CSCI 150 Discrete Mathematics
Homework 2
Solution

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Problem 0

- Understand the connection between the product rule, $n!$, and permutations
- Understand the meaning of "n choose k", and in particular "n choose 2"
- Make yourself comfortable with summation and product notations, Σ and $\prod$
- Review power series and the definition of logarithm (chapter 0)
- Watch the videos posted on the course website

Problem 1: Round table (will not be graded)
We have seen in class the problem of seating n people on n chairs. This can be done in $n!$ ways because any seating can be represented by some ordering of the people, i.e. a permutation, and there are $n!$ permutations on n objects.

However, it is known that seating n people on a round table with n chairs can be done in $(n-1)!$ ways. Your job is to learn about this by searching the internet and understanding why $n!$ is an overcount (by n) for the round setting. Try to describe this in your own words.

Solution: There are $n!$ ways to assign people to chairs on the round table. However, since the table is round, any rotation of the seat assignment is equivalent. For instance, if we have four people, A, B, C, and D, all the following assignments are equivalent (A, B, C, D), (B, C, D, A), (C, D, A, B), and (D, A, B, C), where the position of a person in the tuple indicates the chair. In general, there are n such rotations, so we overcount by n, and the answer should be $n!/n = (n-1)!$.

Another way to think about this is the following: The first person will always be on chair 1, because no matter what we do it will be equivalent to rotating everyone until person 1 is on chair 1. So this suggests the following procedure:

1. choose chair 1 for person 1 ... 1 way
2. choose another chair for person 2 ... $(n-1)$ ways
3. choose another chair for person 3 ... $(n-2)$ ways
4. etc...

until we choose another chair for person n in the n^{th} phase, in 1 way.

By the product rule, we get $(n-1)!$. The procedure does not overcount. In other words, put person 1 on chair 1, then seat $(n-1)$ people on $(n-1)$ chairs, which can be done in $(n-1)!$ ways.

Problem 2: An interesting product
Given a fixed number x where $0 \leq x \leq 1$, consider the following quantity:

$$f(x) = \prod_{i=0}^{\infty}(1-(1-x)x^i)$$

(a) Express this quantity by writing the first 4 terms of the product explicitly, followed by "...". Each of the 4 terms must be simplified as much as possible; for instance, there should be no nested parenthesis.

Solution:

$$(1-(1-x)x^0)(1-(1-x)x^1)(1-(1-x)x^2)(1-(1-x)x^3)\dots$$
$$= x(1-x+x^2)(1-x^2+x^3)(1-x^3+x^4)\dots$$

(b) What are $f(0)$ and $f(1)$?

Solution: From above, we can see that $f(0) = 0$ because the first term of the product is 0. When $x = 1$, all terms are 1, so $f(1) = 1$.

(c) If $x \approx 1$ (x is approximately 1), then $(1-x)x^i \approx 0$. Consider the natural logarithm of the above quantity, and use the fact that $\ln(1-\epsilon) \approx -\epsilon$ when $\epsilon \approx 0$ to find $\ln f(x)$ and then $f(x)$ when $x \approx 1$. *Hint*: A useful tool here is power series. Also review what the log of a product is. Both of these concepts are in chapter 0.

Solution: Assume $x \approx 1$.

$$\ln \prod_{i=0}^{\infty}(1-(1-x)x^i) = \sum_{i=0}^{\infty} \ln(1-(1-x)x^i) \approx \sum_{i=0}^{\infty} -(1-x)x^i = -(1-x)\sum_{i=0}^{\infty} x^i$$

Using power series, $\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + x^3 + \ldots = 1/(1-x)$, so we obtain -1. Now,

$$\ln \prod_{i=0}^{\infty}(1-(1-x)x^i) \approx -1 \Rightarrow \prod_{i=}^{\infty}(1-(1-x)x^i \approx e^{-1} = 1/e$$

Problem 3: Unique pairwise sums (will not be graded)
Consider a set of n positive integers $S = \{a_1, a_2, \ldots, a_n\}$. You should know by now that there are $n(n-1)/2$ pairs of integers in this set. However, if we compute $a_i + a_j$ for each of these pairs, we do not necessarily get different results. Here's an example: In $S = \{1, 2, 3, 4, \ldots, n\}$, $(1, 4)$ and $(2, 3)$ are different pairs, but their sums are equal, $1 + 4 = 2 + 3$. It is not hard to construct a set where each pair gives a different sum; for instance, $S = \{10^0, 10^1, 10^2, \ldots, 10^{n-1}\}$, but can you construct a set with unique pairwise sums while keeping the numbers as small as possible? Try...

Solution: Note: This problem has nothing to do with anything that we have learned in class. But there is nothing wrong in exploring, which is what mathematics is all about...

We can grow the set one element at a time, $a_1, a_2, \ldots$, starting with $a_1 = 1$ and $a_2 = 2$. At any point in time t, the largest sum for any given pair is equal $s = a_{t-1} + a_t$, the sum of the last two elements. This means, and since the smallest element is 1, we can add s itself as the next element: The smallest sum that s can make with another element is $1 + s$, which is greater than s, the largest sum so far. So the set will evolve like this: $S = \{1, 2, 3, 5, 8, 13, 21, 34, \ldots\}$ (recognizable?). But is this the best we can do? No, here's an example: $S = \{1, 2, 3, 5, 8, 13, 21, 30, \ldots\}$.

Problem 4: Apples and Oranges and ...
For each of the following, design the generation procedure by explicitly describing the number of phases, what each phase does, and the number of ways each phase can be carried out. Then answer the following questions:

- does my procedure generate all valid outcomes?
- does my procedure generate any outcome that is not valid?
- does my procedure overcount (and by how much)?

Then use the product rule, with possible adjustment, to determine the final count.

There are 12 children. In how many ways can you give them fruits if:

(a) You have an infinite supply of apples, oranges, and bananas (you're lucky!), and you want to give each child one fruit.

Solution: Here's a procedure that gives each child one fruit.

1. give the first child one fruit ... 3 ways
2. give the second child one fruit ... 3 ways
3. give the third child one fruit ... 3 ways

and so on until phase 12. Each phase is carried out in 3 ways, so by the product rule this is 3^{12}. This procedure does not overcount. To see this, consider one possible outcome $(A, O, B, A, A, B, O, O, A, B, B, A)$, which gives the first child an apple, the second an orange, the third a banana, etc... The only way to generate this outcome is by making the corresponding choices in each phase. Any change to the choices will result in a child getting a different fruit, so a different outcome.

(b) You have 4 of each kind, and you want to give each child one fruit.

Solution: Here's a procedure that gives each child one fruit.

1. choose a child for the first apple ... 12 ways
2. choose a child for the second apple ... 11 ways
3. choose a child for the third apple ... 10 ways
4. choose a child for the fourth apple ... 9 ways
5. choose a child for the first orange ... 8 ways
6. choose a child for the second orange ... 7 ways
7. choose a child for the third orange ... 6 ways
8. choose a child for the fourth orange ... 5 ways
9. choose a child for the first banana ... 4 ways
10. choose a child for the second banana ... 3 ways
11. choose a child for the third banana ... 2 ways
12. choose a child for the fourth banana ... 1 ways

By the product rule, this is 12!. However, this procedure overcounts. Consider the following outcome $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$, which gives children $\{1, 2, 3, 4\}$ apples, children $\{5, 6, 7, 8\}$ oranges, and children $\{9, 10, 11, 12\}$ bananas. Any permutation of the first 4, second 4, or third 4 will result in the same outcome. For instance $(2, 4, 3, 1, 5, 7, 8, 6, 11, 10, 12, 9)$ is the same as above. Each group of 4 can be permuted in 4! ways, so we are overcounting by $4! \cdot 4! \cdot 4!$. So the answer is

$$\frac{12!}{4! \cdot 4! \cdot 4!}$$

Here's another procedure that does not overcount:

1. choose 4 children to take apples ... $\binom{12}{4}$ ways
2. choose 4 children to take oranges ... $\binom{8}{4}$ ways
3. choose 4 children to take bananas ... $\binom{4}{4}$ ways

A possible outcome is now this: $(\{1, 2, 3, 4\}, \{5, 6, 7, 8\}, \{9, 10, 11, 12\})$. Permuting the groups of 4 does not change the outcome because it preserves the same sets. So the answer is

$$\binom{12}{4}\binom{8}{4}\binom{4}{4}$$

Verify that this is exactly the same answer as before, and relate this to anagrams from Practice sheet 2.

Problem 5: A 3-letter "word"

Use any strategy to solve this problem, but explain your reasoning. Consider the following three-letter "word", where the first two letters are hidden:

$$\cdot \quad \cdot \quad Z$$

In how many ways can you specify the two hidden letters if:

(a) Letters cannot repeat.

Solution: We can choose any pair of letters that does not contain Z. The order within a pair is relevant because different orders give different words. The following procedure works:

1. choose a letter from $\{A, \ldots, Y\}$ ... 25 ways
2. choose **another** letter from $\{A, \ldots, Y\}$ ... 24 ways

By the product rule this is $25 \cdot 24$. Observe that this is also

$$\frac{25!}{(25-2)!} = 2\frac{25!}{2!(25-2)!} = 2\binom{25}{2}$$

because it amounts to choosing a pair of letters **with order**.

(b) Letters can repeat.

Solution: When the letters can repeat, we can choose any of the 26 letters for each hidden spot. This is $26 \cdot 26 = 26^2$ by the product rule. Also illustrated by the following procedure that does not overcount:

1. choose any letter in $\{A, \ldots, Z\}$ ... 26 ways
2. choose any letter in $\{A, \ldots, Z\}$ ... 26 ways

(c) Letters cannot repeat, and the three letters must appear in alphabetical order.

Solution: This is similar to part (a) except that order is irrelevant. Once the pair of letters is chosen, the order is implicitly determined as the alphabetical order. So this is $\binom{25}{2}$.

(d) Letters can repeat, and the three letters must appear in alphabetical order.

Solution: We can use the addition rule. If we start with A, the second letter can be any letter in $\{A, \ldots, Z\}$. If we start with B, the second letter can be any letter in $\{B, \ldots, Z\}$, and so on until if we start with Y, the second letter can be any letter in $\{Y, Z\}$, and if we start with Z, the second letter can be any letter in $\{Z\}$. Since each of these categories start with different letters, we can add them up to get

$$26 + 25 + 24 + \ldots + 1 = \frac{26 \cdot 27}{2} = \binom{27}{2}$$

Why $\left(\begin{array}{c}27\\2\end{array}\right)$ as if the alphabet has 27 letters? One way to think about it is this: Consider $\{A, B, C, \ldots, Z, *\}$. We select a pair of letters (implicitly alphabetical as in part (b)), and when the pair is $(x, *)$, we interpret this as (x, x). This covers all possibilities including repetitions. This idea can be generalized to say (verify):

$$n + \left(\begin{array}{c}n\\2\end{array}\right) = \left(\begin{array}{c}n+1\\2\end{array}\right)$$

Something to think about: It should be obvious why the answer for (c) is half the answer for (a). But why isn't the answer for (d) half the answer for (b)?