Technische Hochschule Rosenheim
Fakultät ANG
Prof. Dr. André Herzwurm

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**Exercise Sheet 8**

# Linear Algebra (AAI)

## Exercise 8.1 (H)

Let

$$V = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 \colon\ x_1 + x_2 + x_3 = 0\},$$
$$W = \{(y_1, y_2, y_3, y_4)^\top \in \mathbb{R}^4 \colon\ y_1 + y_2 + y_3 + y_4 = 0\}$$

be subspaces of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively.

a) Determine $n_1, m_1, n_2, m_2 \in \mathbb{N}$ and linear maps $G_1 \colon \mathbb{R}^{n_1} \to \mathbb{R}^{m_1}$ and $G_2 \colon \mathbb{R}^{n_2} \to \mathbb{R}^{m_2}$ such that $V = \ker G_1$ and $W = \ker G_2$.

b) Let $\mathcal{A} = \big((1,-1,0)^\top, (1,0,-1)^\top\big)$ and $\mathcal{B} = \big((1,-1,0,0)^\top, (1,0,-1,0)^\top, (1,0,0,-1)^\top\big)$.

   i) Show that $\mathcal{A}$ and $\mathcal{B}$ are bases of $V$ and $W$, respectively.
   *Hint: Use the Rank-Nullity Theorem to determine* $\dim V$ *and* $\dim W$.

   ii) Compute $F(v)$ for $F \in L(V, W)$ given by

$$\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ 3 & 5 \end{pmatrix}$$

   and $v = (2, -1, -1)^\top$.

## Exercise 8.2 (H)

Let $\mathcal{A} = \big((1,1,0)^\top, (-1,1,1)^\top, (0,1,2)^\top\big)$ and $\mathcal{B} = \big((2,1,1)^\top, (0,0,1)^\top, (-1,1,1)^\top\big)$ be bases of $\mathbb{R}^3$.

a) Determine the change-of-basis matrix $\mathcal{T}_{\mathcal{B}}^{\mathcal{A}}$.

b) Let $F \in L(\mathbb{R}^3, \mathbb{R}^3)$ be given by the transformation matrix

$$\mathcal{M}_{\mathcal{B}}^{\mathcal{A}}(F) = \begin{pmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \\ 1 & 4 & -2 \end{pmatrix}.$$

   Compute $\mathcal{M}_{\mathcal{E}}^{\mathcal{E}}(F)$ for the standard basis $\mathcal{E}$ of $\mathbb{R}^3$.

## Exercise 8.3 (H)

Let $A, B \in \mathbb{R}^{3\times 2}$ be given by

$$A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}.$$

a) Show that $A$ and $B$ are equivalent.

b) Let $F \in L(\mathbb{R}^2, \mathbb{R}^3)$ be given by $F = \mathcal{F}_A$. Construct bases $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively, such that
$$\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F) = B.$$

c) Construct matrices $S \in \mathrm{Gl}(3, \mathbb{R})$ and $T \in \mathrm{Gl}(2, \mathbb{R})$ such that $A = S \cdot B \cdot T^{-1}$.
*Hint: Note that $\mathcal{M}^{\mathcal{A}}_{\mathcal{B}}(F) = B$ and $\mathcal{M}^{\mathcal{E}}_{\mathcal{E}'}(F) = A$, where $\mathcal{E}$ and $\mathcal{E}'$ denote the standard bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively.*

## Exercise 8.4 (H)

Let $n \in \mathbb{N}$ and $A = (a_{i,j})_{1\le i,j\le n} \in \mathbb{R}^{n\times n}$ be given by

$$a_{i,j} = \begin{cases} 1, & \text{if } i \ge j, \\ 0, & \text{else.} \end{cases}$$

a) Show that $A$ is invertible.

b) Determine $A^{-1}$.
*Hint: First, consider the case $n = 4$ and transform $A$ into $E_4$ by using elementary row operations of type II (start with the last row).*