Problem Set #1

You should try to solve these problems by yourself. I recommend that you start early and get help in office hours if needed. If you find it helpful to discuss problems with other students, go for it. **You do not need to turn these problems in. The goal is to be ready for the in class quiz that will cover the same or similar problems.**

Problem 1: Sets and Functions

Let $f : A \to B$, $S \subseteq A$, and $T \subseteq A$. Prove or disprove:

(a) $f(S \cap T) = f(S) \cap f(T)$

> **Solution**
>
> The answer is No. The following is a simple counterexample. Let $f(a) = f(b) = 1$, $f(c) = f(d) = 2$, $S = \{a, c\}$, and $T = \{b, d\}$. Then $f(S \cap T) = f(\emptyset) = \emptyset$, but $f(S) \cap f(T) = \{1, 2\} \cap \{1, 2\} = \{1, 2\}$.

(b) $f(S \cap T) = f(S) \cap f(T)$ if f is a one-to-one function.

> **Solution**
>
> The answer is Yes. Suppose $y \in f(S) \cap f(T)$. Then $y = f(s)$ for some $s \in S$ and $y = f(t)$ for some $t \in T$. Since f is one-to-one, it must be that $s = t$. Thus y is the image of an element that lies in both S and T and so $y \in f(S \cap T)$. Thus $y \in f(S) \cap f(T) \Rightarrow y \in f(S \cap T)$ and so $f(S) \cap f(T) \subseteq f(S \cap T)$. Now suppose $y \in f(S \cap T)$. Then y is the image of an element that lies in both S and T. Thus $y \in f(S)$ and $y \in f(T)$, i.e. $y \in f(S) \cap f(T)$. Thus $y \in f(S \cap T) \Rightarrow y \in f(S) \cap f(T)$ and so $f(S \cap T) \subseteq f(S) \cap f(T)$. Since $f(S) \cap f(T) \subseteq f(S \cap T)$ and $f(S \cap T) \subseteq f(S) \cap f(T)$, we will have $f(S \cap T) = f(S) \cap f(T)$.

Problem 2: Proofs by Contradiction

Prove each of the following.

(a) $\sqrt{2}$ is irrational.

> **Solution**
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> Suppose that $\sqrt{2}$ was rational. Then $\sqrt{2} = a/b$ where a and b are positive integers. Also assume that a/b is simplified to its lowest terms. It follows that $2 = a^2/b^2$, or $a^2 = 2b^2$. So the square of a is an even number. Then a is also even (prove, or see **(c)** below). So $a = 2k$ for some k. Then
>
> $$\begin{aligned} 2 &= (2k)^2/b^2 \\ 2 &= 4k^2/b^2 \\ 2b^2 &= 4k^2 \\ b^2 &= 2k^2 \end{aligned}$$
>
> So b^2 is even, and it follows that b is even. But if both a and b are even, then a/b is not simplified to lowest terms, which is a contradiction.

(b) The sum of an irrational number and a rational number is irrational.

> **Solution**
>
> Suppose that r is rational and i is irrational and $s = i + r$ is rational. Then r can be expressed as $r = p/q$ and s can be expressed as $s = t/u$. Then
>
> $$\begin{aligned} r + i &= s \\ p/q + i &= t/u \\ i &= t/u - p/q \\ i &= tq/qu - pu/qu \\ i &= (tq - pu)/qu \end{aligned}$$
>
> But $(tq - pu)$ and qu are rational by definition, which is a contradiction.

(c) If n^2 is even, n is even.

> **Solution**
>
> Assume this is not the case, i.e., that n^2 is even, but n is odd. Because n is odd, you can write it as $2k + 1$. Then, squared, n is $(2k + 1)^2 = 4k^2 + 4k + 1$, which one can rewrite as $2j + 1$, which is odd, not even, a contradiction.

Problem 3: Graphs

(a) Show that in any undirected graph, there is a path from any vertex with odd degree to some other vertex of odd degree.

Solution

For some graph G, pick an arbitrary vertex v of odd degree. Take a subset of the graph that is connected to v via some path (i.e., the connected component of G that contains v), call this G_v. Since G_v is itself a graph, it must have total degree even (because every edge connects two vertices). Because v is of odd degree, there must be another vertex of odd degree in G_v. Since G_v is connected, there must exist a path between v and this other vertex. (There is also a totally legitimate, though longer, proof by induction.)

(b) Show that an undirected graph G with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Solution

Suppose that G is not connected. Then it has a component of k vertices for some k, $1 \leq k \leq n-1$. The most edges G could have is

$$\begin{aligned} C(k,2) + C(n-k,2) &= \frac{k!}{2(k-2)!} + \frac{(n-k)!}{2(n-k-2)!} \\ &= \frac{k(k-1) + (n-k)(n-k-1)}{2} \\ &= k^2 - nk - \frac{n^2 - n}{2} \end{aligned}$$

This quadratic function is minimized at $k = n/2$ and maximized at $k = 1$ or $k = n-1$. Hence, if G is not connected, the number of edges does not exceed the value of this function at 1 and $n-1$, namely $\frac{(n-1)(n-2)}{2}$. This is a contradiction to the proposition.

Problem 4: Proofs by Induction

Prove each of the following.

(a) $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Solution

Any odd positive integer n can be expressed as $n = 2k - 1$, where k is a positive integer and hence $n^2 - 1 = (2k-1)^2 - 1$. Let $P(k)$ be the proposition that $(2k-1)^2 - 1$ is divisible by 8. *Base Case*: $P(1)$ is trivially true. *Inductive Step*: Assume that $P(k)$ is true. Then for $P(k+1)$ we will have

$$\begin{aligned}
P(k+1) &= (2(k+1)-1)^2 - 1, \\
&= (2k+2-1)^2 - 1, \\
&= ((2k-1)+2)^2 - 1, \\
&= ((2k-1)^2 + 4(2k-1) + 4) - 1, \\
&= ((2k-1)^2 - 1) + (4(2k-1) + 4), \\
&= ((2k-1)^2 - 1) + (8k - 4 + 4), \\
&= ((2k-1)^2 - 1) + 8k.
\end{aligned}$$

$P(k+1)$ is thus true since both terms on the right-hand side are divisible by 8 (the first one by the inductive hypothesis).

(b) Any postage that is a positive integer number of cents greater than 7 cents can be formed using just 3-cent stamps and 5-cent stamps.

Solution

Let $P(n)$ be the proposition that a postage of n cents can be formed using 3-cent and 5-cent stamps. *Base Case*: $P(8)$ is true because a 8 cents postage can be formed from a single 3-cent stamp and a single 5-cent stamp. *Inductive Step*: Assume that $P(n)$ is true (that is a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps). Show that $P(n+1)$ is true. By the inductive hypothesis, a postage of n cents can be formed using just 3-cent stamps and 5-cent stamps. If this included a 5-cent stamp, replace it with two 3-cent stamps to get a postage of $n+1$ cents. Otherwise, only 3-cent stamps were used, and $n \geq 9$. Remove three of these 3-cent stamps and replace them with two 5-cent stamps to get a postage of $n+1$ cents.

Problem 5: Trees

Show by induction that the number of degree-2 nodes in any non-empty binary tree is 1 fewer than the number of leaves.

Solution

Base Case: Consider a binary tree with one node. There is 1 leaf node, and 0 nodes with two children; 0 is exactly one less than 1.

Inducitive Step: Assume the proposition is true for a binary tree of size n. Show that it remains true for a binary tree of size $n+1$. Start with a binary tree with n nodes. To make it a binary tree with $n+1$ nodes, we can add our new node either as the first child of a leaf node or as a second child of a non-leaf node that does not already have two children. In the first case, we removed one leaf node, added a leaf node, and didn't change the number of nodes with two children. Therefore, if the proposition holds for the tree of size n, it still holds for the tree of size $n+1$. In the second case, we added a leaf node, but we also "completed" a node, increasing the number of nodes with two children by one, thereby maintaining the proposition.

Problem 6: Stable Marriage

The stable matching problem, as described in the text, assumes that all men and women have a fully ordered list of preferences. In this problem, we will consider a version of the problem in which men and women can be *indifferent* between certain options. As before, we have a set M of n men and a set W of n women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that m_1 is ranked in first place; second place is a tie between m_2 and m_3 (she has no preference between them); and m_4 is in last place. We will say that w *prefers* m to m' if m is ranked higher than m' on her preference list (they are not tied).

With indifferences in the rankings, there could be two natural notions of stability. And for each, we can ask about the existence of stable matchings.

(a) A *strong instability* in a perfect matching S consists of a man m and a woman w, such that each of m and w prefers the other to their partner in S. Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a strong instability or give an algorithm that is guaranteed to find a perfect matching with no strong instability.

Solution

The answer is Yes. A simple way to think about it is to break the ties in some fashion and then run the stable matching algorithm on the resulting preference lists. We can, for example, break the ties lexicographically—that is, if a man m is indifferent between two women w_i and w_j, then w_i appears on m's preference list before w_j if $i < j$ and if $j < i$, w_j appears before w_i. Similarly, if w is indifferent between two men m_i and m_j, then m_i appears on w's preference list before m_j if $i < j$ and if $j < i$, m_j appears before m_i.
Now that we have concrete preference lists, we run the stable matching algorithm. We claim that the matching produced would have no strong instability. But this claim is true because any strong instability would be an instability for the match produced by the original algorithm in the original situations, yet we know that this is not the case.

(b) A *weak instability* in a perfect matching S consists of a man m and a woman w, such that their partners in S are w' and m', respectively, and one of the following holds:

- m prefers w to w', and w either prefers m to m' or is indifferent between these two choices; or
- w prefers m to m', and m either prefers w to w' or is indifferent between these two choices.

In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

Solution

The answer is No. The following is a simple counterexample. Let $n = 2$ and m_1, m_2 be the two men and w_1, w_2 the two women. Let m_1 be indifferent between w_1 and w_2 and let both women prefer m_1 to m_2. The choices of m_2 are insignificant. There is no mathcing without weak stability in this example, since regardless of who was matched with m_1, the other woman together with m_1 would form a weak instability.

Problem 7: Colorings

Given an undirected graph $G = (V, E)$, a **k-coloring** of G is a function $c : V \to \{0, 1, \dots, k-1\}$ such that $c(u) \neq c(v)$ for every edge $(u, v) \in E$. In other words, the numbers $0, 1, \dots, k-1$ represent the k colors, and adjacent vertices must have different colors.

a. Show that any tree is 2-colorable.

> **Solution**
>
> Prove by induction on the height of a tree.
> **Base Case**: Consider a tree of height 1. Color the node at depth 0 the first color and color the node at depth 1 the second color.
> **Inductive Step**: Assume that any tree of height n is 2-colorable. Show that a tree of height n+1 is also 2-colorable. Start with the colored tree of height n. Whatever color the nodes are at depth n, make the nodes in our new tree at depth n+1 the *other* color.
> *I'm also ok with a discussion that just says that all of the nodes at an even depth are one color and all of the nodes at an odd depth are the other color.*

b. Show that the following are equivalent:

1. G is bipartite.
2. G is 2-colorable.
3. G has no cycles of odd length.

> **Solution**
>
> **You need to be careful to prove everything: e.g.,** $1 \Leftrightarrow 2$**,** $2 \Leftrightarrow 3$**, and** $1 \Leftrightarrow 3$**. There could also be** $1 \Rightarrow 2$**,** $2 \Rightarrow 3$**, and** $3 \Rightarrow 1$**.**
> From the definition of bipartite, the nodes V in G can be partitioned into two sets V_1 and V_2 such that no edges go between nodes in the same set (i.e., all edges go between the two sets). The G is is 2-colorable just by coloring all of the nodes in V_1 one color and all of the nodes in V_2 a different color. because every edge goes from a node $u \in V_1$ to a node $v \in V_2$, and u and v are different colors, no edge connects two nodes of the same color. In the other direction, if G is 2-colorable, it must also be bipartite. The argument is similar; divide the vertices in G in two sets such that all of the nodes of one color make up V_1 and all of the nodes of the other color make up V_2. Since the graph was 2-colorable, no edges go between two nodes both in V_1 or both in V_2, and therefore the graph is bipartite.
> If G is bipartite, then every edge in G goes from a node of one set to a node of the other set. Having a cycle requires going from one node, through other nodes, and ending up back at the original node. Any path in a bipartite graph must alternate in taking an edge from a node in V_1 to a node in V_2, back to a node in V_1, and so on. So a cycle p from u back to u in a bipartite graph must take the form $\langle u, v_1, u_1, v_2, u_2, \dots, u_{k-1}, v_k, u \rangle$. Any cycle of this form must have an even number of edges. In the other direction, if a graph has no cycles of odd lengths, then the graph can be partitioned into two sets such that, for any cycle (of even length), the cycle starts and ends in the same set and alternates visiting each set, defining the partitions for bipartite-ness.
> If G is 2-colorable, then every edge in G goes from a node of one color to a node of the other color... Repeat the above paragraph, replacing “set” with “color” and “bipartite” with “2-colorable.”

c. Let d be the maximum degree of any vertex in graph G. Prove that we can color G with $d+1$ colors.

Solution

Prove by induction on the number of nodes in the graph.

Base case: consider a graph of size 1. the maximum degree of any vertex is 0; this graph can be colored with 1 color.

Inductive step: Assume a graph of size n in which the maximum degree of any vertex is d that we can color with $d+1$ colors. We need to prove that a graph with one more node in it that has a maximum degree of d can be colored with $d+1$ colors. Take any graph of size n for which the assumption holds (that's all of them, by the way). Add another node that has at most degree d. Color it with the $(d+1)$-st color that is different from the colors used on the d neighbors. The original graph on n vertices has a valid $d+1$ coloring by the inductive hypothesis, in other words the edges in the original graph have different color endpoints. The new graph contains up to d new edges, all of which have different color endpoints as well, by the coloring we used (since the new node is not adjacent to any node with the same color).

d. Show that if G has $O(|V|)$ edges, then we can color G with $O(\sqrt{|V|})$ colors.

Solution

Prove by induction on the number of edges in the graph.

Base case: consider a graph with 1 edge. Then we can color G with 2 colors; $1 = O(1)$ and $2 = O(1)$.

Inductive step: Assume a graph with $O(|V|)$ edges. Then by the inductive step, we can color it with $O(\sqrt{|V|})$ colors. This means that $\exists a, c : a, c$ are constants, G has $a|V|$ edges and can be colored with $c\sqrt{|V|}$ colors. We need to show that, by adding one edge (i.e., we now have $a(|V|+1)$ edges), we can color the graph with $d(\sqrt{|V|+1})$ colors. When we add one edge to the graph, we may need to add a color and make one of the nodes adjacent to that edge that new color. That is, when we change the graph from having $a|V|$ edges to having $a(|V|+1)$ edges, we may need to increase the number of colors to $c\sqrt{|V|}+1$. So now we simply need to show that a constant d exists such that $d(\sqrt{|V|+1}) = c\sqrt{|V|}+1$. Squaring both sides, we have $d^2(|V|+1) = (c\sqrt{|V|}+1)^2$, which extends to $d^2(|V|+1) = c^2|V| + 2c\sqrt{|V|}+1$. Dividing through by $|V|+1$, we get $d^2 = \frac{c^2|V|}{|V|+1} + \frac{2c\sqrt{|V|}}{|V|+1} + \frac{1}{|V|+1}$. For any positive values of $|V|$, $\frac{|V|}{|V|+1}$, $\frac{\sqrt{|V|}}{|V|+1}$, and $\frac{1}{|V|+1}$ are all less than or equal to 1, so d is a constant for any real values of $|V|$.