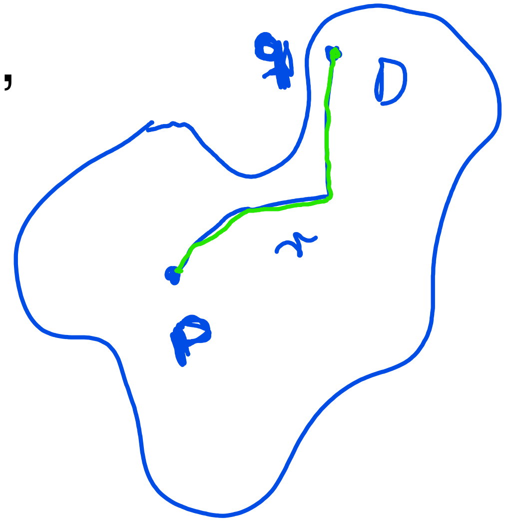
Curves

A set $D \subseteq \mathbb{R}^n$ is called **(path-)connected**, if for any $\mathbf{p}, \mathbf{q} \in D$ there exists an interval $[a, b]$ and a continuous function $r : [a, b] \to \mathbb{R}^n$, such that:



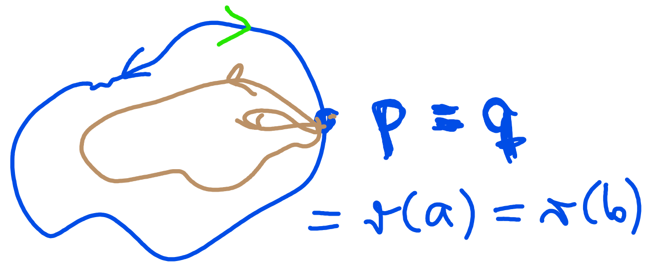
- $r(t) \in D \quad$ for all $t \in [a, b]$
- $r(a) = \mathbf{p}$ and $r(b) = \mathbf{q}$

The set $r([a, b])$ (the range of r) is called a **curve**.

The mapping r itself is called **parametrization of the curve** or **path** from $\mathbf{p}$ to $\mathbf{q}$ in D.

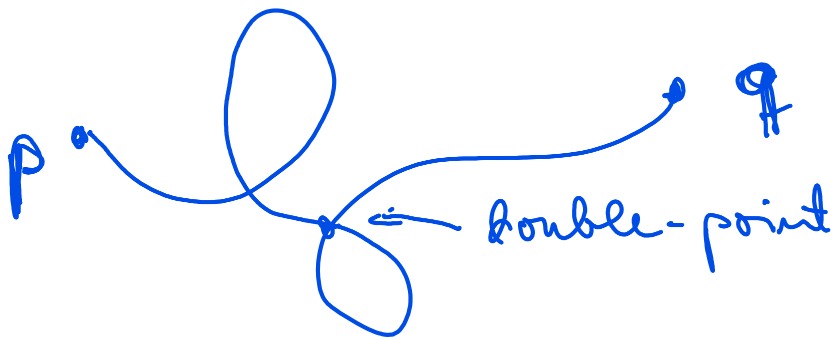
Curves and Connectedness

A curve is called **closed**, if $r(a) = r(b)$.



A set D is called **simply connected**, if any closed curve maybe continuously contracted (however this is defined rigorously) to a single point.

A **region** or domain[1] is an open and connected subset of $\mathbb{R}^n$.



a curve without a double point (injective) is called a simple curve

[1] not to be confused with a domain of definition

Examples for Connected Sets

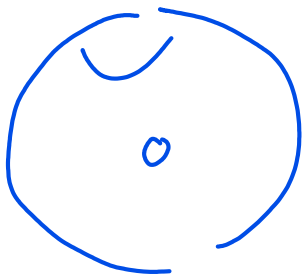
1) $\mathbb{R}^n$ is simply connected

2) $U_\varepsilon(x) \subset \mathbb{R}^n$ for any $\varepsilon > 0$ is simply connected

3) 

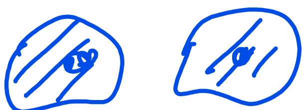
$D \subset \mathbb{R}^2$

D is connected, but not simply connected

4) 

$D = U_2(0) \setminus U_1(0)$

$\mathbb{R}^3 \supset D$ connected & simply connected

5) 

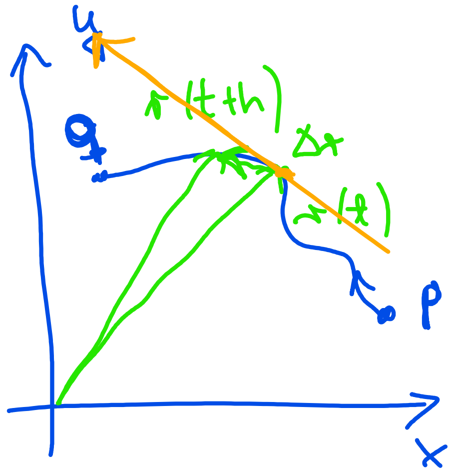
P Q

is not (path-) connected

6) $\mathbb{N}$ is not connected

Differentiation of Curves

We start with 2d:



Consider a parametrization $\mathbf{r}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

with x, y differentiable functions.

We call

the vector Δr is a secant

$$\dot{\mathbf{r}}(t) = \lim_{h \to 0} \frac{1}{h} (\mathbf{r}(t+h) - \mathbf{r}(t)) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix}$$

a tangential vector of the curve.

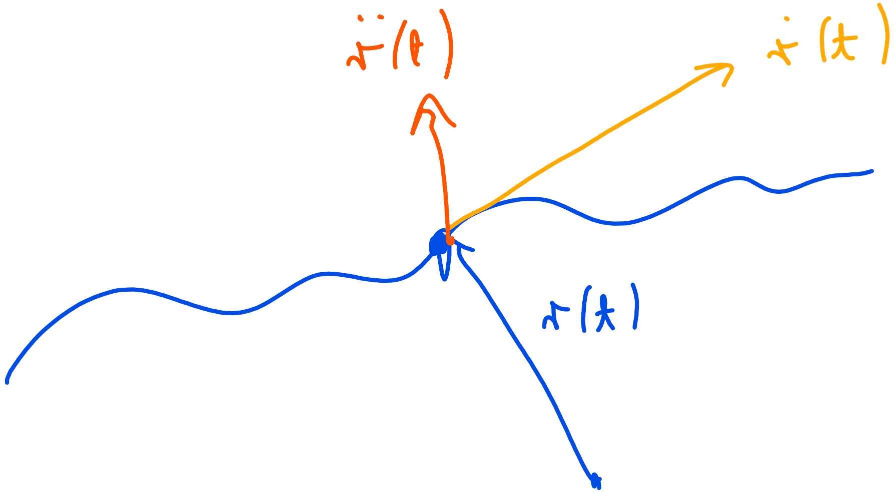
It may interpreted as the limit vector of the secant vectors.

Analogously this may be generalized to 3d and higher dimensions.

Differentiation of Curves - Example

Motion of a mass point in 3d space

Position	Velocity	Acceleration
$r(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$	$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}$	$\begin{pmatrix} \ddot{x}(t) \\ \ddot{y}(t) \\ \ddot{z}(t) \end{pmatrix}$



Outline

Curve Integrals, Areas of Sectors & Solids of Revolution

Before turning to iterated integrals of functions of more than 1 variable,
we consider integrals of functions in 1 variable:

- Vector-valued functions:
  Curve integrals
- Real-valued functions of vectors, but with symmetry:
  Areas of sectors,
  Solids of revolution

Regular Parametrization of a Curve

Let $D \subseteq \mathbb{R}^n$ open.

A mapping $\mathbf{r} : [a, b] \to D : t \mapsto \mathbf{r}(t) = (r_1(t), \dots, r_n(t))^\top$ is called **regular parametrization** of a curve in D, if

- $\mathbf{r}$ is continuously differentiable and
- $\mathbf{r}'(t) \neq \mathbf{0} \in \mathbb{R}^n$ for all $t \in [a, b]$.

Different regular parametrizations exist.

The sense of circulation may be different.

Curve Integrals

Let $D \subseteq \mathbb{R}^n$ open.

For any continuous function $\mathbf{f} : D \to \mathbb{R}^n$ and any regular parametrization $\mathbf{r} : [a, b] \to D$ of a curve $K := \mathbf{r}([a, b])$ we call

$$/\!/ \quad \frac{d\mathbf{r}(t)}{dt} = \frac{dx}{dt}$$

$$\int_K \mathbf{f}(\mathbf{x}) \bullet d\mathbf{x} := \int_a^b \mathbf{f}(\mathbf{r}(t)) \bullet \mathbf{r}'(t)\, dt$$

the **curve integral** of **f** along the curve K w.r.t. **r**.

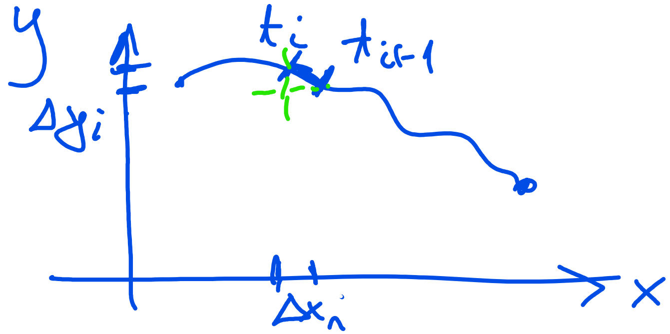
The curve integral is independent from the parametrization, but its sign depends on the sense of circulation.

Arc Length - Parametrized Curve

Let K be a curve with a regular parametrization $(x(t), y(t))^\top$, $t \in [a, b]$,
i.e. with $x(t)$, $y(t)$ cont. diff.able on $[a, b]$ and

$$(\dot{x}(t))^2 + (\dot{y}(t))^2 \neq 0 \quad \text{for all } t \in [a, b].$$

$a = t_0 < t_1 < t_2 < \ldots < t_n = b$



Then the arc length is given by

$$L = \int_a^b \sqrt{\dot{x}(t))^2 + (\dot{y}(t))^2} \, dt.$$

limit process, using the mean value theorem of diff.

In higher dimensions with $\mathbf{x}(t) \in \mathbb{R}^n$ there holds under analoguous assumptions:

$$\sum_{i=1}^{n} \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2} \cdot \Delta t$$

Arc length in 2d - Cartesian Coordinates

For the graph $y = f(\mathbf{x})$ of a cont. differentiable function $f : [a, b] \to \mathbb{R}$
the arc length is given by

$$L = \int_a^b \sqrt{1 + \underbrace{(y'(x))^2}_{= (f'(x))^2}}\, dx.$$

Proof

set $x(t) = t$ $\qquad \dot{x}(t) = 1$

$y(t) = f(t)$ $\qquad \dot{y}(t) = f'(t)$

then change of variable $t \rightsquigarrow x$

Arc length in 2d - Polar Coordinates

For a curve with a regular parametrization $r(\phi)$, $\phi \in [\alpha, \beta]$, i.e. with $r(\phi)$ cont. differentiable on $[\alpha, \beta]$ and

$$r'(\phi) \neq 0 \quad \text{for all } \phi \in [\alpha, \beta].$$

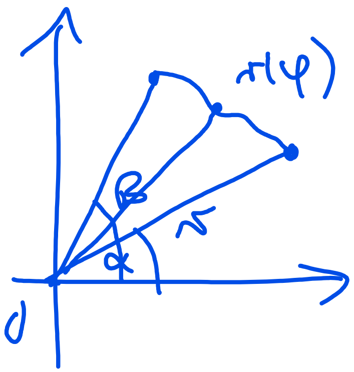
$$L = \int_\alpha^\beta \sqrt{r(\phi)^2 + (\dot{r}(\phi))^2}\, d\phi.$$

$x(t) = r(\varphi)\cos(\varphi)$

$\dot{x}(t) = \dot{r}(\varphi)\cos(\varphi) - r(\varphi)\sin(\varphi)$

$y(t) = r(\varphi)\sin(\varphi)$

$\dot{y}(t) = \dot{r}(\varphi)\sin(\varphi) + r(\varphi)\cos(\varphi)$



Proof: $L \overset{\text{def.}}{=} \int_\alpha^\beta \sqrt{(\dot{x}(t))^2 + (\dot{y}(t))^2}\, dt$ $\quad$ [$t \hat{=} \varphi$, $\alpha \hat{=} a$, $\beta \hat{=} b$]

$= \int_\alpha^\beta \Big[\dot{r}^2\cos^2(\varphi) + r^2\sin^2(\varphi) - 2r\dot{r}\sin(\varphi)\cos(\varphi) + \dot{r}^2\sin^2(\varphi) + r^2\cos^2(\varphi) + 2r\dot{r}\sin(\varphi)\cos(\varphi)\Big]^{1/2} dt$

$= \int_\alpha^\beta \sqrt{(\dot{r}(\varphi))^2 + (r(\varphi))^2}\, d\varphi$

Areas of Sectors in Polar Coordinates

Consider $r : [\alpha, \beta] \to \mathbb{R}_0^+$ continuous, where $\alpha < \beta$.
The sector enclosed by the 3 curves $r(\phi)$, $\phi = \alpha$, and $\phi = \beta$ has the area

$$A = \frac{1}{2} \int_\alpha^\beta (r(\phi))^2 \, d\phi.$$

Motivation:

area of a full circle with r(α)

$$\Delta A \approx \cancel{\pi} r^2 \cdot \frac{\Delta\phi}{2\cancel{\pi}}$$

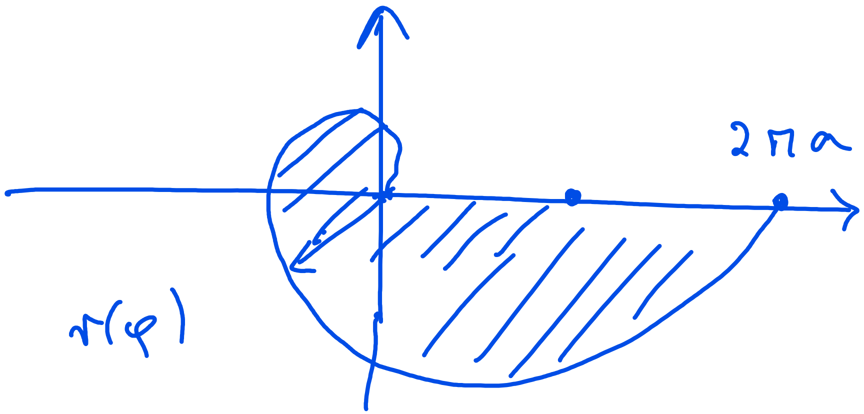
for circular sector with r(φ)

$$A = \lim_{\Delta\phi \to 0} \sum \frac{1}{2} r^2 \Delta\phi$$

Areas of Sectors in Polar Coordinates - Example

Archimedean spiral (1 turn)

$$r(\phi) = a\phi, \quad a > 0, \quad \text{for } 0 \le \phi \le 2\pi$$



$$A = \frac{1}{2}\int_0^{2\pi} (r(\phi))^2 \, d\phi = \frac{1}{2} a^2 \int_0^{2\pi} \phi^2 \, d\phi = \frac{a^2}{2}\left[\frac{\phi^3}{3}\right]_0^{2\pi}$$

$$= \frac{a^2}{2} \cdot \frac{8\pi^3}{3} = \underline{\underline{\frac{4}{3} a^2 \pi^3}}$$

Areas of Sectors in Cartesian Coordinates

Leibniz sector formula

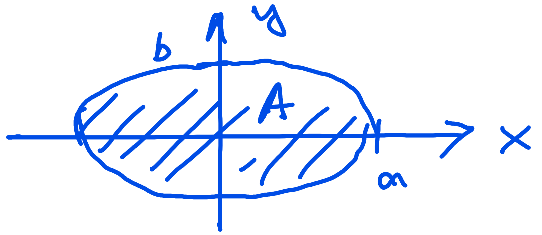
$$A = \frac{1}{2}\left|\int_{t_1}^{t_2} (x(t)y'(t) - y(t)x'(t))\, dt\right|$$

for a parametrized curve $K = \{(x(t), y(t)), t_1 \leq t \leq t_2\}$, where $x(t)$, $y(t)$ cont. differentiable

Example: Ellipse

$x(t) = a\cos(t)$

$y(t) = b\sin(t)$



$x'(t) = -a\sin(t)$

$y'(t) = b\cos(t)$

$a, b > 0 \quad , \; 0 \leq t < 2\pi$

$$A = \frac{1}{2}\left|\int_0^{2\pi} a\cos(t)\, b\cos(t) + b\sin(t)\cdot(+a)\sin(t)\, dt\right|$$

$$= \frac{ab}{2}\left|\int_0^{2\pi} \underbrace{\cos^2(t) + \sin^2(t)}_{=1}\, dt\right| = \frac{ab}{2}\cdot 2\pi = \underline{\underline{\pi ab}}$$