Outline

Differential Calculus in $\mathbb{R}^n$

In Analysis 1 we have discussed differentiation of functions of 1 variable.

Now we consider a real-valued function in several variables

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$$
$$\mathbf{x} = (x_1, \ldots, x_n)^\top \mapsto f(x_1, \ldots, x_n) = f(\mathbf{x})$$

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Later we are going to extend the differential calculus to vector-valued functions (of several variables)

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$$
$$\mathbf{x} = (x_1, \ldots, x_n)^\top \mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))^\top$$

$$= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix}$$

Special Case: n = 2

A real-valued function in 2 variables

$$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$$

x, y indep. variables

$$\mathbf{x} = (x_1, x_2)^\top \mapsto f(x_1, x_2) = z$$

$$= (x, y)^T$$

dependent variable

We may plot the function value as 3rd coordinate over the real plane $\mathbb{R}^2$.
The graph of f is a subset of $\mathbb{R}^3$: a "landscape" or "mountains".

Contour Line, Equipotential Surface, ...

Definition (Level set)

We define the **level set** of a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n)^\top \mapsto f(x_1, \ldots, x_n)$ for the function value $c \in \mathbb{R}$ as the set

$$N_c := \{\mathbf{x} \in D \,|\, f(\mathbf{x}) = c\}.$$

The structure of N_c may be "complicated", it might also be the empty set.

For $n = 2$ the level set is called a contour line or level curve (though it may be an area, e.g.),
for $n = 3$ the level set is called an equipotential surface (though it may be a solid, e.g.).

Partial Function: Definition

A partial function is a "cross section"-function that is obtained by freezing all but 1 variables, e.g. x_i:

$$
\begin{aligned}
g_i : D_i \subseteq \mathbb{R} &\to \mathbb{R} \\
x_i &\mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n),
\end{aligned}
$$

with $(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) \in D$ for all $x_i \in D_i$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ fixed.

Partial Function: Example

$$f(x_1, x_2) = x_1^2 + x_2^2$$

→ see blackboard

Partial Derivatives: Definition

Definition (Partial derivative)

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $\mathbf{x} = (x_1, \ldots, x_n)^\top \mapsto f(x_1, \ldots, x_n)$ and $a \in D$, D an open set.

If the derivative of the partial function

$$x_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) =: f_i(\ldots)$$

exists at $x_i = a_i$,
then we call it the **partial derivative** of f w.r.t. x_i at a.

We write:

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) \quad \text{or} \quad f'_{x_i}(\mathbf{a}) \quad \text{or} \quad f_{x_i}(a) \text{ or } \frac{\partial}{\partial x_i} f(a)$$

We say f is **partially differentiable** in $\mathbf{a}$, if all $\frac{\partial f}{\partial x_i}(\mathbf{a})$ exist.
We say f is partially differentiable in $E \subseteq D$, if f is partially differentiable at any $\mathbf{a} \in E$.

Partial Derivatives: Gradient

We assume that f is partially differentiable in D.
It is convenient to arrange the (first) partial derivatives in a vector.

We define the **gradient** of $f : D \subseteq \mathbb{R} \to \mathbb{R}$ as the column vector (function):

$$\operatorname{grad} f(\mathbf{x}) = \nabla f(\mathbf{x}) := \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_i}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

2d: $\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix}$

The gradient is

- orthogonal to level sets and
- points into the direction of the steepest ascent.

Partial Derivatives: Examples

$$f(x,y) = (x^2+y^2)\exp(\sin(x+y))$$

$$\nabla f(x,y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x,y) \\ \frac{\partial f}{\partial y}(x,y) \end{pmatrix} = \exp(\sin(x+y))\left(2\begin{pmatrix} x \\ y \end{pmatrix} + \dots\right.$$

$$\left. + (x^2+y^2)\cos(x+y)\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$g(x,y) = 100(y-x^2)^2 + (1-x)^2$$

Rosenbrock or banana function

$$\nabla g(x,y) = \dots = \begin{pmatrix} -400x(y-x^2) - 2(1-x) \\ 200(y-x^2) \end{pmatrix}$$

Higher Partial Derivatives: Definition

Partial derivatives w.r.t. some x_j are again functions of x_1, x_2, ..., x_n and, again, we may search for partial derivatives w.r.t. some x_i, yielding **second partial derivatives**. $(i, j = 1, \ldots, n)$

We write:

$$\frac{\partial}{\partial x_i}\frac{\partial f}{\partial x_j}(\mathbf{a}) \quad \text{or} \quad \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{a}) \quad \text{or} \quad f''_{x_i,x_j}(\mathbf{a}) \quad \text{or} \quad f''_{i,j}(\mathbf{a}) \quad \ldots$$

In 2d with $(x, y) = (x_1, x_2)$ this reads, e.g.:

$$\frac{\partial}{\partial x}\frac{\partial f}{\partial y}(\mathbf{a}) \quad \text{or} \quad \frac{\partial^2}{\partial x \partial y} f(\mathbf{a}) \quad \text{or} \quad f''_{x,y}(\mathbf{a}) \quad \text{or} \quad f''_{1,2}(\mathbf{a}) \quad \ldots$$

Higher Partial Derivatives: Hesse Matrix

We assume that f is 2x partially differentiable in D. Again it is convenient to order the second partial derivatives in a (quadratic) matrix.

We define the **Hesse matrix** of $f : D \subseteq \mathbb{R} \to \mathbb{R}$ as the quadratic matrix (function):

$$H_f(\mathbf{x}) := \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f(\mathbf{x}) & \cdots & \cdots & \frac{\partial^2}{\partial x_1 \partial x_n} f(\mathbf{x}) \\ \vdots & \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_n \partial x_1} f(\mathbf{x}) & \cdots & \cdots & \frac{\partial^2}{\partial x_n \partial x_n} f(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n \times n}$$

In 2d with $(x, y) = (x_1, x_2)$ this reads, e.g.:

$$H_f(x, y) := \begin{pmatrix} \frac{\partial^2}{\partial x \partial x} f(x, y) & \frac{\partial^2}{\partial x \partial y} f(x, y) \\ \frac{\partial^2}{\partial y \partial x} f(x, y) & \frac{\partial^2}{\partial y \partial y} f(x, y) \end{pmatrix} \quad = \frac{\partial^2}{\partial y^2} f(x,y)$$

Schwarz-Clairaut Theorem

According to the Schwarz-Clairaut theorem there holds:

Suppose f is defined on a disk $D \subseteq \mathbb{R}^2$ with $(x_0, y_0) \in D$.
If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous ~~on D~~, in (x_0, y_0)
then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ at (x_0, y_0).

This shows that the Hesse matrix is symmetric.

The order of the partial derivatives doesn't matter in this case!

Higher Partial Derivatives: Examples

$$H_f(x,y) = \ldots = \exp(\sin(x+y)) \cdot \ldots$$

$$\ldots \cdot \Big[(x^2+y^2)(\cos^2(x+y) - \sin(x+y)) \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \ldots$$

$$\ldots + 2\cos(x+y) \begin{pmatrix} 2x, & x+y \\ x+y, & 2y \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Big]$$

$$H_g(x,y) = \ldots = \begin{pmatrix} -400(y-3x^2)+2, & -400x \\ -400x, & 200 \end{pmatrix}$$

Limits and Continuity in $\mathbb{R}^n$

Let $D \subseteq \mathbb{R}^n$.

- ε-neighbourhood of $\mathbf{a} \in \mathbb{R}^n$:

$$U_\varepsilon(\mathbf{a}) := \left\{ \mathbf{x} \in \mathbb{R}^n \,\middle|\, \|\mathbf{x} - \mathbf{a}\|_2 := \sqrt{\sum_{i=1}^{n} (x_i - a_i)^2} < \varepsilon \right\}$$

- $\mathbf{a} \in D$ is called an **interior point** of D $:\Longleftrightarrow$ there exists $U_\epsilon(\mathbf{a})$ with $U_\epsilon(\mathbf{a}) \subseteq D$.
- A set D is called **open** $:\Longleftrightarrow$ any point of D is an interior point. A set D is called **closed** $:\Longrightarrow \mathbb{R}^n \setminus D$ is open.
- $\mathbf{b} \in \mathbb{R}^n$ is called a **boundary point** of D $:\Longrightarrow$ for all $U_\varepsilon(\mathbf{b})$ holds, there exists a $\mathbf{x}, \mathbf{y} \in U_\varepsilon(\mathbf{b})$ with $\mathbf{x} \in D$ and $\mathbf{y} \notin D$. The set ∂D of all boundary points $\mathbf{b}$ is called **boundary** of D.

Examples, see blackboard

Revision/Definition: Limits and Continuity

Let $f : \mathbb{R}^n \subseteq D \to \mathbb{R}^m$, $\mathbf{a} \in D \cup \partial D$.

$$\lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = c \in \mathbb{R}^m :\Longleftrightarrow$$

for all $\varepsilon > 0$ exists $U_\varepsilon(\mathbf{a}) : \|f(\mathbf{x}) - c\| \leq \varepsilon$ for all $\mathbf{x} \in D \cap U_\varepsilon(\mathbf{a})$.

f is called **continuous** in $\mathbf{a} \in D :\Longleftrightarrow \lim_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.
f is called continuous in $D :\Longleftrightarrow f$ is continuous for all $\mathbf{a} \in D$.

Example: For $n = 2, m = 1$, f is continuous means that there are no “jumps” or “ridges”.

Pathologic Example for (Dis)continuity in 2d

Functions of severable variables exhibit phenomena that we do not encounter in 1d:

Consider $f : \mathbb{R}^2 \to \mathbb{R}$,

$$f(x,y) := \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } x \neq 0 \text{ or } y \neq 0, \\ 0, & \text{if } (x,y)^\top = (0,0)^\top. \end{cases}$$

A limit in 2d has to hold for any "path" $(x,y) \to (x_0, y_0)$.
It is not enough to consider each coordinate separately!