Homework 5: revision & integrating/differentiating Fourier series

To submit: on **Friday, 29.04.2022**, 9:00 a.m., online by the learning campus

Exercise 1 (6 pts.)
Let for $a < c < b$ a function

$$f(x) = \begin{cases} 2\frac{x-a}{(b-a)(c-a)} & \text{for } a \leq x \leq c, \\ 2\frac{b-x}{(b-a)(b-c)} & \text{for } c < x \leq b, \\ 0 & \text{otherwise,} \end{cases}$$

given.

a) Compute $F(x) := \int_{-\infty}^{x} f(\tilde{x})\, d\tilde{x}$. [4 pts.]

b) Show that $F : \mathbb{R} \to \mathbb{R}$ is continuous. [1 pt.]

c) Check that $\lim_{x\to\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$. [1 pt.]

Remark: The function f serves as a probability density function of the so-called triangle distribution. F is the corresponding cumulative density function.

Exercise 2 (5 pts.)
Compute the partial fraction expansion of

$$f(x) = \frac{x^3 - 6x^2 + 11x - 6}{x^2 + 3x - 2}$$

and then derive a primitive for f.

Exercise 3 (7 pts.)
We recall the Fourier series

$$F(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{x}$$

that corresponds to the saw tooth function

$$f : [0, 2\pi) \to \mathbb{R}, x \mapsto f(x) = \begin{cases} 0 & \text{for } x = 0, \\ \frac{\pi - x}{2} & \text{for } 0 < x < 2\pi, \end{cases},$$

periodically extended with period 2π.

a) Compute the Fourier series $\tilde{F}$ of the 2π-periodically extended function

$$\tilde{f} : [0, 2\pi), x \mapsto \tilde{f}(x) = -\frac{(\pi - x)^2}{4}$$

by an integration term by term. [3 pts.]

b) Use the result from a) to proof

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \quad \text{[2 pts.].}$$

c) Differentiate the Fourier series of F term by term. What happens? Explain that this is no contradiction to the representation result stated in the lecture (slide 66). [2 pts.]