

Stochastic Models and Optimization: Problem Set 4

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Q1

Q2

Q3

Asset selling w/offer estimation

Primitives

- x_k current offer.
- w_0, w_1, \dots, w_{n-1} of iid offers with unknown distribution
- an underlying distribution of the offers w (i.e. the hidden state x_k) F_1 or F_2 , thus $y_k = y^1$ if true distribution is F_1 and y^2 if the true distribution is F_2
- constraints (if seller sells (u_1) or not (u_2)): $\left\{ \begin{array}{l} u^1, u^2 \text{ if } x_k \neq T \\ 0, \text{ otherwise} \end{array} \right\}$
- rewards: $g_n(x_N) = \left\{ \begin{array}{l} x_N, \text{ if } x_N \neq T \\ 0, \text{ otherwise} \end{array} \right\}$
 $g_k(x_k, u_k, w_k) = \left\{ \begin{array}{l} (1+r)^{N-k} x_k, \text{ if } x_k \neq T \text{ and if } u_k = u^1 \\ 0, \text{ otherwise} \end{array} \right\}$
- q = prior belief that F_1 is true
- $q_{k+1} = \frac{\mathbb{P}\{y_k=y^1|w_0,\dots,w_k\}}{\mathbb{P}(w_1=w_1)} = \frac{q_k F_1(w_k)}{q_k F_1(w_k) + (1-q_k) F_2(w_k)}$

Now, we can apply the DP algorithm to find an optimal asset selling policy

$$J_{N-1}(P_{N-1}) = \left\{ \begin{array}{l} (P_{N-1} \mathbb{E}_{F_1}[w_{N-1}] + (1 - P_{N-1}) \mathbb{E}_{F_2}[w_{N-1}]) (1+r)^{N-k} \\ 0, \text{ otherwise} \end{array} \right\} \text{ if } x_{N-1} \neq T$$

$$J_k(x_k) = \left\{ \begin{array}{l} \max(P_k \mathbb{E}_{F_1}[w_k] + (1 - P_k) \mathbb{E}_{F_2}[w_k]) (1+r)^k, \mathbb{E}[J_{k+1}(w_k)] \\ 0, \text{ otherwise} \end{array} \right\} \text{ if } x_k \neq T$$

Thus, the threshold for selling an asset will be: $P_k \mathbb{E}_{F_1}(w_k) + (1 - P_k) \mathbb{E}_{F_2}(w_k) \geq \frac{\mathbb{E}[J_{k+1}(w_k)]}{(1+r)^{n-k}}$

$$\text{And the optimal asset selling policy: } \mu^*(x_k) = \left\{ \begin{array}{l} u^*, \frac{\mathbb{E}[J_{k+1}(w_k)]}{(1+r)^{n-k}} \\ u^2, \text{ otherwise} \end{array} \right\}$$

Q4

This problem is basically the same as the inventory management considering the demand as a random variable following an unknown distribution. It is a case with imperfect state information, in which the distribution of demand will be either F_1 or F_2 . The probability that the demand follows F_1 is updated at each period k after observing the realization of the demand. That will effect the way the expectation of the demand is computed.

Primitives:

x_k : items in the inventory at period k .

u_k : quantity ordered at period k .

w_k : demand during period k . w_k are iid with probability distribution either F_1 or F_2 .

q_k : probability that w_k follows distribution F_1 .

$q_0 = q$: a priori probability that demand follows the distribution F_1 .

Dynamics:

$$x_{k+1} = x_k + u_k - w_k$$

$$q_{k+1} = \frac{q_k f_1(w_k)}{q_k f_1(w_k) + (1 - q_k) f_2(w_k)}, \text{ where } f_i(w) \text{ is the pdf of the distribution } F_i.$$

Cost:

$$g_N(x_N) = 0.$$

$$g_k(x_k, u_k, w_k) = cu_k + h \max\{0, w_k - x_k - u_k\} + p \max\{0, x_k + u_k - w_k\}, \text{ where } c, h, p \text{ are positive and } p > c.$$

DP algorithm:

$$J_N(x_N) = 0$$

$$J_k(x_k) = \min_{u_k \geq 0} \mathbb{E} [cu_k + h \max\{0, w_k - x_k - u_k\} + p \max\{0, x_k + u_k - w_k\} + J_{k+1}(x_{k+1})]$$

In order to solve it we can introduce the variable $y_k = x_k + u_k$, and then we have

$$J_k(y_k) = \min_{u_k \geq x_k} G_k(y_k) - cx_k, \text{ where}$$

$$G_k(y_k) = cy_k + h \mathbb{E}[\max\{0, w_k - y_k\}] + p \mathbb{E}[\max\{0, y_k - w_k\}] + \mathbb{E}[J_{k+1}(y_k - w_k)].$$

Now, since w_k is drawn from F_1 with probability q_k and from F_2 with probability F_2 we can apply the law of total probabilities, leading to

$$G(y_k) = cy_k + q_k(h \mathbb{E}_{w_k|w \sim F_1}[\max\{0, w_k - y_k\}] + p \mathbb{E}_{w_k|w \sim F_1}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w \sim F_1}[J_{k+1}(y_k - w_k)]) + (1 - q_k)(h \mathbb{E}_{w_k|w \sim F_2}[\max\{0, w_k - y_k\}] + p \mathbb{E}_{w_k|w \sim F_2}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w \sim F_2}[J_{k+1}(y_k - w_k)]).$$

We saw in class that $cy_k + h \mathbb{E}_{w_k|w \sim F_i}[\max\{0, w_k - y_k\}] + p \mathbb{E}_{w_k|w \sim F_i}[\max\{0, y_k - w_k\}] + \mathbb{E}_{w_k|w \sim F_i}[J_{k+1}(y_k - w_k)]$ is convex, since we have a sum of convex, our $G(y_k)$ will also be convex. So, there exists a S_k that will represent the optimal stock we seek at period k . However, S_k could be smaller than x_k , so it would not be reachable (in which case we would not buy stock). Then, the policy will be

$$\mu_k^*(x_k) = \begin{cases} S_k - x_k & \text{if } S_k > x_k \\ 0 & \text{otherwise.} \end{cases}$$