Stochastic Models and Optimization: Problem Set 2

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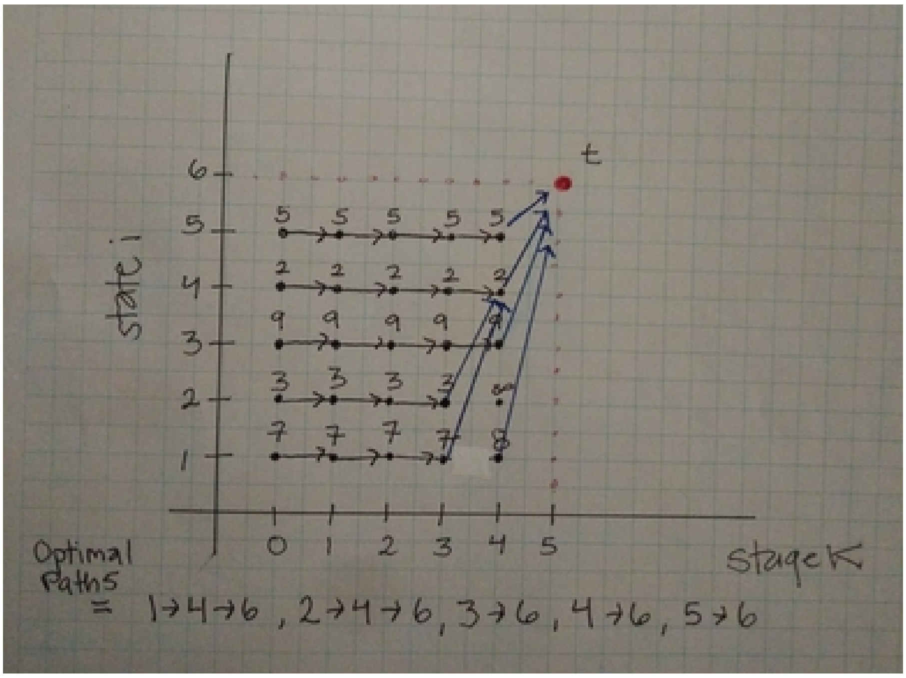
Problem 1 (Shortest Path using DP):

$$
\begin{aligned}
J_K(i) &= min[a_{ij} + J_{k+1}(j)] \\
K &= 0, 1, ..., N-2 \\
j &= 1, ..., N \\
J_{N-1}(i) &= a_{it}, i = 1, 2, ...N \\
J_K(i) &= \text{optimal cost of getting from i to t in N-k moves}
\end{aligned}
$$



Problem 2 (Shortest Path via Label Correcting Methods):

Table 1: Bellman Ford Algorithm

Iteration	Exiting Open	Open at end of Iteration	Upper
0	-	1	∞
1	1	1-2 (2), 1-3(1)	∞
2	1-2	1-3(1), 1-2-4(3)	2
3	1-3	1-2-4(3)	2
4	1-2-4	$\varnothing$	2

Table 2: Dijkstra's Algorithm

Iteration	Exiting Open	Open at end of Iteration	Upper
0	-	1	∞
1	1	1-2 (2), 1-3(1)	∞
2	1-3	1-2(2), 1-3-4(4)	∞
3	1-2	1-3-4(4), 1-2-4(3)	2
4	1-2-4	1-3-4(4)	2
5	1-3-4	$\varnothing$	2

Problem 3. Clustering: *We have a set of N objects, denoted $1, 2, ..., N$, which we want to group in clusters that consist of consecutive objects. For each cluster $i, i+1, ..., j$, there is an associated cost a_{ij}. We want to find a grouping of the objects in clusters such that the total cost is minimum. Formulate the problem as a shortest path problem, and write a DP algorithm for its solution.*

The primitives of the problem are:
x_k is the last node of a cluster, with $x_k \in S = 0, 1, ..., N$ for $k = 0, 1, ..., N$
u_k is the decision made at every step k over all objects i such that $i \geqslant x$.
a_{ij} is the cost of a cluster running from i to j.
Dynamics:
$x_{k+1} = u_k$ and
$x_0 = 0$
$u_k \in U_k(x) = i \in S | i \geqslant x$ if $x \neq N$ for $k = 0, 1, ..., N-1$ and
$u_k \in U_k(x) = N$ if $x = N$

$g_k(x, u) = a_{x+1,u}$ if $x \neq N$ for $k = 0, 1, ..., N-1$, and
$g_k(x, u) = 0$ if $x = N$

We then set up the DP algorithm as follows:
$J_N(N) = 0$
$J_k(i) = \min_{j \in S | j \geqslant i} [a_{i+1,j} + J_{k+1}(j)]$ if $x \neq N$ and for $k = 0, 1, ..., N-1$
$J_k(i) = 0$ if $i = N$
Return $J_0(0)$ as the lowest cost.

Problem 4 (Path Bottleneck Problem): *Consider the framework of the shortest path problem. For any path P, define the* ***bottleneck*** *arc of P as an arc that has maximum length over all arcs of P. We wish to find a path whose length of bottleneck arc is minimum, among all paths connecting the origin node to the destination node. Develop and justify an analog of the label correcting algorithm that solves this problem.*

The formulation of the Path Bottleneck problem can be done analogously to the label correcting method just changing the minimum length of the path from s to i to the minimum bottleneck from s to i.

<u>Formulation:</u>
s: origin.
t: destination.
α_{ij}: length i-j.
$$\begin{cases} b_i: & \text{length of the minimum bottleneck arc from } s \text{ to } i. \\ p_i: & \text{parent of node } i. \end{cases}$$
Open: set of nodes whose labels may need correction.
Upper: length of the minimum bottleneck from s to t.

<u>Initialization:</u> $s = 0$, $b_i = \infty$, $\forall i \neq s$. Open=$\{s\}$, Upper=∞.

The algorithm can also be defined analogously but in this case instead of comparing the paths from s to j, the new path and the Upper, we will compare their bottlenecks. That is, we will find the new possible bottleneck of the path from s to j going through i, and then we will compare it with the best bottleneck we have until now from s to j and from s to t in a similar way we did for the label correcting algorithm. Observe that the bottleneck for the new possible best path that goes from s to j passing through i is now $\max\{b_i, \alpha_{ij}\}$, where b_i is the minimum bottleneck we have found from s to i and α_{ij} the length $i - j$.

<u>Algorithm:</u>
1) Choose i ∈Open and remove it.
$\forall j$ child of i execute 2)

2) If $\max\{b_i, \alpha_{ij}\} < \min\{b_j, \text{Upper}\}$ then,
$b_j = \max\{b_i, \alpha_{ij}\}$
$p_j = i$
If $j \neq t$, put j in Open.
If $j = t$, Upper=$\max\{b_i, \alpha_{ij}\}$.

3) If Open=$\varnothing$ terminate,
else, go to 1).

By using this algorithm we will find at each step 2) a new possibly best bottleneck from s to j, where j is a child of i. In case this new bottleneck is not better than the best bottleneck of the paths that go from s to t until now, we discard the path. That makes sense because as we advance through a path we can only find bottlenecks that are greater than the one we have until now (it is always the maximum). So at the end we will have found the best one by pruning and discarding all the others just as we have by the label correcting algorithm we studied.

Problem 5. TSP Computational Assignment:
Visit the website: http://www.math.uwaterloo.ca/tsp/world/countries.html. Solve the Traveling Salesman Problem for Uruguay based on the dataset provided. You can use your favorite programming language and solution method for the TSP. Provide a printout of your code with detailed documentation, and compare the optimal solution you obtain to the one available at the website.

The code has been done in R. We used 3 heuristic approaches to find approximate the problem: The nearest neighbor, the greedy algorithm and the simulation anneling. We can see that the best approach (anneling) is above the optimal solution by 12%, however comparing to the second best it just 1% below. Furthermore, this 1% represented an important loose in terms of efficiency. In the following table you can see some important results:

	optimal	**nearest neighbor**	**greedy**	**anneling**
distance	79114.00	100056.45	89559.29	88985.51
distance/optimal		1.26	1.13	1.12
run time (min)		0.19	2.55	11.69

```
library(fields)
library(dplyr)

# Read data and estimate distances between cities
data_uy734        ← read.csv("/home/chpmoreno/Dropbox/Documents/BGSE/Second_Term/SMO/Problemsets/PS2/uy734.csv")[, -1]
cities_distances ← rdist(data_uy734) # euclidean distance estimation

# ///////////////////////////////
# nearest Neighbor approach ####
# ///////////////////////////////
city_path_nearest_neighbor ← function(cities_distances, city = round(runif(1, 1, nrow(cities_distances)))) {
  # Create an auxiliar distance matrix for eliminating selected cities
  cities_distances_aux ← cities_distances
  # Impose big distances for 0 diagonal values of distance matrix. If we do not do this the diagonal will be
  # the minimum distance for each city.
  cities_distances_aux[cities_distances_aux == 0] ← 1000000000
  n_cities ← nrow(cities_distances_aux) # number of cities

  city_path ← city # initial city (by default usually random)

  # nearest neighbor O(n^2) algorithm:
  # 1. Select a random city.
  # 2. Find the nearest unvisited city and go there.
  # 3. Are there any unvisitied cities left? If yes, repeat step 2.
  # 4. Return to the first city.
  i = 1
  while(length(city_path) < (n_cities + 1)) {
    current_city_distances  ← cities_distances_aux[, city_path[i]] # current city
    nearest_city_to_current ← which.min(current_city_distances) # find the minimum available distance
    city_path ← c(city_path, nearest_city_to_current) # add the nearest city to
```

```
      the path
    cities_distances_aux[city_path, city_path[i + 1]] ← 1000000000 # eliminate the new current city distance
    i = i + 1
  }
  city_path ← c(city_path, city_path[1]) # return to the first city

  # Calculate the total distance of the path
  total_distance ← 0
  for(i in 1:(length(city_path) - 1)){
    total_distance ← total_distance + cities_distances[city_path[i], city_path[i + 1]]
  }

  # return the path and its distance
  return(list(path = city_path, distance = total_distance))
}

# Compute the best nearest Neighbor path from all the cities as initial ones
best_path_nearest_neighbor ← function(cities_distances) {
  nearest_neighbor_paths ← NULL
  nearest_neighbor_distances ← NULL
  for(i in 1:nrow(cities_distances)) {
    estimator_aux ← city_path_nearest_neighbor(cities_distances, i)
    nearest_neighbor_paths     ← cbind(nearest_neighbor_paths, estimator_aux$path)
    nearest_neighbor_distances ← c(nearest_neighbor_distances, estimator_aux$distance)
  }

  return(list(best_path = nearest_neighbor_paths[, which.min(nearest_neighbor_distances)],
              distance = min(nearest_neighbor_distances)))
}

# //////////////////////////////
# Greedy Algorithm approach ####
# //////////////////////////////
city_path_greedy ← function(cities_distances) {
  n_cities ← nrow(cities_distances)
  # Take all the edges and weights from distance matrix
  edges_and_weights_matrix ← NULL
  for(i in 1:n_cities) {
    city_distance_vector     ← cities_distances[i:n_cities,i][-1]
    if(length(city_distance_vector) > 0)
      edges_and_weights_matrix ← rbind(edges_and_weights_matrix, cbind(rep(i, length(city_distance_vector)),
                                                                        seq(i+1, n_cities),
                                                                        city_distance_
```

```
vector))
  }
  # Order the edges by weights
  edges_and_weights_df         <- as.data.frame(edges_and_weights_matrix)
  edges_and_weights_ordered_df <- arrange(edges_and_weights_df, city_distance_
    vector)

  # greedy O(n2log_2(n)) algorithm:
  # Constrains: gradually constructs the by
  # repeatedly selecting the shortest edge and adding it to
  # the path as long as it does not create a cycle with less
  # than N edges, or increases the degree of any node to
  # more than 2. We must not add the same edge twice. Then:
  # 1. Sort all edges.
  # 2. Select the shortest edge and add it to our
  # path if it does not violate any of the constraints.
  # 3. Do we have N edges in our tour? If no, repeat
  # step 2.
  city_path <- edges_and_weights_ordered_df[1, 1:2]
  total_distance <- 0
  for(i in 2:nrow(edges_and_weights_ordered_df)) {
    # Constrains
    if((sum(city_path == edges_and_weights_ordered_df[i, 1]) < 2 &
        sum(city_path == edges_and_weights_ordered_df[i, 2]) < 2) &
       sum((city_path[edges_and_weights_ordered_df[i, 1] == city_path[, 1], 2]
          ==
            city_path[edges_and_weights_ordered_df[i, 2] == city_path[, 2], 1]))
              == 0) {
      # path fill
      city_path <- rbind(city_path, edges_and_weights_ordered_df[i, 1:2])
      # compute the distance
      total_distance <- total_distance + edges_and_weights_ordered_df[i, 3]
    }
  }
  return(list(best_path = city_path, distance = total_distance))
}

# ////////////////////////////////
# Simulated annealing approach ####
# ////////////////////////////////

# This approach is based on Todd W. Schneider code and his blog post, availables
    on:
# * http://toddwschneider.com/posts/traveling-salesman-with-simulated-annealing-
    r-and-shiny/
# * https://github.com/toddwschneider/shiny-salesman

# Calculate the path distance
calculate_path_distance = function(path, distance_matrix) {
  sum(distance_matrix[embed(c(path, path[1]), 2)])
}

# Compute the current temperature
```

```r
current_temperature = function(iter, s_curve_amplitude, s_curve_center, s_curve_width) {
  s_curve_amplitude * s_curve(iter, s_curve_center, s_curve_width)
}

s_curve = function(x, center, width) {
  1 / (1 + exp((x - center) / width))
}

# simulation anneling O() algorithm:
# 1. Start with a random path through the selected cities.
# 2. Pick a new candidate path at random from all neighbors of the existing path.
# This candidate path might be better or worse compared to the existing one.
# 3. If the candidate path is better than the existing path, accept it as the new path. If the candidate
# path is worse than the existing tour, still maybe accept it, according to some probability. The probability
# of accepting an inferior tour is a function of how much longer the candidate is compared to the current tour,
# and the temperature of the annealing process. A higher temperature makes you more likely to accept an inferior
# path.
# 4. Go back to step 2 and repeat as many times as you want or can.
city_path_annealing_process = function(distance_matrix, path, path_distance, best_path = c(), best_distance = Inf,
                                       starting_iteration = 0, number_of_iterations = 10000000,
                                       s_curve_amplitude = 400000, s_curve_center = 0, s_curve_width = 300000) {

  n_cities = nrow(distance_matrix) # number of cities

  for(i in 1:number_of_iterations) {
    iter = starting_iteration + i
    # computation of temperature
    temp = current_temperature(iter, s_curve_amplitude, s_curve_center, s_curve_width)

    candidate_path = path # initial path
    swap = sample(n_cities, 2) # new path
    candidate_path[swap[1]:swap[2]] = rev(candidate_path[swap[1]:swap[2]])
    candidate_dist = calculate_path_distance(candidate_path, distance_matrix) # compute the distance for new path

    # ratio indicator
    if (temp > 0) {
      ratio = exp((path_distance - candidate_dist) / temp)
    } else {
      ratio = as.numeric(candidate_dist < path_distance)
    }
    # probabilistic decision
    if (runif(1) < ratio) {
      path = candidate_path
```

```r
        path_distance = candidate_dist
        # best path and best distance
        if (path_distance < best_distance) {
          best_path = path
          best_distance = path_distance
        }
      }
    }
  return(list(path=path, path_distance=path_distance,
              best_path=best_path, distance=best_distance))
}

# //////////////////////
# Code execution #######
# /////////////////////
# Optimal solution given by http://www.math.uwaterloo.ca/tsp/world/uytour.html
optimal = 79114
# nearest Neighbor
nearest_neighbor_time <- Sys.time()
nearest_neighbor_distance <- best_path_nearest_neighbor(cities_distances)$distance
nearest_neighbor_time <- Sys.time() - nearest_neighbor_time
# Greedy
greedy_time <- Sys.time()
greedy_distance <- city_path_greedy(cities_distances)$distance
greedy_time <- Sys.time() - greedy_time
# Anneling
distance_matrix = cities_distances
path = sample(nrow(distance_matrix))
path_distance = calculate_path_distance(path, distance_matrix)
anneling_time <- Sys.time()
anneling_distance <- city_path_annealing_process(distance_matrix = distance_matrix,
                                                 path = path,
                                                 path_distance = path_distance)$distance
anneling_time <- Sys.time() - anneling_time
# Comparison table
comparison_table <- rbind(c(optimal, nearest_neighbor_distance, greedy_distance, anneling_distance),
                          c(NA, nearest_neighbor_distance / optimal, greedy_distance / optimal,
                            anneling_distance / optimal),
                          c(NA, nearest_neighbor_time / 60, greedy_time, anneling_time))
comparison_table <- round(as.data.frame(comparison_table), 2)
colnames(comparison_table) <- c("optimal", "nearest_neighbor", "greedy", "anneling")
rownames(comparison_table) <- c("distance", "distance/optimal", "run time (min)")
```