HW3 /1

a) $\sum_{n=1}^{\infty} (-1)^n \cdot b_n \qquad b_n = \frac{z^{2k+1}}{2k+1}$

$$\lim_{k \to \infty} b_n = \left| \frac{z^{2(k+1)+1}}{2(k+1)+1} \cdot \frac{2k+1}{z^{2k+1}} \right| = \lim_{k \to \infty} \left| \frac{z^{2k+2+1}}{2k+2+1} \cdot \frac{2k+1}{z^{2k+1}} \right|$$

$$= \lim_{k \to \infty} = \left| \frac{z^2}{2} \right| = 0.5 \lim_{k \to \infty} (z^2) \qquad \text{Radius} = 1$$

b) $\lim_{k \to \infty} = \left| \frac{z^{2(k+1)+1}}{(2(k+1)+1)!} \cdot \frac{(2k+1)!}{z^{2k+1}} \right|$

$$= \lim_{k \to \infty} \left| \frac{z^{2k+2+1}}{(2k+2+1)!} \cdot \frac{(2k+1)!}{z^{2k+1}} \right| = \lim_{k \to \infty} \left| \frac{z^2}{(2k+3) \cdot (2k+2)} \right| \qquad \begin{aligned}&(2k+3)! \\ &= (2k+3) \cdot (2k+2) \\ &\cdot (2k+1)!\end{aligned}$$

$$= \lim_{k \to \infty} \left| \frac{z^2}{4k^2 + 10k + 6} \right| = 0 \qquad " \frac{z^2}{\infty} "$$

Radius $= \infty$

c) $\sum_{n=1}^{\infty} (-1)^n \cdot b_n \qquad b_n = \frac{z^{2k}}{k!}$

$$\lim_{k \to \infty} b_n = \left| \frac{z^{2(k+1)}}{(k+1)!} \cdot \frac{k!}{z^{2k}} \right| = \lim_{k \to \infty} \left| \frac{z^{2k+2}}{(k+1) \cdot k!} \cdot \frac{k!}{z^{2k}} \right|$$

$$= \lim_{k \to \infty} \left| \frac{z^2}{k+1} \right| = 0 \qquad \text{Radius} = \infty$$

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for $\alpha \in \mathbb{N}_0$ we get $(1+z)^\alpha = \sum_{n=0}^{\alpha} \binom{\alpha}{n} z^n$

and then $\binom{\alpha}{n} = 0$ for $n > \alpha$. Radius $= \infty$

b)

$$\lim_{n\to\infty}\left|\frac{\binom{\alpha}{n+1}\cdot z^{n+1}}{\binom{\alpha}{n}\cdot z^{n}}\right| = \lim_{n\to\infty}\left|\frac{\frac{\alpha\cdot(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)(\alpha-n)}{(n+1)!}\,z^{n+1}}{\frac{\alpha\cdot(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)}{n!}\,z^{n}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\frac{(\alpha-n)}{(n+1)}\cdot z}{\frac{1}{1}\cdot 1}\right| = \lim_{n\to\infty}\left|\frac{z(\alpha-n)}{(n+1)}\right| = |z|\lim_{n\to\infty}\left|\frac{(\alpha-n)}{(n+1)}\right|$$

$$= |z|\lim_{n\to\infty}\left|\frac{\frac{\alpha}{n}-\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right| = |z| \qquad \text{Radius} = 1$$

c)

$$f(z) = (1-z)^{\alpha}$$
$$f'(z) = \alpha(1-z)^{(\alpha-1)}$$
$$f''(z) = \alpha(\alpha-1)\cdot(1-z)^{(\alpha-2)}$$
$$f'''(z) = \underbrace{\alpha(\alpha-1)(\alpha-2)}_{\text{same as in } \binom{\alpha}{n}}\cdot(1-z)^{(\alpha-3)}$$

$$f(z_0) = f(0)$$
$$f(0) = (1-0)^{\alpha} = 1$$
$$f'(0) = \alpha(1-0)^{(\alpha-1)} = \alpha((1-0)^{\alpha}\cdot(1-0)^{-1}) = \alpha(1\cdot 1) = \alpha$$
$$f''(0) = \alpha(\alpha-1)\cdot(1-0)^{(\alpha-2)} = \alpha(\alpha-1) = \alpha^2-\alpha$$
$$f'''(0) = \alpha(\alpha-1)(\alpha-2)\cdot(1-0)^{(\alpha-3)} = (\alpha^2-\alpha)(\alpha-2) = \alpha^3-2\alpha^2-\alpha^2+2\alpha = \alpha^3-3\alpha^2+2\alpha$$

– Pascal's triangle pattern

$$T_3(f,z,0) = \frac{1}{0!}\cdot(z-0)^0 + \frac{\alpha}{1!}\cdot(z-0)^1 + \frac{\alpha^2-\alpha}{2!}\cdot(z-0)^2 + \frac{\alpha^3-3\alpha^2+2\alpha}{3!}\cdot(z-0)^3$$
$$= (z)^0 + \alpha\cdot z^1 + \frac{\alpha^2-\alpha}{2}\cdot z^2 + (\alpha^3-3\alpha^2+2\alpha)\cdot\frac{1}{6}\cdot z^3$$

Lagrange remainder:

$$f''''(z) = \alpha(\alpha-1)(\alpha-2)(\alpha-3)\cdot(1-z)^{(\alpha-4)}$$

$$\frac{f''''(0+\Theta_{z,z_0}\cdot(z-0))}{4!}\cdot(z-0)^4$$

$$= \frac{f^{(4)}(z\cdot\Theta_{z,z_0})}{4!}\cdot z^4$$

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$$\sum_{r=0}^{\infty}\left(\sum_{l=0}^{r} a_l b_{r-l}\right) z^r = 1$$

for $r=0$:

$$a_0 \cdot b_0 \cdot z^0 = 1$$
$$a_0 \cdot b_0 \cdot 1 = 1$$
$$a_0 = \frac{1}{b_0}$$
$$b_0 = \frac{1}{a_0}$$

for $r=1$:

$$(a_0 \cdot b_1 + a_1 \cdot b_0) \cdot z^1 + (a_0 \cdot b_0) \cdot z^0 = 1$$
$$(a_0 \cdot b_1 + a_1 \cdot b_0) \cdot z^1 = 1 - (a_0 \cdot b_0 \cdot z^0)$$
$$(a_0 \cdot b_1 + a_1 \cdot b_0) = \frac{1 - a_0 \cdot b_0}{z}$$
$$a_0 \cdot b_1 = \frac{1 - a_0 \cdot b_0}{z} - a_1 \cdot b_0$$
$$b_1 = \frac{\frac{1 - a_0 \cdot b_0}{z} - a_1 \cdot b_0}{a_0}$$

for $r=2$: $(a_0 \cdot b_2 + a_1 \cdot b_1 + a_2 \cdot b_0) \cdot z^2 + (a_0 \cdot b_1 + a_1 \cdot b_0) \cdot z^1 + (a_0 \cdot b_0) \cdot z^0 = 1$

$$(a_0 \cdot b_2 + a_1 \cdot b_1 + a_2 \cdot b_0) \cdot z^2 = 1 - (a_0 \cdot b_1 + a_1 \cdot b_0) \cdot z^1 - (a_0 \cdot b_0) \cdot z^0$$
$$(a_0 \cdot b_2 + a_1 \cdot b_1 + a_2 \cdot b_0) = \frac{1 - (a_0 \cdot b_1 + a_1 \cdot b_0) \cdot z^1 - (a_0 \cdot b_0) \cdot z^0}{z^2}$$
$$a_0 \cdot b_2 = \frac{1 - (a_0 \cdot b_1 + a_1 \cdot b_0) \cdot z^1 - (a_0 \cdot b_0) \cdot z^0}{z^2} - a_1 \cdot b_1 - a_2 \cdot b_0$$
$$b_2 = \frac{\frac{1 - (a_0 \cdot b_1 + a_1 \cdot b_0) \cdot z^1 - (a_0 \cdot b_0) \cdot z^0}{z^2} - a_1 \cdot b_1 - a_2 \cdot b_0}{a_0}$$

$$r=3: (a_3\cdot b_0 + a_2\cdot b_1 + a_1\cdot b_2 + a_0\cdot b_3)\cdot z^3 + (a_2\cdot b_0 + a_1\cdot b_1 + a_0\cdot b_2)\cdot z^2$$
$$+(a_1\cdot b_0 + a_0\cdot b_1)\cdot z^1 + (a_0\cdot b_0)\, z^0 = 1$$

$$(a_3\cdot b_0 + a_2\cdot b_1 + a_1\cdot b_2 + a_0\cdot b_3)\cdot z^3 = 1-(a_2\cdot b_0 + a_1\cdot b_1 + a_0\cdot b_2)\cdot z^2 - (a_1\cdot b_0 + a_0\cdot b_1)\cdot z^1 - (a_0\cdot b_0)\, z^0$$

$$(a_3\cdot b_0 + a_2\cdot b_1 + a_1\cdot b_2 + a_0\cdot b_3) = \left(1-(a_2\cdot b_0 + a_1\cdot b_1 + a_0\cdot b_2)\cdot z^2 - (a_1\cdot b_0 + a_0\cdot b_1)\cdot z^1 - (a_0\cdot b_0)\, z^0\right) : z^3$$

$$a_0\cdot b_3 = \frac{1-(a_2\cdot b_0 + a_1\cdot b_1 + a_0\cdot b_2)\cdot z^2 - (a_1\cdot b_0 + a_0\cdot b_1)\cdot z^1 - (a_0\cdot b_0)\cdot z^0}{z^3} - a_3\cdot b_0 - a_2\cdot b_1 - a_1\cdot b_2$$

$$b_3 = \frac{\frac{1-(a_2\cdot b_0 + a_1\cdot b_1 + a_0\cdot b_2)\cdot z^2 - (a_1\cdot b_0 + a_0\cdot b_1)\cdot z^1 - (a_0\cdot b_0)\cdot z^0}{z^3} - a_3\cdot b_0 - a_2\cdot b_1 - a_1\cdot b_2}{a_0}$$

b) $f(z) = (1+z)^3 = (1+2z+z^2)\cdot(1+z) = 1+2z+z^2+z+2z^2+z^3$
$= z^3+3z^2+3z+1$

$f'(z) = 3(1+z)^2 = 3(1+2z+z^2) = 3+6z+3z^2$

$f''(z) = 6(1+z)^1 = 6+6z$

$f'''(z) = 6(1+z)^0 = 6$

$$g_3(z) = b_0 z^0 + b_1 z^1 + b_2 z^2 + b_3 z^3$$
$$= 1 - \frac{3}{1^2} z + \left(\frac{3^2}{1^3} - \frac{3}{1^2}\right) z^2 + \left(\frac{1}{1^2} - \frac{3^3}{1^4}\right) z^3$$
$$= 1 - 3z + 6z^2 - 26z^3$$

rewriting b0-b3:

$$b_0 = \frac{1}{a_0}$$

$$b_1 = \frac{\frac{1-a_0\cdot\frac{1}{a_0}}{z} - a_1\cdot\frac{1}{a_0}}{a_0} = \frac{\frac{1-1}{z} - \frac{a_1}{a_0}}{a_0} = -\frac{\frac{a_1}{a_0}}{\frac{a_0}{1}} = -\frac{a_1}{a_0^2}$$

$$b_2 = \frac{\frac{1-\left(a_0\cdot\left(-\frac{a_1}{a_0^2}\right) + \left(a_1\cdot\frac{1}{a_0}\right)\right)\cdot z^1 - \left(a_0\cdot\frac{1}{a_0}\right)\cdot z^0}{z^2} - a_1\cdot\left(-\frac{a_1}{a_0^2}\right) - \frac{a_2}{a_0}}{a_0}$$

$$= \frac{\frac{1-\left(-\frac{a_1}{a_0} + \frac{a_1}{a_0}\right)\cdot z^1 - 1}{z^2} + \frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}}{a_0}$$

$$= \frac{\frac{0}{z^2} + \frac{a_1^2}{a_0^2} - \frac{a_2}{a_0}}{a_0}$$

$$= \frac{\frac{a_1^2}{a_0^2}}{\frac{a_0}{1}} - \frac{\frac{a_2}{a_0}}{\frac{a_0}{1}} = \frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2} = b_2$$

$b_3 =$ $(((1-(a2*b0 + a1*b1 + a0*b2)*z^2 - (a1*b0 + a0*b1)*z - (a0*b0)*1)/z^3) - a3*b0 - a2*b1 - a1*b2)/a0$

$= (((1-(a2*(1/a0) + a1*(-(a1/{a^2}0)) + a0*(({a^2}1/{a^3}0)-(a2/{a^2}0)))*z^2 - (a1*(1/a0) + a0*(-(a1/{a^2}0)))*z - (a0*(1/a0))*1)/z^3) - a3*(1/a0) - a2*(-(a1/{a^2}0)) - a1*(({a^2}1/{a^3}0)-(a2/{a^2}0)))/a0$

$= (((1 - 0 - 0 - 1)/z^3) - (a3/a0) - ({a^3}1/{a^3}0))/a0$

$$b_3 = \frac{\frac{0}{z^3} - \frac{a_3}{a_0} - \frac{a_1^3}{a_0^3}}{a_0} = -\frac{\frac{a_3}{a_0}}{\frac{a_0}{1}} - \frac{\frac{a_1^3}{a_0^3}}{\frac{a_0}{1}} = \frac{a_3}{a_0^2} - \frac{a_1^3}{a_0^4}$$