1) $f(x,y) = y^2(x-1) + x^2(x+1) = xy^2 - y^2 + x^3 + x^2$

$\frac{\partial f}{\partial x} = y^2 + 3x^2 + 2x$ $\longrightarrow$ $P_1\ (0|0)$

$\frac{\partial f}{\partial y} = 2xy - 2y$

$$\left| \begin{array}{l} 0 = y^2 + 3x^2 + 2x \\ 0 = 2xy - 2y \end{array} \right.$$

$\frac{\partial^2 f}{\partial^2 x} = 6x + 2$

$2y = 2xy$
$2 = 2x$
$1 = x$

$-y^2 = 3 \cdot 1^2 + 2 \cdot 1$
$y^2 = -5$
$y = \pm\sqrt{-5} = \pm\sqrt{5} \cdot i$

$\frac{\partial^2 f}{\partial^2 y} = 2x - 2$

$\frac{\partial^2 f}{\partial x \partial y} = 2y$

$\longrightarrow P_2\ (1|\sqrt{5}\,i)\quad P_3\ (1|-\sqrt{5}\,i)$

let $y = 0$

$0 = 3x^2 + 2x = x(3x+2)$

$P_1(0|0)$ $\quad P_4(-\frac{2}{3}|0)$

$$D = \frac{\partial^2 f}{\partial^2 x} \cdot \frac{\partial^2 f}{\partial^2 y} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$= (6x+2)(2x-2) - (2y)^2$$

$$= 12x^2 - 12x + 4x - 4 - 4y^2 = 12x^2 - 4y^2 - 8x - 4$$

$D_1 = 0 - 0 - 0 - 4 = -4 < 0 \ \longrightarrow$ saddle point P_1

$D_2 = 12 - 4(\sqrt{5}\,i)^2 - 8 - 4 = 20 > 0 \ \longrightarrow$ min/max

$\frac{\partial^2 f}{\partial^2 x}$: $\quad 6 \cdot 1 + 2 = 8 > 0 \ \longrightarrow$ minima P_2

$D_3 = 12 - 4(-\sqrt{5}\,i)^2 - 8 - 4 = 20 > 0 \ \longrightarrow$ min/max

$\frac{\partial^2 f}{\partial^2 x}$: $\quad 6 \cdot 1 + 2 = 8 > 0 \ \longrightarrow$ minima P_3

$D_4 = 12 \cdot (-\frac{2}{3})^2 - 0 - 8(-\frac{2}{3}) - 4 = -8 + \frac{16}{3} - 4 = \frac{20}{3} > 0$

$\frac{\partial^2 f}{\partial^2 x}$: $\quad 6 \cdot (-\frac{2}{3}) + 2 = -2 < 0 \ \longrightarrow$ maxima P_4

/2

The sufficient condition of second order is just a sufficient condition since in some cases you can tell that all Points are above the minima. In the example x^2 + y^4, the minima is the origin and every other possible point P(x,y) is greater than 0 on the z-achsis. Therefore you don't need any Hessian to determine the type of (0|0).

/3 By the necessary condition of first order, you can tell where a stationary point is if you know where the gradient of x equals zero.

$$h(x,y) = x^2y^2 + x^4 - 2x^2 + 1$$

$$\nabla h(x,y) = \begin{pmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{pmatrix} = \begin{pmatrix} 2xy^2 + 4x^3 - 4x \\ 2yx^2 \end{pmatrix}$$

$$\left| \begin{array}{l} 0 = 2xy^2 + 4x^3 - 4x \\ 0 = 2yx^2 \end{array} \right. \quad \rightarrow \quad 0 = 2 \cdot 0 \cdot 0^2$$

$$y = 0 \qquad \rightarrow P_1(0|0)$$
$$x = 0$$

Let $x, y = 0$:

$$0 = 2 \cdot 0 \cdot 0^2 + 4 \cdot 0^3 - 4 \cdot 0 \quad ✓$$

Let $x = 0$:

$$0 = 2 \cdot 0 \cdot y^2 + 4 \cdot 0^3 - 4 \cdot 0 \quad ✓$$

Let $y = 0$:

$$0 = 2x \cdot 0^2 + 4x^3 - 4x$$

$$0 = 4x^3 - 4x$$

$$0 = x(4x^2 - 4)$$

$\rightarrow P_2(1|0)$, $P_3(-1|0)$

$\rightarrow P_1(0|0)$

By the sufficient condition, calculating the Hessian determines the type of these points.

$$H = \begin{pmatrix} 2y^2 + 12x^2 - 4 & 4xy \\ 4xy & 2x^2 \end{pmatrix}$$

$$D = 4x^2y^2 + 24x^4 - 8x^2 - 16x^2y^2$$

$$= -12x^2y^2 + 24x^4 - 8x^2$$

$D_1 = 0 + 0 - 0 \longrightarrow$ inconclusive, P_1

possibly different types

$D_2 = 24 - 8 > 0$,

$\frac{\partial^2 h}{\partial^2 x}$: $12 - 4 > 0 \longrightarrow$ local min P_2

$D_3 = 24 - 8 > 0$,

$\frac{\partial^2 h}{\partial^2 x}$: $12 - 4 > 0 \longrightarrow$ local min P_3