Linear Algebra

Bachelor Applied Artificial Intelligence (AAI-B2)

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Literature

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Contents

Chapter I

Algebraic Structures

In this chapter we introduce the notion of groups and fields. We also present simple examples and elementary properties of groups and fields.

1 Groups

In the sequel let G be a non-empty set.

Definition 1. A map

$$*\colon G \times G \to G$$

is called *operation* on G. An operation is *associative* if[1]

$$\forall\, a, b, c \in G\colon (a * b) * c = a * (b * c),$$

and *commutative* if

$$\forall\, a, b \in G\colon a * b = b * a.$$

Notation: Brackets may be dropped in case of an associative operation with multiple elements. Depending on the context we may also write $a + b$, $a \cdot b$, ab etc. instead of $a * b$.

Definition 2. A set G with an operation $*$ is called *group* if $*$ is associative and there exists $e \in G$ such that

$$\forall\, a \in G\colon e * a = a \tag{1}$$

and

$$\forall\, a \in G\ \exists\, a' \in G\colon a' * a = e. \tag{2}$$

Furthermore, a group is called *commutative (abelian)* if $*$ is commutative.

Notation: We sometimes write $(G, *)$ to emphasize the operation $*$.

[1] In a definition one typically uses “if” instead of “iff” (if and only if).

Example 3.

(i) Additive groups:

a) $(\mathbb{N}_0, +)$ is not a group since (2) does not hold.

b) $(\mathbb{Z}, +)$ is a commutative group with $e = 0$ and

$$\forall\, m \in \mathbb{Z}\colon \big(-m \in \mathbb{Z} \ \wedge\ (-m) + m = 0\big).$$

Clearly, $+$ is associative and commutative and

$$\forall\, m, n \in \mathbb{Z}\colon m + n \in \mathbb{Z}.$$

c) $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$, and $(\mathbb{C}, +)$ are commutative groups.

(ii) Multiplicative groups:

a) $(\mathbb{Z}, \cdot)$ is not a group since (2) does not hold. In particular, we have

$$\forall\, m \in \mathbb{Z}\colon m \cdot 0 = 0 \neq 1.$$

b) $(\mathbb{Q} \setminus \{0\}, \cdot)$ is a commutative group with $e = 1$ and

$$\forall\, q \in \mathbb{Q} \setminus \{0\}\colon \big(1/q \in \mathbb{Q} \setminus \{0\} \ \wedge\ 1/q \cdot q = 1\big).$$

Clearly, $\cdot$ is associative and commutative and

$$\forall\, q_1, q_2 \in \mathbb{Q} \setminus \{0\}\colon q_1 \cdot q_2 \in \mathbb{Q} \setminus \{0\}.$$

c) $(\mathbb{R} \setminus \{0\}, \cdot)$ and $(\mathbb{C} \setminus \{0\}, \cdot)$ are commutative groups.

(iii) $G = \{0, 1\}$ together with

$*$	0	1
0	0	1
1	1	0

is a commutative group. See `Exercise` 0.1.

Definition 4. Let $*$ be an operation on G and let $\varnothing \neq G' \subseteq G$ satisfy

$$\forall\, a, b \in G'\colon a * b \in G'.$$

Then the operation $*'\colon G' \times G' \to G'$, defined by $a *' b = a * b$, is called the *induced operation* on G'.

Henceforth we do not distinguish between $*$ and $*'$.

1.1 Permutations

Notation: The set of mappings from a set X to a set Y is denoted by Y^X.

Proposition 5. Let $X \neq \varnothing$ and

$$G = \{f \in X^X \colon f \text{ bijective}\}.$$

G together with function composition $\circ$ is a group. Moreover, G is not commutative if $|X| \geq 3$.

Definition 6. G from Proposition 5 is called the *symmetric group* of the set X. Its elements are called *permutations*.

1.2 Elementary Properties of Groups

In the sequel let G be a group.

Lemma 7. Let $e \in G$ satisfy (1) and (2) and let $a, a' \in G$. Then we have

(i) $a'a = e \;\Rightarrow\; aa' = e$,

(ii) $ae = a$.

Proof. ad (i): According to (2) there exists $a'' \in G$ such that

$$a''a' = e.$$

Using (1) we obtain

$$aa' = eaa' = a''a'aa' = a''ea' = a''a' = e.$$

ad (ii): Let $a' \in G$ such that $a'a = e$. From part (i) we get

$$ae = aa'a \overset{\text{(i)}}{=} ea = a.$$ □

Proposition 8. There exists a unique $e \in G$ satisfying (1) and (2).

Proof. Let (1) and (2) be satisfied for $e = e_1$ and $e = e_2$. Then we have $e_1e_2 = e_2$ and $e_2e_1 = e_1$. Lemma 7.(i) shows $e_2e_1 = e_2$ and hence $e_1 = e_2e_1 = e_2$. □

Definition 9. $e \in G$ satisfying (1) and (2) is called the *neutral element* of G.

In the sequel let e be the neutral element of G.

Proposition 10. For every $a \in G$ there exists a unique $a' \in G$ such that $a'a = e$.

Proof. Let $a', a'' \in G$ such that $a'a = a''a = e$. Lemma 7.(i) shows $aa' = e$. Using Lemma 7.(ii) we obtain

$$a'' \overset{\text{(ii)}}{=} a''e = a''aa' = ea' = a'.$$

Cf. `Exercise` 0.2. □

Definition 11. For $a \in G$ the element $a' \in G$ satisfying $a'a = e$ is called the *inverse element* of a.
Notation: $a' = a^{-1}$, $a' = 1/a$, or $a' = -a$.

Lemma 12. For $a, b, c \in G$ we have

(i) $ab = ac \;\Rightarrow\; b = c$,

(ii) $ba = ca \;\Rightarrow\; b = c$,

(iii) $(a^{-1})^{-1} = a$,

(iv) $(ab)^{-1} = b^{-1}a^{-1}$,

(v) $e^{-1} = e$,

(vi) $\exists_1\, x \in G\colon ax = b$.

Proof. ad (i): Assume $ab = ac$. Then we have $b = eb = a^{-1}ab = a^{-1}ac = ec = c$.
ad (ii): Assume $ba = ca$. By Lemma 7, we have $b = be = baa^{-1} = caa^{-1} = ce = c$.
ad (iii): By Lemma 7.(ii), we have $a = ea = (a^{-1})^{-1}a^{-1}a = (a^{-1})^{-1}e = (a^{-1})^{-1}$.
ad (iv): We have $(b^{-1}a^{-1})(ab) = b^{-1}(a^{-1}a)b = b^{-1}eb = b^{-1}b = e$.
ad (v): We have $ee = e$.
ad (vi): Put $x = a^{-1}b \in G$. By Lemma 7.(i), we have $ax = aa^{-1}b = eb = b$. Uniqueness follows from part (i).
Cf. `Exercise` 0.3. □

1.3 Subgroups

Definition 13. $G' \subseteq G$ is a *subgroup* of G if the following conditions hold:

(i) $G' \neq \varnothing$,

(ii) $\forall\, a, b \in G'\colon ab \in G'$,

(iii) $\forall\, a \in G'\colon a^{-1} \in G'$.

Remark 14.

(i) Every subgroup G' of G satisfies $e \in G'$.
(Proof: Choose $a \in G'$. Then we obtain $a^{-1} \in G'$ and $e = a^{-1}a \in G'$.)

(ii) $\{e\}$ and G are the smallest and the largest subgroup of G, respectively, i.e., every subgroup G' of G satisfies $\{e\} \subseteq G' \subseteq G$.

Example 15. (i) $G' = \mathbb{Z}$ is a subgroup of $G = \mathbb{Q}$ w.r.t. the addition.

(ii) $G' = \{q \in \mathbb{Q}\colon q > 0\}$ is a subgroup of $G = \mathbb{Q} \setminus \{0\}$ w.r.t. the multiplication.

(iii) Let G be the symmetric group of X. For $X_0 \subseteq X$ put

$$G' = \{f \in G \colon \forall\, x \in X_0 \colon f(x) = x\}.$$

Then G' is a subgroup of G.

Proposition 16. Every subgroup of G (together with the induced operation) is a group with neutral element e.

Proof. Definition 13.(i)-(ii) ensure that the induced operation $*'$ on a subgroup G' is well-defined. Obviously, the induced operation $*'$ on G' inherits the associativity of $*$ on G. According to Remark 14.(i) we have $e \in G'$, and for $a \in G'$ we have $e *' a = e * a = a$. Moreover, Definition 13.(iii) ensures $a^{-1} \in G'$ for $a \in G'$ such that $a^{-1} *' a = a^{-1} * a = e$. □

Lemma 17. Let G_1 and G_2 be subgroups of G. Then $G_1 \cap G_2$ is a subgroup of G.

Proof. Note that $e \in G_1 \cap G_2$. Let $a, b \in G_1 \cap G_2$. Then we have $ab \in G_i$ and $a^{-1} \in G_i$ for $i = 1, 2$ since the G_i are subgroups of G. This shows $ab \in G_1 \cap G_2$ and $a^{-1} \in G_1 \cap G_2$. □

Example 18. $G_1 = \{2k \colon k \in \mathbb{Z}\}$ and $G_2 = \{3k \colon k \in \mathbb{Z}\}$ are subgroups of $G = \mathbb{Z}$ w.r.t. the addition. We have $G_1 \cap G_2 = \{6k \colon k \in \mathbb{Z}\}$.
Note that $-2, 3 \in G_1 \cup G_2$, but $(-2) + 3 = 1 \notin G_1 \cup G_2$. Thus $G_1 \cup G_2$ is not a subgroup of G, cf. `Exercise` 0.4.

2 Fields

Definition 1. A set K together with two operations

$$\begin{aligned} +\colon K \times K &\to K \quad \textit{(addition)} \\ \cdot\colon K \times K &\to K \quad \textit{(multiplication)} \end{aligned}$$

is a *field* if the following conditions hold:

(i) $(K, +)$ is a commutative group.

(ii) Let 0 be the neutral element in $(K, +)$. For $K^* = K \setminus \{0\}$ we have

$$\forall\, a, b \in K^* \colon a \cdot b \in K^*,$$

and $(K^*, \cdot)$ is a commutative group.

(iii) $\forall\, a, b, c \in K \colon \big(a \cdot (b + c) = a \cdot b + a \cdot c \wedge (a + b) \cdot c = a \cdot c + b \cdot c\big)$ (*distributivity*).

Notation: The neutral element and the inverse element of $a \in K$ in $(K, +)$ are denoted by 0 and $-a$, respectively. The neutral element in $(K^*, \cdot)$ is denoted by 1. The inverse element of $a \in K^*$ is denoted by a^{-1} or $1/a$.

We often write ab instead of $a \cdot b$ and $a - b$ instead of $a + (-b)$ for $a, b \in K$, and a/b instead of $a \cdot b^{-1}$ for $a \in K$ and $b \in K^*$.

Convention: $\cdot$ has precedence over $+$.

Remark 2. Per definition we have $0 \neq 1$ in every field.

Example 3. (i) $\mathbb{Q}$ with the usual operations $+$ and $\cdot$ is a field.

(ii) $K = \{0, 1\}$ with the addition according to Example 3.(iii) and the multiplication given by

$$\begin{array}{c|cc} \cdot & 0 & 1 \\ \hline 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}$$

is a field. Note that $1 + 1 = 0$ in this case.

2.1 Elementary Properties of Fields

In the sequel let K be a field.

Lemma 4. For $a, b, c \in K$ we have

(i) $0 \cdot a = a \cdot 0 = 0$,

(ii) $a \cdot b = 0 \;\Rightarrow\; a = 0 \vee b = 0$,

(iii) $a \cdot (-b) = (-a) \cdot b = -(ab)$,

(iv) $(-a) \cdot (-b) = ab$,

(v) $a \cdot b = a \cdot c \wedge a \neq 0 \;\Rightarrow\; b = c$,

(vi) $a \neq 0 \;\Rightarrow\; (\exists_1\, x \in K \colon a \cdot x = b)$.

Proof. ad (i): We have

$$0 + 0 \cdot a = 0 \cdot a = (0 + 0) \cdot a = 0 \cdot a + 0 \cdot a.$$

Lemma 1.12.(ii) ensures $0 = 0 \cdot a$. Analogously, we obtain $0 = a \cdot 0$.
ad (ii): Note that (ii) is equivalent to (proof by contraposition)

$$a \neq 0 \wedge b \neq 0 \;\Rightarrow\; a \cdot b \neq 0,$$

which is clearly satisfied since $(K^*, \cdot)$ is a group, see Definition 1.(ii).

ad (iii): By using (i) we obtain

$$a \cdot b + a \cdot (-b) = a \cdot (b + (-b)) = a \cdot 0 \overset{\text{(i)}}{=} 0.$$

Hence the unique inverse element (see Proposition 1.10) is given by $-(a \cdot b) = a \cdot (-b)$. Analogously, we obtain $-(a \cdot b) = (-a) \cdot b$.

ad (iv): Using (iii) we have

$$(-a) \cdot (-b) \overset{\text{(iii)}}{=} -(a \cdot (-b)) \overset{\text{(iii)}}{=} -(-(a \cdot b)) = a \cdot b,$$

where the last equality holds due to Lemma 1.12.(iii).

ad (v): Assume $a \cdot b = a \cdot c$. Using (iii) we thus have

$$a \cdot (b - c) = a \cdot b + a \cdot (-c) \overset{\text{(iii)}}{=} a \cdot b - a \cdot c = 0.$$

According to (ii) we obtain $a = 0 \vee (b - c) = 0$.

ad (vi): Let $a \neq 0$. If $b = 0$, then (ii) implies $x = 0$, and according to (i) we have

$$a \cdot x = a \cdot 0 \overset{\text{(i)}}{=} 0 = b,$$

i.e., $x = 0$ is the unique solution to $ax = b$. If $b \neq 0$, then $x = a^{-1}b$ is the unique solution to $ax = b$ in K^*, see Lemma 1.12.(vi). Moreover, note that $x = 0$ is not a solution to $ax = b$ in this case due to (i).

Cf. `Exercise` 1.1. □

Remark 5. Due to Lemma 4.(i) the multiplication is associative and commutative on $K = K^* \cup \{0\}$ and

$$\forall\, a \in K \colon 1 \cdot a = a.$$

Note that 0 does not have an inverse element w.r.t. the multiplication.

2.2 Pointwise Addition and Multiplication of Functions

In the sequel let $X \neq \varnothing$.

Definition 6. *Addition* and *multiplication* on K^X are defined as follows: For $f, g \in K^X$ and $x \in X$ we put

$$(f + g)(x) = f(x) + g(x)$$

and

$$(f \cdot g)(x) = f(x) \cdot g(x).$$

Terminology: pointwise addition (or sum) and pointwise multiplication (or product) of functions, respectively.

Lemma 7.

(i) $(K^X, +)$ is a commutative group with neutral element $0 \in K^X$.

(ii) The multiplication is associative and commutative on K^X, and for $1 \in K^X$ we have
$$\forall\, f \in K^X \colon 1 \cdot f = f.$$

(iii) Distributivity holds.

(iv) For all $f \in K^X$ we have
$$\big(\exists\, g \in K^X \colon g \cdot f = 1\big) \Leftrightarrow \big(\forall\, x \in X \colon f(x) \neq 0\big).$$

Proof. `Exercise`. □

Notation: $f - g$ instead of $f + (-g)$ for $f, g \in K^X$ and f/g instead of $f \cdot h$ for $f \in K^X$ and $g \in K^X$ provided that $g(x) \neq 0$ and $h(x) = (g(x))^{-1}$ for all $x \in X$.

Chapter II

Vector Spaces and Linear Maps

Vector spaces and linear maps are the fundamental objects of linear algebra. In this context we study, for instance, systems of linear equations.

1 Vector Spaces

In the sequel let K be a field. We use the standard notation and conventions in the context of fields, see Definition I.2.1.

Definition 1. A set V together with two mappings

$$\begin{aligned} \oplus &: V \times V \to V \quad \textit{(vector addition)}, \\ \odot &: K \times V \to V \quad \textit{(scalar multiplication)} \end{aligned}$$

is a *K-vector space* (*vector space over K*) if the following conditions hold:

(i) $(V, \oplus)$ is a commutative group.

(ii) For all $\lambda, \mu \in K$ and $v, w \in V$ we have

$$\begin{aligned} (\lambda + \mu) \odot v &= \lambda \odot v \oplus \mu \odot v, \\ \lambda \odot (v \oplus w) &= \lambda \odot v \oplus \lambda \odot w, \\ \lambda \odot (\mu \odot w) &= (\lambda \cdot \mu) \odot w, \\ 1 \odot v &= v. \end{aligned}$$

The elements of V are called *vectors* and the elements of K are called *scalars*.
Notation: $\mathbf{0}$ for the neutral element and $\ominus v$ for the inverse element of $v \in V$ in $(V, \oplus)$.
Convention: $\odot$ has precedence over $\oplus$.

Example 2 (Coordinate space)**.** For $n \in \mathbb{N}$ let

$$V = K^n = \{(x_1, \dots, x_n) \colon x_i \in K \text{ for } i \in \{1, \dots, n\}\}.$$

Define

$$\begin{aligned}(x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n) &= (x_1 + y_1, \ldots, x_n + y_n), \\ \lambda \odot (x_1, \ldots, x_n) &= (\lambda \cdot x_1, \ldots, \lambda \cdot x_n)\end{aligned} \tag{1}$$

for $\lambda \in K$ and $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in V$. Then V is a K-vector space, see `Exercise` 1.2. In particular, $V = K$ is a K-vector space.

Convention: If not stated otherwise we regard $V = K^n$ as a K-vector space with the vector space operations $\oplus$ and $\odot$ defined in (1).

Example 3 (Function spaces)**.** Let $X \neq \varnothing$ and $V = K^X$ (set of mappings from X to K). Define

$$\begin{aligned}(f \oplus g)(x) &= f(x) + g(x), \\ (\lambda \odot f)(x) &= \lambda \cdot f(x)\end{aligned} \tag{2}$$

for $f, g \in K^X$, $\lambda \in K$ and $x \in X$. Note that the vector addition is given by the pointwise addition from Definition I.2.6. Then V is a K-vector space, see `Exercise` 1.3.

Special cases:

(i) For $K = \mathbb{R}$ and $X = [0,1]$ we obtain the $\mathbb{R}$-vector space $\mathbb{R}^{[0,1]}$ of real-valued mappings on the unit interval $[0,1]$.

(ii) For $K = \mathbb{R}$ and $X = \mathbb{N}$ we obtain the $\mathbb{R}$-vector space $\mathbb{R}^{\mathbb{N}}$ of real-valued sequences.

Convention: We always regard $V = K^X$ as a K-vector space with the vector space operations $\oplus$ and $\odot$ defined in (2).

In the sequel let V be a K-vector space.

Lemma 4. For $v \in V$ and $\lambda \in K$ we have

(i) $0 \odot v = \mathbf{0}$,

(ii) $\lambda \odot \mathbf{0} = \mathbf{0}$,

(iii) $\lambda \odot v = \mathbf{0} \Rightarrow \lambda = 0 \vee v = \mathbf{0}$,

(iv) $(-1) \odot v = \ominus v$.

Proof. ad (i): We have $0 \odot v = (0 + 0) \odot v = 0 \odot v \oplus 0 \odot v$.

ad (ii): We have $\lambda \odot \mathbf{0} = \lambda \cdot (\mathbf{0} \oplus \mathbf{0}) = \lambda \odot \mathbf{0} \oplus \lambda \odot \mathbf{0}$.

ad (iii): From $\lambda \odot v = \mathbf{0}$ and $\lambda \neq 0$ we obtain using (ii) that

$$v = 1 \odot v = (\lambda^{-1} \cdot \lambda) \odot v = \lambda^{-1} \odot \mathbf{0} = \mathbf{0}.$$

ad (iv): Using (i) we obtain

$$v \oplus (-1) \odot v = 1 \odot v \oplus (-1) \odot v = (1 + (-1)) \odot v = \mathbf{0}. \qquad \square$$

Notation: Henceforth we use 0, $+$, and $\cdot$ instead of $\mathbf{0}$, $\oplus$, and $\odot$, respectively. Moreover, we often use λv instead of $\lambda \cdot v$ for $\lambda \in K$ and $v \in V$.

1.1 Subspaces

Definition 5. $U \subseteq V$ is a *subspace* (*linear subspace, vector subspace*) of V if the following conditions hold:

(i) $U \neq \varnothing$,

(ii) $\forall\, v, w \in U \colon v + w \in U$,

(iii) $\forall\, \lambda \in K\ \forall\, v \in U \colon \lambda \cdot v \in U$.

Remark 6. (i) Every subspace U of V satisfies $0 \in U$.

(ii) $\{0\}$ and V are the smallest and the largest subspace of V, respectively, i.e., every subspace U of V satisfies $\{0\} \subseteq U \subseteq V$.

Cf. Remark I.1.14.

Example 7. (i) For $a \in \mathbb{R}^n$ und $b \in \mathbb{R}$ put

$$U = \Big\{x \in \mathbb{R}^n \colon \sum_{i=1}^{n} a_i \cdot x_i = b\Big\}.$$

U is a subspace of $\mathbb{R}^n$ if and only if $b = 0$, see `Exercise` 1.4.

(ii) The set of convergent sequences in $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{N}}$.

(iii) The set $C([0, 1])$ of continuous functions from $[0, 1]$ to $\mathbb{R}$ is a subspace of $\mathbb{R}^{[0,1]}$, see `Exercise` 1.3.

Proposition 8. Every subspace of V (together with the induced vector addition and the induced scalar multiplication) is a K-vector space.

Proof. Analogous to subgroups, see Proposition I.1.16. □

Lemma 9. Let U_1 and U_2 be subspaces of V. Then $U_1 \cap U_2$ is a subspace of V.

Proof. Analogous to subgroups, see Lemma I.1.17. □

Remark 10. The analogue of Lemma 9 for the union of subspaces is false, see `Exercise` 1.4.

1.2 Span and Linear Combinations

Definition 11. For $W \subseteq V$ let

$$\mathfrak{U} = \{U \colon U \text{ subspace of } V,\, W \subseteq U\}.$$

$\bigcap_{U \in \mathfrak{U}} U$ is called *span* (*linear span, linear hull*) of W. *Notation*:

$$\operatorname{span}(W) = \bigcap_{U \in \mathfrak{U}} U.$$

Remark 12. $\operatorname{span}(W)$ is the smallest subspace of V that contains W. In particular, $\operatorname{span}(\varnothing) = \{0\}$.

Definition 13. For $n \in \mathbb{N}$ we call $v \in V$ a *linear combination* of $v_1, \dots, v_n \in V$ if

$$\exists\, \lambda_1, \dots, \lambda_n \in K \colon v = \sum_{i=1}^{n} \lambda_i v_i.$$

Notation: $L(v_1, \dots, v_n)$ is the set of all linear combinations of $v_1, \dots, v_n$.

Example 14. For $V = \mathbb{R}^2$ and $v_1 = (1, 0)$, $v_2 = (0, 1)$, and $v_3 = (3, 2)$ we have

$$L(v_1, v_2) = L(v_1, v_3) = L(v_2, v_3) = L(v_1, v_2, v_3) = \mathbb{R}^2.$$

Lemma 15. $L(v_1, \dots, v_n)$ is a subspace of V for all $n \in \mathbb{N}$ and $v_1, \dots, v_n \in V$.

Proof. We have $0 \in L(v_1, \dots, v_n)$. For $v, w \in L(v_1, \dots, v_n)$ there exist $\lambda_1, \dots, \lambda_n \in K$ and $\mu_1, \dots, \mu_n \in K$ such that

$$v = \sum_{i=1}^{n} \lambda_i v_i, \qquad w = \sum_{i=1}^{n} \mu_i v_i.$$

Hence we get

$$v + w = \sum_{i=1}^{n} (\lambda_i + \mu_i) v_i \in L(v_1, \dots, v_n)$$

and

$$\lambda v = \sum_{i=1}^{n} \lambda \lambda_i v_i \in L(v_1, \dots, v_n)$$

for all $\lambda \in K$. $\square$

Proposition 16. For $\varnothing \neq W \subseteq V$ we have

$$\operatorname{span}(W) = \bigcup_{n \in \mathbb{N}} \bigcup_{(v_1, \dots, v_n) \in W^n} L(v_1, \dots, v_n).$$

Corollary 17. For $n \in \mathbb{N}$ and $v_1, \dots, v_n \in V$ we have

$$\operatorname{span}(\{v_1, \dots, v_n\}) = L(v_1, \dots, v_n).$$

Proof. "$\supseteq$": We have $\{v_1, \dots, v_n\} \subseteq \operatorname{span}(\{v_1, \dots, v_n\})$. Since $\operatorname{span}(\{v_1, \dots, v_n\})$ is a subspace of V, see Remark 12, we obtain $L(v_1, \dots, v_n) \subseteq \operatorname{span}(\{v_1, \dots, v_n\})$.

"$\subseteq$": We have $\{v_1, \dots, v_n\} \subseteq L(v_1, \dots, v_n)$. Moreover, $L(v_1, \dots, v_n)$ is a subspace according to Lemma 15, and hence $\operatorname{span}(\{v_1, \dots, v_n\}) \subseteq L(v_1, \dots, v_n)$. $\square$

Example 18. Let $V = \mathbb{R}^3$ and $v_1, v_2 \in V$. Corollary 17 shows

$$\operatorname{span}(\{v_1\}) = \{\lambda v_1 \colon \lambda \in \mathbb{R}\}.$$

If $v_1 \neq 0$, $\operatorname{span}(\{v_1\})$ is the straight line passing through 0 and v_1. Furthermore, we have

$$\operatorname{span}(\{v_1, v_2\}) = \{\lambda v_1 + \mu v_2 \colon \lambda, \mu \in \mathbb{R}\}.$$

If $v_1 \neq 0$ and $v_2 \notin \operatorname{span}(\{v_1\})$, $\operatorname{span}(\{v_1, v_2\})$ is the plane passing through 0, v_1, and v_2.

Example 19. Let $V = \mathbb{R}^{\mathbb{R}}$. Define $v_i \in V$ for $i \in \{0,1,2,3\}$ by $v_i(x) = x^i$ for $x \in \mathbb{R}$. Then $\operatorname{span}(\{v_i\colon i \in \{0,1,2,3\}\})$ is the subspace of polynomial functions where the degree is at most 3.

2 Bases and Dimension

In the sequel let V be a K-vector space and $(v_i)_{i\in I}$ be a family in V for a set I. We put $V_0 = \{v_i\colon i \in I\}$. If $I \neq \varnothing$ is finite, we may assume w.l.o.g. that $I = \{1,\dots,n\}$.
Notation: Often $(v_1,\dots,v_n)$ or $(v_i)_{i\in\{1,\dots,n\}}$ instead of $(v_i)_{i\in I}$.

2.1 Linear Independence

Definition 1. A family $(v_1,\dots,v_n)$ is *linearly independent* if for all $(\lambda_i)_{i\in\{1,\dots,n\}} \in K^n$ it holds

$$\sum_{i=1}^{n} \lambda_i v_i = 0 \;\Rightarrow\; (\forall\, i \in \{1,\dots,n\}\colon \lambda_i = 0).$$

Otherwise the family $(v_1,\dots,v_n)$ is *linearly dependent*. The empty family $(v_i)_{i\in\varnothing}$ is linearly independent.

Remark 2.

(i) The family (v_1) is linearly independent if and only if $v_1 \neq 0$.

(ii) If $v_i = 0$ for some $i \in \{1,\dots,n\}$, then $(v_1,\dots,v_n)$ is linearly dependent.
(Proof: $1 \cdot v_i = 0$.)

(iii) If $v_i = v_j$ for $i \neq j$, then $(v_1,\dots,v_n)$ is linearly dependent.
(Proof: $1 \cdot v + (-1) \cdot v = 0$.)

(iv) For $J \subseteq \{1,\dots,n\}$ it holds: $(v_i)_{i\in\{1,\dots,n\}}$ linearly independent $\Rightarrow$ $(v_i)_{i\in J}$ linearly independent.

Lemma 3. Let $n \geq 2$. The following statements are equivalent:

(i) $(v_1,\dots,v_n)$ linearly dependent,

(ii) $\exists\, i \in \{1,\dots,n\}\colon v_i \in L((v_j)_{j\in\{1,\dots,n\}\setminus\{i\}})$.

Proof. "(i) $\Rightarrow$ (ii)": There exist $(\lambda_j)_{j\in\{1,\dots,n\}} \in K^n$ and $i \in \{1,\dots,n\}$ such that $\sum_{j=1}^{n} \lambda_j v_j = 0$ and $\lambda_i \neq 0$. This shows

$$v_i = \sum_{j\in\{1,\dots,n\}\setminus\{i\}} (-\lambda_j/\lambda_i) \cdot v_j \in L((v_j)_{j\in\{1,\dots,n\}\setminus\{i\}}).$$

"(ii) $\Rightarrow$ (i)": There exist $i \in \{1,\dots,n\}$ and $(\lambda_j)_{j\in\{1,\dots,n\}\setminus\{i\}} \in K^{n-1}$ such that $v_i = \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \lambda_j v_j$. Put $\lambda_i = -1$ to obtain

$$\sum_{j=1}^{n} \lambda_j v_j = \lambda_i v_i + \sum_{j\in\{1,\dots,n\}\setminus\{i\}} \lambda_j v_j = -v_i + v_i = 0.$$ □

Example 4. (i) Consider the situation from Example 1.14. Then (v_1, v_2), (v_1, v_3), and (v_2, v_3) are linearly independent. E.g., for (v_1, v_3) we have

$$\lambda_1 v_1 + \lambda_3 v_3 = 0 \ \Leftrightarrow\ \lambda_1 \cdot (1,0) + \lambda_3 \cdot (3,2) = 0 \ \Leftrightarrow\ \begin{cases} \lambda_1 + 3\lambda_3 = 0 \\ \qquad 2\lambda_3 = 0 \end{cases}$$

for $\lambda_1, \lambda_3 \in \mathbb{R}$, and hence $\lambda_1 = \lambda_3 = 0$. Furthermore, (v_1, v_2, v_3) is linearly dependent since $v_3 = 3v_1 + 2v_2$.

(ii) Consider the situation from Example 1.19 with $v_0(x) = 1$ and $v_1(x) = x$ for $x \in \mathbb{R}$. Then (v_0, v_1) is linearly independent. To see this, let $\lambda_0 v_0 + \lambda_1 v_1 = 0$ for $\lambda_0, \lambda_1 \in \mathbb{R}$. At the evaluation sites $x = 0$ and $x = 1$ we then have

$$\begin{aligned} 0 &= (\lambda_0 v_0 + \lambda_1 v_1)(0) = \lambda_0 \cdot v_0(0) + \lambda_1 \cdot v_1(0) = \lambda_0 \cdot 1 + \lambda_1 \cdot 0 = \lambda_0, \\ 0 &= (\lambda_0 v_0 + \lambda_1 v_1)(1) = \lambda_0 \cdot v_0(1) + \lambda_1 \cdot v_1(1) = \lambda_0 \cdot 1 + \lambda_1 \cdot 1 = \lambda_0 + \lambda_1, \end{aligned}$$

and hence $\lambda_0 = \lambda_1 = 0$.

Cf. `Exercise` 2.2 and `Exercise` 2.4.

Proposition 5. The following statements are equivalent:

(i) $(v_1, \dots, v_n)$ is linearly independent,

(ii) for all $v \in \operatorname{span}(\{v_1, \dots, v_n\})$ there exists a unique $(\lambda_i)_{i \in \{1,\dots,n\}} \in K^n$ such that

$$v = \sum_{i=1}^{n} \lambda_i v_i.$$

Proof. "(i) $\Rightarrow$ (ii)": Existence follows from Corollary 1.17. Moreover, the fact

$$\sum_{i=1}^{n} \lambda_i v_i = \sum_{i=1}^{n} \mu_i v_i \ \Leftrightarrow\ \sum_{i=1}^{n} (\lambda_i - \mu_i) v_i = 0$$

for $(\lambda_i)_{i \in \{1,\dots,n\}}, (\mu_i)_{i \in \{1,\dots,n\}} \in K^n$ and the linear independence of $(v_1, \dots, v_n)$ shows uniqueness.

"(ii) $\Rightarrow$ (i)": The unique representation of $v = 0$ corresponds to $\lambda_1 = \ldots = \lambda_n = 0$. $\square$

2.2 Generating Sets and Bases

Terminology: For a finite index set I we call $(v_i)_{i \in I}$ finite and $|I|$ the length of $(v_i)_{i \in I}$. For an infintie index set I we call $(v_i)_{i \in I}$ infinite.

Definition 6.

(i) $(v_i)_{i \in I}$ is a *generating set* (*spanning set*) of V if $V = \operatorname{span}(\{v_i \colon i \in I\})$.

(ii) A linearly independent generating set of V is called *basis* of V.

(iii) V is *finitely generated* if there exists a finite generating set of V.

Example 7. Let $V = \mathbb{R}^n$ and $e_1, \ldots, e_n \in \mathbb{R}^n$ be given by

$$e_i(j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{else,} \end{cases}$$

where $e_i(j)$ denotes the j-th component of e_i for $i, j \in \{1, \ldots, n\}$. Clearly, $(e_1, \ldots, e_n)$ is linearly independent and every $x \in \mathbb{R}^n$ can be expressed by

$$x = (x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \cdot e_i.$$

Hence $(e_1, \ldots, e_n)$ is a basis of $\mathbb{R}^n$. It is called the *standard basis* or the *canonical basis* of the coordinate space $\mathbb{R}^n$.

Proposition 8. Let $n \in \mathbb{N}$. The following statements are equivalent:

(i) $(v_1, \ldots, v_n)$ is a basis of V.

(ii) $(v_1, \ldots, v_n)$ is a generating set of V, and $(v_j)_{j \in \{1,\ldots,n\} \setminus \{i\}}$ is not a generating set of V for every $i \in \{1, \ldots, n\}$.

(iii) For all $v \in V$ there exists a unique $(\lambda_1, \ldots, \lambda_n) \in K^n$ such that $\sum_{i=1}^{n} \lambda_i v_i = v$.

(iv) $(v_1, \ldots, v_n)$ is linearly independent and every family $(v_1, \ldots, v_n, v)$ in V with $v \in V$ is linearly dependent.

Proof. We consider the non-trivial case $n \geq 2$.

"(i) $\Rightarrow$ (ii)": Proof by contradiction: Let $i \in \{1, \ldots, n\}$ such that $(v_j)_{j \in \{1,\ldots,n\} \setminus \{i\}}$ is a generating set of V. Then we have $v_i \in L((v_j)_{j \in \{1,\ldots,n\} \setminus \{i\}})$, see Corollary 1.17. Lemma 3 shows that $(v_1, \ldots, v_n)$ is linearly dependent, which is a contradiction to (i).

"(ii) $\Rightarrow$ (iii)": Existence is obvious. Proof by contradiction to show uniqueness: Let $(\lambda_1, \ldots, \lambda_n), (\mu_1, \ldots, \mu_n) \in K^n$ and $i \in \{1, \ldots, n\}$ such that $\lambda_i \neq \mu_i$ and

$$\sum_{j=1}^{n} \lambda_j v_j = \sum_{j=1}^{n} \mu_j v_j.$$

Then we have $v_i \in L((v_j)_{j \in \{1,\ldots,n\} \setminus \{i\}})$, see Proposition 5 and Lemma 3, and hence $L(v_1, \ldots, v_n) = L((v_j)_{j \in \{1,\ldots,n\} \setminus \{i\}})$, which is a contradiction to (ii).

"(iii) $\Rightarrow$ (iv)": The linear independence of $(v_1, \ldots, v_n)$ follows from Proposition 5, the linear dependence of $(v_1, \ldots, v_n, v)$ follows from Lemma 3 since $v \in L(v_1, \ldots, v_n)$.

"(iv) $\Rightarrow$ (i)": It remains to show $V \subseteq \operatorname{span}(\{v_1, \ldots, v_n\})$. For $v \in V$ there exist $(\lambda_1, \ldots, \lambda_n) \in K^n$ and $\lambda \in K$ such that $\sum_{i=1}^{n} \lambda_i v_i + \lambda v = 0$ and additionally $(\lambda_1, \ldots, \lambda_n, \lambda) \neq 0 \in K^{n+1}$. Since $(v_1, \ldots, v_n)$ is linearly independent, we have $\lambda \neq 0$ and hence

$$v = -1/\lambda \cdot \sum_{i=1}^{n} \lambda_i v_i \in \operatorname{span}(\{v_1, \ldots, v_n\}). \qquad \square$$

Corollary 9 ("Basisauswahlsatz")**.** Let $(v_i)_{i\in I}$ be a finite generating set of V. Then there exists $I_0 \subseteq I$ such that $(v_i)_{i\in I_0}$ is a basis of V. In particular, every finitely generated vector space has a basis.

Proof. We consider the non-trivial case $V \neq \{0\}$. Let $V = L((v_i)_{i\in I})$ with a finite set $I \neq \varnothing$.
If $|I| = 1$, then $(v_i)_{i\in I}$ is a basis, see Remark 2.(i). If $|I| \geq 2$, we either have

$$\forall\, i \in I\colon L((v_j)_{j\in I\setminus\{i\}}) \neq V,$$

which implies that $(v_i)_{i\in I}$ is a basis according to Proposition 8, or there exists $i \in I$ such that

$$L((v_j)_{j\in I\setminus\{i\}}) = V.$$

In this case we consider $(v_j)_{j\in I\setminus\{i\}}$ and start over. □

2.3 Dimension of a Vector Space

Lemma 10 ("Basisaustauschlemma")**.** Let $n \in \mathbb{N}$ and let $(v_1, \ldots, v_n)$ be a basis of V. Moreover, let $w = \sum_{j=1}^n \lambda_j v_j$ with $(\lambda_1, \ldots, \lambda_n) \in K^n$ such that $\lambda_i \neq 0$ for some $i \in \{1, \ldots, n\}$. Then $(\tilde{v}_1, \ldots, \tilde{v}_n)$ given by

$$\tilde{v}_j = \begin{cases} v_j, & \text{if } j \neq i, \\ w, & \text{if } j = i, \end{cases}$$

is also a basis of V.

Proposition 11. If V has a finite basis, then every basis of V is finite and the lengths of all bases coincide.

Definition 12. If V has a finite basis, the length of a basis of V is called *dimension* of V and V is called *finite-dimensional*. Otherwise, V is called *infinite-dimensional*.
Notation: $\dim V$ for the dimension of V.

Example 13. For $n \in \mathbb{N}$ we have $\dim \mathbb{R}^n = n$, see Example 7.

Definition 14. A subspace U of $\mathbb{R}^n$ is a *(straight) line* passing through zero if $\dim U = 1$. A subspace U of $\mathbb{R}^n$ is a *plane* passing through zero if $\dim U = 2$.

Corollary 15. Let V be finitely generated and let U be a subspace of V. Then we have

(i) $\dim U \leq \dim V$,

(ii) $\dim U = \dim V \;\Rightarrow\; U = V$.

Corollary 16 ("Basisergänzungssatz")**.** Let V be finite-dimensional with $\dim V = n \geq 2$. Moreover, let $(w_1, \ldots, w_r)$ be linearly independent with $r \in \{1, \ldots, n-1\}$. Then there exist $w_{r+1}, \ldots, w_n \in V$ such that $(w_1, \ldots, w_n)$ is a basis of V.

Proposition 17. Let U_1, U_2 be finite-dimensional subspaces of V. Then we have

$$\dim(U_1 \cap U_2) + \dim(U_1 + U_2) = \dim(U_1) + \dim(U_2).$$

Proof. According to Corollary 15 the subspace $U_1 \cap U_2$ of U_1 is finitely generated and hence there exists a basis $(u_1, \dots, u_n)$ of $U_1 \cap U_2$ due to Corollary 9. Using Corollary 16 we may extend the basis $(u_1, \dots, u_n)$ of $U_1 \cap U_2$ to a basis $(u_1, \dots, u_n, u_1', \dots, u_m')$ of U_1 and a basis $(u_1, \dots, u_n, u_1'', \dots, u_p'')$ of U_2 for $m, p \in \mathbb{N}_0$.
We show that $(u_1, \dots, u_n, u_1', \dots, u_m', u_1'', \dots, u_p'')$ is a basis of $U_1 + U_2$: Since, by construction, $(u_1, \dots, u_n, u_1', \dots, u_m', u_1'', \dots, u_p'')$ is a generating set of $U_1 + U_2$, it remains to show the linear independence. Let $\lambda_1, \dots, \lambda_n, \mu_1, \dots, \mu_m, \nu_1, \dots, \nu_p \in K$ such that

$$\sum_{i=1}^{n} \lambda_i u_i + \sum_{i=1}^{m} \mu_i u_i' + \sum_{i=1}^{p} \nu_i u_i'' = 0$$

Then we have

$$\underbrace{\sum_{i=1}^{n} \lambda_i u_i + \sum_{i=1}^{m} \mu_i u_i'}_{\in U_1} = \underbrace{-\sum_{i=1}^{p} \nu_i u_i''}_{\in U_2} \in U_1 \cap U_2.$$

Hence there exist $\lambda_1', \dots, \lambda_n' \in K$ such that

$$\sum_{i=1}^{n} \lambda_i u_i + \sum_{i=1}^{m} \mu_i u_i' = \sum_{i=1}^{n} \lambda_i' u_i.$$

Since $(u_1, \dots, u_n, u_1', \dots, u_m')$ is a basis of U_1, Proposition 8.(iii) shows

$$\forall\, i \in \{1, \dots, n\} \colon \lambda_i = \lambda_i' \quad \wedge \quad \forall\, i \in \{1, \dots, m\} \colon \mu_i = 0.$$

This yields

$$\sum_{i=1}^{n} \lambda_i u_i + \sum_{i=1}^{p} \nu_i u_i'' = 0.$$

Since $(u_1, \dots, u_n, u_1'', \dots, u_p'')$ is a basis of U_2, we obtain

$$\forall\, i \in \{1, \dots, n\} \colon \lambda_i = 0 \quad \wedge \quad \forall\, i \in \{1, \dots, p\} \colon \nu_i = 0,$$

i.e., $(u_1, \dots, u_n, u_1', \dots, u_m', u_1'', \dots, u_p'')$ is linearly independent and hence a basis of $U_1 + U_2$. Finally, we obtain

$$\begin{aligned} \dim(U_1 \cap U_2) + \dim(U_1 + U_2) &= n + (n + m + p) \\ &= (n + m) + (n + p) = \dim(U_1) + \dim(U_2). \end{aligned}$$ □

Outlook 18. Based on Zorn's lemma (axiom of choice) one can show that every vector space has a basis.

3 Matrices

In the sequel let K be a field and $m, n, p \in \mathbb{N}$.
Given: $v_1, \dots, v_m \in K^n$.
Aim: Find a basis of $\operatorname{span}(\{v_1, \dots, v_m\})$, cf. Corollary 2.9.

Definition 1. An $m \times n$-*matrix* $A = (a_{i,j})_{1 \le i \le m,\, 1 \le j \le n}$ (over K) is a rectangular array of elements of K. Its elements $a_{i,j} \in K$ are called *entries* of A.
Notation: $K^{m \times n}$ denotes the set of all $m \times n$-matrices over K. We write

$$A = (a_{i,j})_{1 \le i \le m,\, 1 \le j \le n} = (a_{i,j})_{i,j} \in K^{m \times n}.$$

Henceforth we identify[1] K^m with $K^{m \times 1}$. We put ("*transposed*")

$$(x_1, \dots, x_m)^\top = \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} \in K^m$$

and

$$\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}^\top = (x_1, \dots, x_m) \in K^{1 \times m}$$

for $x_1, \dots, x_m \in K$.

Definition 2.

(i) For $a_1, \dots, a_n \in K^m$, $A = (a_1, \dots, a_n) \in K^{m \times n}$, and $1 \le j \le n$ we call a_j the j-*th column* of A.

(ii) For $a_1, \dots, a_m \in K^n$,

$$A = \begin{pmatrix} a_1^\top \\ \vdots \\ a_m^\top \end{pmatrix} \in K^{m \times n}, \tag{1}$$

and $1 \le i \le m$ we call $a_i^\top \in K^{1 \times n}$ the i-*th row* of A.

3.1 Row Space and Rank

Definition 3.

(i) For A according to (1) the *row space* of A is given by

$$\operatorname{RS}(A) = \operatorname{span}(\{a_1, \dots, a_m\}) \subseteq K^n.$$

The *rank* of A is given by $\operatorname{rank} A = \dim \operatorname{RS}(A)$.

[1] Elements of K^m are "column vectors".

(ii) $A \in K^{m\times n}$ is in *row echelon form* if there exists $r \in \{0, \dots, \min(m,n)\}$ such that:

(a) $\forall\, 1 \le i \le r \;\exists\, 1 \le j \le n\colon a_{i,j} \neq 0$,

(b) $\forall\, r < i \le m \;\forall\, 1 \le j \le n\colon a_{i,j} = 0$,

(c) if $r \ge 2$, we have $j_1 < \dots < j_r$ for $j_i = \min\{j\colon a_{i,j} \neq 0\}$.

The leading non-zero entries a_{i,j_i} for $i \in \{1, \dots, r\}$ are called *pivots*.

Example 4. Consider the matrix $A \in \mathbb{R}^{4\times 6}$ given by

$$A = \begin{pmatrix} 0 & 2 & -3 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then, A is in row echolon form with $m = 4$, $n = 6$, $r = 3$, $j_1 = 2$, $j_2 = 3$, and $j_3 = 6$, see Definition 3.(ii). Moreover, the pivots are $a_{1,2} = 2$, $a_{2,3} = -1$, and $a_{3,6} = 3$.

Remark 5. Let A be given by (1) and in row echolon form, and let r be according to Definition 3.(ii). Then we have $r = \operatorname{rank} A$. Moreover, if $r > 0$, we have $\mathrm{RS}(A) = \operatorname{span}(\{a_1, \dots, a_r\})$, and $(a_1, \dots, a_r)$ is a basis of the subspace $\mathrm{RS}(A)$.

Definition 6. There are three types of *elementary row operations*, which can be applied to $A \in K^{m\times n}$ with $\lambda \in K \setminus \{0\}$:

(i) multiplication of a row of A by λ (type I),

(ii) addition of the λ-fold of a row of A to another row of A (type II),

(iii) switching two rows within A (type III).

Lemma 7. Let $\tilde{A} \in K^{m\times n}$ result from finitely many elementary row operations applied to $A \in K^{m\times n}$. Then we have $\mathrm{RS}(\tilde{A}) = \mathrm{RS}(A)$.

Proof. For a single elementary row operation we clearly have $\mathrm{RS}(\tilde{A}) \subseteq \mathrm{RS}(A)$. Since every elementary row operation is reversible using an elementary row operation of the same type, we also obtain $\mathrm{RS}(A) \subseteq \mathrm{RS}(\tilde{A})$. The general case follows by induction. □

Proposition 8. Every matrix $A \in K^{m\times n}$ can be transformed into row echelon form by using fintely many elementary row operations.

Proof. Gaussian elimination for $A = (a_1, \dots, a_n) = (a_{i,j})_{i,j} \in K^{m\times n}$:

1.) If $A = 0$ or $m = 1$: STOP.

2.) Let $j^* = \min\{j \in \{1, \dots, n\}\colon a_j \neq 0\}$.

3.) Choose $i \in \{1, \dots, m\}$ such that $a_{i,j^*} \neq 0$.

4.) If $i \neq 1$: Switch rows 1 and i.

5.) For $i = 2, \dots, m$: Subtract the $a_{i,j^*}/a_{1,j^*}$-fold of row 1 from row i.

6.) If $j^* = n$: STOP.

7.) Remove row 1 and columns $1, \dots, j^*$. Take the resulting submatrix and continue with step 1.).

By induction, this algorithm transforms every $A \in K^{m\times n}$ into row echelon form. □

Example 9. For $m = n = 3$ we consider $v_1, v_2, v_3 \in \mathbb{R}^3$ given by

$$v_1 = (0,0,1)^\top, \quad v_2 = (0,1,0)^\top, \quad v_3 = (0,1,1)^\top.$$

Gaussian elimination for A consisting of the rows $v_1^\top, v_2^\top, v_3^\top$ yields[2]

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{(I)}\leftrightarrow\text{(II)}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xrightarrow{\text{(III')=(III)}-\text{(I)}} \begin{pmatrix} 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & 1 \\ \cdot & \cdot & 1 \end{pmatrix} \xrightarrow{\text{(II')=(II)}-\text{(I)}} \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \boxed{1} \\ \cdot & \cdot & 0 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence we get $\operatorname{rank} A = 2$ and

$$\operatorname{span}(\{v_1, v_2, v_3\}) = \operatorname{span}(\{(0,1,0)^\top, (0,0,1)^\top)\}) = \{x \in \mathbb{R}^3 \colon x_1 = 0\}.$$

Moreover, $((0,1,0)^\top, (0,0,1)^\top)$ is a basis of this subspace.

3.2 Matrix Multiplication

Remark 10. $K^{m\times n}$ is a K-vector space with the coordinatewise operations from Example 1.2. We have $\dim K^{m\times n} = \dim K^{m\cdot n} = m \cdot n$, see Example 2.13.

Definition 11. The *matrix product* $C = (c_{i,j})_{i,j} \in K^{m\times p}$ of two matrices $A = (a_{i,\ell})_{i,\ell} \in K^{m\times n}$ and $B = (b_{\ell,j})_{\ell,j} \in K^{n\times p}$ is defined by

$$c_{i,j} = \sum_{\ell=1}^{n} a_{i,\ell} \cdot b_{\ell,j}, \qquad 1 \le i \le m,\ 1 \le j \le p.$$

[2] The elementary row operation, e.g., of type II where the "new" third row results from the sum of the previous third row and the λ-fold of the first row is indicated by "(III')=(III)+$\lambda\cdot$(I)".

The matrix product can be illustrated by the following scheme:

$$
\begin{array}{cc}
 & \left.\begin{pmatrix} b_{1,1} & \dots & b_{1,j} & \dots & b_{1,p} \\ \vdots & & \vdots & & \vdots \\ b_{n,1} & \dots & b_{n,j} & \dots & b_{n,p} \end{pmatrix}\right\} n \\
\underbrace{\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{i,1} & \dots & a_{i,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{pmatrix}}_{n} & \left.\underbrace{\begin{pmatrix} c_{1,1} & \dots & c_{1,j} & \dots & c_{1,p} \\ \vdots & & \vdots & & \vdots \\ c_{i,1} & \dots & \boxed{c_{i,j}} & \dots & c_{i,p} \\ \vdots & & \vdots & & \vdots \\ c_{m,1} & \dots & c_{m,j} & \dots & c_{m,p} \end{pmatrix}}_{p}\right\} m
\end{array}
$$

Notation: $C = A \cdot B$ or $C = AB$.

Convention: $\cdot$ has precedence over $+$.

Remark 12. The matrix multiplication is associative, see `Exercise` 4.2. However, it is neither commutative nor does $AB = 0$ imply $A = 0$ or $B = 0$, see, e.g.,

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Remark 13. Let

$$A = \begin{pmatrix} a_1^\top \\ \vdots \\ a_m^\top \end{pmatrix} \in K^{m \times n}, \qquad B = (b_1, \dots, b_p) \in K^{n \times p}.$$

Then we have

$$A \cdot B = (A \cdot b_1, \dots, A \cdot b_p) = \begin{pmatrix} a_1^\top \cdot B \\ \vdots \\ a_m^\top \cdot B \end{pmatrix}.$$

Special cases:

(i) For $A = (a_1, \dots, a_n) = (a_{i,j})_{i,j} \in K^{m \times n}$ and $x = (x_1, \dots, x_n)^\top \in K^n$ we have

$$Ax = \begin{pmatrix} \sum_{\ell=1}^n a_{1,\ell} \cdot x_\ell \\ \vdots \\ \sum_{\ell=1}^n a_{m,\ell} \cdot x_\ell \end{pmatrix} = \sum_{\ell=1}^n x_\ell \cdot a_\ell \in K^m.$$

(ii) For $y \in K^n$ and

$$B = \begin{pmatrix} b_1^\top \\ \vdots \\ b_n^\top \end{pmatrix} = (b_{i,j})_{i,j} \in K^{n \times p}$$

we have

$$y^\top \cdot B = \left(\sum_{\ell=1}^n y_\ell \cdot b_{\ell,1}, \dots, \sum_{\ell=1}^n y_\ell \cdot b_{\ell,p} \right) = \sum_{\ell=1}^n y_\ell \cdot b_\ell^\top \in K^{1 \times p}.$$

Definition 14. $E_n = (e_{i,j})_{i,j} \in K^{n\times n}$ given by

$$e_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{else}, \end{cases}$$

is called $n \times n$-*identity matrix*.

Lemma 15. For $A, A_1, A_2 \in K^{m\times n}$, $B, B_1, B_2 \in K^{n\times p}$, and $\lambda \in K$ we have[3]

(i) $(A_1 + A_2) \cdot B = A_1 \cdot B + A_2 \cdot B$,

(ii) $A \cdot (B_1 + B_2) = A \cdot B_1 + A \cdot B_2$,

(iii) $(\lambda \cdot A) \cdot B = A \cdot (\lambda \cdot B) = \lambda \cdot (A \cdot B)$,

(iv) $A \cdot E_n = A$ and $E_n \cdot B = B$.

Proof. `Exercise`. □

Notation: Due to Remark 12 and Lemma 15.(iii) we may omit brackets in the context of scalar and matrix multiplication.

4 Linear Maps

In the sequel let U, V, and W be K-vector spaces.

Definition 1. A map $F\colon V \to W$ is *linear* if

(i) $\forall\, v, w \in V\colon F(v + w) = F(v) + F(w)$,

(ii) $\forall\, v \in V\ \forall\, \lambda \in K\colon F(\lambda v) = \lambda F(v)$.

F is a *linear isomorphism* if F is linear and bijective. Two vector spaces V and W are *isomorphic* if there exists a linear isomorphism from V to W.

Example 2.

(i) For $A \in K^{m\times n}$ let $\mathcal{F}_A\colon K^n \to K^m$ be given by

$$\mathcal{F}_A(v) = Av, \qquad v \in K^n.$$

According to Lemma 3.15.(ii) and (iii) we have

$$\begin{aligned} \mathcal{F}_A(v + w) &= A(v + w) = Av + Aw = \mathcal{F}_A(v) + \mathcal{F}_A(w), \\ \mathcal{F}_A(\lambda v) &= A(\lambda v) = \lambda Av = \lambda \mathcal{F}_A(v) \end{aligned}$$

for all $v, w \in K^n$ and $\lambda \in K$. Hence $\mathcal{F}_A$ is a linear map from K^n to K^m.

[3] $(K^{n\times n}, +, \cdot)$ is a ring with unity E_n.

(ii) For $a, b \in \mathbb{R}$ with $a < b$ let $F\colon C([a,b]) \to \mathbb{R}$ be given by

$$F(v) = \int_a^b v(x)\,dx, \qquad v \in C([a,b]).$$

Note that

$$\begin{aligned} F(v+w) &= \int_a^b v(x) + w(x)\,dx = \int_a^b v(x)\,dx + \int_a^b w(x)\,dx = F(v) + F(w), \\ F(\lambda v) &= \int_a^b \lambda \cdot v(x)\,dx = \lambda \cdot \int_a^b v(x)\,dx = \lambda F(v) \end{aligned}$$

for all $v, w \in C([a,b])$ and $\lambda \in \mathbb{R}$, see Analysis I. Hence F is a linear map from $C([a,b])$ to $\mathbb{R}$.

(iii) For $x \in [0,1]$ let $F\colon \mathbb{R}^{[0,1]} \to \mathbb{R}$ be given by

$$F(\varphi) = \varphi(x), \qquad \varphi \in \mathbb{R}^{[0,1]}.$$

Note that

$$\begin{aligned} F(\varphi + \psi) &= (\varphi + \psi)(x) = \varphi(x) + \psi(x) = F(\varphi) + F(\psi), \\ F(\lambda\varphi) &= (\lambda\varphi)(x) = \lambda \cdot \varphi(x) = \lambda F(\varphi) \end{aligned}$$

for all $\varphi, \psi \in \mathbb{R}^{[0,1]}$ and $\lambda \in \mathbb{R}$, see Definition I.2.6. Hence F is a linear map from $\mathbb{R}^{[0,1]}$ to $\mathbb{R}$. It is called function *evaluation* at x.

Lemma 3. Let $F\colon V \to W$ be linear. Then we have:

(i) $F(0) = 0$.

(ii) F is injective if and only if $F(v) \neq 0$ for all $v \in V \setminus \{0\}$.

(iii) $F(\sum_{i=1}^n \lambda_i v_i) = \sum_{i=1}^n \lambda_i F(v_i)$ for $\lambda_1, \dots, \lambda_n \in K$ and $v_1, \dots, v_n \in V$.

(iv) If $(v_1, \dots, v_n)$ is linearly dependent in V, then $(F(v_i), \dots, F(v_n))$ is also linearly dependent in W.

(v) If V' is a subspace of V, then $F(V')$ is a subspace of W.

(vi) If W' is a subspace of W, then $F^{-1}(W')$ is a subspace of V.

(vii) If V is finite-dimensional, then $F(V)$ is also finite-dimensional and $\dim F(V) \leq \dim V$.

Proof. ad (i): $F(0) = F(0 \cdot 0) = 0 \cdot F(0) = 0$.
ad (ii): For $v, w \in V$ we have $F(v - w) = F(v) - F(w)$ and hence

$$F(v-w) = 0 \;\Leftrightarrow\; F(v) = F(w).$$

Thus F is injective if and only if

$$F(v-w)=0 \;\Leftrightarrow\; F(v)=F(w) \;\Leftrightarrow\; v-w=0.$$

ad (iii): Induction.

ad (iv): There exists $i \in \{1,\ldots,n\}$ such that

$$v_i = \sum_{j\in\{1,\ldots,n\}\setminus\{i\}} \lambda_j v_j$$

for $\lambda_j \in K$, see Lemma 2.3. According to part (iii) we obtain

$$F(v_i) = \sum_{j\in\{1,\ldots,n\}\setminus\{i\}} \lambda_j F(v_j).$$

According to Lemma 2.3 the family $(F(v_i),\ldots,F(v_n))$ is linearly dependent in W. Cf. `Exercise` 5.1.

ad (v): We have $0=F(0)\in F(V')$. Moreover, for $v,w\in V'$ and $\lambda \in K$ we have $v+w\in V'$ and $\lambda v \in V'$. Hence we get $F(v)+F(w)=F(v+w)\in F(V')$ and $\lambda F(v)=F(\lambda v)\in F(V')$. Cf. `Exercise` 5.1.

ad (vi): We have $F(0)=0\in W'$. Moreover, for $v,w\in V$ with $F(v),F(w)\in W'$ and $\lambda\in K$ we have $F(v)+F(w)\in W'$ and $\lambda F(v)\in W'$. Hence we get $F(v+w)=F(v)+F(w)\in W'$ and $F(\lambda v)=\lambda F(v)\in W'$.

ad (vii): Let $(v_1,\ldots,v_n)$ be a basis of V. Then $(F(v_1),\ldots,F(v_n))$ is a generating system of $F(V)$ according to part (iii). Apply Corollary 2.9 (Basisauswahlsatz). □

Remark 4. Let $F\colon V\to W$.

(i) F is linear if and only if $F(\lambda v+w)=\lambda F(v)+F(w)$ for all $v,w\in V$ and $\lambda\in K$, cf. `Exercise` 5.2.
Proof: "$\Rightarrow$": Let F be linear. We have

$$F(\lambda v+w)=F(\lambda v)+F(w)=\lambda F(v)+F(w)$$

for $v,w\in V$ and $\lambda\in K$.
"$\Leftarrow$": For $w\in V$ we have

$$F(w)=F(1\cdot 0+w)=1\cdot F(0)+F(w)=F(0)+F(w),$$

which shows $F(0)=0$. Hence we have

$$F(v+w)=F(1\cdot v+w)=1\cdot F(v)+F(w)=F(v)+F(w)$$

and

$$F(\lambda v)=F(\lambda v+0)=\lambda F(v)+F(0)=\lambda F(v),$$

respectively, for $v,w\in V$ and $\lambda\in K$.

(ii) If F is a linear isomorphism, then F^{-1} is also linear.
Proof: Let $w, w' \in W$ and $\lambda \in K$. For $v = F^{-1}(w)$ and $v' = F^{-1}(w')$ we have

$$F(\lambda v + v') = \lambda w + w'$$

and thus

$$\lambda F^{-1}(w) + F^{-1}(w') = \lambda v + v' = F^{-1}(\lambda w + w').$$

Proposition 5. Let $F\colon V \to W$ be linear and $n \in \mathbb{N}$.

(i) If F is injective and $(v_1, \ldots, v_n)$ is a linearly independent family in V, then $(F(v_1), \ldots, F(v_n))$ is a linear independent family in W.

(ii) If F is surjective and $(v_1, \ldots, v_n)$ is a generating set of V, then $(F(v_1), \ldots, F(v_n))$ is a generating set of W.

(iii) If F is a linear isomorphism and V is finite-dimensional, then $\dim V = \dim W$.

Proof. ad (i): Let $\sum_{i=1}^n \lambda_i F(v_i) = 0$ for $\lambda_1, \ldots, \lambda_n \in K$. Then we have $F(\sum_{i=1}^n \lambda_i v_i) = 0$ and thus $\sum_{i=1}^n \lambda_i v_i = 0$ since F is injective, see Lemma 3.(ii). The linear independence of $(v_1, \ldots, v_n)$ implies $\lambda_i = 0$ for all $i \in \{1, \ldots, n\}$.

ad (ii): Let $w \in W$. By assumption, there exist $v \in V$ and $\lambda_1, \ldots, \lambda_n \in K$ such that $F(v) = w$ and $v = \sum_{i=1}^n \lambda_i v_i$. Then we have $w = \sum_{i=1}^n \lambda_i F(v_i)$.

ad (iii): Let $(v_1, \ldots, v_n)$ be a basis of V. According to (i) and (ii) the family $(F(v_1), \ldots, F(v_n))$ is a basis of W. $\square$

4.1 Construction of Linear Maps

In the sequel let $n, m \in \mathbb{N}$ and $\mathcal{A} = (v_1, \ldots, v_n)$ be a basis of V.

Lemma 6. For linear maps $F, G\colon V \to W$ we have

$$F = G \;\Leftrightarrow\; \forall\, i \in \{1, \ldots, n\}\colon F(v_i) = G(v_i).$$

Proof. "$\Rightarrow$": Obvious.

"$\Leftarrow$": Let $v \in V$. Since $\mathcal{A}$ is a basis of V and thus a generating set of V, there exist $\lambda_1, \ldots, \lambda_n \in K$ such that $v = \sum_{i=1}^n \lambda_i v_i$. By using Lemma 3.(iii) we obtain

$$F(v) = F\left(\sum_{i=1}^n \lambda_i v_i\right) = \sum_{i=1}^n \lambda_i F(v_i) = \sum_{i=1}^n \lambda_i G(v_i) = G\left(\sum_{i=1}^n \lambda_i v_i\right) = G(v). \qquad \square$$

Proposition 7. Let $(w_1, \ldots, w_n)$ be a family in W. Then there exists a unique linear map $F\colon V \to W$ such that

$$\forall\, i \in \{1, \ldots, n\}\colon F(v_i) = w_i.$$

Moreover, F is injective if and only if $(w_1, \ldots, w_n)$ is linearly independent, and F is surjective if and only if $(w_1, \ldots, w_n)$ is a generating set of W.

Proof. "Uniqueness": See Lemma 6.
"Existence": For $v \in V$ there exist unique $\lambda_1, \ldots, \lambda_n \in K$ such that $v = \sum_{i=1}^n \lambda_i v_i$. We define $F\colon V \to W$ by

$$F(v) = \sum_{i=1}^n \lambda_i w_i.$$

For $w \in V$ there exist unique $\mu_1, \ldots, \mu_n \in K$ such that $w = \sum_{i=1}^n \mu_i v_i$ and hence $v + w = \sum_{i=1}^n (\lambda_i + \mu_i) v_i$. This shows

$$F(v+w) = \sum_{i=1}^n (\lambda_i + \mu_i) w_i = \sum_{i=1}^n \lambda_i w_i + \sum_{i=1}^n \mu_i w_i = F(v) + F(w).$$

Moreover, for $\lambda \in K$ we have $\lambda v = \sum_{i=1}^n (\lambda \cdot \lambda_i) v_i$ and hence

$$F(\lambda v) = \sum_{i=1}^n (\lambda \cdot \lambda_i) w_i = \lambda \cdot \sum_{i=1}^n \lambda_i w_i = \lambda F(v).$$

This shows that F is linear.
"Injectivity": By Lemma 3.(ii), F is injective if and only if

$$F(v) = 0 \;\Rightarrow\; v = 0.$$

Finally, note that $F(v) = \sum_{i=1}^n \lambda_i w_i$, $v = \sum_{i=1}^n \lambda_i v_i$, and

$$v = 0 \;\Leftrightarrow\; (\forall\, i \in \{1, \ldots, n\}\colon \lambda_i = 0),$$

since $\mathcal{A}$ is a basis and thus linearly independent.
"Surjectivity": Obvious. □

Example 8. Let $V = W = \mathbb{R}^2$.

(i) For $\alpha \in \mathbb{R}$ let

$$\begin{aligned} v_1 &= (\cos(0), \sin(0))^\top = (1,0)^\top, \\ v_2 &= (\cos(\pi/2), \sin(\pi/2))^\top = (0,1)^\top, \\ w_1 &= (\cos(\alpha), \sin(\alpha))^\top, \\ w_2 &= (\cos(\pi/2 + \alpha), \sin(\pi/2 + \alpha))^\top = (-\sin(\alpha), \cos(\alpha))^\top. \end{aligned}$$

The unique linear map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $F(v_1) = w_1$ and $F(v_2) = w_2$ represents a rotation of α (counterclockwise).

(ii) Let $v_1 = (1,1)^\top$ and $v_2 = (-1,1)^\top$. The unique linear map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $F(v_1) = v_1$ and $F(v_2) = -v_2$ represents a reflection through the axis $\operatorname{span}(\{v_1\})$.

Corollary 9. Let V and W be finite-dimensional. V and W are isomorphic if and only if $\dim V = \dim W$.

Proof. "$\Rightarrow$": See Proposition 5.(iii).
"$\Leftarrow$": Choose bases $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_n)$ of V and W, respectively, and apply Proposition 7. □

4.2 Coordinate Systems

Example 10. Consider the standard basis $\mathcal{E} = (e_1, \dots, e_n)$ of $\mathbb{R}^n$, see Example 2.7, and let $\mathcal{A} = (v_1, \dots, v_n)$ be a basis of the $\mathbb{R}$-vector space V. According to Proposition 7 we define a linear isomorphism $\Phi_\mathcal{A} \colon \mathbb{R}^n \to V$ by

$$\Phi_\mathcal{A}(e_i) = v_i.$$

For $(\lambda_1, \dots, \lambda_n)^\top \in \mathbb{R}^n$ we have

$$\Phi_\mathcal{A}\left((\lambda_1, \dots, \lambda_n)^\top\right) = \Phi_\mathcal{A}\left(\sum_{i=1}^n \lambda_i e_i\right) = \sum_{i=1}^n \lambda_i v_i.$$

Definition 11. $\Phi_\mathcal{A}$ from Example 10 is the *coordinate system* in V that is induced by $\mathcal{A}$, and $\Phi_\mathcal{A}^{-1}(v)$ is the *family of coordinates* of $v \in V$ w.r.t. $\mathcal{A}$.

Corollary 12. Every finite-dimensional K-vector space $V \neq \{0\}$ is isomorphic to K^n for $n = \dim V$.

Example 13. Consider the $\mathbb{R}$-vector space V of polynomial functions where the degree is at most 3, i.e.,

$$V = \operatorname{span}(\{v_0, \dots, v_3\}) \subseteq \mathbb{R}^\mathbb{R}$$

with $v_i(x) = x^i$ for $x \in \mathbb{R}$ and $i \in \{0, 1, 2, 3\}$, see Example 1.19. Moreover, the family $\mathcal{A} = (v_0, \dots, v_3)$ is linearly independent, see Example 2.4.(ii) for (v_0, v_1). Hence $\mathcal{A}$ is a basis and $\dim V = 4$. By Corollary 12, V is isomorphic to $\mathbb{R}^4$. The polynomial function $v \in V$ given by

$$v(x) = 4x^3 - 2x^2 + x - 7, \qquad x \in \mathbb{R},$$

satisfies $v = -7v_0 + 1v_1 - 2v_2 + 4v_3$ such that the family of coordinates of v w.r.t. the basis $\mathcal{A}$ is given by $\Phi_\mathcal{A}^{-1}(v) = (-7, 1, -2, 4)^\top \in \mathbb{R}^4$.

4.3 Kernel and Image

In the sequel let $F \colon V \to W$ be linear.

Definition 14. The *kernel* of F is defined by

$$\ker F = F^{-1}(\{0\}) = \{v \in V \colon F(v) = 0\}.$$

The *image* of F is defined by

$$\operatorname{im} F = F(V) = \{F(v) \colon v \in V\}.$$

Remark 15.

(i) $\ker F$ and $\operatorname{im} F$ are subspaces of V and W, respectively, see Lemma 3.(v)-(vi).

(ii) F is injective if and only if $\ker F = \{0\}$, see Lemma 3.(ii).

Definition 16. If $\operatorname{im} F$ is finite-dimensional, the *rank* of F is given by $\operatorname{rank} F = \dim \operatorname{im} F$.

Example 17. For $V = W = \mathbb{R}^2$ and $A = (a_1, a_2) \in \mathbb{R}^{2\times 2}$ with $a_1 = (2,1)^\top$ and $a_2 = (-2,-1)^\top$ let $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $F(x) = Ax$ for $x = (x_1, x_2)^\top \in \mathbb{R}^2$. Then we have

$$Ax = x_1 a_1 + x_2 a_2 = (x_1 - x_2) a_1,$$

see Remark 3.13, such that $\operatorname{im} F = \operatorname{span}(\{a_1\})$, which implies $\operatorname{rank} F = 1$, and $\ker F = \operatorname{span}(\{(1,1)^\top\})$.

Notation: For $u \in V$ and $V_0 \subseteq V$ let

$$u + V_0 = \{u + v_0 \colon v_0 \in V_0\}.$$

It describes a *translation* of V_0 in direction u.

Definition 18. $X \subseteq V$ is an *affine subspace* of V if $X = \varnothing$ or if there exists a subspace V_0 of V and $u \in V$ such that $X = u + V_0$. If V_0 is finite-dimensional, the *dimension* of X is defined by $\dim X = \dim V_0$ and X is called *finite-dimensional*. Otherwise X is called *infinite-dimensional*.

Definition 19. An affine subspace X of $\mathbb{R}^n$ is a *(straight) line* if $\dim X = 1$. An affine subspace X of $\mathbb{R}^n$ is a *plane* if $\dim X = 2$.

Proposition 20. Let $w \in W$. Then $F^{-1}(\{w\})$ is an affine subspace of V. If $w \in \operatorname{im} F$ we have

$$F^{-1}(\{w\}) = u + \ker F$$

for every $u \in F^{-1}(\{w\})$.

Proof. If $w \notin \operatorname{im} F$, we have $F^{-1}(\{w\}) = \varnothing$. Let $w \in \operatorname{im} F$ and $u \in F^{-1}(\{w\})$, i.e., $F(u) = w$. For $v \in V$ we have $v = u + (v - u)$ and $F(v) = F(u) + F(v - u)$. Hence we get

$$v \in F^{-1}(\{w\}) \;\Leftrightarrow\; u - v \in \ker F. \qquad \square$$

Example 21. Consider the situation from Example 17. For every $\lambda \in \mathbb{R}$ we have $\lambda a_1 \in \operatorname{im} F$ and

$$\begin{aligned} F^{-1}(\{\lambda a_1\}) = \{v \in V \colon F(v) = \lambda a_1\} &= \lambda \cdot (1,0)^\top + \{\mu \cdot (1,1)^\top \colon \mu \in \mathbb{R}\} \\ &= \{\lambda \cdot (1,0)^\top + \mu \cdot (1,1)^\top \colon \mu \in \mathbb{R}\}. \end{aligned}$$

Proposition 22. Let $k, p \in \mathbb{N}_0$, let $(v_1, \dots, v_k)$ be a basis of $\ker F$, and let $(w_1, \dots, w_p)$ be a basis of $\operatorname{im} F$. Moreover, let $v_{k+j} \in F^{-1}(\{w_j\})$ for $j \in \{1, \dots, p\}$. Then $(v_1, \dots, v_{k+p})$ is a basis of V.

Proof. If $p = 0$, we have $\operatorname{im} F = \{0\}$. This shows $F = 0$ and $\ker F = V$. In the following let $p \in \mathbb{N}$.

We show that the family $(v_1, \ldots, v_{k+p})$ is linearly independent: Let $\sum_{i=1}^{k+p} \lambda_i v_i = 0$ for $\lambda_1, \ldots, \lambda_{k+p} \in K$. Applying F shows

$$0 = F(0) = F\left(\sum_{i=1}^{k+p} \lambda_i v_i\right) = \sum_{i=1}^{k+p} \lambda_i F(v_i) = \sum_{i=1}^{p} \lambda_{k+i} w_i$$

and thus $\lambda_{k+1} = \ldots = \lambda_p = 0$ since $(w_1, \ldots, w_p)$ is a basis of $\operatorname{im} F$. If $k \in \mathbb{N}$, we have $\sum_{i=1}^{k} \lambda_i v_i = 0$, which implies $\lambda_1 = \ldots = \lambda_k = 0$ since $(v_1, \ldots, v_k)$ is a basis of $\ker F$.

Let $v \in V$. We show $v \in \operatorname{span}(\{v_1, \ldots, v_{k+p}\})$: There exists $w = \sum_{i=1}^{p} \lambda_i w_i$ such that $F(v) = w$. Proposition 20 shows

$$v \in F^{-1}(\{w\}) = u + \operatorname{span}(\{v_1, \ldots, v_k\})$$

with $u = \sum_{i=1}^{p} \lambda_i v_{k+i} \in \operatorname{span}(\{v_{k+1}, \ldots, v_{k+p}\})$. □

Example 23. Consider the situation of Example 17. Then $((1,1)^\top, (1,0)^\top)$ is a basis of $\mathbb{R}^2$ according to Proposition 22.

Corollary 24 (Rank-Nullity Theorem)**.** Let V be finite-dimensional. Then we have

$$\dim V = \dim \ker F + \dim \operatorname{im} F.$$

Corollary 25. Let V and W be finite-dimensional with $\dim V = \dim W$. Then, F is injective if and only if F is surjective.

Proof. "$\Rightarrow$": Let F be injective. Corollary 24 shows $\dim W = \dim V = 0 + \dim \operatorname{im} F$ and thus $W = \operatorname{im} F$ by Corollary 2.15.

"$\Leftarrow$": Let F be surjective. Corollary 24 shows $\dim \ker F = 0$ and thus $\ker F = \{0\}$. □

Notation: Let $L(V, W)$ be the set of all linear maps from V to W.

Proposition 26. For $F \in L(V, W)$ and $G \in L(U, V)$ we have $F \circ G \in L(U, W)$.

Proof. See `Exercise` 5.4 a). □

5 Transformation Matrices of Linear Maps

In the sequel let V and W K-vector spaces with $\dim V = n \in \mathbb{N}$ and $\dim W = m \in \mathbb{N}$. Moreover, let $\mathcal{A} = (v_1, \ldots, v_n)$ and $\mathcal{B} = (w_1, \ldots, w_m)$ be bases of V and W, respectively, and let $\mathcal{E} = (e_1, \ldots, e_n)$ and $\tilde{\mathcal{E}} = (\tilde{e}_1, \ldots, \tilde{e}_m)$ be the standard bases of K^n and K^m, respectively, see Example 2.7.

Remark 1. For every linear map $F\colon V \to W$ there exists a unique matrix $A = (a_{i,j})_{i,j} \in K^{m\times n}$ such that

$$\forall\, 1 \le j \le n\colon F(v_j) = \sum_{i=1}^{m} a_{i,j} w_i, \tag{1}$$

see Proposition 2.8.(iii). For $1 \le j \le n$ we have

$$\begin{pmatrix} a_{1,j} \\ \vdots \\ a_{m,j} \end{pmatrix} = \Phi_{\mathcal{B}}^{-1}(F(v_j)).$$

Definition 2. The matrix $A \in K^{m\times n}$ from Remark 1 is denoted by $\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F)$ and is called *transformation matrix* of $F \in L(V,W)$ w.r.t. the bases $\mathcal{A}$ and $\mathcal{B}$.

Example 3. Consider the situation of Example 4.8.(ii). Since $e_1 = 1/2 \cdot (v_1 - v_2)$ and $e_2 = 1/2 \cdot (v_1 + v_2)$, we have

$$\begin{aligned} F(e_1) &= 1/2 \cdot (v_1 + v_2) = e_2, \\ F(e_2) &= 1/2 \cdot (v_1 - v_2) = e_1. \end{aligned}$$

Hence we get

$$\mathcal{M}_{\mathcal{E}}^{\mathcal{E}}(F) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

For $\mathcal{A} = (v_1, v_2)$ we have

$$\mathcal{M}_{\mathcal{A}}^{\mathcal{A}}(F) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Proposition 4. For $F \in L(V,W)$ and $v \in V$ we have

$$F(v) = \Phi_{\mathcal{B}}\big(\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) \cdot \Phi_{\mathcal{A}}^{-1}(v)\big).$$

Commutative diagram:

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "\Phi_{\mathcal{A}}^{-1}"'] & W \\ K^n \arrow[r, "x \mapsto \mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F)\cdot x"'] & K^m \arrow[u, "\Phi_{\mathcal{B}}"'] \end{tikzcd}$$

Proof. The map $G\colon V \to W$ given by

$$G(v) = \Phi_{\mathcal{B}}\big(\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) \cdot \Phi_{\mathcal{A}}^{-1}(v)\big)$$

is linear due to Example 4.2.(i), Remark 4.4.(ii), and Proposition 4.26. It remains to show that $F(v_j) = G(v_j)$ for all $1 \le j \le n$, see Lemma 4.6. Indeed we have

$$G(v_j) = \Phi_{\mathcal{B}}\big(\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) \cdot e_j\big) = \Phi_{\mathcal{B}}\big(\Phi_{\mathcal{B}}^{-1}(F(v_j))\big) = F(v_j). \qquad \square$$

Remark 5.

(i) For $A, B \in K^{m\times n}$ we have

$$(\forall x \in K^n \colon Ax = Bx) \;\Rightarrow\; A = B.$$

(ii) For every $F \in L(V,W)$ there exists a unique matrix $A \in K^{m\times n}$ such that $F = \Phi_{\mathcal{B}} \circ \mathcal{F}_A \circ \Phi_{\mathcal{A}}^{-1}$.

In the sequel let U be a finite-dimensional K-vector space with $\dim U = p \in \mathbb{N}$ and basis $\mathcal{C} = (u_1, \dots, u_p)$.

Proposition 6. For $F \in L(V,W)$ and $G \in L(W,U)$ we have

$$\mathcal{M}_{\mathcal{C}}^{\mathcal{A}}(G \circ F) = \mathcal{M}_{\mathcal{C}}^{\mathcal{B}}(G) \cdot \mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F).$$

Proof. Let $v \in V$. Remark 3.12 and Proposition 4 show

$$\begin{aligned} G \circ F(v) &= G\big(\Phi_{\mathcal{B}}(\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) \cdot \Phi_{\mathcal{A}}^{-1}(v))\big) \\ &= \Phi_{\mathcal{C}}\Big(\mathcal{M}_{\mathcal{C}}^{\mathcal{B}}(G) \cdot \Phi_{\mathcal{B}}^{-1}\Big(\Phi_{\mathcal{B}}\big(\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) \cdot \Phi_{\mathcal{A}}^{-1}(v)\big)\Big)\Big) \\ &= \Phi_{\mathcal{C}}\Big(\mathcal{M}_{\mathcal{C}}^{\mathcal{B}}(G) \cdot \mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) \cdot \Phi_{\mathcal{A}}^{-1}(v)\Big). \end{aligned}$$

Apply Remark 5.(ii). □

6 Change of Basis

In the sequel let $n, m \in \mathbb{N}$ and K be a field. Moreover, let $V \neq \{0\}$ and $W \neq \{0\}$ be finite-dimensional K-vector spaces.

6.1 Invertible Matrices

Definition 1. $A \in K^{n\times n}$ is *invertible* if there exists $A' \in K^{n\times n}$ such that $A'A = E_n$.
Notation: $\mathrm{Gl}(n,K) = \{A \in K^{n\times n} \colon A \text{ invertible}\}$.

Lemma 2. For $A = (a_1, \dots, a_n) \in K^{n\times n}$ the following statements are equivalent:

(i) A is invertible.

(ii) $\mathcal{F}_A \colon K^n \to K^n$ is bijective.

(iii) $(a_1, \dots, a_n)$ is a basis of K^n.

Proof. Put $F = \mathcal{F}_A$.
"(i) ⇒ (ii)": Let $x \in K^n$ such that $F(x) = 0$, and let $A' \in K^{n\times n}$ such that $A'A = E_n$. Then we have

$$x = E_n x = A'Ax = A'\,0 = 0.$$

Hence F is injective and thus bijective due to Corollary 4.25.

"(ii) ⇒ (i)": Remark 4.4.(ii) shows $F^{-1} \in L(K^n, K^n)$. According to `Exercise` 5.3 a) there exists $A' \in K^{n \times n}$ such that

$$\forall\, x \in K^n \colon F^{-1}(x) = A'x.$$

This yields

$$E_n x = x = F^{-1}(F(x)) = A'Ax$$

for all $x \in K^n$. Remark 5.5.(i) shows $A'A = E_n$.
"(ii) ⇔ (iii)": See Proposition 4.7. □

Proposition 3. Let $F \colon V \to W$ be linear. The following statements are equivalent:

(i) F is a linear isomorphism.

(ii) For all bases $\mathcal{A}$ and $\mathcal{B}$ of V and W, respectively, the matrix $\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F)$ is invertible.

(iii) There exist bases $\mathcal{A}$ and $\mathcal{B}$ of V and W, respectively, such that the matrix $\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F)$ is invertible.

Proof. "(i) ⇒ (ii)": Due to Proposition 5.4 we have

$$\mathcal{F}_{\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F)} = \Phi_{\mathcal{B}}^{-1} \circ F \circ \Phi_{\mathcal{A}}.$$

In particular, $\mathcal{F}_{\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F)}$ is bijective since it is the composition of bijective maps. Apply Lemma 2.
"(ii) ⇒ (iii)": Obvious.
"(iii) ⇒ (i)": Due to Proposition 5.4 we have

$$F = \Phi_{\mathcal{B}} \circ \mathcal{F}_{\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F)} \circ \Phi_{\mathcal{A}}^{-1},$$

and $\mathcal{F}_{\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F)}$ is bijective according to Lemma 2. Hence F is bijective since it is the composition of bijective maps. □

Proposition 4. $(\mathrm{Gl}(n, K), \cdot)$ is a group with neutral element E_n.

Notation: A^{-1} is the *inverse matrix* of $A \in \mathrm{Gl}(n, K)$ w.r.t. $\cdot$.

Remark 5. For $A, B \in \mathrm{Gl}(n, K)$ we have $A \cdot A^{-1} = E_n$, $(A^{-1})^{-1} = A$, and $(A \cdot B)^{-1} = B^{-1} \cdot A^{-1}$, see Lemma I.1.7 and Lemma I.1.12.

Lemma 6. Let $F \colon V \to W$ be a linear isomorphism, and let $\mathcal{A}$ and $\mathcal{B}$ be bases of V and W, respectively. Then we have

$$\mathcal{M}^{\mathcal{B}}_{\mathcal{A}}(F^{-1}) = \left(\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F)\right)^{-1}.$$

Proof. Proposition 5.6 shows $\mathcal{M}^{\mathcal{B}}_{\mathcal{A}}(F^{-1}) \cdot \mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F) = \mathcal{M}^{\mathcal{A}}_{\mathcal{A}}(\mathrm{id}_V) = E_n$. □