Problem Set #2

You should try to solve these problems by yourself. I recommend that you start early and get help in office hours if needed. If you find it helpful to discuss problems with other students, go for it. **You do not need to turn these problems in. The goal is to be ready for the in class quiz that will cover the same or similar problems.**

Problem 1: Sets and Counterexamples

Show that for arbitrary sets A, B, and C, taken from the universe $\{1, 2, 3, 4, 5\}$ that the following two claims are not always true by using a simple counter example for each:

(a) if $A \cap B \subseteq C$, then $C \subseteq A \cup B$

> **Solution**
>
> Let $A = \{1\}$, $B = \{2\}$, and $C = \{3\}$. $A \cap B = \emptyset$, so $A \cap B \subseteq C$ is automatically true for this example. However, we also know $3 \in C$, but $3 \notin A \cup B$, and therefore $C \subseteq A \cup B$ is **false** for this example.

(b) if $C \subseteq A \cup B$, then $A \cap B \subseteq C$

> **Solution**
>
> Let $A = \{1, 2\}$, $B = \{1, 2\}$, and $C = \{1\}$. In this example, $C \subseteq A \cup B$ is true since $1 \in A \cup B$. However, we also know $A \cap B = \{1, 2\}$, where $2 \in A \cap B$ but $2 \notin C$, and therefore $A \cap B \subseteq C$ is **false** for this example.

Problem 2: Stable Marriage (from last HW)

The stable matching problem, as described in the text, assumes that all men and women have a fully ordered list of preferences. In this problem, we will consider a version of the problem in which men and women can be *indifferent* between certain options. As before, we have a set M of n men and a set W of n women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that m_1 is ranked in first place; second place is a tie between m_2 and m_3 (she has no preference between them); and m_4 is in last place. We will say that *w prefers m to m'* if m is ranked higher than m' on her preference list (they are not tied).

With indifferences in the rankings, there could be two natural notions of stability. And for each, we can ask about the existence of stable matchings.

(a) A *strong instability* in a perfect matching S consists of a man m and a woman w, such that each of m and w prefers the other to their partner in S. Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a strong instability or give an algorithm that is guaranteed to find a perfect matching with no strong instability.

Solution

The answer is Yes. A simple way to think about it is to break the ties in some fashion and then run the stable matching algorithm on the resulting preference lists. We can, for example, break the ties lexicographically—that is, if a man m is indifferent between two women w_i and w_j, then w_i appears on m's preference list before w_j if $i < j$ and if $j < i$, w_j appears before w_i. Similarly, if w is indifferent between two men m_i and m_j, then m_i appears on w's preference list before m_j if $i < j$ and if $j < i$, m_j appears before m_i.
Now that we have concrete preference lists, we run the stable matching algorithm. We claim that the matching produced would have no strong instability. But this claim is true because any strong instability would be an instability for the match produced by the original algorithm in the original situations, yet we know that this is not the case.

(b) A *weak instability* in a perfect matching S consists of a man m and a woman w, such that their partners in S are w' and m', respectively, and one of the following holds:

- m prefers w to w', and w either prefers m to m' or is indifferent between these two choices; or
- w prefers m to m', and m either prefers w to w' or is indifferent between these two choices.

In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

Solution

The answer is No. The following is a simple counterexample. Let $n = 2$ and m_1, m_2 be the two men and w_1, w_2 the two women. Let m_1 be indifferent between w_1 and w_2 and let both women prefer m_1 to m_2. The choices of m_2 are insignificant. There is no mathcing without weak stability in this example, since regardless of who was matched with m_1, the other woman together with m_2 would form a weak instability.

Problem 3: Stable Marriage

Decide whether the following statement is true or false. If it is true, give a short proof. If it is false, give a counter example.

In every instance of the Stable Marriage problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

Solution

False. Counter-example. Consider the case with two women (w_1 and w_2) and two men (m_1 and m_2). w_1 prefers m_2 over m_1; w_2 prefers m_1 over m_2. m_1 prefers w_1 over w_2; m_2 prefers w_2 over w_1. There are two possible stable matches (there are two perfect matchings, and theyre both stable): $\{(m_1, w_1), (m_2, w_2)\}$ and $\{(m_1, w_2), (m_2, w_1)\}$. By inspection, in all four pairings in these two matchings, one of the participants did get their top priority, while the other one did not.

Problem 4: Running Times

You are given six algorithms capable of solving the same problem with the running times listed below. You are also given a computer capable of performing 6^{16} operations per second. You have been alotted an hour of computing time on this machine. For each algorithm, give the largest input size (n) for which you can complete the computation within an hour.

(a) n^4

Solution

The operations that the computer can perform within an hour is $6^{16} * 3600$.

$n^4 = 6^{16} * 3600$, so we have $n = 10,038.7728$

(b) n^5

Solution

$n^5 = 6^{16} * 3600$, so we have $n = 1,589.80735$

(c) $6000n^3$

Solution

$6000n^3 = 6^{16} * 3600$, so we have $n = 11,917.6443$

(d) $n^2 \log n$

Solution

$n^2 \log n = 6^{16} * 3600$, so we have $n = 3.664x10^7$

(e) 2^n

Solution

$2^n = 6^{16} * 3600$, so we have $n = 53.1731812$

(f) n^{2^n}

Solution

$n^{2^n} = 6^{16} * 3600$, so we have $n = 4.5963$

Problem 5: Bounds

For each of the following pairs of functions $f(n)$ and $g(n)$, give an appropriate positive constant c such that $f(n) \leq cg(n)$ for all $n > 1$.

(a) $f(n) = n^2 + n + 1$, $g(n) = 2n^3$

Solution

$n^2 + n + 1 \leq c(2n^3)$. Then $c \geq \frac{n^2+n+1}{2n^3}$. As n approaches 1, $c \geq \frac{3}{2}$. As n approaches infinity, $c \geq 0$. $\frac{3}{2}$ is the smallest appropriate constant.

(b) $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$

Solution

$n\sqrt{n} + n^2 \leq cn^2$. Then $c \geq \frac{n\sqrt{n}+n^2}{n^2}$. As n approaches 1, $c \geq 2$. As n approaches infinity, $c \geq 1$. 2 is the smallest appropriate constant.

(c) $f(n) = n^2 - n + 1$, $g(n) = n^2/2$

Solution

$n^2 - n + 1 \leq c(n^2/2)$. Then $c \geq \frac{2(n^2-n+1)}{n^2}$. As n approaches 1, $c \geq 2$. As n approaches infinity, $c \geq 2$. 2 is the only appropriate constant.