15周上课补充

$$x \sim N(0,\sigma_1^2) \quad y \sim N(0,\sigma_2^2) \quad \begin{pmatrix} x \\ y \end{pmatrix} \sim N(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma)$$

$$g_{\sigma_x,\sigma_y}(x,y) = \frac{1}{2\pi|\Sigma|^{\frac{1}{2}}} \exp\left(-\frac{\begin{pmatrix} x & y \end{pmatrix}\Sigma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}}{2}\right)$$

$$\Sigma = \begin{bmatrix} E(x-\mu_x)^2 & E(x-\mu_x)(y-\mu_y) \\ E(x-\mu_x)(y-\mu_y) & E(y-\mu_y)^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

$$\rho = \frac{E(x-\mu_x)(y-\mu_y)}{\sqrt{E(x-\mu_x)^2 E(y-\mu_y)^2}} \quad , \quad |\rho| \le 1$$

$$\begin{aligned} g(x,y) &= \frac{1}{\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)} \exp\left[-\frac{1}{2\left(1-\rho^2\right)}\left(\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho\left(x-\mu_1\right)\left(y-\mu_2\right)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right)\right] \\ &= \frac{1}{\left(2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}\right)} \exp\left[-\frac{\left(\sigma_2^2(x-\mu_1)^2 - 2\sigma_1\sigma_2\rho\left(x-\mu_1\right)\left(y-\mu_2\right) + \sigma_1^2(y-\mu_2)^2\right)}{2\left(1-\rho^2\right)\sigma_1^2\sigma_2^2}\right] \end{aligned}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\Sigma^{-1} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}^{-1} = \frac{1}{(1-\rho^2)\sigma_1^2\sigma_2^2}\begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_1\sigma_2 & \sigma_1^2 \end{bmatrix}$$

空间旋转

$$A(\Sigma^{-1}) = \begin{bmatrix} a & c \\ c & b \end{bmatrix}$$

A的二次型为$L(x,y) = ax^2 + by^2 + 2cxy$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} (负角表示逆时针旋转)$$

我们用的是这个，(x,y)旋转到(x',y')

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} (\text{正角表示逆时针旋转})$$

$$L(x', y') = a(x'\cos\theta - y'\sin\theta)^2 + b(x'\sin\theta + y'\cos\theta)^2 + \\ 2c(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta)$$

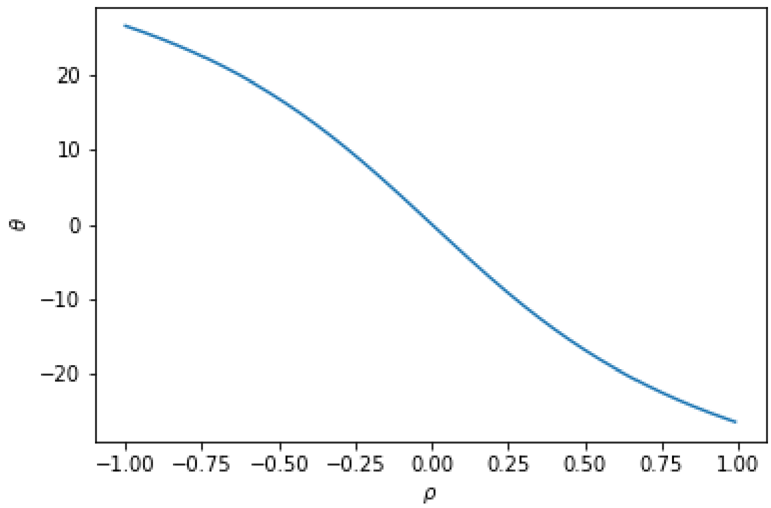
交叉项

$$-2a\cos\theta\sin\theta + 2b\sin\theta\cos\theta + 2c\cos^2\theta - 2c\sin^2\theta \\ = -a\sin 2\theta + b\sin 2\theta + 2c\cos 2\theta$$

让$L(x', y')$没有交叉项，得到

$$-a\sin 2\theta + b\sin 2\theta + 2c\cos 2\theta = 0 \Rightarrow \tan 2\theta = \frac{2c}{a-b} = \frac{2\rho\sigma_1\sigma_2}{\sigma_1^2 - \sigma_2^2}, \quad a \neq b$$

$$\theta = \begin{cases} \frac{1}{2}\arctan(\frac{2\rho\sigma_1\sigma_2}{\sigma_1^2-\sigma_2^2}) & \text{if } a \neq b \\ \frac{\pi}{4} & \text{if } a = b \end{cases}$$



ρ	$-1 \rightarrow +1$
θ	$+ \rightarrow -$
图	顺时针 $\rightarrow$ 逆时针

Gabor

$$\int_{-\infty}^{+\infty} exp\left[-ax^2\right] exp\left[-2\pi i x\xi\right] dx = \sqrt{\frac{\pi}{a}} exp\left[-\frac{\pi^2}{a}\xi^2\right]$$

f_0频点调制

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} exp\left[-ax^2\right] exp\left[2\pi i x f_0\right] exp\left[-2\pi i x \xi\right] dx \\
&= \int_{-\infty}^{+\infty} exp\left[-ax^2\right] exp\left[-2\pi i x(\xi - f_0)\right] dx \\
&= \sqrt{\frac{\pi}{a}} exp\left[-\frac{\pi^2}{a}(\xi - f_0)^2\right]
\end{aligned}
$$

空间旋转	频点调制	频率旋转
$exp\left[-\frac{1}{2}(\frac{x'^2}{a^2} + \frac{y'^2}{b^2})\right]$	$exp\left[2\pi i(x' f_0 + y' f_1)\right]$	$exp\left[-2\pi i(x' \xi' + y' \eta')\right]$