Complex Representation

By the complex exponential function we may rewrite

$$F(x) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega x) \quad \text{with } \omega := \frac{2\pi}{T}$$

where the complex Fourier coefficient c_k, $k \in \mathbb{Z}$, is given by

$$c_k = \frac{1}{T} \int_0^T f(x) \exp(-ik\omega x)\, dx.$$

Note that the limit is to be understood symmetrically:

$$\sum_{k=-\infty}^{\infty} c_k \exp(ik\omega x) := \lim_{n\to\infty} \sum_{k=-n}^{n} c_k \exp(ik\omega x).$$

Orthonormality

Theorem (Orthonormality Relations)

If $m, n \in \mathbb{N}$, then:

$$\frac{2}{T}\int_0^T \sin(m\omega x)\sin(n\omega x)\,dx = \delta_{m,n},$$

$$\frac{2}{T}\int_0^T \cos(m\omega x)\cos(n\omega x)\,dx = \delta_{m,n},$$

$$\frac{2}{T}\int_0^T \sin(m\omega x)\cos(n\omega x)\,dx = 0,$$

and if $m, n \in \mathbb{Z}$, then:

$$\frac{1}{T}\int_0^T \exp(im\omega x)\exp(-in\omega x)\,dx = \delta_{m,n}.$$

Piecewise Continuous Differentiability

Definition (Piecewise Continuously Differentiable Function)

Let $f : \mathbb{R} \to \mathbb{R}$ with

(i) Only at a finite number of **singularities** the function f is not continuously differentiable.

(ii) At any singularity x_0 there exist the following one-sided limits:

$$f(x_0+) := \lim_{x \to x_0+} f(x) \qquad f(x_0-) := \lim_{x \to x_0-} f(x),$$
$$f'(x_0+) := \lim_{x \to x_0+} f'(x) \qquad f'(x_0-) := \lim_{x \to x_0-} f'(x).$$

Then f is called **piecewise continuously differentiable**.

Discontinuities are singularities, but not any singularity is a discontinuity.

Convergence of Fourier Series

Theorem (Convergence of Fourier Series)

Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function that is piecewise continuously differentiable. Then:

- *The Fourier series F converges at any x that is not a singularity to f.*
- *At any singularity x_0 the Fourier series converges to the "mean value" of the jump*

$$\frac{1}{2}\left(f(x_0+) - f(x_0-)\right).$$

- *In any compact interval that does not contain a discontinuity, the convergence of F to f is uniform.*

Note that there exist periodic, continuous functions, whose Fourier series does not converge to f!

Properties of Fourier Series I

Let f and g be piecewise continuous, periodic functions with Fourier series $F = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega x)$, $G = \sum_{k=-\infty}^{\infty} d_k \exp(ik\omega x)$, resp. There holds:

- For any $\alpha, \beta \in \mathbb{R}$:

$$\alpha F(x) + \beta G(x) = \sum_{k=-\infty}^{\infty} (\alpha c_k + \beta d_k) \exp(ik\omega x)$$

- $F(-x) = \sum_{k=-\infty}^{\infty} c_{-k} \exp(ik\omega x)$
- For any $\alpha \in \mathbb{R}$

$$F(\alpha x) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega\alpha x)$$

- For any $\alpha \in \mathbb{R}$

$$F(\alpha + x) = \sum_{k=-\infty}^{\infty} (c_k \exp(ik\omega\alpha)) \exp(ik\omega x)$$

Properties of Fourier Series II

Let f be a piecewise continuous differentiable, periodic function with Fourier series $F = \sum_{k=-\infty}^{\infty} c_k \exp(in\omega x)$. There holds:

- The Fourier series F' represents f':

$$F'(x) = \sum_{k=-\infty}^{\infty} (ik\omega c_k) \exp(ik\omega x)$$

- $F(-x) = \sum_{k=-\infty}^{\infty} c_{-k} \exp(ik\omega x)$
- Suppose $c_0 = 0$, then the Fourier series $\tilde{F} := \int F(\xi)\, d\xi$ represents $\tilde{f} := \int f(\xi)\, d\xi$:

$$\tilde{F}(x) = \frac{2}{T} \int_0^T \tilde{f}(\xi)\, d\xi + \sum_{k=-\infty, k\neq 0}^{\infty} \frac{c_k}{ik\omega} \exp(ik\omega x)$$