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Exercise Sheet 12

Linear Algebra (AAI)

Exercise 12.1 (H)

For $A \in \mathbb{R}^{2\times 2}$ we define $\langle \cdot, \cdot \rangle_A \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\langle x, y \rangle_A = \langle x, Ay \rangle, \qquad x, y \in \mathbb{R}^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product on $\mathbb{R}^2$. If $\langle \cdot, \cdot \rangle_A$ is an inner product, the corresponding induced norm and angles are denoted by $\| \cdot \|_A$ and $\measuredangle_A$, respectively.

a) Let $A_1, A_2, A_3 \in \mathbb{R}^{2\times 2}$ be given by

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Determine for every $i = 1, 2, 3$ whether $\langle \cdot, \cdot \rangle_{A_i}$ is an inner product on $\mathbb{R}^2$.

b) Compute $\|x\|$ and $\|x\|_{A_3}$ for $x \in \{(-1, 1)^\top, (1, 0)^\top\}$. Moreover, compute $\measuredangle(x, y)$ and $\measuredangle_{A_3}(x, y)$ for $x = (-1, 1)^\top$ and $y = (1, 0)^\top$.

c) Determine the orthogonal complement of $\{(1, 0)^\top\}$ w.r.t. $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_A$.

Exercise 12.2 (H)

Let $V = C([0, 1])$. We define $\langle \cdot, \cdot \rangle \colon V \times V \to \mathbb{R}$ by

$$\langle f, g \rangle = \int_0^1 f(x) \cdot g(x)\, dx, \qquad f, g \in V.$$

a) Show that $\langle \cdot, \cdot \rangle$ is an inner product on V.
Hint: Use ε-δ-characterization of continuity to show positive-definiteness.

b) Determine an orthonormal basis of the subspace $\Pi_2 \subseteq V$ (where the domain is restricted to $[0, 1]$).

c) Let $f \in V$ be given by $f(x) = \exp(x)$ for $x \in [0, 1]$. Determine the orthogonal projection of f onto Π_2.

Exercise 12.3 (H)

Prove Remark IV.1.19 and Remark IV.1.27.

Exercise 12.4 (H)

Let $m, n \in \mathbb{N}$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$ with $n \leq m$.

a) Show that $\operatorname{rank}(A^\top A) = \operatorname{rank} A$.
Hint: Use the Rank-Nullity Theorem and show that $\ker A = \ker(A^\top A)$. *Note that* $Ax = 0$ *if and only if* $\langle Ax, Ax \rangle = 0$.

b) Let $\operatorname{rank} A = n$. Show[1] that $\langle x, A^\top A x \rangle > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.
Hint: Is $\mathcal{F}_A$ *injective?*

c) Let $\operatorname{rank} A = n$. Show that the function

$$\Delta \colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \|Ax - b\|$$

has a unique minimizer $x^* \in \mathbb{R}^n$ satisfying $A^\top A x^* = A^\top b$.
Hint: The function $f \colon \mathbb{R}^n \to \mathbb{R}$ *given by* $f(x) = \|Ax - b\|^2$ *is differentiable.*

d) Consider the following data $(x_i, y_i) \in \mathbb{R}^2$ for $i = 1, \ldots, 4$:

i	1	2	3	4
x_i	0	1	2	3
y_i	2	0.5	3	4.5

Determine $f \in \Pi_1$ such that

$$\left(\tilde{\Delta}(f)\right)^2 = \sum_{i=1}^{4} \left(y_i - f(x_i)\right)^2$$

is minimal.

[1] A matrix $B \in \mathbb{R}^{n \times n}$ is called *symmetric* if $B = B^\top$. Moreover, a symmetric matrix $B \in \mathbb{R}^{n \times n}$ is called *positive-definite* if $\langle x, Bx \rangle > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.