Example (Radius of Convergence)

1) $f(z) = \sum_{j=0}^{\infty} 2^{-j-(-1)^j} z^j$, so $a_j := 2^{-j-(-1)^j}$, $z_0 = 0$

$$= \frac{1}{2^{j+(-1)^j}} = \frac{1}{2^j} \cdot \frac{1}{2^{(-1)^j}}$$

Root test: $\overline{\lim_{j\to\infty}} \sqrt[j]{|a_j|} = \lim_{j\to\infty} \sqrt[j]{2^{-j-1}} = \lim_{j\to\infty} \left(\frac{1}{2^{j+1}}\right)^{1/j} = \frac{1}{2} \implies \rho = 2$

Ratio test: $\overline{\lim_{j\to\infty}} \left|\frac{a_{j+1}}{a_j}\right| = \overline{\lim_{j\to\infty}} \frac{2^{\cancel{-j}-1+(-1)^j}}{2^{\cancel{-j}-(-1)^j}} = \lim_{j\to\infty} 2^{-1} \cdot \frac{2^{1}}{2^{-1}} = 2$

$$\liminf_{j\to\infty} \left|\frac{a_{j+1}}{a_j}\right| = \underline{\lim_{j\to\infty}} \left|\frac{a_{j+1}}{a_j}\right| = \lim_{j\to\infty} 2^{-1} \cdot \frac{2^{-1}}{2^{+1}} = \frac{1}{8}$$

2) Modified Geometric series $f(z) = \sum_{j=0}^{\infty} 2^{-j} z^j = \frac{1}{1-z/2}$; $\rho = ?$

Root test $\sqrt[j]{|2^{-j}|} \xrightarrow{j\to\infty} \frac{1}{2}$ or ratio test $\left|\frac{a_{j+1}}{a_j}\right| = \frac{2^{-j-1}}{2^{-j}} = \frac{2^{\cancel{j}}}{2^{\cancel{j}+1}} = \frac{1}{2} \implies \rho = 2$

3) Logarithm as series

Spoiler: $-\ln(1-z) = \sum_{j=1}^{\infty} \frac{z^j}{j}$, $a_j = \frac{1}{j} \implies \lim_{j\to\infty} \frac{j}{j+1} = 1 \implies \rho = 1$
$\begin{cases} \implies \text{conv. for } |z| < 1 \\ \implies \text{div. for } |z| > 1 \\ z = 1 \implies \text{div. to } +\infty \\ z = -1 \implies \text{conv. to } -\ln(2) \text{ (but no abs. conv.)} \end{cases}$