CSCI 150 Discrete Mathematics
Homework 1
Solution

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Problem 1: Snakes and Ladders
Consider a regular 8×8 chessboard. The chessboard is to be used to play snakes and ladders, but there is an extra condition on how to place snakes on the chessboard: The head and tail of a snake must not occupy squares of the same color. Ladders remain unconstrained. In how many ways can we:

(a) Place one snake.

Solution: Here's a procedure for generating a snake (here we assume n is even, and in our example $n = 64$).

1. pick a black square ... $n/2$ ways
2. pick a white square ... $n/2$ ways

Every snake can be generated this way, with one black square and one white square. The head and the tail of the snake are uniquely determined once the two squares are given. There is no way to overcount because we generate the black square first, and the white square next. This means that any combination of 2-colored pair of squares is generated exactly once. Therefore, the number of ways we can place a snake is $n^2/4 = 64 \cdot 64/4$.

(b) Place a ladder and a snake.

Here's a procedure to generate a snake and a ladder. We will first generate the snake, because that makes the second phase independent of the first. If on the other hand we choose the ladder first, then the number of black and white color squares might be different.

1. choose a black square for the snake ... $n/2$ ways
2. choose a white square for the snake ... $n/2$ ways
3. choose the first square for the ladder ... $n - 2$ ways
4. choose the second square for the ladder ... $n - 3$ ways

It is not hard to see that this procedure generates all possible valid outcomes of one snake and one ladder. To check for overcouting, assume we choose squares

(a, b, c, d). Since square a is black and square b is white, they cannot be permuted. However, squares c and d are not constrained. So (a, b, d, c) represent the same snake/ladder combination. Other types of permuations will result in different snake and ladder. So each combination is counted exactly twice. By the product rule, and after adjustment due to overcounting, the answer should be $n^2(n-2)(n-3)/8$.

(c) Place two snakes.

Solution: Here's a procedure to generate two snakes:

1. choose a black square for the first snake ... $n/2$ ways
2. choose a white square for the first snake ... $n/2$ ways
3. choose a black square for the second snake ... $(n-2)/2$ ways
4. choose a white square for the second snake ... $(n-2)/2$ ways

Again, this procedure generates all possible valid pairs of snakes, but it overcounts by 2. For instance (a, b, c, d) would be the same as (c, d, a, b). By the product rule, the answer is $n^2(n-2)^2/32$ after adjusting for overcounting.

In answering the questions above, show the reasoning using the product rule, and explicitly describe the different stages and in how many ways they can be carried out. In addition, justify any adjustment you make due to possible overcounting.

Problem 2: Mathematical identities

We have seen in class that $\binom{n}{2} = n(n-1)/2$. It is tempting, based on this notation, to conclude that:

$$\binom{n}{k} = n(n-1)/k \quad \text{or} \quad \binom{n}{k} = n(n-k/2)/k$$

or make any kind of generalization based on the pattern. Making this kind of hasty generalizations based on patterns is similar to saying that if $5^2 = 25$ and $6^2 = 36$, then 7^2 must be 47. It turns out that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

(a) Show using algebraic manipulation that

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

Solution:

$$k\binom{n}{k} = k\frac{n!}{k!(n-k)!} = \frac{n!}{(k-1)!(n-k)!} = n\frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= n\frac{(n-1)!}{(k-1)![(n-1)-(k-1)]!} = n\binom{n-1}{k-1}$$

Consider the sum

$$\sum_{i=a}^{b}(ix+y)$$

where a, b, x, and y are given.

(b) Show using algebraic manipulations that this sum is equal to

$$(b-a+1)\Big(\frac{a+b}{2}x+y\Big)$$

Solution:

$$\sum_{i=a}^{b}(ix+y)=\sum_{i=a}^{b}ix+\sum_{i=a}^{b}y=x\sum_{i=a}^{b}i+(b-a+1)y$$

The equality $\sum_i ix = x\sum i$ is obtained by the distributive law, and $\sum_i y$ simply adds y a number of times equal to the range of i, which is $(b-a+1)$.

We know that $\sum_{i=a}^{b} i = (b-a+1)\frac{a+b}{2}$ (look at arithmetic series). Therefore, by distributive law again, we have

$$(b-a+1)[\frac{a+b}{2}x+y]$$

Problem 3: Schedule

A department offers three courses at 10:00 AM, six courses at 12:00 PM, and five courses at 2:00 PM.

(a) Use the multiplication rule to figure out the number of ways you can register for three courses.

Solution: Here's a procedure to generate the schedule:

1. choose a morning course ... 3 ways
2. choose a noon course ... 6 ways
3. choose an afternoon course ... 5 ways

This procedure generates all valid schedules and does not overcout (any given schedule can only be generated in one way). Therefore, by the product rule there are $3 \cdot 6 \cdot 5$ possible schedules.

(b) Use the addition rule and the multiplication rule to figure out the number of ways you can register for two courses.

Solution: There are 3 categories of possible 2-course schedules: (morning, noon), (morning, afternoon), and (noon, afternoon). For each category, and using a procedure like the outlined above, the number of possibilities is simply the product of the two corresponding numbers of courses. So we get $3 \cdot 6 + 3 \cdot 5 + 6 \cdot 5 = 18 + 15 + 30 = 63$. Here we have used the addition rule because the categories are exclusive (disjoint).