Solids of Revolution

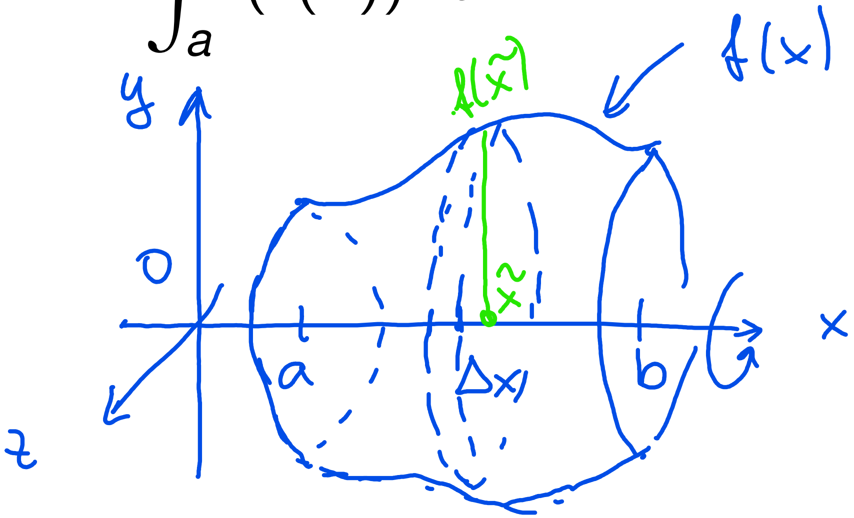
lateral area:

$$A = 2\pi \int_a^b f(x) \sqrt{1+(f'(x))^2}\, dx$$

Let $f : [a, b] \to \mathbb{R}$ continuous.

The solid of revolution generated by rotating the curve $y = f(x)$, $a \le x \le b$, around the x-axis has the volume

$$V = \pi \int_a^b (f(x))^2\, dx$$



Motivation:

$$\Delta V = A(x) \cdot \Delta x \approx \pi (f(x))^2 \cdot \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum \pi (f(x))^2 \cdot \Delta x$$

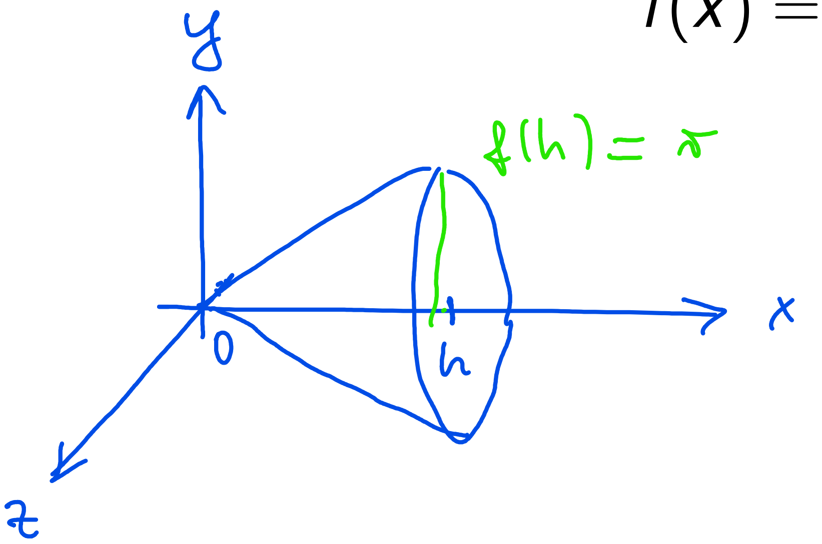
Remark: Rotations around other axes yield analogous formulas.

Solids of Revolution - Example

Circular cone

$f : [0, h] \to \mathbb{R}$,

$$f(x) = \frac{r}{h}x \quad , \quad r, h > 0$$



Expect: $\frac{1}{3}\underbrace{\pi r^2}_{\text{base area}} \cdot \overset{\text{height}}{\overset{\downarrow}{h}}$

Check:

$$V = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 \, dx$$

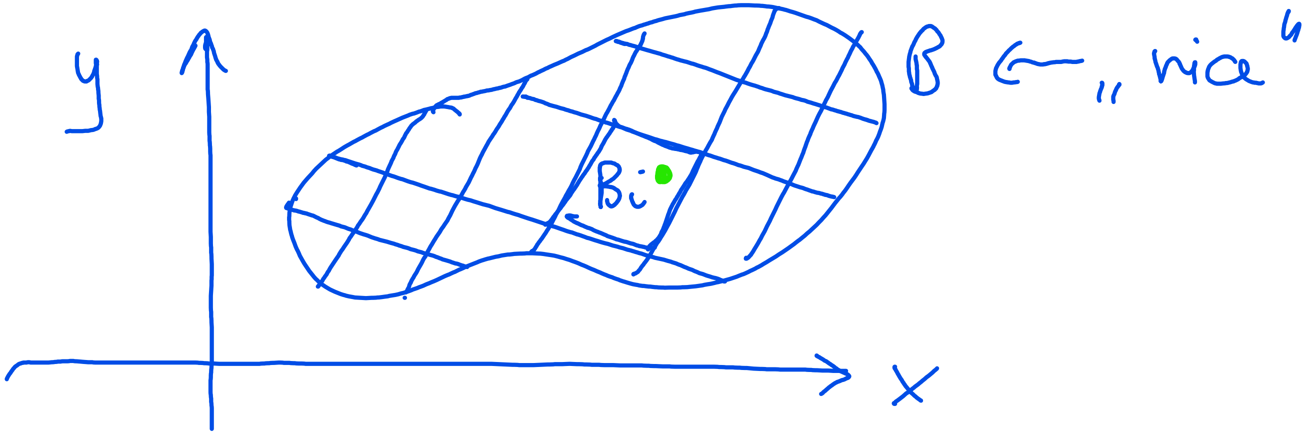
$$= \pi \frac{r^2}{h^2} \int_0^h x^2 \, dx = \pi \frac{r^2}{h^2} \left[\frac{x^3}{3}\right]_{x=0}^{h} = \frac{\pi}{3} \frac{r^2}{\cancel{h^2}} (h^{\cancel{3}} - 0)$$

$$= \frac{1}{3} \pi r^2 h$$

Integration on $\mathbb{R}^n$

Up to now we have considered different cases of functions depending in principle on <u>one</u> variable, although the range might be higher dimensional, i.e. in $\mathbb{R}^n$.
We wish to enlarge this to several variables in $\mathbb{R}^m$.

We consider $f : B \subseteq \mathbb{R}^2 \to \mathbb{R}$.



$$F = |B| \xleftarrow{n\to\infty} S_n := \sum_{i=1}^{n} f(x_*^{[i]}, y_*^{[i]}) \cdot \Delta F_i \quad \text{with } (x_*^{[i]}, y_*^{[i]}) \in B$$

Small domains B_i with area $|B_i| = \Delta F_i$ with fineness δ

Double Integral

Idea: iterated Riemann integral

But why should we restrict us to an approximation by rectangles?

Like Riemann construction

, t in a parameter is possible

$$\iint_B f(x,y,t)\, \underbrace{dF}_{dx\,dy} := \lim_{\delta\to 0, n\to\infty} \sum_{i=1}^{n} f(x_*^{[i]}, y_*^{[i]}) \cdot \Delta F_i$$

$$=: \int_B f(x,y)\, dF = \int \underbrace{\left(\int_{-\sqrt{y}}^{\sqrt{y}} f(x,y)\, dx \right)}_{g(y)} dy$$

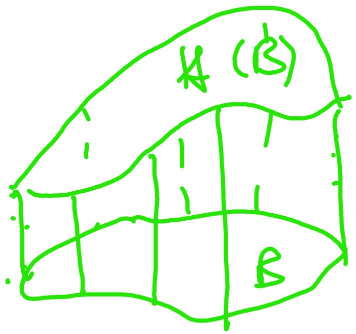
Important Double Integrals

Area integrals

$$A = \iint\limits_{B} 1 \, dA = \text{ area of } B$$

Volume integrals

$$V = \iiint\limits_{C} 1 \, dV = \text{ volume of } C$$



$$\overset{\text{special case}}{=} \iint\limits_{B} f(x,y) \, dA$$

$$= \text{volume of solid with base area } B$$

and top area described by ~~C~~ $f(B)$

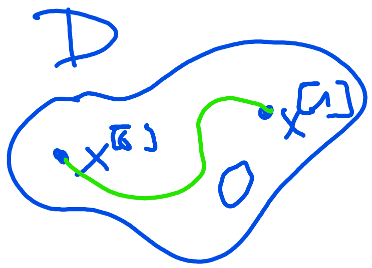
Note $A = |B|$, $V = |C|$

(Regular) Domains

For a more formal approach, we need some preparation:

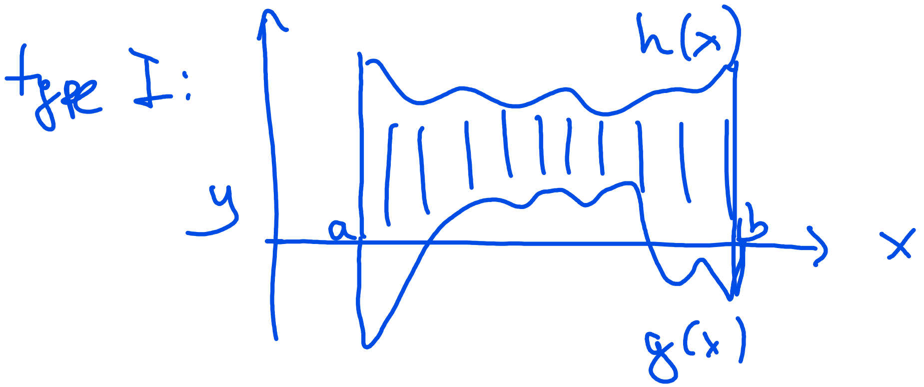
- Recall: A set $D \subseteq \mathbb{R}^n$ is called connected, iff for any 2 points $\mathbf{x}^{[0]}$ and $\mathbf{x}^{[1]}$ there exists a curve $k : [a, b] \to \mathbb{R}^n$ with $k(a) = \mathbf{x}^{[0]}$ and $k(b) = \mathbf{x}^{[1]}$.

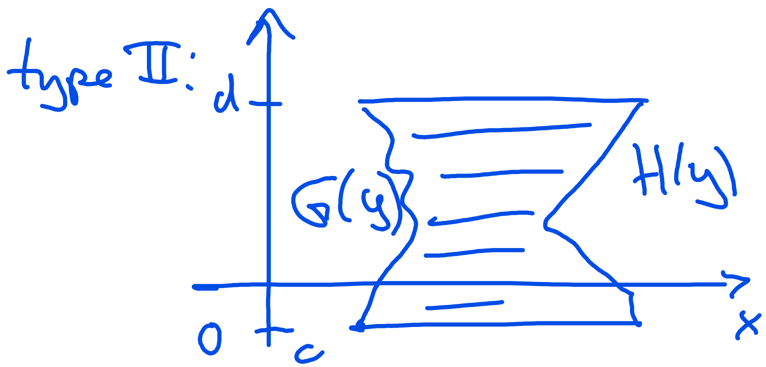
  A **region** or domain[2] is an open and connected subset of $\mathbb{R}^n$.



D is connected, but not simply connected

- Normal areas (regular domains) $\rightsquigarrow$ see next slide





[2]not to be confused with a domain of definition

Normal areas

Normal areas are in 2D:

- Type I:
  $B_I = \{(x, y) \mid a \leq x \leq b \text{ and } g(x) \leq y \leq h(x)\}$
  with $a, b \in \mathbb{R}$, where $a < b$, and
  $g : [a, b] \to \mathbb{R}$, $h : [a, b] \to \mathbb{R}$ cont. differentiable.
- Type II:
  $B_{II} = \{(x, y) \mid c \leq y \leq d \text{ and } G(y) \leq y \leq H(y)\}$
  with $c, d \in \mathbb{R}$, where $c < d$, and
  $G : [c, d] \to \mathbb{R}$, $H : [c, d] \to \mathbb{R}$ cont. differentiable.

The roles of x and y are reversed.

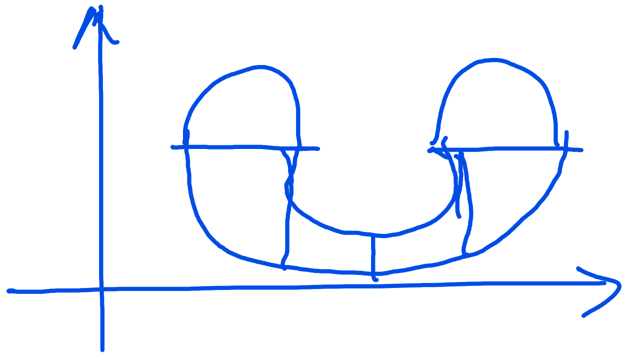
This may be extended to higher dimensions.

Practical Computation of Integrals

- We perform the integrations "from inside to outside". The integral and the differential work like a “bracket”.

$$\iint f(x,y)\,dy\,dx = \int \left(\int f(x,y)\,dy \right) dx$$

- For practical computations it may be helpful to split the normal area B by cuts (that are parallel to the axes) into smaller (disjunct) normal areas, e.g. B_1 and B_2. The whole integral is then obtained by the additivity.



Fubini Theorem

The order of integration may not be exchanged in general, but:

Theorem (Fubini Theorem)

Let $R = \{(x, y) \,|\, a \le x \le b \text{ and } c \le y \le d\}$, $a, b, c, d \in \mathbb{R}$ a rectangle, and $f : \mathbb{R}^2 \to \mathbb{R}$ continuous.

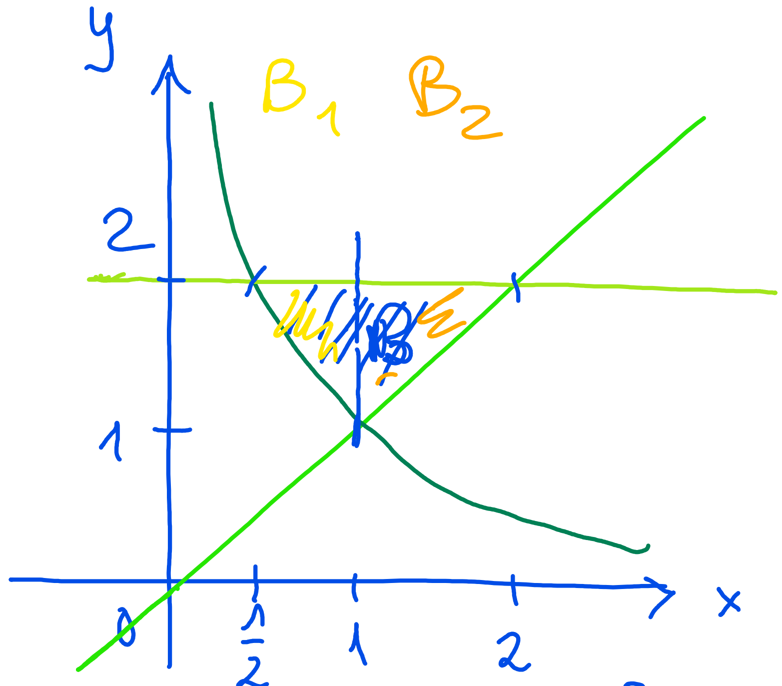
Then there holds

$$\iint_R f \, dA = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx = \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy.$$

Fubini Theorem - Example

Consider ~~X~~ B bounded by the curves $y = x$, $y = \frac{1}{x}$, and $y = 2$.

$$A = \iint_B dx\,dy = \iint_B dy\,dx$$



Normal area of type II:

$$B_{II} = \{(x,y) \in \mathbb{R}^2 \mid 1 \le y \le 2, \text{ and } \tfrac{1}{y} \le x \le y\}$$

$$A = \int_1^2 \int_{1/y}^{y} 1\,dx\,dy$$

$$= \int_1^2 [x]_{x=1/y}^{y}\,dy$$

$$= \int_1^2 \left(y - \frac{1}{y}\right) dy = \left[\frac{y^2}{2} - \ln(y)\right]_{y=1}^{2} = \frac{3}{2} - \ln 2$$