Technische Hochschule Rosenheim
Fakultät ANG
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**Exercise Sheet 0**

# Linear Algebra (AAI)

## Exercise 0.1 (H)

Verify that $G = \{0, 1\}$ together with

$$\begin{array}{c|cc} * & 0 & 1 \\ \hline 0 & 0 & 1 \\ 1 & 1 & 0 \end{array}$$

is a commutative group, see Example 3(iii) in Section I.1.

*Solution:*
By definition, we have $*\colon G \times G \to G$. Moreover, $(G, *)$ satisfies:

i) Associativity: We have

$$\forall\, a, b, c \in G\colon\ (a * b) * c = a * (b * c)$$

since (there are $2^3$ cases)

$$\begin{aligned}
(0 * 0) * 0 = 0 = 0 * (0 * 0),\\
(0 * 0) * 1 = 1 = 0 * (0 * 1),\\
(0 * 1) * 0 = 1 = 0 * (1 * 0),\\
(0 * 1) * 1 = 0 = 0 * (1 * 1),\\
(1 * 0) * 0 = 1 = 1 * (0 * 0),\\
(1 * 0) * 1 = 0 = 1 * (0 * 1),\\
(1 * 1) * 0 = 0 = 1 * (1 * 0),\\
(1 * 1) * 1 = 1 = 1 * (1 * 1).
\end{aligned}$$

ii) The neutral element is given by 0 since $0 * 0 = 0$ and $0 * 1 = 1$.

iii) The inverse elements are given by $-1 = 1$ and $-0 = 0$ since

$$\begin{aligned} 1 * 1 = 0,\\ 0 * 0 = 0. \end{aligned}$$

iv) Commutativity: We have

$$\forall\, a, b \in G\colon\ a * b = b * a$$

since $0 * 1 = 1 = 1 * 0$.

## Exercise 0.2 (H)

Prove Proposition 10 from Section I.1.

*Solution:*
See Proposition I.1.10 in the lecture notes.

## Exercise 0.3 (H)

Prove Lemma 12 from Section I.1.

*Solution:*
See Lemma I.1.12 in the lecture notes.

## Exercise 0.4 (H)

Let $G_1$ and $G_2$ be subgroups of a group $G$. Show that $G_1 \cup G_2$ is not a subgroup in general.
Hint: Cf. Example 18 from Section I.1.

*Solution:*
See Example I.1.18 in the lecture notes.

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**Exercise Sheet 1**

# Linear Algebra (AAI)

## Exercise 1.1 (H)

Prove Lemma I.2.4.

*Solution:*
See Lemma I.2.4 in the lecture notes.

## Exercise 1.2 (H)

a) Verify that $V = \mathbb{R}^n$ is a $\mathbb{R}$-vector space.

*Solution:*
By using the field properties of $\mathbb{R}$ in every coordinate we obtain:

1.) $(\mathbb{R}^n, \oplus)$ is a commutative group, i.e., we have $\oplus\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and the following properties:

i) Associativity: We have

$$\forall\, x, y, z \in \mathbb{R}^n \colon (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

since

$$\begin{aligned}(x \oplus y) \oplus z &= \Big((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n\Big),\\ &= \Big(x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)\Big)\\ &= x \oplus (y \oplus z).\end{aligned}$$

ii) The neutral element is given by $\mathbf{0} = (0, \dots, 0)$ since

$$\mathbf{0} \oplus x = (0 + x_1, \dots, 0 + x_n) = (x_1, \dots, x_n) = x$$

for all $x \in \mathbb{R}^n$.

iii) The inverse element of $x \in \mathbb{R}^n$ is given by $-x = (-x_1, \dots, -x_n)$ since

$$\begin{aligned}-x \oplus x &= (-x_1, \dots, -x_n) \oplus (x_1, \dots, x_n)\\ &= (-x_1 + x_1, \dots, -x_n + x_n)\\ &= (0, \dots, 0) = \mathbf{0}.\end{aligned}$$

iv) Commutativity: We have

$$\forall\, x, y \in \mathbb{R}^n \colon x \oplus y = y \oplus x$$

since

$$\begin{aligned}x \oplus y &= (x_1 + y_1, \dots, x_n + y_n)\\ &= (y_1 + x_1, \dots, y_n + x_n)\\ &= y \oplus x.\end{aligned}$$

2.) For all $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}
(\lambda + \mu) \odot x &= \big((\lambda + \mu) \cdot x_1, \dots, (\lambda + \mu) \cdot x_n\big) \\
&= (\lambda x_1 + \mu x_1, \dots, \lambda x_n + \mu x_n) \\
&= (\lambda \cdot x_1, \dots, \lambda \cdot x_n) \oplus (\mu \cdot x_1, \dots, \mu \cdot x_n) \\
&= \lambda \odot x \oplus \mu \odot x, \\
\lambda \odot (x \oplus y) &= \big(\lambda \cdot (x_1 + y_1), \dots, \lambda \cdot (x_n + y_n)\big) \\
&= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n) \\
&= (\lambda \cdot x_1, \dots, \lambda \cdot x_n) \oplus (\lambda \cdot y_1, \dots, \lambda \cdot y_n) \\
&= \lambda \odot x \oplus \lambda \odot y, \\
\lambda \odot (\mu \odot x) &= \lambda \odot (\mu \cdot x_1, \dots, \mu \cdot x_n) \\
&= (\lambda \cdot \mu \cdot x_1, \dots, \lambda \cdot \mu \cdot x_n) \\
&= \big((\lambda \cdot \mu) \cdot x_1, \dots, (\lambda \cdot \mu) \cdot x_n\big) \\
&= (\lambda \cdot \mu) \odot x, \\
1 \odot x &= (1 \cdot x_1, \dots, 1 \cdot x_n) \\
&= (x_1, \dots, x_n) \\
&= x.
\end{aligned}$$

b) Is $\mathbb{C}^n$ a $\mathbb{R}$-vector space (based on (1) from Example II.1.2)?

*Solution:*
The scalar multiplication

$$\odot \colon \mathbb{R} \times \mathbb{C}^n \to \mathbb{C}^n$$

is given by the correpsonding operation on $\mathbb{C} \times \mathbb{C}^n$ ($\mathbb{C}^n$ seen as a $\mathbb{C}$-vector space), where the scalars are restricted to the subset $\mathbb{R} \subseteq \mathbb{C}$. Since $C^n$ is a $\mathbb{C}$-vector space, all vector space conditions are also met for this restriction.

## Exercise 1.3 (H)

Let $V = \mathbb{R}^{[0,1]}$.

a) Verify that $V$ is a $\mathbb{R}$-vector space.

*Solution:*
By using the field properties of $\mathbb{R}$ in every coordinate (i.e., for every evaluation site $x \in [0,1]$) we obtain the analogous properties as in Exercise 1.2 a). [...]

b) Show that $C([0,1])$ is a subspace of $V$.

*Solution:*
We verify the subspace conditions from Definition II.1.5:

i) $\mathbf{0} \in \mathbb{R}^{[0,1]}$ given by $\mathbf{0}(x) = 0$ for all $x \in [0,1]$ (= zero function) is continuous and hence $C([0,1]) \neq \emptyset$.

ii) Since the sum of two continuous functions is again a continuous function (see Analysis I), the set $C([0,1])$ is closed w.r.t. the vector addition.

iii) Since a continuous function multiplied by a constant is again a continuous function (see Analysis I), the set $C([0,1])$ is closed w.r.t. the scalar multiplication.

**Exercise 1.4 (H)**

Consider Example II.1.7.(i) with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and

$$U = \Big\{x \in \mathbb{R}^n \colon \sum_{i=1}^{n} a_i \cdot x_i = b\Big\}.$$

a) Show that $U$ is a subspace of $\mathbb{R}^n$ if and only if $b = 0$.

*Solution:*
"$\Rightarrow$": Let $U$ be a subspace and $x \in U$, i.e., we have

$$\sum_{i=1}^{n} a_i \cdot x_i = b$$

and hence

$$\sum_{i=1}^{n} a_i \cdot 2x_i = 2 \cdot \sum_{i=1}^{n} a_i \cdot x_i = 2b. \tag{1}$$

Since $U$ is a subspace, we also have $2x \in U$ (see Definition II.1.5.(iii)) and thus

$$\sum_{i=1}^{n} a_i \cdot 2x_i = b. \tag{2}$$

Combining (1) and (2) we obtain $2b = b$, which implies $b = 0$.
"$\Leftarrow$": Let $b = 0$. We verify the subspace conditions from Definition II.1.5:

i) We have

$$\sum_{i=1}^{n} a_i \cdot 0 = 0$$

and hence $\mathbf{0} = (0, \ldots, 0) \in U$ such that $U \neq \emptyset$.

ii) Let $x, y \in U$. Then we have

$$\sum_{i=1}^{n} a_i \cdot (x_i + y_i) = \sum_{i=1}^{n} a_i \cdot x_i + \sum_{i=1}^{n} a_i \cdot y_i = 0 + 0 = 0$$

and hence $x + y \in U$.

iii) Let $x \in U$ and $\lambda \in \mathbb{R}$. Then we have

$$\sum_{i=1}^{n} a_i \cdot (\lambda \cdot x_i) = \lambda \cdot \sum_{i=1}^{n} a_i \cdot x_i = \lambda \cdot 0 = 0$$

and hence $\lambda \cdot x \in U$.

b) Let $n = 2$. Determine and illustrate $U$ for all combinations of $a \in \{(1,1), (-1,0)\}$ and $b \in \{0, 1, -2\}$.

*Solution:*
...

c) Show Remark II.1.10 (regarding the union of subspaces).

*Solution:*
Consider $U_1 = \{x \in \mathbb{R}^2 \colon x_1 = 0\}$ and $U_2 = \{x \in \mathbb{R}^2 \colon x_2 = 0\}$. Then $U_1$ and $U_2$ are subspaces of $\mathbb{R}^2$ according to part a). However, $U_1 \cup U_2$ is not a subspace since

$$(0,1) \in U_1 \subseteq U_1 \cup U_2,$$
$$(1,0) \in U_2 \subseteq U_1 \cup U_2,$$

and

$$(0,1) + (1,0) = (1,1) \notin U_1 \cup U_2.$$

**Exercise Sheet 2**

# Linear Algebra (AAI)

## Exercise 2.1 (H)

Determine whether $U$ is a subspace of $\mathbb{R}^3$:

a) $U = \{(x, y, z) \in \mathbb{R}^3 \colon z = 3x - y\}$,

b) $U = \{(x, y, z) \in \mathbb{R}^3 \colon x \cdot y \cdot z = 0\}$,

c) $U = \{(x, y, z) \in \mathbb{R}^3 \colon x \cdot (\exp(y) + z) = 0\}$,

d) $U = \{(x, y, z) \in \mathbb{R}^3 \colon x, y, z \geq 0\}$,

e) $U = \{(\lambda, 2\lambda, 4\lambda) \in \mathbb{R}^3 \colon \lambda \in \mathbb{R}\}$.

*Solution:*
The sets from a) and e) are subspaces of $\mathbb{R}^3$.
... (Verify the subspace conditions from Definition II.1.5.)

The sets from b), c), and d) are not subspaces of $\mathbb{R}^3$. As an example we show that b) is not a subspace: Note that $(1, 1, 0), (0, 0, 1) \in U$, but

$$(1, 1, 0) + (0, 0, 1) = (1, 1, 1) \notin U.$$

Hence condition (ii) from Definition II.1.5 is not satisfied.

## Exercise 2.2 (H)

Consider $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ given by

$$v_1 = (1, 2, 1, 2),\ v_2 = (1, 1, 1, 1),\ v_3 = (0, 1, 1, 0),\ v_4 = (0, 1, 0, 1),\ v_5 = (1, 0, 0, 1).$$

a) Express each vector $v_i$ as a linear combination of the remaining vectors $v_j$ with $j \neq i$ (if possible).

*Solution:*
We have $v_1 = v_2 + v_4$ and $v_2 = v_3 + v_5$. Using these two equalities we obtain, e.g.,

$$\begin{aligned} v_1 &= v_2 + v_4, \\ v_2 &= v_3 + v_5, \\ v_3 &= v_2 - v_5, \\ v_4 &= v_1 - v_2, \\ v_5 &= v_2 - v_3. \end{aligned}$$

b) Prove or disprove:

i) $\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_3, v_5\})$,

ii) $\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_1, v_3, v_5\})$,

iii) $\operatorname{span}(\{v_2, v_5\}) = \operatorname{span}(\{v_3, v_5\})$,

iv) $\operatorname{span}(\{v_1, v_5\}) = \operatorname{span}(\{v_1, v_2, v_3, v_4, v_5\})$,

v) $\operatorname{span}(\{v_1, v_2, v_4\}) = \operatorname{span}(\{v_2, v_3, v_5\})$.

*Solution:*

Statements i) and iii) are correct. Statements ii), iv), and v) are false.

ad i): We clearly have

$$\operatorname{span}(\{v_2, v_3, v_5\}) = L(v_2, v_3, v_5) \supseteq L(v_3, v_5) = \operatorname{span}(\{v_3, v_5\}),$$

see Corollary II.1.17. Moreover, every linear combination of $v_2, v_3, v_5$ can be expressed as a linear combination of $v_3, v_5$ since $v_2 = v_3 + v_5$ and thus

$$\lambda_2 v_2 + \lambda_3 v_3 + \lambda_5 v_5 = \lambda_2(v_3 + v_5) + \lambda_3 v_3 + \lambda_5 v_5 = (\lambda_2 + \lambda_3)v_3 + (\lambda_2 + \lambda_5)v_5$$

for $\lambda_2, \lambda_3, \lambda_5 \in \mathbb{R}$. Hence we get

$$\operatorname{span}(\{v_2, v_3, v_5\}) = L(v_2, v_3, v_5) \subseteq L(v_3, v_5) = \operatorname{span}(\{v_3, v_5\}).$$

ad ii): Using part (i) we obtain

$$\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_3, v_5\}) \subseteq \operatorname{span}(\{v_1, v_3, v_5\}).$$

Moreover, we have

$$\begin{aligned} \operatorname{span}(\{v_3, v_5\}) = L(v_3, v_5) &= \{\lambda v_3 + \mu v_5 \colon \lambda, \mu \in \mathbb{R}\} \\ &= \{(\mu, \lambda, \lambda, \mu) \colon \lambda, \mu \in \mathbb{R}\}. \end{aligned}$$

This shows $v_1 = (1, 2, 1, 2) \notin \operatorname{span}(\{v_3, v_5\})$ (Note that the first and the last entry of $v_1$ are not equal.) and hence

$$\operatorname{span}(\{v_2, v_3, v_5\}) = \operatorname{span}(\{v_3, v_5\}) \not\supseteq \operatorname{span}(\{v_1, v_3, v_5\}).$$

ad iii): ...
ad iv): ...
ad v): ...

c) Determine all linearly independent families $(v_i)_{i \in I}$ with $I \subseteq \{1, \ldots, 5\}$ and $|I| \leq 3$.

*Solution:*

$|I| = 0$: $(v_i)_{i \in \emptyset}$ is linearly independent per definition.
$|I| = 1$: $(v_i)$ is linearly independent if and only if $v_i \neq 0$. Hence $(v_i)$ is linearly independent for every $i \in \{1, \ldots, 5\}$.
$|I| = 2$: By Lemma II.2.3 we have

$$(v_i, v_j) \text{ linearly independent } \Leftrightarrow v_i \notin L(v_j) \wedge v_j \notin L(v_i).$$

Hence $(v_i, v_j)$ is linearly independent for all $i, j \in \{1, \ldots, 5\}$ with $i \neq j$.
$|I| = 3$: Use Lemma II.2.3 for each of the 10 combinations: The families $(v_1, v_2, v_4)$ and $(v_2, v_3, v_5)$ are linearly dependent according to part a). E.g., $(v_3, v_4, v_5)$ is linearly independent since for $\lambda_3, \lambda_4, \lambda_5 \in \mathbb{R}$ we have

$$\lambda_3 v_3 + \lambda_4 v_4 + \lambda_5 v_5 = 0 \;\Leftrightarrow\; \begin{Bmatrix} & & \lambda_5 & = 0 \\ \lambda_3 + \lambda_4 & & & = 0 \\ \lambda_3 & & & = 0 \\ & \lambda_4 + \lambda_5 & & = 0 \end{Bmatrix} \;\Leftrightarrow\; \lambda_3 = \lambda_4 = \lambda_5 = 0.$$

## Exercise 2.3 (H)

Let $U \subseteq V$ be a subspace of $V$ and let $x, y \in V$. Show that

$$y \in \operatorname{span}(U \cup \{x\}) \;\wedge\; y \notin U \;\Rightarrow\; x \in \operatorname{span}(U \cup \{y\}).$$

*Solution:*
Let $y \in \operatorname{span}(U \cup \{x\})$. Then there exist $\lambda \in K$ and $u \in U$ such that

$$y = \lambda x + u,$$

see Proposition II.1.16 and note that $U$ is a subspace. If additionally $y \notin U$, we have $\lambda \neq 0$. Hence we get

$$x = \lambda^{-1} \cdot (y - u) = \lambda^{-1} y - \lambda^{-1} u \in \operatorname{span}(U \cup \{y\}).$$

## Exercise 2.4 (H)

a) Let $v_1, v_2, v_3 \in \mathbb{R}^{[0,\infty)}$ be given by

$$v_1(x) = 1, \quad v_2(x) = \sqrt{x}, \quad v_3(x) = \sin(x)$$

for $x \in [0, \infty)$. Show that $(v_1, v_2, v_3)$ is linearly independent.

*Solution:*
... (Cf. Example II.2.4.(ii) and consider, e.g., the evaluation sites $x_1 = 0$, $x_2 = \pi$, and $x_3 = \pi/2$.)

b) Let $V$ be a $\mathbb{R}$-vector space and let $(v_1, v_2)$ be linearly independent for $v_1, v_2 \in V$. Show that $(v_1 - v_2, v_1 + v_2)$ is linearly independent.

*Solution:*
Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$\lambda_1(v_1 - v_2) + \lambda_2(v_1 + v_2) = 0.$$

Then we obtain

$$(\lambda_1 + \lambda_2)\, v_1 + (-\lambda_1 + \lambda_2)\, v_2 = 0.$$

Since $(v_1, v_2)$ is linearly independent by assumption, we have

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0, \\ -\lambda_1 + \lambda_2 &= 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} &(\mathrm{I}) & \lambda_1 + \lambda_2 &= 0, \\ &(\mathrm{I}) + (\mathrm{II}) & 2\lambda_2 &= 0. \end{aligned}$$

Hence we get $\lambda_1 = \lambda_2 = 0$.

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**Exercise Sheet 3**

# Linear Algebra (AAI)

## Exercise 3.1 (H)

a) Show that $(v_1, v_2, v_3)$ given by

$$v_1 = (1,0,1), \quad v_2 = (1,1,0), \quad v_3 = (0,1,1)$$

is a basis of $\mathbb{R}^3$.

*Solution:*
Since $\dim \mathbb{R}^3 = 3$, it suffices to show that $(v_1, v_2, v_3)$ is linearly independent.
... (Cf. Exercise 2.2 c).)

b) Let $V$ be an $\mathbb{R}$-vector space with $\dim V = 3$, and let $(v_1, v_2, v_3)$ be a basis of $V$. Show that

$$(v_1 - v_3, v_1 + v_2 - v_3, v_1 + v_2 + v_3)$$

is a basis of $V$, too.

*Solution:*
... (Cf. Exercises 3.1 a) and 2.4 b).)

## Exercise 3.2 (H)

Determine a basis and the dimension of the following subspaces of $\mathbb{R}^3$:

a) $U_1 = \{(x,y,z) \in \mathbb{R}^3 \colon x = y = z = 0\}$,

*Solution:*
We have $U_1 = \{(0,0,0)\} = \operatorname{span}(\emptyset)$ and $\dim U_1 = 0$.

b) $U_2 = \{(x,y,z) \in \mathbb{R}^3 \colon x + y - z = 0\}$.

*Solution:*
For $(x,y,z) \in \mathbb{R}^3$ we have

$$(x,y,z) \in U_2 \;\Leftrightarrow\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - y \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} + \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

This shows

$$\begin{aligned} U_2 &= \{\lambda \cdot (1,0,1) + \mu \cdot (-1,1,0) \colon \lambda, \mu \in \mathbb{R}\} \\ &= \operatorname{span}\{(1,0,1), (-1,1,0)\}. \end{aligned}$$

Moreover, the family $((1,0,1), (-1,1,0))$ is linearly independent [since ...] and hence a basis of $U_2$.

## Exercise 3.3 (H)

Let $V$ be a $K$-vector space, let $(v_1, \dots, v_n)$ be a basis of $V$, and let $(w_1, \dots, w_m)$ be a generating set of $V$. Prove or disprove:

a) $(v_1, w_2, \dots, w_m)$ is a generating set of $V$.

*Solution:*
False: Choose $V = \mathbb{R}^2$, $n = m = 2$, and

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad w_1 = v_2, \quad w_2 = v_1.$$

Then, $(v_1, w_2) = (v_1, v_1)$ is not a generating set of $\mathbb{R}^2$.

b) $(v_1 + w_1, \dots, v_n + w_n)$ is a basis of $V$.

*Solution:*
False: ...

c) There exists $i \in \{1, \dots, n\}$ such that $(v_1, \dots, v_{i-1}, w_1, v_{i+1}, \dots, v_n)$ is a basis of $V$.

*Solution:*
False: Choose $m = n + 1$ and $w_1 = 0$ as well as

$$(w_2, \dots, w_{n+1}) = (v_1, \dots, v_n).$$

A family containing the zero vector is linearly dependent and hence not a basis. (Note that the statement is correct for $w_1 \neq 0$ due to Lemma II.2.10.)

## Exercise 3.4 (H)

Let $U_1, U_2$ be subspaces of $V$, and let

$$U_1 + U_2 = \{u_1 + u_2 \colon u_1 \in U_1, u_2 \in U_2\}.$$

a) Show that $U_1 + U_2$ is a subspace of $V$.

*Solution:*
... (Verify the subspace conditions from Definition II.1.5.)

b) Let $V = \mathbb{R}^n$ and $\dim U_1 = \dim U_2 = n - 1$. Determine all possible cases of $\dim U_1 \cap U_2$ and provide an explicit example for each case with $n = 3$. Cf. Exercise 1.4.

*Solution:*
By Proposition II.2.17 we have

$$\begin{aligned} \dim(U_1 \cap U_2) &= \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) \\ &= (n - 1) + (n - 1) - \dim(U_1 + U_2) \end{aligned}$$

Moreover, we have

$$0 \leq \dim(U_1 \cap U_2) \leq \min(\dim(U_1), \dim(U_2)) = n - 1$$

and

$$n - 1 = \max(\dim(U_1), \dim(U_2)) \leq \dim(U_1 + U_2) \leq \dim V = n,$$

see Corollary II.2.15. This yields the following possible cases:

| | $\dim(U_1 \cap U_2)$ | $\dim(U_1 + U_2)$ |
|---|---|---|
| case 1 | $n - 1$ | $n - 1$ |
| case 2 | $n - 2$ | $n$ |

... (examples for $n = 3$)

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**Exercise Sheet 4**

# Linear Algebra (AAI)

## Exercise 4.1 (H)

a) Consider $A \in \mathbb{R}^{4\times 3}$ and $B \in \mathbb{R}^{3\times 4}$ given by

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Transform $A$ and $B$ into row echelon form and determine row space and rank.

*Solution:*
... (Cf. `sheet4.m`.)

b) Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^5$ be given by

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Determine a basis of $\operatorname{span}(\{v_1, v_2, v_3, v_4\})$.

*Solution:*
Write $v_1^\top, v_2^\top, v_3^\top, v_4^\top$ as rows into a matrix $C \in \mathbb{R}^{4\times 5}$ and use Gaussian elimination.
... (Cf. `sheet4.m`.)

## Exercise 4.2 (H)

a) Compute $A \cdot B$ and $B \cdot A$ for $A, B$ given by Exercise 4.1a).

*Solution:*
... (Cf. `sheet4.m`.)

b) Compute every possible matrix product $C_i \cdot C_j$ for

$$C_1 = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \\ 1 & 8 & -7 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 \\ 0 \\ 8 \\ -7 \end{pmatrix},$$

$$C_4 = \begin{pmatrix} -1 & 2 & 0 & 8 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{pmatrix}.$$

*Solution:*
... (Cf. `sheet4.m`.)

c) Let $m, n, p, q \in \mathbb{N}$ and $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{p\times q}$. Verify that
$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

*Solution:*
...

## Exercise 4.3 (H)

a) Let $A \in \mathbb{R}^{4\times 4}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Express the following elementary row operations on $A$ as a matrix product $L \cdot A$ with $L \in \mathbb{R}^{4\times 4}$:
   i) multiplication of the third row of A by $\lambda$,
   ii) addition of the $\lambda$-fold of the first row of $A$ to the fourth row of $A$,
   iii) switching the second and the fourth row of $A$.

   *Solution:*
   ... (Cf. `sheet4.m`.)

b) Let $A = (a_{i,j})_{i,j} \in \mathbb{R}^{3\times 3}$ be given by
$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{pmatrix}.$$
Transform the matrix
$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 1 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3\times 6}$$
into
$$\begin{pmatrix} 1 & 0 & 0 & b_{1,1} & b_{1,2} & b_{1,3} \\ 0 & 1 & 0 & b_{2,1} & b_{2,2} & b_{2,3} \\ 0 & 0 & 1 & b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} \in \mathbb{R}^{3\times 6}$$
by using elementary row operations. Compute the matrix products $A \cdot B$ and $B \cdot A$ for $B = (b_{i,j})_{i,j} \in \mathbb{R}^{3\times 3}$.

   *Solution:*
   ... (Cf. `sheet4.m`.)

## Exercise 4.4 (H)

Let $A, B \in \mathbb{R}^{n\times n}$. Prove or disprove:

a) $\operatorname{rank}(A + B) = \operatorname{rank} A + \operatorname{rank} B$.

   *Solution:*
   False: Choose $n = 2$ and $A = B = E_2$ (identity matrix).
   ...

b) $\operatorname{rank}(A \cdot B) = \operatorname{rank} A \cdot \operatorname{rank} B$.

   *Solution:*
   False: ...

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**Exercise Sheet 5**

# Linear Algebra (AAI)

## Exercise 5.1 (H)

Prove Lemma II.4.3.(iv)-(v).

*Solution:*
See Lemma II.4.3 in the lecture notes.

## Exercise 5.2 (H)

a) Show Remark II.4.4.(i).

*Solution:*
See Remark II.4.4 in the lecture notes.

b) Let $V$ and $W$ be $K$-vector spaces, and let $F\colon V \to W$ be linear. Moreover, let $v_1, v_2 \in V$ such that $v_1 \neq v_2$ and $F(v_1) = F(v_2) \neq 0$. Show that $(v_1, v_2)$ is linearly independent.
*Hint: Consider $\lambda_1 v_1 + \lambda_2 v_2 = 0$, apply $F$ to both sides of the equation, and use Lemma II.1.4.(iii).*

*Solution:*
Let $\lambda_1, \lambda_2 \in K$ with $\lambda_1 v_1 + \lambda_2 v_2 = 0$. By applying $F$ and using $F(v_1) = F(v_2)$ we have

$$0 = F(0) = F(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 F(v_1) + \lambda_2 F(v_2) = (\lambda_1 + \lambda_2) \cdot F(v_1).$$

Since $F(v_1) \neq 0$, Lemma II.1.4.(iii) yields $\lambda_1 + \lambda_2 = 0$, or equivalently, $\lambda_2 = -\lambda_1$. Using this we obtain

$$0 = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 v_1 - \lambda_1 v_2 = \lambda_1 \cdot (v_1 - v_2).$$

Since $v_1 \neq v_2$, Lemma II.1.4.(iii) shows $\lambda_1 = 0$. Hence we get $\lambda_1 = \lambda_2 = 0$.

## Exercise 5.3 (H)

a) Let $F\colon \mathbb{R}^n \to \mathbb{R}^m$ be linear. Show that there exists a unique matrix $A \in \mathbb{R}^{m\times n}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^n$.
*Hint: According to Proposition II.4.7 a linear map is uniquely determined by the image of a basis. Use the standard basis $(e_1, \ldots, e_n)$ in $\mathbb{R}^n$ to obtain $A$. Then show that $A$ satisfies $F(x) = Ax$ for all $x \in \mathbb{R}^n$.*

*Solution:*
...

b) Determine the matrix $A \in \mathbb{R}^{3\times 3}$ from part a) for the following choices of $F$:

i) $F\big((1,0,0)^\top\big) = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad F\big((0,1,0)^\top\big) = \begin{pmatrix}2\\0\\0\end{pmatrix}, \quad F\big((0,0,1)^\top\big) = \begin{pmatrix}0\\1\\0\end{pmatrix}.$

ii) $F\Big((1,0,0)^\top\Big) = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad F\Big((1,0,1)^\top\Big) = \begin{pmatrix}2\\0\\0\end{pmatrix}, \quad F\Big((0,1,1)^\top\Big) = \begin{pmatrix}0\\1\\0\end{pmatrix}.$

*Hint: Note that the $j$-th column $a_j \in \mathbb{R}^3$ of the matrix $A$ satisfies $a_j = Ae_j$.*

*Solution:*
... (Cf. `sheet5.m`.)

## Exercise 5.4 (H)

a) Let $U$, $V$, and $W$ be $K$-vector spaces, and let $F\colon V \to W$ and $G\colon U \to V$ be linear. Show that $F \circ G$ is linear.

*Solution:*
Since $F$ and $G$ are linear, we have

$$F(G(\lambda u_1 + u_2)) = F(\lambda G(u_1) + G(u_2)) = \lambda F(G(u_1)) + F(G(u_2))$$

for $\lambda \in K$ and $u_1, u_2 \in U$.

b) Consider the situation from Exercise 5.3.

i) Determine the matrix $B \in \mathbb{R}^{3\times 3}$ that represents the linear map $F\colon \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$F\Big((1,0,0)^\top\Big) = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad F\Big((0,1,0)^\top\Big) = \begin{pmatrix}1\\0\\1\end{pmatrix}, \quad F\Big((0,0,1)^\top\Big) = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

ii) Compute $B^{-1}$ using the algorithm from Exercise 4.3 b).

iii) Compute $B^{-1}v_i$ for $i \in \{1,2,3\}$ and

$$v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \quad v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

iv) Let $A \in \mathbb{R}^{3\times 3}$ be the matrix from Exercise 5.3 b).i). Compute $AB^{-1}$ and compare the result with the matrix from Exercise 5.3 b).ii).

*Solution:*
... (Cf. `sheet5.m`.)

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**Exercise Sheet 12**

# Linear Algebra (AAI)

## Exercise 12.1 (H)

For $A \in \mathbb{R}^{2\times 2}$ we define $\langle \cdot, \cdot \rangle_A \colon \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\langle x, y\rangle_A = \langle x, Ay\rangle, \qquad x, y \in \mathbb{R}^2,$$

where $\langle \cdot, \cdot \rangle$ denotes the canonical inner product on $\mathbb{R}^2$. If $\langle \cdot, \cdot \rangle_A$ is an inner product, the corresponding induced norm and angles are denoted by $\| \cdot \|_A$ and $\measuredangle_A$, respectively.

a) Let $A_1, A_2, A_3 \in \mathbb{R}^{2\times 2}$ be given by

$$A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}, \qquad A_3 = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Determine for every $i = 1, 2, 3$ whether $\langle \cdot, \cdot \rangle_{A_i}$ is an inner product on $\mathbb{R}^2$.

*Solution:*
Note that

$$\langle \lambda x + z, y\rangle_A = (\lambda x + z)^\top Ay = \lambda \cdot x^\top Ay + z^\top Ay = \lambda \langle x, y\rangle_A + \langle z, y\rangle_A$$

for $x, y, z \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$. Hence $\langle \cdot, \cdot \rangle_A$ satisfies the linearity property for every matrix $A \in \mathbb{R}^{2\times 2}$.

We have

$$\langle x, y\rangle_A = x^\top Ay = (x^\top Ay)^\top = y^\top A^\top x = \langle y, x\rangle_{A^\top}$$

for $x, y \in \mathbb{R}^2$ and

$$\langle e_i, e_j\rangle_A = e_i^\top Ae_j = a_{ij}$$

for $i, j \in \{1, 2\}$. Thus $\langle \cdot, \cdot \rangle_A$ is symmetric if and only if $A$ is symmetric, i.e., $A = A^\top$.

If $A$ is diagonalizable with $A = SDS^\top$ (this is possible for symmetric matrices), we have

$$\langle x, x\rangle_A = x^\top SDS^\top x = (S^\top x)^\top D(S^\top x) = \langle S^\top x, S^\top x\rangle_D$$

for $x \in \mathbb{R}^{2\times 2}$. In this case $\langle \cdot, \cdot \rangle_A$ is positive definite if and only if $\sigma(A) \subseteq \,]0, \infty[$, i.e., all eigenvalues of $A$ are positive.

...

For $x = (x_1, x_2)^\top \in \mathbb{R}^2$ we have

$$\begin{aligned}
\langle x, x\rangle_{A_3} &= 3x_1^2 + 2x_1x_2 + 3x_2^2 \\
&= 2x_1^2 + 2x_2^2 + (x_1 + x_2)^2 \\
&\geq 2x_1^2 + 2x_2^2 \\
&= 2\langle x, x\rangle.
\end{aligned}$$

Hence $\langle \cdot, \cdot \rangle_{A_3}$ is positive-definite.

b) Compute $\|x\|$ and $\|x\|_{A_3}$ for $x \in \{(-1,1)^\top, (1,0)^\top\}$. Moreover, compute $\measuredangle(x,y)$ and $\measuredangle_{A_3}(x,y)$ for $x = (-1,1)^\top$ and $y = (1,0)^\top$.

*Solution:*

...

We have

$$\measuredangle_{A_3}(x,y) = \arccos\left(\frac{\langle x,y\rangle_{A_3}}{\|x\|_{A_3} \cdot \|y\|_{A_3}}\right) = \arccos\left(\frac{-2}{2\cdot\sqrt{3}}\right) \approx 125{,}3^\circ.$$

c) Determine the orthogonal complement of $\{(1,0)^\top\}$ w.r.t. $\langle\cdot,\cdot\rangle$ and $\langle\cdot,\cdot\rangle_{A_3}$.

*Solution:*

...

For $x = (x_1, x_2)^\top \in \mathbb{R}^2$ we have

$$0 = \langle e_1, x\rangle_{A_3} = e_1^\top \left(x_1 \cdot (3,1)^\top + x_2 \cdot (1,3)^\top\right) = 3x_1 + x_2$$

and hence $\{e_1\}^\perp = \operatorname{span}(\{(-1,3)^\top\})$ w.r.t. $\langle\cdot,\cdot\rangle_{A_3}$.

## Exercise 12.2 (H)

Let $V = C([0,1])$. We define $\langle\cdot,\cdot\rangle \colon V \times V \to \mathbb{R}$ by

$$\langle f, g\rangle = \int_0^1 f(x)\cdot g(x)\,dx, \qquad f,g \in V.$$

a) Show that $\langle\cdot,\cdot\rangle$ is an inner product on $V$.
*Hint: Use $\varepsilon$-$\delta$-characterization of continuity to show positive-definiteness.*

*Solution:*
Standard integration rules [...] show that $\langle\cdot,\cdot\rangle$ is a symmetric bilinear form.
Positive-definiteness: Let $f \in V$. We clearly have

$$\langle f, f\rangle = \int_0^1 (f(x))^2\,dx \geq 0.$$

Let $f \neq 0$. Then there exists $x_0 \in [0,1]$ and $\varepsilon_0 > 0$ such that $(f(x_0))^2 = \varepsilon_0$. Since $f$ (and thus $f \cdot f$) is continuous, we have

$$\exists\,\delta > 0\ \forall\, x \in [0,1] \colon (|x - x_0| \leq \delta \ \Rightarrow\ |(f(x))^2 - (f(x_0))^2| \leq \varepsilon_0/2).$$

Thus we have $(f(x))^2 \geq \varepsilon_0/2 > 0$ for all $x \in [x_0 - \delta, x_0 + \delta] \cap [0,1]$ and

$$\langle f, f\rangle \geq \int_{[x_0-\delta,x_0+\delta]\cap[0,1]} (f(x))^2\,dx \geq \delta \cdot \varepsilon_0/2 > 0.$$

b) Determine an orthonormal basis of the subspace $\Pi_2 \subseteq V$ (where the domain is restricted to $[0,1]$).

*Solution:*
Use Gram-Schmidt orthonormalization for the basis $(w_0, w_1, w_2)$ of $\Pi_2$ given by

$$w_0(x) = 1, \qquad w_1(x) = x, \qquad w_2(x) = x^2$$

for $x \in [0,1]$:

$$\begin{aligned}
v_0(x) &= w_0(x) \cdot (\langle w_0, w_0\rangle)^{-1/2} = \Big(\int_0^1 1^2\,dx\Big)^{-1/2} = 1,\\
\tilde v_1(x) &= w_1(x) - \langle w_1, v_0\rangle \cdot v_0(x) = x - \int_0^1 x\cdot 1\,dx = x - 1/2,\\
v_1(x) &= \tilde v_1(x) \cdot (\langle \tilde v_1, \tilde v_1\rangle)^{-1/2} = (x-1/2)\cdot \Big(\int_0^1 (x-1/2)^2\,dx\Big)^{-1/2}\\
&= (x-1/2)\cdot\sqrt{12},\\
\tilde v_2(x) &= w_2(x) - \langle w_2, v_0\rangle \cdot v_0(x) - \langle w_2, v_1\rangle \cdot v_1(x) = \dots,\\
v_2(x) &= \tilde v_2(x) \cdot (\langle \tilde v_2, \tilde v_2\rangle)^{-1/2} = \dots.
\end{aligned}$$

Then $(v_0, v_1, v_2)$ is an ONB of $\Pi_2$.

c) Let $f \in V$ be given by $f(x) = \exp(x)$ for $x \in [0,1]$. Determine the orthogonal projection of $f$ onto $\Pi_2$.

*Solution:*
According to Proposition IV.1.20 the orthogonal projection of $f$ onto $\Pi_2$ is given by

$$F(f) = \sum_{i=0}^{2} \langle f, v_i\rangle\, v_i,$$

where $(v_0, v_1, v_2)$ is an ONB of $\Pi_2$. Taking the ONB $(v_0, v_1, v_2)$ from part b) we obtain

$$\begin{aligned}
F(f)(x) &= \langle f, v_0\rangle\, v_0(x) + \langle f, v_1\rangle\, v_1(x) + \langle f, v_2\rangle\, v_2(x)\\
&= \int_0^1 \exp(x)\cdot 1\,dx\\
&\quad + (x-1/2)\cdot\sqrt{12}\cdot \int_0^1 \exp(x)\cdot (x-1/2)\cdot\sqrt{12}\,dx\\
&\quad + \dots\\
&= \dots
\end{aligned}$$

## Exercise 12.3 (H)

Prove Remark IV.1.19 and Remark IV.1.27.

*Solution:*
...

## Exercise 12.4 (H)

Let $m, n \in \mathbb{N}$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m\times n}$ with $n \le m$.

a) Show that $\operatorname{rank}(A^\top A) = \operatorname{rank} A$.
*Hint: Use the Rank-Nullity Theorem and show that* $\ker A = \ker(A^\top A)$. *Note that* $Ax = 0$ *if and only if* $\langle Ax, Ax\rangle = 0$.

*Solution:*
The Rank-Nullity Theorem applied to $\mathcal{F}_A\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathcal{F}_{A^\top A}\colon \mathbb{R}^n \to \mathbb{R}^n$ shows

$$\dim\ker A + \operatorname{rank} A = n = \dim\ker(A^\top A) + \operatorname{rank}(A^\top A)$$

and hence

$$\operatorname{rank}(A^\top A) = \operatorname{rank} A \;\Leftrightarrow\; \dim\ker A = \dim\ker(A^\top A).$$

We will show $\ker A = \ker(A^\top A)$:

"$\ker A \subseteq \ker(A^\top A)$": Let $x \in \mathbb{R}^n$ such that $Ax = 0$. Then we have

$$(A^\top A)x = A^\top(Ax) = A^\top 0 = 0.$$

"$\ker A \supseteq \ker(A^\top A)$": Let $x \in \mathbb{R}^n$ such that $A^\top Ax = 0$. Then we have

$$\langle Ax, Ax\rangle = (Ax)^\top(Ax) = x^\top A^\top Ax = x^\top 0 = 0.$$

Due to the positive-definiteness of $\langle\cdot,\cdot\rangle$ we have $Ax = 0$.

b) Let $\operatorname{rank} A = n$. Show[1] that $\langle x, A^\top Ax\rangle > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.
*Hint: Is $\mathcal{F}_A$ injective?*

*Solution:*
The Rank-Nullity Theorem applied to $\mathcal{F}_A\colon \mathbb{R}^n \to \mathbb{R}^m$ shows

$$\dim\ker A = n - \operatorname{rank} A = n - n = 0$$

such that $\ker A = \{0\}$. For $x \in \mathbb{R}^n \setminus \{0\}$ we thus have

$$\langle x, A^\top Ax\rangle = x^\top A^\top Ax = (Ax)^\top Ax = \langle Ax, Ax\rangle > 0$$

due to the positive-definiteness of $\langle\cdot,\cdot\rangle$.

c) Let $\operatorname{rank} A = n$. Show that the function

$$\Delta\colon \mathbb{R}^n \to \mathbb{R}, \quad x \mapsto \|Ax - b\|$$

has a unique minimizer $x^* \in \mathbb{R}^n$ satisfying $A^\top Ax^* = A^\top b$.
*Hint: The function $f\colon \mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \|Ax - b\|^2$ is differentiable.*

*Solution:*
We have

$$\begin{aligned} f(x) = \langle Ax - b, Ax - b\rangle &= \langle Ax, Ax\rangle - 2\langle Ax, b\rangle + \langle b, b\rangle \\ &= \langle x, A^\top Ax\rangle - 2\langle x, A^\top b\rangle + \langle b, b\rangle. \end{aligned}$$

A necessary condition for a minimizer $x^* \in \mathbb{R}^n$ is given by

$$0 = (\operatorname{grad} f)(x^*) = 2A^\top Ax^* - 2A^\top b.$$

Hence we have

$$A^\top Ax^* = A^\top b,$$

or equivalently,

$$x^* = (A^\top A)^{-1}A^\top b$$

due to the invertibility of $A^\top A \in \mathbb{R}^{n\times n}$, see part a).
... [Hessian matrix: Note that $A^\top A$ is positive-definite due to part b).]

d) Consider the following data $(x_i, y_i) \in \mathbb{R}^2$ for $i = 1, \dots, 4$:

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $x_i$ | 0 | 1 | 2 | 3 |
| $y_i$ | 2 | 0.5 | 3 | 4.5 |

Determine $f \in \Pi_1$ such that

$$\left(\tilde{\Delta}(f)\right)^2 = \sum_{i=1}^{4}\left(y_i - f(x_i)\right)^2$$

is minimal.

[1] A matrix $B \in \mathbb{R}^{n\times n}$ is called *symmetric* if $B = B^\top$. Moreover, a symmetric matrix $B \in \mathbb{R}^{n\times n}$ is called *positive-definite* if $\langle x, Bx\rangle > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$.

*Solution:*
Let $f(x) = a_0 + a_1 x$ for $a_0, a_1 \in \mathbb{R}$. We have

$$\left(\tilde{\Delta}(f)\right)^2 = \|Ay - b\|^2$$

with

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \qquad y = \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 0.5 \\ 3 \\ 4.5 \end{pmatrix}.$$

Note that $\operatorname{rank} A = 2$. According to part c) the unique minimizer $(a_0^*, a_1^*)^\top \in \mathbb{R}^2$ is given by

$$\begin{pmatrix} a_0^* \\ a_1^* \end{pmatrix} = (A^\top A)^{-1} \cdot A^\top b = \frac{1}{10} \begin{pmatrix} 7 & -3 \\ -3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 10 \\ 20 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

i.e., $f(x) = 1 + x$.