Technische Hochschule Rosenheim
Fakultät ANG
Prof. Dr. André Herzwurm

SoSe 2022
12.04.2022

**Exercise Sheet 4**

# Linear Algebra (AAI)

## Exercise 4.1 (H)

a) Consider $A \in \mathbb{R}^{4\times 3}$ and $B \in \mathbb{R}^{3\times 4}$ given by

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Transform $A$ and $B$ into row echelon form and determine row space and rank.

b) Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^5$ be given by

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Determine a basis of $\operatorname{span}(\{v_1, v_2, v_3, v_4\})$.

## Exercise 4.2 (H)

a) Compute $A \cdot B$ and $B \cdot A$ for $A, B$ given by Exercise 4.1a).

b) Compute every possible matrix product $C_i \cdot C_j$ for

$$C_1 = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \\ 1 & 8 & -7 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 \\ 0 \\ 8 \\ -7 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -1 & 2 & 0 & 8 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{pmatrix}.$$

c) Let $m, n, p, q \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{p\times q}$. Verify that

$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

## Exercise 4.3 (H)

a) Let $A \in \mathbb{R}^{4\times 4}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Express the following elementary row operations on $A$ as a matrix product $L \cdot A$ with $L \in \mathbb{R}^{4\times 4}$:

   i) multiplication of the third row of A by $\lambda$,

   ii) addition of the $\lambda$-fold of the first row of $A$ to the fourth row of $A$,

   iii) switching the second and the fourth row of $A$.

b) Let $A = (a_{i,j})_{i,j} \in \mathbb{R}^{3\times 3}$ be given by

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{pmatrix}.$$

Transform the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 1 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 6}$$

into

$$\begin{pmatrix} 1 & 0 & 0 & b_{1,1} & b_{1,2} & b_{1,3} \\ 0 & 1 & 0 & b_{2,1} & b_{2,2} & b_{2,3} \\ 0 & 0 & 1 & b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} \in \mathbb{R}^{3\times 6}$$

by using elementary row operations. Compute the matrix products $A \cdot B$ and $B \cdot A$ for $B = (b_{i,j})_{i,j} \in \mathbb{R}^{3\times 3}$.

## Exercise 4.4 (H)

Let $A, B \in \mathbb{R}^{n\times n}$. Prove or disprove:

a) $\operatorname{rank}(A + B) = \operatorname{rank} A + \operatorname{rank} B$,

b) $\operatorname{rank}(A \cdot B) = \operatorname{rank} A \cdot \operatorname{rank} B$.