EE 360C - Algorithms
The University of Texas at Austin

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Shortest Paths: For each of the following statements decide whether it is true of false. If it is true, give a short proof. If it is false give a counterexample.
Suppose we are given a directed graph $G = (V, E)$ and we assume all edge weights are positive and distinct. Let P be a shortest path from node s to node t with a weight of w_P.

a) Suppose the weight of each edge is increased by a constant, that is, the new edge weights are $w'_e = w_e + c$, for all $e \in E$, where c is some positive constant. True or false, P must still be the shortest path from node s to node t in the graph with the new weights?

b) Now, suppose the weight of each edge is doubled, that is, the new edge weights are $w'_e = 2w_e$, for all $e \in E$. True or false, P must still be the shortest path from node s to node t in the graph with the new weights?

c) Finally, suppose the weight of each edge is squared, that is, the new edge weights are $w'_e = w_e^2$, for all $e \in E$. True or false, P must still be the shortest path from node s to node t in the graph with the new weights?

Solution

a) False. Consider a graph with 3 nodes, $\{s, u, t\}$ and 3 edges, $\{(s, u), (u, t), (s, t)\}$. Let the weight of the edges be $w_{(s,u)} = 1, w_{(u,t)} = 1, w_{(s,t)} = 5$. The weight of the shortest path is 2 and the path is $< s, u, t >$. Now increase all edge weights by 10, i.e., let $c = 10$. The new weights are $w'_{(s,u)} = 11, w'_{(u,t)} = 11, w'_{(s,t)} = 15$ and the shortest path is $< s, t >$ with is a weight of 15. The shortest path is not the same after the constant increase.

b) True. Let the shortest path from s to t be $P =< s, v_1, v_{,2}, ..., v_k, t >$, and $w_P = w_{(s,v_1)} + w_{(v_1,v_2)} + ... + w_{(v_k,t)}$. Now, double the weight of each edge. The new weight of the original path is $w_P^{new} = 2w_{(s,v_1)} + 2w_{(v_1,v_2)} + ... + 2w_{(v_k,t)} = 2w_P$. Suppose P is not the shortest path after the edge doubling, therefore there is a path P' with weight $w_{P'}^{new} < w_P^{new}$, but because all the edges were doubled the original weight of path P' was such that $w_{P'}^{new} = 2w_{P'}$. Since $w_{P'}^{new} < w_P^{new}$, then $2w_{P'} < 2w_P$ which implies $w_{P'} < w_P$, since all edges have positive weights. This is a contradiction with the fact that P was a shortest path in the original graph.

c) False. Consider a graph with 3 nodes, $\{s, u, t\}$ and 3 edges, $\{(s, u), (u, t), (s, t)\}$. Let the weight of the edges be $w_{(s,u)} = 3, w_{(u,t)} = 3, w_{(s,t)} = 5$. The weight of the shortest path is 5 and the path is $< s, t >$. Now square all edge weights. The new weights are $w'_{(s,u)} = 9, w'_{(u,t)} = 9, w'_{(s,t)} = 25$ and the shortest path is $< s, u, t >$ with is a weight of 18. The shortest path is not the same after the sqauring.