Technische Hochschule Rosenheim — SoSe 2022
Applied Artificial Intelligence - Bachelor
Analysis 2

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Homework 3: power series

To submit: on **Friday, 08.04.2022**, 9:00 a.m., online by the learning campus

Exercise 1 (4 pts.)

Compute the radius of convergence of the following power series, $z \in \mathbb{C}$:

$$\text{a) } \arctan(z) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k+1}}{2k+1}, \quad \text{b) } \sinh(z) = \sum_{k=0}^{\infty} \frac{z^{2k+1}}{(2k+1)!}, \quad \text{c) } \exp(-z^2) = \sum_{k=0}^{\infty} (-1)^k \frac{z^{2k}}{k!}.$$

Can you justify, why the power series in c) holds?

Exercise 2 (9 pts.)

Let $\alpha \in \mathbb{R}, n \in \mathbb{N}_0$ and we extend the binomial coefficients as

$$\binom{\alpha}{n} := \frac{\alpha(\alpha-1)(\alpha-2)\cdot\ldots\cdot(\alpha-n+1)}{n!}, \quad \binom{\alpha}{0} = 1.$$

We define the so-called binomial series (with exponent α) as

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n, \quad z \in \mathbb{C}.$$

a) What happens if $\alpha \in \mathbb{N}_0$? What is the radius of convergence? [2 pts.]

b) Assume that $\alpha \notin \mathbb{N}_0$. Determine the radius of convergence by the ratio test! [2 pts.]

c) Compute the Taylor polynomial of the function $(1+z)^\alpha$ of degree 3 with center $z_0 = 0$. What do you observe? Give the Lagrange form of the remainder. [5 pts.]

Exercise 3 (5 pts.)

We consider a power series for a complex-valued function f and its reciproke $g := \frac{1}{f}$, given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad g(z) = \frac{1}{f(z)} = \sum_{m=0}^{\infty} b_m z^m.$$

Applying the Cauchy product there holds (under suitable assumptions)

$$1 = f(z)g(z) = \sum_{r=0}^{\infty} \left(\sum_{l=0}^{r} a_l b_{r-l} \right) z^r.$$

By equating coefficients we find, for instance, $b_0 = \frac{1}{a_0}$.

Compute analogously the coefficients b_1, b_2 and b_3 in terms of the coefficients $\{a_j\}_j \in \mathbb{N}$. [3 pts.]

Apply your result by computing an reciproke power series up to order 3 for $f(z) = (1+z)^3$. Note that for f we have $a_0 = 1$, $a_1 = 3$, $a_2 = 3$, $a_3 = 1$, and $a_k = 0$ for $k \geq 4$. [2 pts.]