EE 360C - ALGORITHMS Dr. Pedro Santacruz
The University of Texas at Austin Due: **September 11, 2014** (*in class quiz*)

Homework #1

You should try to solve these problems by yourself. I recommend that you start early and get help in office hours if needed. If you find it helpful to discuss problems with other students, go for it. **The goal is to be ready for the in class quiz that will cover the same or similar problems.**

Problem 1: Sets and Functions
Let $f : A \to B$, $S \subseteq A$, and $T \subseteq A$. Prove or disprove:

(a) $f(S \cap T) = f(S) \cap f(T)$

(b) $f(S \cap T) = f(S) \cap f(T)$ if f is a one-to-one function.

Problem 2: Proofs by Contradiction
Prove each of the following:

(a) $\sqrt{2}$ is irrational.

(b) The sum of an irrational number and a rational number is irrational.

(c) If n^2 is even, n is even.

Problem 3: Graphs
Show the following:

(a) Show that in any undirected graph, there is a path from any vertex with odd degree to some other vertex of odd degree.

(b) Show that an undirected graph G with n vertices is connected if it has more than $\frac{(n-1)(n-2)}{2}$ edges.

Problem 4: Proofs by Induction
Prove each of the following:

(a) $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

(b) Any postage that is a positive integer number of cents greater than 7 cents can be formed using just 3-cent stamps and 5-cent stamps.

Problem 5: Trees
Show by induction that the number of degree-2 nodes in any non-empty binary tree is 1 fewer than the number of leaves.

Problem 6: Stable Marriage
The stable matching problem, as described in the text, assumes that all men and women have a fully ordered list of preferences. In this problem, we will consider a version of the problem in which men and women can be *indifferent* between certain options. As before, we have a set M of n men and a set W of n women. Assume each man and each woman ranks the members of the opposite gender, but now we allow ties in the ranking. For example (with $n = 4$), a woman could say that m_1 is ranked in first place; second place is a tie between m_2 and m_3 (she has no preference between them); and m_4 is in last place. We will say that w *prefers* m to m' if m is ranked higher than m' on her preference list (they are not tied).
With indifferences in the rankings, there could be two natural notions of stability. And for each, we can ask about the existence of stable matchings.

(a) A *strong instability* in a perfect matching S consists of a man m and a woman w, such that each of m and w prefers the other to their partner in S. Does there always exist a perfect matching with no strong instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a strong instability or give an algorithm that is guaranteed to find a perfect matching with no strong instability.

(b) A *weak instability* in a perfect matching S consists of a man m and a woman w, such that their partners in S are w' and m', respectively, and one of the following holds:

- m prefers w to w', and w either prefers m to m' or is indifferent between these two choices; or
- w prefers m to m', and m either prefers w to w' or is indifferent between these two choices.

In other words, the pairing between m and w is either preferred by both, or preferred by one while the other is indifferent. Does there always exist a perfect matching with no weak instability? Either give an example of a set of men and women with preference lists for which every perfect matching has a weak instability; or give an algorithm that is guaranteed to find a perfect matching with no weak instability.

Problem 7: Colorings

Given an undirected graph $G = (V, E)$, a **k-coloring** of G is a function $c : V \rightarrow \{0, 1, \ldots, k-1\}$ such that $c(u) \neq c(v)$ for every edge $(u, v) \in E$. In other words, the numbers $0, 1, \ldots, k-1$ represent the k colors, and adjacent vertices must have different colors.

(a) Show that any tree is 2-colorable.

(b) Show that the following are equivalent:

1. G is bipartite.
2. G is 2-colorable.
3. G has no cycles of odd length.

(c) Let d be the maximum degree of any vertex in graph G. Prove that we can color G with $d + 1$ colors.

(d) Show that if G has $O(|V|)$ edges, then we can color G with $O(\sqrt{|V|})$ colors.