Stochastic Models and Optimization: Problem Set 4

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1 Problem 1 - linear-quadratic problem with forecasts

We are given a linear-quadratic problem with perfect state information but with a forecast. We first set up the primitives of the system:
x_k: the state in period k
u_k: the decision variable in period k
w_k: the disturbances in period k
y_k: an accurate prediction that w_k will be selected according to a particular probability distribution $P_{k|y_k}$

We set up the dynamics of the problem with a linear system, as follows:

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

Where A and B are $n \times n$ matrices.
We also have a quadratic cost:

$$g_N(x_N) = x_N' Q_N x_N$$

$$g_k(x_k) = x_k' Q_k x_k + u_k' R_k u_k$$

Where Q_k and R_k are $n \times n$ positive definite matrices.

Our problem is thus to minimise:

$$E[\sum_{k=0}^{N-1} (x_k' Q_k x_k + u_k' R_k u_k) + x_N' Q_N x_N]$$

Subject to:

$$x_{k+1} = A_k x_k + B_k u_k + w_k$$

We can now set up our DP-algorithm:

$$J_N(x_N, y_N) = x_N' Q_N x_N$$

$$J_k(x_k, y_k) = \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + J_{k+1}(x_{k+1}, y_{k+1})]$$

We can see that $J_N(x_N)$ is of the form $J(x_k, y_k) = x_k' K_k x_k + x_k' b_k(y_k) + c(y_k)$, with$Q_n = K_n$, $x_N' b_N(y_N) = 0$ and $c(y_k) = 0$.
We assume that this is also true at stage k+1, so that:

$$J(x_{k+1}, y_{k+1}) = x_{k+1}' K_{k+1} x_{k+1} + x_{k+1}' b_{k+1}(y_{k+1}) + c(y_{k+1})$$

Where $b_{k+1}(y_{k+1})$ is an n-dimensional vector and $c(y_{k+1})$ is a scalar. Using this, we can compute $J_k(x_k, y_k)$:

$$J_k(x_k, y_k) = \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + J_{k+1}(x_{k+1}, y_{k+1})]$$

$$= \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + x_{k+1}' K_{k+1} x_{k+1} + x_{k+1}' b_{k+1}(y_{k+1}) + c(y_{k+1})$$

$$= \min_{u_k \in R^n} E_{w_k}[x_k' Q_k x_k + u_k' R_k u_k + (A_k x_k + B_k u_k + w_k)' K_{k+1}(A_k x_k + B_k u_k + w_k) + (A_k x_k + B_k u_k + w_k)' b_{k+1}(y_{k+1}) + c(y_{k+}$$

$$= x_k'(Q_k + A_k' K_{k+1} A_k) x_k + E[w_k' K_{k+1} w_k] + \min_{u_k \in R^n}[u_k'(R_k + B_k' K_{k+1} B_k) u_k + 2x_k' A_k K_{k+1} B_k u_k + 2u_k B_k K_{k+1} E[w_k | y_k] +$$

We then find our optimal decision by taking the derivative of $J_k(x_k, y_k)$ with respect to u_k and set it equal to zero in order to solve for our optimal solution $u*$:

$$2(R_k + B_k' K_{k+1} B_k) u_k^* + 2B_k' K_{k+1} A_k x_k + 2B_k K_{k+1} B_k + 2B_k K_{k+1} E[w_k | y_k] + B_k' b_{k+1}(y_{k+1}) = 0$$

$$u_k^* = (R_k + B_k' K_{k+1} B_k)^{-1} B_k' K_{k+1}(A_k x_k + E[w_k | y_k]) + \alpha_k$$

Where $\alpha_k = B_k' b_{k+1}(y_{k+1})$

Q2

Q3

Asset selling w/offer estimation

Primitives

- x_k current offer.
- $w_0, w_1, ...w_{n-1}$ of iid offers with unknown distribution
- an underlying distribution of the offers w (i.e. the hidden state x_k) F_1 or F_2, thus $y_k = y^1$ if true distribution is F_1 and y^2 if the true distribution is F_2
- constraints (if seller sells (u_1) or not (u_2)): $\left\{ \begin{array}{l} u^1, u^2 \text{ if } x_k \neq \text{T} \\ 0, \text{otherwise} \end{array} \right\}$
- rewards: $g_n(x_N) = \left\{ \begin{array}{l} x_N, \text{ if } x_N \neq \text{T} \\ 0, \text{otherwise} \end{array} \right\}$
  $g_k(x_k, u_k, w_k) = \left\{ \begin{array}{l} (1+r)^{N-k} x_k, \text{ if } x_k \neq \text{T and if } u_k = u^1 \\ 0, \text{otherwise} \end{array} \right\}$
- q = prior belief that F_1 is true
- $q_{k+1} = \frac{\mathbb{P}\{y_k = y^1 | w_0, ..., w_k\}}{\mathbb{P}(w_1 = w_1)} = \frac{q_k F_1(w_k)}{q_k F_1(w_k) + (1-q) F_2(w_k)}$

Now, we can apply the DP algorithm to find an optimal asset selling policy

$$J_{N-1}(P_{N-1}) = \left\{ \begin{array}{l} (P_{N-1}\mathbb{E}_{F_1}[w_{N-1}] + (1 - P_{N-1})\mathbb{E}_{F_2}[w_{N-1}])(1+r)^{N-k} \\ 0, \text{otherwise} \end{array} \right\} \text{if } x_{N-1} \neq T$$

$$J_k(x_k) = \left\{ \begin{array}{l} \max(P_k \mathbb{E}_{F_1}[w_k] + (1 - P_k)\mathbb{E}_{F_2}[w_k])(1+r)^k, \mathbb{E}[J_{k+1}(w_k)] \\ 0, \text{otherwise} \end{array} \right\} \text{if } x_k \neq \text{T}$$

Thus, the threshold for selling an asset will be: $P_k \mathbb{E}_{F_1}(w_k) + (1 - P_k)\mathbb{E}_{F_2}(w_k) \geqslant \frac{\mathbb{E}[J_{k+1}(w_k)]}{(1+r)^{n-k}}$

And the optimal asset selling policy: $\mu^*(x_k) = \left\{ \begin{array}{l} u^*, \frac{\mathbb{E}[J_{k+1}(w_k)]}{(1+r)^{n-k}} \\ u^2, \text{ otherwise} \end{array} \right\}$

Q4

This problem is basically the same as the inventory management considering the demand as a random variable following an unknown distribution. It is a case with imperfect state information, in which the distribution of demand will be either F_1 or F_2. The probability that the demand follows F_1 is updated at each period k after observing the realization of the demand. That will effect the way the expectation of the demand is computed.

Primitives:
x_k: items in the inventory at period k.
u_k: quantity ordered at period k.
w_k: demand during period k. w_k are iid with probability distribution either F_1 or F_2.
q_k: probability that w_k follows distribution F_1.
$q_0 = q$: a priori probability that demand follows the distribution F_1.

Dynamics:
$x_{k+1} = x_k + u_k - w_k$
$q_{k+1} = \frac{q_k f_1(w_k)}{q_k f_1(w_k) + (1-q_k) f_2(w_k)}$, where $f_i(w)$ is the pdf of the distribution F_i.

Cost:
$g_N(x_N) = 0$.
$g_k(x_k, u_k, w_k) = cu_k + h \max\{0, w_k - x_k - u_k\} + p \max\{0, x_k + u_k - w_k\}$, where c, h, p are positive and $p > c$.

DP algorithm:

$J_N(x_N) = 0$
$J_k(x_k) = \min_{u_k \geqslant 0} \mathbb{E}\left[cu_k + h \max\{0, w_k - x_k - u_k\} + p \max\{0, x_k + u_k - w_k\} + J_{k+1}(x_{k+1})\right]$

In order to solve it we can introduce the variable $y_k = x_k + u_k$, and the we have
$J_k(y_k) = \min_{u_k \geqslant x_k} G_k(y_k) - cx_k$, where

$$G_k(y_k) = cy + h\mathbb{E}[\max\{0, w_k - y_k\}] + p\mathbb{E}[\max\{0, y_k - w_k\}] + \mathbb{E}[J_{k+1}(y_k - w_k)].$$

Now, since w_k is drawn from F_1 with probability q_k and from F_2 with probability F_2 we can apply the law of total probabilities, leading to

$$\begin{aligned} G(y_k) = cy_k &+ q_k(h\mathbb{E}_{w_k|w\sim F_1}[\max\{0, w_k - y_k\}] + p\mathbb{E}_{w_k|w\sim F_1}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w\sim F_1}[J_{k+1}(y_k - w_k)]) + \\ &+ (1 - q_k)(h\mathbb{E}_{w_k|w\sim F_2}[\max\{0, w_k - y_k\}] + p\mathbb{E}_{w_k|w\sim F_2}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w\sim F_2}[J_{k+1}(y_k - w_k)]). \end{aligned}$$

We saw in class that $cy_k + h\mathbb{E}_{w_k|w\sim F_i}[\max\{0, w_k - y_k\}] + p\mathbb{E}_{w_k|w\sim F_i}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w\sim F_i}[J_{k+1}(y_k - w_k)]$ is convex, since we have a sum of convex, our $G(y_k)$ will also be convex. So, there exists a S_k

that will represent the optimal stock we seek at period k. However, S_k could be smaller than x_k, so it would not be reachable (in which case we would not by stock). Then, the policy will be

$$\mu_k^*(x_k) = \begin{cases} S_k - x_k & \text{if } S_k > x_k \\ 0 & \text{otherwise.} \end{cases}$$