

## Preface

本笔记旨在分享笔者对 **Riemann-Hilbert** 方法的理解, 因仅为一家之言, 仓促间完成又后期无他人审校, 难免存在疏漏, 错误和挂一漏万之处. 望各位读者在阅读时能指出不足, 共同探讨与完善.

本笔记为在老家过年期间编写, 整个过程并不算顺利 [1], 老家的房子年久失修, 塌了外墙和两间屋顶, 白天需要和父亲一起修房子, 应付家庭琐事; 只有夜间才能缓慢的推进写作. 另外农村没有暖气, 零下十余度的环境实难称舒适, 需要字面意义上的争分夺秒. 尤其深夜伏案时, 恍若独行于漫长甬道, 徘徊在黑暗迷宫之中. 然正如 **A. Zee** 所言, 夜航人自有夜行法 [2]. 历时一月的艰辛写作中, 笔者感到一种难以言喻的, 某种漠然的相互理解, 如越高墙, 漫步在满月下的林地, 并在微茫中获得慰藉. 是邪, 非邪? 解释或属妄诞, 感受毕竟真实.

序曲将终, 敬无穷的远方, 与无尽的人们.

本笔记存档于 <https://github.com/IceySwan/Notes>, 更新记录请看 <https://blog.icey.one/RH-Note>, 如发现任何错误请提 [issue](#) 或联系 [hi@icey.one](mailto:hi@icey.one)

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2025 年 2 月

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# 第 1 章 预备知识

## 1.1 Plemelj 公式

### 定理 1.1 (Abel 定理)

设  $A \in \mathbb{C}$ , 对矩阵微分方程  $Y_x = A(x)Y$ , 可得标量微分方程

$$(\det Y)_x = \operatorname{tr}(A) \det Y \quad (1.1.1)$$

从而有

$$\det(Y(x)) = \det(Y(x_0)) \cdot \exp \int_{x_0}^x \operatorname{tr}(A(t)) dt \quad (1.1.2)$$

### 定理 1.2 (Morera)

如果  $f(z)$  在单连通区域  $D$  内连续, 且对于  $D$  内任意闭曲线  $\sum$ , 有  $\int_{\sum} f(z) dz = 0$ , 则有  $f(z)$  在  $D$  内解析

### 定义 1.1 (Schwartz 空间)

欧几里得空间  $\mathbb{R}^n$  上的 Schwartz 空间  $S(\mathbb{R}^n)$  定义为

$$S(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) : \|f\|_{\alpha, \beta} = \sum_{x \in \mathbb{R}^n} |x^\alpha \partial_\beta f(x)| < \infty, \alpha, \beta \in \mathbb{Z} \right\} \quad (1.1.3)$$

其中  $\alpha, \beta$  为多重指标,  $\partial_\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$ .  $f$  称为速降函数或 Schwartz 函数.

简单来说, 速降函数是指当  $x \rightarrow \infty$  时趋近于零的速度比所有的多项式的倒数都快, 并且任意阶的导数都有这种性质的函数。

### 定理 1.3 (Painleve 开拓定理)

设  $D_1, D_2$  为两个没有公共点的区域, 边界为  $\Gamma$ , 并设  $f_1(z), f_2(z)$  分别在  $D_1, D_2$  内解析, 在  $D_1 + \Gamma, D_2 + \Gamma$  上连续, 且  $f_1(z) = f_2(z), \forall z \in \Gamma$ , 则

$$f(z) = \begin{cases} f_1(z), & z \in D_1, \\ f_1(z) = f_2(z), & z \in \Gamma, \\ f_2(z), & z \in D_2 \end{cases} \quad (1.1.4)$$

在  $D_1 + D_2 + \Gamma$  内解析.

**证明** 显然  $f(Z)$  在  $D_1 + D_2 + \Gamma$  上连续, 根据定理 1.2, 只需证明  $f(z)$  沿任何闭曲线  $\sum$  的积分为 0.

如果  $\sum \subset D_1$ , 或  $\sum \subset D_2$ , 则由  $f(z)$  的解析性可得

$$\int_{\sum} f(z) dz = 0 \quad (1.1.5)$$

如果  $\sum$  同时包含于  $D_1, D_2$  内, 把  $\Gamma$  在  $\sum$  内的曲线记为  $C_\gamma$ , 则

$$\int_{C_1 + C_\gamma} f(z) dz = 0, \quad \int_{C_1 + C_\gamma}^- f(z) dz = 0 \quad (1.1.6)$$

故

$$\int_{\sum} f(z) dz = \int_{C_1 + C_2} f(z) dz = \int_{C_1 + C_\gamma} f(z) dz = \int_{C_1 + C_\gamma}^- f(z) dz = 0. \quad (1.1.7)$$

□

## 引理 1.1

设  $f(\xi)$  在  $z \in \Sigma$  满足  $\mu$  次 Hölder 条件, 且  $z' \rightarrow z$  时,  $h/d$  有界, 其中

$$h = |z' - z|, \quad d = \min_{\xi \in \Sigma} |\xi - z'|, \quad (1.1.8)$$

则

$$\lim_{z' \rightarrow z} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z'} d\xi = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z} d\xi \quad (1.1.9)$$

证明

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z'} d\xi - \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{\Sigma} \frac{(\xi - z)(f(\xi) - f(z)) - (\xi - z')(f(\xi) - f(z))}{(\xi - z')(\xi - z)} d\xi = \frac{1}{2\pi i} \int_{\Sigma} \frac{z' - z}{\xi - z} \cdot \frac{f(\xi) - f(z)}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{\Sigma} \frac{h}{d} M \frac{(\xi - z)^{\mu}}{\xi - z} d\xi = \Delta_1 + \Delta_2 \end{aligned} \quad (1.1.10)$$

其中

$$\begin{aligned} |\Delta_1| &\leq \frac{hM}{2\pi d} \int_{C_{\delta}} \frac{(\xi - z)^{\mu}}{\xi - z} d|\xi| \leq \frac{hM}{2\pi d} \int_0^{\delta} t^{\mu-1} dt = \frac{hM}{2\pi d \delta} \delta^{\mu} (C_{\delta} = \{|\xi - z| \leq \delta\}) \\ |\Delta_2| &= \left| \frac{1}{2\pi} \int_{\Sigma \setminus C_{\delta}} \frac{(f(\xi) - f(z))(z - z')}{\xi - z} d\xi \right| \end{aligned} \quad (1.1.11)$$

由于  $\Sigma \setminus C_{\delta}$  不包含  $z$ , 故  $\Delta_2$  为关于  $z'$  的连续函数, 则

$$|\Delta_2| = \left| \frac{1}{2\pi i} \int_{\Sigma \setminus C_{\delta}} \frac{f(\xi) - f(z)}{\xi - z'} d\xi - \frac{1}{2\pi} \int_{\Sigma \setminus C_{\delta}} \frac{f(\xi) - f(z)}{\xi - z} d\xi \right| < |\Delta_1| \quad (1.1.12)$$

取  $\delta \rightarrow 0$ , 可令  $|\Delta_1| < \frac{\epsilon}{2}$ , 故  $|\Delta_1| + |\Delta_2| < \epsilon$  □

## 定理 1.4 (Plemelj)

设  $z \in \Sigma$  为正则点, 且不为边界点,  $f(\xi)$  在  $z$  点满足  $\mu$  次 Hölder 条件, 且  $z' \rightarrow z$  时,  $h/d$  有界, 其中  $h = |z' - z|$ ,  $d = \min_{\xi \in \Sigma} |\xi - z'|$ , 则

$$F_+ = \lim_{\substack{z' \rightarrow z \\ z' \in \Sigma_+}} F(z') = F(z) + \frac{1}{2} f(z) \quad (1.1.13)$$

$$F_- = \lim_{\substack{z' \rightarrow z \\ z' \in \Sigma_-}} F(z') = F(z) - \frac{1}{2} f(z) \quad (1.1.14)$$

证明 首先证明闭区线情形:

$$\begin{aligned} F_+ &= \lim_{\substack{z' \rightarrow z \\ z' \in \Sigma_+}} F(z') = \lim_{z' \rightarrow z} \int_{\Sigma} \frac{1}{2\pi i} \frac{f(\xi)}{\xi - z'} d\xi \\ &= \lim_{z' \rightarrow z} \int_{\Sigma} \frac{1}{2\pi i} \frac{f(\xi) - f(z)}{\xi - z'} d\xi + \frac{f(z)}{2\pi i} \int_{\Sigma} \frac{1}{\xi - z'} d\xi \\ &= \lim_{z' \rightarrow z} \frac{1}{2\pi i} \int_{\Sigma_+} \frac{f(\xi) - f(z)}{\xi - z'} d\xi + f(z) \\ &= \lim_{z' \rightarrow z} \frac{1}{2\pi i} \int_{\Sigma_+} \frac{f(\xi)}{\xi - z'} d\xi - \frac{f(z)}{2\pi i} \int_{\Sigma_+} \frac{1}{\xi - z'} d\xi + f(z) \\ &= F(z) + \frac{1}{2} f(z) \end{aligned} \quad (1.1.15)$$

$F_-(z')$  同理, 下证  $\Sigma$  为开曲线情形, 则可补充  $\Sigma'$ , s.t.  $\Sigma \cup \Sigma'$  为闭曲线, 且定义  $\forall \xi \in \Sigma', f(\xi) = 0$ , 则

$$F(z) = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(z)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\Sigma \cup \Sigma'} \frac{f(z)}{\xi - z} d\xi. \quad (1.1.16)$$

则由开曲线情形可得

$$\begin{aligned} F_+(z) &= \lim_{\substack{z' \rightarrow z \\ z' \in \Sigma}} \int_{\Sigma \cup \Sigma'} \frac{f(\xi)}{\xi - z} d\xi = \int_{\Sigma \cup \Sigma'} \frac{f(\xi) - f(z)}{\xi - z} d\xi + \frac{f(z)}{2\pi i} \int_{\Sigma \cup \Sigma'} \frac{1}{\xi - z} d\xi \\ &= \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z} d\xi - \frac{f(z)}{2\pi i} \int_{\Sigma'} \frac{1}{\xi - z} d\xi + f(z) = F(z) + \frac{1}{2}f(z) \end{aligned} \quad (1.1.17)$$

$F_-(z')$  同理

**注** 如果  $z \in \Sigma$  为一个角点, 在其两切线的夹角为  $\alpha$ , 则可

$$\begin{aligned} F_+(z) &= \lim_{\substack{z' \rightarrow z \\ z' \in D}} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z} d\xi + \frac{\alpha}{2\pi} f(z) + (1 - \frac{\alpha}{2\pi}) f(z) \\ &= F(z) - \frac{\alpha}{2\pi} f(z) \end{aligned} \quad (1.1.18)$$

$$\begin{aligned} F_-(z) &= \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z} d\xi = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi) - f(z)}{\xi - z} d\xi + \frac{\alpha}{2\pi} f(z) - \frac{\alpha}{2\pi} f(z) \\ &= F(z) - \frac{\alpha}{2\pi} f(z) \end{aligned} \quad (1.1.19)$$

### 定义 1.2 (Plemelj 公式)

由 (1.1.13) - (1.1.14), (1.1.18) - (1.1.19) 可以看出, 无论  $z$  是正则点还是角点, 都有

$$F_+(z) - F_-(z) = f(z), \quad z \in \Sigma \quad (1.1.20)$$

$$F_+(z) - F_-(z) = \frac{1}{\pi i} \int_{\Sigma} \frac{f(\xi)}{\xi - z} d\xi =: H(f)(z) \quad (1.1.21)$$

称为标量 RH 问题, 其解可用 Cauchy 积分给出

$$F(z) = A + \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi)}{\xi - z} d\xi, \quad z \in \mathbb{C} \quad (1.1.22)$$

其中  $A$  为任意常数, 一般被边值或渐进条件决定, 这一公式被称为 Plemelj 公式.

对于如下 RH 问题

$$G_+(z) = G_-(z)v(z), \quad z \in \Sigma \quad (1.1.23)$$

只需两边取  $\log$  变换则有

$$\log G_+(z) - \log G_-(z) = \log v(z), \quad z \in \Sigma \quad (1.1.24)$$

由 Plemelj 公式, 则有

$$\log G(z) = A + \frac{1}{2\pi i} \int_{\Sigma} \frac{v(\xi)}{\xi - z} d\xi \implies G(z) = B \exp \left( \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi)}{\xi - z} d\xi \right), \quad z \in \Sigma \quad (1.1.25)$$

其中  $B$  为任意常数.

## 1.2 矩阵 RH 问题

### 定义 1.3 (RH 问题)

设  $\Sigma$  为复平面  $\mathbb{C}$  内的有向路径, 假设存在一个  $\Sigma^0$  上的光滑映射  $v(z) : \Sigma \rightarrow GL(n, \mathbb{C})$ , 则  $(\Sigma, v)$  决定了一个 RH 问题, 寻找一个  $n$  阶矩阵  $M(z)$  满足

$$M(z) \in C, \quad (\mathbb{C} \setminus \Sigma) \quad (1.2.1)$$

$$M_+(z) = M_-(z)v(z), \quad z \in \Sigma \quad (1.2.2)$$

$$M(z) \rightarrow I, \quad z \rightarrow \infty \quad (1.2.3)$$

其中  $M_{\pm}$  表示在正负区域内  $z' \rightarrow z$  时的极限,  $\Sigma$  称为跳跃曲线,  $v(z)$  称为跳跃矩阵.

根据 Beadls-Coifman 定理, 如上 RH 问题的解可通过如下方式构造: 不妨设跳跃矩阵  $v(z)$  具有如下分解

$$v = (b_-)^{-1} b_+ \quad (1.2.4)$$

由此, 可构造

$$w_+ = b_+ - I, \quad w_- = I - b_- \quad (1.2.5)$$

进一步可定义 Cauchy 投影算子

$$(C_{\pm} f)(z) = \lim_{\substack{z' \rightarrow z \in \Sigma \\ z' \in \pm \Sigma}} \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\xi)}{\xi - z'} d\xi \quad (1.2.6)$$

则可以证明如果  $f(z) \in L^2(\Sigma)$ , 则  $C_{\pm} : L^2 \rightarrow L^2$  的有界算子, 且  $C_+ - C_- = 1$ , 再定义算子

$$C_w f = C_+(f w_-) - C_-(f w_+) \quad (1.2.7)$$

则  $C_w : L^2 \cap L^{\infty} \rightarrow L^2$  的有界算子.

#### 定理 1.5

设  $\det v = 1$ , 算子  $I - C_w$  在  $L^2(\Sigma)$  上可逆,  $\mu \in I + L^2(\Sigma)$  为下列方程

$$(I - C_w)\mu = I \quad (1.2.8)$$

的解, 且

$$(I - C_w)(\mu - I) = C_w I + C_+ w_- + C_- w_+ \in L^2(\Sigma) \quad (1.2.9)$$

则

$$M(z) = I + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mu(\xi) w(\xi)}{\xi - z} d\xi \quad (1.2.10)$$

为上述 RH 问题的唯一解, RH 问题  $M(z)$  的可解等价于奇异积分方程 (1.2.8).

**证明** 只需证明 (1.2.10) 满足 (1.2.8)

$$\begin{aligned} M_+ &= I + \lim_{z' \rightarrow z \in \Sigma} \frac{1}{2\pi i} \int_{\Sigma} \frac{\mu(\xi) w(\xi)}{\xi - z'} d\xi \stackrel{(1.2.6)}{=} I + C_+(\mu w) \\ &= I + C_+(\mu w_-) + C_+(\mu w_+) = I + C_+(\mu w_-) + C_-(\mu w_+) + C_+(\mu w_+) + C_-(\mu w_+) \\ &\stackrel{(1.2.7)}{=} I + C_w(\mu) + \mu w_+ \stackrel{(1.2.8)}{=} \mu + \mu w_+ = \mu(I + w_+) = \mu b_+ \end{aligned} \quad (1.2.11)$$

同理有

$$M_-(z) = \mu b_- \quad (1.2.12)$$

故有

$$M_+(z) = \mu b_+ = \mu b_-(b_-)^{-1} b_+ = M_-(z) v(z) \quad (1.2.13)$$

下证唯一性, 先证  $M$  可逆, 对 (1.2.2) 取行列式, 并注意到  $\det v(z) = 1$ , 则  $\det M_+(z) = \det M_-(z)$ . 故由 Painleve 开拓定理得  $\det M(z)$  在  $\mathbb{C}$  上解析. 由 (1.2.3) 得

$$\det M(z) \rightarrow 1, z \rightarrow \infty \quad (1.2.14)$$

故由 Liouville 定理可知  $\det M(z) = c$ , 再由渐进条件得  $c = 1$ , 故  $M(z)$  可逆.

设  $\tilde{M}$  为上述 RH 问题的另一个解, 则  $\tilde{M}$  可逆, 且在  $\mathbb{C} \setminus \Sigma$  上解析, 且在  $\Sigma$  上满足

$$(M \tilde{M}^{-1})_+ = M_+ \tilde{M}_+^{-1} = M_- v(\tilde{M}_- v)^{-1} = M_- (\tilde{M}_-)^{-1} = (M \tilde{M}^{-1})_- \quad (1.2.15)$$

由 Painleve 开拓定理,  $M \tilde{M}^{-1}$  在  $\mathbb{C}$  上解析. 另外由  $M, \tilde{M} \rightarrow I$ , 故  $M \tilde{M}^{-1}$  有界, 故为常矩阵. 由渐进性得

$$M \tilde{M}^{-1} = I \implies M = \tilde{M} \quad (1.2.16)$$

□

**注** 由 RH 问题解的唯一性, RH 问题 (1.2.1) - (1.2.3) 的解与跳跃矩阵  $v(z)$  的分解无关, 因此可以考虑  $v(z)$  的平凡解

$$b_- = I, \quad b_+ = v \quad (1.2.17)$$

故可构造

$$\begin{aligned} w_- &= 0, \quad w_+ = v - I, \quad w = v - I \\ C_w f &= C_+(f w_-) + C_-(f w_+) = C_+(f(v - I)), \quad \mu = (I - C_w)^{-1} I \end{aligned} \quad (1.2.18)$$

则 RH 问题的解可表示为

$$M(z) = I + \frac{1}{2\pi i} \int_{+\Sigma} \frac{\mu(\xi)(v(\xi) - I)}{\xi - z} d\xi \quad (1.2.19)$$

**注** RH 方法的关键思想就是改变积分路径, 通过跳跃矩阵的分解情况, 确定对积分路径进行一系列形变, 再取极限去除跳跃矩阵为单位阵的情形, 将其化解为可解的 RH 问题, 所以一个自然的问题就是: “为什么可以扔掉跳跃矩阵为单位阵的路径? 或者说为什么跳跃矩阵对 RH 问题的解不产生贡献.”

这很容易通过表达式 (1.2.19) 看出, 我们将积分路径分解为  $\Sigma = \Sigma_1 + \Sigma_2$  且

$$v = \begin{cases} v_1 \neq I & \Sigma_1, \\ v_1 = I & \Sigma_2 \end{cases} \quad (1.2.20)$$

则在  $\Sigma_1$  上,  $v - I = v_1 - I \neq 0$ , 在  $\Sigma_2$  上,  $v - I = v_1 - I = 0$ , 故

$$\begin{aligned} M(z) &= I + \frac{1}{2\pi i} \int_{\Sigma} \frac{\mu(\xi)(v(\xi) - I)}{\xi - z} d\xi \\ &= I + \frac{1}{2\pi i} \int_{\Sigma_1} \frac{\mu(\xi)(v_1(\xi) - I)}{\xi - z} d\xi + \frac{1}{2\pi i} \int_{\Sigma_2} \frac{\mu(\xi)(v_2(\xi) - I)}{\xi - z} d\xi \\ &= I + \frac{1}{2\pi i} \int_{\Sigma_1} \frac{\mu(\xi)(v_1(\xi) - I)}{\xi - z} d\xi \end{aligned} \quad (1.2.21)$$

易得 RH 问题  $(\Sigma, v(z))$  的解与  $(\Sigma_1, v_1(z))$  的解相同, 即跳跃矩阵为单位阵的路径  $\Sigma_2$  对 RH 问题的解无贡献, 可以舍去. 只求  $\Sigma_1$  上的 RH 问题

## 第 2 章 RH 方法求解零边界的 NLS 方程

### 2.1 聚焦 NLS 方程

#### 2.1.1 特征函数

考虑聚焦 NLS 方程的初值问题

$$iq_t(x, t) + q_{xx}(x, t) + 2q(x, t)|q(x, t)|^2 = 0 \quad (2.1.1)$$

$$q(x, 0) = q_0(x) \in \mathcal{S}(\mathbb{R}) \quad (2.1.2)$$

其中  $\mathcal{S}(\mathbb{R})$  表示 Schwartz 空间. NLS 方程 (2.1.1) 具有如下矩阵形式的 Lax 对

$$\psi_x + iz\sigma_3\psi = P\psi \quad (2.1.3)$$

$$\psi_t + 2iz^2\sigma_3\psi = Q\psi \quad (2.1.4)$$

其中

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad P = \begin{pmatrix} & q \\ -q^* & \end{pmatrix}, \quad Q = \begin{pmatrix} i|q|^2 & iq_x \\ iq_x^* & -i|q|^2 \end{pmatrix} + 2zP$$

满足零曲率方程

$$(P - iz\sigma_3)_t - (Q - 2iz^2\sigma_3)_x + [P - iz\sigma_3, Q - 2iz^2\sigma_3] = 0 \quad (2.1.5)$$

#### 2.1.2 渐进性

由于  $q_0(x) \in \mathcal{S}(\mathbb{R})$ , 且  $q(x, t), q_x(x, t) \rightarrow 0, (x \rightarrow \infty)$ , 故  $x$  足够大时,  $P, Q$  可忽略, 从而 Lax 对 (2.1.3) - (2.1.4) 可近似为

$$\psi_x \sim -iz\sigma_3\psi, \quad \psi_t \sim -2iz^2\sigma_3\psi$$

可得渐进形式的 Jost 解  $\psi = e^{-i\theta(z)\sigma_3}, (x \rightarrow \infty)$ , 其中  $\theta(z) = zx + 2z^2t$ . 下面为表示方便, 不妨记  $E := e^{i\theta(z)\sigma_3}$ , 做变换  $\mu(x, t, z) = \psi(x, t, z)E$ , 则有

$$E^{-1} = e^{-i\theta(z)\sigma_3}, \quad \mu(x, t, z) \rightarrow I (x \rightarrow \infty) \quad (2.1.6)$$

则可得

$$\mu_x = \psi_x E + \psi E(i\sigma_3 z) \quad (2.1.7)$$

$$\mu_t = \psi_t E + \psi E(2i\sigma_3 z^2) \quad (2.1.8)$$

带入 Lax 对 (2.1.3) - (2.1.4) 可得

$$\mu_x = (P\psi - iz\sigma_3\psi)E + \psi E(i\sigma_3 z) \quad (2.1.9)$$

$$\mu_t = (Q\psi - 2iz^2\sigma_3\psi)E + \psi E(2i\sigma_3 z^2) \quad (2.1.10)$$

故有

$$\mu_x = P\mu - iz[\sigma_3, \mu] \implies \mu_x + iz[\sigma_3, \mu] = P\mu \quad (2.1.11)$$

同理可得

$$\mu_t + 2iz^2[\sigma_3, \mu] = Q\mu \quad (2.1.12)$$



由 (2.1.11) 式可得

$$\mu_x + iz\hat{\sigma}_3\mu = P\mu \quad (2.1.13)$$

$$e^{i\theta(z)\hat{\sigma}_3}\mu_x + iz\hat{\sigma}_3e^{i\theta(z)\hat{\sigma}_3}\mu = e^{i\theta(z)\hat{\sigma}_3}P \quad (2.1.14)$$

其中  $\hat{\sigma}_3 X = [\sigma_3, X]$ ,  $e^{i\theta(z)\hat{\sigma}_3}X = EXE^{-1}$ , 为表示方便不妨记  $\hat{E} = e^{i\theta(z)\hat{\sigma}_3}$ . 易得 Lax 对的全微分形式

$$d(\hat{E}\mu) = \hat{E}[(Pdx + Qdt)\mu] \quad (2.1.15)$$

为构造规范的 Riemann-Hilbert 问题, 即其解在  $z \rightarrow \infty$  时渐进于单位阵, 现只需证明  $\mu \rightarrow I(z \rightarrow \infty)$ .

**证明** 将  $\mu$  在无穷远点 Taylor 展开

$$\mu = \mu^{(0)} + \frac{1}{z}\mu^{(1)} + \cdots = \mu^{(0)} + O(z^{-1}) \quad (2.1.16)$$

其中  $\mu^{(0)}, \mu^{(1)}$  与  $z$  无关, 将上式代入 (2.1.11), (2.1.12) 比较  $z$  的次数可得

$$\begin{aligned} [\mu^{(0)} + O(z^{-1})]_x + iz \left[ \sigma_3, \mu^{(0)} + \frac{\mu^{(1)}}{z} + O(z^{-2}) \right] &= P [\mu^{(0)} + O(z^{-1})] \\ [\mu^{(0)} + O(z^{-1})]_t + 2iz^2 [\sigma_3, \mu^{(0)} + \frac{\mu^{(1)}}{z} + O(z^{-2})] &= (Q_0 + 2zP) [\mu^{(0)} + O(z^{-1})] \end{aligned} \quad (2.1.17)$$

可得  $x$  部分:

$$\begin{aligned} z: [\sigma_3, \mu^{(0)}] &= 0 \\ z^0: \mu^{(0)} + 2iz[\sigma_3, \mu^{(1)}] &= P\mu^{(0)} \implies \mu^{(0)} \text{ 为对角阵} \end{aligned} \quad (2.1.18)$$

$t$  部分:

$$\begin{aligned} z^2: [\sigma_3, \mu^{(0)}] &= 0 \\ z: i[\sigma_3, \mu^{(1)}] &= P\mu^{(1)} \implies \mu_x^{(0)} = 0 \end{aligned} \quad (2.1.19)$$

故  $\mu^{(0)}$  为与  $x$  无关的对角矩阵, 因此 (2.1.16) 式对  $x, z$  同时取极限, 并交换极限顺序可得

$$\lim_{z \rightarrow \infty} \lim_{x \rightarrow \infty} \mu = \lim_{x \rightarrow \infty} \lim_{z \rightarrow \infty} (\mu^{(0)} + O(z^{-1})) \quad (2.1.20)$$

利用 (2.1.6), (2.1.16) 式可得  $\mu^{(0)} = I$ , 故  $\mu \rightarrow I(z \rightarrow \infty)$ .  $\square$

再将  $\mu^{(0)} = I$  代入 (2.1.19) 式比较矩阵对角元素得

$$q(x, t) = 2i(\mu^{(1)})_{12} = 2i \lim_{z \rightarrow \infty} (z\mu)_{12} \quad (2.1.21)$$

**注** 上式可将 NLS 方程与特征函数联系起来, 接下来将特征函数与 RH 问题建立联系. 从而 NLS 方程的解可用 RH 问题的解表示, 然后通过 RH 问题反接触 NLS 方程的解, 即

$$NLS \rightleftharpoons \text{特征函数} \rightleftharpoons RH \text{问题}$$

## 2.2 解析性与对称性

由于 (2.1.15) 为全微分形式, 积分与路径无关, 故选择两个特殊路径:

$$(-\infty, t) \rightarrow (x, t), \quad (+\infty, t) \rightarrow (x, t)$$

因此可获得 Lax 对 (2.1.11), (2.1.12) 的两个特征函数

$$\mu_1 = I - \int_{-\infty}^x e^{-iz(x-y)\sigma_3} P \mu_1 dy, \quad \mu_2 = I - \int_{+\infty}^x e^{-iz(x-y)\sigma_3} P \mu_2 dy \quad (2.2.1)$$

**证明** 由 (2.1.15) 式, 由  $(\hat{E}\mu)_x = \hat{E}P\mu$  可得

$$\begin{aligned} \hat{E}\mu|_{-\infty}^x &= \int_{-\infty}^x \hat{E}P\mu dy \\ LHS &= \hat{E}\mu - \hat{E}I, \quad RHS = \int_{-\infty}^x e^{izy+z^2t\hat{\sigma}_3} P \mu dy \end{aligned} \quad (2.2.2)$$

两边同时乘以  $\hat{E}$ , 即得  $\mu_1, \mu_2$  同理可得.  $\square$

显然, 其仍具有如下性质

$$\mu_1, \mu_2 \rightarrow I(x \rightarrow \pm\infty), \quad \mu_1, \mu_2 \rightarrow I(z \rightarrow +\infty) \quad (2.2.3)$$

由于  $\psi_1 = \mu_1 E^{-1}, \psi_2 = \mu_2 E^{-1}$ , 为 Lax 对的两个解, 而 Lax 对为一节齐次线性方程组, 故这两个解线性相关, 故有

$$\mu_1(x, t, z) = \mu_2(x, t, z) \hat{E} S(z), \quad S(z) = \begin{pmatrix} S_{11}(z) & S_{12}(z) \\ S_{21}(z) & S_{22}(z) \end{pmatrix} \quad (2.2.4)$$

其中矩阵  $S(z)$  与  $x, t$  无关, 称为谱函数矩阵.

再由  $\mu = \psi E$  可知  $\det(\mu_j) = \det(\psi_j)$ . 又因为  $\text{tr}(P - iz\sigma_3) = \text{tr}(Q - 2iz^2\sigma_3) = 0$ , 由 Abel 定理可得

$$\det(\psi_j)_x = \det(\psi_j)_t = 0 \quad (2.2.5)$$

故有  $\det(\mu_j)_x = \det(\mu_j)_t = 0$ , 这说明  $\det(\mu_j)$  与  $x, t$  无关, 再由渐进性可得

$$\det(\mu_j) = \lim_{|x| \rightarrow \infty} \det(\mu_j) = 1, \quad (j = 1, 2) \quad (2.2.6)$$

故对 (2.2.4) 两边取行列式有  $\det S(z) = 1$ .

### 2.2.1 解析性

下面我们考虑特征函数  $\mu_1, \mu_1$  和谱矩阵  $S(z)$  的解析性, 记  $\mu_1$  的第一二列分别为

$$\mu_1 = \begin{pmatrix} \mu_1^{11} & \mu_1^{12} \\ \mu_1^{21} & \mu_1^{22} \end{pmatrix} = (\mu_1^+, \mu_1^-) \quad (2.2.7)$$

则由积分方程 (2.2.1) 可得如下 Volterra 积分方程

$$\mu_1^+(x, t, z) = (1, 0)^T - \int_{-\infty}^x \begin{pmatrix} 1 & \\ & e^{iz(x-y)} \end{pmatrix} P \mu_1^+ dy \quad (2.2.8)$$

$$\mu_1^-(x, t, z) = (0, 1)^T - \int_{-\infty}^x \begin{pmatrix} e^{-iz(x-y)} & \\ & 1 \end{pmatrix} P \mu_1^- dy \quad (2.2.9)$$

对于上面两方程, 由于积分变量  $y < x$ , 可得

$$e^{2iz(x-y)} = e^{2i(x-y)\Re(z)} e^{-2i(x-y)\Im(z)}, \quad e^{-2iz(x-y)} = e^{-2i(x-y)\Re(z)} e^{2i(x-y)\Im(z)}$$

因此当  $q(x) \in L^1(\mathbb{R})$  时, 通过构造序列与 Neumann 级数, 可得  $\mu_1^+, \mu_1^-$  分别在  $\mathbb{C}_\pm$  解析性.

**注** 上面 Volterra 积分的推导: 考虑到  $e^{\hat{\sigma}_3 X} = e^{\sigma_3 X} e^{-\sigma_3}$ , 故有

$$\begin{aligned} \mu_1 &= I + \int_{-\infty}^x e^{-iz(x-y)\hat{\sigma}_3} P \mu_1 dy = I + \int_{-\infty}^x e^{-iz(x-y)\sigma_3} P e^{iz(x-y)\sigma_3} \mu_1 dy \\ &= I + \int_{-\infty}^x \begin{pmatrix} & qe^{2iz(x-y)} \\ -q^*e^{-2iz(x-y)} & \end{pmatrix} \mu dy \end{aligned}$$

故  $\mu_1^+ =, \mu_1^- =$

**证明** 下证  $\mu_1^+$  的解析性.

Step1. 解的存在性: 事实上方程 (2.2.8) 有如下 Neumann 级数分解

$$\mu_1^+ = \sum_{n=0}^{\infty} c_n(x, z), \quad c_{n+1} = \int_{-\infty}^x \begin{pmatrix} 1 & \\ & e^{iz(x-y)} \end{pmatrix} P c_n(y, z) dy \quad (2.2.10)$$

其分量形式为

$$c_{n+1}^{(1)}(x, z) = \int_{-\infty}^x q(y) c_n^{(2)}(y, z) dy, \quad c_{n+1}^{(2)}(x, z) = \int_{-\infty}^x e^{2iz(x-y)} p(y) c_n^{(1)}(y, z) dy \quad (2.2.11)$$

其中  $c_{n+1} = (c_{n+1}^{(1)}, c_{n+1}^{(2)})^T, p = -q^*$ . 由  $c_0^{(2)} = 0 \implies c_{2n+1}^{(1)} = c_{2n}^{(2)} = 0$ , 故上面方程可简化为

$$c_{2n}^{(1)}(x, z) = \int_{-\infty}^x q(y) c_{2n-1}^{(2)}(y, z) dy, \quad c_{2n+1}^{(2)}(x, z) = \int_{-\infty}^x e^{2iz(x-y)} p(y) c_{2n}^{(1)}(y, z) dy \quad (2.2.12)$$

对上面方程组进一步简化, 引入如下恒等式

$$\begin{aligned} \frac{1}{j!} \int_{-\infty}^x |f(\xi)| \left[ \int_{-\infty}^{\xi} |f(\eta)| \right]^j d\xi &= \frac{1}{(j+1)!} \int_{-\infty}^x \frac{d}{d\xi} \left[ \int_{-\infty}^{\xi} |f(\eta)| \right]^{j+1} d\xi \\ &= \frac{1}{(j+1)!} \left[ \int_{-\infty}^x |f(\xi)| d\xi \right]^{j+1} \quad (f \in L^1(\mathbb{R})) \end{aligned} \quad (2.2.13)$$

利用归纳法可证明当  $\Im(z) > 0$  时, 有

$$|c_{2n+1}^{(2)}| = \frac{u^{n+1}(x)}{(n+1)!} \frac{v^n(x)}{n!}, \quad |c_{2n}^{(1)}| = \frac{u^n(x)}{n!} \frac{v^n(x)}{n!} \quad (2.2.14)$$

其中

$$u(x) = \int_{-\infty}^x |p(y)| dy, \quad v(x) = \int_{-\infty}^x |q(y)| dy \quad (2.2.15)$$

事实上, 当  $\Im(z) > 0$  时,  $|e^{2iz(x-y)}| \leq 1$ , 利用  $c_0^{(1)} = 1$ , 由 (2.2.14) 可得

$$c_1^{(2)}(x, y) = \int_{-\infty}^x e^{2iz(x-y)} p(y) dy, \quad c_2^{(1)}(x, y) = \int_{-\infty}^x e^{2iz(x-y)} q(y) c_1^{(2)} dy \quad (2.2.16)$$

又由  $u(x) \geq 0, v(x) \geq 0$ , 有  $c_1^{(2)} \leq v(x)$

$$\begin{aligned} |c_2^{(1)}(x, y)| &\leq \int_{-\infty}^x |q| c_1^{(2)}(y, z) dy \leq \int_{-\infty}^x |q| v(y) dy = \int_{-\infty}^x v'(y) u(y) dy \\ &= u(x) v(x) - \int_{-\infty}^x u'(y) v(y) dy \leq u(x) v(x) \end{aligned} \quad (2.2.17)$$

进而可得

$$\begin{aligned} |c_3^{(2)}(x, z)| &\leq \int_{-\infty}^x |p| c_2^{(1)}(y, z) dy \leq \int_{-\infty}^x |p| u(y) v(y) dy \\ &\leq \int_{-\infty}^x u'(y) u(y) v(y) dy = \frac{1}{2} u^2(x) v(x) - \int_{-\infty}^x \frac{1}{2} u^2(x) v'(x) dy \leq \frac{1}{2} u^2(x) v(x) \end{aligned} \quad (2.2.18)$$

不妨设对于  $k \leq l$  时满足  $|c_{2l+1}^{(2)}| \leq \frac{u^{l+1}(x)}{(l+1)!} \frac{v^l(x)}{l!}, |c_{2l}^{(1)}| \leq \frac{u^l(x)}{l!} \frac{v^l(x)}{l!}$ , 下证对于  $k = l+1$  仍成立, 利用 (2.2.13) 式可得

$$|c_{2l+2}^{(1)}| \leq \int_{-\infty}^x |q(y)| \frac{u^{l+1}(y)}{(l+1)!} \frac{v^l(y)}{l!} dy \leq \int_{-\infty}^x v'(y) \frac{u^{l+1}(y)}{(l+1)!} \frac{v^l(y)}{l!} dy = \frac{u^{l+1}}{(l+1)!} \frac{v^{l+1}}{(l+1)!} \quad (2.2.19)$$

及

$$|c_{2l+3}^{(2)}| \leq \int_{-\infty}^x |p(y)| \frac{u^{l+1}(y)}{(l+1)!} \frac{v^l(y)}{l!} dy \leq \int_{-\infty}^x u'(y) \frac{u^{l+1}(y)}{(l+1)!} \frac{v^l(y)}{l!} dy = \frac{u^{l+2}}{(l+2)!} \frac{v^{l+1}}{(l+1)!} \quad (2.2.20)$$

从而上述 (2.2.12) 式得证. 注意到  $|q(x)| = |p(x)|$ , 则  $v(x) = u(x)$ , 故 (2.2.12) 式可化简为

$$|c_{2n+1}^{(2)}| \leq \frac{u^{2n+1}}{n!(n+1)!}, \quad |c_{2n}^{(1)}| \leq \frac{u^{2n}}{n!n!} \quad (2.2.21)$$

当  $q \in L^1(\mathbb{R})$  时, 有  $u(x)$  的上述级数均收敛, 故 Neumann 级数  $\sum_{n=1}^{\infty} c_n(x, z)$  绝对收敛, 此时  $\mu_1^+$  在  $\text{Im} z > 0$  上解析, 且在  $\text{Im} z \geq 0$  上连续.

Step2. 解的唯一性: 不妨设  $\tilde{\mu}_1^+$  为方程 (2.2.8) 的另一个解, 不妨设  $h = \mu_1^+ - \tilde{\mu}_1^+$ , 则有

$$\|h(x, t, z)\| = \left\| \int_{-\infty}^x \begin{pmatrix} 1 \\ e^{iz(x-y)} \end{pmatrix} P h dy \right\| \leq 2 \int_{-\infty}^x |q| \|h\| dy \implies \|h(x, t, z)\| \equiv 0 \quad (2.2.22)$$

故  $\mu_1^+$  为方程的唯一解, 同理可得  $\mu_1^-$  的解析性.  $\square$

所以可得  $\mu_1^+, \mu_1^-$  分别在  $\mathbb{C}_+, \mathbb{C}_-$  上解析. 同理可得  $\mu_2$  的第一二列分别在  $\mathbb{C}_-, \mathbb{C}_+$  上解析. 记作

$$\mu_2 = \begin{pmatrix} \mu_2^{(11)} & \mu_2^{(12)} \\ \mu_2^{(21)} & \mu_2^{(22)} \end{pmatrix} = (\mu_2^-, \mu_2^+)$$

由 (2.2.6) 可知,  $\mu_1, \mu_2$  可逆, 且逆矩阵为其伴随阵, 另外基于  $\mu_1, \mu_2$  的列向量函数的解析性, 可得  $\mu_1^{-1}$  的第一二行

在  $\mathbb{C}_-, \mathbb{C}_+$  上解析,  $\mu_2^{-1}$  的第一二行在  $\mathbb{C}_+, \mathbb{C}_-$  上解析. 记为

$$\mu_1^{-1} = \begin{pmatrix} \mu_1^{(22)} & -\mu_1^{(21)} \\ -\mu_1^{(12)} & \mu_1^{(11)} \end{pmatrix} = \begin{pmatrix} -\hat{\mu}_1^+ \\ \hat{\mu}_1^- \end{pmatrix}, \quad \mu_2^{-1} = \begin{pmatrix} \mu_2^{(22)} & -\mu_2^{(21)} \\ -\mu_2^{(12)} & \mu_2^{(11)} \end{pmatrix} = \begin{pmatrix} -\hat{\mu}_2^+ \\ \hat{\mu}_2^- \end{pmatrix} \quad (2.2.23)$$

利用 (2.2.4), (2.2.7), (2.2.23) 可得谱函数  $S(z)$  的解析性

$$S(z) = \mu_2^{-1} \mu_1 = \begin{pmatrix} -\hat{\mu}_2^+ \\ \hat{\mu}_2^- \end{pmatrix} (\mu_1^+, \mu_1^-) = \begin{pmatrix} -\hat{\mu}_2^+ \mu_1^+ & -\hat{\mu}_2^+ \mu_1^- \\ \hat{\mu}_2^- \mu_1^+ & \hat{\mu}_2^- \mu_1^- \end{pmatrix} \quad (2.2.24)$$

可见  $S_{11}(z)$  在  $\mathbb{C}_+$  解析,  $S_{22}(z)$  在  $\mathbb{C}_-$  解析,  $S_{12}, S_{21}$  在  $\mathbb{C}_+, \mathbb{C}_-$  均不解析, 但连续到边界.

## 2.2.2 对称性

### 定理 2.1

如上构造的特征函数  $\mu_1, \mu_2$  与谱函数  $S(z)$  具有如下对称性

$$\mu_j^\dagger(x, t, z^*) = \mu_j^{-1}(x, t, z), \quad S^\dagger(z^*) = S^{-1}(z) \quad (j = 1, 2) \quad (2.2.25)$$

**证明** 由 (2.1.11) 有

$$\mu_{j,x}(x, t, z) = iz[\sigma_3, \mu_j(x, t, z)] = P\mu_j(x, t, z) \quad (2.2.26)$$

将  $z$  替换为  $z^*$ , 并在两边同时取 Hermite 共轭, 则有

$$\mu_{j,x}^\dagger(x, t, z^*) + iz[\sigma_3, \mu_j^\dagger(x, t, z^*)] = \mu_j^\dagger(x, t, z^*)P^\dagger \quad (2.2.27)$$

由于  $P^\dagger = -P$ , 故上式可化为

$$\mu_{j,x}^\dagger(x, t, z^*) + iz[\sigma_3, \mu_j^\dagger(x, t, z^*)] = -\mu_j^\dagger(x, t, z^*)P \quad (2.2.28)$$

另外对于  $\mu_j \cdot \mu_j^{-1} = I$  对  $x$  求偏导有

$$\mu_{j,x} \mu_j^{-1} + \mu_j \mu_{j,x}^{-1} = 0 \implies \mu_{j,x}^{-1} = -\mu_j^{-1} \mu_{j,x} \mu_j^{-1} \quad (2.2.29)$$

将 (2.2.26) 带入上式有

$$\mu_{j,x}^{-1} = -\mu_j^{-1}(P\mu_j(x, t, z) - iz[\sigma_3, \mu_j(x, t, z)])\mu_j^{-1}(x, t, z) \implies \mu_{j,x}^{-1} + iz[\sigma_3, \mu_j^{-1}] = \mu_j^{-1}P \quad (2.2.30)$$

由 (2.2.8), (2.2.10) 可得  $\mu_j^\dagger, \mu_j^{-1}$  满足相同的一次线性微分方程, 具有相同的渐进性, 又因为

$$\mu_j^\dagger(x, t, z^*), \mu_j^{-1}(x, t, z) \rightarrow I, \quad |x| \rightarrow \infty \quad (2.2.31)$$

因此两者相等, 得到对称关系  $\mu_j^\dagger(x, t, z^*) = \mu_j^{-1}(x, t, z), (j = 1, 2)$ . 接下来考虑  $S$  的对称性, 将 (2.2.4) 式改写为

$$S(z) = e^{i\theta(z)\hat{\sigma}_3}(\mu_2^{-1}\mu_1) = \hat{E}(\mu_2^{-1}\mu_1) \quad (2.2.32)$$

由 (2.2.31) 可得如下

$$S(z^*)^\dagger = e^{i\theta(z^*)\sigma_3}(\mu_2^{-1}\mu_1)e^{-i\theta(z^*)} = E\mu_2^{-1}\mu_1E^{-1} = E\mu_1^\dagger(\mu_2^{-1})^\dagger E^{-1} = E\mu_1^\dagger\mu_2E^{-1} = S(z)^{-1} \quad (2.2.33)$$

比较对应元素, 有

$$S_{11}^*(z^*) = S_{22}(z), S_{12}^*(z^*) = -S_{21}(z) \quad (2.2.34)$$

□

## 2.3 相关的 RH 问题

### 2.3.1 规范 RH 问题

基于 2.2 节的结论, 我们构造 RH 问题, 引入记号

$$H_1 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \quad (2.3.1)$$

并定义两个矩阵

$$\begin{aligned} P_+(x, t, z) &= \mu_1 H_1 + \mu_2 H_2 = \begin{pmatrix} \mu_1^{(11)} & \mu_2^{(12)} \\ \mu_1^{(21)} & \mu_2^{(22)} \end{pmatrix} = (\mu_1^+, \mu_2^+) \\ P_-(x, t, z) &= H_1 \mu_1^{-1} + H_2 \mu_2^{-1} = \begin{pmatrix} \mu_1^{(22)} & -\mu_1^{(12)} \\ -\mu_2^{(21)} & \mu_2^{(11)} \end{pmatrix} = (\hat{\mu}_1^-, \hat{\mu}_2^-) \end{aligned} \quad (2.3.2)$$

则由  $\mu_j, \mu_j^{-1} (j = 1, 2)$  的解析性与渐进性, 直接可得  $P_+$  在  $\mathbb{C}_+$  上解析,  $P_-$  在  $\mathbb{C}_-$  上解析, 且具有如下渐进性

$$P_+, P_- \rightarrow I \quad |x| \rightarrow \infty \quad (2.3.3)$$

由此可证明如下对称关系

#### 定理 2.2

$P_+(x, t, z), P_-(x, t, z)$  具有如下对称性

$$P_+^\dagger(x, t, z^*) = P_-(x, t, z) \quad (2.3.4)$$

**证明** 利用 (2.2.31) 可得

$$\begin{aligned} P_+^\dagger(x, t, z^*) &= (\mu_1(x, t, z)H_1 + \mu_2(x, t, z)H_2)^\dagger = H_1 \mu_1^\dagger(z^*) + H_2 \mu_2^\dagger(z^*) \\ &= H_1 \mu_1^{-1}(x, t, z) + H_2 \mu_2^{-1}(x, t, z) = P_-(x, t, z) \end{aligned} \quad (2.3.5)$$

□

#### 定义 2.1

概括上面结果, 我们可以得到下面 RH 问题

$$P_\pm(x, t, z) \in C(\mathbb{C}_\pm) \quad (2.3.6)$$

$$p_- p_+ = G(x, t, z) \quad z \in \mathbb{R} \quad (2.3.7)$$

$$P_\pm \rightarrow I \quad z \rightarrow \infty \quad (2.3.8)$$

其中跳跃矩阵为

$$G(x, t, z) = \hat{E}^{-1} \begin{pmatrix} 1 & -S_{12}(z) \\ S_{21}(z) & 1 \end{pmatrix} \quad (2.3.9)$$

进一步 NLS 方程的解  $q(x, t)$  可由 RH 问题的解给出

$$q(x, t) = 2i \lim_{z \rightarrow \infty} (z P_+)_{12} = 2i (P_+^{(1)})_{12} \quad (2.3.10)$$

其中  $P_+ = I + \frac{P_+^{(1)}}{z} + O(z^{-2})$

**证明** 只需证明跳跃关系即可, 注意到  $\hat{E} H_j = H_j, (j = 1, 2)$  可得

$$\begin{aligned} G &= P_- P_+ = (H_1 \mu_1^{-1} + H_2 \mu_2^{-1})(\mu_1 H_1 + \mu_2 H_2) \\ &\stackrel{(2.2.4)}{=} [H_1(\hat{E}^{-1} S^{-1})\mu_2^{-1} + H_2 \mu_2^{-1}][\mu_2(\hat{E}^{-1} S)H_1 + \mu_2 H_2] \\ &= \hat{E}^{-1}[(H_1 S^{-1} + H_2)(S H_1 + H_2)] = \hat{E}^{-1} \begin{pmatrix} 1 & -S_{12}(z) \\ S_{21}(z) & 1 \end{pmatrix} \end{aligned} \quad (2.3.11)$$

直接计算可得

$$\begin{aligned}\det(P_+) &= \det(\mu_1 H_1 + \mu_2 H_2) = \det(\mu_2 \hat{E}^{-1} S) H_1 + \mu_2 H_2 \\ &= \det(\mu_2) \det((\hat{E}^{-1} S) H_1 + \mu_2 H_2) = S_{11}(z)\end{aligned}\quad (2.3.12)$$

同理, 有  $\det(P_-) = S_{22}(z)$  □

### 2.3.2 RH 问题的可解性

下面分两种情况讨论 RH 问题 (2.3.6) - (2.3.8) 的解

Case1. 如果

$$\det P_{\pm}(z) \neq 0 (\forall z \in \mathbb{C}), \quad (2.3.13)$$

则称 RH 问题 (2.3.6) - (2.3.8) 为正则的, 将方程 (2.3.7) 改写为

$$P_+^{-1} - P_- = (I - G)P_+^{-1} := \hat{C}P_+^{-1} \quad (2.3.14)$$

由 Plemelj 公式可得

$$P_+ = I + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\hat{C}(x, t, z) P_+^{-1}(x, t, z)}{s - z} ds, \quad z \in \mathbb{C}_+ \quad (2.3.15)$$

Case2. 如果条件 (2.3.13) 不满足, 则称 RH 问题为非正则的, 假设  $\det P_{\pm}$  在某些离散的点处为零, 由谱函数对称性 (2.2.34) 有

$$\det P_+(z) = S_{11}(z) = S_{22}^*(z^*) = \det P_-^*(z^*) = \det P_-(z^*) \quad (2.3.16)$$

故

$$\det P_+(z) = 0 \iff \det P_-(z^*) = 0, \quad (2.3.17)$$

因此  $\det P_+(z)$  与  $\det P_-(z)$  有相同的零点个数, 且彼此共轭. 即若设  $z_j (j = 1, 2, \dots, N)$  为  $\det P_+(z)$  在  $\mathbb{C}_+$  上的单零点, 则  $z_j^* (j = 1, 2, \dots, N)$  为  $\det P_-(z)$  在  $\mathbb{C}_-$  上的单零点.

由于  $\det P_+(z_j) = \det P_-(z_j^*) = 0$ , 假设  $w_j, w_j^*$  分别为下列线性方程组的解

$$P_+(z_j)w_j(z_j) = 0, \quad w_j^*(z_j^*)P_-(z_j^*) = 0 \quad (2.3.18)$$

对上式取共轭转置, 则有  $w_j^\dagger(z_j)P_+^\dagger(z_j) = 0$ , 再利用对称性 (2.3.4) 有

$$w_j^\dagger(z_j)P_-(z_j^*) = 0 \quad (2.3.19)$$

比较可得  $w_j^\dagger(z) = w_j^*(z^*)$ .

非正则 RH 问题 (2.3.6) - (2.3.8) 的解可由下面定理给出

#### 定理 2.3 (Zakharov, Shabat, 1979)

带有零点结构 (2.3.18) 的非正则 RH 问题 (2.3.6) - (2.3.8) 可分解为

$$P_+(z) = \hat{P}_+(z)\Gamma(z), \quad P_-(z) = \Gamma^{-1}(z)\hat{P}_-(z) \quad (2.3.20)$$

其中

$$\Gamma(z) = I + \sum_{k,j=1}^N \frac{w_k(M^{-1})_{kj}w_j^*}{z - z_j^*}, \quad \Gamma(z)^- = I - \sum_{k,j=1}^N \frac{w_k(M^{-1})_{kj}w_j^*}{z - z_k} \quad (2.3.21)$$

这里  $M$  为  $N \times N$  矩阵, 其  $(k, j)$  元素由下式给定

$$M_{kj} = \frac{w_k^* w_j}{z_k^* - z_j}, \quad k, j = 1, 2, \dots, N, \quad \det \Gamma(z) = \prod_{k=1}^N \frac{z - z_k}{z - z_k^*} \quad (2.3.22)$$

而  $\hat{P}_{\pm}$  为正则 RH 问题的唯一解, 且

1.  $\hat{P}_{\pm}$  在  $\mathbb{C}_+$  上解析,
2.  $\hat{P}_- \hat{P}_+ = \Gamma(z)G\Gamma^{-1}(z), z \in \mathbb{R}$

$$3. \hat{P}_{\pm}(z) \rightarrow I, \quad z \rightarrow \infty,$$

**证明** 非正则 RH 问题 (2.3.6) - (2.3.8) 是由于  $2N$  个离散谱上的  $\det P_+(z_j) = \det P_-(z_j^*) = 0 (j = 1, 2, \dots, N)$  造成的. 所以主要任务是消除这些零点结构, 并且去除  $P_+(z_j), P_-(z_j^*)$  在  $z_j, z_j^*$  上的零点结构, 为此需定义单极点矩阵

$$\Gamma_1(z) = I + \frac{z_1^* - z_1}{z - z_1^*} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \quad (2.3.23)$$

其具有如下性质

$$\begin{aligned} F_1^{-1}(z) &= I - \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \\ \det \Gamma_1(z) &= \frac{z - z_1}{z - z_1^*}, \quad \det \Gamma_1^{-1}(z) = \frac{z - z_1^*}{z - z_1} \\ \Gamma_1(z_1) w_1 &= w_1 + \frac{z_1^* - z_1}{z_1 - z_1^*} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \cdot w_1 = \frac{z_1^* - z_1}{z_1 - z_1^*} \frac{w_1 (w_1^* w_1)}{w_1^* w_1} = w_1 - w_1 = 0 \\ w_1^* \Gamma_1^{-1}(z_1^*) &= w_1^* - \frac{z_1^* - z_1}{z_1^* - z_1} \cdot \frac{w_1^* w_1}{w_1^* w_1} \cdot w_1^* = w_1^* - w_1^* = 0 \end{aligned} \quad (2.3.24)$$

令  $x_j = \frac{w_1 w_1^*}{w_1^* w_1}$ , 则  $x_j$  为投影算子, 即  $x_j^2 = x_j$ , 由上定义知  $x_j$  为一阶矩阵, 故与矩阵  $\text{diga}\{1, 0\}$  相似, 既有可逆阵  $T_j$  使得  $T_j^{-1} x_j T_j = \text{diag}\{1, 0\}$ , 从而有

$$\Gamma_1(z) = \det \left( I + \frac{z_1^* - z_1}{z - z_1^*} T_j^{-1} x_j T_j \right) = \begin{vmatrix} 1 + \frac{z_1^* - z_1}{z - z_1^*} & 0 \\ 0 & 1 \end{vmatrix} = \frac{z - z_1}{z - z_1^*} \quad (2.3.25)$$

定义矩阵函数  $R_1^+(z) = P_+(z) \Gamma_1^{-1}(z)$ ,  $R_1^-(z) = \Gamma_1(z) P_-(z)$ , 由 (2.3.18) - (2.3.19) 知

$$\begin{aligned} \text{Res}_{z=z_1} R_1^+(z) &= \text{Res}_{z=z_1} \left( P_+(z) - P_+(z) \frac{z_1^* - z_1}{z - z_1} \frac{w_1 w_1^*}{w_1^* w_1} \right) = -\frac{z_1^* - z_1}{w_1 w_1^*} (P_+(z) w_1) w_1^* = 0 \\ \text{Res}_{z=z_1^*} R_1^-(z) &= \text{Res}_{z=z_1^*} \left( P_1(z) + \frac{z_1^* - z_1}{z - z_1} \frac{w_1 w_1^*}{w_1^* w_1} P_-(z) \right) = -\frac{z_1^* - z_1}{w_1^* w_1} w_1 (w_1^* P_-(z_1^*)) = 0 \end{aligned} \quad (2.3.26)$$

因此  $R_1^+(z), R_1^-(z)$  分别在  $\mathbb{C}_+, \mathbb{C}_-$  上解析, 且

$$\begin{aligned} \det R_1^+(z_1) &= \lim_{z \rightarrow z_1} [\det P_+(z) \cdot \det \Gamma_1^{-1}(z)] = \lim_{z \rightarrow z_1} S_{11}(z) \frac{z - z_1^*}{z - z_1} \\ &= \lim_{z \rightarrow z_1} \frac{S_{11}(z) - S_{11}(z_1)}{z - z_1} (z - z_1^*) \Leftarrow (S_{11}(z) = 0) \\ &= \lim_{z \rightarrow z_1} S'_{11}(z) (z - z_1^*) = 0 \end{aligned} \quad (2.3.27)$$

$$\det R_1^-(z_1^*) = \lim_{z \rightarrow z_1^*} [\det \Gamma_1(z) \cdot \det P_-(z)] = \lim_{z \rightarrow z_1^*} \frac{z - z_1}{z - z_1^*} S_{22}(z) = \lim_{z \rightarrow z_1^*} S'_{22}(z) (z - z_1) = 0$$

这说明  $R_1^+(z), R_1^-(z)$  分别在  $z = z_1, z = z_1^*$  不再具有零点结构, 然后去除其在  $z_2, z_2^*$  上的零点结构, 由于

$$\begin{aligned} \det R_1^+(z_2) &= \det P_+(z_2) \det \Gamma_1^{-1}(z_2) = S_{11}(z_2) \frac{z_2 - z_1^*}{z_2 - z_1} = 0 \\ \det R_1^-(z_2^*) &= \det \Gamma_1(z_2^*) \det P_-(z_2^*) = \frac{z_2^* - z_1}{z_2^* - z_1} S_{22}(z_2^*) = 0 \end{aligned} \quad (2.3.28)$$

因此, 下列齐次线性方程组有非零解, 即存在  $v_2(z_2), v_2^*(z_2^*)$ , 使得

$$R_1^+(z_2) v_2(z_2) = P_+(z_2) \Gamma_1^{-1}(z_2) v_2(z_2) = 0, \quad v_2^*(z_2^*) R_1^-(z_2^*) = v_2^*(z_2^*) \Gamma_1(z_2^*) P_-(z_2^*) = 0 \quad (2.3.29)$$

与 (2.3.18) 比较可得

$$w_2(z_2) = \Gamma_1^{-1}(z_2) v_2(z_2), \quad w_2^*(z_2^*) = v_2^*(z_2^*) \Gamma_1(z_2^*) \quad (2.3.30)$$

为去除  $R_1^+(z_2), R_1^-(z_2^*)$  在  $z_2, z_2^*$  上的零点结构, 令

$$\begin{aligned}\Gamma_2(z) &= I + \frac{z_2^* - z_2}{z - z_2^*} \cdot \frac{v_2 v_2^*}{v_2^* v_2}, \quad \det \Gamma_2(z) = \frac{z - z_2}{z - z_2^*} \\ \Gamma_2^{-1}(z) &= I - \frac{z_2^* - z_2}{z - z_2} \cdot \frac{v_2 v_2^*}{v_2^* v_2}, \quad \det \Gamma_2^{-1} = \frac{z - z_2^*}{z - z_2}\end{aligned}\quad (2.3.31)$$

$$\begin{aligned}R_2^+(z) &= P_1^+(z) \Gamma_2^{-1}(z) = P_1^+(z) \Gamma_1^{-1}(z) \Gamma_2^{-1}(z) \\ R_2^{-1}(z) &= \Gamma_2(z) P_1^{-1}(z) = \Gamma_2(z) \Gamma_1(z) P_1^{-1}(z)\end{aligned}$$

则

$$\operatorname{Res}_{z=z_2} R_2^+(z) = \operatorname{Res}_{z=z_2} \left[ \left( P_+(z) - P_+(z) \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{w_2^* w_1} \right) \left( I - \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{v_2^* v_1} \right) \right] = -\frac{z_2^* - z_1 R_1(z_2) v_2 v_2^*}{v_2^* v_2} \quad (2.3.32)$$

同理  $\operatorname{Res}_{z=z_2^*} R_2^{-1}(z) = 0$ , 由此可得  $R_2^+(z), R_2^{-1}(z)$  分别在在  $z = z_1, z_2, z = z_1^*, z_2^*$  上无零点结构, 则

$$\begin{aligned}\det R_2^+(z_j) &= \lim_{z \rightarrow z_j} [\det P_+(z) \det \Gamma_1^{-1}(z) \det \Gamma_2^{-1}(z)] \\ &= \lim_{z \rightarrow z_j} S_{11}(z) \frac{z - z_1^*}{z - z_1} \cdot \frac{z - z_2^*}{z - z_2} \\ &= s'_{11}(z_j) \frac{(z_j - z_1^*)(z_j - z_2^*)}{z_j - z_k} \neq 0\end{aligned}\quad (2.3.33)$$

$$\det R_2^-(z_j^*) \neq 0 \quad (k = 1, 2, k \neq j)$$

更一般的, 可得

$$\begin{aligned}w_j &= \Gamma_1^{-1}(z_j) \cdots \Gamma_{j-1}^{-1}(z_j) \cdot v_j(z_j) \\ w_j^* &= v_j^*(z_j^*) \cdot \Gamma_1(z_j^*) \cdots \Gamma_{j-1}(z_j^*) \\ R_j^+(z) &= P_+(z) \cdot \Gamma_1^{-1}(z) \cdots \Gamma_j^{-1}(z) \\ R_j^{-1}(z) &= \Gamma_j(z) \cdots \Gamma_1(z) \cdot P_-(z)\end{aligned}\quad (2.3.34)$$

其中

$$\Gamma_j(z) = I + \frac{z_j^* - z_j}{z - z_j^*} \cdot \frac{v_j v_j^*}{v_j^* v_j}, \quad \Gamma_j^{-1}(z) = I - \frac{z_j^* - z_j}{z - z_j} \cdot \frac{v_j v_j^*}{v_j^* v_j} \quad (2.3.35)$$

则  $R_j^+(z), R_j^{-1}(z)$  在  $\mathbb{C}_+, \mathbb{C}_-$  上解析, 且在  $z_k, z_k^* (k = 1, 2, \dots, j)$  上无零点结构, 即

$$\begin{aligned}\det R_j^+(z_k) &= s'_{11}(z_k) \frac{\prod_{l=1}^j (z_k - z_l^*)}{\prod_{l=1, l \neq k}^j (z_k - z_l)} \neq 0 \\ \det R_j^{-1}(z_k^*) &= s'_{22}(z_k^*) \frac{\prod_{l=1}^j (z_k^* - z_l)}{\prod_{l=1, l \neq k}^j (z_k^* - z_l)} \neq 0\end{aligned}\quad (k = 1, 2, \dots, j) \quad (2.3.36)$$

而

$$\begin{aligned}\det R_j^+(z_{j+1}) &= S_{11}(z_{j+1}) \prod_{j=1}^j \frac{z_{j+1} - z_k^*}{z_{j+1} - z_k} = 0 \\ \det R_j^{-1}(z_{j+1}^*) &= S_{22}(z_{j+1}^*) \prod_{j=1}^j \frac{z_{j+1}^* - z_k}{z_{j+1}^* - z_k} = 0\end{aligned}\quad (2.3.37)$$

最后令

$$\begin{aligned}\Gamma_z &= \Gamma_N(z) \cdots \Gamma_1(z), \quad \Gamma_z^{-1} = \Gamma_1^{-1}(z) \cdots \Gamma_N^{-1}(z) \\ \hat{P}_+(z) &= P_+(z) \Gamma_z^{-1}(z), \quad \hat{P}_-(z) = \Gamma_z(z) P_-(z)\end{aligned}\quad (2.3.38)$$

则  $\hat{P}_+(z), \hat{P}_-(z)$  在  $\mathbb{C}_+, \mathbb{C}_-$  上解析, 且在  $z_j, z_j^* (j = 1, 2, \dots, N)$  上无零点结构 (实际上在解析区域内无零点结构), 即

$$\det \hat{P}_+(z_k) \neq 0, \quad \det \hat{P}_-(z_k^*) \neq 0 \quad (k = 1, 2, \dots, N) \quad (2.3.39)$$

得到分解 (2.3.20). 下证 (2.3.21). 注意到  $\Gamma(z), \Gamma^{-1}(z)$  分别为具有单极点  $z_j, z_j^* (1 \leq j \leq N)$  的亚纯函数, 以及分解



(2.3.35), (2.3.38), 寻找的向量使得

$$\Gamma(Z) = I + \sum_{j=1}^N \frac{\xi_j w_j^*}{z - z_j^*}, \quad \Gamma^{-1}(z) = I - \sum_{j=1}^N \frac{w_j^* \xi_j}{z - z_j} \quad (2.3.40)$$

注意到  $\Gamma(z)\Gamma^{-1}(z) = I$  对任意  $z$  都成立, 当然也在  $z = z_k$  处成立, 即  $\Gamma(z)\Gamma^{-1}(z) = I$  在  $z = z_k$  处正则, 为保证  $\Gamma(z)\Gamma^{-1}(z) = I$  在  $z = z_k$  处成立, 只需让其留数为 0, 因此利用 (2.3.40) 可得

$$\begin{aligned} 0 = \text{Res}_{z=z_k} [\Gamma(z)\Gamma^{-1}(z)] &= \text{Res}_{z=z_k} \left( \Gamma(z) - \sum_{j=1}^N \Gamma(z) \frac{w_j^* \xi_j}{z - z_j} \right) = -\Gamma(z_k) w_k \xi_k^* \\ &- \left( I + \sum_{j=1}^N \frac{\xi_j w_j^*}{z_k - z_j^*} \right) w_k \xi_k^* = \left( -w_k + \sum_{j=1}^N \frac{w_j^* w_k}{z_j^* - z_k} \xi_j^* \right) \xi_k^* \end{aligned} \quad (2.3.41)$$

对上式两边同时作用  $\xi_k$ , 则有

$$\left( -w_k + \sum_{j=1}^N \frac{w_j^* w_k}{z_j^* - z_k} \xi_j \right) |\xi_k|^2 = 0 \implies -w_k + \sum_{j=1}^N \frac{w_j^* w_k}{z_j^* - z_k} \xi_j = 0 \quad (1 \leq k \leq N) \quad (2.3.42)$$

将上式改写为  $\xi_1, \xi_2, \dots, \xi_N$  的分块矩阵形式的线性方程组, 则有

$$(\xi_1, \xi_2, \dots, \xi_N) M = (w_1^*, w_2^*, \dots, w_N^*) \quad (2.3.43)$$

其中  $M = (M_{kj})_{N \times N}$ , 其中  $M_{kj} = \frac{w_k^* w_j}{z_k^* - z_j}$  则

$$\begin{cases} \frac{w_1^* w_1}{z_1^* - z_1} \xi_1 + \frac{w_2^* w_1}{z_2^* - z_1} \xi_2 + \dots + \frac{w_N^* w_1}{z_N^* - z_1} \xi_N = w_1 \\ \frac{w_1^* w_2}{z_1^* - z_2} \xi_1 + \frac{w_2^* w_2}{z_2^* - z_2} \xi_2 + \dots + \frac{w_N^* w_2}{z_N^* - z_2} \xi_N = w_2 \\ \vdots \\ \frac{w_1^* w_N}{z_1^* - z_N} \xi_1 + \frac{w_2^* w_N}{z_2^* - z_N} \xi_2 + \dots + \frac{w_N^* w_N}{z_N^* - z_N} \xi_N = w_N \end{cases} \implies M = \begin{pmatrix} \frac{w_1^* w_1}{z_1^* - z_1} & \frac{w_1^* w_2}{z_1^* - z_2} & \dots & \frac{w_1^* w_N}{z_1^* - z_N} \\ \frac{w_2^* w_1}{z_2^* - z_1} & \frac{w_2^* w_2}{z_2^* - z_2} & \dots & \frac{w_2^* w_N}{z_2^* - z_N} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{w_N^* w_1}{z_N^* - z_1} & \frac{w_N^* w_2}{z_N^* - z_2} & \dots & \frac{w_N^* w_N}{z_N^* - z_N} \end{pmatrix} \quad (2.3.44)$$

可得  $\xi_j = \sum_{k=1}^N (M^{-1})_{kj} w_k$ . 最后将其代入 (2.3.40) 得到 (2.3.21) □

## 2.4 NLS 方程的 N 孤子解

### 2.4.1 矩阵向量解的时空演化

对方程 (2.3.18) 第一个式子两边分别对  $x, t$  求导, 可得

$$P_{+,x} w_j + P_+ w_{j,x} = 0, \quad P_{+,t} w_j + P_+ w_{j,t} = 0 \quad (2.4.1)$$

利用  $P_+$  的定义与 Lax 对 (2.1.11), (2.1.12) 可得

$$\begin{aligned} P_{+,x} &= \mu_{1,x} H_1 + \mu_{2,x} H_2 = (iz_j [\sigma_3, \mu_1] + P \mu_1) H_1 + (-iz_j [\sigma_3, \mu_2] + P \mu_2) H_2 \\ &- iz_j (\mu_1 \sigma_3 H_1 - \mu_2 \sigma_3 H_2 + \sigma_3 \mu_1 H_1 + \sigma_3 \mu_2 H_2) + P \mu_1 H_1 + P \mu_2 H_2 \\ &= -iz [\sigma_3, P_+] + P P_+ \end{aligned} \quad (2.4.2)$$

同理有

$$P_{+,t} = -iz_j^2 [\sigma_3, P_+] + Q P_+ \quad (2.4.3)$$

将 (2.4.2), (2.4.3) 代入 (2.4.1) 且由于  $P_+ w_j = 0, w_j P = 0$ , 有

$$(-iz_j [\sigma_3, P_+] + P P_+) w_j + P_+ w_{j,x} = 0 \implies iz_j P_+ \sigma_3 w_j + P_+ w_{j,x} = 0 \implies P_+ (w_{j,x} + iz_j \sigma_3 w_j) = 0 \quad (2.4.4)$$

同理有  $P_+ (w_{j,t} + iz_j^2 \sigma_3 w_j) = 0$ .

$$\begin{cases} w_{j,x} + iz_j \sigma_3 w_j = 0 \\ w_{j,t} + iz_j^2 \sigma_3 w_j = 0 \end{cases} \implies w_j = e^{-i\theta(z_j) \sigma_3} w_{j,0}, \quad (j = 1, 2, \dots, N) \quad (2.4.5)$$

其中  $w_{j,0}$  为 2 维常向量, 从而  $w_j^* = w_{j,0}^\dagger e^{i\theta(z_j^*)\sigma_3}$ .

### 2.4.2 N 维孤子解公式

已知  $P_- P_+ = G \implies P_+^{-1} - P_- = \hat{G} P_+$  (其中  $I - G = \hat{G}$ ), 且  $\hat{P}_-(z) \hat{P}_+(z) = \Gamma(z) G \Gamma^{-1}(z) (z \in \mathbb{R})$ , 可得

$$\begin{aligned} \hat{P}_+^{-1} - \hat{P}_- &= (I - \hat{P}_- \hat{P}_+) \hat{P}_+^{-1} = \left( I - \Gamma(z) G \Gamma^{-1}(z) \right) \hat{P}_+^{-1} = (\Gamma(z) \Gamma^{-1}(z) - \Gamma(z) G \Gamma^{-1}(z)) \hat{P}_+^{-1} \\ &= \Gamma(z) (I - G) \Gamma^{-1}(z) \hat{P}_+^{-1} = \Gamma(z) \hat{G} \Gamma^{-1}(z) \hat{P}_+^{-1} \end{aligned} \quad (2.4.6)$$

由 Taylor 公式  $\frac{1}{s-z} = -\frac{1}{z} \left( \frac{1}{1-s/z} \right) = -\frac{1}{z} \left( 1 + \frac{s}{z} + \dots \right)$ , 故 Plemelj 公式可写为

$$\begin{aligned} \hat{P}_+^{-1} &= I + \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\Gamma(z) \hat{G} \Gamma^{-1}(z) \hat{P}_+^{-1}}{s-z} ds \\ &= I - \frac{1}{2\pi i z} \int_{-\infty}^{+\infty} \left( 1 + \frac{s}{z} + \left( \frac{s}{z} \right)^2 + \dots \right) \Gamma(z) \hat{G} \Gamma^{-1}(z) \hat{P}_+^{-1} ds \\ &= I - \frac{1}{2\pi i z} \int_{-\infty}^{+\infty} \Gamma(z) \hat{G} \Gamma^{-1}(z) \hat{P}_+^{-1} ds + O(z^{-2}) \end{aligned} \quad (2.4.7)$$

由  $(I - A)^{-1} = I + A + A^2 + \dots$ ,  $(P^{-1} = I + \frac{A}{z} + \dots, P = \frac{B}{z} + \dots)$ , 可得

$$\hat{P}_+ = I + \frac{1}{2\pi i z} \int_{-\infty}^{+\infty} \Gamma(z) \hat{G} \Gamma^{-1}(z) \hat{P}_+ ds + O(z^{-2}) \quad (2.4.8)$$

再由

$$\Gamma(z) = i + \sum_{k,j=1}^N \frac{w_k (M^{-1})_{kj} w_j^*}{z - z_j^*} \implies \Gamma(z) = I + \frac{1}{z} \sum_{k,j=1}^N w_k (M^{-1})_{kj} w_j^* + O(z^{-2}) \quad (2.4.9)$$

将上渐进式代入 (2.3.20), 比较  $z^{-1}$  的次数可得

$$P_+^{(1)} = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Gamma(z) \hat{G} \Gamma^{-1}(z) \hat{P}_+^{-1} ds = \sum_{k,j=1}^N w_k (M^{-1})_{kj} w_j^* \quad (2.4.10)$$

特别的, 当散射数据  $S_{12} = S_{21} = 0$  时, 有  $\hat{G} = I - G = 0$ . 故上式 (2.4.10) 可简化为

$$P_+^{-1} = \sum_{k,j=1}^N w_k (M^{-1})_{kj} w_j^* \quad (2.4.11)$$

不妨取  $\lambda_j = -i(z_j x + 2z_j^2 t)$ , 并取  $w_{j,0} = (c_j, 1)^T$ , 则有

$$w_j = \begin{pmatrix} e^{\lambda_j} \\ e^{-\lambda_j} \end{pmatrix} \begin{pmatrix} c_j \\ 1 \end{pmatrix} = \begin{pmatrix} c_j e^{\lambda_j} \\ e^{-\lambda_j} \end{pmatrix} \quad (2.4.12)$$

从而  $w_j^* = w_{j,0}^* \cdot e_j^{\lambda} \sigma_3 = (c_j^* e^{\lambda_j^*}, e^{-\lambda_j})$ . 故

$$M_{k,j} = \frac{w_k^* w_j}{z_k^* - z_j} = \frac{1}{z_k^* - z_j} (c_k^* c_j e^{\lambda_k^* + \lambda_j} + e^{-\lambda_k - \lambda_j}) \quad (2.4.13)$$

再由 (2.4.9)

$$\begin{aligned} q(x, t) &= 2i \lim_{z \rightarrow \infty} (z P_+)_{12} = 2i (P_+^{(1)})_{12} \\ &= 2i \sum_{k,j}^N (w_k w_j^*)_{12} (M^{-1})_{kj} = 2i \sum_{k,j}^N c_k e^{\lambda_k - \lambda_j^*} (M^{-1})_{kj} \end{aligned} \quad (2.4.14)$$

令

$$R = \begin{pmatrix} 0 & c_1 e^{\lambda_1} & \dots & c_N e^{\lambda_N} \\ e^{-\lambda_1^*} & M_{11} & \dots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\lambda_N^*} & M_{N1} & \dots & M_{NN} \end{pmatrix} \quad (2.4.15)$$

则

$$\begin{aligned}
\det R &= \sum_{k=1}^N (-1)^{k+2} c_k e^{\lambda_k} \det \begin{pmatrix} e^{-\lambda_1^*} & M_{11} & \cdots & M_{1,k-1} & M_{1,k+1} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e^{-\lambda_N^*} & M_{N1} & \cdots & M_{N,k-1} & M_{N,k+1} & \cdots & M_{NN} \end{pmatrix} \\
&= \sum_{k=1}^N (-1)^{k+2} c_k e^{\lambda_k} \sum_{j=1}^N e^{-\lambda_j^*} \det \begin{pmatrix} M_{11} & \cdots & M_{1,k-1} & M_{1,k+1} & \cdots & M_{1N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{N1} & \cdots & M_{N,k-1} & M_{N,k+1} & \cdots & M_{NN} \end{pmatrix} := \Delta \\
&= \sum_{k=1}^N (-1)^{k+j+3} c_k e^{\lambda_k - \lambda_j^*} \det(\Delta) \\
&= -c_k e^{\lambda_k - \lambda_j^*} (M^*)_{jk} \Leftrightarrow \left( (M^*)_{jk} - (-1)^{k+j} \det(\Delta) \right) \\
&= -\det M \sum c_k e^{\lambda_k - \lambda_j^*} \Leftrightarrow \left( (M^{-1})_{jk} (M^*) = |M| M^{-1} \right)
\end{aligned} \tag{2.4.16}$$

因此

$$\sum_{k=1}^N c_k e^{\lambda_k - \lambda_j^*} (M^{-1})_{jk} = -\frac{\det R}{\det M} \tag{2.4.17}$$

将 (2.4.17) 代入 (2.4.14) 可得 NLS 方程的 N 孤子解

$$q = -2i \frac{\det R}{\det M} \tag{2.4.18}$$

## 第 3 章 Reverse Time Space NLS

This chapter mainly introduces three inverse problems of nonlocal NLS, mainly referring to Yang's article[3]

### 3.1 The coupled Schrödinger equations

Consider the rverse-space NLS equation

$$iq_t(x, t) + q_{xx}(x, t) + 2q^2(x, t)q^*(-x, t) = 0 \quad (3.1.1)$$

reverse-time NLS equation

$$iq_t(x, t) + q_{xx}(x, t) + 2q^2(x, t)q^*(x, -t) = 0 \quad (3.1.2)$$

and the reverse-space-time NLS equation

$$iq_t(x, t) + q_{xx}(x, t) + 2q^2(x, t)q^*(-x, -t) = 0 \quad (3.1.3)$$

This equations can be derived form the following member of the AKNS hierarchy - the coupled NLS equations

$$iq_t + q_{xx} - 2q^2r = 0, \quad ir_t - r_{xx} - 2r^2q = 0. \quad (3.1.4)$$

Under reductions

$$r(x, t) = -q^*(-x, t), \quad (3.1.5)$$

$$r(x, t) = -q(x, -t), \quad (3.1.6)$$

$$r(x, t) = -q(-x, -t), \quad (3.1.7)$$

these coupled equations reduce to the reverse-space NLS equation (3.1.1), the reverse-time NLS equation (3.1.2) and the reverse-space-time NLS equation (3.1.3) respectively.

### 3.2 Symetry relations of scattering data in the nonlocal NLS equations

We first present symmetry relations of the secattering data for the reverse-space NLS equation (3.1.1) and the reverse-time NLS equation (3.1.2). The symmetry relations of the scattering data for the reverse-space-time NLS equation (3.1.3) can be obtained in a similar way. For this pourpose, we first introduce some notations. We define

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (3.2.1)$$

which is a Paili spin matrix, and use the

#### 3.2.1 The reverse-space NLS equation

##### 定理 3.1

For the reverser-space NLS equation (3.1.1), if  $\zeta$  is an eigenvalue, so is  $-\zeta^*$ . Thus, non-pirely-imaginary eigenvalues appear as pairs  $(\zeta, -\zeta^*)$ , which lie in the same half of the complex plane. Symmetry ralations on the eigenvectors are given as follows:

1. If  $(\zeta_k, \hat{\zeta}_k) \in \mathbb{C}_+$ , then  $\hat{\zeta}_k = -\zeta_k^*$ , their column eigenvectors are related as  $\hat{\mathbf{v}}_{k0} = \sigma_1 \mathbf{v}_{k0}^*$ .
2. If  $\zeta_k \in i\mathbb{R}_+$ , its eigenvectors is of the form  $\mathbf{v}_{k0} = \begin{bmatrix} 1 & e^{i\theta_k} \end{bmatrix}^T$ , where  $\theta_k$  is a real constant.
3. If  $(\bar{\zeta}_k, \hat{\bar{\zeta}}_k) \in \mathbb{C}_-$ , then  $\hat{\bar{\zeta}}_k = -\bar{\zeta}_k^*$ , their row eigenvectors are related as  $\hat{\mathbf{v}}_{k0} = \bar{\mathbf{v}}_{k0}^* \sigma_1$ .
4. If  $\bar{\zeta}_k \in i\mathbb{R}_-$ , its eigenvectors is of the form  $\bar{\mathbf{v}}_{k0} = \begin{bmatrix} 1 & e^{i\bar{\theta}_k} \end{bmatrix}$ , where  $\bar{\theta}_k$  is a real constant.



To proof these results in perspective, we recall that for the local NLS equation,

$$iq_t + q_{xx} \pm 2q^2 q^* = 0 \quad (3.2.2)$$

which is obtained from the coupled Schrödinger equations (3.1.4) under the reduction of  $r(x, t) = -q^*(x, t)$ , the symmetry of its scattering data are  $\bar{\zeta}_k = -\zeta_k^*$  and  $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^*$ .

Thus, symmetry relations for the nonlocal NLS equations are different from those local NLS equations. In particular, for the reverse-space and reverse-space-time NLS equations, eigenvalues in the upper and lower halves of the complex plane are completely independent. This independence allows for novel eigenvalue configurations, which will give rise to new types of multi-solitons. This will be demonstrated in the next section.

Before proving this theorem, we first establish a connection between the discrete scattering data for N-solitons  $\{\zeta_k, \bar{\zeta}_k, a_k, b_k, \bar{a}_k, \bar{b}_k\} (1 \leq k \leq N)$  and discrete eigenmodes in the eigenvalue problem  $Y_x = MY$  and its adjoint problem  $K_x = -KM$ , where we set

$$Y_x = -i\zeta \Lambda Y + QY, \quad (3.2.3)$$

$$K_x = i\zeta \Lambda K + KQ \quad (3.2.4)$$

where the potential matrix  $Q$  is given by

$$Q = \begin{pmatrix} 0 & q(x, 0) \\ r(x, 0) & 0 \end{pmatrix} \quad (3.2.5)$$

and  $q(x, 0), r(x, 0)$  are the initial conditions of functions  $q(x, t), r(x, t)$  at  $t = 0$ . Indeed  $\forall \{\zeta_k, a_k, b_k\}$  of the discrete scattering data, where  $\zeta \in \mathbb{C}_+$  is the eigenvalue of (3.2.3), whose discrete eigenfunction  $Y_k$  has the following asymptotics

$$Y_k(x) \rightarrow \begin{bmatrix} a_k e^{-i\zeta_k x} \\ 0 \end{bmatrix}, \quad x \rightarrow -\infty, \quad Y_k(x) \rightarrow \begin{bmatrix} 0 \\ -b_k e^{i\zeta_k x} \end{bmatrix}, \quad x \rightarrow +\infty \quad (3.2.6)$$

Analogously, for the eigenvalue  $\bar{\zeta} \in \mathbb{C}_-$  of the adjoint eigenvalue problem (3.2.4), the discrete eigenfunction  $K_k$  has the following asymptotics

$$K_k(x) \rightarrow \begin{bmatrix} \bar{a}_k e^{-i\bar{\zeta}_k x} & 0 \end{bmatrix}, \quad x \rightarrow -\infty, \quad K_k(x) \rightarrow \begin{bmatrix} 0 & -\bar{b}_k e^{-i\bar{\zeta}_k x} \end{bmatrix}, \quad x \rightarrow +\infty \quad (3.2.7)$$

In view of this connection, in order to derive symmetry relations on the (discrete) scattering data

### 3.2.2 The reverse-time NLS equation

### 3.2.3 The reverse-space-time NLS equation

## 参考文献

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