1 Q2. Inventory Pooling

Primitives
D = demands
Q = quantity ordered
P = price
h = inventory costs = c-s
b = backholding costs = p-c

First we will show that $Q_p^* = \sqrt{n}Q^* + \mu(n - \sqrt{n})$

$$\begin{aligned} G(Q) &= hE[(Q-D)]^{+} bE[(D-Q)]^{+} \\ G'(Q^*) &= hP(D \leqslant Q^*) - b(1 - P[(D \leqslant Q^*)]) = 0 \\ P(D \leqslant Q^*) &= \frac{b}{h+b} \end{aligned}$$

Now, for the pooling we would obtain the same:

$$P(\sum_{L=1}^{n} D_i \leqslant Q_p^*) = \frac{b}{b+R}$$

Since this $\sum_{i=1}^{n} D_i = \sqrt{n}D_i + \mu(n - \sqrt{n})$,

$$P(\sqrt{n}D_1 + \mu(n - \sqrt{n}) \leqslant Q_p^*) = \frac{b}{b+R} \Longleftrightarrow P(D_1 \leqslant \frac{1}{\sqrt{n}}(Q_p^* - \mu(n - \sqrt{n}))) = \frac{b}{b+h}$$

which implies that $\frac{1}{\sqrt{n}}(Q_p^* - \mu(n - \sqrt{n}))) = Q^*$. Thus, $Q_P^* = \sqrt{n}Q^* + \mu(n - \sqrt{n})$. Next, we will apply the hint to prove the desired result:

$$\begin{aligned} nG(Q^*) &= n[hE[(Q^* - D)^{+}] + bE[(D - Q^*)^{+}]] \\ G(Q_p^*) &= [hE[(Q_p^* - \sum_{i=1}^{n} D_i)^{+}] + bE[(\sum_{i=1}^{n} D_i - Q_p^*)^{+}]] \end{aligned}$$

Since $\sum_{i=1}^{n} D_i = \sqrt{n}D_i + \mu(n - \sqrt{n})$ and $Q_P^* = \sqrt{n}Q^* + \mu(n - \sqrt{n})$, we can show that:

$$\begin{aligned} G(Q_P^*) = [&hE[(\sqrt{n}Q^* + \mu(n - \sqrt{n}) - \sqrt{n}D_i - \mu(n - \sqrt{n}))^{+}] \\ &+bE[\sqrt{n}D_i + \mu(n - \sqrt{n}) - \sqrt{n}Q^* + \mu(n - \sqrt{n})^{+}]] \\ &= \sqrt{n}hE[(Q^* - D_i)^{+}] + \sqrt{n}\, bE[(D_i - Q^*)^{+}] = \frac{nG(Q^*)}{\sqrt{n}} \end{aligned}$$

Q.E.D.
Problem 4 (An Investment Problem)