CSCI 150 Discrete Mathematics
Practice 1
Solution

Saad Mneimneh
Computer Science
Hunter College of CUNY

Problem 1: Practice sum and product notation

Mathematical notation is important for conveying mathematical ideas. For instance, consider the sum of the first n positive integers. This can be written as

$$1 + 2 + 3 + \ldots + n$$

More concisely, we can also express this using the summation notation:

$$\sum_{i=1}^{n} i$$

For each of the following, provide two ways of expressing it similar to the example given above:

- the sum of the first n positive odd numbers.

$$1 + 3 + \ldots + (2n - 1)$$

$$\sum_{i=1}^{n}(2i - 1) = 2\sum_{i=1}^{n} i - n$$

- the product of the first n positive even numbers.

$$2 \cdot 4 \cdot \ldots \cdot 2n$$

$$\prod_{i=1}^{n} 2i = 2^n \prod_{i=1}^{n} i = 2^n n!$$

- the sum of the first n positive perfect squares.

$$1 + 4 + 9 + \ldots + n^2$$

$$\sum_{i=1}^{n} i^2$$

- the equivalent of s in the following C++ program:

```
//assume n is given
int s=0;
for (int i=1; i<=n; i++) {
  s=s+i*(n-i+1);
}
```

$$1 \cdot n + 2(n-1) + 3(n-2) + \ldots + n \cdot 1$$
$$\sum_{i=1}^{n} i(n-i+1)$$

Problem 2: Series

(a) Find the 168$^{\text{th}}$ term of the sequence

$$68, 79, 90, \ldots, 2257$$

Solution: We have to make 167 steps from the first number to get to the 168^{th} number. Each step add 11, so $68 + 167 \cdot 11 = 1905$. Another way to see this is to express each number in terms of a counter i. For instance, the i^{th} number is $68 + 11(i-1)$. When $i = 168$ we get the above answer.

(b) A triangle is made out of a row of 100 stars, followed by a row of 97 stars, followed by a row of 94 stars, and so on. Write a formula for the total number of stars using Σ notation, and evaluate it.

Solution:

$$\sum_{i=0}^{33}(3i+1) = 3\sum_{i=0}^{33} i + \sum_{i=0}^{33} 1 = 3\frac{33 \cdot 34}{2} + 34 = 1717$$

(c) Find $\sum_{i=10}^{20}(i-15)$.

Solution:

$$\sum_{i=10}^{20} i - \sum_{i=10}^{20} 15 = \frac{(10+20)[(20-10)/1+1]}{2} - 11 \cdot 15 = 165 - 165 = 0$$

Another way is to compute $\sum_{i=10}^{20} i$ as $\sum_{i=1}^{20} i - \sum_{i=1}^{9} i$.

Problem 3

(a) Is it possible to come up with two graphs that have the same number of edges but different set of degrees? If yes, show an example, if no, explain why.

Solution: Yes, it is easy to provide two graphs that have the same number of edges, but different vertex degrees.

(b) Is it possible to come up with two graphs that have the same set of degrees but different number of edges? If yes, show an example, if no, explain why.

Solution: No, if the two graphs have the same set of degrees, then they both have the same sum of degrees, and we know that this is equal to twice the number of edges by the handshake lemma, so both must have the same number of edges.

Problem 3: Permutations
We have seen in class that the number of permutations on n objects is $n!$. Explain this result by making use of the multiplication rule. State clearly what the stages are, and in how many ways each stage can be carried out.

Solution: The permutation can be defined by ordering the objects. We consider a procedure with n stages. In each stage, we choose one of the remaining objects. An object chosen in stage i will be considered the i^{th} object. Stage i, $1 \leq i \leq n$ can be carried out in $n - i + 1$ ways. By the multiplication rule, the number of permutations is given by $n(n-1)\ldots 1 = n!$. There is no overcounting because every order can be generated by the above procedure in exactly one way. Therefore, the number of permutations is $n!$.

Problem 4: Snakes and Ladders
Use the approach we have seen in class to determine the following:

(a) In how many ways can we place one snake and one ladder on a grid with n squares?

Solution: A snake and a ladder can be defined by 4 squares: the first pair is our snake, the second pair is our ladder. Therefore, we can choose 4 squares in 4 stages. By the multiplication rule, this can be done in $n(n-1)(n-2)(n-3)$ ways. We are overcounting, however. To show this, consider one possible outcome given by 4 squares (a, b, c, d). We can permute the first two, and the last two, and still obtain the same snake and ladder. There are 2! ways to permute the first pair, and 2! ways to permute the second pair. Therefore, (a, b, c, d) has 4 equivalent outcomes. Our answer should be $n(n-1)(n-2)(n-3)/4$.

(b) In how many ways can we place two snakes on a grid with n squares?

Solution: The same reasoning applies, we pick four squares, the first pair defines one snake, and the second pair defines the other. But this time we are overcounting more. The two pairs can now be permuted and this will still produce the same two snakes. Given (a, b, c, d), the outcome (c, d, a, b) is equivalent. Therefore, we are now overcounting by 8. The answer is $n(n-1)(n-2)(n-3)/8$.

(c) Verify your answers (in terms of n) for the case of a 2×2 grid, i.e. when $n = 4$, by enumerating all possibilities.

Solution: On a 2x2 grid, the above answers give $4!/4 = 6$ for part (a) and $4!/8 = 3$ for part (b).

(a): (1,2,3,4), (1,3,2,4), (1,4,2,3), (2,3,1,4), (2,4,1,3), (3,4,1,2).

(b): (1,2,3,4), (1,3,2,4), (1,4,2,3)