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Exercise Sheet 6

Linear Algebra (AAI)

Exercise 6.1 (H)

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ be linear and given by

$$\begin{aligned} F\left((1,0,0)^\top\right) &= (1/\sqrt{2}, 1/\sqrt{2}, 0)^\top, \\ F\left((0,1,0)^\top\right) &= (-1/\sqrt{2}, 1/\sqrt{2}, 0)^\top, \\ F\left((0,0,1)^\top\right) &= (0,0,1)^\top. \end{aligned}$$

a) Determine $A \in \mathbb{R}^{3\times 3}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^3$.

b) Determine the dimension and a basis of $\ker F$ and $\operatorname{im} F$.

c) Is F bijective? If so, determine the inverse map $F^{-1} \colon \mathbb{R}^3 \to \mathbb{R}^3$ and $A^{-1} \in \mathbb{R}^{3\times 3}$ such that $F^{-1}(x) = A^{-1}x$ for all $x \in \mathbb{R}^3$. What does F represent geometrically?

Exercise 6.2 (H)

Let $A \in \mathbb{R}^{3\times 3}$ be given by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \\ 7 & 8 & 15 \end{pmatrix}.$$

a) Determine the dimension and a basis of $\ker F$ and $\operatorname{im} F$ for $F = \mathcal{F}_A$.

b) Is F bijective? If so, determine the inverse map $F^{-1} \colon \mathbb{R}^3 \to \mathbb{R}^3$ and $A^{-1} \in \mathbb{R}^{3\times 3}$ such that $F^{-1}(x) = A^{-1}x$ for all $x \in \mathbb{R}^3$.

c) Determine $F^{-1}(\{b\})$ for $b = (-1,-4,-7)^\top$ and $b = (1,0,0)^\top$, i.e., find all $x \in \mathbb{R}^3$ such that $Ax = b$.

Exercise 6.3 (H)

Let V be a K-vector space and let $F \colon V \to V$ be linear. Show that

$$\ker F \cap \operatorname{im} F = \{0\} \;\Leftrightarrow\; \ker(F \circ F) = \ker F.$$

Hint: For "$\Rightarrow$" assume $\ker F \cap \operatorname{im} F = \{0\}$ *and show both statements* $\ker(F \circ F) \subseteq \ker F$ *and* $\ker(F \circ F) \supseteq \ker F$ *separately. For "$\Leftarrow$" consider an arbitrary* $x \in \ker F \cap \operatorname{im} F$ *and deduce that* $x = 0$.

Exercise 6.4 (H)

Let Π_2 be the $\mathbb{R}$-vector space of polynomial functions where the degree is at most 2, see Example II.4.13. Moreover, let $\mathcal{B} = (f_0, f_1, f_2)$ be given by

$$f_0(x) = 1, \qquad f_1(x) = 1 + x, \qquad f_2(x) = 1 + x + x^2$$

for $x \in \mathbb{R}$.

a) Show that $\mathcal{B}$ is a basis of Π_2.

b) Let $v \in \Pi_2$ be given by $v(x) = 2x^2 + x - 3$ for $x \in \mathbb{R}$. Determine the family of coordinates $\Phi_{\mathcal{B}}^{-1}(v) \in \mathbb{R}^3$ of v w.r.t. the basis $\mathcal{B}$.
Hint: Express v in terms of $\mathcal{B}$.