Fourier Series

Motivation:

- Periodic signals, image processing (jpeg/mpeg compression)
- Generation of signals by periodic means
- Harmonic oscillations, acoustics
- Periodic orbits
- Revolving motors

Many periodic functions may be approximated by a sum of trigonometric base functions as

$$1, \quad \cos(kx), \quad \sin(kx), \quad k \in \mathbb{N}$$

with suitable coefficients a_0, a_k, b_k.

Frequently, it may make sense to replace the finite sum by a series.

Example: Rotational Force of an Engine I

A rotational force f (due to gas pressure and mass inertia) acts on the crankshaft of a single-cylinder two-stroke engine.

At the points $t_k = k \cdot \pi/8$, $k = 0, 1, \dots, 15$, the following values of f_k were determined experimentally:

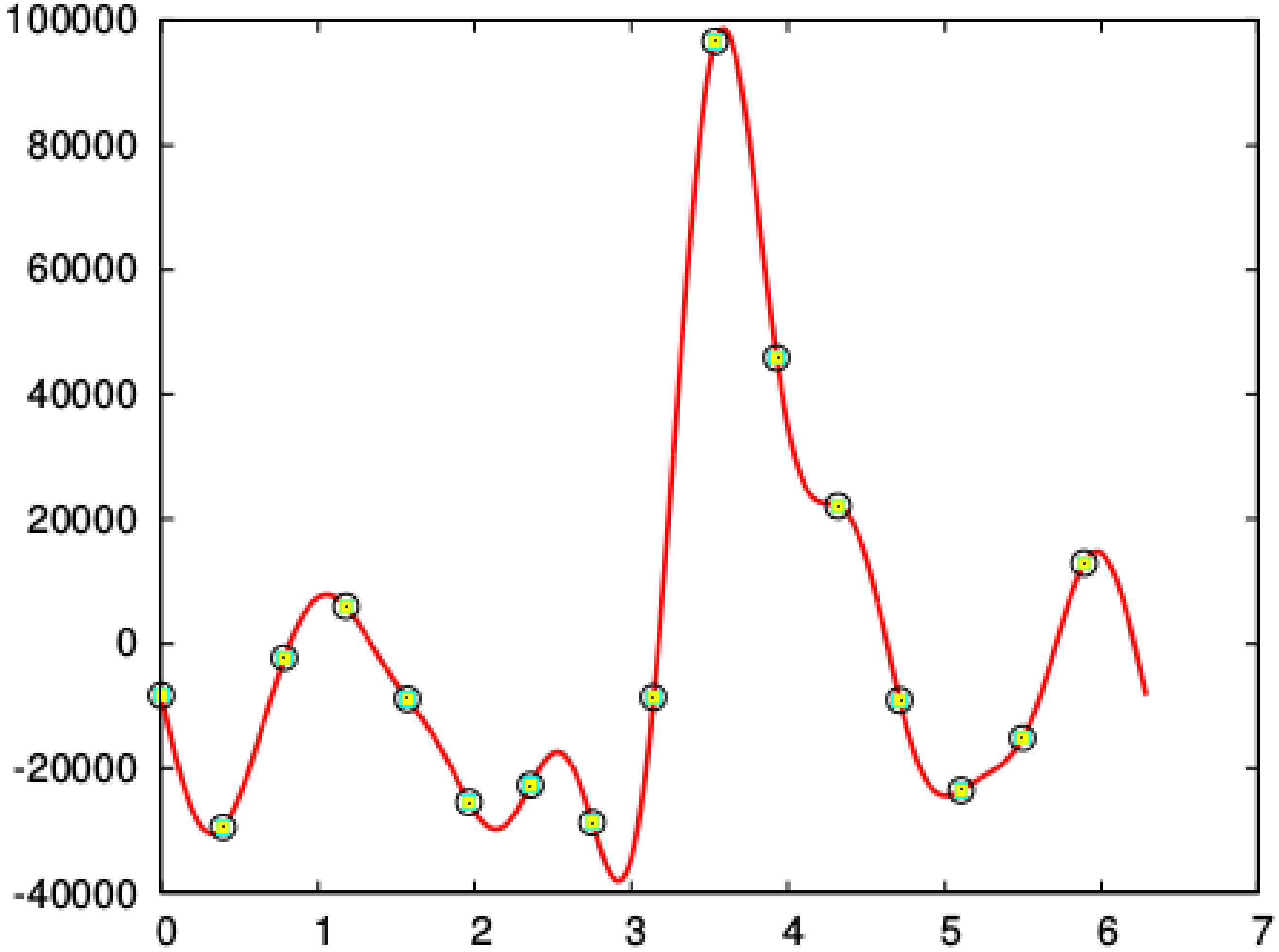
f_0	f_1	f_2	f_3	f_4	f_5
−8250	−29430	−2286	5974	−8829	−25408

f_6	f_7	f_8	f_9	f_{10}
−22681	−28655	−8564	96560	45862

f_{11}	f_{12}	f_{13}	f_{14}	f_{15}
22092	−9025	−23514	−15127	12880

Example: Rotational Force of an Engine II

The resulting trigonometric approximation of f looks as follows:



Periodicity

Definition (Periodic function)

A function $f : \mathbb{R} \to \mathbb{C}$ is called T**-periodic**, if there exists a $T > 0$ s.t.

$$f(x+T) = f(x) \quad \text{for all } x \in \mathbb{R}.$$

T is called a **period** of f.

Note that a T-periodic function f is completely defined by its values on the interval $[0, T)$. Thus it is enough to consider f only on $[0, T)$.

Properties of Periodic Functions

Let f and g be T-periodic.

- Then $\alpha f + \beta g$ is T-periodic for any $\alpha, \beta \in \mathbb{C}$.
- Then there holds for all $c \in \mathbb{R}$:

$$\int_c^{c+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

- Then the function $\tilde{f} : \mathbb{R} \to \mathbb{C}$, defined by

$$\tilde{f}(x) := f\left(\frac{T}{2\pi}x\right)$$

is 2π-periodic, since:

Definition (Fourier Series)

The function series

$$F(x) := \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega x) + b_k \sin(k\omega x) \quad \text{with } \omega := \frac{2\pi}{T}$$

is called **Fourier series.** The coefficients $a_0, a_1, \ldots$ and $b_1, b_2, \ldots$ are called **Fourier coefficients.** The representation of a function by its Fourier series is called **Fourier** (or harmonic) **analysis.** The finite sum is called **Fourier sum**:

$$F_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(k\omega x) + b_k \sin(k\omega x).$$

Computing Fourier Coefficients

Theorem (Fourier)

Assume a T-periodic function f may be represented as a Fourier series

$$F(x) := \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\omega x) + b_k \sin(k\omega x) \quad \text{with } \omega := \frac{2\pi}{T},$$

then the Fourier coefficients are given by

$$\begin{aligned} a_0 &= \frac{2}{T} \int_0^T f(x)\,dx, \\ a_k &= \frac{2}{T} \int_0^T f(x)\cos(k\omega x)\,dx, \\ b_k &= \frac{2}{T} \int_0^T f(x)\sin(k\omega x)\,dx. \end{aligned}$$

Symmetry & Fourier Coefficients

The following symmetries may be exploited when computing Fourier coefficients, $k \in \mathbb{N}$:

- If f is an odd function, i.e. $f(x) = -f(-x)$ for all $x \in \mathbb{R}$, then

$$b_k = \frac{2}{T}\int_0^T f(x)\sin(k\omega x)\,dx = \frac{4}{T}\int_0^{T/2} f(x)\sin(k\omega x)\,dx,$$

$$a_k = \frac{2}{T}\int_0^T f(x)\cos(k\omega x)\,dx = \qquad\qquad = 0.$$

- If f is an even function, i.e. $f(x) = f(-x)$ for all $x \in \mathbb{R}$, then

$$b_k = \frac{2}{T}\int_0^T f(x)\sin(k\omega x)\,dx = \qquad\qquad = 0,$$

$$a_k = \frac{2}{T}\int_0^T f(x)\cos(k\omega x)\,dx = \frac{4}{T}\int_0^{T/2} f(x)\cos(k\omega x)\,dx.$$

Example: Saw-tooth Function revisited

Warning: In general $f(x) \neq F(x)$ for all $x \in \mathbb{R}$!

Gibbs phenomenon: overshooting & undershooting (up to 18% for large n) at points of discontinuity

Revision: Trigonometric Functions

Definition (Cosine and sine)

For $x \in \mathbb{R}$ we define:

$$\cos(x) := \mathrm{Re}\,(\exp(ix))$$
$$\sin(x) := \mathrm{Im}\,(\exp(ix))$$

We see that the Euler formula holds:

$$\exp(ix) = \cos(x) + i\sin(x), \quad x \in \mathbb{R}$$