Preface

Introduction

本笔记旨在分享笔者对 Riemann-Hilbert 方法的理解, RH 方法作为一种强大的数学工具, 在非线性偏微分方程的求解, 尤其是可积系统中具有重要的应用价值. 通过对其理论背景, 基本方法及具体应用的学习, 笔者希望为读者提供一个清晰的入门指引。

主要内容如下:

第一章主要介绍 RH 方法的背景知识, 包括 Plemelj 定理, RH 问题等.

第二章主要介绍利用 RH 方法求解零边界的 NLS 方程, 通过构造特征函数, 分析其解析性与对称性, 以及建立相关的 RH 问题，最终推导出 NLS 方程的 N 孤子解. 本章节主要参考了复旦大学范恩贵老师的讲义 [1].

第三章主要介绍利用 RH 方法求解反时间, 反空间, 及反时空的反演, 这些方程是耦合 NLS 方程在不同约束条件下的特殊形式。通过分析其散射数据的对称性，进一步推导出这些非局部方程的 N 孤子解. 本章节主要参考了杨建科老师的论文 [2]

感想

本笔记为在老家过年期间编写, 整个过程并不算顺利, 老家的房子年久失修, 塌了外墙和两间屋顶, 白天需要和父亲一起修房子, 应付家庭琐事; 只有夜间才能缓慢的推进写作. 另外农村没有暖气, 零下十余度的环境实难称舒适, 需要字面意义上的争分夺秒. 尤其深夜伏案时, 恍若独行于漫长甬道, 徘徊在黑暗迷宫之中. 然正如 A. Zee 所言, 夜航人自有夜行法 [3]. 历时一月的艰辛写作中, 笔者感到一种难以言喻的, 某种漠然的相互理解, 如越过高墙, 漫步在满月下的林地, 并在夜色中获得慰籍. 是邪，非邪? 解释或属妄诞，感受毕竟真实.

因仅为一家之言, 仓促间完成又后期无他人审校, 难免存在疏漏, 错误和挂一漏万之处. 望各位读者在阅读时能指出不足, 共同探讨与完善.

序曲将终，敬无穷的远方，与无尽的人们.

本笔记存档于 https://github.com/IceySwan/Notes, 如发现任何错误请提 issue 或联系 hi@icey.one

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2025 年 2 月

目录

第 1 章 预备知识

1.1 Plemelj 公式

定理 1.1 (Abel 定理)

设 $A \in \mathbb{C}$, 对矩阵微分方程 $Y_x = A(x)Y$, 可得标量微分方程

$$(\det Y)_x = \operatorname{tr}(A)\det Y \tag{1.1.1}$$

从而有

$$\det(Y(x)) = \det(Y(x_0)) \cdot \exp\int_{x_0}^{x} \operatorname{tr}(\mathrm{A(t)})\mathrm{dt} \tag{1.1.2}$$

♡

定理 1.2 (Morera)

如果 $f(z)$ 在单连通区域 D 内连续, 且对于 D 内任意闭曲线 $\sum$, 有 $\int_{\sum} f(z)\mathrm{d}z = 0$, 则有 $f(z)$ 在 D 内解析

♡

定义 1.1 (Schwartz 空间)

欧几里得空间 $\mathbb{R}^n$ 上的 Schwartz 空间 $\mathcal{S}(\mathbb{R}^n)$ 定义为

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^\kappa) : \|f\|_{\alpha,\beta} = \sum_{x\in\mathbb{R}^n} |x^\alpha \partial_\beta f(x)| < \infty, \alpha, \beta \in \mathbb{Z} \right\} \tag{1.1.3}$$

其中 α, β 为多重指标, $\partial_\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$. f 称为速降函数或 Schwartz 函数.

♣

简单来说，速降函数是指当 $x \to \infty$ 时趋近于零的速度比所有的多项式的倒数都快，并且任意阶的导数都有这种性质的函数。

定理 1.3 (Painleve 开拓定理)

设 D_1, D_2 为两个没有公共点的区域, 边界为 Γ, 并设 $f_1(z), f_2(z)$ 分别在 D_1, D_2 内解析, 在 $D_1+\Gamma, D_2+\Gamma$ 上连续, 且 $f_1(z) = f_2(z), \forall z \in \Gamma$, 则

$$f(z) = \begin{cases} f_1(z), & z \in D_1, \\ f_1(z) = f_2(z), & z \in \Gamma, \\ f_2(z), & z \in D_2 \end{cases} \tag{1.1.4}$$

在 $D_1 + D_2 + \Gamma$ 内解析.

♡

证明 显然 $f(Z)$ 在 $D_1 + D_2 + \Gamma$ 上连续, 根据定理 1.2, 只需证明 $f(z)$ 沿任何闭曲线 $\sum$ 的积分为 0.

如果 $\sum \subset D_1$, 或 $\sum \subset D_1$, 则由 $f(z)$ 的解析性可得

$$\int_{\sum} f(z)\mathrm{d}z = 0 \tag{1.1.5}$$

如果 $\sum$ 同时包含于 D_1, D_2 内, 把 Γ 在 $\sum$ 内的曲线记为 C_γ, 则

$$\int_{C_1+C_\gamma} f(z)\mathrm{d}z = 0, \quad \int_{C_1+C_\gamma}^{-} f(z)\mathrm{d}z = 0 \tag{1.1.6}$$

故

$$\int_{\sum} f(z)\mathrm{d}z = \int_{C_1+C_2} f(z)\mathrm{d}z = \int_{C_1+C_\gamma} f(z)\mathrm{d}z = \int_{C_1+C_\gamma^-} f(z)\mathrm{d}z = 0. \tag{1.1.7}$$

□

引理 1.1

设 $f(\xi)$ 在 $z\in\sum$ 满足 μ 次 Hölder 条件, 且 $z'\to z$ 时, h/d 有界, 其中

$$h=|z'-z|,\quad d=\min_{\xi\in\sum}|\xi-z'|, \tag{1.1.8}$$

则

$$\lim_{z'\to z}\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z'}\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z}\mathrm{d}\xi \tag{1.1.9}$$

证明

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z'}\,\mathrm{d}\xi-\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z}\,\mathrm{d}\xi\\
=&\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{(\xi-z)(f(\xi)-f(z))-(\xi-z')(f(\xi)-f(z))}{(\xi-z')(\xi-z)}\,\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{z'-z}{\xi-z}\cdot\frac{f(\xi)-f(z)}{\xi-z}\,\mathrm{d}\xi\\
=&\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{h}{d}M\frac{(\xi-z)^{\mu}}{\xi-z}\,\mathrm{d}\xi=\Delta_1+\Delta_2
\end{aligned} \tag{1.1.10}$$

其中

$$\begin{aligned}
|\Delta_1|&\le\frac{hM}{2\pi d}\int_{C_\delta}\frac{(\xi-z)^{\mu}}{\xi-z}\,\mathrm{d}|\xi|\le\frac{hM}{2\pi d}\int_0^{\delta}t^{\mu-1}\,\mathrm{d}t=\frac{hM}{2\pi d\delta}\delta^{\mu}(C_\delta=\{|\xi-z|\le\delta\})\\
|\Delta_2|&=\left|\frac{1}{2\pi}\int_{\sum\backslash C_\delta}\frac{(f(\xi)-f(z))(z-z')}{\xi-z}\,\mathrm{d}\xi\right|
\end{aligned} \tag{1.1.11}$$

由于 $\sum\backslash C_\delta$ 不包含 z, 故 Δ_2 为关于 z' 的连续函数, 则

$$|\Delta_2|=\left|\frac{1}{2\pi\mathrm{i}}\int_{\sum\backslash C_\delta}\frac{f(\xi)-f(z)}{\xi-z'}\,\mathrm{d}\xi-\frac{1}{2\pi}\int_{\sum\backslash C_\delta}\frac{f(\xi)-f(z)}{\xi-z}\,\mathrm{d}\xi\right|<|\Delta_1| \tag{1.1.12}$$

取 $\delta\to0$, 可令 $|\Delta_1|<\frac{\epsilon}{2}$, 故 $|\Delta_1|+|\Delta_2|<\epsilon$ □

定理 1.4 (Plemelj)

设 $z\in\sum$ 为正则点, 且不为边界点, $f(\xi)$ 在 z 点满足 μ 次 Hölder 条件, 且 $z'\to z$ 时, h/d 有界, 其中 $h=|z'-z|,\quad d=\min_{\xi\in\sum}|\xi-z'|$, 则

$$F_+=\lim_{\substack{z'\to z\\ z'\in\sum_+}}F(z')=F(z)+\frac{1}{2}f(z) \tag{1.1.13}$$

$$F_-=\lim_{\substack{z'\to z\\ z'\in\sum_-}}F(z')=F(z)-\frac{1}{2}f(z) \tag{1.1.14}$$

证明 首先证明闭区线情形:

$$\begin{aligned}
F_+=&\lim_{\substack{z'\to z\\ z'\in\sum_+}}F(z')=\lim_{z'-z}\int_{\sum}\frac{1}{2\pi\mathrm{i}}\frac{f(\xi)}{\xi-z'}\,\mathrm{d}\xi\\
=&\lim_{z'-z}\int_{\sum}\frac{1}{2\pi\mathrm{i}}\frac{f(\xi)-f(z)}{\xi-z'}\,\mathrm{d}\xi+\frac{f(\xi)}{2\pi\mathrm{i}}\int_{\sum}\frac{1}{\xi-z'}\,\mathrm{d}\xi\\
=&\lim_{z'-z}\frac{1}{2\pi\mathrm{i}}\int_{\sum_+}\frac{f(\xi)-f(z)}{\xi-z'}\mathrm{d}\xi+f(z)\\
=&\lim_{z'-z}\frac{1}{2\pi\mathrm{i}}\int_{\sum_+}\frac{f(\xi)}{\xi-z'}\mathrm{d}\xi-\frac{f(z)}{2\pi\mathrm{i}}\int_{\sum_+}\frac{1}{\xi-z'}\mathrm{d}\xi+f(z)\\
=&F(z)+\frac{1}{2}f(z)
\end{aligned} \tag{1.1.15}$$

$F_-(z')$ 同理, 下证 $\sum$ 为开曲线情形, 则可补充 $\sum'$, s.t. $\sum\cup\sum'$ 为闭曲线, 且定义 $\forall\xi\in\sum', f(\xi)=0$, 则

$$F(z)=\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(z)}{\xi-z}\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\int_{\sum\cup\sum'}\frac{f(z)}{\xi-z}\mathrm{d}\xi. \tag{1.1.16}$$

则由开曲线情形可得

$$\begin{aligned} F_+(z) &= \lim_{\substack{z'\to z\\ z'\in+\sum}} \int_{\sum\cup\sum'} \frac{f(\xi)}{\xi-z}\mathrm{d}\xi = \int_{\sum\cup\sum'} \frac{f(\xi)-f(z)}{\xi-z}\mathrm{d}\xi + \frac{f(z)}{2\pi\mathrm{i}}\int_{\sum\cup\sum'} \frac{1}{\xi-z'}\mathrm{d}\xi \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)-f(z)}{\xi-z}\mathrm{d}\xi - \frac{f(z)}{2\pi\mathrm{i}}\int_{\sum'} \frac{1}{\xi-z}\mathrm{d}\xi + f(z) = F(z) + \frac{1}{2}f(z) \end{aligned} \tag{1.1.17}$$

$F_-(z')$ 同理 □

注 如果 $z\in\sum$ 为一个角点, 在其两切线的夹角为 α, 则可

$$\begin{aligned} F_+(z) &= \lim_{\substack{z'\to z\\ z'\in D}} \frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)}{\xi-z}\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)-f(z)}{\xi-z}\mathrm{d}\xi + \frac{\alpha}{2\pi}f(z) + (1-\frac{\alpha}{2\pi})f(z) \\ &= F(z) - \frac{\alpha}{2\pi}f(z) \end{aligned} \tag{1.1.18}$$

$$\begin{aligned} F_-(z) &= \frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)-f(z)}{\xi-z}\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)-f(z)}{\xi-z}\mathrm{d}\xi + \frac{\alpha}{2\pi}f(z) - \frac{\alpha}{2\pi})f(z) \\ &= F(z) - \frac{\alpha}{2\pi}f(z) \end{aligned} \tag{1.1.19}$$

定义 1.2 (Plemelj 公式)

由 (1.1.13) - (1.1.14), (1.1.18) - (1.1.19) 可以看出, 无论 z 是正则点还是角点, 都有

$$F_+(z) - F_-(z) = f(z), \quad z\in\sum \tag{1.1.20}$$

$$F_+(z) - F_-(z) = \frac{1}{\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)}{\xi-z}\mathrm{d}\xi =: H(f)(z) \tag{1.1.21}$$

称为标量 RH 问题, 其解可用 Cauchy 积分给出

$$F(z) = A + \frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)}{\xi-z}\mathrm{d}\xi, \quad z\in\mathbb{C} \tag{1.1.22}$$

其中 A 为任意常数, 一般被边值或渐进条件决定, 这一公式被称为 Plemelj 公式. ♣

对于如下 RH 问题

$$G_+(z) = G_-(z)v(z), \quad z\in\sum \tag{1.1.23}$$

只需两边取 log 变换则有

$$\log G_+(z) - \log G_-(z) = \log v(z), \quad z\in\sum \tag{1.1.24}$$

由 Plemelj 公式, 则有

$$\log G(z) = A + \frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{v(\xi)}{\xi-z}\mathrm{d}\xi \implies G(z) = B\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\sum} \frac{f(\xi)}{\xi-z}\mathrm{d}\xi\right), \quad z\in\sum \tag{1.1.25}$$

其中 B 为任意常数.

1.2 矩阵 RH 问题

定义 1.3 (RH 问题)

设 $\sum$ 为复平面 $\mathbb{C}$ 内的有向路径, 假设存在一个 $\sum^0$ 上的光滑映射 $v(z):\sum\to GL(n,\mathbb{C})$, 则 $(\sum, v)$ 决定了一个 RH 问题, 寻找一个 n 阶矩阵 $M(z)$ 满足

$$M(z)\in C, \quad (\mathbb{C}\setminus\sum) \tag{1.2.1}$$

$$M_+(z) = M_-(z)v(z), \quad z\in\sum \tag{1.2.2}$$

$$M(z)\to I, \quad z\to\infty \tag{1.2.3}$$

其中 $M_\pm$ 表示在正负区域内 $z' \to z$ 时的极限, $\sum$ 称为跳跃曲线, $v(z)$ 称为跳跃矩阵.

根据 Beadls-Coifman 定理, 如上 RH 问题的解可通过如下方式构造: 不妨设跳跃矩阵 $v(z)$ 具有如下分解

$$v = (b_-)^{-1} b_+ \tag{1.2.4}$$

由此, 可构造

$$w_+ = b_+ - I, \quad w_- = I - b_- \tag{1.2.5}$$

进一步可定义 Cauchy 投影算子

$$(C_\pm f)(z) = \lim_{\substack{z' \to z \in \sum \\ z' \in \pm \sum}} \frac{1}{2\pi \mathrm{i}} \int_\sum \frac{f(\xi)}{\xi - z'} \mathrm{d}\xi \tag{1.2.6}$$

则可以证明如果 $f(z) \in L^2(\sum)$, 则 $C_\pm : L^2 \to L^2$ 的有界算子, 且 $C_+ - C_- = 1$, 再定义算子

$$C_w f = C_+(f w_-) - C_-(f w_+) \tag{1.2.7}$$

则 $C_w : L^2 \cap L^\infty \to L^2$ 的有界算子.

定理 1.5

设 $\det v = 1$, 算子 $I - C_w$ 在 $L^2(\sum)$ 上可逆, $\mu \in I + L^2(\sum)$ 为下列方程

$$(I - C_w)\mu = I \tag{1.2.8}$$

的解, 且

$$(I - C_w)(\mu - I) = C_w I + C_+ w_- + C_- w_+ \in L^2\left(\sum\right) \tag{1.2.9}$$

则

$$M(z) = I + \frac{1}{2\pi \mathrm{i}} \int_\sum \frac{\mu(\xi) w(\xi)}{\xi - z} \mathrm{d}\xi \tag{1.2.10}$$

为上述 RH 问题的唯一解, RH 问题 $M(z)$ 的可解等价于奇异积分方程 (1.2.8).

证明 只需证明 (1.2.10) 满足 (1.2.8)

$$\begin{aligned} M_+ &= I + \lim_{z' \to z \in \sum} \frac{1}{2\pi \mathrm{i}} \int_\sum \frac{\mu(\xi) w(\xi)}{\xi - z} \mathrm{d}\xi \overset{(1.2.6)}{=} I + C_+(\mu w) \\ &= I + C_+(\mu w_-) + C_+(\mu w_+) = I + C_+(\mu w_-) + C_-(\mu w_+) + C_+(\mu w_+) + C_-(\mu w_+) \\ &\overset{(1.2.7)}{=} I + C_w(\mu) + \mu w_+ \overset{(1.2.8)}{=} \mu + \mu w_+ = \mu(I + w_+) = \mu b_+ \end{aligned} \tag{1.2.11}$$

同理有

$$M_-(z) = \mu b_- \tag{1.2.12}$$

故有

$$M_+(z) = \mu b_+ = \mu b_-(b_-)^{-1} b_+ = M_-(z) v(z) \tag{1.2.13}$$

下证唯一性, 先证 M 可逆, 对 (1.2.2) 取行列式, 并注意到 $\det v(z) = 1$, 则 $\det M_+(z) = \det M_-(z)$. 故由 Painleve 开拓定理得 $\det M(z)$ 在 $\mathbb{C}$ 上解析. 由 (1.2.3) 得

$$\det M(z) \to 1, z \to \infty \tag{1.2.14}$$

故由 Liouville 定理可知 $\det M(z) = c$, 再由渐进条件得 $c = 1$, 故 $M(z)$ 可逆.

设 $\widetilde{M}$ 为上述 RH 问题的另一个解, 则 $\widetilde{M}$ 可逆, 且在 $\mathbb{C} \setminus \sum$ 上解析, 且在 $\sum$ 上满足

$$(M\widetilde{M}^{-1}) = M_+ \widetilde{M}_+^{-1} = M_- v (\widetilde{M}_- v)^{-1} = M_-(\widetilde{M}_-)^{-1} = (M\widetilde{M}^{-1})_- \tag{1.2.15}$$

由 Painleve 开拓定理, $M\widetilde{M}^{-1}$ 在 $\mathbb{C}$ 上解析. 另外由 $M, \widetilde{M} \to I$, 故 $M\widetilde{M}^{-1}$ 有界, 故为常矩阵. 由渐进性得

$$M\widetilde{M}^{-1} = I \implies M = \widetilde{M} \tag{1.2.16}$$

□

注 由 RH 问题解的唯一性, RH 问题 (1.2.1) - (1.2.3) 的解与跳跃矩阵 $v(z)$ 的分解无关, 因此可以考虑 $v(z)$ 的平凡解

$$b_- = I, \quad b_+ = v \tag{1.2.17}$$

故可构造

$$\begin{aligned} w_- = 0, \quad w_+ = v - I, \quad w = v - I \\ C_w f = C_+(f w_-) + C_-(f w_+) = C_+(f(v - I)), \quad \mu = (I - C_w)^{-1} I \end{aligned} \tag{1.2.18}$$

则 RH 问题的解可表示为

$$M(z) = I + \frac{1}{2\pi \mathrm{i}} \int_{+\sum} \frac{\mu(\xi)(v(\xi) - I)}{\xi - z} \mathrm{d}\xi \tag{1.2.19}$$

注 RH 方法的关键思想就是改变积分路径, 通过跳跃矩阵的分解情况, 确定对积分路径进行一系列形变, 再取极限去除跳跃矩阵为单位阵的情形, 将其化解为可解的 RH 问题, 所以一个自然的问题就是: “为什么可以扔掉跳跃矩阵为单位阵的路径? 或者说为什么跳跃矩阵对对 RH 问题的解不产生贡献.”

这很容易通过表达式 (1.2.19) 看出, 我们将积分路径分解为 $\sum = \sum_1 + \sum_2$ 且

$$v = \begin{cases} v_1 \neq I & \sum_1, \\ v_1 = I & \sum_2 \end{cases} \tag{1.2.20}$$

则在 $\sum_1$ 上, $v - I = v_1 - I \neq 0$, 在 $\sum_2$ 上, $v - I = v_1 - I = 0$, 故

$$\begin{aligned} M(z) &= I + \frac{1}{2\pi \mathrm{i}} \int_{\sum} \frac{\mu(\xi)(v(\xi) - I)}{\xi - z} \mathrm{d}\xi \\ &= I + \frac{1}{2\pi \mathrm{i}} \int_{\sum_1} \frac{\mu(\xi)(v_1(\xi) - I)}{\xi - z} \mathrm{d}\xi + \frac{1}{2\pi \mathrm{i}} \int_{\sum_2} \frac{\mu(\xi)(v_2(\xi) - I)}{\xi - z} \mathrm{d}\xi \\ &= I + \frac{1}{2\pi \mathrm{i}} \int_{\sum_1} \frac{\mu(\xi)(v_1(\xi) - I)}{\xi - z} \mathrm{d}\xi \end{aligned} \tag{1.2.21}$$

易得 RH 问题 $(\sum, v(z))$ 的解与 $(\sum_1, v_1(z))$ 的解相同, 即跳跃矩阵为单位阵的路径 $\sum_2$ 对 RH 问题的解无贡献, 可以舍去. 只求 $\sum_1$ 上的 RH 问题

第 2 章 RH 方法求解零边界的 NLS 方程

2.1 聚焦 NLS 方程

2.1.1 特征函数

考虑聚焦 NLS 方程的初值问题

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q(x,t)|q(x,t)|^2 = 0 \tag{2.1.1}$$

$$q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}) \tag{2.1.2}$$

其中 $\mathcal{S}(\mathbb{R})$ 表示 Schwartz 空间. NLS 方程 (2.1.1) 具有如下矩阵形式的 Lax 对

$$\psi_x + \mathrm{i}z\sigma_3\psi = P\psi \tag{2.1.3}$$

$$\psi_t + 2\mathrm{i}z^2\sigma_3\psi = Q\psi \tag{2.1.4}$$

其中

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad P = \begin{pmatrix} & q \\ -q^* & \end{pmatrix}, \quad Q = \begin{pmatrix} \mathrm{i}|q|^2 & \mathrm{i}q_x \\ \mathrm{i}q_x^* & -\mathrm{i}|q|^2 \end{pmatrix} + 2zP$$

满足零曲率方程

$$(P - \mathrm{i}z\sigma_3)_t - \left(Q - 2\mathrm{i}z^2\sigma_3\right)_x + \left[P - \mathrm{i}z\sigma_3, Q - 2\mathrm{i}z^2\sigma_3\right] = 0 \tag{2.1.5}$$

2.1.2 渐进性

由于 $q_0(x) \in \mathcal{S}(\mathbb{R})$, 且 $q(x,t), q_x(x,t) \to 0, (x \to \infty)$, 故 x 足够大时, P, Q 可忽略, 从而 Lax 对 (2.1.3) - (2.1.4) 可近似为

$$\psi_x \sim -\mathrm{i}z\sigma_3\psi, \quad \psi_t \sim -2\mathrm{i}z^2\sigma_3\psi$$

可得渐进形式的 Jost 解 $\psi = e^{-\mathrm{i}\theta(z)\sigma_3}, (x \to \infty)$, 其中 $\theta(z) = zx + 2z^2t$. 下面为表示方便, 不妨记 $E := e^{\mathrm{i}\theta(z)\sigma_3}$, 做变换 $\mu(x,t,z) = \psi(x,t,z)E$, 则有

$$E^{-1} = e^{-\mathrm{i}\theta(z)\sigma_3}, \quad \mu(x,t,z) \to I(x \to \infty) \tag{2.1.6}$$

则可得

$$\mu_x = \psi_x E + \psi E(\mathrm{i}\sigma_3 z) \tag{2.1.7}$$

$$\mu_t = \psi_t E + \psi E(2\mathrm{i}\sigma_3 z^2) \tag{2.1.8}$$

带入 Lax 对 (2.1.3) - (2.1.4) 可得

$$\mu_x = (P\psi - \mathrm{i}z\sigma_3\psi)E + \psi E(\mathrm{i}\sigma_3 z) \tag{2.1.9}$$

$$\mu_t = (Q\psi - 2\mathrm{i}z^2\sigma_3\psi)E + \psi E(2\mathrm{i}\hat{\sigma}_3 z^2) \tag{2.1.10}$$

故有

$$\mu_x = P\mu - \mathrm{i}z[\sigma_3, \mu] \implies \mu_x + \mathrm{i}z[\sigma_3, \mu] = P\mu \tag{2.1.11}$$

同理可得

$$\mu_t + 2\mathrm{i}z^2[\sigma_3, \mu] = Q\mu \tag{2.1.12}$$

由 (2.1.11) 式可得

$$\mu_x + \mathrm{i}z\hat{\sigma}\mu = P\mu \tag{2.1.13}$$

$$e^{\mathrm{i}\theta(z)\hat{\sigma}_3}\mu_x + \mathrm{i}z\hat{\sigma}_3 e^{\mathrm{i}\theta(z)\hat{\sigma}_3}\mu = e^{\mathrm{i}\theta(z)\hat{\sigma}_3}P \tag{2.1.14}$$

其中 $\hat{\sigma}_3 X = [\sigma_3, X], e^{\mathrm{i}\theta(z)\hat{\sigma}_3}X = EXE^{-1}$, 为表示方便不妨记 $\hat{E} = e^{\mathrm{i}\theta(z)\hat{\sigma}_3}$. 易得 Lax 对的全微分形式

$$\mathrm{d}(\hat{E}\mu) = \hat{E}[(P\mathrm{d}x + Q\mathrm{d}t)\mu] \tag{2.1.15}$$

为构造规范的 Riemann-Hilbert 问题, 即其解在 $z \to \infty$ 时渐进于单位阵, 现只需证明 $\mu \to I(z \to \infty)$.

证明 将 μ 在无穷远点 Taylor 展开

$$\mu = \mu^{(0)} + \frac{1}{z}\mu^{(1)} + \cdots = \mu^{(0)} + O(z^{-1}) \tag{2.1.16}$$

其中 $\mu^{(0)}, \mu^{(1)}$ 与 z 无关, 将上式代入 (2.1.11), (2.1.12) 比较 z 的次数可得

$$\begin{aligned} &\left[\mu^{(0)} + O(z^{-1})\right]_x + \mathrm{i}z\left[\sigma_3, \mu^{(0)} + \frac{\mu^{(1)}}{z} + O(z^{-2})\right] = P\left[\mu^{(0)} + O(z^{-1})\right] \\ &[\mu^{(0)} + O(z^{-1})]_t + 2\mathrm{i}z^2[\sigma_3, \mu^{(0)} + \frac{\mu^{(1)}}{z} + O(z^{-2})] = (Q_0 + 2zP)[\mu^{(0)} + O(z^{-1})] \end{aligned} \tag{2.1.17}$$

可得 x 部分:

$$\left.\begin{aligned} z:&\quad [\sigma_3, \mu^{(0)}] = 0 \\ z^0:&\quad \mu^{(0)} + 2\mathrm{i}z[\sigma_3, \mu^{(1)}] = P\mu^{(0)} \end{aligned}\right. \implies \mu^{(0)}\text{为对角阵} \tag{2.1.18}$$

t 部分:

$$\left.\begin{aligned} z^2:&\quad [\sigma_3, \mu^{(0)}] = 0 \\ z:&\quad \mathrm{i}[\sigma_3, \mu^{(1)}] = P\mu^{(1)} \end{aligned}\right. \implies \mu_x^{(0)} = 0 \tag{2.1.19}$$

故 $\mu^{(0)}$ 为与 x 无关的对角矩阵, 因此 (2.1.16) 式对 x, z 同时取极限, 并交换极限顺序可得

$$\lim_{z\to\infty}\lim_{x\to\infty}\mu = \lim_{x\to\infty}\lim_{z\to\infty}\left(\mu^{(0)} + O(z^{(-1)})\right) \tag{2.1.20}$$

利用 (2.1.6), (2.1.16) 式可得 $\mu^{(0)} = I$, 故 $\mu \to I(z \to \infty)$. □

再将 $\mu^{(0)} = I$ 带入 (2.1.19) 式比较矩阵对角元素得

$$q(x,t) = 2\mathrm{i}(\mu^{(1)})_{12} = 2\mathrm{i}\lim_{z\to\infty}(z\mu)_{12} \tag{2.1.21}$$

注 上式可将 NLS 方程与特征函数联系起来, 接下来将特征函数与 RH 问题建立联系. 从而 NLS 方程的解可用 RH 问题的解表示, 然后通过 RH 问题反接触 NLS 方程的解, 即

$$NLS \rightleftharpoons \text{特征函数} \rightleftharpoons RH\text{问题}$$

2.2 解析性与对称性

由于 (2.1.15) 为全微分形式, 积分与路径无关, 故选择两个特殊路径:

$$(-\infty, t) \to (x, t), \quad (+\infty, t) \to (x, t)$$

因此可获得 Lax 对 (2.1.11), (2.1.12) 的两个特征函数

$$\mu_1 = I - \int_{-\infty}^{x} e^{-\mathrm{i}z(x-y)\sigma_3}P\mu_1\mathrm{d}x, \quad \mu_2 = I - \int_{+\infty}^{x} e^{-\mathrm{i}z(x-y)\sigma_3}P\mu_2\mathrm{d}x \tag{2.2.1}$$

证明 由 (2.1.15) 式, 由 $(\hat{E}\mu)_x = \hat{E}P\mu$ 可得

$$\hat{E}\mu|_{-\infty}^{x} = \int_{-\infty}^{x}\hat{E}P\mu\mathrm{d}y \tag{2.2.2}$$

$$LHS = \hat{E}\mu - \hat{E}I, \quad RHS = \int_{-\infty}^{x} e^{\mathrm{i}zy + z^2t\hat{\sigma}_3}P\mu\mathrm{d}y$$

两边同时乘以 $\hat{E}$, 即得 μ_1, μ_2 同理可得. □

显然, 其仍具有如下性质

$$\mu_1, \mu_2 \to I(x \to \pm\infty), \quad \mu_1, \mu_2 \to I(z \to +\infty) \tag{2.2.3}$$

由于 $\psi_1 = \mu_1 E^{-1}, \psi_2 = \mu_2 E^{-1}$, 为 Lax 对的两个解, 而 Lax 对为一节齐次线性方程组, 故这两个解线性相关, 故有

$$\mu_1(x,t,z) = \mu_2(x,t,z)\hat{E}S(z), \quad S(z) = \begin{pmatrix} S_{11}(z) & S_{12}(z) \\ S_{21}(z) & S_{22}(z) \end{pmatrix} \tag{2.2.4}$$

其中矩阵 $S(z)$ 与 x,t 无关, 称为谱函数矩阵.

再由 $\mu = \psi E$ 可知 $\det(\mu_j) = \det(\psi_j)$. 又因为 $tr(P - \mathrm{i}z\sigma_3) = tr(Q - 2\mathrm{i}z^2\sigma_3) = 0$, 由 Abel 定理可得

$$\det(\psi_j)_x = \det(\psi_j)_t = 0 \tag{2.2.5}$$

故有 $\det(\mu_j)_x = \det(\mu_j)_t = 0$, 这说明 $\det(\mu_j)$ 与 x,t 无关, 再由渐进性可得

$$\det(\mu_j) = \lim_{|x|\to\infty} \det(\mu_j) = 1, \quad (j = 1, 2) \tag{2.2.6}$$

故对 (2.2.4) 两边取行列式有 $\det S(z) = 1$.

2.2.1 解析性

下面我们考虑特征函数 μ_1, μ_1 和谱矩阵 $S(z)$ 的解析性, 记 μ_1 的第一二列分别为

$$\mu_1 = \begin{pmatrix} \mu_1^{11} & \mu_1^{12} \\ \mu_1^{21} & \mu_1^{22} \end{pmatrix} = (\mu_1^+, \mu_1^-) \tag{2.2.7}$$

则由积分方程 (2.2.1) 可得如下 Volterra 积分方程

$$\mu_1^+(x,t,z) = (1,0)^T - \int_{-\infty}^{x} \begin{pmatrix} 1 & \\ & e^{\mathrm{i}z(x-y)} \end{pmatrix} P\mu_1^+ \mathrm{d}y \tag{2.2.8}$$

$$\mu_1^-(x,t,z) = (0,1)^T - \int_{-\infty}^{x} \begin{pmatrix} e^{-\mathrm{i}z(x-y)} & \\ & 1 \end{pmatrix} P\mu_1^- \mathrm{d}y \tag{2.2.9}$$

对于上面两方程, 由于积分变量 $y < x$, 可得

$$e^{2\mathrm{i}Z(x-y)} = e^{2\mathrm{i}(x-y)\mathrm{Re}(z)}e^{-2\mathrm{i}(x-y)\mathrm{Im}(z)}, \quad e^{-2\mathrm{i}Z(x-y)} = e^{-2\mathrm{i}(x-y)\mathrm{Re}(z)}e^{2\mathrm{i}(x-y)\mathrm{Im}(z)}$$

因此当 $q(x) \in L^1(\mathbb{R})$ 时, 通过构造序列与 Neumann 级数, 可得 $\mu_1^{\pm}, \mu_1^-$ 分别在 $\mathbb{C}_\pm$ 解析性.

注 上面 Volterra 积分的推导: 考虑到 $e^{\hat{\sigma}_3}X = e^{\sigma_3}Xe^{-\sigma_3}$, 故有

$$\begin{aligned} \mu_1 &= I + \int_{-\infty}^{x} e^{-\mathrm{i}z(x-y)\hat{\sigma}_3} P\mu_1 \mathrm{d}y = I + \int_{-\infty}^{x} e^{-\mathrm{i}z(x-y)\sigma_3} P e^{\mathrm{i}z(x-y)\sigma_3} \mu_1 \mathrm{d}y \\ &= I + \int_{-\infty}^{x} \begin{pmatrix} & qe^{2\mathrm{i}z(x-y)} \\ -q^* e^{-2\mathrm{i}z(x-y)} & \end{pmatrix} \mu \mathrm{d}y \end{aligned}$$

故 $\mu_1^+ =, \quad \mu_1^- =$

证明 下证 μ_1^+ 的解析性.

Step1. 解的存在性: 事实上方程 (2.2.8) 有如下 Neumann 级数分解

$$\mu_1^+ = \sum_{n=0}^{\infty} c_n(x,z), \quad c_{n+1} = \int_{-\infty}^{x} \begin{pmatrix} 1 & \\ & e^{\mathrm{i}z(x-y)} \end{pmatrix} Pc_n(y,z)\mathrm{d}y \tag{2.2.10}$$

其分量形式为

$$c_{n+1}^{(1)}(x,z) = \int_{-\infty}^{x} q(y)c_n^{(2)}(y,z)\mathrm{d}y, \quad c_{n+1}^{(2)}(x,z) = \int_{-\infty}^{x} e^{2\mathrm{i}z(x-y)}p(y)c_n^{(1)}(y,z)\mathrm{d}y \tag{2.2.11}$$

其中 $c_{n+1} = (c_{n+1}^{(1)}, c_{n+1}^{(2)})^T, p = -q^*$. 由 $c_0^{(2)} = 0 \implies c_{2n+1}^{(1)} = c_{2n}^{(2)} = 0$, 故上面方程可简化为

$$c_{2n}^{(1)}(x,z) = \int_{-\infty}^{x} q(y)c_{2n-1}^{(2)}(y,z)\mathrm{d}y, \quad c_{2n+1}^{(2)}(x,z) = \int_{-\infty}^{x} e^{2\mathrm{i}z(x-y)}p(y)c_{2n}^{(1)}(y,z)\mathrm{d}y \tag{2.2.12}$$

对上面方程组进一步简化, 引入如下恒等式

$$\begin{aligned}&\frac{1}{j!}\int_{-\infty}^{x}|f(\xi)|\left[\int_{-\infty}^{\xi}|f(\eta)|\right]^{j}\mathrm{d}\xi=\frac{1}{(j+1)!}\int_{-\infty}^{x}\frac{d}{d\xi}\left[\int_{-\infty}^{\xi}|f(\eta)|\right]^{j+1}\mathrm{d}\xi\\&=\frac{1}{(j+1)!}\left[\int_{-\infty}^{x}|f(\xi)|d\xi\right]^{j+1}\qquad(f\in L^{1}(\mathbb{R}))\end{aligned}\tag{2.2.13}$$

利用归纳法可证明当 $\mathrm{Im}(z)>0$ 时, 有

$$|c_{2n+1}^{(2)}|=\frac{u^{n+1}(x)}{(n+1)!}\frac{v^{n}(x)}{n!},\quad |c_{2n}^{(1)}|=\frac{u^{n}(x)}{n!}\frac{v^{n}(x)}{n!}\tag{2.2.14}$$

其中

$$u(x)=\int_{-\infty}^{x}|p(y)|\mathrm{d}y,\quad v(x)=\int_{-\infty}^{x}|q(y)|\mathrm{d}y\tag{2.2.15}$$

事实上, 当 $\mathrm{Im}(z)>0$ 时, $|e^{2\mathrm{i}z(x-y)}|\le 1$, 利用 $c_0^{(1)}=1$, 由 (2.2.14) 可得

$$c_{1}^{(2)}(x,y)=\int_{-\infty}^{x}e^{2\mathrm{i}z(x-y)}p(y)\mathrm{d}y,\quad c_{2}^{(1)}(x,y)=\int_{-\infty}^{x}e^{2\mathrm{i}z(x-y)}q(y)c_{1}^{(2)}\mathrm{d}y\tag{2.2.16}$$

又由 $u(x)\ge 0, v(x)\ge 0$, 有 $c_1^{(2)}\le v(x)$

$$\begin{aligned}|c_{2}^{(1)}(x,y)\le\int_{-\infty}^{x}|q|c_{1}^{(2)}(y,z)\mathrm{d}y\le\int_{-\infty}^{x}|q|v(y)\mathrm{d}y=\int_{-\infty}^{x}v'(y)u(y)\mathrm{d}y\\=u(x)v(x)-\int_{-\infty}^{x}u'(y)v(y)\mathrm{d}y\le u(x)v(x)\end{aligned}\tag{2.2.17}$$

进而可得

$$\begin{aligned}|c_{3}^{(2)}(x,z)|&\le\int_{-\infty}^{x}|p|c_{2}^{(1)}(y,z)\mathrm{d}y\le\int_{-\infty}^{x}|p|u(y)v(y)\mathrm{d}y\\&\le\int_{-\infty}^{x}u'(y)u(y)v(y)\mathrm{d}y=\frac{1}{2}u^{2}(x)v(x)-\int_{-\infty}^{x}\frac{1}{2}u^{2}(x)v'(x)\mathrm{d}y\le\frac{1}{2}u^{2}(x)v(x)\end{aligned}\tag{2.2.18}$$

不妨设对于 $k\le l$ 时满足 $|c_{2l+1}^{(2)}|\le\frac{u^{l+1}(x)}{(l+1)!}\frac{v^{l}(x)}{l!}, |c_{2l}^{(1)}|\le\frac{u^{l}(x)}{l!}\frac{v^{l}(x)}{l!}$, 下证对于 $k=l+1$ 仍成立, 利用 (2.2.13) 式可得

$$|c_{2l+2}^{(1)}|\le\int_{-\infty}^{x}|q(y)|\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y\le\int_{-\infty}^{x}v'(y)\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y=\frac{u^{l+1}}{(l+1)!}\frac{v^{l+1}}{(l+1)!}\tag{2.2.19}$$

及

$$|c_{2l+3}^{(2)}|\le\int_{-\infty}^{x}|p(y)|\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y\le\int_{-\infty}^{x}u'(y)\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y=\frac{u^{l+2}}{(l+2)!}\frac{v^{l+1}}{(l+1)!}\tag{2.2.20}$$

从而上述 (2.2.12) 式得证. 注意到 $|q(x)|=|p(x)|$, 则 $v(x)=u(x)$, 故 (2.2.12) 式可化简为

$$|c_{2n+1}^{(2)}|\le\frac{u^{2n+1}}{n!(n+1)!},\quad |c_{2n}^{(1)}|\le\frac{u^{2n}}{n!n!}\tag{2.2.21}$$

当 $q\in L^1(\mathbb{R})$ 时, 有 $u(x)$ 的上述级数均收敛, 故 Neumann 级数 $\sum_{n=1}^{\infty}c_n(x,z)$ 绝对收敛, 此时 μ_1^+ 在 $Imz>0$ 上解析, 且在 $Imz\ge 0$ 上连续.

Step2. 解的唯一性: 不妨设 $\tilde{\mu}_1^+$ 为方程 (2.2.8) 的另一个解, 不妨设 $h=\mu_1^+-\tilde{\mu}_1^+$, 则有

$$\|h(x,t,z)\|=\left\|\int_{-\infty}^{x}\begin{pmatrix}1&\\&e^{\mathrm{i}z(x-y)}\end{pmatrix}Ph\mathrm{d}y\right\|\le 2\int_{-\infty}^{x}|q|\|h\|\mathrm{d}y\implies\|h(x,t,z)\|\equiv 0\tag{2.2.22}$$

故 μ_1^+ 为方程的唯一解, 同理可得 μ_1^- 的解析性. □

所以可得 μ_1^+,μ_1^- 分别在 $\mathbb{C}_+,\mathbb{C}_-$ 上解析. 同理可得 μ_2 的第一二列分别在 $\mathbb{C}_-,\mathbb{C}_+$ 上解析. 记作

$$\mu_2=\begin{pmatrix}\mu_2^{(11)}&\mu_2^{(12)}\\\mu_2^{(21)}&\mu_2^{(22)}\end{pmatrix}=(\mu_2^-,\mu_2^+)$$

由 (2.2.6) 可知, μ_1,μ_2 可逆, 且逆矩阵为其伴随阵, 另外基于 μ_1,μ_2 的列向量函数的解析性, 可得 μ_1^{-1} 的第一二行

在 $\mathbb{C}_-,\mathbb{C}_+$ 上解析, μ_2^{-1} 的第一二行在 $\mathbb{C}_+,\mathbb{C}_-$ 上解析. 记为

$$\mu_1^{-1}=\begin{pmatrix}\mu_1^{(22)} & -\mu_1^{(21)}\\ -\mu_1^{(12)} & \mu_1^{(11)}\end{pmatrix}=\begin{pmatrix}-\hat{\mu_1^-}\\ \hat{\mu_1^+}\end{pmatrix},\quad \mu_2^{-1}=\begin{pmatrix}\mu_2^{(22)} & -\mu_2^{(21)}\\ -\mu_2^{(12)} & \mu_2^{(11)}\end{pmatrix}=\begin{pmatrix}-\hat{\mu_2^+}\\ \hat{\mu_2^-}\end{pmatrix}\tag{2.2.23}$$

利用 (2.2.4), (2.2.7), (2.2.23) 可得谱函数 $S(z)$ 的解析性

$$S(z)=\mu_2^{-1}\mu_1=\begin{pmatrix}-\hat{\mu_2^+}\\ \hat{\mu_2^-}\end{pmatrix}(\mu_1^+,\mu_1^-)=\begin{pmatrix}-\hat{\mu_2^+}\mu_1^+ & -\hat{\mu_2^+}\mu_1^-\\ \hat{\mu_2^-}\mu_1^+ & \hat{\mu_2^-}\mu_1^-\end{pmatrix}\tag{2.2.24}$$

可见 $S_{11}(z)$ 在 $\mathbb{C}_+$ 解析, $S_{22}(z)$ 在 $\mathbb{C}_-$ 解析, S_{12},S_{21} 在 $\mathbb{C}_+,\mathbb{C}_-$ 均不解析, 但连续到边界.

2.2.2 对称性

定理 2.1

如上构造的特征函数 μ_1,μ_2 与谱函数 $S(z)$ 具有如下对称性

$$\mu_j^\dagger(x,t,z^*)=\mu_j^{-1}(x,t,z),\quad S^\dagger(z^*)=S^{-1}(z)\quad (j=1,2)\tag{2.2.25}$$

证明 由 (2.1.11) 有

$$\mu_{j,x}(x,t,z)=\mathrm{i}z[\sigma_3,\mu_j(x,t,z)]=P\mu_j(x,t,z)\tag{2.2.26}$$

将 z 替换为 z^*, 并在两边同时取 Hermite 共轭, 则有

$$\mu_{j,x}^\dagger(x,t,z^*)+\mathrm{i}z[\sigma_3,\mu_j^\dagger(X,t,z^*)]=\mu_j^\dagger(x,t,z^*)P^\dagger\tag{2.2.27}$$

由于 $P^\dagger=-P$, 故上式可化为

$$\mu_{j,x}^\dagger(x,t,z^*)+\mathrm{i}z[\sigma_3,\mu_j^\dagger(X,t,z^*)]=-\mu_j^\dagger(x,t,z^*)P\tag{2.2.28}$$

另外对于 $\mu_j\cdot\mu_j^{-1}=I$ 对 x 求偏导有

$$\mu_{j,x}\mu_j^{-1}+\mu_j\mu_{j,x}^{-1}=0\implies \mu_{j,x}^{-1}=-\mu_j^{-1}\mu_{j,x}\mu_j^{-1}\tag{2.2.29}$$

将 (2.2.26) 带入上式有

$$\mu_{j,x}^{-1}=-\mu_j^{-1}(P\mu_j(x,t,z)-\mathrm{i}z[\sigma_3,\mu_j(x,t,z)]\mu_j^{-1}(x,t,z)\implies \mu_{j,x}^{-1}+\mathrm{i}z[\sigma_3,\mu_j^{-1}]=\mu_j^{-1}P\tag{2.2.30}$$

由 (2.2.8), (2.2.10) 可得 $\mu_j^\dagger,\mu_j^{-1}$ 满足相同的一次线性微分方程, 具有相同的渐进性, 又因为

$$\mu_j^\dagger(x,t,z^*),\mu_j^{-1}(x,t,z)\to I,\quad |x|\to\infty\tag{2.2.31}$$

因此两者相等, 得到对称关系 $\mu_j^\dagger(x,t,z^*)=\mu_j^{-1}(x,t,z),(j=1,2)$. 接下来考虑 S 的对称性, 将 (2.2.4) 式改写为

$$S(z)=e^{\mathrm{i}\theta(z)\hat{\sigma}_3}(\mu_2^{-1}\mu_1)=\hat{E}(\mu_2^{-1}\mu_1)\tag{2.2.32}$$

由 (2.2.31) 可得如下

$$S(z^*)^\dagger=e^{\mathrm{i}\theta(z^*)\sigma_3}(\mu_2^{-1}\mu_1)e^{-\mathrm{i}\theta(z^*)}=E\mu_2^{-1}\mu_1E^{-1}=E\mu_1^\dagger(\mu_2^{-1})^\dagger E^{-1}=E\mu_1^\dagger\mu_2E^{-1}=S(z)^{-1}\tag{2.2.33}$$

比较对应元素, 有

$$S_{11}^*(z^*)=S_{22}(z),S_{12}^*(z^*)=-S_{21}(z)\tag{2.2.34}$$

□

2.3 相关的 RH 问题

2.3.1 规范 RH 问题

基于 2.2 节的结论, 我们构造 RH 问题, 引入记号

$$H_1=\begin{pmatrix}1 & \\ & 0\end{pmatrix},\quad H_2=\begin{pmatrix}0 & \\ & 1\end{pmatrix} \tag{2.3.1}$$

并定义两个矩阵

$$\begin{aligned}P_+(x,t,z)&=\mu_1H_1+\mu_2H_2=\begin{pmatrix}\mu_1^{(11)} & \mu_2^{(12)}\\ \mu_1^{(21)} & \mu_2^{(22)}\end{pmatrix}=(\mu_1^+,\mu_2^+)\\ P_-(x,t,z)&=H_1\mu_1^{-1}+H_2\mu_2^{-1}=\begin{pmatrix}\mu_1^{(22)} & -\mu_1^{(12)}\\ -\mu_2^{(21)} & \mu_2^{(11)}\end{pmatrix}=\begin{pmatrix}\hat{\mu}_1^-\\ \hat{\mu}_2^-\end{pmatrix}\end{aligned} \tag{2.3.2}$$

则由 $\mu_j,\mu_j^{-1}(j=1,2)$ 的解析性与渐进性, 直接可得 P_+ 在 $\mathbb{C}_+$ 上解析, P_- 在 $\mathbb{C}_-$ 上解析, 且具有如下渐进性

$$P_+,P_-\to I\quad |x|\to\infty \tag{2.3.3}$$

由此可证明如下对称关系

定理 2.2 $P_+(x,t,z),P_-(x,t,z)$ 具有如下对称性

$$P_+^\dagger(x,t,z^*)=P_-(x,t,z) \tag{2.3.4}$$

♡

证明 利用 (2.2.31) 可得

$$\begin{aligned}P_+^\dagger(x,t,z^*)&=(\mu_1(x,t,z)H_1+\mu_2(x,t,z)H_2)^\dagger=H_1\mu_1^\dagger(z^*)+H_2\mu_2^\dagger(z^*)\\ &=H_1\mu_1^{-1}(x,t,z)+H_2\mu_2^{-1}(x,t,z)=P_-(x,t,z)\end{aligned} \tag{2.3.5}$$

□

定义 2.1 概括上面结果, 我们可以得到下面 RH 问题

$$P_\pm(x,t,z)\in C(\mathbb{C}_\pm) \tag{2.3.6}$$

$$p_-p_+=G(x,t,z)\quad z\in\mathbb{R} \tag{2.3.7}$$

$$P_\pm\to I\quad z\to\infty \tag{2.3.8}$$

其中跳跃矩阵为

$$G(x,t,z)=\hat{E}^{-1}\begin{pmatrix}1 & -S_{12}(z)\\ S_{21}(z) & 1\end{pmatrix} \tag{2.3.9}$$

进一步 NLS 方程的解 $q(x,t)$ 可由 RH 问题的解给出

$$q(x,t)=2\mathrm{i}\lim_{z\to\infty}(zP_+)_{12}=2\mathrm{i}(P_+^{(1)})_{12} \tag{2.3.10}$$

其中 $P_+=I+\frac{P_+^{(1)}}{z}+O(z^{-2})$

♣

证明 只需证明跳跃关系即可, 注意到 $\hat{E}H_j=H_j,(j=1,2)$ 可得

$$\begin{aligned}G&=P_-P_+=(H_1\mu_1^{-1}+H_2\mu_2^{-1})(\mu_1H_1+\mu_2H_2)\\ &\overset{(2.2.4)}{=}[H_1(\hat{E}^{-1}S^{-1})\mu_2^{-1}+H_2\mu_2^{-1}][\mu_2(\hat{E}^{-1}S)H_1+\mu_2H_2]\\ &=\hat{E}^{-1}[(H_1S^{-1}+H_2)(SH_1+H_2)]=\hat{E}^{-1}\begin{pmatrix}1 & -S_{12}(z)\\ S_{21}(z) & 1\end{pmatrix}\end{aligned} \tag{2.3.11}$$

直接计算可得

$$\begin{aligned}\det(P_+) &= \det(\mu_1 H_1 + \mu_2 H_2) = \det(\mu_2 \hat{E}^{-1} S) H_1 + \mu_2 H_2) \\ &= \det(\mu_2)\det((\hat{E}^{-1}S)H_1 + \mu_2 H_2) = S_{11}(z)\end{aligned} \tag{2.3.12}$$

同理, 有 $\det(P_-) = S_{22}(z)$ □

2.3.2 RH 问题的可解性

下面分两种情况讨论 RH 问题 (2.3.6) - (2.3.8) 的解

Case1. 如果

$$\det P_\pm(z) \neq 0 (\forall z \in \mathbb{C}), \tag{2.3.13}$$

则称 RH 问题 (2.3.6) - (2.3.8) 为正则的, 将方程 (2.3.7) 改写为

$$P_+^{-1} - P_- = (I - G) P_+^{-1} := \hat{C} P_+^{-1} \tag{2.3.14}$$

由 Plemelj 公式可得

$$P_+ = I + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\hat{C}(x,t,z) P_+^{-1}(x,t,z)}{s - z} \mathrm{d}s, \quad z \in \mathbb{C}_+ \tag{2.3.15}$$

Case2. 如果条件 (2.3.13) 不满足, 则称 RH 问题为非正则的, 假设 $\det P_\pm$ 在某些离散的点处为零, 由谱函数对称性 (2.2.34) 有

$$\det P_+(z) = S_{11}(z) = S_{22}^*(z^*) = \det P_-^*(z^*) = \det P_-(z^*) \tag{2.3.16}$$

故

$$\det P_+(z) = 0 \iff \det P_-(Z^*) = 0, \tag{2.3.17}$$

因此 $\det P_+(z)$ 与 $\det P_-(z)$ 有相同的零点个数, 且彼此共轭. 即若设 $z_j (j = 1, 2, \ldots, N)$ 为 $\det P_+(z)$ 在 $\mathbb{C}_+$ 上的单零点, 则 $z_j^* (j = 1, 2, \ldots, N)$ 为 $\det P_-(z)$ 在 $\mathbb{C}_-$ 上的单零点.

由于 $\det P_+(z_j) = \det P_-(z_j^*) = 0$, 假设 w_j, w_j^* 分别为下列线性方程组的解

$$P_+(z_j) w_j(z_j) = 0, \quad w_j^*(z_j^*) P_-(z_j^*) = 0 \tag{2.3.18}$$

对上式取共轭转置, 则有 $w_j^\dagger(z_j) P_+^\dagger(z_j) = 0$, 再利用对称性 (2.3.4) 有

$$w_j^\dagger(z_j) P_-(z_j^*) = 0 \tag{2.3.19}$$

比较可得 $w_j^\dagger(z) = w_j^*(z^*)$.

非正则 RH 问题 (2.3.6) - (2.3.8) 的解可由下面定理给出

定理 2.3 (Zakharov, Shabat, 1979)

带有零点结构 (2.3.18) 的非正则 RH 问题 (2.3.6) - (2.3.8) 可分解为

$$P_+(z) = \hat{P}_+(z)\Gamma(z), \quad P_-(z) = \Gamma^{-1}(z)\hat{P}_-(z) \tag{2.3.20}$$

其中

$$\Gamma(z) = I + \sum_{k,j=1}^{N} \frac{w_k (M^{-1})_{kj} w_j^*}{z - z_j^*}, \quad \Gamma(z)^- = I - \sum_{k,j=1}^{N} \frac{w_k (M^{-1})_{kj} w_j^*}{z - z_k} \tag{2.3.21}$$

这里 M 为 $N \times N$ 矩阵, 其 $(k.j)$ 元素由下式给定

$$M_{kj} = \frac{w_k^* w_j}{z_k^* - z_j}, \quad k, j = 1, 2, \ldots, N, \quad \det \Gamma(z) = \prod_{k=1}^{N} \frac{z - z_k}{z - z_k^*} \tag{2.3.22}$$

而 $\hat{P}_\pm$ 为正则 RH 问题的唯一解, 且

1. $\hat{P}_\pm$ 在 $\mathbb{C}_+$ 上解析,
2. $\hat{P}_- \hat{P}_+ = \Gamma(z) G \Gamma^{-1}(z), z \in \mathbb{R}$

3. $\hat{P_{\pm}}(z) \to I, \quad z \to \infty,$

证明 非正则 RH 问题 (2.3.6) - (2.3.8) 是由于 $2N$ 个离散谱上的 $\det P_+(z_j) = \det P_-(Z_{j^*}) = 0(j = 1, 2, \dots N)$ 造成的. 所以主要任务是消除这些零点结构, 并且去除 $P_+(z_j), P_-(z_j^*)$ 在 z_j, z_j^* 上的零点结构, 为此需定义单极点矩阵

$$\Gamma_1(z) = I + \frac{z_1^* - z_1}{z - z_1^*} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \tag{2.3.23}$$

其具有如下性质

$$\begin{aligned}
F_1^{-1}(z) &= I - \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \\
\det \Gamma_1(z) &= \frac{z - z_1}{z - z_1^*}, \quad \det \Gamma_1^{-1}(z) = \frac{z - z_1^*}{z - z_1} \\
\Gamma_1(z_1) w_1 &= w_1 + \frac{z_1^* - z_1}{z_1 - z_1^*} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \cdot w_1 = \frac{z^* - z_1}{z - z_1^*} \frac{w_1(w_1^* w_1)}{w^* w_1} = w_1 - w_1 = 0 \\
w_1^* \Gamma_1^{-1}(z_1^*) &= w_1^* - \frac{z_1^* - z_1}{z^* - z_1} \cdot \frac{w_1^* w_1}{w_1^* w_1} \cdot w_1^* = w_1^* - w_1^* = 0
\end{aligned} \tag{2.3.24}$$

令 $x_j = \frac{w_1 w_1^*}{w_1^* w_1}$, 则 x_j 为投影算子, 即 $x_j^2 = x_j$, 由上定义知 x_j 为一阶矩阵, 故与矩阵 diga$\{1, 0\}$ 相似, 既有可逆阵 T_j 使得 $T_j^{-1} x_j T_j = \mathrm{diag}\{1, 0\}$, 从而有

$$\Gamma_1(z) = \det \left(I + \frac{z_1^* - z_1}{z - z_1^*} T_j^{-1} X_j T_j \right) = \begin{vmatrix} 1 + \frac{z_1^* - z_1}{z - z_1^*} & 0 \\ 0 & 1 \end{vmatrix} = \frac{z - z_1}{z - z_1^*} \tag{2.3.25}$$

定义矩阵函数 $R_1^+(z) = P_+(z)\Gamma_+^{-1}(z), R_1^{-1} = \Gamma_1(z) P_-(z)$, 由 (2.3.18) - (2.3.19) 知

$$\begin{aligned}
\operatorname*{Res}_{z=z_1} R_1^+(z) &= \operatorname*{Res}_{z=z_1} \left(P_+(z) - P_+(z) \frac{z_1^* - z_1}{z - z_1} \frac{w_1 w_1^*}{w_1^* w_1} \right) = -\frac{z_1^* - z_1}{w_1 w_1^*} (P_+(z) w_1) w_1^* = 0 \\
\operatorname*{Res}_{z=z_1^*} R_1^{-1}(z) &= \operatorname*{Res}_{z=z_!^*} \left(P_1(z) + \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{w_1^* w_1} P_-(z) \right) = -\frac{z_1^* - z_1}{w_1^* w_1} w_1 (w_1^* P_-(z_1^*)) = 0
\end{aligned} \tag{2.3.26}$$

因此 $R_1^+(z), R_1^-(z)$ 分别在 $\mathbb{C}_+, \mathbb{C}_-$ 上解析, 且

$$\begin{aligned}
\det R_1^+(z_1) &= \lim_{z \to z_1} \left[\det P_+(z) \cdot \det \Gamma_1^{-1}(z) \right] = \lim_{z \to z_1} S_{11}(z) \frac{z - z_1^*}{z - z_1} \\
&= \lim_{z \to z_1} \frac{S_{11}(z) - S_{11}(z_1)}{z - z_1} (z - z^*) \Leftarrow (S_{11}(z) = 0) \\
&= \lim_{z \to z_1} S_{11}'(z)(z - z_1^*) = 0 \\
\det R_1^{-1}(z_1^*) &= \lim_{z \to z_1^*} \left[\det \Gamma_1(z) \cdot \det P_-(z) \right] = \lim_{z \to z_1^*} \frac{z - z_1}{z - z_1^*} S_{22}(z) = \lim_{z \to z_1^*} S_{22}'(z)(z - z_1) = 0
\end{aligned} \tag{2.3.27}$$

这说明 $R_1^+(z), R_1^-(z)$ 分别在 $z = z_1, z = z_1^*$ 不再具有零点结构, 然后去除其在 z_2, z_2^* 上的零点结构, 由于

$$\begin{aligned}
\det R_1^+(z_2) &= \det P_+(z_2) \det \Gamma_1^{-1}(z_2) = S_{11}(z_2) \frac{z_2 - z_1^*}{z_2 - z_1} = 0 \\
\det R_1^{-1}(z_2^*) &= \det \Gamma_1(z_2^*) \det P_-(z_2^*) = \frac{z_2^* - z_1}{z_2^* - z_1} S_{22}(z_2^*) = 0
\end{aligned} \tag{2.3.28}$$

因此, 下列齐次线性方程组有非零解, 即存在 $v_2(z_2), v_2^*(z_2^*)$, 使得

$$R_1^+(z_2) v(z_2) = P_+(z_2) \Gamma_1^{-1}(z_2) v_2(z_2) = 0, \quad v_2^*(z_2^*) R_1^{-1}(z_2^*) = v_2^*(z_2^*) \Gamma_1(z_2^*) P_-(z_2^*) = 0 \tag{2.3.29}$$

与 (2.3.18) 比较可得

$$w_2(z_2) = \Gamma_1^{-1}(z_2) v_2(z_2), \quad w_2^*(z_2^*) = v_2^*(z_2^*) \Gamma_1(z_2^*) \tag{2.3.30}$$

为去除 $R_1^+(z_2), R_1^-(z_2^*)$ 在 z_2, z_2^* 上的零点结构, 令

$$\begin{aligned}
\Gamma_2(z) &= I + \frac{z_2^* - z_2}{z - z_2^*} \cdot \frac{v_2 v_2^*}{v_2^* v_2}, \quad \det \Gamma_2(z) = \frac{z - z_2}{z - z_2^*} \\
\Gamma_2^{-1}(z) &= I - \frac{z_2^* - z_2}{z - z_2} \cdot \frac{v_2 v_2^*}{v_2^* v_2}, \quad \det \Gamma_2^{-1} = \frac{z - z_2^*}{z - z_2} \\
R_2^+(z) &= P_1^+(z)\Gamma_2^{-1}(z) = P_1^+(z)\Gamma_1^{-1}(z)\Gamma_2^{-1}(z) \\
R_2^{-1}(z) &= \Gamma_2(z) P_1^{-1}(z) = \Gamma_2(z)\Gamma_1(z) P_1^{-1}(z)
\end{aligned} \tag{2.3.31}$$

则

$$\operatorname*{Res}_{z=z_2} R_2^+(z) = \operatorname*{Res}_{z=z_2} \left[\left(P_+(z) - P_+(z) \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{w_2^* w_1} \right) \left(I - \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{v_2^* v_1} \right) \right] = -\frac{z_2^* - z_1 R_1(z_2) v_2 v_2^*}{v_2^* v_2} \tag{2.3.32}$$

同理 $\operatorname{Res}_{z=z_2^*} R_2^{-1}(z) = 0$, 由此可得 $R_2^+(z), R_2^{-1}(z)$ 分别在在 $z = z_1, z_2, z = z_1^*, z_2^*$ 上无零点结构, 则

$$\begin{aligned}
\det R_2^+(z_j) &= \lim_{z \to z_j} \left[\det P_+(z) \det \Gamma_1^{-1}(z) \det \Gamma_2^{-1}(z) \right] \\
&= \lim_{z \to z_j} S_{11}(z) \frac{z - z_1^*}{z - z_1} \cdot \frac{z - z_2^*}{z - z_2} \\
&= s_{11}'(z_j) \frac{(z_j - z_1^*)(z_j - z_2^*)}{z_j - z_k} \neq 0 \\
\det R_2^-(z_j^*) &\neq 0 \quad (k = 1, 2, k \neq z)
\end{aligned} \tag{2.3.33}$$

更一般的, 可得

$$\begin{aligned}
w_j &= \Gamma_1^{-1}(z_j) \cdots \Gamma_{j-1}^{-1}(z_j) \cdot v_j(z_j) \\
w_j^* &= v_j^*(z_j^*) \cdot \Gamma_1(z_j^*) \cdots \Gamma_{j-1}(z_j^*) \\
R_j^+(z) &= P_+(z) \cdot \Gamma_1^{-1}(z) \cdots \Gamma_j^{-1}(z) \\
R_j^{-1}(z) &= \Gamma_j(z) \cdots \Gamma_1(z) \cdot P_-(z)
\end{aligned} \tag{2.3.34}$$

其中

$$\Gamma_j(z) = I + \frac{z_j^* - z_j}{z - z_j^*} \cdot \frac{v_j v_j^*}{v_j^* v_j}, \quad \Gamma_j^{-1}(z) = I - \frac{z_j^* - z_j}{z - z_j} \cdot \frac{v_j v_j^*}{v_j^* v_j} \tag{2.3.35}$$

则 $R_j^+(z), R_j^{-1}(z)$ 在 $\mathbb{C}_+, \mathbb{C}_-$ 上解析, 且在 $z_k, z_k^* (k = 1, 2, \ldots j)$ 上无零点结构, 即

$$\left.\begin{aligned}
\det R_j^+(z_k) &= s_{11}'(z_k) \frac{\prod_{l=1}^j (z_k - z_l^*)}{\prod_{l=1, l \neq k}^j (z_k - z_l)} \neq 0 \\
\det R_j^{-1}(z_k^*) &= s_{22}'(z_k^*) \frac{\prod_{l=1}^j (z_k^* - z_l)}{\prod_{l=1, l \neq k}^j (z_k^* - z_l)} \neq 0
\end{aligned}\right\} \quad (k = 1, 2, \ldots, j) \tag{2.3.36}$$

而

$$\begin{aligned}
\det R_j^+(z_{j+1}) &= S_{11}(z_{j+1}) \prod_{j=1}^{j} \frac{z_{j+1} - z_k^*}{z_{j+1} - z_k} = 0 \\
\det R_j^{-1}(z_{j+1}^*) &= S_{22}(z_{j+1}^*) \prod_{j=1}^{j} \frac{z_{j+1}^* - z_k}{z_{j+1}^* - z_k} = 0
\end{aligned} \tag{2.3.37}$$

最后令

$$\begin{aligned}
\Gamma_z &= \Gamma_N(z) \cdots \Gamma_1(z), \quad \Gamma_z^{-1} = \Gamma_1^{-1}(z) \cdots \Gamma_N^{-1}(z) \\
\hat{P}_+(z) &= P_+(z) \Gamma_z^{-1}(z), \quad \hat{P}_-(z) = \Gamma_z(z) P_-(z)
\end{aligned} \tag{2.3.38}$$

则 $\hat{P}_+(z), \hat{P}_-(z)$ 在 $\mathbb{C}_+, \mathbb{C}_-$ 上解析, 且在 $z_j, z_j^* (j = 1, 2, \ldots, N)$ 上无零点结构 (实际上在解析区域内无零点结构), 即

$$\det \hat{P}_+(z_k) \neq 0, \quad \det \hat{P}_-(z_k^*) \neq 0 \quad (k = 1, 2, \ldots, N) \tag{2.3.39}$$

得到分解 (2.3.20). 下证 (2.3.21). 注意到 $\Gamma(z), \Gamma^{-1}(z)$ 分别为具有单极点 $z_j, z_j^* (1 \leq j \leq N)$ 的亚纯函数, 以及分解

(2.3.35), (2.3.38), 寻找的向量使得

$$\Gamma(Z) = i + \sum_{j=1}^{N} \frac{\xi_j w_j^*}{z - z_j^*}, \quad \Gamma^{-1}(z) = I - \sum_{j=1}^{N} \frac{w_j^* \xi_j}{z - z_j} \tag{2.3.40}$$

注意到 $\Gamma(z)\Gamma^{-1}(z) = I$ 对任意 z 都成立, 当然也在 $z = z_k$ 处成立, 即 $\Gamma(z)\Gamma^{-1}(z) = I$ 在 $z = z_k$ 处正则, 为保证 $\Gamma(z)\Gamma^{-1}(z) = I$ 在 $z = z_k$ 处成立, 只需让其留数为 0, 因此利用 (2.3.40) 可得

$$\begin{aligned} 0 &= \operatorname*{Res}_{z=z_k}[\Gamma(z)\Gamma^{-1}(z)] = \operatorname*{Res}_{z=z_k}\left(\Gamma(z) - \sum_{j=1}^{N} \Gamma(z)\frac{w_j \xi_j^*}{z - z_j}\right) = -\Gamma(z_k) w_k \xi_k^* \\ &-\left(I + \sum_{j=1}^{N} \frac{\xi_j w_j^*}{z_k - z_j^*}\right) w_k \xi_k^* = \left(-w_k + \sum_{j=1}^{N} \frac{w_j^* w_k}{z_j^* - z_k}\xi_j^*\right)\xi_k^* \end{aligned} \tag{2.3.41}$$

对上式两边同时作用 ξ_k, 则有

$$\left(-w_k + \sum_{j=1}^{N} \frac{w_j^* w_k}{z_j^* - z_k}\xi_j\right)|\xi_k|^2 = 0 \implies -w_k + \sum_{j=1}^{N} \frac{w_j^* w_k}{z_j^* - z_k}\xi_j = 0 \quad (1 \le k \le N) \tag{2.3.42}$$

将上式改写为 $\xi_1, \xi_2, \ldots, \xi_N$ 的分块矩阵形式的线性方程组, 则有

$$(\xi_1, \xi_2, \ldots, \xi_N)M = (w_1^*, w_2^*, \ldots, w_N^*) \tag{2.3.43}$$

其中 $M = (M_{kj})_{N\times N}$, 其中 $M_{kj} = \frac{w_k^* w_j}{z_k^* - z_j}$ 则

$$\begin{cases} \frac{w_1^* w_1}{z_1^*-z_1}\xi_1 + \frac{w_2^* w_1}{z_2^*-z_1}\xi_2 + \cdots + \frac{w_N^* w_1}{z_N^*-z_1}\xi_N = w_1 \\ \frac{w_1^* w_2}{z_1^*-z_2}\xi_1 + \frac{w_2^* w_2}{z_2^*-z_2}\xi_2 + \cdots + \frac{w_N^* w_2}{z_N^*-z_2}\xi_N = w_2 \\ \vdots \\ \frac{w_1^* w_N}{z_1^*-z_N}\xi_1 + \frac{w_2^* w_N}{z_2^*-z_N}\xi_2 + \cdots + \frac{w_N^* w_N}{z_N^*-z_N}\xi_N = w_N \end{cases} \implies M = \begin{pmatrix} \frac{w_1^* w_1}{z_1^*-z_1} & \frac{w_1^* w_2}{z_1^*-z_2} & \cdots & \frac{w_1^* w_N}{z_1^*-z_N} \\ \frac{w_2^* w_1}{z_2^*-z_1} & \frac{w_2^* w_2}{z_2^*-z_2} & \cdots & \frac{w_2^* w_N}{z_2^*-z_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_N^* w_1}{z_N^*-z_1} & \frac{w_N^* w_2}{z_N^*-z_2} & \cdots & \frac{w_N^* w_N}{z_N^*-z_N} \end{pmatrix} \tag{2.3.44}$$

可得 $\xi_j = \sum_{k=1}^{N}(M^{-1})_{kj} w_k$. 最后将其带入 (2.3.40) 得到 (2.3.21) □

2.4 NLS 方程的 N 孤子解

2.4.1 矩阵向量解的时空演化

对方程 (2.3.18) 第一个式子两边分别对 x, t 求导, 可得

$$P_{+,x} w_j + P_+ w_{j,x} = 0, \quad P_{+,t} w_j + P_+ w_{j,t} = 0 \tag{2.4.1}$$

利用 P_+ 的定义与 Lax 对 (2.1.11), (2.1.12) 可得

$$\begin{aligned} P_{+,x} &= \mu_{1,x} H_1 + \mu_{2,x} H_2 = \left(\mathrm{i}z_j[\sigma_3, \mu_1] + P\mu_1\right) H_1 + \left(-\mathrm{i}z_j[\sigma_3, \mu_2] + P\mu_2\right) H_2 \\ &- \mathrm{i}z_j\left(\mu_1\sigma_3 H_1 - \mu_2\sigma_3 H_2 + \sigma_3\mu_1 H_1 + \sigma_3\mu_2 H_2\right) + P\mu_1 H_1 + P\mu_2 H_2 \\ &= -\mathrm{i}z[\sigma_3, p_+] + PP_+ \end{aligned} \tag{2.4.2}$$

同理有

$$P_{+,t} = -\mathrm{i}z_j^2[\sigma_3, P_+] + QP_+ \tag{2.4.3}$$

将 (2.4.2), (2.4.3) 代入 (2.4.1) 且由于 $P_+ w_j = 0, w_j P = 0$, 有

$$(-\mathrm{i}z_j[\sigma_3, P_+] + PP_+)w_j + P_+ w_{j,x} = 0 \implies \mathrm{i}z_j P_+ \sigma_3 w_j + P_+ w_{j,x} = 0 \implies P_+(w_{j,x} + \mathrm{i}z_j\sigma_3 w_j) = 0 \tag{2.4.4}$$

同理有 $P_+(w_{j,t} + \mathrm{i}z_j^2\sigma_3 w_j) = 0$.

$$\begin{cases} w_{j,x} + \mathrm{i}z_j\sigma_3 w_j = 0 \\ w_{j,t} + \mathrm{i}z_j^2\sigma_3 w_j = 0 \end{cases} \implies w_j = e^{-\mathrm{i}\theta(z_j)\sigma_3} w_{j,0}, \quad (j = 1, 2, \ldots, N) \tag{2.4.5}$$

其中 $w_{j,0}$ 为 2 维常向量, 从而 $w_j^* = w_{j,0}^\dagger e^{i\theta(z_j^*)\sigma_3}$.

2.4.2 N 维孤子解公式

已知 $P_-P_+ = G \implies P_+^{-1} - P_- = \hat{G}P_+$(其中 $I - G = \hat{G}$), 且 $\hat{P_-}(z)\hat{P_+}(z) = \Gamma(z)G\Gamma^{-1}(z)(z\in\mathbb{R})$, 可得

$$\begin{aligned} \hat{P_+^{-1}} - \hat{P_-} &= (I - \hat{P_-}\hat{P_+})\hat{P_+^{-1}} = \left(I - \Gamma(z)G\Gamma^{-1}(z)\right)\hat{P_+^{-1}} = (\Gamma(z)\Gamma^{-1}(z) - \Gamma(z)G\Gamma^{-1}(z))\hat{P_+^{-1}} \\ &= \Gamma(z)(I-G)\Gamma^{-1}(z)\hat{P_+^{-1}} = \Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}} \end{aligned} \tag{2.4.6}$$

由 Taylor 公式 $\frac{1}{s-z} = -\frac{1}{z}(\frac{1}{1-s/z}) = -\frac{1}{z}\left(1+\frac{s}{z}+\cdots\right)$, 故 Plemelj 公式可写为

$$\begin{aligned} \hat{P_+^{-1}} &= I + \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{+\infty}\frac{\Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}}}{s-z}\mathrm{d}s \\ &= I - \frac{1}{2\pi \mathrm{i}z}\int_{-\infty}^{+\infty}\left(1+\frac{s}{z}+(\frac{s}{z})^2+\cdots\right)\Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}}\mathrm{d}s \\ &= I - \frac{1}{2\pi \mathrm{i}z}\int_{-\infty}^{+\infty}\Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}}\mathrm{d}s + O(z^{-2}) \end{aligned} \tag{2.4.7}$$

由 $(I-A)^{-1} = I + A + A^2 + \cdots$, $(P^{-1} = I + \frac{A}{z} + \dots, P = \frac{B}{z} + \dots)$, 可得

$$\hat{P_+} = I + \frac{1}{2\pi \mathrm{i}z}\int_{-\infty}^{+\infty}\Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+}\mathrm{d}s + O(z^{-2}) \tag{2.4.8}$$

再由

$$\Gamma(z) = i + \sum_{k,j=1}^{N}\frac{w_k(M^{-1})_{kj}w_j^*}{z-z_j^*} \implies \Gamma(z) = I + \frac{1}{z}\sum_{k,j=1}^{N}w_k(M^{-1})_{kj}w_j^* + O(z^{-2}) \tag{2.4.9}$$

将上渐进式带入 (2.3.20), 比较 z^{-1} 的次数可得

$$P_+^{(1)} = \frac{1}{2\pi \mathrm{i}}\int_{-\infty}^{+\infty}\Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}}\mathrm{d}s = \sum_{k,j=1}^{N}w_k(M^{-1})_{kj}w_j^* \tag{2.4.10}$$

特别的, 当散射数据 $S_{12} = S_{21} = 0$ 时, 有 $\hat{G} = I - G = 0$. 故上式 (2.4.10) 可简化为

$$P_+^{-1} = \sum_{k,j=1}^{N}w_k(M^{-1})_{kj}w_j^* \tag{2.4.11}$$

不妨取 $\lambda_j = -\mathrm{i}(z_jx + 2z_j^2t)$, 并取 $w_{j,0} = (c_j, 1)^T$, 则有

$$w_j = \begin{pmatrix} e^{\lambda_j} & \\ & e^{-\lambda_j}\end{pmatrix}\begin{pmatrix} c_j \\ 1\end{pmatrix} = \begin{pmatrix} c_je^{\lambda_j} \\ e^{-\lambda_j}\end{pmatrix} \tag{2.4.12}$$

从而 $w_j^* = w_{j,0}^* \cdot e_j^\lambda\sigma_3 = (c_j^*e^{\lambda_j^*}, e^{-\lambda_j})$. 故

$$M_{k,j} = \frac{w_k^*w_j}{z_k^*-z_j} = \frac{1}{z_k^*-z_j}(c_k^*c_je^{\lambda_k^*+\lambda_j} + e^{-\lambda_k-\lambda_j}) \tag{2.4.13}$$

再由 (2.4.9)

$$\begin{aligned} q(x,t) &= 2\mathrm{i}\lim_{z\to\infty}(zP_+)_{12} = 2\mathrm{i}(P_+^{(1)})_{12} \\ &= 2\mathrm{i}\sum_{k,j}^{N}(w_kw_j^*)_{12}(M^{-1})_{kj} = 2\mathrm{i}\sum_{k,j}^{N}c_ke^{\lambda_k-\lambda_j^*}(M^{-1})_{kj} \end{aligned} \tag{2.4.14}$$

令

$$R = \begin{pmatrix} 0 & c_1e^{\lambda_1} & \cdots & c_Ne^{\lambda_N} \\ e^{-\lambda_1^*} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\lambda_N^*} & M_{N1} & \cdots & M_{NN}\end{pmatrix} \tag{2.4.15}$$

则

$$
\begin{aligned}
\det R &= \sum_{k=1}^{N}(-1)^{k+2}c_k e^{\lambda_k}\det\begin{pmatrix} e^{-\lambda_1^*} & M_{11} & \cdots & M_{1,k-1} & M_{1,k+1} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e^{-\lambda_N^*} & M_{N1} & \cdots & M_{N,k-1} & M_{N,k+1} & \cdots & M_{NN} \end{pmatrix} \\
&= \sum_{k=1}^{N}(-1)^{k+2}c_k e^{\lambda_k}\sum_{j=1}^{N} e^{-\lambda_j^*}\det\underline{\begin{pmatrix} M_{11} & \cdots & M_{1,k-1} & M_{1,k+1} & \cdots & M_{1N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{N1} & \cdots & M_{N,k-1} & M_{N,k+1} & \cdots & M_{NN} \end{pmatrix}} := \Delta \\
&= \sum_{k=1}^{N}(-1)^{k+j+3}c_k e^{\lambda_k-\lambda_j^*}\det(\Delta) \\
&= -c_k e^{\lambda_k-\lambda_j^*(M^*)_{jk}} \qquad \Leftarrow \left((M^*)_{jk} - (-1)^{k+j}\det(\Delta)\right) \\
&= -\det M\sum c_k e^{\lambda_k-\lambda_j^*} \qquad \Leftarrow \left((M^{-1})_{jk}(M^*) = |M|M^{-1}\right)
\end{aligned}
\tag{2.4.16}
$$

因此

$$
\sum_{k=1}^{N} c_k e^{\lambda_k-\lambda_j^*}(M^{-1})_{jk} = -\frac{\det R}{\det M} \tag{2.4.17}
$$

将 (2.4.17) 带入 (2.4.14) 可得 NLS 方程的 N 孤子解

$$
q = -2\mathrm{i}\frac{\det R}{\det M} \tag{2.4.18}
$$

第 3 章 Reverse Time Space NLS

This chapter mainly introduces three inverse problems of nonlocal NLS, mainly referring to Yang's article[2]

3.1 The coupled Schrödinger equations

Consider the rverse-space NLS equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(-x,t) = 0 \tag{3.1.1}$$

reverse-time NLS equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(x,-t) = 0 \tag{3.1.2}$$

and the reverse-space-time NLS equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(-x,-t) = 0 \tag{3.1.3}$$

This equations can be derived form the following member of the AKNS hierarchy - the coupled NLS equations

$$\mathrm{i}q_t + q_{xx} - 2q^2 r = 0, \quad \mathrm{i}r_t - r_{xx} - 2r^2 q = 0. \tag{3.1.4}$$

Under reductions

$$r(x,t) = -q^*(-x,t), \tag{3.1.5a}$$

$$r(x,t) = -q(x,-t), \tag{3.1.5b}$$

$$r(x,t) = -q(-x,-t), \tag{3.1.5c}$$

these coupled equations reduce to the reverse-space NLS equation (3.1.1), the reverse-time NLS equation (3.1.2) and the reverse-space-time NLS equation (3.1.3) respectively.

3.2 N-solitions for general coupled Shorödinger equations

Our basic idea for deriving N-soliton of the reverse-space, reverse-time, and reverse-space-time NLS equations (3.1.1)−(3.1.3) is to recognize that these equations are reductions of the coupled Schrödinger equations (3.1.4). To this end, we begin with the Riemann-Hilbert formulation of N-soliton for the coupled Schrödinger equations, based on given scattering data. By imposing suitable symmetry conditions on the scattering data, we obtain N-soliton solutions for the corresponding nonlocal equations. Specifically, we consider the coupled Schrödinger equations (3.1.4), which belong to the AKNS hierarchy. Their Lax pair is given by:

$$Y_x = MY, \quad Y_t = NY \tag{3.2.1}$$

where

$$M = \begin{pmatrix} \mathrm{i}\zeta & 0 \\ 0 & -\mathrm{i}\zeta \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -\mathrm{i}qr - 2\mathrm{i}\zeta^2 & \mathrm{i}q_x + 2\zeta q \\ -\mathrm{i}r_x + 2\zeta r & \mathrm{i}qr + 2\mathrm{i}\zeta^2 \end{pmatrix} \tag{3.2.2}$$

Following this Riemann-Hilbert mothed, N-solitons in this system were explictly written down in chapter2 as

$$q(x,t) = -2\mathrm{i}\frac{\det F}{\det M}, \quad r(x,t) = 2\mathrm{i}\frac{\det G}{\det M} \tag{3.2.3}$$

where M is a $N \times N$ matrix, and F, G are $N+1 \times N+1$ matrices. The elements of the matrix M are given by

$$M_{jk} = \frac{\bar{\mathbf{v}}_j \mathbf{v}_k}{\bar{\zeta}_j - \zeta_k}, \quad \mathbf{v}_k(x,t) = \mathrm{e}^{\theta_k \Lambda}\mathbf{v}_{k0}, \quad \bar{\mathbf{v}}_k(x,t) = \bar{\mathbf{v}}_{k0}\mathrm{e}^{\bar{\theta}_k \Lambda} \tag{3.2.4}$$

where $\zeta_k \in \mathbb{C}_+, \bar{\zeta}_k \in \mathbb{C}_-$ is the eigenvalues and $\mathbf{v}_{k0}, \bar{\mathbf{v}}_{k0}$ is the eigenvalues $\theta_k = -\mathrm{i}\zeta_k x - 2\mathrm{i}\zeta_k^2 t, \bar{\theta}_k = \mathrm{i}\bar{\zeta}_k x + 2\mathrm{i}\bar{\zeta}_k^2 t$ and

$$\mathbf{v}_{k0} = \begin{bmatrix} a_k \\ b_k \end{bmatrix}, \quad \bar{\mathbf{v}}_{k0} = \begin{bmatrix} \bar{a}_k & \bar{b}_k \end{bmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2.5}$$

and

$$F = \begin{pmatrix} 0 & a_1 \mathrm{e}^{\theta_1} & \cdots & a_N \mathrm{e}^{\theta_N} \\ \bar{b}_1 \mathrm{e}^{-\bar{\theta}_1} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{b}_N \mathrm{e}^{-\bar{\theta}_N} & M_{N1} & \cdots & M_{NN} \end{pmatrix} \quad G = \begin{pmatrix} 0 & b_1 \mathrm{e}^{-\bar{\theta}_1} & \cdots & b_N \mathrm{e}^{-\bar{\theta}_N} \\ \bar{a}_1 \mathrm{e}^{\theta_1} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_N \mathrm{e}^{\theta_N} & M_{N1} & \cdots & M_{NN} \end{pmatrix} \tag{3.2.6}$$

3.3 Symetry relations of scattering data in the nonlocal NLS equations

We first present symmetry relations of the secattering data for the reverse-space NLS equation (3.1.1) and the reverse-time NLS equation (3.1.2). The symmetry relations of the scattering data for the reverse-space-time NLS equation (3.1.3) can be obtained in a similar way. For this pourpose, we first introduce some notations. We define

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.3.1}$$

which is a Paili spin matrix.

3.3.1 The reverse-space NLS equation

定理 3.1

For the reverser-space NLS equation (3.1.1), if ζ is an eigenvalue, so is $-\zeta^*$. Thus, non-pirely-imaginary eigenvalues appear as pairs $(\zeta, -\zeta^*)$, which lie in the same half of the complex plane. Symmetry ralations on the eigenvactors are given as follows:

1. If $(\zeta_k, \hat{\zeta}_k) \in \mathbb{C}_+$, then $\hat{\zeta}_k = -\zeta_k^*$, their column eigenvectors are related as $\hat{\mathbf{v}}_{k0} = \sigma_1 \mathbf{v}_{k0}^*$.
2. If $\zeta_k \in i\mathbb{R}_+$, its eigenvectors is of the form $\mathbf{v}_{k0} = (1, e^{\mathrm{i}\theta_k})^T$, where θ_k is a real constant.
3. If $(\bar{\zeta}_k, \hat{\bar{\zeta}}_k) \in \mathbb{C}_-$, then $\hat{\bar{\zeta}}_k = -\bar{\zeta}_k^*$, their row eigenvectors are related as $\hat{\bar{\mathbf{v}}}_{k0} = \bar{\mathbf{v}}_{k0}^* \sigma_1$.
4. If $\bar{\zeta}_k \in i\mathbb{R}_-$, its eigenvectors is of the form $\bar{\mathbf{v}}_{k0} = (1, e^{\mathrm{i}\bar{\theta}_k})$, where $\bar{\theta}_k$ is a real constant.

♡

To proof these results in perspective, we recall that for the local NLS equation,

$$\mathrm{i}q_t + q_{xx} \pm 2q^2 q^* = 0 \tag{3.3.2}$$

which is obatined from the coupled Schrödinger equations (3.1.4) under the reduction of $r(x,t) = -q^*(x,t)$, the symmetry of its scattering data are $\bar{\zeta}_k = -\zeta_k^*$ and $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^*$.

Thus, symmetry relations for the nonlocal NLS equations are different from those local NLS equations. In particular, for the reverse-space and reverse-space-time NLS equations, eigenvalues in the upper and lower halves of the complex plane are completely independent. This independence allows for novel eigenvalue con-figurations, which will give rise to new types of multi-solitons. This will be demonstrated in the next section.

Before proving this theorem, we first establish a connection between the discrete scattering datda for N-solitons $\{\zeta_k, \bar{\zeta}_k, a_k, b_k, \bar{a}_k, \bar{b}_k\}(1 \le k \le N)$ and discrete eigenmodes in the eigenvalue problem $Y_x = MY$ and its adjoint problem $K_x = -KM$, where we set

$$Y_x = -\mathrm{i}\zeta\Lambda Y + QY, \tag{3.3.3a}$$

$$K_x = \mathrm{i}\zeta\Lambda K + KQ \tag{3.3.3b}$$

where the potenitial matrix Q is given by

$$Q = \begin{pmatrix} 0 & q(x,0) \\ r(x,0) & 0 \end{pmatrix} \tag{3.3.4}$$

and $q(x,0), r(x,0)$ are the initial conditions of functions $q(x,t), r(x,t)$ at $t = 0$. Indeed $\forall\{\zeta_k, a_k, b_k\}$ of the discrete scattering data, where $\zeta \in \mathbb{C}_+$ is the eigenvalue of (3.3.3a), whose discrete eigenfunction Y_k has the following asymptotics

$$Y_k(x) \to \begin{bmatrix} a_k e^{-i\zeta_k x} \\ 0 \end{bmatrix}, x \to -\infty, \quad Y_k(x) \to \begin{bmatrix} 0 \\ -b_k e^{i\zeta_k x} \end{bmatrix}, x \to +\infty \tag{3.3.5}$$

Analogously, for the eigenvalue $\bar{\zeta} \in \mathbb{C}_-$ of the adjoint eigenvalue problem (3.3.3b), the discrete eigenfunction K_k has the following asymptotics

$$K_k(x) \to \begin{bmatrix} \bar{a}_k e^{-i\bar{\zeta}_k x} & 0 \end{bmatrix}, x \to -\infty, \quad K_k(x) \to \begin{bmatrix} 0 & -\bar{b}_k e^{-i\bar{\zeta}_k x} \end{bmatrix}, x \to +\infty \tag{3.3.6}$$

In view of this connection, in order to derive symmetry relations on the (discrete) scattering data, we will use symmetry relations of discrete eigenmodes in the eigenvalue problems (3.3.3a)-(3.3.3b).

证明 The reverse-space NLS equation(3.1.1) was derived from the coupled Schrödinger equations (3.1.4) under the reduction of (3.1.5a), and the potential matrix Q is

$$Q = \begin{pmatrix} 0 & q(x,0) \\ -q^*(-x,0) & 0 \end{pmatrix} \tag{3.3.7}$$

Obviously we have $Q^*(-x) = -\sigma_1^{-1} Q \sigma_1$, so

$$\begin{aligned} Y_x = -i\zeta\Lambda Y + QY &\implies -Y_x(-x) = -i\zeta\Lambda Y(-x) + Q(-x)Y(-x) \\ &\implies -Y^*(-x) = i\zeta^*\Lambda Y^*(-x) + Q^*(-x)Y^*(-x) \\ &\implies -\alpha\sigma_1 Y^*(-x) = i\alpha\sigma_1\zeta^*\Lambda Y^*(-x) - \alpha\sigma_1(\sigma_1^{-1}Q\sigma_1)Y^*(-x) \\ &\implies \alpha\sigma_1 Y^*(-x) = -i\alpha(-\zeta^*)\Lambda\sigma_1 Y^*(-x) + Q\alpha\sigma_1 Y^*(-x) \end{aligned} \tag{3.3.8}$$

We get $\hat{Y}_x = -i\hat{\zeta}\Lambda\hat{Y} + Q\hat{Y}$, where

$$\hat{\zeta} - -\zeta^*, \hat{Y} = \alpha\sigma_1 Y^*(-x), \quad \forall\alpha \in \mathbb{C} \tag{3.3.9}$$

This equation shows that: if $\zeta_k \in \mathbb{C}_+$ is an eigenvalue of the scattering problem (3.3.3a), then $\hat{\zeta}_k = -\zeta_k^* \in \mathbb{C}_+$ is also, and

$$\hat{\mathbf{v}}_{k0} = -\alpha\sigma_1\mathbf{v}_{k0}^* = \begin{pmatrix} -\alpha b_k^* \\ -\alpha a_k^* \end{pmatrix} \tag{3.3.10}$$

If $\mathrm{Re}(\zeta_k) \neq 0 \implies \hat{\zeta}_k = -\zeta_k^* \neq \zeta_k$. In this case, when the above $\hat{\mathbf{v}}_{k0}$ expression is inserted into the N-soliton formulae (3.2.3), then constant $-\alpha$ cancels out and does not contribute to the soluition. Thus we can set $-\alpha = 1$ without loss og generality. Then $\hat{\mathbf{v}}_{k0} = \sigma_1\mathbf{v}_{k0}^*$, hence the part 1 is proved.

If $\mathrm{Re}(\zeta_k) = 0 \implies \hat{\zeta}_k = -\zeta_k^* = \zeta_k$. Thus, their eigenvactors are also the same. Without loss of generality, we can scale the eigenvector $\mathbf{v}_{k0}$ so that $a_k = 1$, inserting this into (3.3.10), we have $\alpha = 1, \mathbf{v}_{k0} = (1, -\alpha)^T$, denoting $-\alpha = e^{i\theta_k}, \theta_k \in \mathbb{R}$, we get $\mathbf{v}_{k0} = (1, e^{i\theta_k})^T$, hence the part 2 is proved.

Repeating the above arguments on the adjoint eigenvalue problem (3.3.3b), parts 3 and 4 can be similarly proved. □

3.3.2 The reverse-time NLS equation

定理 3.2

For the reverse-time NLS equation (3.1.2). If ζ is a discrete eigenvalue of the associated Lax pair, then so is $-\zeta$. Hence, the discrete spectrum is symmetric with respect to the origin, and eigenvalues always appear in pairs $(\zeta, -\zeta)$, located in opposite halves of the complex plane.

For each such pair $(\zeta_k, \bar{\zeta}_k)$ with $\zeta_k \in \mathbb{C}_+$ and $\bar{\zeta}_k = -\zeta_k \in \mathbb{C}_-$, the associated eigenvectors $\mathbf{v}_{k0}$ and $\bar{\mathbf{v}}_{k0}$ satisfy

$\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^T$.

证明 The reverse-time NLS equation (3.1.2) was derived from the coupled Schrödinger equations (3.1.4) under the reduction of (3.1.5a), and the potential matrix Q is

$$Q = \begin{pmatrix} 0 & q(x,0) \\ -q(x,0) & 0 \end{pmatrix} \tag{3.3.11}$$

which fearures the following syummetry $Q^T(x) = -Q(x)$. Then, taking the transpose of the eigenvalue problem (3.3.3a), we have

$$Y_x = -\mathrm{i}\zeta\Lambda Y + QY \implies Y_x^T = -\mathrm{i}\zeta Y^T\Lambda^T + Y^TQ^T \implies Y_x^T = -\mathrm{i}\zeta Y^T\Lambda - Y^TQ \tag{3.3.12}$$

We get $\bar{Y}_x = \mathrm{i}\bar{Y}\Lambda - Y^TQ$, where

$$\bar{\zeta} = -\zeta, \bar{Y}(x) = Y^T(x) \tag{3.3.13}$$

It means that $[\bar{\zeta}, \bar{Y}(x)]$ satifies the adjoint eigenvalue equation (3.3.3b).

Thus, if $\zeta_k \in \mathbb{C}_+$ is an eigenvalue of the scattering problem (3.3.3a), then $\bar{\zeta}_k = -\zeta_k \in \mathbb{C}_-$ is an eigenvalue of the adjoint scattering problem (3.3.3b). Utilizing this eigenfunction relation as well as the large-x asymptotics of the eigenfunctions and adjoint eigenfunctions in (3.3.5)-(3.3.6), we readily find that $\bar{a}_k = a_k, \bar{b}_k = b_k$ and $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^T$. This completes the proof of the theorem. □

3.3.3 The reverse-space-time NLS equation

定理 3.3 For the reverse-space-time NLS equation (3.1.3), eigenvalues ζ can be anywhere in $\mathbb{C}_+$, and eigenvalues $\bar{\zeta}_k$ can be anywhere in $\mathbb{C}_-$. However, their eigenvectors must be of the forms

$$\mathbf{v}_{k0} = (1, \omega_k), \quad \bar{\mathbf{v}}_{k0} = (1, \bar{\omega}_k) \tag{3.3.14}$$

where $\omega_k = \pm 1, \bar{\omega}_k = \pm 1$

证明 The reverse-space-time NLS equation (3.1.3) was derived from the coupled Schrödinger equations (3.1.4) under the reduction of (3.1.5c), and the potential matrix Q is

$$Q = \begin{pmatrix} 0 & q(x,0) \\ -q(-x,0) & 0 \end{pmatrix} \tag{3.3.15}$$

which features the following symmetry $Q^*(-x) = -\sigma_1^{-1}Q(x)\sigma_1$. Then, taking the adjoint of the eigenvalue problem (3.3.3a), we have

$$\begin{aligned} Y_x = -\mathrm{i}\zeta\Lambda Y + QY &\implies -Y_x(-x) = -\mathrm{i}\zeta\Lambda Y(-x) + Q(-x)Y(-x) \\ &\implies -\sigma_1 Y_x(-x) = -\mathrm{i}\zeta\sigma_1\Lambda Y(-x) + \sigma_1 Q(-x)Y(-x) \\ &\implies \sigma_1 Y_x(-x) = -\mathrm{i}\zeta\sigma_1\Lambda Y(-x) + \sigma_1 Q(-x)Y(-x) \end{aligned} \tag{3.3.16}$$

We get $\hat{Y}_x(x) = -\mathrm{i}\zeta\Lambda Y(x) + QY(x)$, where

$$\hat{Y}(x) = \sigma_1 Y(-x) \tag{3.3.17}$$

This equation means that for any eigenvalue $\zeta_k \in \mathbb{C}_+$ and $Y_k(x)$ is its eigenfunction, so is $\hat{Y}_k(x) = \sigma_1 Y(-x)$. Thus $\hat{Y}_k$ amd Y_k are rlinearly dependent as

$$Y_k(x) = \omega_k\sigma_1 Y_k(x) \tag{3.3.18}$$

where ω_k is some constant. Utilizing this relation and the large-x asymptotics of the eigenfunction $Y_k(x)$ in (3.3.5), we readily find that $a_k = \omega_k b_k, b_k = \omega_k a_k$, so $\omega_k = \pm 1$. Without loss of generality, we can scale the eigenvector $\mathbf{v}_{k0}$ so that

$a_k = 1$, then $\mathbf{v}_{k0} = (1, \omega_k)^T$.

Since (3.3.17) also means that for any eigenvalue $\bar{\zeta}_k \in \mathbb{C}_-$, if $K_k(x)$ is its adjoint eigenfunction, so is $\hat{K}_k(x) = \sigma_1 K(-x)$. Hence utilizing this relation and the large-x asymptotics of the adjoint eigenfunction $\hat{K}_k(x)$ in (3.3.6), we can similarly show that $\bar{\mathbf{v}}_{k0} = (1, \bar{\omega}_k)$, where $\bar{\omega}_k = \pm 1$. This complests the proof of the theorem. □

Before concluding this section, we point that it is also possible to impose (q, r) reductions (3.1.5a)-(3.1.5c) directly on the determinant solutions (3.2.3) in order to extract symmetry relations on the scattering data $\{\zeta_k, \bar{\zeta}_k, \mathbf{v}_{k0}, \bar{\mathbf{v}}_{k0}, 1 \leq k \leq N\}$. However, our derivation of these relations above is easier. In addition, this derivation is more insightful since it is in the inverse-scattering and Riemann-Hilbert framework.

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