

Introduction

Power series

**Differentiation in
Higher Dimensions**

Limits and Continuity in \mathbb{R}^n

Differentiability in \mathbb{R}^n

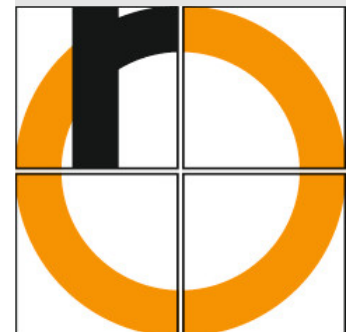
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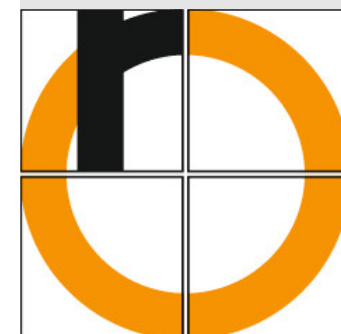


In Analysis 1 we have discussed differentiation of functions of 1 variable.

Now we consider a real-valued function in several variables

$$f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$$

$$\mathbf{x} = (x_1, \dots, x_n)^\top \mapsto f(x_1, \dots, x_n) = f(\mathbf{x})$$



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Now we consider a real-valued function in several variables

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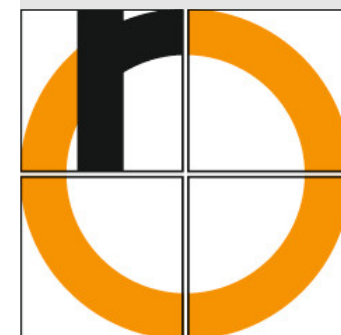
$$\mathbf{x} = (x_1, \dots, x_n)^\top \mapsto f(x_1, \dots, x_n) = f(\mathbf{x})$$

Later we are going to extend the differential calculus to vector-valued functions (of several variables)

$$f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}^m$$

$$\mathbf{x} = (x_1, \dots, x_n)^\top \mapsto (f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n))^\top$$

$$= \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix}$$



Special Case: $n = 2$

A real-valued function in 2 variables

$$f : D \subseteq \mathbb{R}^2 \rightarrow \mathbb{R}$$

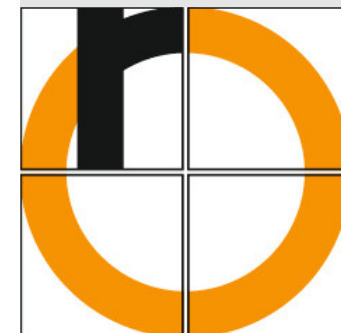
*x, y indep.
variables*

$$\mathbf{x} = (x_1, x_2)^T \mapsto f(x_1, x_2) = z \\ \approx (x, y)^T$$

*dependent
variable*

We may plot the function value as 3rd coordinate over the real plane \mathbb{R}^2 .

The graph of f is a subset of \mathbb{R}^3 : a "landscape" or "mountains".



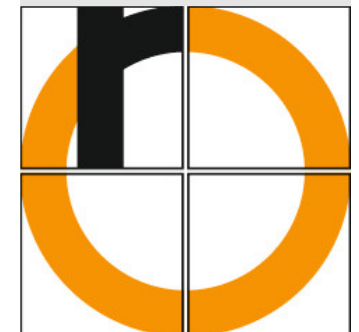
Definition (Level set)

We define the **level set** of a function $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $(x_1, \dots, x_n)^\top \mapsto f(x_1, \dots, x_n)$ for the function value $c \in \mathbb{R}$ as the set

$$N_c := \{\mathbf{x} \in D \mid f(\mathbf{x}) = c\}.$$

The structure of N_c may be "complicated", it might also be the empty set.

For $n = 2$ the level set is called a contour line or level curve (though it may be an area, e.g.), for $n = 3$ the level set is called an equipotential surface (though it may be a solid, e.g.).

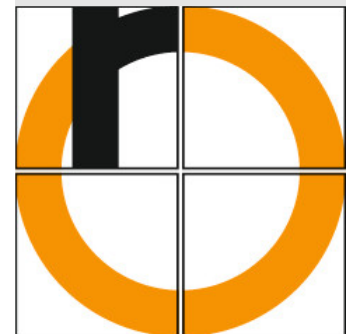


A partial function is a "cross section"-function that is obtained by freezing all but 1 variables, e.g. x_j :

$$g_j : D_j \subseteq \mathbb{R} \rightarrow \mathbb{R}$$

$$x_j \mapsto f(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n),$$

with $(a_1, \dots, a_{j-1}, x_j, a_{j+1}, \dots, a_n) \in D$ for all $x_j \in D_j$,
 $a_1, \dots, a_{j-1}, a_{j+1}, \dots, a_n$ fixed.



Partial Function: Example

$$f(x_1, x_2) = x_1^2 + x_2^2$$

→ see blackboard

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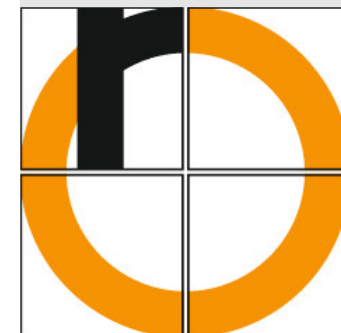
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Definition (Partial derivative)

Let $f : D \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$, $\mathbf{x} = (x_1, \dots, x_n)^\top \mapsto f(x_1, \dots, x_n)$ and $a \in D$, D an open set.

If the derivative of the partial function

$$x_i \mapsto f(a_1, \dots, a_{i-1}, x_i, a_{i+1}, \dots, a_n) =: f_i(\dots)$$

exists at $x_i = a_i$,

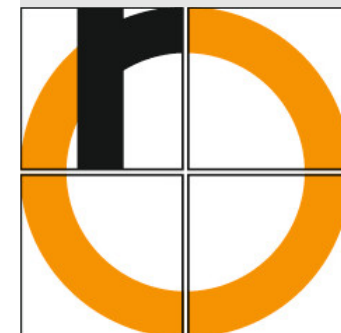
then we call it the **partial derivative** of f w.r.t. x_i at a .

We write:

$$\frac{\partial f}{\partial x_i}(\mathbf{a}) \quad \text{or} \quad f'_{x_i}(\mathbf{a}) \quad \text{or} \quad f_{x_i}(\mathbf{a}) \quad \text{or} \quad \frac{\partial}{\partial x_i} f(\mathbf{a})$$

We say f is **partially differentiable** in \mathbf{a} , if all $\frac{\partial f}{\partial x_i}(\mathbf{a})$ exist.

We say f is partially differentiable in $E \subseteq D$, if f is partially differentiable at any $\mathbf{a} \in E$.



We assume that f is partially differentiable in D .

It is convenient to arrange the (first) partial derivatives in a vector.

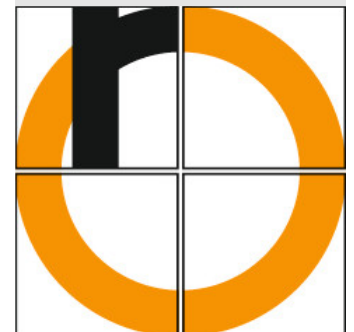
We define the **gradient** of $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ as the column vector (function):

$$\text{grad } f(\mathbf{x}) = \nabla f(\mathbf{x}) := \begin{pmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_i}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{pmatrix}$$

$$2d: \nabla f(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x, y) \\ \frac{\partial f}{\partial y}(x, y) \end{pmatrix}$$

The gradient is

- orthogonal to level sets and
- points into the direction of the steepest ascent.



Partial Derivatives: Examples

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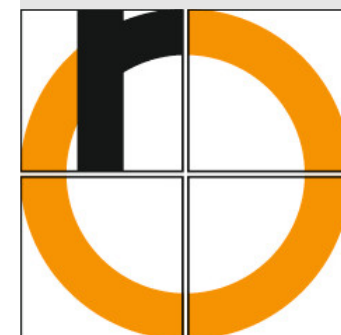
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$$f(x, y) = (x^2 + y^2) \exp(\sin(x + y))$$

$$\nabla f(x, y) = \begin{pmatrix} \frac{\partial f}{\partial x}(x, y) \\ \frac{\partial f}{\partial y}(x, y) \end{pmatrix} = \exp(\sin(x + y)) \left(2 \begin{pmatrix} x \\ y \end{pmatrix} + \dots \right. \\ \left. + (x^2 + y^2) \cos(x + y) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$$g(x, y) = 100(y - x^2)^2 + (1 - x)^2 \quad \text{Rosenbrock or banana function}$$

$$\nabla g(x, y) = \dots = \begin{pmatrix} -400x(y - x^2) - 2(1 - x) \\ 200(y - x^2) \end{pmatrix}$$



Higher Partial Derivatives: Definition

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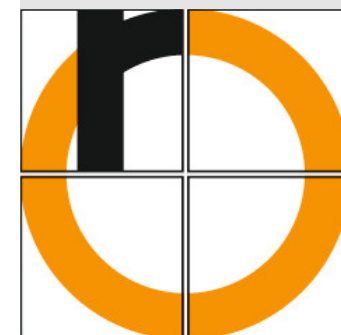
Partial derivatives w.r.t. some x_j are again functions of x_1, x_2, \dots, x_n and, again, we may search for partial derivatives w.r.t. some x_i , yielding **second partial derivatives**. ($i, j = 1, \dots, n$)

We write:

$$\frac{\partial}{\partial x_i} \frac{\partial f}{\partial x_j}(\mathbf{a}) \quad \text{or} \quad \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{a}) \quad \text{or} \quad f''_{x_i, x_j}(\mathbf{a}) \quad \text{or} \quad f''_{i,j}(\mathbf{a}) \quad \dots$$

In 2d with $(x, y) = (x_1, x_2)$ this reads, e.g.:

$$\frac{\partial}{\partial x} \frac{\partial f}{\partial y}(\mathbf{a}) \quad \text{or} \quad \frac{\partial^2}{\partial x \partial y} f(\mathbf{a}) \quad \text{or} \quad f''_{x,y}(\mathbf{a}) \quad \text{or} \quad f''_{1,2}(\mathbf{a}) \quad \dots$$



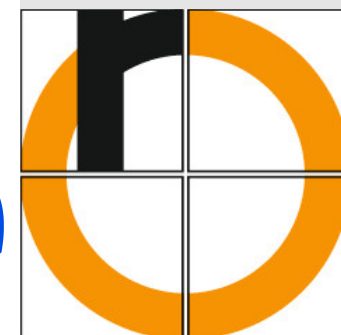
We assume that f is 2x partially differentiable in D .
Again it is convenient to order the second partial
derivatives in a (quadratic) matrix.

We define the **Hesse matrix** of $f : D \subseteq \mathbb{R} \rightarrow \mathbb{R}$ as the
quadratic matrix (function):

$$H_f(\mathbf{x}) := \begin{pmatrix} \frac{\partial^2}{\partial x_1 \partial x_1} f(\mathbf{x}) & \dots & \dots & \frac{\partial^2}{\partial x_1 \partial x_n} f(\mathbf{x}) \\ \vdots & \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \frac{\partial^2}{\partial x_n \partial x_1} f(\mathbf{x}) & \dots & \dots & \frac{\partial^2}{\partial x_n \partial x_n} f(\mathbf{x}) \end{pmatrix} \in \mathbb{R}^{n \times n}$$

In 2d with $(x, y) = (x_1, x_2)$ this reads, e.g.:

$$H_f(x, y) := \begin{pmatrix} \frac{\partial^2}{\partial x \partial x} f(x, y) & \frac{\partial^2}{\partial x \partial y} f(x, y) \\ \frac{\partial^2}{\partial y \partial x} f(x, y) & \frac{\partial^2}{\partial y \partial y} f(x, y) \end{pmatrix} = \frac{\partial^2}{\partial y^2} f(x, y)$$



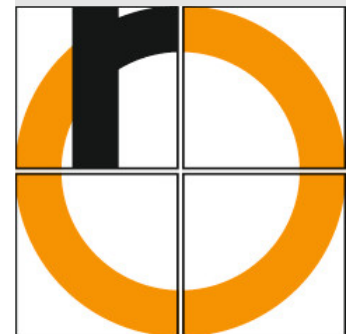
According to the Schwarz-Clairaut theorem there holds:

Suppose f is defined on a disk $D \subseteq \mathbb{R}^2$ with $(x_0, y_0) \in D$.

If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are both continuous on D ,
then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ *at (x_0, y_0) in (x_0, y_0)* .

This shows that the Hesse matrix is symmetric.

The order of the partial derivatives doesn't matter in this case!



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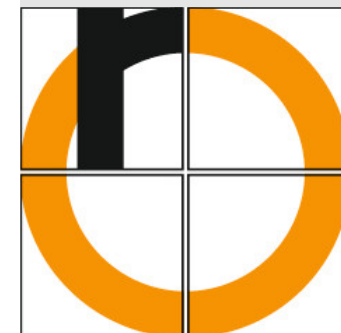
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$$H_f(x, y) = \dots = \exp(\sin(x + y)) \cdot \dots$$

$$\cdot \left[(x^2 + y^2)(\cos^2(x + y) - \sin(x + y)) \cdot \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \dots \right.$$

$$\left. \dots + 2 \cos(x + y) \begin{pmatrix} 2x & x + y \\ x + y & 2y \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right]$$

$$H_g(x, y) = \dots = \begin{pmatrix} -400(y - 3x^2) + 2, & -400x \\ -400x, & 200 \end{pmatrix}$$



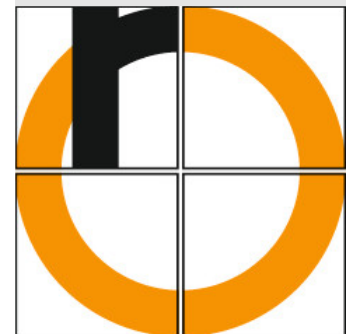
Let $D \subseteq \mathbb{R}^n$.

- ε -neighbourhood of $\mathbf{a} \in \mathbb{R}^n$:

$$U_\varepsilon(\mathbf{a}) := \left\{ \mathbf{x} \in \mathbb{R}^n \mid \|\mathbf{x} - \mathbf{a}\|_2 := \sqrt{\sum_{i=1}^n (x_i - a_i)^2} < \varepsilon \right\}$$

- $\mathbf{a} \in D$ is called an **interior point** of $D : \Longleftrightarrow$ there exists $U_\varepsilon(\mathbf{a})$ with $U_\varepsilon(\mathbf{a}) \subseteq D$.
- A set D is called **open** : \Longleftrightarrow any point of D is an interior point.
A set D is called **closed** : $\Longrightarrow \mathbb{R}^n \setminus D$ is open.
- $\mathbf{b} \in \mathbb{R}^n$ is called a **boundary point** of $D : \Longrightarrow$ for all $U_\varepsilon(\mathbf{b})$ holds, there exists a $\mathbf{x}, \mathbf{y} \in U_\varepsilon(\mathbf{b})$ with $\mathbf{x} \in D$ and $\mathbf{y} \notin D$.
The set ∂D of all boundary points \mathbf{b} is called **boundary** of D .

Examples, see blackboard



Revision/Definition: Limits and Continuity

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Let $f : \mathbb{R}^n \subseteq D \rightarrow \mathbb{R}^m$, $\mathbf{a} \in D \cup \partial D$.

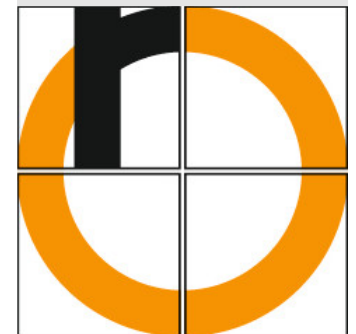
$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = \mathbf{c} \in \mathbb{R}^m : \Longleftrightarrow$$

for all $\varepsilon > 0$ exists $U_\varepsilon(\mathbf{a}) : \|f(\mathbf{x}) - \mathbf{c}\| \leq \varepsilon$ for all $\mathbf{x} \in D \cap U_\varepsilon(\mathbf{a})$.

f is called **continuous** in $\mathbf{a} \in D : \Longleftrightarrow \lim_{\mathbf{x} \rightarrow \mathbf{a}} f(\mathbf{x}) = f(\mathbf{a})$.

f is called continuous in $D : \Longleftrightarrow f$ is continuous for all $\mathbf{a} \in D$.

Example: For $n = 2, m = 1$, f is continuous means that there are no “jumps” or “ridges”.



Pathologic Example for (Dis)continuity in 2d

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Functions of severable variables exhibit phenomena that we do not encounter in 1d:

Consider $f : \mathbb{R}^2 \rightarrow \mathbb{R}$,

$$f(x, y) := \begin{cases} \frac{2xy}{x^2+y^2}, & \text{if } x \neq 0 \text{ or } y \neq 0, \\ 0, & \text{if } (x, y)^\top = (0, 0)^\top. \end{cases}$$

A limit in 2d has to hold for any “path” $(x, y) \rightarrow (x_0, y_0)$.
It is not enough to consider each coordinate separately!

