

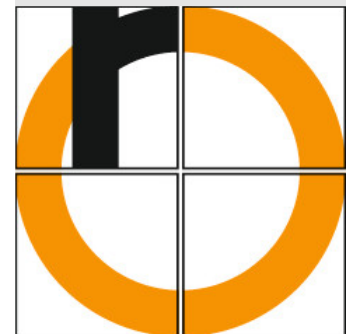
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We know the concepts of  
functions,  
sequences, and  
series.

We are interested in series representing a function  $f(x)$  at  
every  $x$ :

$$f(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j$$

For this purpose we need the concept of a sequence of  
functions at first.



A sequence of elements of  $\mathbb{R}$  (cf. Analysis 1):

$$f : \mathbb{N} \rightarrow \mathbb{R}, n \mapsto a_n =: f(n)$$

## Definition (Sequences of Functions)

Let  $D$  be a set. A mapping

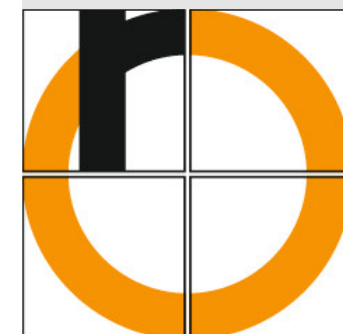
$$f : D \times \mathbb{N} \rightarrow \mathbb{R}, (x, n) \mapsto f_n =: f(n)$$

is called a **sequence** of functions  $f_k : D \rightarrow \mathbb{R}, k \in \mathbb{N}$ .

Other notations by writing the functions, e.g., are:

$$\{f_k\}_{k \in \mathbb{N}} = \{f_k\}_{k \geq 1} = \{f_1, f_2, f_3, \dots, f_k, \dots\}$$

The domain of definition  $D$  and the target area, here  $\mathbb{R}$ , have to be identical for all functions  $f_k$ .

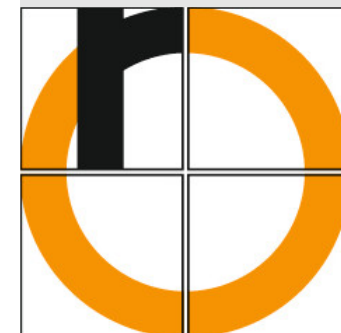


# Example (Pointwise Convergence)

Consider  $\{f_k\}_{k \in \mathbb{N}}$ ,  $f_k : [0, 2] \rightarrow \mathbb{R}$  with

$$f_k(x) = \begin{cases} n^2 x, & 0 \leq x \leq \frac{1}{n}, \\ 2n - n^2 x, & \frac{1}{n} \leq x \leq \frac{2}{n}, \\ 0, & \frac{2}{n} \leq x \leq 2. \end{cases}$$

This example exhibits that we may not swap the limit and the integral (another limit process) in general!



## Definition (Pointwise Convergence)

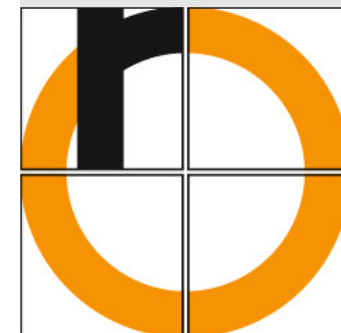
Let  $D$  be a set. A sequence  $\{f_n\}_{n \geq k_0}$  of functions  $f_k : D \rightarrow \mathbb{R}$  is called **pointwise convergent** to a function  $f : D \rightarrow \mathbb{R}$ , if and only if

$$\lim_{k \rightarrow \infty} f_k(x) = f(x) \quad \text{for any } x \text{ in } D.$$

Equivalently,

For any  $x \in D$  and  $\varepsilon > 0$   
there exists a  $N = N(x, \varepsilon)$  s.t.:

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for any } x \text{ in } D \text{ and all } n \geq N.$$



## Definition (Uniform Convergence)

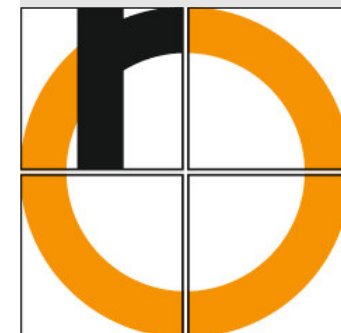
Let  $D$  be a set. A sequence  $\{f_n\}_{k \geq k_0}$  of functions  $f_k : D \rightarrow \mathbb{R}$  is called **uniformly convergent** to a function  $f : D \rightarrow \mathbb{R}$ , if and only if

For any  $x \in D$  and  $\varepsilon > 0$  there exists a  $N = N(\varepsilon)$  s.t.:

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for any } x \text{ in } D \text{ and all } n \geq N.$$

Notice that  $N$  may depend only on  $\varepsilon$  but not on the point  $x$ .

Pointwise convergence always implies uniform convergence, the opposite is not true (see last example).



# Example Uniform Convergence

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S.-J. Kimmerle

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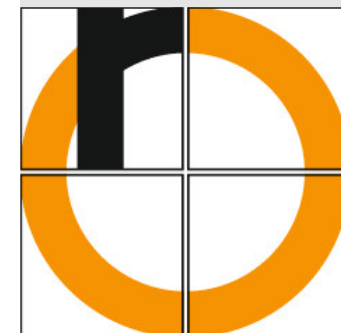
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Let  $x \in D = [0, 1)$ . The sequence

$$\left\{ \frac{1}{2^{x+n}} \right\}_{n \in \mathbb{N}}$$

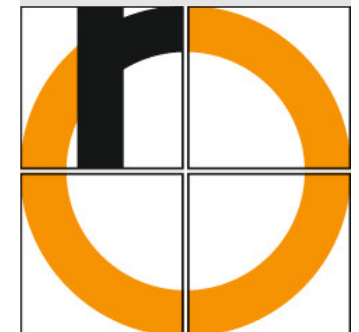
converges uniformly:



## Theorem (Uniform convergence preserves continuity)

*Let  $D \subseteq \mathbb{C}$  and  $f_n : D \rightarrow \mathbb{C}$ ,  $n \in \mathbb{N}$ , a sequence of continuous functions, that uniformly converge to a function  $f : D \rightarrow \mathbb{C}$ , then  $f$  is continuous.*

The limit of a uniformly convergent sequence of continuous functions, is again continuous.





# Example: Saw-tooth function

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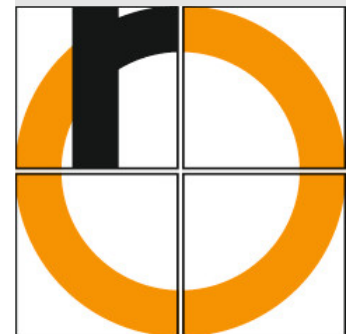
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## Definition (Uniform norm or sup norm)

Let  $D$  be a set and  $f : D \rightarrow \mathbb{C}$ .

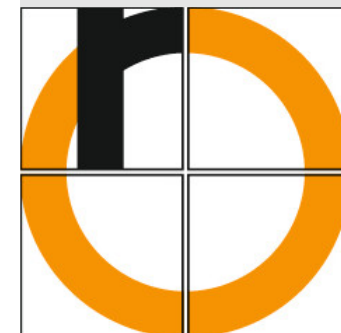
We set

$$\|f\|_D := \sup_{x \in D} |f(x)|.$$

$\|\cdot\|_D$  defines a norm on  $D$ .

A function  $f$  is bounded iff  $\|f\| < \infty$ .

When misunderstandings are excluded, we just write  $\|f\|$  instead of  $\|f\|_D$ .



By this notation we may reformulate the uniform convergence:

$\{f_n\}_{n \in \mathbb{N}}$  converges uniformly on  $D$

$$\iff \lim_{n \rightarrow \infty} \|f_n - f\|_D = 0$$

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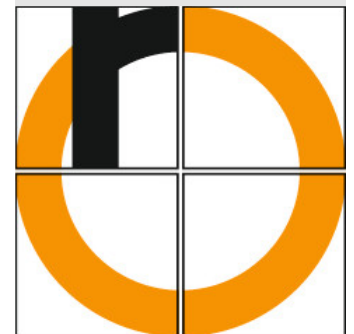
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## Theorem (Weierstrass Convergence Criterion)

Let  $f_n : D \rightarrow \mathbb{C}$ ,  $n \in \mathbb{N}$ .

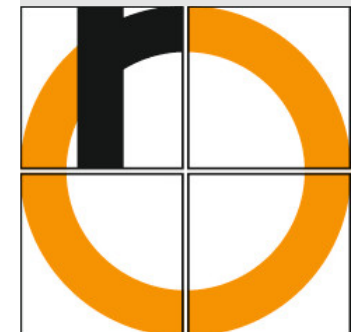
If

$$\sum_{n=0}^{\infty} \|f_n\|_D < \infty$$

then the series

$$\sum_{n=0}^{\infty} f_n$$

converges absolutely and uniformly on  $D$  to a function  $F : D \rightarrow \mathbb{C}$ .



# Example: Convergence of a Power Series

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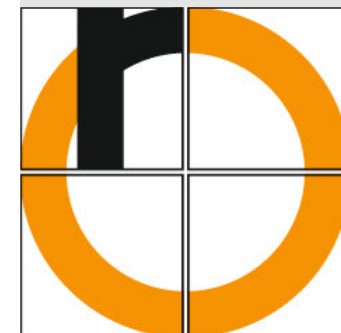
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The series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

converges uniformly on  $\mathbb{R}$ .



Polynomials are among the functions that are easy to handle (e.g. for a machine).

Many other functions become manageable, when they are approximated by polynomials, i.e. as power series.

As for polynomials it is helpful to consider power series on  $\mathbb{C}$  from the start.

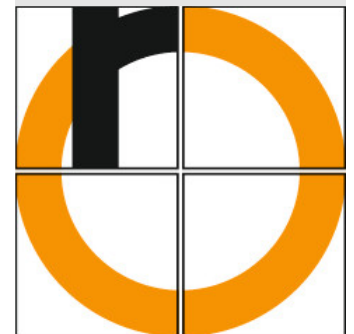
## Definition (Power Series)

Let  $\{a_j\}_{j \in \mathbb{N}}$  a sequence of complex numbers and  $z_0 \in \mathbb{C}$ .

Then

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \dots$$

is called a **(complex) power series** with the **center** of the series  $z_0$ .



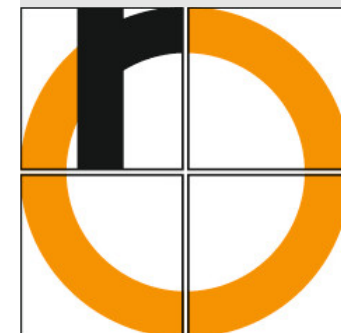
The set of points  $z$  where the power series converges form a set  $M \subseteq \mathbb{C}$ .

Note that  $z_0 \in M$ .

By this the power series defines a function  $f : M \rightarrow \mathbb{C}$ .

The partial sums of power series are polynomials (multiply out!).

Power series have excellent properties of convergence.



## Theorem (Radius of Convergence)

Let

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

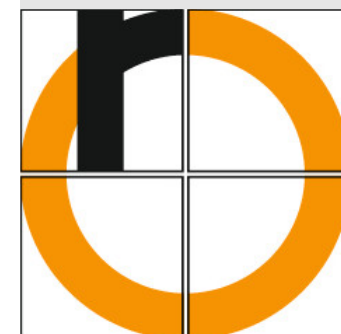
be a complex power series.

Then exactly one of the following 2 cases holds:

- There exists a  $\rho \in \mathbb{R}_0^+$  s.t. the series converges absolutely for all  $z \in O_\rho(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \rho\}$  and diverges for all  $z$  with  $|z - z_0| > \rho$ .
- The series converges absolutely for all  $z \in \mathbb{C}$ .

$\rho \in \mathbb{R}_0^+ \cup \{+\infty\}$  is called **radius of convergence**,  
 $O_\rho(z_0)$  is called **circle of convergence**.

For  $|z| = \rho$  no general statement on convergence/divergence is possible.





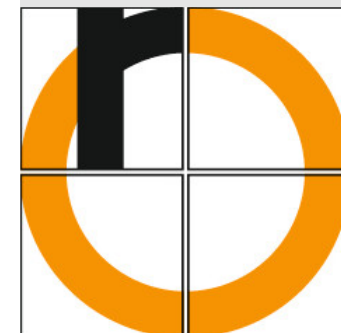
Assume for a power series holds

$$\overline{\lim}_{j \rightarrow \infty} \sqrt[j]{|a_j|} = b \quad \text{or} \quad \lim_{j \rightarrow \infty} \left| \frac{a_{j+1}}{a_j} \right| = b,$$

where  $b \in \mathbb{R}_0^+ \cup \infty$ , then:

- If  $b = 0$ , then  $\rho = +\infty$ .
- If  $b = +\infty$ , then  $\rho = 0$ .
- If  $0 < b$ , then  $\rho = \frac{1}{b}$ .

The limes superior (or inferior) of the quotient is not helpful in general.



# Example: Geometric Series

Geometric series:

$$\sum_{j=0}^{\infty} z^n$$

Set  $a_j = 1$  for all  $j \in \mathbb{N}$  (and  $z_0 = 0$ ).

Since  $b = \lim_{j \rightarrow \infty} \sqrt[j]{a_j} = 1$ , we find  $\rho = 1/b = 1$ .

Thus we have (as expected) convergence for  $|z| < 1$  and divergence for  $|z| > 1$ .

What happens for  $|z| = 1$  ? Divergence, since we do not encounter a zero sequence  $z^n$ .

Moreover  $\sum_{j=0}^{\infty} z^n = \frac{1}{1-z}$  for  $|z| < 1$ .

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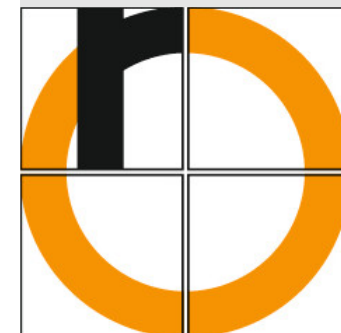
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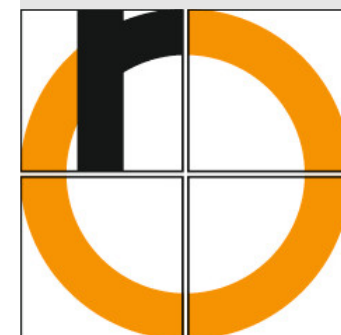
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## Theorem (Addition & scaling of Power Series)

*Consider two power series*

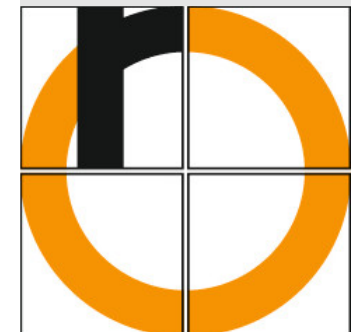
$f(z) = \sum_{n=0}^{\infty} a_n z^n$  with radius of convergence  $\rho_f$  and  
 $g(z) = \sum_{n=0}^{\infty} b_n z^n$  with radius of convergence  $\rho_g$ .

*Then the sum/difference is given for all  $z$  with  
 $|z| < \min(\rho_f, \rho_g)$  by:*

$$\sum_{n=0}^{\infty} (a_n \pm b_n) z^n = \sum_{n=0}^{\infty} a_n z^n \pm \sum_{n=0}^{\infty} b_n z^n = f(z) \pm g(z)$$

*Further, the scaling i.e. multiplication with a factor  $c \in \mathbb{C}$  is  
given for all  $z$  with  $|z| < \rho_f$  by:*

$$\sum_{n=0}^{\infty} c a_n z^n = c \sum_{n=0}^{\infty} a_n z^n = c f(z).$$



## Theorem ((Cauchy) Product of Power Series)

*Consider two power series*

$f(z) = \sum_{n=0}^{\infty} a_n z^n$  with radius of convergence  $\rho_f$  and

$g(z) = \sum_{n=0}^{\infty} b_n z^n$  with radius of convergence  $\rho_g$ .

*Then the product  $f(z_1)g(z_2)$  is given for all  $z_1, z_2$  with  $|z_1|, |z_2| < \min(\rho_f, \rho_g)$  by:*

$$f(z_1)g(z_2) = \sum_{n=0}^{\infty} \sum_{k=0}^n a_k b_{n-k} z_1^k z_2^{n-k}.$$

*In particular, if  $z = z_1 = z_2$ :*

$$f(z)g(z) = \sum_{n=0}^{\infty} \left( \sum_{k=0}^n a_k b_{n-k} \right) z^n.$$

The latter 2 theorems also hold for  $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ ,  $g(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$  by a shift.

## Theorem (Series of polynomials is continuous)

Let

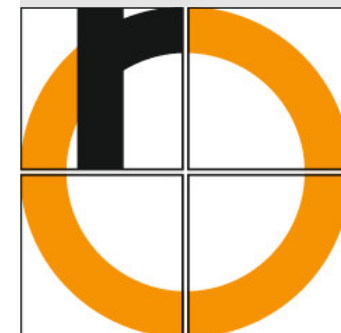
$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

be a complex power series with radius of convergence  $\rho$ .

Then the function

$$f : O_{\rho}(z_0) \rightarrow \mathbb{C} : z \mapsto \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

is continuous.



# More Examples: Exponential Series etc.

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The series  
for the exponential function and  
for (co)sine, resp.,  
have the radius of convergence  $\rho = \infty$ .

The latter 3 series yield a continuous function.

