Technische Hochschule Rosenheim
Fakultät ANG
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Exercise Sheet 0

Linear Algebra (AAI)

Exercise 0.1 (H)

Verify that $G = \{0, 1\}$ together with

$$\begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

is a commutative group, see Example 3(iii) in Section I.1.

Solution:
By definition, we have $*\colon G \times G \to G$. Moreover, $(G, *)$ satisfies:

i) Associativity: We have

$$\forall\, a, b, c \in G\colon (a * b) * c = a * (b * c)$$

since (there are 2^3 cases)

$$\begin{aligned}
(0 * 0) * 0 = 0 = 0 * (0 * 0), \\
(0 * 0) * 1 = 1 = 0 * (0 * 1), \\
(0 * 1) * 0 = 1 = 0 * (1 * 0), \\
(0 * 1) * 1 = 0 = 0 * (1 * 1), \\
(1 * 0) * 0 = 1 = 1 * (0 * 0), \\
(1 * 0) * 1 = 0 = 1 * (0 * 1), \\
(1 * 1) * 0 = 0 = 1 * (1 * 0), \\
(1 * 1) * 1 = 1 = 1 * (1 * 1).
\end{aligned}$$

ii) The neutral element is given by 0 since $0 * 0 = 0$ and $0 * 1 = 1$.

iii) The inverse elements are given by $-1 = 1$ and $-0 = 0$ since

$$\begin{aligned}
1 * 1 = 0, \\
0 * 0 = 0.
\end{aligned}$$

iv) Commutativity: We have

$$\forall\, a, b \in G\colon a * b = b * a$$

since $0 * 1 = 1 = 1 * 0$.

Exercise 0.2 (H)

Prove Proposition 10 from Section I.1.

Solution:
See Proposition I.1.10 in the lecture notes.

Exercise 0.3 (H)

Prove Lemma 12 from Section I.1.

Solution:
See Lemma I.1.12 in the lecture notes.

Exercise 0.4 (H)

Let G_1 and G_2 be subgroups of a group G. Show that $G_1 \cup G_2$ is not a subgroup in general.
Hint: Cf. Example 18 from Section I.1.

Solution:
See Example I.1.18 in the lecture notes.

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Exercise Sheet 1

Linear Algebra (AAI)

Exercise 1.1 (H)

Prove Lemma I.2.4.

Solution:
See Lemma I.2.4 in the lecture notes.

Exercise 1.2 (H)

a) Verify that $V = \mathbb{R}^n$ is a $\mathbb{R}$-vector space.

Solution:
By using the field properties of $\mathbb{R}$ in every coordinate we obtain:

1.) $(\mathbb{R}^n, \oplus)$ is a commutative group, i.e., we have $\oplus\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the following properties:

i) Associativity: We have

$$\forall\, x, y, z \in \mathbb{R}^n\colon (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

since

$$\begin{aligned}(x \oplus y) \oplus z &= \Big((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n\Big),\\ &= \Big(x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)\Big)\\ &= x \oplus (y \oplus z).\end{aligned}$$

ii) The neutral element is given by $\mathbf{0} = (0, \dots, 0)$ since

$$\mathbf{0} \oplus x = (0 + x_1, \dots, 0 + x_n) = (x_1, \dots, x_n) = x$$

for all $x \in \mathbb{R}^n$.

iii) The inverse element of $x \in \mathbb{R}^n$ is given by $-x = (-x_1, \dots, -x_n)$ since

$$\begin{aligned}-x \oplus x &= (-x_1, \dots, -x_n) \oplus (x_1, \dots, x_n)\\ &= (-x_1 + x_1, \dots, -x_n + x_n)\\ &= (0, \dots, 0) = \mathbf{0}.\end{aligned}$$

iv) Commutativity: We have

$$\forall\, x, y \in \mathbb{R}^n\colon x \oplus y = y \oplus x$$

since

$$\begin{aligned}x \oplus y &= (x_1 + y_1, \dots, x_n + y_n)\\ &= (y_1 + x_1, \dots, y_n + x_n)\\ &= y \oplus x.\end{aligned}$$

2.) For all $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}
(\lambda + \mu) \odot x &= \big((\lambda + \mu) \cdot x_1, \ldots, (\lambda + \mu) \cdot x_n\big) \\
&= (\lambda x_1 + \mu x_1, \ldots, \lambda x_n + \mu x_n) \\
&= (\lambda \cdot x_1, \ldots, \lambda \cdot x_n) \oplus (\mu \cdot x_1, \ldots, \mu \cdot x_n) \\
&= \lambda \odot x \oplus \mu \odot x, \\
\lambda \odot (x \oplus y) &= \big(\lambda \cdot (x_1 + y_1), \ldots, \lambda \cdot (x_n + y_n)\big) \\
&= (\lambda x_1 + \lambda y_1, \ldots, \lambda x_n + \lambda y_n) \\
&= (\lambda \cdot x_1, \ldots, \lambda \cdot x_n) \oplus (\lambda \cdot y_1, \ldots, \lambda \cdot y_n) \\
&= \lambda \odot x \oplus \lambda \odot y, \\
\lambda \odot (\mu \odot x) &= \lambda \odot (\mu \cdot x_1, \ldots, \mu \cdot x_n) \\
&= (\lambda \cdot \mu \cdot x_1, \ldots, \lambda \cdot \mu \cdot x_n) \\
&= \big((\lambda \cdot \mu) \cdot x_1, \ldots, (\lambda \cdot \mu) \cdot x_n\big) \\
&= (\lambda \cdot \mu) \odot x, \\
1 \odot x &= (1 \cdot x_1, \ldots, 1 \cdot x_n) \\
&= (x_1, \ldots, x_n) \\
&= x.
\end{aligned}$$

b) Is $\mathbb{C}^n$ a $\mathbb{R}$-vector space (based on (1) from Example II.1.2)?

Solution:
The scalar multiplication

$$\odot \colon \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$$

is given by the correpsonding operation on $\mathbb{C} \times \mathbb{C}^n$ ($\mathbb{C}^n$ seen as a $\mathbb{C}$-vector space), where the scalars are restricted to the subset $\mathbb{R} \subseteq \mathbb{C}$. Since C^n is a $\mathbb{C}$-vector space, all vector space conditions are also met for this restriction.

Exercise 1.3 (H)

Let $V = \mathbb{R}^{[0,1]}$.

a) Verify that V is a $\mathbb{R}$-vector space.

Solution:
By using the field properties of $\mathbb{R}$ in every coordinate (i.e., for every evaluation site $x \in [0,1]$) we obtain the analogous properties as in Exercise 1.2 a). [...]

b) Show that $C([0,1])$ is a subspace of V.

Solution:
We verify the subspace conditions from Definition II.1.5:

i) $\mathbf{0} \in \mathbb{R}^{[0,1]}$ given by $\mathbf{0}(x) = 0$ for all $x \in [0,1]$ (= zero function) is continuous and hence $C([0,1]) \neq \emptyset$.

ii) Since the sum of two continuous functions is again a continuous function (see Analysis I), the set $C([0,1])$ is closed w.r.t. the vector addition.

iii) Since a continuous function multiplied by a constant is again a continuous function (see Analysis I), the set $C([0,1])$ is closed w.r.t. the scalar multiplication.

Exercise 1.4 (H)

Consider Example II.1.7.(i) with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and

$$U = \left\{x \in \mathbb{R}^n \colon \sum_{i=1}^{n} a_i \cdot x_i = b\right\}.$$

a) Show that U is a subspace of $\mathbb{R}^n$ if and only if $b = 0$.

Solution:
"$\Rightarrow$": Let U be a subspace and $x \in U$, i.e., we have

$$\sum_{i=1}^{n} a_i \cdot x_i = b$$

and hence

$$\sum_{i=1}^{n} a_i \cdot 2x_i = 2 \cdot \sum_{i=1}^{n} a_i \cdot x_i = 2b. \tag{1}$$

Since U is a subspace, we also have $2x \in U$ (see Definition II.1.5.(iii)) and thus

$$\sum_{i=1}^{n} a_i \cdot 2x_i = b. \tag{2}$$

Combining (1) and (2) we obtain $2b = b$, which implies $b = 0$.
"$\Leftarrow$": Let $b = 0$. We verify the subspace conditions from Definition II.1.5:

i) We have

$$\sum_{i=1}^{n} a_i \cdot 0 = 0$$

and hence $\mathbf{0} = (0, \ldots, 0) \in U$ such that $U \neq \emptyset$.

ii) Let $x, y \in U$. Then we have

$$\sum_{i=1}^{n} a_i \cdot (x_i + y_i) = \sum_{i=1}^{n} a_i \cdot x_i + \sum_{i=1}^{n} a_i \cdot y_i = 0 + 0 = 0$$

and hence $x + y \in U$.

iii) Let $x \in U$ and $\lambda \in \mathbb{R}$. Then we have

$$\sum_{i=1}^{n} a_i \cdot (\lambda \cdot x_i) = \lambda \cdot \sum_{i=1}^{n} a_i \cdot x_i = \lambda \cdot 0 = 0$$

and hence $\lambda \cdot x \in U$.

b) Let $n = 2$. Determine and illustrate U for all combinations of $a \in \{(1, 1), (-1, 0)\}$ and $b \in \{0, 1, -2\}$.

Solution:
...

c) Show Remark II.1.10 (regarding the union of subspaces).

Solution:
Consider $U_1 = \{x \in \mathbb{R}^2 \colon x_1 = 0\}$ and $U_2 = \{x \in \mathbb{R}^2 \colon x_2 = 0\}$. Then U_1 and U_2 are subspaces of $\mathbb{R}^2$ according to part a). However, $U_1 \cup U_2$ is not a subspace since

$$(0, 1) \in U_1 \subseteq U_1 \cup U_2,$$
$$(1, 0) \in U_2 \subseteq U_1 \cup U_2,$$

and

$$(0, 1) + (1, 0) = (1, 1) \notin U_1 \cup U_2.$$

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Exercise Sheet 2

Linear Algebra (AAI)

Exercise 2.1 (H)

Determine whether U is a subspace of $\mathbb{R}^3$:

a) $U = \{(x, y, z) \in \mathbb{R}^3 \colon z = 3x - y\}$,

b) $U = \{(x, y, z) \in \mathbb{R}^3 \colon x \cdot y \cdot z = 0\}$,

c) $U = \{(x, y, z) \in \mathbb{R}^3 \colon x \cdot (\exp(y) + z) = 0\}$,

d) $U = \{(x, y, z) \in \mathbb{R}^3 \colon x, y, z \geq 0\}$,

e) $U = \{(\lambda, 2\lambda, 4\lambda) \in \mathbb{R}^3 \colon \lambda \in \mathbb{R}\}$.

Solution:
The sets from a) and e) are subspaces of $\mathbb{R}^3$.
... (Verify the subspace conditions from Definition II.1.5.)

The sets from b), c), and d) are not subspaces of $\mathbb{R}^3$. As an example we show that b) is not a subspace: Note that $(1, 1, 0), (0, 0, 1) \in U$, but

$$(1, 1, 0) + (0, 0, 1) = (1, 1, 1) \notin U.$$

Hence condition (ii) from Definition II.1.5 is not satisfied.

Exercise 2.2 (H)

Consider $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ given by

$$v_1 = (1, 2, 1, 2),\ v_2 = (1, 1, 1, 1),\ v_3 = (0, 1, 1, 0),\ v_4 = (0, 1, 0, 1),\ v_5 = (1, 0, 0, 1).$$

a) Express each vector v_i as a linear combination of the remaining vectors v_j with $j \neq i$ (if possible).

Solution:
We have $v_1 = v_2 + v_4$ and $v_2 = v_3 + v_5$. Using these two equalities we obtain, e.g.,

$$\begin{aligned} v_1 &= v_2 + v_4, \\ v_2 &= v_3 + v_5, \\ v_3 &= v_2 - v_5, \\ v_4 &= v_1 - v_2, \\ v_5 &= v_2 - v_3. \end{aligned}$$

b) Prove or disprove:

i) $\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_3, v_5\})$,

ii) $\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_1, v_3, v_5\})$,

iii) $\operatorname{span}(\{v_2, v_5\}) = \operatorname{span}(\{v_3, v_5\})$,

iv) $\operatorname{span}(\{v_1, v_5\}) = \operatorname{span}(\{v_1, v_2, v_3, v_4, v_5\})$,

v) $\operatorname{span}(\{v_1, v_2, v_4\}) = \operatorname{span}(\{v_2, v_3, v_5\})$.

Solution:

Statements i) and iii) are correct. Statements ii), iv), and v) are false.

ad i): We clearly have

$$\operatorname{span}(\{v_2, v_3, v_5\}) = L(v_2, v_3, v_5) \supseteq L(v_3, v_5) = \operatorname{span}(\{v_3, v_5\}),$$

see Corollary II.1.17. Moreover, every linear combination of v_2, v_3, v_5 can be expressed as a linear combination of v_3, v_5 since $v_2 = v_3 + v_5$ and thus

$$\lambda_2 v_2 + \lambda_3 v_3 + \lambda_5 v_5 = \lambda_2(v_3 + v_5) + \lambda_3 v_3 + \lambda_5 v_5 = (\lambda_2 + \lambda_3)v_3 + (\lambda_2 + \lambda_5)v_5$$

for $\lambda_2, \lambda_3, \lambda_5 \in \mathbb{R}$. Hence we get

$$\operatorname{span}(\{v_2, v_3, v_5\}) = L(v_2, v_3, v_5) \subseteq L(v_3, v_5) = \operatorname{span}(\{v_3, v_5\}).$$

ad ii): Using part (i) we obtain

$$\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_3, v_5\}) \subseteq \operatorname{span}(\{v_1, v_3, v_5\}).$$

Moreover, we have

$$\begin{aligned} \operatorname{span}(\{v_3, v_5\}) = L(v_3, v_5) &= \{\lambda v_3 + \mu v_5 \colon \lambda, \mu \in \mathbb{R}\} \\ &= \{(\mu, \lambda, \lambda, \mu) \colon \lambda, \mu \in \mathbb{R}\}. \end{aligned}$$

This shows $v_1 = (1, 2, 1, 2) \notin \operatorname{span}(\{v_3, v_5\})$ (Note that the first and the last entry of v_1 are not equal.) and hence

$$\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_3, v_5\}) \not\supseteq \operatorname{span}(\{v_1, v_3, v_5\}).$$

ad iii): ...

ad iv): ...

ad v): ...

c) Determine all linearly independent families $(v_i)_{i\in I}$ with $I \subseteq \{1, \ldots, 5\}$ and $|I| \leq 3$.

Solution:

$|I| = 0$: $(v_i)_{i\in\emptyset}$ is linearly independent per definition.

$|I| = 1$: (v_i) is linearly independent if and only if $v_i \neq 0$. Hence (v_i) is linearly independent for every $i \in \{1, \ldots, 5\}$.

$|I| = 2$: By Lemma II.2.3 we have

$$(v_i, v_j) \text{ linearly independent } \Leftrightarrow\ v_i \notin L(v_j) \wedge v_j \notin L(v_i).$$

Hence (v_i, v_j) is linearly independent for all $i, j \in \{1, \ldots, 5\}$ with $i \neq j$.

$|I| = 3$: Use Lemma II.2.3 for each of the 10 combinations: The families (v_1, v_2, v_4) and (v_2, v_3, v_5) are linearly dependent according to part a). E.g., (v_3, v_4, v_5) is linearly independent since for $\lambda_3, \lambda_4, \lambda_5 \in \mathbb{R}$ we have

$$\lambda_3 v_3 + \lambda_4 v_4 + \lambda_5 v_5 = 0 \ \Leftrightarrow\ \begin{Bmatrix} & & \lambda_5 & = 0 \\ \lambda_3 + \lambda_4 & & & = 0 \\ \lambda_3 & & & = 0 \\ & \lambda_4 + \lambda_5 & & = 0 \end{Bmatrix} \ \Leftrightarrow\ \lambda_3 = \lambda_4 = \lambda_5 = 0.$$

Exercise 2.3 (H)

Let $U \subseteq V$ be a subspace of V and let $x, y \in V$. Show that

$$y \in \operatorname{span}(U \cup \{x\}) \ \wedge\ y \notin U \ \Rightarrow\ x \in \operatorname{span}(U \cup \{y\}).$$

Solution:
Let $y \in \operatorname{span}(U \cup \{x\})$. Then there exist $\lambda \in K$ and $u \in U$ such that

$$y = \lambda x + u,$$

see Proposition II.1.16 and note that U is a subspace. If additionally $y \notin U$, we have $\lambda \neq 0$. Hence we get

$$x = \lambda^{-1} \cdot (y - u) = \lambda^{-1} y - \lambda^{-1} u \in \operatorname{span}(U \cup \{y\}).$$

Exercise 2.4 (H)

a) Let $v_1, v_2, v_3 \in \mathbb{R}^{[0,\infty)}$ be given by

$$v_1(x) = 1, \quad v_2(x) = \sqrt{x}, \quad v_3(x) = \sin(x)$$

for $x \in [0, \infty)$. Show that (v_1, v_2, v_3) is linearly independent.

Solution:
... (Cf. Example II.2.4.(ii) and consider, e.g., the evaluation sites $x_1 = 0$, $x_2 = \pi$, and $x_3 = \pi/2$.)

b) Let V be a $\mathbb{R}$-vector space and let (v_1, v_2) be linearly independent for $v_1, v_2 \in V$. Show that $(v_1 - v_2, v_1 + v_2)$ is linearly independent.

Solution:
Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$\lambda_1(v_1 - v_2) + \lambda_2(v_1 + v_2) = 0.$$

Then we obtain

$$(\lambda_1 + \lambda_2)\, v_1 + (-\lambda_1 + \lambda_2)\, v_2 = 0.$$

Since (v_1, v_2) is linearly independent by assumption, we have

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0, \\ -\lambda_1 + \lambda_2 &= 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(\mathrm{I}) & \lambda_1 + \lambda_2 &= 0, \\ &(\mathrm{I}) + (\mathrm{II}) & 2\lambda_2 &= 0. \end{aligned}$$

Hence we get $\lambda_1 = \lambda_2 = 0$.

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Exercise Sheet 3

Linear Algebra (AAI)

Exercise 3.1 (H)

a) Show that (v_1, v_2, v_3) given by

$$v_1 = (1, 0, 1), \quad v_2 = (1, 1, 0), \quad v_3 = (0, 1, 1)$$

is a basis of $\mathbb{R}^3$.

Solution:
Since $\dim \mathbb{R}^3 = 3$, it suffices to show that (v_1, v_2, v_3) is linearly independent.
... (Cf. Exercise 2.2 c).)

b) Let V be an $\mathbb{R}$-vector space with $\dim V = 3$, and let (v_1, v_2, v_3) be a basis of V. Show that

$$(v_1 - v_3, v_1 + v_2 - v_3, v_1 + v_2 + v_3)$$

is a basis of V, too.

Solution:
... (Cf. Exercises 3.1 a) and 2.4 b).)

Exercise 3.2 (H)

Determine a basis and the dimension of the following subspaces of $\mathbb{R}^3$:

a) $U_1 = \{(x, y, z) \in \mathbb{R}^3 \colon x = y = z = 0\}$,

Solution:
We have $U_1 = \{(0, 0, 0)\} = \operatorname{span}(\emptyset)$ and $\dim U_1 = 0$.

b) $U_2 = \{(x, y, z) \in \mathbb{R}^3 \colon x + y - z = 0\}$.

Solution:
For $(x, y, z) \in \mathbb{R}^3$ we have

$$(x, y, z) \in U_2 \;\Leftrightarrow\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - y \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} + \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

This shows

$$\begin{aligned} U_2 &= \{\lambda \cdot (1, 0, 1) + \mu \cdot (-1, 1, 0) \colon \lambda, \mu \in \mathbb{R}\} \\ &= \operatorname{span}\{(1, 0, 1), (-1, 1, 0)\}. \end{aligned}$$

Moreover, the family $((1, 0, 1), (-1, 1, 0))$ is linearly independent [since ...] and hence a basis of U_2.

Exercise 3.3 (H)

Let V be a K-vector space, let $(v_1, \dots, v_n)$ be a basis of V, and let $(w_1, \dots, w_m)$ be a generating set of V. Prove or disprove:

a) $(v_1, w_2, \dots, w_m)$ is a generating set of V.

Solution:
False: Choose $V = \mathbb{R}^2$, $n = m = 2$, and

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad w_1 = v_2, \quad w_2 = v_1.$$

Then, $(v_1, w_2) = (v_1, v_1)$ is not a generating set of $\mathbb{R}^2$.

b) $(v_1 + w_1, \dots, v_n + w_n)$ is a basis of V.

Solution:
False: ...

c) There exists $i \in \{1, \dots, n\}$ such that $(v_1, \dots, v_{i-1}, w_1, v_{i+1}, \dots, v_n)$ is a basis of V.

Solution:
False: Choose $m = n + 1$ and $w_1 = 0$ as well as

$$(w_2, \dots, w_{n+1}) = (v_1, \dots, v_n).$$

A family containing the zero vector is linearly dependent and hence not a basis. (Note that the statement is correct for $w_1 \neq 0$ due to Lemma II.2.10.)

Exercise 3.4 (H)

Let U_1, U_2 be subspaces of V, and let

$$U_1 + U_2 = \{u_1 + u_2 \colon u_1 \in U_1, u_2 \in U_2\}.$$

a) Show that $U_1 + U_2$ is a subspace of V.

Solution:
... (Verify the subspace conditions from Definition II.1.5.)

b) Let $V = \mathbb{R}^n$ and $\dim U_1 = \dim U_2 = n-1$. Determine all possible cases of $\dim U_1 \cap U_2$ and provide an explicit example for each case with $n = 3$. Cf. Exercise 1.4.

Solution:
By Proposition II.2.17 we have

$$\begin{aligned} \dim(U_1 \cap U_2) &= \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) \\ &= (n-1) + (n-1) - \dim(U_1 + U_2) \end{aligned}$$

Moreover, we have

$$0 \le \dim(U_1 \cap U_2) \le \min(\dim(U_1), \dim(U_2)) = n - 1$$

and

$$n - 1 = \max(\dim(U_1), \dim(U_2)) \le \dim(U_1 + U_2) \le \dim V = n,$$

see Corollary II.2.15. This yields the following possible cases:

	$\dim(U_1 \cap U_2)$	$\dim(U_1 + U_2)$
case 1	$n-1$	$n-1$
case 2	$n-2$	n

... (examples for $n = 3$)