N \\ \infty & \text{otherwise} \end{cases}$$

$$J_{k+1} = \min_{\mu_k(x_k) \in U_k} \max_{w_k \in W_k(x_k, \mu_k(x_k))} \begin{cases} g_k(\mu(x_k)) + J_{k+1}(x_{k+1}) & \text{if } x_k \in X_k \\ \infty & \text{otherwise} \end{cases},$$

where $x_{k+1} = f_k(x_k, \mu_k(x_k)) + h_k(w_k)$. Then, we will need to compute recursively the $\bar{X}_k$ that allow to continue in the sets $X_{k+1}, \ldots, X_N$. In order to do so, x_k will need to be in a certain set Y_{k+1} such that for any $w_k \in W_k$, $f(x_k, \mu_k(x_k)) + h(w_k)$ belong to $\bar{X}_{k+1}$. Initializing $\bar{X}_N = X_N$, $\forall k = 1, \ldots, N-1$, the recursion would be the next:

$$Y_{k+1} = \{z \in \mathcal{X} : z + h_k(w_k) \in \bar{X}_{k+1}, \forall w_k \in W_k(x_k, \mu_k(x_k))\}$$

$$\bar{X}_k = \{x \in X_k : \exists \mu_k(x) \in U_k \text{ such that } f_k(x, \mu_k(x_k)) \in Y_{k+1}\}$$