Technische Hochschule Rosenheim
Fakultät ANG
Prof. Dr. André Herzwurm

SoSe 2022
28.04.2022

Exercise Sheet 5

Linear Algebra (AAI)

Exercise 5.1 (H)

Prove Lemma II.4.3.(iv)-(v).

Exercise 5.2 (H)

a) Show Remark II.4.4.(i).

b) Let V and W be K-vector spaces, and let $F\colon V \to W$ be linear. Moreover, let $v_1, v_2 \in V$ such that $v_1 \neq v_2$ and $F(v_1) = F(v_2) \neq 0$. Show that (v_1, v_2) is linearly independent.
Hint: Consider $\lambda_1 v_1 + \lambda_2 v_2 = 0$, apply F to both sides of the equation, and use Lemma II.1.4.(iii).

Exercise 5.3 (H)

a) Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. Show that there exists a unique matrix $A \in \mathbb{R}^{m\times n}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^n$.
Hint: According to Proposition II.4.7 a linear map is uniquely determined by the image of a basis. Use the standard basis $(e_1, \dots, e_n)$ in $\mathbb{R}^n$ to obtain A. Then show that A satisfies $F(x) = Ax$ for all $x \in \mathbb{R}^n$.

b) Determine the matrix $A \in \mathbb{R}^{3\times 3}$ from part a) for the following choices of F:

i) $$F\big((1,0,0)^\top\big) = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad F\big((0,1,0)^\top\big) = \begin{pmatrix}2\\0\\0\end{pmatrix}, \quad F\big((0,0,1)^\top\big) = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

ii) $$F\big((1,0,0)^\top\big) = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad F\big((1,0,1)^\top\big) = \begin{pmatrix}2\\0\\0\end{pmatrix}, \quad F\big((0,1,1)^\top\big) = \begin{pmatrix}0\\1\\0\end{pmatrix}.$$

Hint: Note that the j-th column $a_j \in \mathbb{R}^3$ of the matrix A satisfies $a_j = Ae_j$.

Exercise 5.4 (H)

a) Let U, V, and W be K-vector spaces, and let $F\colon V \to W$ and $G\colon U \to V$ be linear. Show that $F \circ G$ is linear.

b) Consider the situation from Exercise 5.3.

   i) Determine the matrix $B \in \mathbb{R}^{3\times 3}$ that represents the linear map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F\big((1,0,0)^\top\big) = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad F\big((0,1,0)^\top\big) = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad F\big((0,0,1)^\top\big) = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

   ii) Compute B^{-1} using the algorithm from Exercise 4.3 b).

   iii) Compute $B^{-1}v_i$ for $i \in \{1,2,3\}$ and

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

   iv) Let $A \in \mathbb{R}^{3\times 3}$ be the matrix from Exercise 5.3 b).i). Compute AB^{-1} and compare the result with the matrix from Exercise 5.3 b).ii).