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Vector Analysis

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Let $D \subseteq \mathbb{R}^n$.
A function $\mathbf{F} : D \to \mathbb{R}^n$ with $n \geq 2$ is also called a **vector field**.

Example:
Let $f : D \to \mathbb{R}$, then ∇f is a vector field.

A vector field F is called **conservative** or **gradient field**, if there exists a function ϕ s.t.:

$$\mathbf{F} = \nabla \phi.$$

with a „−" in physics

Then we call ϕ a (scalar) **potential (function)**.

Application of Potential Functions

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Path independence:

Let D be a simply connected domain.

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{s} = \phi(\mathbf{b}) - \phi(\mathbf{a})$$

for a sufficiently smooth curve from **a** to **b**.

Note that the integral does not depend on the curve, but only on the start and end points, **a** and **b**.

How can we decide whether a vector field is conservative?

Characterization of Potential Functions

Consider a simply connected domain D.

The following statements are equivalent:

- $\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{F} \cdot d\mathbf{s} = \phi(\mathbf{b}) - \phi(\mathbf{a})$ for a curve from **a** to **b**.
- The curve integral over **F** along any closed curve is zero.

integrability condition

- $\nabla \times \mathbf{F} = 0$ $\overset{n=3}{\Longleftrightarrow}$ $\frac{\partial}{\partial x_i} F_j = \frac{\partial}{\partial x_j} F_i, \quad 1 \leq i, j \leq n$

We suppose that the involved curves are sufficiently smooth.

Note that ϕ is unique up to an additive constant.

Examples: Potential

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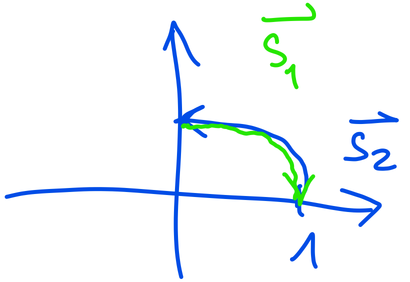
Consider $\vec{v}: \mathbb{R}^2 \to \mathbb{R}^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} x \\ 1 \end{pmatrix}$

$\vec{s}_1 : [0,1] \to \mathbb{R}^2, \quad t \longmapsto \begin{pmatrix} t \\ \sqrt{1-t^2} \end{pmatrix}$

$\vec{s}_2 : [0, \frac{\pi}{2}] \to \mathbb{R}^2, \quad t \longmapsto \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$



$\vec{v}$ is conservative, i.e. there exists a potential ϕ

$$\frac{\partial}{\partial x} v_2 = \frac{\partial}{\partial x} 1 = 0 = \frac{\partial}{\partial y} v_1 = \frac{\partial}{\partial y} x \quad \checkmark$$

$$-\int_{\vec{s}_1} \vec{v} \cdot d\vec{s} \overset{\text{cons.}}{=} \int_{\vec{s}_2} \vec{v} \cdot d\vec{s} = \int_0^{\pi/2} \vec{v}(r(t)) \cdot r'(t)\, dt = \ldots = \frac{1}{2}$$

Fundamental Theorem (of Analysis)

The fundamental theorem of Analysis 1

$$F(x) := \int_a^x F'(\tilde{x})\, d\tilde{x}$$

or

$$\int_a^b F'(x)\, dx = F(b) - F(a)$$

cannot be easily generalized to several dimensions.

Integral Theorems

Some generalizations are:

- Gradient theorem ("1d")

$$\int_\gamma \nabla f(\mathbf{r}) \cdot d\mathbf{r} = f(\mathbf{r}(\mathbf{b})) - f(\mathbf{r}(\mathbf{a}))$$

(γ a curve from **a** to **b** in $\mathbb{R}^n$ with parametrization **r**)

- Stokes theorem (2d)

$$\iint_M (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \int_{\partial M} \mathbf{F} \cdot d\gamma$$

(M a 2d submanifold of $\mathbb{R}^3$, bounded by a curve γ representing ∂M)

- Gauss divergence theorem in 3d

outer normal

$$\iiint_K \underbrace{\nabla \cdot \mathbf{F}}_{\text{sinks/sources}} \, dV = \iint_{\partial K} \mathbf{F} \cdot \overbrace{d\mathbf{A}}^{= \vec{n} \cdot da}$$

($K \subseteq \mathbb{R}^3$ compact. May be generalized to $\mathbb{R}^n$.)

Important Differential Operators

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Recall: gradient

$$\mathit{grad}\, f(\mathbf{x}) = \nabla f(\mathbf{x}) = \left(\frac{\partial}{\partial x_1} f(\mathbf{x}) \ldots, \frac{\partial}{\partial x_n} f(\mathbf{x})\right)^\top$$

Divergence (sources or sinks)

$$\mathit{div}\, \mathbf{F}(\mathbf{x}) = \nabla \cdot \mathbf{F}(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} F_i(\mathbf{x})$$

Curl or rot(ation), in $\mathbb{R}^3$ ($\mathbb{R}^2$ also possible):

$$\mathit{curl}\, \mathbf{F}(\mathbf{x}) = \nabla \times \mathbf{F}(\mathbf{x}) = \begin{pmatrix} \frac{\partial}{\partial x_2} F_3(\mathbf{x}) - \frac{\partial}{\partial x_3} F_2(\mathbf{x}) \\ \frac{\partial}{\partial x_3} F_1(\mathbf{x}) - \frac{\partial}{\partial x_1} F_3(\mathbf{x}) \\ \frac{\partial}{\partial x_1} F_2(\mathbf{x}) - \frac{\partial}{\partial x_2} F_1(\mathbf{x}) \end{pmatrix}$$

Laplace operator

$$\mathit{div}\, \mathit{grad}\, f(\mathbf{x}) = \nabla \cdot \nabla f(\mathbf{x}) = \Delta f(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} f(\mathbf{x})$$

Integration by Parts

Integration by parts

$$\iiint_V g\nabla\cdot\mathbf{f}\,dV = \iint_{\partial V} g\mathbf{f}\cdot d\mathbf{A} - \iiint_V \nabla g\cdot\mathbf{f}\,dV$$

(May be generalized to $\mathbb{R}^n$.)

$$\nabla\cdot(g\vec{f}) = \nabla g\cdot\vec{f} + g(\nabla\cdot\vec{f})$$

Integral Transforms

Fourier and Laplace transforms are useful tools for solving (partial) differential equations

Applications of the Fourier transform

- Information technology/electrical engineering: Low-pass filter
- Spectroscopy (NMR, FTIR, . . .)
- Acoustics
- Quantum mechanics
- ...
- Generalizes Fourier series from periodic to (some) non-periodic functions

Fourier Transform: Motivation I

Suppose

$$f(x) = \sum_{k=-\infty}^{\infty} c_k \exp(ik(\Delta\omega)x), \quad \Delta\omega = \frac{2\pi}{T}$$

with complex Fourier coefficients

$$\begin{aligned} c_k &= \frac{1}{T}\int_0^T f(x)\exp(-ik\overset{\Delta\omega}{\cancel{\omega}}x)\,dx \\ &= \frac{1}{T}\int_{-T/2}^{T/2} f(x)\overline{\exp(ik\underset{\Delta\omega}{\cancel{\omega}}x)}\,dx, \quad k \in \mathbb{Z} \end{aligned}$$

Introduce $\omega_k := k\Delta\omega$ and insert c_k into the series:

$$f(x) = \sum_{k=-\infty}^{\infty}\left(\frac{1}{T}\int_{-T/2}^{T/2} f(x)\overline{\exp(i\underset{k}{\omega}x)}\underset{dx}{\uparrow}\right)\exp(i\omega_k x) \quad = \dots$$

Fourier Transform: Motivation II

$$f(x) = \sum_{k=-\infty}^{\infty} \overbrace{\left(\frac{1}{T} \int_{-T/2}^{T/2} f(\xi) \overline{\exp(i\omega_k \xi)} \, d\xi \right)}^{\text{for clarity: } x \rightsquigarrow \xi} \exp(i\omega_k x)$$

$$= \cdot\cdot \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \left(\int_{-T/2}^{T/2} f(\xi) \overline{\exp(i\omega_k \xi)} \, d\xi \right) \exp(i\omega_k x)$$

$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \left(\int_{-T/2}^{T/2} f(\xi) \overline{\exp(i\omega_k \xi)} \, d\xi \right) \exp(i\omega_k \cancel{t}\, x) \Delta\omega$$

$$\Delta\omega \to 0, \quad T \to \infty$$

The series may be interpreted as a limit of Riemann sums that approximate an integral on $[-\infty, \infty]$ with refinement $\Delta\omega$ and grid points ω_k, $k \in \mathbb{Z}$.

Fourier Transform - Definition

For $T \to \infty$ (and $\Delta\omega \to 0$) we may expect convergence to an improper integral:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega x) \, d\omega$$

with

$$F(\omega) = \int_{-\infty}^{\infty} f(x) \overline{\exp(i\omega x)} \, dx$$

Definition:

$$F(\omega) := \mathcal{F}\{f(x)\} := \int_{-\infty}^{\infty} f(x) \exp(-i\omega x) \, dx$$

is called **Fourier transform** (or spectral function) of $f : \mathbb{R} \to \mathbb{C}$, if the integral exists for all $\omega \in \mathbb{R}$.

$$\mathcal{F}^{-1}\{F(\omega)\} := f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega x) \, d\omega$$

is called **inverse Fourier transform** of $F : \mathbb{R} \to \mathbb{C}$, if the integral exists for all $x \in \mathbb{R}$.

Fourier Transform - Properties

- Linearity

$$\mathcal{F}\{c_1 f_1(x) + c_2 f_2(x)\} = c_1 \mathcal{F}\{f_1(x)\} + c_2 \mathcal{F}\{f_2(x)\} = c_1 F_1(\omega) + c_2 F_2(\omega)$$

$$c_1, c_2 \in \mathbb{R}$$

- Displacement
- Similarity
- Derivatives

$$\mathcal{F}\{f'(t)\} = i\omega \mathcal{F}(f(t)) = i\omega F(\omega)$$

- Modulation

Laplace Transform

Laplace transform

$$F(s) := \mathcal{L}\{f(t)\} := \int_0^\infty f(t)\exp(-st)\,dt,$$

is called **Laplace transform** of $f : \mathbb{R}^+ \to \mathbb{R}$,
if the integral exists for some $s \in \mathbb{R}$.

The inverse Laplace transform is hard to calculate.
Usually, reference tables are used.

The Laplace transform has similar properties as the Fourier transform,
in particular

$$\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0) = sY(s) - y(0)$$

transforming differential equations (incl. initial values) into algebraic equations

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Outlook and insight: What we did in Analysis 2

- **Differential calculus in higher dimensions (mainly 2d, 3d)**
- **Integral calculus in higher dimensions (mainly 2d, 3d)**
- Sequences of functions
- **Power series,**

  Taylor series/Taylor expansions

  Fourier series
- ...

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What we did in Analysis 2 - More Details

- Sequences of functions, uniform convergence
- Power series,
  especially Taylor series, Fourier series,
  discrete Fourier transform
- Limits and continuity (for any d), curves (application: ECC)
- Differential calculus (for any d),
  especially optimization, linear regression
- Integrals along curves, on sectors; solids of revolution
- Integral calculus (for any d)
- Vector analysis (brief insight)
- Fourier transform (very brief insight)

What we have not discussed here

- Total differential, implicit functions, implicit differentiation
- Optimization with constraints, Lagrange multipliers
- Curvature
- **Differential equations**
- Integrals over curved surfaces, . . .
- Laplace transform
- Fundamental theorem of Algebra
  to be proved by analytical methods ;)
- . . .

. . . and all topics of this lecture may be discussed with all proofs and more details and extensions

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References: recommended literature

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Formula collections are also available in English.

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Software packages

General mathematical software:

- SCILAB, free software, `scilab.org`)
- MATLAB, campus licence
- Wolfram Mathematica
- ...

Computer algebra, i.e. “symbolic calculators”:

- MAPLE
- MATLAB, symbolic toolbox, campus licence ???
- Wolfram Alpha
- ...

For optimization, finite element methods, machine learning, . . . specialized software packages exist.

Programming languages as Python, Java, C, . . . are always useful.