Multiple Integrals

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Let $f : B \subseteq \mathbb{R}^n \to \mathbb{R}$,
B a normal area.

$$
\begin{aligned}
\int_B f(\mathbf{x})\, dF &:= \int \cdots \int_B f(\mathbf{x})\, d\mathbf{x} \\
&:= \int \cdots \int_B f(\mathbf{x})\, dx_1 \ldots dx_n \\
&= \lim_{\delta \to 0, n \to \infty} \sum_{i=1}^{n} f(\mathbf{x}^{[i]}) \cdot \Delta F_i
\end{aligned}
$$

Properties of Integrals I

The following properties transfer to multiple integrals.

Let $f, g : B \to \mathbb{R}$ integrable functions and $B \subseteq \mathbb{R}^n$ a normal area.

Linearity:

$$\int_B f(\mathbf{x}) + g(\mathbf{x})\, d\mathbf{x} = \int_B f(\mathbf{x})\, d\mathbf{x} + \int_B g(\mathbf{x})\, d\mathbf{x}$$

and (factor rule) $\int_B c\, f(\mathbf{x})\, d\mathbf{x} = c \int_B f(\mathbf{x})\, d\mathbf{x}$, $c \in \mathbb{R}$

Additivity for $B_1 \cup B_2 = B$ with $B_1 \cap B_2 = \emptyset$:

$$\int_{B_1} f(\mathbf{x})\, d\mathbf{x} + \int_{B_2} f(\mathbf{x})\, d\mathbf{x} = \int_B f(\mathbf{x})\, dx$$

Monotonicity: $f \leq g \implies \int_B f(\mathbf{x})\, d\mathbf{x} \leq \int_B g(\mathbf{x})\, d\mathbf{x}$

Properties of Integrals II

Inequalities:

"triangle inequality"

$$\left|\int_B f(\mathbf{x})\,d\mathbf{x}\right| \le \int_B |f(\mathbf{x})|\,d\mathbf{x}$$

Cauchy-Schwarz

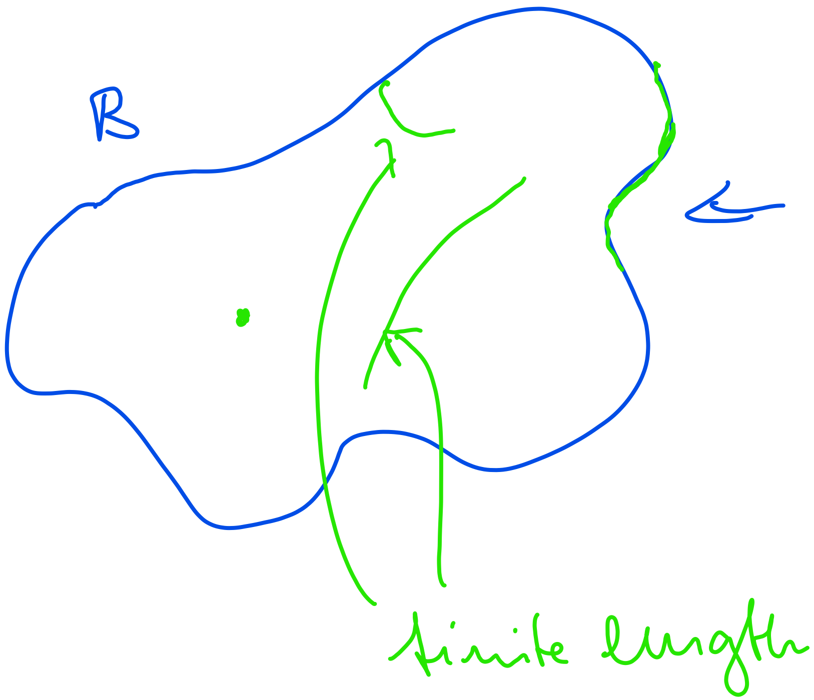
$$\left(\int_B f(\mathbf{x})g(\mathbf{x})\,d\mathbf{x}\right)^2 \le \left(\int_B f(\mathbf{x})^2\,d\mathbf{x}\right)\left(\int_B g(\mathbf{x})^2\,d\mathbf{x}\right)$$

Integration over a set N of measure "zero":

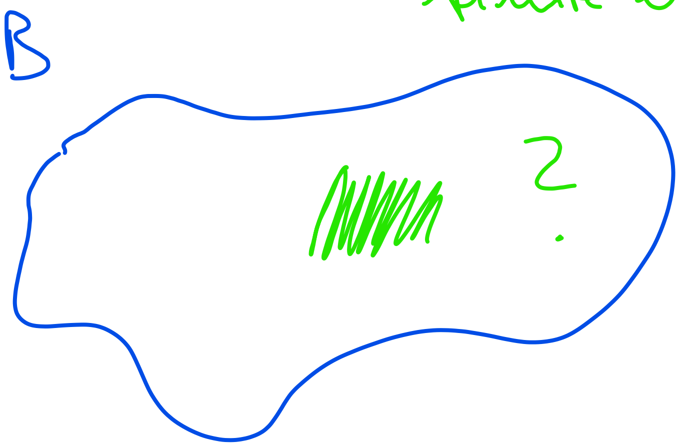
$$\int_N f(\mathbf{x})\,d\mathbf{x} = 0$$

Sets of Measure Zero

We only give some examples in 2d:



doesn't matter if you integrate over an open or a closed set



Examples in 3d:

- plane
- point
- line

Another Application: Barycenter

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The **center of mass s** of a mass distribution (mass density) $\rho : B \subseteq \mathbb{R}^n \to \mathbb{R}^+$ in space is the unique point where the weighted relative position of the distributed mass sums to zero:

$$\mathbf{s} := \frac{1}{M} \int_B \mathbf{x}\, \rho(\mathbf{x})\, d\mathbf{x}$$

where $M := \int_B \rho(\mathbf{x})\, d\mathbf{x}$.

Remarks:
A center of mass may be translated to any distribution ρ or a data set.
If $\rho = 1$, we obtain the so-called centroid (geometric center). The word **barycenter** comprises the terms "center of mass" and "centroid".

Example: Barycenter of a quarter circle

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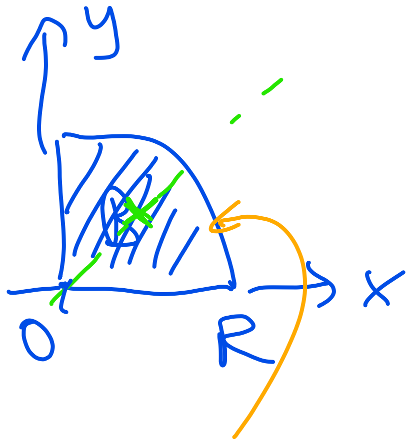

$x^2 + y^2 = R^2$

Due to symmetry

$s_x = s_y$.

Homogeneous mass density

$\rho(x,y) = \rho_0$

Normal area of type I:

$B = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le R,\ 0 \le y \le \sqrt{R^2 - x^2}\}$

$$M = \rho_0 A = \rho_0 \frac{\pi R^2}{4}$$

$$s_x = \frac{1}{M} \iint_B \rho_0 \, x \, dx \, dy =$$

$$= \frac{1}{M} \rho_0 \int_0^R x \int_0^{\sqrt{R^2 - x^2}} 1 \, dy \, dx = \frac{\rho_0}{M} \int_0^R x \sqrt{R^2 - x^2} \, dx \overset{\text{Subs.}}{=}$$

$$= \frac{\rho_0}{M} \left[-\frac{1}{3} (R^2 - x^2)^{3/2} \right]_{x=0}^{R} = \frac{\rho_0}{3M} R^3 = \frac{\cancel{\rho_0} R^3}{3 \cdot \frac{\cancel{\rho_0} \pi R^2}{4}} = \frac{4}{3\pi} R.$$

Area of a Surface in Space

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If $f : B \subseteq \mathbb{R}^2 \to \mathbb{R}$ continuously partial differentiable, $B \subseteq \mathbb{R}^2$ regular,

then the surface (the graph of f) has the area

$$\int\int_B \sqrt{1 + \left(\frac{\partial f}{\partial x}(x,y)\right)^2 + \left(\frac{\partial f}{\partial y}(x,y)\right)^2}\, dx\, dy\,.$$

Remarks:

- For a $f : I \subseteq \mathbb{R} \to \mathbb{R}$ this reduces to the length of a curve.
- Surface integrals are a topic of their own.

Transformation Formula: Coordinate Transformation

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We consider a domain $S \subseteq \mathbb{R}^2$ generated by a **coordinate transformation**

$$x = x(u, v), y = y(u, v)$$

(being continuously differentiable, bijective, ... see literature) from a domain $B \subseteq \mathbb{R}^2$.

Let

$$g : \mathbb{R}^2 \to \mathbb{R}^2, \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x(u, v) \\ y(u, v) \end{pmatrix}.$$

Analogously, we define coordinate transformations for 3d and higher dimensions.

Transformation Formula 2d

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For such a coordinate transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$,

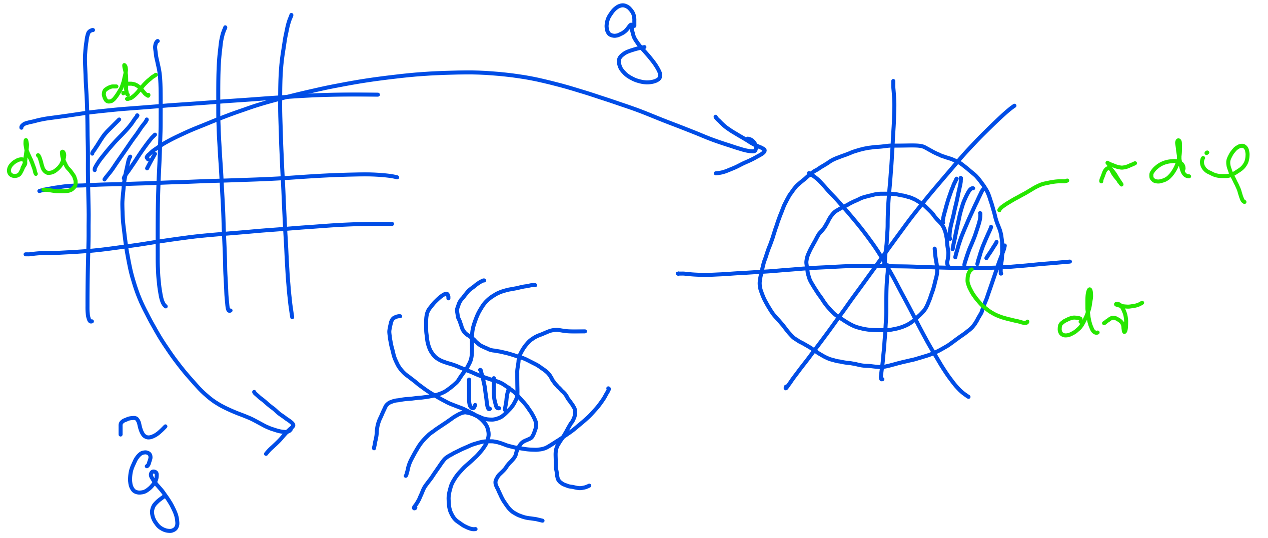
we have for $f : S \subseteq \mathbb{R}^2 \to \mathbb{R}$ continuous $\quad g(B) \subseteq S$

Jacobian (or functional) matrix

$$\iint_S f(x,y)\,dx\,dy$$

$$= \iint_B f(x(u,v), y(u,v)) \cdot \underbrace{\left|\det(J_g(u,v))\right|}_{\frac{\partial(x,y)}{\partial(u,v)}}\,du\,dv\,.$$

Jacobian (or functional) determinant



Transformation from Polar Coordinates

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For polar coordinates there holds

$$x = r\cos(\phi),$$
$$y = r\sin(\phi),$$

thus

$$g : \mathbb{R}^+ \times [0, 2\pi) \to \mathbb{R}^2, \begin{pmatrix} r \\ \phi \end{pmatrix} \mapsto \begin{pmatrix} x(r,\phi) \\ y(r,\phi) \end{pmatrix} = \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \end{pmatrix}.$$

Hence the Jacobian determinant (functional determinant) is:

$$\det \begin{pmatrix} \cos(\phi) & -r\sin(\phi) \\ \sin(\phi) & r\cos(\phi) \end{pmatrix} = r\cos(\phi)^2 + r\sin(\phi)^2 = r$$

Since $r > 0$, we replace the “area element” $dx\,dy$ by $r\,dr\,d\phi$.

Example: Polar Coordinates

Let $S = \{x^2 + y^2 \leq 1\}$.
We apply the transformation formula to compute

$$I = \iint_S \rho \, x^2 \, dx \, dy$$

(the moment of inertia of the full circular disc w.r.t. the y-axis).

We assume $\rho = const$.

$$I = \rho \frac{\pi}{4}$$

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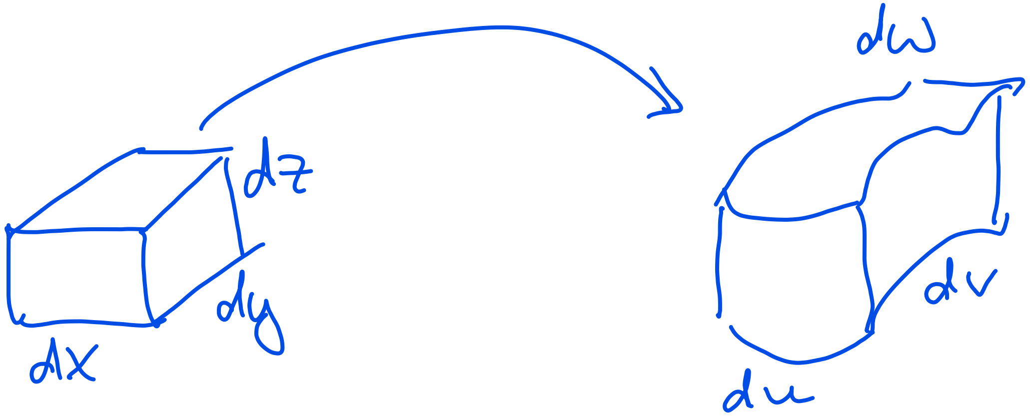
Transformation Formula 3d

For a suitable (see 2d case) coordinate transformation $g : \mathbb{R}^3 \to \mathbb{R}^3$,

we have for $f : S \subseteq \mathbb{R}^3 \to \mathbb{R}$ continuous

$$\iiint_S f(x, y, z)\, dx\, dy\, dz$$
$$= \iiint_B f(x(u, v, w), y(u, v, w)) \cdot \left|\det(J_g(u, v, w))\right| du\, dv\, dw\,.$$



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For cylinder coordinates there holds

$$\begin{aligned} x &= r\cos(\phi), \\ y &= r\sin(\phi), \\ z &= z \end{aligned}$$

thus

$$g : \mathbb{R}^+ \times (-\pi, \pi] \times \mathbb{R} \to \mathbb{R}^3, \begin{pmatrix} r \\ \phi \\ z \end{pmatrix} \mapsto \begin{pmatrix} x(r,\phi,z) \\ y(r,\phi,z) \\ z(r,\phi,z) \end{pmatrix} = \begin{pmatrix} r\cos(\phi) \\ r\sin(\phi) \\ z \end{pmatrix}.$$

Hence the Jacobian determinant (functional determinant) is:

$$\det \begin{pmatrix} \cos(\phi) & -r\sin(\phi) & 0 \\ \sin(\phi) & r\cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(r\cos(\phi)^2 + r\sin(\phi)^2\right) \cdot 1 = r$$

Since $r > 0$, we replace the “volume element” $dx\,dy\,dz$ by $r\,dr\,d\phi\,dz$.

Example: Cylinder Coordinates

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We compute the volume of a cylinder with radius $R > 0$ and height $h > 0$:

$$V = \ldots = \pi R^2 h$$

see blackboard, next lecture!

Transformation from Spherical Coordinates

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For spherical coordinates there holds

$$\begin{aligned} x &= r\cos(\phi)\sin(\theta), \\ y &= r\sin(\phi)\sin(\theta), \\ z &= r\cos(\theta) \end{aligned}$$

thus

$$g : \mathbb{R}^+ \times (-\pi, \pi] \times [0, \pi) \to \mathbb{R}^3, \begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} \mapsto \begin{pmatrix} x(r,\phi,\theta) \\ y(r,\phi,\theta) \\ z(r,\phi,\theta) \end{pmatrix} = \begin{pmatrix} r\cos(\phi)\sin(\theta) \\ r\sin(\phi)\sin(\theta) \\ r\cos(\theta) \end{pmatrix}.$$

Hence the Jacobian determinant (functional determinant) is:

$$\det \begin{pmatrix} \cos(\phi)\sin(\theta) & -r\sin(\phi)\sin(\theta) & r\cos(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) & r\cos(\phi)\sin(\theta) & r\sin(\phi)\cos(\theta) \\ \cos(\theta) & 0 & -r\sin(\theta) \end{pmatrix} = -r^2\sin(\theta)$$

Since $\sin(\theta) > 0$, we replace the “volume element” $dx\,dy\,dz$ by $r^2\,\sin(\theta)\,dr\,d\phi\,d\theta$.

Example: Spherical Coordinates

We compute the mass of a homogeneous sphere S with radius $R > 0$.
Let $\rho(x, y, z) = \rho_0 = const$ be the constant mass density.

$$M = \int_S \rho(x,y,z)\,dx\,dy\,dz \overset{\text{trafo. formula}}{=} \rho_0 \int_{-\pi}^{\pi}\int_0^{\pi}\int_0^R r^2 \sin(\theta)\,dr\,d\theta\,d\varphi$$

$$= \rho_0 \int_{-\pi}^{\pi} d\varphi \cdot \int_0^{\pi} \sin(\theta)\,d\theta \cdot \int_0^R r^2\,dr$$

$$= \rho_0 \cdot 2\pi \cdot \underbrace{\Big[-\cos(\theta)\Big]_{\theta=0}^{\pi}}_{=(-(-1))-(-(+1))=2} \cdot \Big[\frac{1}{3}r^3\Big]_{r=0}^{R} = \rho_0 \frac{4\pi R^3}{3}$$

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