第二节 积 分 法

一、第一类换元法

二、第二类换元法

三、分部积分法

§ 5.2.1 第一类换元法

例 计算不定积分$\int \frac{2x}{1+x^2}\mathrm{d}x$.

第一类换元法的基本思想: 把要计算的不定积分通过变量代换，转换成基本积分公式中的某一形式. 求出原函数后, 再换回原来的变量, 即为复合函数求导法的逆运算.

定理1(第一换元法(凑微分法)) 设 $f(u)$有原函数$F(u)$，

$u=\varphi(x)$可导, 则有公式

$$\underbrace{\int f[\varphi(x)]\varphi'(x)dx}_{\text{难求}} = \int f[\varphi(x)]\,d\varphi(x) \overset{u=\varphi(x)}{=} \underbrace{\int f(u)du}_{\text{易求}}\Big|_{u=\varphi(x)}$$

$$= F(u)+C\,\Big|_{u=\varphi(x)} = F(\varphi(x))+C$$

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第一类换元法的难点：如何将被积函数凑为一个复合函数 $f[\varphi(x)]$与微分 $\varphi'(x)dx$的乘积.

$$\int g(x)dx = \int f[\varphi(x)]\varphi'(x)\,dx = \int f(\varphi(x))\,d\varphi(x).$$

$$= \int f(u)\,du\,\Big|_{u=\varphi(x)}$$

例1 求$\int 2x\cos x^2 dx$.

解 原式$=\int \cos x^2 (x^2)' \mathrm{d}x$

$$=\int \cos x^2 \, d(x^2)$$

$$=\int \cos u \, du \Big|_{u=x^2}$$

$$=\sin u + C \Big|_{u=x^2} = \sin x^2 + C$$

例2 计算不定积分$\int \frac{2x}{1+x^2}\mathrm{d}x$.

解 $\int \frac{2x}{1+x^2}\mathrm{d}x = \int \frac{1}{1+x^2}(1+x^2)'\mathrm{d}x$

$$= \int \frac{1}{1+x^2}\mathrm{d}(1+x^2)$$

$$= \ln(1+x^2)+C$$

基本积分公式(续)

(14) $\int \tan x \, \mathrm{d}x = -\ln|\cos x| + C$

(15) $\int \cot x \mathrm{d}x = \ln|\sin x| + C$

(18) $\int \frac{1}{a^2 + x^2} \mathrm{d}x = \frac{1}{a} \arctan \frac{x}{a} + C$

(21) $\int \frac{1}{\sqrt{a^2 - x^2}} \mathrm{d}x = \arcsin \frac{x}{a} + C (a > 0)$

(14) $\int \tan x \, \mathrm{d}x = -\ln|\cos x| + C$

证明 $\int \tan x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\int \frac{(\cos x)'}{\cos x} \mathrm{d}x$

$$= -\int \frac{1}{\cos x} \mathrm{d}\cos x$$

$$= -\ln|\cos x| + C.$$

(18) $$\int \frac{1}{a^2+x^2}\mathrm{d}x = \frac{1}{a^2}\int \frac{1}{1+(\frac{x}{a})^2}\mathrm{d}x = \frac{1}{a}\int \frac{(\frac{x}{a})'}{1+(\frac{x}{a})^2}\mathrm{d}x$$

$$= \frac{1}{a}\int \frac{1}{1+(\frac{x}{a})^2}\mathrm{d}(\frac{x}{a})$$

$$= \frac{1}{a}\arctan(\frac{x}{a}) + C.$$

$$\int \frac{1}{1+u^2}\mathrm{d}u = \arctan u + C$$

(21) $$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \int \frac{1}{a\sqrt{1-(\frac{x}{a})^2}}\,\mathrm{d}x$$

$$= \int \frac{1}{\sqrt{1-(\frac{x}{a})^2}}\,\mathrm{d}(\frac{x}{a}) = \arcsin\frac{x}{a} + C.$$

$$\int \frac{\mathrm{d}u}{\sqrt{1-u^2}} = \arcsin u + C.$$

基本积分公式(续)

(16) $\int \sec x \mathrm{d}x = \ln\left|\sec x + \tan x\right| + C$

(17) $\int \csc x \mathrm{d}x = \ln\left|\csc x - \cot x\right| + C$

(19) $\int \frac{1}{x^2 - a^2} \mathrm{d}x = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$

(20) $\int \frac{1}{a^2 - x^2} \mathrm{d}x = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C$

$$= \frac{1}{2a} \ln\left|\frac{x+a}{x-a}\right| + C$$

(19) $$\int \frac{1}{x^2-a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

证明 $$\int \frac{1}{x^2-a^2}dx = \int \frac{1}{(x-a)(x+a)}dx$$

$$= \frac{1}{2a}\int \left(\frac{1}{x-a} - \frac{1}{x+a}\right)dx$$

$$= \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right]$$

$$= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

(16) $\int \sec x dx = \ln\left|\sec x + \tan x\right| + C$

证明 $\int \sec x dx = \int \frac{1}{\cos x} dx = \int \frac{\cos x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} d\sin x$

$$= \int \frac{1}{1-\sin^2 x} d\sin x = \frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right| + C$$

$$= \frac{1}{2}\ln\frac{(1+\sin x)^2}{\cos^2 x} + C = \ln\left|\frac{1+\sin x}{\cos x}\right| + C$$

$$= \ln\left|\sec x + \tan x\right| + C$$

例3 求$\int (ax+b)^m \mathrm{d}x \quad (m \neq -1)$.

解 $\int (ax+b)^m dx = \frac{1}{a}\int (ax+b)^m d(ax+b)$

$$= \frac{1}{a(m+1)}(ax+b)^{m+1} + C.$$

注：当$m=-1$时，

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\int \frac{\mathrm{d}(ax+b)}{ax+b} = \frac{1}{a}\ln|ax+b| + C.$$

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$$

例4 求$\int \frac{\mathrm{d}x}{x\ln x}$.

解: 原式 $=\int \frac{(\ln x)'}{\ln x}dx = \int \frac{\mathrm{d}(\ln x)}{\ln x} = \ln|\ln x| + C$.

例4 求$\int \frac{\mathrm{d}x}{x(1+2\ln x)}$.

解: 原式 $= \frac{1}{2}\int \frac{(1+2\ln x)'}{1+2\ln x}dx = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2}\ln\left|1+2\ln x\right| + C.$$

$$\int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\,\mathrm{d}\ln x$$

例5 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx$.

解: 原式 $= \frac{2}{3}\int e^{3\sqrt{x}} d(3\sqrt{x})$

$$= \frac{2}{3} e^{3\sqrt{x}} + C.$$

$$\int f(\sqrt{x}) \frac{1}{\sqrt{x}} dx = 2\int f(\sqrt{x})\, d\sqrt{x}$$

例8 求$\int \frac{dx}{1+e^x}$.

解法1 $\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{\mathrm{e}^x}{\mathrm{e}^x+(\mathrm{e}^x)^2}\mathrm{d}x = \int \frac{1}{\mathrm{e}^x+(\mathrm{e}^x)^2}\mathrm{d}\mathrm{e}^x$

$$\underline{\underline{u=\mathrm{e}^x}} \int \frac{1}{u+u^2}\mathrm{d}u = \int \left(\frac{1}{u} - \frac{1}{1+u}\right)\mathrm{d}u$$

$$= \ln\left|\frac{u}{1+u}\right| + C = \ln\frac{e^x}{1+e^x} + C$$

$$= x - \ln(1+\mathrm{e}^x) + C.$$

$$\int f(\mathrm{e}^x)\mathrm{e}^x\,dx = \int f(\mathrm{e}^x)\,\mathrm{d}\mathrm{e}^x$$

例8 求$\int \frac{dx}{1+e^x}$.

解法2 $$\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}\,\mathrm{d}x = -\int \frac{\mathrm{d}(1+\mathrm{e}^{-x})}{1+\mathrm{e}^{-x}}$$

$$= -\ln(1+\mathrm{e}^{-x}) + C.$$

解法3 $$\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{1+\mathrm{e}^x-\mathrm{e}^x}{1+\mathrm{e}^x}\mathrm{d}x = \int (1-\frac{\mathrm{e}^x}{1+\mathrm{e}^x})\,\mathrm{d}x$$

$$= x - \int \frac{1}{1+e^x}\mathrm{d}(1+e^x)$$

$$= x - \ln(1+e^x) + C$$

有关三角函数积分的一些常见方法

(1) 一般地，对于 $\sin^{2k+1}x\cos^n x$ 或 $\sin^n x\cos^{2k+1}x$ 型函数的积分，常作变形 $\sin x dx = -d\cos x$ 或 $\cos x dx = d\sin x$，将其转换成中间变量为 $\cos x$ 或 $\sin x$ 的不定积分.

例10 求 $\int \sin^2 x\cos^5 x dx$.

解 $\int \sin^2 x\cos^5 x dx = \int \sin^2 x\cos^4 x\cos x dx$

$$= \int \sin^2 x(1-\sin^2 x)^2 d(\sin x)$$

$$= \int (\sin^2 x - 2\sin^4 x + \sin^6 x)d(\sin x)$$

$$= \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C.$$

(2) 一般地，对于$\sin^{2^k} x$或$\cos^{2^l} x$型函数的积分，使用倍角公式化成$\cos 2nx$的多项式，然后积分.

例11　求$\int \cos^4 x \, dx$.

解：$\because \quad \cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}(1 + 2\cos 2x + \frac{1+\cos 4x}{2})$$

$$= \frac{1}{4}(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)$$

$$\therefore \quad \int \cos^4 x\,\mathrm{d}x = \frac{1}{4}\int \left(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x\right)\mathrm{d}x$$

$$= \frac{1}{4}\left[\frac{3}{2}\int \mathrm{d}x + \int \cos 2x\,\mathrm{d}(2x) + \frac{1}{8}\int \cos 4x\,\mathrm{d}(4x)\right]$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C.$$

(3) 一般地，对于$\tan^n x\sec^{2k} x$或$\tan^{2k-1} x\sec^n x(k\in Z^+)$型函数的积分，可作变形

$$\sec^2 xdx = d\tan x\text{或}\sec x\tan xdx = d\sec x,$$

将其转换成中间变量为$\tan x$或$\sec x$的不定积分.

例12 求$\int\sec^6 xdx$.

解: 原式 $= \int(\tan^2 x+1)^2\cdot\sec^2 xdx$

$$= \int(\tan^2 x+1)^2 \mathrm{d}\tan x$$

$$= \int(\tan^4 x+2\tan^2 x+1)\,\mathrm{d}\tan x$$

$$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C.$$

常用的凑微分形式:

1) $\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$

2) $\int f(x^n)\,x^{n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\mathrm{d}x^n$

3) $\int f(x^n)\frac{1}{x}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\frac{1}{x^n}\,\mathrm{d}x^n$

4) $\int f(\sin x)\cos x\,\mathrm{d}x = \int f(\sin x)\,\mathrm{d}\sin x$

5) $\int f(\cos x)\sin x\,\mathrm{d}x = -\int f(\cos x)\mathrm{d}\cos x$

6) $\int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\,\mathrm{d}\tan x$

$$7)\ \int f(e^x)e^x\,dx = \int f(e^x)\,de^x$$

$$8)\ \int f(\ln x)\frac{1}{x}\,dx = \int f(\ln x)\,d\ln x$$

$$9)\ \int f(\sqrt{x})\frac{1}{\sqrt{x}}\,dx = 2\int f(\sqrt{x})\,d\sqrt{x}$$

$$10)\ \int f(\arctan x)\frac{1}{1+x^2}\,dx = \int f(\arctan x)\,d\arctan x$$

$$11)\ \int f(\sqrt{1+x^2})\frac{x}{\sqrt{1+x^2}}\,dx = \int f(\sqrt{1+x^2})\,d\sqrt{1+x^2}$$

$$12)\ \int f(x+\frac{1}{x})(1-\frac{1}{x^2})\,dx = \int f(x+\frac{1}{x})\,d(x+\frac{1}{x})$$

$$12')\ \int f(x-\frac{1}{x})(1+\frac{1}{x^2})\,dx = \int f(x-\frac{1}{x})\,d(x-\frac{1}{x})$$

思考题 求 $I=\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解 原式 $=\int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}}=\arcsin\frac{2x-1}{\sqrt{5}}+C$.

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(21) $\int \frac{1}{\sqrt{a^2-x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a}+C$