Preface

Contents

本笔记旨在分享笔者对 Riemann-Hilbert 方法的理解, RH 方法作为一种强大的数学工具, 在非线性偏微分方程的求解, 尤其是可积系统中具有重要的应用价值. 通过对其理论背景, 基本方法及具体应用的学习, 笔者希望为读者提供一个清晰的入门指引。

主要内容如下:

第一章主要介绍 RH 方法的背景知识, 包括 Plemelj 定理, RH 问题等.

第二章主要介绍利用 RH 方法求解零边界的 NLS 方程, 通过构造特征函数, 分析其解析性与对称性, 以及建立相关的 RH 问题，最终推导出 NLS 方程的 N 孤子解. 本章节主要参考了复旦大学范恩贵老师的讲义 [4]. 这一章节感谢我的同门韩刻蓉的帮助与讲解.

第三章主要介绍利用 RH 方法求解反时间, 反空间, 及反时空的反演, 这些方程是耦合 NLS 方程在不同约束条件下的特殊形式。通过分析其散射数据的对称性，进一步推导出这些非局部方程的 N 孤子解. 本章节主要参考了杨建科老师的论文 [2]

第四章为在第三章的基础上, 进一步推广到三维的反时空 NLS 方程. 通过分析其散射数据的对称性，推导出三维反时空 NLS 方程的 N 孤子解. 本章节构成了我的第一篇论文.

感想

本笔记为在老家过年期间编写, 整个过程并不算顺利, 老家的房子年久失修, 塌了外墙和两间屋顶, 白天需要和父亲一起修房子, 应付家庭琐事; 只有夜间才能缓慢的推进写作. 另外农村没有暖气, 零下十余度的环境实难称舒适, 需要字面意义上的争分夺秒. 尤其深夜伏案时, 恍若独行于漫长甬道, 徘徊在黑暗迷宫之中. 然正如 A. Zee 所言, 夜航人自有夜行法 [3]. 历时一月的艰辛写作中, 笔者感到一种难以言喻的, 某种漠然的相互理解, 如越过高墙, 漫步在满月下的林地, 并在夜色中获得慰籍. 是邪，非邪？解释或属妄诞，感受毕竟真实.

因仅为一家之言, 仓促间完成又后期无他人审校, 难免存在疏漏, 错误和挂一漏万之处. 望各位读者在阅读时能指出不足, 共同探讨与完善.

序曲将终，敬无穷的远方，与无尽的人们.

本笔记存档于 https://github.com/IceySwan/Notes, 如发现任何错误请提 issue 或联系 hi@icey.one

Icey Swan
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目录

第 1 章 预备知识

1.1 Plemelj 公式

定理 1.1 (Abel 定理)

设 $A \in \mathbb{C}$, 对矩阵微分方程 $Y_x = A(x)Y$, 可得标量微分方程

$$(\det Y)_x = \operatorname{tr}(A) \det Y \tag{1.1.1}$$

从而有

$$\det\left(Y(x)\right) = \det\left(Y(x_0)\right) \cdot \exp \int_{x_0}^{x} \mathrm{tr(A(t))dt} \tag{1.1.2}$$

定理 1.2 (Morera)

如果 $f(z)$ 在单连通区域 D 内连续, 且对于 D 内任意闭曲线 $\sum$, 有 $\int_{\sum} f(z)\mathrm{d}z = 0$, 则有 $f(z)$ 在 D 内解析

定义 1.1 (Schwartz 空间)

欧几里得空间 $\mathbb{R}^n$ 上的 Schwartz 空间 $\mathcal{S}(\mathbb{R}^n)$ 定义为

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^\kappa) : \|f\|_{\alpha,\beta} = \sum_{x\in\mathbb{R}^n} |x^\alpha \partial_\beta f(x)| < \infty, \alpha, \beta \in \mathbb{Z} \right\} \tag{1.1.3}$$

其中 α, β 为多重指标, $\partial_\beta = \partial_{x_1}^{\beta_1} \cdots \partial_{x_n}^{\beta_n}$. f 称为速降函数或 Schwartz 函数.

简单来说，速降函数是指当 $x \to \infty$ 时趋近于零的速度比所有的多项式的倒数都快，并且任意阶的导数都有这种性质的函数。

定理 1.3 (Painleve 开拓定理)

设 D_1, D_2 为两个没有公共点的区域, 边界为 Γ, 并设 $f_1(z), f_2(z)$ 分别在 D_1, D_2 内解析, 在 $D_1 + \Gamma, D_2 + \Gamma$ 上连续, 且 $f_1(z) = f_2(z), \forall z \in \Gamma$, 则

$$f(z) = \begin{cases} f_1(z), & z \in D_1, \\ f_1(z) = f_2(z), & z \in \Gamma, \\ f_2(z), & z \in D_2 \end{cases} \tag{1.1.4}$$

在 $D_1 + D_2 + \Gamma$ 内解析.

证明 显然 $f(Z)$ 在 $D_1 + D_2 + \Gamma$ 上连续, 根据定理 1.2, 只需证明 $f(z)$ 沿任何闭曲线 $\sum$ 的积分为 0.

如果 $\sum \subset D_1$, 或 $\sum \subset D_1$, 则由 $f(z)$ 的解析性可得

$$\int_{\sum} f(z)\mathrm{d}z = 0 \tag{1.1.5}$$

如果 $\sum$ 同时包含于 D_1, D_2 内, 把 Γ 在 $\sum$ 内的曲线记为 C_γ, 则

$$\int_{C_1+C_\gamma} f(z)\mathrm{d}z = 0, \quad \int_{C_1+C_\gamma}^{-} f(z)\mathrm{d}z = 0 \tag{1.1.6}$$

故

$$\int_{\sum} f(z)\mathrm{d}z = \int_{C_1+C_2} f(z)\mathrm{d}z = \int_{C_1+C_\gamma} f(z)\mathrm{d}z = \int_{C_1+C_\gamma^-} f(z)\mathrm{d}z = 0. \tag{1.1.7}$$

□

引理 1.1

设 $f(\xi)$ 在 $z\in\sum$ 满足 μ 次 Hölder 条件, 且 $z'\to z$ 时, h/d 有界, 其中

$$h=|z'-z|,\quad d=\min_{\xi\in\sum}|\xi-z'|, \tag{1.1.8}$$

则

$$\lim_{z'\to z}\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z'}\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z}\mathrm{d}\xi \tag{1.1.9}$$

证明

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z'}\,\mathrm{d}\xi-\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi-z}\,\mathrm{d}\xi\\
=&\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{(\xi-z)(f(\xi)-f(z))-(\xi-z')(f(\xi)-f(z))}{(\xi-z')(\xi-z)}\,\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{z'-z}{\xi-z}\cdot\frac{f(\xi)-f(z)}{\xi-z}\,\mathrm{d}\xi\\
=&\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{h}{d}M\frac{(\xi-z)^{\mu}}{\xi-z}\,\mathrm{d}\xi=\Delta_1+\Delta_2
\end{aligned}\tag{1.1.10}$$

其中

$$\begin{aligned}
|\Delta_1|&\le\frac{hM}{2\pi d}\int_{C_\delta}\frac{(\xi-z)^{\mu}}{\xi-z}\,\mathrm{d}|\xi|\le\frac{hM}{2\pi d}\int_0^{\delta}t^{\mu-1}\,\mathrm{d}t=\frac{hM}{2\pi d\delta}\delta^{\mu}(C_\delta=\{|\xi-z|\le\delta\})\\
|\Delta_2|&=\left|\frac{1}{2\pi}\int_{\sum\backslash C_\delta}\frac{(f(\xi)-f(z))(z-z')}{\xi-z}\,\mathrm{d}\xi\right|
\end{aligned}\tag{1.1.11}$$

由于 $\sum\backslash C_\delta$ 不包含 z, 故 Δ_2 为关于 z' 的连续函数, 则

$$|\Delta_2|=\left|\frac{1}{2\pi\mathrm{i}}\int_{\sum\backslash C_\delta}\frac{f(\xi)-f(z)}{\xi-z'}\,\mathrm{d}\xi-\frac{1}{2\pi}\int_{\sum\backslash C_\delta}\frac{f(\xi)-f(z)}{\xi-z}\,\mathrm{d}\xi\right|<|\Delta_1| \tag{1.1.12}$$

取 $\delta\to 0$, 可令 $|\Delta_1|<\frac{\epsilon}{2}$, 故 $|\Delta_1|+|\Delta_2|<\epsilon$ □

定理 1.4 (Plemelj)

设 $z\in\sum$ 为正则点, 且不为边界点, $f(\xi)$ 在 z 点满足 μ 次 Hölder 条件, 且 $z'\to z$ 时, h/d 有界, 其中 $h=|z'-z|,\quad d=\min_{\xi\in\sum}|\xi-z'|$, 则

$$F_+=\lim_{\substack{z'\to z\\ z'\in\sum_+}}F(z')=F(z)+\frac{1}{2}f(z) \tag{1.1.13}$$

$$F_-=\lim_{\substack{z'\to z\\ z'\in\sum_-}}F(z')=F(z)-\frac{1}{2}f(z) \tag{1.1.14}$$

证明 首先证明闭区线情形:

$$\begin{aligned}
F_+&=\lim_{\substack{z'\to z\\ z'\in\sum_+}}F(z')=\lim_{z'-z}\int_{\sum}\frac{1}{2\pi\mathrm{i}}\frac{f(\xi)}{\xi-z'}\,\mathrm{d}\xi\\
&=\lim_{z'-z}\int_{\sum}\frac{1}{2\pi\mathrm{i}}\frac{f(\xi)-f(z)}{\xi-z'}\,\mathrm{d}\xi+\frac{f(\xi)}{2\pi\mathrm{i}}\int_{\sum}\frac{1}{\xi-z'}\,\mathrm{d}\xi\\
&=\lim_{z'-z}\frac{1}{2\pi\mathrm{i}}\int_{\sum_+}\frac{f(\xi)-f(z)}{\xi-z'}\mathrm{d}\xi+f(z)\\
&=\lim_{z'-z}\frac{1}{2\pi\mathrm{i}}\int_{\sum_+}\frac{f(\xi)}{\xi-z'}\mathrm{d}\xi-\frac{f(z)}{2\pi\mathrm{i}}\int_{\sum_+}\frac{1}{\xi-z'}\mathrm{d}\xi+f(z)\\
&=F(z)+\frac{1}{2}f(z)
\end{aligned}\tag{1.1.15}$$

$F_-(z')$ 同理, 下证 $\sum$ 为开曲线情形, 则可补充 $\sum'$, s.t. $\sum\cup\sum'$ 为闭曲线, 且定义 $\forall\xi\in\sum'$, $f(\xi)=0$, 则

$$F(z)=\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(z)}{\xi-z}\mathrm{d}\xi=\frac{1}{2\pi\mathrm{i}}\int_{\sum\cup\sum'}\frac{f(z)}{\xi-z}\mathrm{d}\xi. \tag{1.1.16}$$

则由开曲线情形可得

$$\begin{aligned}F_+(z) &= \lim_{\substack{z'\to z\\ z'\in+\sum}} \int_{\sum\cup\sum'} \frac{f(\xi)}{\xi - z}\mathrm{d}\xi = \int_{\sum\cup\sum'} \frac{f(\xi)-f(z)}{\xi - z}\mathrm{d}\xi + \frac{f(z)}{2\pi\mathrm{i}}\int_{\sum\cup\sum'}\frac{1}{\xi - z'}\mathrm{d}\xi \\ &= \frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi - z}\mathrm{d}\xi - \frac{f(z)}{2\pi\mathrm{i}}\int_{\sum'}\frac{1}{\xi - z}\mathrm{d}\xi + f(z) = F(z) + \frac{1}{2}f(z)\end{aligned} \tag{1.1.17}$$

$F_-(z')$ 同理 □

注 如果 $z \in \sum$ 为一个角点, 在其两切线的夹角为 α, 则可

$$\begin{aligned}F_+(z) &= \lim_{\substack{z'\to z\\ z'\in D}} \frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)}{\xi - z}\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi - z}\mathrm{d}\xi + \frac{\alpha}{2\pi}f(z) + (1-\frac{\alpha}{2\pi})f(z) \\ &= F(z) - \frac{\alpha}{2\pi}f(z)\end{aligned} \tag{1.1.18}$$

$$\begin{aligned}F_-(z) &= \frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi - z}\mathrm{d}\xi = \frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)-f(z)}{\xi - z}\mathrm{d}\xi + \frac{\alpha}{2\pi}f(z) - \frac{\alpha}{2\pi})f(z) \\ &= F(z) - \frac{\alpha}{2\pi}f(z)\end{aligned} \tag{1.1.19}$$

定义 1.2 (Plemelj 公式)

由 (1.1.13) - (1.1.14), (1.1.18) - (1.1.19) 可以看出, 无论 z 是正则点还是角点, 都有

$$F_+(z) - F_-(z) = f(z), \quad z \in \sum \tag{1.1.20}$$

$$F_+(z) - F_-(z) = \frac{1}{\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)}{\xi - z}\mathrm{d}\xi =: H(f)(z) \tag{1.1.21}$$

称为标量 RH 问题, 其解可用 Cauchy 积分给出

$$F(z) = A + \frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)}{\xi - z}\mathrm{d}\xi, \quad z \in \mathbb{C} \tag{1.1.22}$$

其中 A 为任意常数, 一般被边值或渐进条件决定, 这一公式被称为 Plemelj 公式.

对于如下 RH 问题

$$G_+(z) = G_-(z)v(z), \quad z \in \sum \tag{1.1.23}$$

只需两边取 log 变换则有

$$\log G_+(z) - \log G_-(z) = \log v(z), \quad z \in \sum \tag{1.1.24}$$

由 Plemelj 公式, 则有

$$\log G(z) = A + \frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{v(\xi)}{\xi - z}\mathrm{d}\xi \implies G(z) = B\exp\left(\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{f(\xi)}{\xi - z}\mathrm{d}\xi\right), \quad z \in \sum \tag{1.1.25}$$

其中 B 为任意常数.

1.2 矩阵 RH 问题

定义 1.3 (RH 问题)

设 $\sum$ 为复平面 $\mathbb{C}$ 内的有向路径, 假设存在一个 $\sum^0$ 上的光滑映射 $v(z): \sum \to GL(n, \mathbb{C})$, 则 $(\sum, v)$ 决定了一个 RH 问题, 寻找一个 n 阶矩阵 $M(z)$ 满足

$$M(z) \in C, \quad (\mathbb{C} \setminus \sum) \tag{1.2.1}$$

$$M_+(z) = M_-(z)v(z), \quad z \in \sum \tag{1.2.2}$$

$$M(z) \to I, \quad z \to \infty \tag{1.2.3}$$

其中 $M_{\pm}$ 表示在正负区域内 $z' \to z$ 时的极限, $\sum$ 称为跳跃曲线, $v(z)$ 称为跳跃矩阵.

根据 Beadls-Coifman 定理, 如上 RH 问题的解可通过如下方式构造: 不妨设跳跃矩阵 $v(z)$ 具有如下分解

$$v = (b_-)^{-1} b_+ \tag{1.2.4}$$

由此, 可构造

$$w_+ = b_+ - I, \quad w_- = I - b_- \tag{1.2.5}$$

进一步可定义 Cauchy 投影算子

$$(C_{\pm} f)(z) = \lim_{\substack{z' \to z \in \sum \\ z' \in \pm \sum}} \frac{1}{2\pi \mathrm{i}} \int_{\sum} \frac{f(\xi)}{\xi - z'} \mathrm{d}\xi \tag{1.2.6}$$

则可以证明如果 $f(z) \in L^2(\sum)$, 则 $C_{\pm} : L^2 \to L^2$ 的有界算子, 且 $C_+ - C_- = 1$, 再定义算子

$$C_w f = C_+(f w_-) - C_-(f w_+) \tag{1.2.7}$$

则 $C_w : L^2 \cap L^\infty \to L^2$ 的有界算子.

定理 1.5

设 $\det v = 1$, 算子 $I - C_w$ 在 $L^2(\sum)$ 上可逆, $\mu \in I + L^2(\sum)$ 为下列方程

$$(I - C_w)\mu = I \tag{1.2.8}$$

的解, 且

$$(I - C_w)(\mu - I) = C_w I + C_+ w_- + C_- w_+ \in L^2\left(\sum\right) \tag{1.2.9}$$

则

$$M(z) = I + \frac{1}{2\pi \mathrm{i}} \int_{\sum} \frac{\mu(\xi) w(\xi)}{\xi - z} \mathrm{d}\xi \tag{1.2.10}$$

为上述 RH 问题的唯一解, RH 问题 $M(z)$ 的可解等价于奇异积分方程 (1.2.8).

证明 只需证明 (1.2.10) 满足 (1.2.8)

$$\begin{aligned} M_+ &= I + \lim_{z' \to z \in \sum} \frac{1}{2\pi \mathrm{i}} \int_{\sum} \frac{\mu(\xi) w(\xi)}{\xi - z} \mathrm{d}\xi \overset{(1.2.6)}{=} I + C_+(\mu w) \\ &= I + C_+(\mu w_-) + C_+(\mu w_+) = I + C_+(\mu w_-) + C_-(\mu w_+) + C_+(\mu w_+) + C_-(\mu w_+) \\ &\overset{(1.2.7)}{=} I + C_w(\mu) + \mu w_+ \overset{(1.2.8)}{=} \mu + \mu w_+ = \mu(I + w_+) = \mu b_+ \end{aligned} \tag{1.2.11}$$

同理有

$$M_-(z) = \mu b_- \tag{1.2.12}$$

故有

$$M_+(z) = \mu b_+ = \mu b_- (b_-)^{-1} b_+ = M_-(z) v(z) \tag{1.2.13}$$

下证唯一性, 先证 M 可逆, 对 (1.2.2) 取行列式, 并注意到 $\det v(z) = 1$, 则 $\det M_+(z) = \det M_-(z)$. 故由 Painleve 开拓定理得 $\det M(z)$ 在 $\mathbb{C}$ 上解析. 由 (1.2.3) 得

$$\det M(z) \to 1, z \to \infty \tag{1.2.14}$$

故由 Liouville 定理可知 $\det M(z) = c$, 再由渐进条件得 $c = 1$, 故 $M(z)$ 可逆.

设 $\widetilde{M}$ 为上述 RH 问题的另一个解, 则 $\widetilde{M}$ 可逆, 且在 $\mathbb{C} \setminus \sum$ 上解析, 且在 $\sum$ 上满足

$$(M\widetilde{M}^{-1}) = M_+ \widetilde{M}_+^{-1} = M_- v (\widetilde{M}_- v)^{-1} = M_- (\widetilde{M}_-)^{-1} = (M\widetilde{M}^{-1})_- \tag{1.2.15}$$

由 Painleve 开拓定理, $M\widetilde{M}^{-1}$ 在 $\mathbb{C}$ 上解析. 另外由 $M,\widetilde{M}\to I$, 故 $M\widetilde{M}^{-1}$ 有界, 故为常矩阵. 由渐进性得

$$M\widetilde{M}^{-1}=I \implies M=\widetilde{M} \tag{1.2.16}$$

□

注 由 RH 问题解的唯一性, RH 问题 (1.2.1) - (1.2.3) 的解与跳跃矩阵 $v(z)$ 的分解无关, 因此可以考虑 $v(z)$ 的平凡解

$$b_-=I,\quad b_+=v \tag{1.2.17}$$

故可构造

$$\begin{gathered} w_-=0,\quad w_+=v-I,\quad w=v-I \\ C_wf=C_+(fw_-)+C_-(fw_+)=C_+(f(v-I)),\quad \mu=(I-C_w)^{-1}I \end{gathered} \tag{1.2.18}$$

则 RH 问题的解可表示为

$$M(z)=I+\frac{1}{2\pi\mathrm{i}}\int_{+\sum}\frac{\mu(\xi)(v(\xi)-I)}{\xi-z}\mathrm{d}\xi \tag{1.2.19}$$

注 RH 方法的关键思想就是改变积分路径, 通过跳跃矩阵的分解情况, 确定对积分路径进行一系列形变, 再取极限去除跳跃矩阵为单位阵的情形, 将其化解为可解的 RH 问题, 所以一个自然的问题就是: “为什么可以扔掉跳跃矩阵为单位阵的路径? 或者说为什么跳跃矩阵对对 RH 问题的解不产生贡献.”

这很容易通过表达式 (1.2.19) 看出, 我们将积分路径分解为 $\sum=\sum_1+\sum_2$ 且

$$v=\begin{cases} v_1\neq I & \sum_1, \\ v_1=I & \sum_2 \end{cases} \tag{1.2.20}$$

则在 $\sum_1$ 上, $v-I=v_1-I\neq 0$, 在 $\sum_2$ 上, $v-I=v_1-I=0$, 故

$$\begin{aligned} M(z)&=I+\frac{1}{2\pi\mathrm{i}}\int_{\sum}\frac{\mu(\xi)(v(\xi)-I)}{\xi-z}\mathrm{d}\xi \\ &=I+\frac{1}{2\pi\mathrm{i}}\int_{\sum_1}\frac{\mu(\xi)(v_1(\xi)-I)}{\xi-z}\mathrm{d}\xi+\frac{1}{2\pi\mathrm{i}}\int_{\sum_2}\frac{\mu(\xi)(v_2(\xi)-I)}{\xi-z}\mathrm{d}\xi \\ &=I+\frac{1}{2\pi\mathrm{i}}\int_{\sum_1}\frac{\mu(\xi)(v_1(\xi)-I)}{\xi-z}\mathrm{d}\xi \end{aligned} \tag{1.2.21}$$

易得 RH 问题 $(\sum,v(z))$ 的解与 $(\sum_1,v_1(z))$ 的解相同, 即跳跃矩阵为单位阵的路径 $\sum_2$ 对 RH 问题的解无贡献, 可以舍去. 只求 $\sum_1$ 上的 RH 问题

第 2 章 RH 方法求解零边界的 NLS 方程

2.1 聚焦 NLS 方程

2.1.1 特征函数

考虑聚焦 NLS 方程的初值问题

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q(x,t)|q(x,t)|^2 = 0 \tag{2.1.1}$$

$$q(x,0) = q_0(x) \in \mathcal{S}(\mathbb{R}) \tag{2.1.2}$$

其中 $\mathcal{S}(\mathbb{R})$ 表示 Schwartz 空间. NLS 方程 (2.1.1) 具有如下矩阵形式的 Lax 对

$$\psi_x + \mathrm{i}z\sigma_3\psi = P\psi \tag{2.1.3}$$

$$\psi_t + 2\mathrm{i}z^2\sigma_3\psi = Q\psi \tag{2.1.4}$$

其中

$$\sigma_3 = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix}, \quad P = \begin{pmatrix} & q \\ -q^* & \end{pmatrix}, \quad Q = \begin{pmatrix} \mathrm{i}|q|^2 & \mathrm{i}q_x \\ \mathrm{i}q_x^* & -\mathrm{i}|q|^2 \end{pmatrix} + 2zP$$

满足零曲率方程

$$(P - \mathrm{i}z\sigma_3)_t - \left(Q - 2\mathrm{i}z^2\sigma_3\right)_x + \left[P - \mathrm{i}z\sigma_3, Q - 2\mathrm{i}z^2\sigma_3\right] = 0 \tag{2.1.5}$$

2.1.2 渐进性

由于 $q_0(x) \in \mathcal{S}(\mathbb{R})$, 且 $q(x,t), q_x(x,t) \to 0, (x \to \infty)$, 故 x 足够大时, P, Q 可忽略, 从而 Lax 对 (2.1.3) - (2.1.4) 可近似为

$$\psi_x \sim -\mathrm{i}z\sigma_3\psi, \quad \psi_t \sim -2\mathrm{i}z^2\sigma_3\psi$$

可得渐进形式的 Jost 解 $\psi = e^{-\mathrm{i}\theta(z)\sigma_3}, (x \to \infty)$, 其中 $\theta(z) = zx + 2z^2t$. 下面为表示方便, 不妨记 $E := e^{\mathrm{i}\theta(z)\sigma_3}$, 做变换 $\mu(x,t,z) = \psi(x,t,z)E$, 则有

$$E^{-1} = e^{-\mathrm{i}\theta(z)\sigma_3}, \quad \mu(x,t,z) \to I(x \to \infty) \tag{2.1.6}$$

则可得

$$\mu_x = \psi_x E + \psi E(\mathrm{i}\sigma_3 z) \tag{2.1.7}$$

$$\mu_t = \psi_t E + \psi E(2\mathrm{i}\sigma_3 z^2) \tag{2.1.8}$$

带入 Lax 对 (2.1.3) - (2.1.4) 可得

$$\mu_x = (P\psi - \mathrm{i}z\sigma_3\psi)E + \psi E(\mathrm{i}\sigma_3 z) \tag{2.1.9}$$

$$\mu_t = (Q\psi - 2\mathrm{i}z^2\sigma_3\psi)E + \psi E(2\mathrm{i}\hat{\sigma}_3 z^2) \tag{2.1.10}$$

故有

$$\mu_x = P\mu - \mathrm{i}z[\sigma_3, \mu] \implies \mu_x + \mathrm{i}z[\sigma_3, \mu] = P\mu \tag{2.1.11}$$

同理可得

$$\mu_t + 2\mathrm{i}z^2[\sigma_3, \mu] = Q\mu \tag{2.1.12}$$

由 (2.1.11) 式可得

$$\mu_x + \mathrm{i}z\hat{\sigma}\mu = P\mu \tag{2.1.13}$$

$$e^{\mathrm{i}\theta(z)\hat{\sigma}_3}\mu_x + \mathrm{i}z\hat{\sigma}_3 e^{\mathrm{i}\theta(z)\hat{\sigma}_3}\mu = e^{\mathrm{i}\theta(z)\hat{\sigma}_3}P\mu \tag{2.1.14}$$

其中 $\hat{\sigma}_3 X = [\sigma_3, X], e^{\mathrm{i}\theta(z)\hat{\sigma}_3}X = EXE^{-1}$, 为表示方便不妨记 $\hat{E} = e^{\mathrm{i}\theta(z)\hat{\sigma}_3}$. 易得 Lax 对的全微分形式

$$\mathrm{d}(\hat{E}\mu) = \hat{E}[(P\mathrm{d}x + Q\mathrm{d}t)\mu] \tag{2.1.15}$$

为构造规范的 Riemann-Hilbert 问题, 即其解在 $z \to \infty$ 时渐进于单位阵, 现只需证明 $\mu \to I(z \to \infty)$.

证明 将 μ 在无穷远点 Taylor 展开

$$\mu = \mu^{(0)} + \frac{1}{z}\mu^{(1)} + \cdots = \mu^{(0)} + O(z^{-1}) \tag{2.1.16}$$

其中 $\mu^{(0)}, \mu^{(1)}$ 与 z 无关, 将上式代入 (2.1.11), (2.1.12) 比较 z 的次数可得

$$\begin{aligned} &\left[\mu^{(0)} + O(z^{-1})\right]_x + \mathrm{i}z\left[\sigma_3, \mu^{(0)} + \frac{\mu^{(1)}}{z} + O(z^{-2})\right] = P\left[\mu^{(0)} + O(z^{-1})\right] \\ &[\mu^{(0)} + O(z^{-1})]_t + 2\mathrm{i}z^2\left[\sigma_3, \mu^{(0)} + \frac{\mu^{(1)}}{z} + O(z^{-2})\right] = (Q_0 + 2zP)[\mu^{(0)} + O(z^{-1})] \end{aligned} \tag{2.1.17}$$

可得 x 部分:

$$\left.\begin{aligned} z: &\quad [\sigma_3, \mu^{(0)}] = 0 \\ z^0: &\quad \mu^{(0)} + 2\mathrm{i}z[\sigma_3, \mu^{(1)}] = P\mu^{(0)} \end{aligned}\right. \Longrightarrow \mu^{(0)} \text{为对角阵} \tag{2.1.18}$$

t 部分:

$$\left.\begin{aligned} z^2: &\quad [\sigma_3, \mu^{(0)}] = 0 \\ z: &\quad \mathrm{i}[\sigma_3, \mu^{(1)}] = P\mu^{(0)} \end{aligned}\right. \Longrightarrow \mu_x^{(0)} = 0 \tag{2.1.19}$$

故 $\mu^{(0)}$ 为与 x 无关的对角矩阵, 因此 (2.1.16) 式对 x, z 同时取极限, 并交换极限顺序可得

$$\lim_{z\to\infty}\lim_{x\to\infty}\mu = \lim_{x\to\infty}\lim_{z\to\infty}\left(\mu^{(0)} + O(z^{(-1)})\right) \tag{2.1.20}$$

利用 (2.1.6), (2.1.16) 式可得 $\mu^{(0)} = I$, 故 $\mu \to I(z \to \infty)$. □

再将 $\mu^{(0)} = I$ 带入 (2.1.19) 式比较矩阵对角元素得

$$q(x,t) = 2\mathrm{i}(\mu^{(1)})_{12} = 2\mathrm{i}\lim_{z\to\infty}(z\mu)_{12} \tag{2.1.21}$$

注 上式可将 NLS 方程与特征函数联系起来, 接下来将特征函数与 RH 问题建立联系. 从而 NLS 方程的解可用 RH 问题的解表示, 然后通过 RH 问题反接触 NLS 方程的解, 即

$$NLS \rightleftharpoons \text{特征函数} \rightleftharpoons RH\text{问题}$$

2.2 解析性与对称性

由于 (2.1.15) 为全微分形式, 积分与路径无关, 故选择两个特殊路径:

$$(-\infty, t) \to (x,t), \quad (+\infty, t) \to (x,t)$$

因此可获得 Lax 对 (2.1.11), (2.1.12) 的两个特征函数

$$\mu_1 = I + \int_{-\infty}^{x} e^{-\mathrm{i}z(x-y)\sigma_3}P\mu_1\mathrm{d}x, \quad \mu_2 = I - \int_{+\infty}^{x} e^{-\mathrm{i}z(x-y)\sigma_3}P\mu_2\mathrm{d}x \tag{2.2.1}$$

证明 由 (2.1.15) 式, 由 $(\hat{E}\mu)_x = \hat{E}P\mu$ 可得

$$\hat{E}\mu|_{-\infty}^{x} = \int_{-\infty}^{x}\hat{E}P\mu\mathrm{d}y \tag{2.2.2a}$$

$$LHS = \hat{E}\mu - \hat{E}I, \quad RHS = \int_{-\infty}^{x} e^{\mathrm{i}(zy+z^2t)\hat{\sigma}_3}P\mu\mathrm{d}y \tag{2.2.2b}$$

两边同时乘以 $\hat{E}^{-1}$, 即得 μ_1, 同理可得 μ_2 . □

显然, 其仍具有如下性质

$$\mu_1, \mu_2 \to I(x \to \pm\infty), \quad \mu_1, \mu_2 \to I(z \to +\infty) \tag{2.2.3}$$

由于 $\psi_1 = \mu_1 E^{-1}, \psi_2 = \mu_2 E^{-1}$, 为 Lax 对的两个解, 而 Lax 对为一节齐次线性方程组, 故这两个解线性相关, 故有

$$\mu_1(x,t,z) = \mu_2(x,t,z)\hat{E}S(z), \quad S(z) = \begin{pmatrix} S_{11}(z) & S_{12}(z) \\ S_{21}(z) & S_{22}(z) \end{pmatrix} \tag{2.2.4}$$

其中矩阵 $S(z)$ 与 x, t 无关, 称为谱函数矩阵.

再由 $\mu = \psi E$ 可知 $\det(\mu_j) = \det(\psi_j)$. 又因为 $tr(P - \mathrm{i}z\sigma_3) = tr(Q - 2\mathrm{i}z^2\sigma_3) = 0$, 由 Abel 定理可得

$$\det(\psi_j)_x = \det(\psi_j)_t = 0 \tag{2.2.5}$$

故有 $\det(\mu_j)_x = \det(\mu_j)_t = 0$, 这说明 $\det(\mu_j)$ 与 x, t 无关, 再由渐进性可得

$$\det(\mu_j) = \lim_{|x|\to\infty} \det(\mu_j) = 1, \quad (j = 1, 2) \tag{2.2.6}$$

故对 (2.2.4) 两边取行列式有 $\det S(z) = 1$.

2.2.1 解析性

下面我们考虑特征函数 μ_1, μ_1 和谱矩阵 $S(z)$ 的解析性, 记 μ_1 的第一二列分别为

$$\mu_1 = \begin{pmatrix} \mu_1^{11} & \mu_1^{12} \\ \mu_1^{21} & \mu_1^{22} \end{pmatrix} = (\mu_1^+, \mu_1^-) \tag{2.2.7}$$

则由积分方程 (2.2.1) 可得如下 Volterra 积分方程

$$\mu_1^+(x,t,z) = (1,0)^T - \int_{-\infty}^{x} \begin{pmatrix} 1 & \\ & e^{\mathrm{i}z(x-y)} \end{pmatrix} P\mu_1^+ \mathrm{d}y \tag{2.2.8}$$

$$\mu_1^-(x,t,z) = (0,1)^T - \int_{-\infty}^{x} \begin{pmatrix} e^{-\mathrm{i}z(x-y)} & \\ & 1 \end{pmatrix} P\mu_1^- \mathrm{d}y \tag{2.2.9}$$

对于上面两方程, 由于积分变量 $y < x$, 可得

$$e^{2\mathrm{i}Z(x-y)} = e^{2\mathrm{i}(x-y)\mathrm{Re}(z)}e^{-2\mathrm{i}(x-y)\mathrm{Im}(z)}, \quad e^{-2\mathrm{i}Z(x-y)} = e^{-2\mathrm{i}(x-y)\mathrm{Re}(z)}e^{2\mathrm{i}(x-y)\mathrm{Im}(z)}$$

因此当 $q(x) \in L^1(\mathbb{R})$ 时, 通过构造序列与 Neumann 级数, 可得 $\mu_1^\pm, \mu_1^-$ 分别在 $\mathbb{C}_\pm$ 解析性.

注 上面 Volterra 积分的推导: 考虑到 $e^{\hat{\sigma}_3}X = e^{\sigma_3}Xe^{-\sigma_3}$, 故有

$$\begin{aligned} \mu_1 &= I + \int_{-\infty}^{x} e^{-\mathrm{i}z(x-y)\hat{\sigma}_3} P\mu_1 \mathrm{d}y = I + \int_{-\infty}^{x} e^{-\mathrm{i}z(x-y)\sigma_3} P e^{\mathrm{i}z(x-y)\sigma_3}\mu_1 \mathrm{d}y \\ &= I + \int_{-\infty}^{x} \begin{pmatrix} & qe^{2\mathrm{i}z(x-y)} \\ -q^* e^{-2\mathrm{i}z(x-y)} & \end{pmatrix} \mu \mathrm{d}y \end{aligned}$$

故 $\mu_1^+ =, \quad \mu_1^- =$

证明 下证 μ_1^+ 的解析性.

Step1. 解的存在性: 事实上方程 (2.2.8) 有如下 Neumann 级数分解

$$\mu_1^+ = \sum_{n=0}^{\infty} c_n(x,z), \quad c_{n+1} = \int_{-\infty}^{x} \begin{pmatrix} 1 & \\ & e^{\mathrm{i}z(x-y)} \end{pmatrix} Pc_n(y,z)\mathrm{d}y \tag{2.2.10}$$

其分量形式为

$$c_{n+1}^{(1)}(x,z) = \int_{-\infty}^{x} q(y)c_n^{(2)}(y,z)\mathrm{d}y, \quad c_{n+1}^{(2)}(x,z) = \int_{-\infty}^{x} e^{2\mathrm{i}z(x-y)}p(y)c_n^{(1)}(y,z)\mathrm{d}y \tag{2.2.11}$$

其中 $c_{n+1} = (c_{n+1}^{(1)}, c_{n+1}^{(2)})^T, p = -q^*$. 由 $c_0^{(2)} = 0 \implies c_{2n+1}^{(1)} = c_{2n}^{(2)} = 0$, 故上面方程可简化为

$$c_{2n}^{(1)}(x,z) = \int_{-\infty}^{x} q(y)c_{2n-1}^{(2)}(y,z)\mathrm{d}y, \quad c_{2n+1}^{(2)}(x,z) = \int_{-\infty}^{x} e^{2\mathrm{i}z(x-y)}p(y)c_{2n}^{(1)}(y,z)\mathrm{d}y \tag{2.2.12}$$

对上面方程组进一步简化，引入如下恒等式

$$
\begin{aligned}
&\frac{1}{j!}\int_{-\infty}^{x}|f(\xi)|\left[\int_{-\infty}^{\xi}|f(\eta)|\right]^{j}\mathrm{d}\xi=\frac{1}{(j+1)!}\int_{-\infty}^{x}\frac{d}{d\xi}\left[\int_{-\infty}^{\xi}|f(\eta)|\right]^{j+1}\mathrm{d}\xi\\
=&\frac{1}{(j+1)!}\left[\int_{-\infty}^{x}|f(\xi)|d\xi\right]^{j+1}\qquad(f\in L^{1}(\mathbb{R}))
\end{aligned}
\tag{2.2.13}
$$

利用归纳法可证明当 $\mathrm{Im}(z)>0$ 时，有

$$
|c_{2n+1}^{(2)}|=\frac{u^{n+1}(x)}{(n+1)!}\frac{v^{n}(x)}{n!},\quad |c_{2n}^{(1)}|=\frac{u^{n}(x)}{n!}\frac{v^{n}(x)}{n!}
\tag{2.2.14}
$$

其中

$$
u(x)=\int_{-\infty}^{x}|p(y)|\mathrm{d}y,\quad v(x)=\int_{-\infty}^{x}|q(y)|\mathrm{d}y
\tag{2.2.15}
$$

事实上，当 $\mathrm{Im}(z)>0$ 时，$|e^{2\mathrm{i}z(x-y)}|\le 1$，利用 $c_0^{(1)}=1$，由 (2.2.14) 可得

$$
c_1^{(2)}(x,y)=\int_{-\infty}^{x}e^{2\mathrm{i}z(x-y)}p(y)\mathrm{d}y,\quad c_2^{(1)}(x,y)=\int_{-\infty}^{x}e^{2\mathrm{i}z(x-y)}q(y)c_1^{(2)}\mathrm{d}y
\tag{2.2.16}
$$

又由 $u(x)\ge 0, v(x)\ge 0$，有 $c_1^{(2)}\le v(x)$

$$
\begin{aligned}
|c_2^{(1)}(x,y)\le&\int_{-\infty}^{x}|q|c_1^{(2)}(y,z)\mathrm{d}y\le\int_{-\infty}^{x}|q|v(y)\mathrm{d}y=\int_{-\infty}^{x}v'(y)u(y)\mathrm{d}y\\
=&u(x)v(x)-\int_{-\infty}^{x}u'(y)v(y)\mathrm{d}y\le u(x)v(x)
\end{aligned}
\tag{2.2.17}
$$

进而可得

$$
\begin{aligned}
|c_3^{(2)}(x,z)|\le&\int_{-\infty}^{x}|p|c_2^{(1)}(y,z)\mathrm{d}y\le\int_{-\infty}^{x}|p|u(y)v(y)\mathrm{d}y\\
\le&\int_{-\infty}^{x}u'(y)u(y)v(y)\mathrm{d}y=\frac{1}{2}u^2(x)v(x)-\int_{-\infty}^{x}\frac{1}{2}u^2(x)v'(x)\mathrm{d}y\le\frac{1}{2}u^2(x)v(x)
\end{aligned}
\tag{2.2.18}
$$

不妨设对于 $k\le l$ 时满足 $|c_{2l+1}^{(2)}|\le\frac{u^{l+1}(x)}{(l+1)!}\frac{v^{l}(x)}{l!}, |c_{2l}^{(1)}|\le\frac{u^{l}(x)}{l!}\frac{v^{l}(x)}{l!}$，下证对于 $k=l+1$ 仍成立，利用 (2.2.13) 式可得

$$
|c_{2l+2}^{(1)}|\le\int_{-\infty}^{x}|q(y)|\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y\le\int_{-\infty}^{x}v'(y)\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y=\frac{u^{l+1}}{(l+1)!}\frac{v^{l+1}}{(l+1)!}
\tag{2.2.19}
$$

及

$$
|c_{2l+3}^{(2)}|\le\int_{-\infty}^{x}|p(y)|\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y\le\int_{-\infty}^{x}u'(y)\frac{u^{l+1}(y)}{(l+1)!}\frac{v^{l}(y)}{l!}\mathrm{d}y=\frac{u^{l+2}}{(l+2)!}\frac{v^{l+1}}{(l+1)!}
\tag{2.2.20}
$$

从而上述 (2.2.12) 式得证. 注意到 $|q(x)|=|p(x)|$，则 $v(x)=u(x)$，故 (2.2.12) 式可化简为

$$
|c_{2n+1}^{(2)}|\le\frac{u^{2n+1}}{n!(n+1)!},\quad |c_{2n}^{(1)}|\le\frac{u^{2n}}{n!n!}
\tag{2.2.21}
$$

当 $q\in L^1(\mathbb{R})$ 时，有 $u(x)$ 的上述级数均收敛，故 Neumann 级数 $\sum_{n=1}^{\infty}c_n(x,z)$ 绝对收敛，此时 μ_1^+ 在 $Imz>0$ 上解析，且在 $Imz\ge 0$ 上连续.

Step2. 解的唯一性：不妨设 $\tilde{\mu}_1^+$ 为方程 (2.2.8) 的另一个解，不妨设 $h=\mu_1^+-\tilde{\mu}_1^+$，则有

$$
\|h(x,t,z)\|=\left\|\int_{-\infty}^{x}\begin{pmatrix}1&\\&e^{\mathrm{i}z(x-y)}\end{pmatrix}Ph\mathrm{d}y\right\|\le 2\int_{-\infty}^{x}|q|\|h\|\mathrm{d}y\implies\|h(x,t,z)\|\equiv 0
\tag{2.2.22}
$$

故 μ_1^+ 为方程的唯一解，同理可得 μ_1^- 的解析性. □

所以可得 μ_1^+,μ_1^- 分别在 $\mathbb{C}_+,\mathbb{C}_-$ 上解析. 同理可得 μ_2 的第一二列分别在 $\mathbb{C}_-,\mathbb{C}_+$ 上解析. 记作

$$
\mu_2=\begin{pmatrix}\mu_2^{(11)}&\mu_2^{(12)}\\\mu_2^{(21)}&\mu_2^{(22)}\end{pmatrix}=(\mu_2^-,\mu_2^+)
$$

由 (2.2.6) 可知，μ_1,μ_2 可逆，且逆矩阵为其伴随阵，另外基于 μ_1,μ_2 的列向量函数的解析性，可得 μ_1^{-1} 的第一二行

在 $\mathbb{C}_-,\mathbb{C}_+$ 上解析, μ_2^{-1} 的第一二行在 $\mathbb{C}_+,\mathbb{C}_-$ 上解析. 记为

$$\mu_1^{-1}=\begin{pmatrix}\mu_1^{(22)} & -\mu_1^{(21)}\\ -\mu_1^{(12)} & \mu_1^{(11)}\end{pmatrix}=\begin{pmatrix}-\hat{\mu_1^-}\\ \hat{\mu_1^+}\end{pmatrix},\quad \mu_2^{-1}=\begin{pmatrix}\mu_2^{(22)} & -\mu_2^{(21)}\\ -\mu_2^{(12)} & \mu_2^{(11)}\end{pmatrix}=\begin{pmatrix}-\hat{\mu_2^+}\\ \hat{\mu_2^-}\end{pmatrix} \tag{2.2.23}$$

利用 (2.2.4), (2.2.7), (2.2.23) 可得谱函数 $S(z)$ 的解析性

$$S(z)=\mu_2^{-1}\mu_1=\begin{pmatrix}-\hat{\mu_2^+}\\ \hat{\mu_2^-}\end{pmatrix}(\mu_1^+,\mu_1^-)=\begin{pmatrix}-\hat{\mu_2^+}\mu_1^+ & -\hat{\mu_2^+}\mu_1^-\\ \hat{\mu_2^-}\mu_1^+ & \hat{\mu_2^-}\mu_1^-\end{pmatrix} \tag{2.2.24}$$

可见 $S_{11}(z)$ 在 $\mathbb{C}_+$ 解析, $S_{22}(z)$ 在 $\mathbb{C}_-$ 解析, S_{12},S_{21} 在 $\mathbb{C}_+,\mathbb{C}_-$ 均不解析, 但连续到边界.

2.2.2 对称性

定理 2.1

如上构造的特征函数 μ_1,μ_2 与谱函数 $S(z)$ 具有如下对称性

$$\mu_j^\dagger(x,t,z^*)=\mu_j^{-1}(x,t,z),\quad S^\dagger(z^*)=S^{-1}(z)\quad (j=1,2) \tag{2.2.25}$$

证明 由 (2.1.11) 有

$$\mu_{j,x}(x,t,z)=\mathrm{i}z[\sigma_3,\mu_j(x,t,z)]=P\mu_j(x,t,z) \tag{2.2.26}$$

将 z 替换为 z^*, 并在两边同时取 Hermite 共轭, 则有

$$\mu_{j,x}^\dagger(x,t,z^*)+\mathrm{i}z[\sigma_3,\mu_j^\dagger(X,t,z^*)]=\mu_j^\dagger(x,t,z^*)P^\dagger \tag{2.2.27}$$

由于 $P^\dagger=-P$, 故上式可化为

$$\mu_{j,x}^\dagger(x,t,z^*)+\mathrm{i}z[\sigma_3,\mu_j^\dagger(X,t,z^*)]=-\mu_j^\dagger(x,t,z^*)P \tag{2.2.28}$$

另外对于 $\mu_j\cdot\mu_j^{-1}=I$ 对 x 求偏导有

$$\mu_{j,x}\mu_j^{-1}+\mu_j\mu_{j,x}^{-1}=0\implies \mu_{j,x}^{-1}=-\mu_j^{-1}\mu_{j,x}\mu_j^{-1} \tag{2.2.29}$$

将 (2.2.26) 带入上式有

$$\mu_{j,x}^{-1}=-\mu_j^{-1}(P\mu_j(x,t,z)-\mathrm{i}z[\sigma_3,\mu_j(x,t,z)]\mu_j^{-1}(x,t,z)\implies \mu_{j,x}^{-1}+\mathrm{i}z[\sigma_3,\mu_j^{-1}]=\mu_j^{-1}P \tag{2.2.30}$$

由 (2.2.8), (2.2.10) 可得 $\mu_j^\dagger,\mu_j^{-1}$ 满足相同的一次线性微分方程, 具有相同的渐进性, 又因为

$$\mu_j^\dagger(x,t,z^*),\mu_j^{-1}(x,t,z)\to I,\quad |x|\to\infty \tag{2.2.31}$$

因此两者相等, 得到对称关系 $\mu_j^\dagger(x,t,z^*)=\mu_j^{-1}(x,t,z),(j=1,2)$. 接下来考虑 S 的对称性, 将 (2.2.4) 式改写为

$$S(z)=e^{\mathrm{i}\theta(z)\hat{\sigma}_3}(\mu_2^{-1}\mu_1)=\hat{E}(\mu_2^{-1}\mu_1) \tag{2.2.32}$$

由 (2.2.31) 可得如下

$$S(z^*)^\dagger=e^{\mathrm{i}\theta(z^*)\sigma_3}(\mu_2^{-1}\mu_1)e^{-\mathrm{i}\theta(z^*)}=E\mu_2^{-1}\mu_1E^{-1}=E\mu_1^\dagger(\mu_2^{-1})^\dagger E^{-1}=E\mu_1^\dagger\mu_2E^{-1}=S(z)^{-1} \tag{2.2.33}$$

比较对应元素, 有

$$S_{11}^*(z^*)=S_{22}(z),S_{12}^*(z^*)=-S_{21}(z) \tag{2.2.34}$$

□

2.3 相关的 RH 问题

2.3.1 规范 RH 问题

基于 2.2 节的结论, 我们构造 RH 问题, 引入记号

$$H_1 = \begin{pmatrix} 1 & \\ & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & \\ & 1 \end{pmatrix} \tag{2.3.1}$$

并定义两个矩阵

$$\begin{aligned} P_+(x,t,z) &= \mu_1 H_1 + \mu_2 H_2 = \begin{pmatrix} \mu_1^{(11)} & \mu_2^{(12)} \\ \mu_1^{(21)} & \mu_2^{(22)} \end{pmatrix} = (\mu_1^+, \mu_2^+) \\ P_-(x,t,z) &= H_1\mu_1^{-1} + H_2\mu_2^{-1} = \begin{pmatrix} \mu_1^{(22)} & -\mu_1^{(12)} \\ -\mu_2^{(21)} & \mu_2^{(11)} \end{pmatrix} = \begin{pmatrix} \hat{\mu}_1^- \\ \hat{\mu}_2^- \end{pmatrix} \end{aligned} \tag{2.3.2}$$

则由 $\mu_j, \mu_j^{-1}(j=1,2)$ 的解析性与渐进性, 直接可得 P_+ 在 $\mathbb{C}_+$ 上解析, P_- 在 $\mathbb{C}_-$ 上解析, 且具有如下渐进性

$$P_+, P_- \to I \quad |x| \to \infty \tag{2.3.3}$$

由此可证明如下对称关系

定理 2.2

$P_+(x,t,z), P_-(x,t,z)$ 具有如下对称性

$$P_+^\dagger(x,t,z^*) = P_-(x,t,z) \tag{2.3.4}$$

证明 利用 (2.2.31) 可得

$$\begin{aligned} P_+^\dagger(x,t,z^*) &= (\mu_1(x,t,z)H_1 + \mu_2(x,t,z)H_2)^\dagger = H_1\mu_1^\dagger(z^*) + H_2\mu_2^\dagger(z^*) \\ &= H_1\mu_1^{-1}(x,t,z) + H_2\mu_2^{-1}(x,t,z) = P_-(x,t,z) \end{aligned} \tag{2.3.5}$$

□

定义 2.1

概括上面结果, 我们可以得到下面 RH 问题

$$P_\pm(x,t,z) \in C(\mathbb{C}_\pm) \tag{2.3.6}$$

$$P_- P_+ = G(x,t,z) \quad z \in \mathbb{R} \tag{2.3.7}$$

$$P_\pm \to I \quad z \to \infty \tag{2.3.8}$$

其中跳跃矩阵为

$$G(x,t,z) = \hat{E}^{-1} \begin{pmatrix} 1 & -S_{12}(z) \\ S_{21}(z) & 1 \end{pmatrix} \tag{2.3.9}$$

进一步 NLS 方程的解 $q(x,t)$ 可由 RH 问题的解给出

$$q(x,t) = 2\mathrm{i} \lim_{z\to\infty} (zP_+)_{12} = 2\mathrm{i}(P_+^{(1)})_{12} \tag{2.3.10}$$

其中 $P_+ = I + \frac{P_+^{(1)}}{z} + O(z^{-2})$

证明 只需证明跳跃关系即可, 注意到 $\hat{E}H_j = H_j, (j=1,2)$ 可得

$$\begin{aligned} G &= P_- P_+ = (H_1\mu_1^{-1} + H_2\mu_2^{-1})(\mu_1 H_1 + \mu_2 H_2) \\ &\overset{(2.2.4)}{=} [H_1(\hat{E}^{-1}S^{-1})\mu_2^{-1} + H_2\mu_2^{-1}][\mu_2(\hat{E}^{-1}S)H_1 + \mu_2 H_2] \\ &= \hat{E}^{-1}[(H_1 S^{-1} + H_2)(SH_1 + H_2)] = \hat{E}^{-1} \begin{pmatrix} 1 & -S_{12}(z) \\ S_{21}(z) & 1 \end{pmatrix} \end{aligned} \tag{2.3.11}$$

直接计算可得

$$\begin{aligned}\det(P_+) &= \det(\mu_1 H_1 + \mu_2 H_2) = \det(\mu_2 \hat{E}^{-1} S) H_1 + \mu_2 H_2) \\ &= \det(\mu_2)\det((\hat{E}^{-1} S) H_1 + \mu_2 H_2) = S_{11}(z)\end{aligned} \tag{2.3.12}$$

同理, 有 $\det(P_-) = S_{22}(z)$ □

2.3.2 RH 问题的可解性

下面分两种情况讨论 RH 问题 (2.3.6) - (2.3.8) 的解

Case1. 如果

$$\det P_\pm(z) \neq 0 (\forall z \in \mathbb{C}), \tag{2.3.13}$$

则称 RH 问题 (2.3.6) - (2.3.8) 为正则的, 将方程 (2.3.7) 改写为

$$P_+^{-1} - P_- = (I - G) P_+^{-1} := \hat{C} P_+^{-1} \tag{2.3.14}$$

由 Plemelj 公式可得

$$P_+ = I + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\hat{C}(x,t,z) P_+^{-1}(x,t,z)}{s - z} \mathrm{d}s, \quad z \in \mathbb{C}_+ \tag{2.3.15}$$

Case2. 如果条件 (2.3.13) 不满足, 则称 RH 问题为非正则的, 假设 $\det P_\pm$ 在某些离散的点处为零, 由谱函数对称性 (2.2.34) 有

$$\det P_+(z) = S_{11}(z) = S_{22}^*(z^*) = \det P_-^*(z^*) = \det P_-(z^*) \tag{2.3.16}$$

故

$$\det P_+(z) = 0 \iff \det P_-(Z^*) = 0, \tag{2.3.17}$$

因此 $\det P_+(z)$ 与 $\det P_-(z)$ 有相同的零点个数, 且彼此共轭. 即若设 $z_j (j = 1, 2, \ldots, N)$ 为 $\det P_+(z)$ 在 $\mathbb{C}_+$ 上的单零点, 则 $z_j^* (j = 1, 2, \ldots, N)$ 为 $\det P_-(z)$ 在 $\mathbb{C}_-$ 上的单零点.

由于 $\det P_+(z_j) = \det P_-(z_j^*) = 0$, 假设 w_j, w_j^* 分别为下列线性方程组的解

$$P_+(z_j) w_j(z_j) = 0, \quad w_j^*(z_j^*) P_-(z_j^*) = 0 \tag{2.3.18}$$

对上式取共轭转置, 则有 $w_j^\dagger(z_j) P_+^\dagger(z_j) = 0$, 再利用对称性 (2.3.4) 有

$$w_j^\dagger(z_j) P_-(z_j^*) = 0 \tag{2.3.19}$$

比较可得 $w_j^\dagger(z) = w_j^*(z^*)$.

非正则 RH 问题 (2.3.6) - (2.3.8) 的解可由下面定理给出

定理 2.3 (Zakharov, Shabat, 1979)

带有零点结构 (2.3.18) 的非正则 RH 问题 (2.3.6) - (2.3.8) 可分解为

$$P_+(z) = \hat{P}_+(z)\Gamma(z), \quad P_-(z) = \Gamma^{-1}(z)\hat{P}_-(z) \tag{2.3.20}$$

其中

$$\Gamma(z) = I + \sum_{k,j=1}^{N} \frac{w_k (M^{-1})_{kj} w_j^*}{z - z_j^*}, \quad \Gamma(z)^- = I - \sum_{k,j=1}^{N} \frac{w_k (M^{-1})_{kj} w_j^*}{z - z_k} \tag{2.3.21}$$

这里 M 为 $N \times N$ 矩阵, 其 $(k.j)$ 元素由下式给定

$$M_{kj} = \frac{w_k^* w_j}{z_k^* - z_j}, \quad k, j = 1, 2, \ldots, N, \quad \det\Gamma(z) = \prod_{k=1}^{N} \frac{z - z_k}{z - z_k^*} \tag{2.3.22}$$

而 $\hat{P}_\pm$ 为正则 RH 问题的唯一解, 且

1. $\hat{P}_\pm$ 在 $\mathbb{C}_+$ 上解析,
2. $\hat{P}_- \hat{P}_+ = \Gamma(z) G \Gamma^{-1}(z), z \in \mathbb{R}$

3. $\hat{P_\pm}(z) \to I, \quad z \to \infty,$

证明 非正则 RH 问题 (2.3.6) - (2.3.8) 是由于 $2N$ 个离散谱上的 $\det P_+(z_j) = \det P_-(Z_{j^*}) = 0 (j = 1, 2, \ldots N)$ 造成的. 所以主要任务是消除这些零点结构, 并且去除 $P_+(z_j), P_-(z_j^*)$ 在 z_j, z_j^* 上的零点结构, 为此需定义单极点矩阵

$$\Gamma_1(z) = I + \frac{z_1^* - z_1}{z - z_1^*} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \tag{2.3.23}$$

其具有如下性质

$$\begin{aligned}
F_1^{-1}(z) &= I - \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \\
\det \Gamma_1(z) &= \frac{z - z_1}{z - z_1^*}, \quad \det \Gamma_1^{-1}(z) = \frac{z - z_1^*}{z - z_1} \\
\Gamma_1(z_1) w_1 &= w_1 + \frac{z_1^* - z_1}{z_1 - z_1^*} \cdot \frac{w_1 w_1^*}{w_1^* w_1} \cdot w_1 = \frac{z^* - z_1}{z - z_1^*} \frac{w_1 (w_1^* w_1)}{w^* w_1} = w_1 - w_1 = 0 \\
w_1^* \Gamma_1^{-1}(z_1^*) &= w_1^* - \frac{z_1^* - z_1}{z^* - z_1} \cdot \frac{w_1^* w_1}{w_1^* w_1} \cdot w_1^* = w_1^* - w_1^* = 0
\end{aligned} \tag{2.3.24}$$

令 $x_j = \frac{w_1 w_1^*}{w_1^* w_1}$, 则 x_j 为投影算子, 即 $x_j^2 = x_j$, 由上定义知 x_j 为一阶矩阵, 故与矩阵 $\mathrm{diga}\{1, 0\}$ 相似, 既有可逆阵 T_j 使得 $T_j^{-1} x_j T_j = \mathrm{diag}\{1, 0\}$, 从而有

$$\Gamma_1(z) = \det\left(I + \frac{z_1^* - z_1}{z - z_1^*} T_j^{-1} X_j T_j\right) = \begin{vmatrix} 1 + \frac{z_1^* - z_1}{z - z_1^*} & 0 \\ 0 & 1 \end{vmatrix} = \frac{z - z_1}{z - z_1^*} \tag{2.3.25}$$

定义矩阵函数 $R_1^+(z) = P_+(z)\Gamma_+^{-1}(z), R_1^{-1} = \Gamma_1(z) P_-(z)$, 由 (2.3.18) - (2.3.19) 知

$$\begin{aligned}
\operatorname*{Res}_{z=z_1} R_1^+(z) &= \operatorname*{Res}_{z=z_1} \left(P_+(z) - P_+(z) \frac{z_1^* - z_1}{z - z_1} \frac{w_1 w_1^*}{w_1^* w_1}\right) = -\frac{z_1^* - z_1}{w_1 w_1^*} (P_+(z) w_1) w_1^* = 0 \\
\operatorname*{Res}_{z=z_1^*} R_1^{-1}(z) &= \operatorname*{Res}_{z=z_!^*} \left(P_1(z) + \frac{z_1^* - z_1}{z - z_1} \frac{w_1 w_1^*}{w_1^* w_1} P_-(z)\right) = -\frac{z_1^* - z_1}{w_1^* w_1} w_1 (w_1^* P_-(z_1^*)) = 0
\end{aligned} \tag{2.3.26}$$

因此 $R_1^+(z), R_1^-(z)$ 分别在 $\mathbb{C}_+, \mathbb{C}_-$ 上解析, 且

$$\begin{aligned}
\det R_1^+(z_1) &= \lim_{z \to z_1} \left[\det P_+(z) \cdot \det \Gamma_1^{-1}(z)\right] = \lim_{z \to z_1} S_{11}(z) \frac{z - z_1^*}{z - z_1} \\
&= \lim_{z \to z_1} \frac{S_{11}(z) - S_{11}(z_1)}{z - z_1} (z - z^*) \Leftarrow (S_{11}(z) = 0) \\
&= \lim_{z \to z_1} S_{11}'(z)(z - z_1^*) = 0 \\
\det R_1^{-1}(z_1^*) &= \lim_{z \to z_1^*} \left[\det \Gamma_1(z) \cdot \det P_-(z)\right] = \lim_{z \to z_1^*} \frac{z - z_1}{z - z_1^*} S_{22}(z) = \lim_{z \to z_1^*} S_{22}'(z)(z - z_1) = 0
\end{aligned} \tag{2.3.27}$$

这说明 $R_1^+(z), R_1^-(z)$ 分别在 $z = z_1, z = z_1^*$ 不再具有零点结构, 然后去除其在 z_2, z_2^* 上的零点结构, 由于

$$\begin{aligned}
\det R_1^+(z_2) &= \det P_+(z_2) \det \Gamma_1^{-1}(z_2) = S_{11}(z_2) \frac{z_2 - z_1^*}{z_2 - z_1} = 0 \\
\det R_1^{-1}(z_2^*) &= \det \Gamma_1(z_2^*) \det P_-(z_2^*) = \frac{z_2^* - z_1}{z_2^* - z_1} S_{22}(z_2^*) = 0
\end{aligned} \tag{2.3.28}$$

因此, 下列齐次线性方程组有非零解, 即存在 $v_2(z_2), v_2^*(z_2^*)$, 使得

$$R_1^+(z_2) v(z_2) = P_+(z_2) \Gamma_1^{-1}(z_2) v_2(z_2) = 0, \quad v_2^*(z_2^*) R_1^{-1}(z_2^*) = v_2^*(z_2^*) \Gamma_1(z_2^*) P_-(z_2^*) = 0 \tag{2.3.29}$$

与 (2.3.18) 比较可得

$$w_2(z_2) = \Gamma_1^{-1}(z_2) v_2(z_2), \quad w_2^*(z_2^*) = v_2^*(z_2^*) \Gamma_1(z_2^*) \tag{2.3.30}$$

为去除 $R_1^+(z_2), R_1^-(z_2^*)$ 在 z_2, z_2^* 上的零点结构, 令

$$\begin{aligned}
\Gamma_2(z) &= I + \frac{z_2^* - z_2}{z - z_2^*} \cdot \frac{v_2 v_2^*}{v_2^* v_2}, \quad \det\Gamma_2(z) = \frac{z - z_2}{z - z_2^*} \\
\Gamma_2^{-1}(z) &= I - \frac{z_2^* - z_2}{z - z_2} \cdot \frac{v_2 v_2^*}{v_2^* v_2}, \quad \det\Gamma_2^{-1} = \frac{z - z_2^*}{z - z_2} \\
R_2^+(z) &= P_1^+(z)\Gamma_2^{-1}(z) = P_1^+(z)\Gamma_1^{-1}(z)\Gamma_2^{-1}(z) \\
R_2^{-1}(z) &= \Gamma_2(z)P_1^{-1}(z) = \Gamma_2(z)\Gamma_1(z)P_1^{-1}(z)
\end{aligned} \tag{2.3.31}$$

则

$$\operatorname*{Res}_{z=z_2} R_2^+(z) = \operatorname*{Res}_{z=z_2}\left[\left(P_+(z) - P_+(z)\frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{w_2^* w_1}\right)\left(I - \frac{z_1^* - z_1}{z - z_1} \cdot \frac{w_1 w_1^*}{v_2^* v_1}\right)\right] = -\frac{z_2^* - z_1 R_1(z_2) v_2 v_2^*}{v_2^* v_2} \tag{2.3.32}$$

同理 $\mathrm{Res}_{z=z_2^*} R_2^{-1}(z) = 0$, 由此可得 $R_2^+(z), R_2^{-1}(z)$ 分别在在 $z = z_1, z_2, z = z_1^*, z_2^*$ 上无零点结构, 则

$$\begin{aligned}
\det R_2^+(z_j) &= \lim_{z\to z_j}\left[\det P_+(z)\det\Gamma_1^{-1}(z)\det\Gamma_2^{-1}(z)\right] \\
&= \lim_{z\to z_j} S_{11}(z)\frac{z - z_1^*}{z - z_1} \cdot \frac{z - z_2^*}{z - z_2} \\
&= s_{11}'(z_j)\frac{(z_j - z_1^*)(z_j - z_2^*)}{z_j - z_k} \neq 0 \\
\det R_2^-(z_j^*) &\neq 0 \quad (k = 1, 2, k \neq z)
\end{aligned} \tag{2.3.33}$$

更一般的, 可得

$$\begin{aligned}
w_j &= \Gamma_1^{-1}(z_j)\cdots\Gamma_{j-1}^{-1}(z_j) \cdot v_j(z_j) \\
w_j^* &= v_j^*(z_j^*) \cdot \Gamma_1(z_j^*)\cdots\Gamma_{j-1}(z_j^*) \\
R_j^+(z) &= P_+(z) \cdot \Gamma_1^{-1}(z)\cdots\Gamma_j^{-1}(z) \\
R_j^{-1}(z) &= \Gamma_j(z)\cdots\Gamma_1(z) \cdot P_-(z)
\end{aligned} \tag{2.3.34}$$

其中

$$\Gamma_j(z) = I + \frac{z_j^* - z_j}{z - z_j^*} \cdot \frac{v_j v_j^*}{v_j^* v_j}, \quad \Gamma_j^{-1}(z) = I - \frac{z_j^* - z_j}{z - z_j} \cdot \frac{v_j v_j^*}{v_j^* v_j} \tag{2.3.35}$$

则 $R_j^+(z), R_j^{-1}(z)$ 在 $\mathbb{C}_+, \mathbb{C}_-$ 上解析, 且在 $z_k, z_k^*(k = 1, 2, \ldots j)$ 上无零点结构, 即

$$\left.\begin{aligned}
\det R_j^+(z_k) &= s_{11}'(z_k)\frac{\prod_{l=1}^{j}(z_k - z_l^*)}{\prod_{l=1,l\neq k}^{j}(z_k - z_l)} \neq 0 \\
\det R_j^{-1}(z_k^*) &= s_{22}'(z_k^*)\frac{\prod_{l=1}^{j}(z_k^* - z_l)}{\prod_{l=1,l\neq k}^{j}(z_k^* - z_l)} \neq 0
\end{aligned}\right. \quad (k = 1, 2, \ldots, j) \tag{2.3.36}$$

而

$$\begin{aligned}
\det R_j^+(z_{j+1}) &= S_{11}(z_{j+1})\prod_{j=1}^{j}\frac{z_{j+1} - z_k^*}{z_{j+1} - z_k} = 0 \\
\det R_j^{-1}(z_{j+1}^*) &= S_{22}(z_{j+1}^*)\prod_{j=1}^{j}\frac{z_{j+1}^* - z_k}{z_{j+1}^* - z_k} = 0
\end{aligned} \tag{2.3.37}$$

最后令

$$\begin{aligned}
\Gamma_z &= \Gamma_N(z)\cdots\Gamma_1(z), \quad \Gamma_z^{-1} = \Gamma_1^{-1}(z)\cdots\Gamma_N^{-1}(z) \\
\hat{P}_+(z) &= P_+(z)\Gamma_z^{-1}(z), \quad \hat{P}_-(z) = \Gamma_z(z)P_-(z)
\end{aligned} \tag{2.3.38}$$

则 $\hat{P}_+(z), \hat{P}_-(z)$ 在 $\mathbb{C}_+, \mathbb{C}_-$ 上解析, 且在 $z_j, z_j^*(j = 1, 2, \ldots, N)$ 上无零点结构 (实际上在解析区域内无零点结构), 即

$$\det\hat{P}_+(z_k) \neq 0, \quad \det\hat{P}_-(z_k^*) \neq 0 \quad (k = 1, 2, \ldots, N) \tag{2.3.39}$$

得到分解 (2.3.20). 下证 (2.3.21). 注意到 $\Gamma(z), \Gamma^{-1}(z)$ 分别为具有单极点 $z_j, z_j^*(1 \leq j \leq N)$ 的亚纯函数, 以及分解

(2.3.35), (2.3.38), 寻找的向量使得

$$\Gamma(Z)=i+\sum_{j=1}^{N}\frac{\xi_j w_j^*}{z-z_j^*},\quad \Gamma^{-1}(z)=I-\sum_{j=1}^{N}\frac{w_j^*\xi_j}{z-z_j} \tag{2.3.40}$$

注意到 $\Gamma(z)\Gamma^{-1}(z)=I$ 对任意 z 都成立, 当然也在 $z=z_k$ 处成立, 即 $\Gamma(z)\Gamma^{-1}(z)=I$ 在 $z=z_k$ 处正则, 为保证 $\Gamma(z)\Gamma^{-1}(z)=I$ 在 $z=z_k$ 处成立, 只需让其留数为 0, 因此利用 (2.3.40) 可得

$$\begin{aligned}0&=\operatorname*{Res}_{z=z_k}[\Gamma(z)\Gamma^{-1}(z)]=\operatorname*{Res}_{z=z_k}\left(\Gamma(z)-\sum_{j=1}^{N}\Gamma(z)\frac{w_j\xi_j^*}{z-z_j}\right)=-\Gamma(z_k)w_k\xi_k^*\\&-\left(I+\sum_{j=1}^{N}\frac{\xi_j w_j^*}{z_k-z_j^*}\right)w_k\xi_k^*=\left(-w_k+\sum_{j=1}^{N}\frac{w_j^*w_k}{z_j^*-z_k}\xi_j^*\right)\xi_k^*\end{aligned} \tag{2.3.41}$$

对上式两边同时作用 ξ_k, 则有

$$\left(-w_k+\sum_{j=1}^{N}\frac{w_j^*w_k}{z_j^*-z_k}\xi_j\right)|\xi_k|^2=0\implies -w_k+\sum_{j=1}^{N}\frac{w_j^*w_k}{z_j^*-z_k}\xi_j=0\quad(1\le k\le N) \tag{2.3.42}$$

将上式改写为 $\xi_1,\xi_2,\dots,\xi_N$ 的分块矩阵形式的线性方程组, 则有

$$(\xi_1,\xi_2,\dots,\xi_N)M=(w_1^*,w_2^*,\dots,w_N^*) \tag{2.3.43}$$

其中 $M=(M_{kj})_{N\times N}$, 其中 $M_{kj}=\frac{w_k^*w_j}{z_k^*-z_j}$ 则

$$\begin{cases}\frac{w_1^*w_1}{z_1^*-z_1}\xi_1+\frac{w_2^*w_1}{z_2^*-z_1}\xi_2+\cdots+\frac{w_N^*w_1}{z_N^*-z_1}\xi_N=w_1\\ \frac{w_1^*w_2}{z_1^*-z_2}\xi_1+\frac{w_2^*w_2}{z_2^*-z_2}\xi_2+\cdots+\frac{w_N^*w_2}{z_N^*-z_2}\xi_N=w_2\\ \vdots\\ \frac{w_1^*w_N}{z_1^*-z_N}\xi_1+\frac{w_2^*w_N}{z_2^*-z_N}\xi_2+\cdots+\frac{w_N^*w_N}{z_N^*-z_N}\xi_N=w_N\end{cases}\implies M=\begin{pmatrix}\frac{w_1^*w_1}{z_1^*-z_1}&\frac{w_1^*w_2}{z_1^*-z_2}&\cdots&\frac{w_1^*w_N}{z_1^*-z_N}\\ \frac{w_2^*w_1}{z_2^*-z_1}&\frac{w_2^*w_2}{z_2^*-z_2}&\cdots&\frac{w_2^*w_N}{z_2^*-z_2}\\ \vdots&\vdots&\ddots&\vdots\\ \frac{w_N^*w_1}{z_N^*-z_1}&\frac{w_N^*w_2}{z_N^*-z_2}&\cdots&\frac{w_N^*w_N}{z_N^*-z_N}\end{pmatrix} \tag{2.3.44}$$

可得 $\xi_j=\sum_{k=1}^{N}(M^{-1})_{kj}w_k$. 最后将其带入 (2.3.40) 得到 (2.3.21) □

2.4 NLS 方程的 N 孤子解

2.4.1 矩阵向量解的时空演化

对方程 (2.3.18) 第一个式子两边分别对 x,t 求导, 可得

$$P_{+,x}w_j+P_+w_{j,x}=0,\quad P_{+,t}w_j+P_+w_{j,t}=0 \tag{2.4.1}$$

利用 P_+ 的定义与 Lax 对 (2.1.11), (2.1.12) 可得

$$\begin{aligned}P_{+,x}&=\mu_{1,x}H_1+\mu_{2,x}H_2=\left(\mathrm{i}z_j[\sigma_3,\mu_1]+P\mu_1\right)H_1+\left(-\mathrm{i}z_j[\sigma_3,\mu_2]+P\mu_2\right)H_2\\&-\mathrm{i}z_j\left(\mu_1\sigma_3H_1-\mu_2\sigma_3H_2+\sigma_3\mu_1H_1+\sigma_3\mu_2H_2\right)+P\mu_1H_1+P\mu_2H_2\\&=-\mathrm{i}z[\sigma_3,p_+]+PP_+\end{aligned} \tag{2.4.2}$$

同理有

$$P_{+,t}=-\mathrm{i}z_j^2[\sigma_3,P_+]+QP_+ \tag{2.4.3}$$

将 (2.4.2), (2.4.3) 代入 (2.4.1) 且由于 $P_+w_j=0, w_jP=0$, 有

$$(-\mathrm{i}z_j[\sigma_3,P_+]+PP_+)w_j+P_+w_{j,x}=0\implies \mathrm{i}z_jP_+\sigma_3w_j+P_+w_{j,x}=0\implies P_+(w_{j,x}+\mathrm{i}z_j\sigma_3w_j)=0 \tag{2.4.4}$$

同理有 $P_+(w_{j,t}+\mathrm{i}z_j^2\sigma_3w_j)=0$.

$$\begin{cases}w_{j,x}+\mathrm{i}z_j\sigma_3w_j=0\\ w_{j,t}+\mathrm{i}z_j^2\sigma_3w_j=0\end{cases}\implies w_j=e^{-\mathrm{i}\theta(z_j)\sigma_3}w_{j,0},\quad(j=1,2,\dots,N) \tag{2.4.5}$$

其中 $w_{j,0}$ 为 2 维常向量, 从而 $w_j^* = w_{j,0}^{\dagger} e^{\mathrm{i}\theta(z_j^*)\sigma_3}$.

2.4.2 N 维孤子解公式

已知 $P_- P_+ = G \implies P_+^{-1} - P_- = \hat{G} P_+$(其中 $I - G = \hat{G}$), 且 $\hat{P_-}(z)\hat{P_+}(z) = \Gamma(z) G \Gamma^{-1}(z)(z \in \mathbb{R})$, 可得

$$\begin{aligned} \hat{P_+^{-1}} - \hat{P_-} &= (I - \hat{P_-}\hat{P_+})\hat{P_+^{-1}} = \left(I - \Gamma(z) G \Gamma^{-1}(z)\right) \hat{P_+^{-1}} = (\Gamma(z)\Gamma^{-1}(z) - \Gamma(z) G \Gamma^{-1}(z))\hat{P_+^{-1}} \\ &= \Gamma(z)(I - G)\Gamma^{-1}(z)\hat{P_+^{-1}} = \Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}} \end{aligned} \tag{2.4.6}$$

由 Taylor 公式 $\frac{1}{s-z} = -\frac{1}{z}(\frac{1}{1-s/z}) = -\frac{1}{z}\left(1 + \frac{s}{z} + \cdots\right)$, 故 Plemelj 公式可写为

$$\begin{aligned} \hat{P_+^{-1}} &= I + \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \frac{\Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}}}{s - z} \mathrm{d}s \\ &= I - \frac{1}{2\pi \mathrm{i} z} \int_{-\infty}^{+\infty} \left(1 + \frac{s}{z} + (\frac{s}{z})^2 + \cdots\right) \Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}} \mathrm{d}s \\ &= I - \frac{1}{2\pi \mathrm{i} z} \int_{-\infty}^{+\infty} \Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}} \mathrm{d}s + O(z^{-2}) \end{aligned} \tag{2.4.7}$$

由 $(I - A)^{-1} = I + A + A^2 + \cdots, (P^{-1} = I + \frac{A}{z} + \ldots, P = \frac{B}{z} + \ldots)$, 可得

$$\hat{P_+} = I + \frac{1}{2\pi \mathrm{i} z} \int_{-\infty}^{+\infty} \Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+} \mathrm{d}s + O(z^{-2}) \tag{2.4.8}$$

再由

$$\Gamma(z) = i + \sum_{k,j=1}^{N} \frac{w_k (M^{-1})_{kj} w_j^*}{z - z_j^*} \implies \Gamma(z) = I + \frac{1}{z} \sum_{k,j=1}^{N} w_k (M^{-1})_{kj} w_j^* + O(z^{-2}) \tag{2.4.9}$$

将上渐进式带入 (2.3.20), 比较 z^{-1} 的次数可得

$$P_+^{(1)} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{+\infty} \Gamma(z)\hat{G}\Gamma^{-1}(z)\hat{P_+^{-1}} \mathrm{d}s = \sum_{k,j=1}^{N} w_k (M^{-1})_{kj} w_j^* \tag{2.4.10}$$

特别的, 当散射数据 $S_{12} = S_{21} = 0$ 时, 有 $\hat{G} = I - G = 0$. 故上式 (2.4.10) 可简化为

$$P_+^{-1} = \sum_{k,j=1}^{N} w_k (M^{-1})_{kj} w_j^* \tag{2.4.11}$$

不妨取 $\lambda_j = -\mathrm{i}(z_j x + 2z_j^2 t)$, 并取 $w_{j,0} = (c_j, 1)^T$, 则有

$$w_j = \begin{pmatrix} e^{\lambda_j} & \\ & e^{-\lambda_j} \end{pmatrix} \begin{pmatrix} c_j \\ 1 \end{pmatrix} = \begin{pmatrix} c_j e^{\lambda_j} \\ e^{-\lambda_j} \end{pmatrix} \tag{2.4.12}$$

从而 $w_j^* = w_{j,0}^* \cdot e_j^{\lambda} \sigma_3 = (c_j^* e^{\lambda_j^*}, e^{-\lambda_j})$. 故

$$M_{k.j} = \frac{w_k^* w_j}{z_k^* - z_j} = \frac{1}{z_k^* - z_j}(c_k^* c_j e^{\lambda_k^* + \lambda_j} + e^{-\lambda_k - \lambda_j}) \tag{2.4.13}$$

再由 (2.4.9)

$$\begin{aligned} q(x,t) &= 2\mathrm{i} \lim_{z \to \infty} (z P_+)_{12} = 2\mathrm{i}(P_+^{(1)})_{12} \\ &= 2\mathrm{i} \sum_{k,j}^{N} (w_k w_j^*)_{12} (M^{-1})_{kj} = 2\mathrm{i} \sum_{k,j}^{N} c_k e^{\lambda_k - \lambda_j^*} (M^{-1})_{kj} \end{aligned} \tag{2.4.14}$$

令

$$R = \begin{pmatrix} 0 & c_1 e^{\lambda_1} & \cdots & c_N e^{\lambda_N} \\ e^{-\lambda_1^*} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ e^{-\lambda_N^*} & M_{N1} & \cdots & M_{NN} \end{pmatrix} \tag{2.4.15}$$

则

$$
\begin{aligned}
\det R &= \sum_{k=1}^{N}(-1)^{k+2}c_k e^{\lambda_k}\det\begin{pmatrix} e^{-\lambda_1^*} & M_{11} & \cdots & M_{1,k-1} & M_{1,k+1} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e^{-\lambda_N^*} & M_{N1} & \cdots & M_{N,k-1} & M_{N,k+1} & \cdots & M_{NN} \end{pmatrix} \\
&= \sum_{k=1}^{N}(-1)^{k+2}c_k e^{\lambda_k}\sum_{j=1}^{N} e^{-\lambda_j^*}\det\underline{\begin{pmatrix} M_{11} & \cdots & M_{1,k-1} & M_{1,k+1} & \cdots & M_{1N} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ M_{N1} & \cdots & M_{N,k-1} & M_{N,k+1} & \cdots & M_{NN} \end{pmatrix}} := \Delta \\
&= \sum_{k=1}^{N}(-1)^{k+j+3}c_k e^{\lambda_k-\lambda_j^*}\det(\Delta) \\
&= -c_k e^{\lambda_k-\lambda_j^*(M^*)_{jk}} \qquad \Leftarrow \left((M^*)_{jk}-(-1)^{k+j}\det(\Delta)\right) \\
&= -\det M\sum c_k e^{\lambda_k-\lambda_j^*} \qquad \Leftarrow \left((M^{-1})_{jk}(M^*) = |M|M^{-1}\right)
\end{aligned}
\tag{2.4.16}
$$

因此

$$
\sum_{k=1}^{N} c_k e^{\lambda_k-\lambda_j^*}(M^{-1})_{jk} = -\frac{\det R}{\det M} \tag{2.4.17}
$$

将 (2.4.17) 带入 (2.4.14) 可得 NLS 方程的 N 孤子解

$$
q = -2\mathrm{i}\frac{\det R}{\det M} \tag{2.4.18}
$$

第 3 章 Reverse Time Space NLS

This chapter mainly introduces three inverse problems of nonlocal NLS, mainly referring to Yang's article[2]

3.1 The coupled Schrödinger equations

Consider the rverse-space NLS equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(-x,t) = 0 \tag{3.1.1}$$

reverse-time NLS equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(x,-t) = 0 \tag{3.1.2}$$

and the reverse-space-time NLS equation

$$\mathrm{i}q_t(x,t) + q_{xx}(x,t) + 2q^2(x,t)q^*(-x,-t) = 0 \tag{3.1.3}$$

This equations can be derived form the following member of the AKNS hierarchy - the coupled NLS equations

$$\mathrm{i}q_t + q_{xx} - 2q^2 r = 0, \quad \mathrm{i}r_t - r_{xx} - 2r^2 q = 0. \tag{3.1.4}$$

Under reductions

$$r(x,t) = -q^*(-x,t), \tag{3.1.5a}$$

$$r(x,t) = -q(x,-t), \tag{3.1.5b}$$

$$r(x,t) = -q(-x,-t), \tag{3.1.5c}$$

these coupled equations reduce to the reverse-space NLS equation (3.1.1), the reverse-time NLS equation (3.1.2) and the reverse-space-time NLS equation (3.1.3) respectively.

3.2 N-solitions for general coupled Shorödinger equations

Our basic idea for deriving N-soliton of the reverse-space, reverse-time, and reverse-space-time NLS equations (3.1.1)–(3.1.3) is to recognize that these equations are reductions of the coupled Schrödinger equations (3.1.4). To this end, we begin with the Riemann-Hilbert formulation of N-soliton for the coupled Schrödinger equations, based on given scattering data. By imposing suitable symmetry conditions on the scattering data, we obtain N-soliton solutions for the corresponding nonlocal equations. Specifically, we consider the coupled Schrödinger equations (3.1.4), which belong to the AKNS hierarchy. Their Lax pair is given by:

$$Y_x = MY, \quad Y_t = NY \tag{3.2.1}$$

where

$$M = \begin{pmatrix} \mathrm{i}\zeta & 0 \\ 0 & -\mathrm{i}\zeta \end{pmatrix} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}, \quad N = \begin{pmatrix} -\mathrm{i}qr - 2\mathrm{i}\zeta^2 & \mathrm{i}q_x + 2\zeta q \\ -\mathrm{i}r_x + 2\zeta r & \mathrm{i}qr + 2\mathrm{i}\zeta^2 \end{pmatrix} \tag{3.2.2}$$

Following this Riemann-Hilbert mothed, N-solitons in this system were explictly written down in chapter2 as

$$q(x,t) = -2\mathrm{i}\frac{\det F}{\det M}, \quad r(x,t) = 2\mathrm{i}\frac{\det G}{\det M} \tag{3.2.3}$$

where M is a $N \times N$ matrix, and F, G are $N+1 \times N+1$ matrices. The elements of the matrix M are given by

$$M_{jk} = \frac{\bar{\mathbf{v}}_j \mathbf{v}_k}{\bar{\zeta}_j - \zeta_k}, \quad \mathbf{v}_k(x,t) = \mathrm{e}^{\theta_k \Lambda}\mathbf{v}_{k0}, \quad \bar{\mathbf{v}}_k(x,t) = \bar{\mathbf{v}}_{k0}\mathrm{e}^{\bar{\theta}_k \Lambda} \tag{3.2.4}$$

where $\zeta_k \in \mathbb{C}_+, \bar{\zeta}_k \in \mathbb{C}_-$ is the eigenvalues and $\mathbf{v}_{k0}, \bar{\mathbf{v}}_{k0}$ is the eigenvalues $\theta_k = -\mathrm{i}\zeta_k x - 2\mathrm{i}\zeta_k^2 t, \bar{\theta}_k = \mathrm{i}\bar{\zeta}_k x + 2\mathrm{i}\bar{\zeta}_k^2 t$ and

$$\mathbf{v}_{k0} = \begin{bmatrix} a_k \\ b_k \end{bmatrix}, \quad \bar{\mathbf{v}}_{k0} = \begin{bmatrix} \bar{a}_k & \bar{b}_k \end{bmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.2.5}$$

and

$$F = \begin{pmatrix} 0 & a_1 \mathrm{e}^{\theta_1} & \cdots & a_N \mathrm{e}^{\theta_N} \\ \bar{b}_1 \mathrm{e}^{-\bar{\theta}_1} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{b}_N \mathrm{e}^{-\bar{\theta}_N} & M_{N1} & \cdots & M_{NN} \end{pmatrix} \quad G = \begin{pmatrix} 0 & b_1 \mathrm{e}^{-\bar{\theta}_1} & \cdots & b_N \mathrm{e}^{-\bar{\theta}_N} \\ \bar{a}_1 \mathrm{e}^{\theta_1} & M_{11} & \cdots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ \bar{a}_N \mathrm{e}^{\theta_N} & M_{N1} & \cdots & M_{NN} \end{pmatrix} \tag{3.2.6}$$

3.3 Symetry relations of scattering data in the nonlocal NLS equations

We first present symmetry relations of the secattering data for the reverse-space NLS equation (3.1.1) and the reverse-time NLS equation (3.1.2). The symmetry relations of the scattering data for the reverse-space-time NLS equation (3.1.3) can be obtained in a similar way. For this pourpose, we first introduce some notations. We define

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{3.3.1}$$

which is a Pauli matrix.

3.3.1 The reverse-space NLS equation

定理 3.1

For the reverser-space NLS equation (3.1.1), if ζ is an eigenvalue, so is $-\zeta^*$. Thus, non-pirely-imaginary eigenvalues appear as pairs $(\zeta, -\zeta^*)$, which lie in the same half of the complex plane. Symmetry ralations on the eigenvactors are given as follows:

1. If $(\zeta_k, \hat{\zeta}_k) \in \mathbb{C}_+$, then $\hat{\zeta}_k = -\zeta_k^*$, their column eigenvectors are related as $\hat{\mathbf{v}}_{k0} = \sigma_1 \mathbf{v}_{k0}^*$.
2. If $\zeta_k \in i\mathbb{R}_+$, its eigenvectors is of the form $\mathbf{v}_{k0} = (1, e^{\mathrm{i}\theta_k})^T$, where θ_k is a real constant.
3. If $(\bar{\zeta}_k, \hat{\bar{\zeta}}_k) \in \mathbb{C}_-$, then $\hat{\bar{\zeta}}_k = -\bar{\zeta}_k^*$, their row eigenvectors are related as $\hat{\bar{\mathbf{v}}}_{k0} = \bar{\mathbf{v}}_{k0}^* \sigma_1$.
4. If $\bar{\zeta}_k \in i\mathbb{R}_-$, its eigenvectors is of the form $\bar{\mathbf{v}}_{k0} = (1, e^{\mathrm{i}\bar{\theta}_k})$, where $\bar{\theta}_k$ is a real constant.

To proof these results in perspective, we recall that for the local NLS equation,

$$\mathrm{i}q_t + q_{xx} \pm 2q^2 q^* = 0 \tag{3.3.2}$$

which is obatined from the coupled Schrödinger equations (3.1.4) under the reduction of $r(x,t) = -q^*(x,t)$, the symmetry of its scattering data are $\bar{\zeta}_k = -\zeta_k^*$ and $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^*$.

Thus, symmetry relations for the nonlocal NLS equations are different from those local NLS equations. In particular, for the reverse-space and reverse-space-time NLS equations, eigenvalues in the upper and lower halves of the complex plane are completely independent. This independence allows for novel eigenvalue con-figurations, which will give rise to new types of multi-solitons. This will be demonstrated in the next section.

Before proving this theorem, we first establish a connection between the discrete scattering datda for N-solitons $\{\zeta_k, \bar{\zeta}_k, a_k, b_k, \bar{a}_k, \bar{b}_k\}(1 \le k \le N)$ and discrete eigenmodes in the eigenvalue problem $Y_x = MY$ and its adjoint problem $K_x = -KM$, where we set

$$Y_x = -\mathrm{i}\zeta\Lambda Y + QY, \tag{3.3.3a}$$

$$K_x = \mathrm{i}\zeta\Lambda K - KQ \tag{3.3.3b}$$

where the potenitial matrix Q is given by

$$Q = \begin{pmatrix} 0 & q(x,0) \\ r(x,0) & 0 \end{pmatrix} \tag{3.3.4}$$

and $q(x,0), r(x,0)$ are the initial conditions of functions $q(x,t), r(x,t)$ at $t=0$.

笔记 这里使用 $Q(x) = Q(x,0)$ 是因为在空间谱问题中，时间 t 被视为固定参数？存疑

Indeed $\forall\{\zeta_k, a_k, b_k\}$ of the discrete scattering data, where $\zeta \in \mathbb{C}_+$ is the eigenvalue of (3.3.3a), whose discrete eigenfunction Y_k has the following asymptotics

$$Y_k(x) \to \begin{bmatrix} a_k \mathrm{e}^{-\mathrm{i}\zeta_k x} \\ 0 \end{bmatrix}, x \to -\infty, \quad Y_k(x) \to \begin{bmatrix} 0 \\ -b_k \mathrm{e}^{\mathrm{i}\zeta_k x} \end{bmatrix}, x \to +\infty \tag{3.3.5}$$

Analogously, for the eigenvalue $\bar{\zeta} \in \mathbb{C}_-$ of the adjoint eigenvalue problem (3.3.3b), the discrete eigenfunction K_k has the following asymptotics

$$K_k(x) \to \begin{bmatrix} \bar{a}_k \mathrm{e}^{-\mathrm{i}\bar{\zeta}_k x} & 0 \end{bmatrix}, x \to -\infty, \quad K_k(x) \to \begin{bmatrix} 0 & -\bar{b}_k \mathrm{e}^{-\mathrm{i}\bar{\zeta}_k x} \end{bmatrix}, x \to +\infty \tag{3.3.6}$$

In view of this connection, in order to derive symmetry relations on the (discrete) scattering data, we will use symmetry relations of discrete eigenmodes in the eigenvalue problems (3.3.3a)-(3.3.3b).

证明 The reverse-space NLS equation(3.1.1) was derived from the coupled Schrödinger equations (3.1.4) under the reduction of (3.1.5a), and the potential matrix Q is

$$Q = \begin{pmatrix} 0 & q(x,0) \\ -q^*(-x,0) & 0 \end{pmatrix} \tag{3.3.7}$$

Obviously we have $Q^*(-x) = -\sigma_1^{-1} Q \sigma_1$, so

$$\begin{aligned} Y_x = -\mathrm{i}\zeta\Lambda Y + QY &\implies -Y_x(-x) = -\mathrm{i}\zeta\Lambda Y(-x) + Q(-x)Y(-x) \\ &\implies -Y^*(-x) = \mathrm{i}\zeta^*\Lambda Y^*(-x) + Q^*(-x)Y^*(-x) \\ &\implies -\alpha\sigma_1 Y^*(-x) = \mathrm{i}\alpha\sigma_1\zeta^*\Lambda Y^*(-x) - \alpha\sigma_1(\sigma_1^{-1}Q\sigma_1)Y^*(-x) \\ &\implies \alpha\sigma_1 Y^*(-x) = -\mathrm{i}\alpha(-\zeta^*)\Lambda\sigma_1 Y^*(-x) + Q\alpha\sigma_1 Y^*(-x) \end{aligned} \tag{3.3.8}$$

We get $\hat{Y}_x = -\mathrm{i}\hat{\zeta}\Lambda\hat{Y} + Q\hat{Y}$, where

$$\hat{\zeta} - -\zeta^*, \hat{Y} = \alpha\sigma_1 Y^*(-x), \quad \forall\alpha \in \mathbb{C} \tag{3.3.9}$$

This equation shows that: if $\zeta_k \in \mathbb{C}_+$ is an eigenvalue of the scattering problem (3.3.3a), then $\hat{\zeta}_k = -\zeta_k^* \in \mathbb{C}_+$ is also, and

$$\hat{\mathbf{v}}_{k0} = -\alpha\sigma_1\mathbf{v}_{k0}^* = \begin{pmatrix} -\alpha b_k^* \\ -\alpha a_k^* \end{pmatrix} \tag{3.3.10}$$

If $\mathrm{Re}(\zeta_k) \neq 0 \implies \hat{\zeta}_k = -\zeta_k^* \neq \zeta_k$. In this case, when the above $\hat{\mathbf{v}}_{k0}$ expression is inserted into the N-soliton formulae (3.2.3), then constant $-\alpha$ cancels out and does not contribute to the soluition. Thus we can set $-\alpha = 1$ without loss og generality. Then $\hat{\mathbf{v}}_{k0} = \sigma_1\mathbf{v}_{k0}^*$, hence the part 1 is proved.

If $\mathrm{Re}(\zeta_k) = 0 \implies \hat{\zeta}_k = -\zeta_k^* = \zeta_k$. Thus, their eigenvactors are also the same. Without loss of generality, we can scale the eigenvector $\mathbf{v}_{k0}$ so that $a_k = 1$, inserting this into (3.3.10), we have $\alpha = 1, \mathbf{v}_{k0} = (1, -\alpha)^T$, denoting $-\alpha = e^{\mathrm{i}\theta_k}, \theta_k \in \mathbb{R}$, we get $\mathbf{v}_{k0} = (1, e^{\mathrm{i}\theta_k})^T$, hence the part 2 is proved.

Repeating the above arguments on the adjoint eigenvalue problem (3.3.3b), parts 3 and 4 can be similarly proved. □

3.3.2 The reverse-time NLS equation

定理 3.2

For the reverse-time NLS equation (3.1.2). If ζ is a discrete eigenvalue of the associated Lax pair, then so is $-\zeta$. Hence, the discrete spectrum is symmetric with respect to the origin, and eigenvalues always appear in pairs $(\zeta, -\zeta)$, located in opposite halves of the complex plane.

For each such pair $(\zeta_k, \bar{\zeta}_k)$ with $\zeta_k \in \mathbb{C}_+$ and $\bar{\zeta}_k = -\zeta_k \in \mathbb{C}_-$, the associated eigenvectors $\mathbf{v}_{k0}$ and $\bar{\mathbf{v}}_{k0}$ satisfy $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^T$.

证明 The reverse-time NLS equation (3.1.2) was derived from the coupled Schrödinger equations (3.1.4) under the reduction of (3.1.5a), and the potential matrix Q is

$$Q = \begin{pmatrix} 0 & q(x,0) \\ -q(x,0) & 0 \end{pmatrix} \tag{3.3.11}$$

which fearures the following syummetry $Q^T(x) = -Q(x)$. Then, taking the transpose of the eigenvalue problem (3.3.3a), we have

$$Y_x = -\mathrm{i}\zeta\Lambda Y + QY \implies Y_x^T = -\mathrm{i}\zeta Y^T\Lambda^T + Y^T Q^T \implies Y_x^T = -\mathrm{i}\zeta Y^T\Lambda - Y^T Q \tag{3.3.12}$$

We get $\bar{Y}_x = \mathrm{i}\bar{Y}\Lambda - Y^T Q$, where

$$\bar{\zeta} = -\zeta, \bar{Y}(x) = Y^T(x) \tag{3.3.13}$$

It means that $[\bar{\zeta}, \bar{Y}(x)]$ satifies the adjoint eigenvalue equation (3.3.3b).

Thus, if $\zeta_k \in \mathbb{C}_+$ is an eigenvalue of the scattering problem (3.3.3a), then $\bar{\zeta}_k = -\zeta_k \in \mathbb{C}_-$ is an eigenvalue of the adjoint scattering problem (3.3.3b). Utilizing this eigenfunction relation as well as the large-x asymptotics of the eigenfunctions and adjoint eigenfunctions in (3.3.5)-(3.3.6), we readily find that $\bar{a}_k = a_k, \bar{b}_k = b_k$ and $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^T$. This completes the proof of the theorem. □

3.3.3 The reverse-space-time NLS equation

定理 3.3

For the reverse-space-time NLS equation (3.1.3), eigenvalues ζ can be anywhere in $\mathbb{C}_+$, and eigenvalues $\bar{\zeta}_k$ can be anywhere in $\mathbb{C}_-$. However, their eigenvectors must be of the forms

$$\mathbf{v}_{k0} = (1, \omega_k), \quad \bar{\mathbf{v}}_{k0} = (1, \bar{\omega}_k) \tag{3.3.14}$$

where $\omega_k = \pm 1, \bar{\omega}_k = \pm 1$

证明 The reverse-space-time NLS equation (3.1.3) was derived from the coupled Schrödinger equations (3.1.4) under the reduction of (3.1.5c), and the potential matrix Q is

$$Q = \begin{pmatrix} 0 & q(x,0) \\ -q(-x,0) & 0 \end{pmatrix} \tag{3.3.15}$$

which features the following symmetry $Q(-x) = -\sigma_1^{-1}Q(x)\sigma_1$. Then, taking the adjoint of the eigenvalue problem (3.3.3a), we have

$$\begin{aligned} Y_x = -\mathrm{i}\zeta\Lambda Y + QY &\implies -Y_x(-x) = -\mathrm{i}\zeta\Lambda Y(-x) + Q(-x)Y(-x) \\ &\implies -\sigma_1 Y_x(-x) = -\mathrm{i}\zeta\sigma_1\Lambda Y(-x) + \sigma_1 Q(-x)Y(-x) \\ &\implies \sigma_1 Y_x(-x) = -\mathrm{i}\zeta\sigma_1\Lambda Y(-x) + \sigma_1 Q(-x)Y(-x) \end{aligned} \tag{3.3.16}$$

We get $\hat{Y}_x(x) = -\mathrm{i}\zeta\Lambda Y(x) + QY(x)$, where

$$\hat{Y}(x) = \sigma_1 Y(-x) \tag{3.3.17}$$

This equation means that for any eigenvalue $\zeta_k \in \mathbb{C}_+$ and $Y_k(x)$ is its eigenfunction, so is $\hat{Y}_k(x) = \sigma_1 Y(-x)$. Thus $\hat{Y}_k$ amd Y_k are rlinearly dependent as

$$Y_k(x) = \omega_k \sigma_1 Y_k(x) \tag{3.3.18}$$

where ω_k is some constant. Utilizing this relation and the large-x asymptotics of the eigenfunction $Y_k(x)$ in (3.3.5), we readily find that $a_k = \omega_k b_k, b_k = \omega_k a_k$, so $\omega_k = \pm 1$. Without loss of generality, we can scale the eigenvector $\mathbf{v}_{k0}$ so that $a_k = 1$, then $\mathbf{v}_{k0} = (1, \omega_k)^T$.

Since (3.3.17) also means that for any eigenvalue $\bar{\zeta}_k \in \mathbb{C}_-$, if $K_k(x)$ is its adjoint eigenfunction, so is $\hat{K}_k(x) = \sigma_1 K(-x)$. Hence utilizing this relation and the large-x asymptotics of the adjoint eigenfunction $\hat{K}_k(x)$ in (3.3.6), we can similarly show that $\bar{\mathbf{v}}_{k0} = (1, \bar{\omega}_k)$, where $\bar{\omega}_k = \pm 1$. This complests the proof of the theorem. $\square$

第 4 章 3D Reverse Time Space NLS

4.1 AKNS's some reductions

In this chapter, let us focus on the 3×3 reverse space nonlocal AKNS system: Consider the Lax pair.

$$\begin{aligned} \Phi_x &= U\Phi = (\mathrm{i}\lambda\sigma_3 + P_0)\Phi, \\ \Phi_t &= V\Phi = -\left[2\mathrm{i}\lambda^2\sigma_3 + 2\lambda P + \mathrm{i}(P_0^2 + P_{0x})\sigma_3\right]\Phi, \end{aligned} \tag{4.1.1}$$

where

$$P_0 = \begin{pmatrix} 0 & p & q \\ -r_1 & 0 & 0 \\ -r_2 & 0 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{4.1.2}$$

where p, q, r_1, r_2 are functions of (x, t), $r_1 = ap_1 + bq_1, r_2 = cp_1 + dq_1$ and $a, b, c, d, \sigma_3 \in \mathbb{C}$. Thus, because the zero curve equation $U_t - V_x + [U, V] = 0$ can be written as

$$\mathrm{i}P_{0t} + P_{0xx}\sigma_3 + 2P_0^3\sigma_3 = 0, \tag{4.1.3}$$

we can obtain the following equations:

$$\mathrm{i}p_t + p_{xx} + 2p(pr_1 + qr_2) = 0, \tag{4.1.4a}$$

$$\mathrm{i}q_t + q_{xx} + 2q(pr_1 + qr_2) = 0, \tag{4.1.4b}$$

$$-\mathrm{i}r_{1t} + r_{1xx} + 2r_1(pr_1 + qr_2) = 0, \tag{4.1.4c}$$

$$-\mathrm{i}r_{2t} + r_{2xx} + 2r_2(pr_1 + qr_2) = 0. \tag{4.1.4d}$$

For the above equations (4.1.4), we can obtain p_1 maybe is $p, p(-x, t), p(x, -t), p(-x, -t)$ and their conjugation, due to the compatibility condition, p_1 can only be $p^*(x, t), p^*(-x, t), p(x, -t), p(-x, -t)$, and q_1 has a similar reductions.

If we condiser $p_1 = p^*(-x, t), q_1 = q^*(-x, t)$, then we can obtain the inverse space equations:

$$\mathrm{i}p_t + p_{xx} + 2p\left[app^*(-x, t) + bpq^*(-x, t) + b^*p^*(-x, t)q + dqq^*(-x, t)\right] = 0, \tag{4.1.5}$$

$$\mathrm{i}q_t + q_{xx} + 2q\left[app^*(-x, t) + bpq^*(-x, t) + b^*p^*(-x, t)q + dqq^*(-x, t)\right] = 0 \tag{4.1.6}$$

If we consider $p_1 = p(x, -t), q_1 = q(x, -t)$, then we can obtain the inverse time equations:

$$\mathrm{i}p_t + p_{xx} + 2p\left[app^*(x, -t) + bpq^*(x, -t) + b^*p^*(x, -t)q + dqq^*(x, -t)\right] = 0, \tag{4.1.7}$$

$$\mathrm{i}q_t + q_{xx} + 2q\left[app^*(x, -t) + bpq^*(x, -t) + b^*p^*(x, -t)q + dqq^*(x, -t)\right] = 0 \tag{4.1.8}$$

If we consider $p_1 = p(-x, -t), q_1 = q(-x, -t)$, then we can obtain the inverse time equations:

$$\mathrm{i}p_t + p_{xx} + 2p\left[app^*(-x, -t) + bpq^*(-x, -t) + b^*p^*(-x, -t)q + dqq^*(-x, -t)\right] = 0, \tag{4.1.9}$$

$$\mathrm{i}q_t + q_{xx} + 2q\left[app^*(-x, -t) + bpq^*(-x, -t) + b^*p^*(-x, -t)q + dqq^*(-x, -t)\right] = 0 \tag{4.1.10}$$

Before we get started, let's consider another Lax pair. This lax pair has better symmetry and ready solutions. And it avoids the disadvantages of $ad - bb^*$.

$$\Phi_x = (-\mathrm{i}\lambda\Lambda + P)\Phi, \tag{4.1.11}$$

$$\Phi_t = \left[2\mathrm{i}\lambda^2\Lambda - 2\mathrm{i}\lambda P + \mathrm{i}(P^2 + P_x\Lambda)\right]\Phi, \tag{4.1.12}$$

where

$$P = \begin{pmatrix} 0 & 0 & p \\ 0 & 0 & q \\ -r_1 & -r_2 & 0 \end{pmatrix} \quad \Lambda = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.1.13}$$

and p, q, r_1, r_2 is same as above. The Lax pair is equivalent to the AKNS system (4.1.1). You can see the zero curve equations's 13, 23, 31, 32 to find that is same to (4.1.4)

$$\Theta_0 = \begin{pmatrix} a & b & \\ b^* & d & \\ & & 1 \end{pmatrix} \tag{4.1.14}$$

and the eigenvalue problem and the adjoint eigenvalue problem are given as follows:

$$Y_x = \mathrm{i}\zeta\sigma_3 Y + PY, \tag{4.1.15a}$$

$$K_x = -\mathrm{i}\zeta^* K\sigma_3 - KP, \tag{4.1.15b}$$

Following previous Chapter's Theorem. 3.1 - Theorem. 3.3 , we give the following three theorems.

4.2 Solution of 3d NLS

Following this Riemann-Hilbert mothed, the solution is given by Wang and Yang[1]. N-solitons in this system were explictly written as:

$$\begin{bmatrix} p(x,t) \\ q(x,t) \end{bmatrix} = 2\mathrm{i} \sum_{j,k=1}^{N} \begin{bmatrix} \alpha_k \\ \beta_j \end{bmatrix} \mathrm{e}^{-(\theta_k^* - \theta_j)} (M^{-1})_{jk}, \tag{4.2.1}$$

where

$$M_{jk} = \frac{1}{\lambda_j^* - \lambda_k} \left[(a\alpha_j^*\alpha_k + b\beta_j^*\alpha_k + b^*\alpha_j^*\beta_k + d\beta_j^*\beta_k)\mathrm{e}^{-(\theta_k + \theta_j^*)} + \mathrm{e}^{\theta_k + \theta_j^*} \right] \tag{4.2.2}$$

and $\theta_k = -\mathrm{i}\lambda_k x + 2\mathrm{i}\lambda_k^2 t$.

证明 It can be written in a more general form

$$p(x,t) = -2\mathrm{i}P_{12} = -2\mathrm{i}\left(\sum_{j,k=1}^{N} v_j (M^{-1})_{jk} \hat{v}_k \right)_{12} \tag{4.2.3}$$

$$q(x,t) = -2\mathrm{i}P_{13} = -2\mathrm{i}\left(\sum_{j,k=1}^{N} v_j (M^{-1})_{jk} \hat{v}_k \right)_{13} \tag{4.2.4}$$

$$r_1(x,t) = -2\mathrm{i}P_{13} = -2\mathrm{i}\left(\sum_{j,k=1}^{N} v_j (M^{-1})_{jk} \hat{v}_k \right)_{31} \tag{4.2.5}$$

$$r_2(x,t) = -2\mathrm{i}P_{13} = -2\mathrm{i}\left(\sum_{j,k=1}^{N} v_j (M^{-1})_{jk} \hat{v}_k \right)_{32} \tag{4.2.6}$$

where

$$M_{jk} = \frac{\hat{v}_j v_k}{\lambda^* - \lambda_k}, \quad v_k = e^{\theta_k \sigma_3} v_{k0}, \quad \hat{v}_k = v_k^\dagger \Theta_0 \tag{4.2.7}$$

If we set $\mathbf{v}_{k0} = (\alpha_k, \beta_k, 1)^T$, then

$$\begin{aligned} M_{jk} &= \frac{\hat{v}_j v_k}{\lambda^* - \lambda_k} = \frac{1}{\lambda_j^* - \lambda_k} v_j^\dagger \Theta_0 v_k \\ &= \frac{1}{\lambda^* - \lambda_k} (\alpha_j^*, \beta_j^*, 1) e^{\theta_j^* \Lambda} \Theta_0 e^{\theta_j \Lambda} (\alpha_k, \beta_k, 1)^T \\ &= \frac{1}{\lambda^* - \lambda_k} (\alpha_j^*, \beta_j^*, 1) \Theta_0 e^{(\theta_j^* + \theta_j)\Lambda} (\alpha_k, \beta_k, 1)^T \Leftarrow (\Theta_0 e^{\theta_j \Lambda} = e^{\theta_j \Lambda} \Theta_0) \\ &= \frac{1}{\lambda_j^* - \lambda_k} \left[(a\alpha_j^*\alpha_k + b\beta_j^*\alpha_k + b^*\alpha_j^*\beta_k + d\beta_j^*\beta_k)\mathrm{e}^{-(\theta_k + \theta_j^*)} + \mathrm{e}^{\theta_k + \theta_j^*} \right] \end{aligned} \tag{4.2.8}$$

The p, q can be similarly obtained. □

4.3 The relation between origin and reductions

笔记 这里完全仿照上一章的写法, 有些地方尚未完善

Indeed $\forall\{\zeta_k, a_k, b_k\}$ of the discrete scattering data, where $\zeta \in \mathbb{C}_+$ is the eigenvalue of (3.3.3a), whose discrete eigenfunction Y_k has the following asymptotics

$$Y_k(x) \to \begin{bmatrix} \alpha_k e^{-i\zeta_k x} \\ \beta_k e^{-i\zeta_k x} \\ 0 \end{bmatrix}, x \to -\infty, \quad Y_k(x) \to \begin{bmatrix} 0 \\ 0 \\ -\gamma_k e^{i\zeta_k x} \end{bmatrix}, x \to +\infty \tag{4.3.1}$$

Analogously, for the eigenvalue $\bar{\zeta} \in \mathbb{C}_-$ of the adjoint eigenvalue problem (3.3.3b), the discrete eigenfunction K_k has the following asymptotics

$$K_k(x) \to \begin{bmatrix} \bar{\alpha}_k e^{-i\bar{\zeta}_k x} & \bar{\beta}_k e^{-i\bar{\zeta}_k x} & 0 \end{bmatrix} \Theta_0, x \to -\infty, \quad K_k(x) \to \begin{bmatrix} 0 & 0 & -\bar{\gamma}_k e^{-i\bar{\zeta}_k x} \end{bmatrix} \Theta_0, x \to +\infty \tag{4.3.2}$$

In view of this connection, in order to derive symmetry relations on the (discrete) scattering data, we will use symmetry relations of discrete eigenmodes in the eigenvalue problems (4.1.15a)-(4.1.15b).

4.3.1 Inverse Space

In this case $a, d \in \mathbb{R}, b = c^*, p_1 = p^*(-x,t), q_1 = q^*(-x,t)$ and $\zeta \in \mathbb{C}_+$.

定理 4.1 (Inverse Space)

For the inverse-space AKNS (4.1.5), if ζ is an eigenvalue, so is $-\zeta^*$. Thus non-pirely-imaginary eigenvalues appear as $(\zeta, -\zeta^*)$, which lie in the same half of the complex plane. Symmetry relations are given as follows:

1. If $(\zeta_k, \hat{\zeta}_k) \in \mathbb{C}_+$, then $\hat{\zeta}_k = -\zeta_k^*$, their column eigenvectors are related as $\hat{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^\dagger \Theta_0 \Theta_1$.
2. If $\zeta_k \in i\mathbb{R}_+$, its eigenvectors is of the form $\mathbf{v}_{k0} = (1, e^{i\theta_k})^T$, where θ_k is a real constant.

证明 The reverse-space NLS equation (4.1.5) was derived from the couped Schrödinger equations under the reduction of $p_1 = p^*(-x,t), q_1 = q^*(-x,t)$ and the potential matrix P and Θ_1 is

$$P = \begin{pmatrix} 0 & 0 & p(x,0) \\ 0 & 0 & q(x,0) \\ -r_1(-x,0) & -r_2(-x,0) & 0 \end{pmatrix} \quad \Theta_1 = \begin{pmatrix} a & b^* & \\ b & d & \\ & & -1 \end{pmatrix} \tag{4.3.3}$$

obviously, we have $P^\dagger(-x) = \Theta_1 P(x) \Theta_1^{-1}$, so

$$\begin{aligned} Y_x = i\zeta\Lambda Y + PY &\implies -Y_x(-x) = i\zeta\Lambda Y(-x) + P(-x)Y(-x) \\ &\implies -Y_x^\dagger(-x) = -i\zeta^* Y^\dagger(-x)\Lambda + Y^\dagger(-x)P^\dagger(-x) = -i\zeta^* Y^\dagger(-x)\Lambda + Y^\dagger \Theta_1 P(x)\Theta_1^{-1} \\ &\implies Y_x^\dagger(-x)\Theta_1 = i\zeta^* Y^\dagger(-x)\Lambda\Theta_1 - Y^\dagger(-x)\Theta_1 P^\dagger(x) \end{aligned} \tag{4.3.4}$$

If set $\hat{Y}(x) = Y(-x)\Theta_1, \hat{\zeta} = -\zeta^*$, and because $\Lambda\Theta_1 = \Theta_1\Lambda$, we have

$$\hat{Y}(x) = -i\hat{\zeta}\hat{Y}(x)\Lambda - \hat{Y}(x)P(x) = -\hat{Y}(i\hat{\zeta}\Lambda + P(x)) \tag{4.3.5}$$

It means that $[\hat{\zeta}, \hat{Y}(x)]$ is satifies the adjoint eigenvalue equation (4.1.15b). □

4.3.2 Inverse Time

In this case $b = c$, $p_1 = p(x,-t), q_1 = q(x,-t)$

定理 4.2 (Inverse Time)

For the reverse-time NLS equation (4.1.7). If ζ is a discrete eigenvalue of the associated Lax pair, then so is $-\zeta$. Hence, the discrete spectrum is symmetric with respect to the origin, and eigenvalues always appear in pairs $(\zeta, -\zeta)$, located in opposite halves of the complex plane.

For each such pair $(\zeta_k, \hat{\zeta}_k)$ with $\zeta_k \in \mathbb{C}_+$ and $\hat{\zeta}_k = -\zeta_k \in \mathbb{C}_-$, the associated eigenvectors $\mathbf{v}_{k0}$ and $\hat{\mathbf{v}}_{k0}$ satisfy $\hat{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^{\dagger}\Theta_0\Theta_1$.

证明 we can get $P^T(x) = -\Theta_2 P(x)\Theta_2^{-1}$, where Θ_2 is

$$\Theta_2 = \begin{pmatrix} a & b & \\ b & d & \\ & & 1 \end{pmatrix} \tag{4.3.6}$$

$$\begin{aligned} Y_x(x) = \mathrm{i}\zeta\Lambda Y + PY &\implies Y_x(x) = \mathrm{i}\zeta\Lambda Y(x) + P(x)Y(x) \\ &\implies Y_x^T(x) = \mathrm{i}\zeta Y^T(x)\Lambda + Y^T(x)P^T(x) = \mathrm{i}\zeta Y^T(x)\Lambda - Y^T(x)\Theta_2 P(x)\Theta_2^{-1} \\ &\implies Y_x^T(x)\Theta_2 = \mathrm{i}\zeta Y^T(x)\Lambda\Theta_2 - Y^T(x)P(x) \end{aligned} \tag{4.3.7}$$

If we set $\hat{Y}(t) = Y(-t)M, \hat{\zeta} = -\zeta$, and because $\sigma_3\Theta_2 = \Theta_2\sigma_3$, we have

$$\hat{Y}_x(t) = -\mathrm{i}\hat{\zeta}\hat{Y}(t)\sigma_3 - \hat{Y}(t)P(t) = -\hat{Y}(t)(\mathrm{i}\hat{\zeta}\sigma_3 + P(t)) \tag{4.3.8}$$

It means that $[\hat{\zeta}, \hat{Y}(t)]$ is satisfies the adjoint eigenvalue equation (4.1.15b).

Thus, if $\zeta_k \in \mathbb{C}_+$ is an eigenvalue of the scattering problem (4.1.15a), then $\hat{\zeta}_k = -\zeta_k \in \mathbb{C}_-$ is an eigenvalue of the adjoint scattering problem (4.1.15b). Utilizing this eigenfunction relation as well as the large-x asymptotics of the eigenfunctions and adjoint eigenfunctions in (3.3.5)-(3.3.6), we readily find that $\bar{a}_k = a_k, \bar{b}_k = b_k$ and $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^T$. This completes the proof of the theorem. □

4.3.3 Inverse Space-Time

In this case $b = c$, $p_1 = p(-x,-t), q_1 = q(-x,-t)$ and $\zeta \in \mathbb{C}_+$.

定理 4.3 (Inverse Time)

For the reverse-time NLS equation (4.1.7). If ζ is a discrete eigenvalue of the associated Lax pair, then so is $-\zeta$. Hence, the discrete spectrum is symmetric with respect to the origin, and eigenvalues always appear in pairs $(\zeta, -\zeta)$, located in opposite halves of the complex plane.

For each such pair $(\zeta_k, \hat{\zeta}_k)$ with $\zeta_k \in \mathbb{C}_+$ and $\hat{\zeta}_k = -\zeta_k \in \mathbb{C}_-$, the associated eigenvectors $\mathbf{v}_{k0}$ and $\hat{\mathbf{v}}_{k0}$ satisfy $\hat{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^T$.

证明 The reverse-space NLS equation (4.1.5) was derived from the couped Schrödinger equations under the reduction of $p_1 = p(-x,-t), q_1 = q(-x,-t)$ and the potential matrix P and Θ_3 is

$$\Theta_3 = \begin{pmatrix} a & b & \\ b & d & \\ & & -1 \end{pmatrix} \tag{4.3.9}$$

obviously, we have $P^T(-x) = \Theta_3 P(x)\Theta_3^{-1}$, so

$$\begin{aligned} Y_x(x) = \mathrm{i}\lambda\sigma_3 Y + PY &\implies -Y_x(-x) = \mathrm{i}\lambda\sigma_3 Y(-x) + P(-x)Y(-x) \\ &\implies -Y_x^T(-x) = \mathrm{i}\lambda Y^T(-x)\sigma_3 + Y^T(-x)P^T(-x) = \mathrm{i}\lambda Y^T(x)\sigma_3 + Y^T(-x)\Theta_2 P(x)\Theta_2^{-1} \\ &\implies Y_x^T(-x)\Theta_2 = -\mathrm{i}\lambda Y^T(x)\sigma_3\Theta_2 - Y^T(-x)P(x) \end{aligned} \tag{4.3.10}$$

If we set $\hat{Y}(x) = Y(-x)M, \hat{\zeta} = \zeta$, and because $\sigma_3\Theta_3 = \Theta_3\sigma_3$, we have

$$\hat{Y}_x(t) = -\mathrm{i}\hat{\zeta}\hat{Y}(t)\sigma_3 - \hat{Y}(t)P(t) = -\hat{Y}(t)(\mathrm{i}\hat{\zeta}\sigma_3 + P(t)) \tag{4.3.11}$$

It means that $[\hat{\zeta}, \hat{Y}(t)]$ is satifies the adjoint eigenvalue equation (4.1.15b).

Thus, if $\zeta_k \in \mathbb{C}_+$ is an eigenvalue of the scattering problem (4.1.15a), then $\hat{\zeta}_k = \zeta_k \in \mathbb{C}_+$ is an eigenvalue of the adjoint scattering problem (4.1.15b). Utilizing this eigenfunction relation as well as the large-x asymptotics of the eigenfunctions and adjoint eigenfunctions in (3.3.5)-(3.3.6), we readily find that $\bar{a}_k = a_k, \bar{b}_k = b_k$ and $\bar{\mathbf{v}}_{k0} = \mathbf{v}_{k0}^T$. This completes the proof of the theorem. $\square$

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