Problem Set #2

You should try to solve these problems by yourself. I recommend that you start early and get help in office hours if needed. If you find it helpful to discuss problems with other students, go for it. **You do not need to turn these problems in. The goal is to be ready for the in class quiz that will cover the same or similar problems.**

Problem 1: Sets and Counterexamples

Show that for arbitrary sets A, B, and C, taken from the universe $\{1, 2, 3, 4, 5\}$ that the following two claims are not aways true by using a simple counter example for each:

(a) if $A \cap B \subseteq C$, then $C \subseteq A \cup B$

(b) if $C \subseteq A \cup B$, then $A \cap B \subseteq C$

Problem 2: Stable Marriage (from last HW)

The stable matching problem, as described in the text, assumes that all men and women have a fully ordered list of preferences. In this problem, we will consider a version of the problem in which men and women can be *indifferent* between certain options. As before, we have a set M of n men and a set W of n women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that m_1 is ranked in first place; second place is a tie between m_2 and m_3 (she has no preference between them); and m_4 is in last place. We will say that w *prefers* m to m' if m is ranked higher than m' on her preference list (they are not tied).

With indifferences in the rankings, there could be two natural notions of stability. And for each, we can ask about the existence of stable matchings.

(a) A *strong instability* in a perfect matching S consists of a man m and a woman w, such that each of m and w prefers the other to their partner in S. Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a strong instability or give an algorithm that is guaranteed to find a perfect matching with no strong instability.

(b) A *weak instability* in a perfect matching S consists of a man m and a woman w, such that their partners in S are w' and m', respectively, and one of the following holds:

- m prefers w to w', and w either prefers m to m' or is indifferent between these two choices; or
- w prefers m to m', and m either prefers w to w' or is indifferent between these two choices.

In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

Problem 3: Stable Marriage

Decide whether the following statement is true or false. If it is true, give a short proof. If it is false, give a counter example.

In every instance of the Stable Marriage problem, there is a stable matching containing a pair (m, w) such that m is ranked first on the preference list of w and w is ranked first on the preference list of m.

Problem 4: Running Times

You are given six algorithms capable of solving the same problem with the running times listed below. You are also given a computer capable of performing 6^{16} operations per second. You have been alotted an hour of computing time on this machine. For each algorithm, give the largest input size (n) for which you can complete the computation within an hour.

(a) n^4

(b) n^5

(c) $6000n^3$

(d) $n^2 \log n$

(e) 2^n

(f) n^{2^n}

Problem 5: Bounds

For each of the following pairs of functions $f(n)$ and $g(n)$, give an appropriate positive constant c such that $f(n) \leq cg(n)$ for all $n > 1$.

(a) $f(n) = n^2 + n + 1$, $g(n) = 2n^3$

(b) $f(n) = n\sqrt{n} + n^2$, $g(n) = n^2$

(c) $f(n) = n^2 - n + 1$, $g(n) = n^2/2$