Technische Hochschule Rosenheim | SoSe 2022
Applied Artificial Intelligence - Bachelor
Analysis 2

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Homework 1: revision differential calculus

To submit: on **Friday, 25.03.2022**, 9:00 a.m., online by the learning campus

Exercise 1 (4 pt.)
Determine

a)
$$\lim_{x\to\infty} \frac{\ln(x)}{e^x}$$

b)
$$\lim_{x\to 1} \frac{1+\cos(\pi x)}{x^2-2x+1}$$

Solution for exercise 1

a)
$$\lim_{x\to\infty} \frac{\ln(x)}{e^x} \overset{\text{“0/0”}}{=} \lim_{x\to\infty} \frac{\ln'(x)}{\exp'(x)} = \lim_{x\to\infty} \frac{1/x}{\exp(x)} = \lim_{x\to\infty} \frac{1}{x\exp(x)} = 0$$

[1 pt. for L'Hôpital, 1 pt. for result]

b)
$$\lim_{x\to 1} \frac{1+\cos(\pi x)}{x^2-2x+1} \overset{\text{“0/0”}}{=} \lim_{x\to 1} \frac{(1+\cos(\pi x))'}{(x^2-2x+1)'} = \lim_{x\to 1} \frac{-\sin(\pi x)\pi}{2x-2} \overset{\text{“0/0”}}{=} \lim_{x\to 1} \frac{-\cos(\pi x)\pi^2}{2} = \frac{\pi^2}{2}$$

[1 pt. for L'Hôpital, 1 pt. for result]

Exercise 2 (3 pts.)
Consider $f(x) = \ln(1+x^2)$. Determine all minima and maxima, without using a second derivative.

Solution for exercise 2
First of all, it is useful to compute

$$f'(x) = \frac{2x}{1+x^2} \quad [0.5 \text{ pt.}].$$

The first derivative is zero iff $x = 0$ [0.5 pt.].

We have $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.
By this change of sign of the derivative from $-$ to $+$ [1 pt.],
we see that $f(x)$ has a minimum in $\hat{x} = 0$ and there are no other extrema. [1 pt.]

Exercise 3 (4 pts.)
Let $A,\omega\in\mathbb{R},\phi_0\in[0,2\pi)$. Compute the first and the second derivative (w.r.t. time t) of the harmonic oscillation described by

$$x(t)=A\sin(\omega t+\phi_0).$$

Check that $x(t)$ is a solution of the following (differential) equation

$$x''(t)+\omega^2x(t)=0$$

and that there holds

$$x(0)=x(2\pi/\omega)=A\sin(\phi_0).$$

Solution for exercise 3
The derivatives read

$$\begin{aligned}
x'(t)&=A\cos(\omega t+\phi_0)\cdot(\omega t+\phi_0)'=A\cos(\omega t+\phi_0)\cdot\omega, \quad \text{[1 pt.]},\\
x''(t)&=-A\sin(\omega t+\phi_0)\cdot\omega^2. \quad \text{[1 pt.]}
\end{aligned}$$

We insert the 2nd derivative and the function itself into the given equation

$$\begin{aligned}
&x''(t)+\omega^2x(t)\\
=&-A\sin(\omega t+\phi_0)\cdot\omega^2+\omega^2\cdot A\sin(\omega t+\phi_0)\\
=&-\omega^2A\sin(\omega t+\phi_0)+\omega^2A\sin(\omega t+\phi_0)\\
=&0 \quad \checkmark \quad \text{[1 pt.]}
\end{aligned}$$

We check the (boundary) conditions

$$\begin{aligned}
x(0)&=A\sin(\omega\cdot 0+\phi_0)=A\sin(\phi_0),\\
x(2\pi/\omega)&=A\sin(2\pi+\phi_0)=A\sin(\phi_0),
\end{aligned}$$

where the latter holds due to the periodicity of the sine. [1 pt.]

Exercise 4 (5 pts.)
Assume the demand N of a product as a function of its price p is given by

$$x=N(p)=100-0,1p-0,2p^2$$

(the so-called demand function).
The costs for the production of x units of the product shall be given by

$$K(x)=100+x.$$

The profit as a function of the price is thus given as the difference between the sales revenue

$$U(p)=pN(p)$$

and the costs K.

Which price $\hat{p}$ maximes the profit function $U(p)-K(x(p))$?

Solution for exercise 4
We summarize what is given:

Demand function: $N(p) = 100 - 0,1p - 0,2p^2$ with variable p [CU = currency units], function value x [PU = product units]

Cost function: $K(x) = 100 + x$ with variable x [PU], function value K [CU]
or $K(x(p)) = 100 + 100 - 0,1p - 0,2p^2 = 200 - 0,1p - 0,2p^2$ with variable p [CU], function value K [CU]

Sales revenue function: $U(p) = pN(p) = p(100 - 0,1p - 0,2p^2)$ [1 pt.] with variable p [CU], function value U [CU]

We search for the:

Profit function: $G(p) = U(p) - K(x(p)) = p(100 - 0,1p - 0,2p^2) - (200 - 0,1p - 0,2p^2) = -0,2p^3 + 0,1p^2 + 100,1p - 200$ [1 pt.] with variable p [CU], function value G [CU]

We would like to maximize $G(p)$:

$$G'(p) = -0,6p^2 + 0,2p + 100,1 \stackrel{!}{=} 0 \quad \text{[1 pt.]}$$

This quadratic equation in p has two zeros:

$$p_{1/2} = \frac{1}{6}\left(1 \pm \sqrt{6007}\right) \approx 0,1667 \pm 12,9174$$

So $p_1 \approx 13,0841$ is a candidate for a maximizer, whereas $p_2 < 0$ is not relevant. [1 pt.]

It is indeed a maximum since

$$G''(p_1) = -1,2p_1 + 0,2 < 0 \quad \text{[1 pt.]}.$$

Thus a price of 13,08 currency units maximizes the profit function $G(p)$.