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Exercise Sheet 7

Linear Algebra (AAI)

Exercise 7.1 (H)

Consider $F : \mathbb{R}^3 \to \mathbb{R}^3$ from Exercise 6.1.

a) Let $\mathcal{E}$ be the standard basis of $\mathbb{R}^3$. Determine $\mathcal{M}^{\mathcal{E}}_{\mathcal{E}}(F)$ and $\mathcal{M}^{\mathcal{E}}_{\mathcal{E}}(F^{-1})$.

b) Determine $\mathcal{M}^{\mathcal{A}}_{\mathcal{E}}(F)$ for the basis

$$\mathcal{A} = \left((0,0,-1)^\top, (1,1,0)^\top, (-1,1,0)^\top\right).$$

Hint: Express the images of the basis vectors of $\mathcal{A}$ in terms of $\mathcal{E}$.

c) Determine bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ such that

$$\mathcal{M}^{\mathcal{B}}_{\mathcal{C}}(F) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

d) Are there bases $\mathcal{B}$ and $\mathcal{C}$ of $\mathbb{R}^3$ such that

$$\mathcal{M}^{\mathcal{B}}_{\mathcal{C}}(F) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}?$$

Exercise 7.2 (H)

Let V be a $\mathbb{R}$-vector space, let (v_1, v_2) be a basis of V, and let $F \in L(V,V)$ be given by

$$F(v_1) = v_1 \qquad \text{and} \qquad F(v_2) = -v_2.$$

Moreover, let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Are there bases $\mathcal{A}$ and $\mathcal{B}$ such that $\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F) = A$ or $\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F) = B$?

Hint: Use $\mathcal{A} = (v_1, v_2)$ and try to find $\mathcal{B}$.

Exercise 7.3 (H)

Let $A, B \in \mathbb{R}^{n \times n}$. Prove or disprove:

a) A, B invertible $\Rightarrow$ $A + B$ invertible.

b) $A \cdot B = B \cdot A \Rightarrow A = B$.

Exercise 7.4 (H)

Let $v_1, v_2, v_3 \in \mathbb{R}^{\mathbb{R}}$ be given by

$$v_1(x) = 1 + x, \qquad v_2(x) = x, \qquad v_3(x) = 1 + \exp(x)$$

for $x \in \mathbb{R}$.

a) Show that $\mathcal{A} = (v_1, v_2, v_3)$ is a basis of $V = \operatorname{span}(\{v_1, v_2, v_3\})$.

b) Let $v \in V$ be given by $v(x) = 4 + 3x - \exp(x)$ for $x \in \mathbb{R}$. Determine the family of coordinates $\Phi_{\mathcal{A}}^{-1}(v) \in \mathbb{R}^3$ of v w.r.t. the basis $\mathcal{A}$.

c) Let $F \colon V \to \mathbb{R}^{\mathbb{R}}$ be given by $F(v) = v'$.

   i) Show that F is linear.

   ii) Show that $\operatorname{im} F \subseteq V$ and that $\mathcal{B} = (F(v_2), F(v_3))$ is a basis of $\operatorname{im} F$.

   iii) Let $G \colon V \to \operatorname{im} F$ be given by $G(v) = F(v)$. Determine $\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(G)$.