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Differential Calculus in $\mathbb{R}^n$

In Analysis 1 we have discussed differentiation of functions of 1 variable.

Now we consider a <u>real-valued</u> function in <u>several variables</u>

$$f : D \subseteq \mathbb{R}^n \to \mathbb{R}$$
$$x = (x_1, \ldots, x_n)^\top \mapsto f(x_1, \ldots, x_n) = f(x)$$

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Later we are going to extend the differential calculus to vector-valued functions (of several variables)

$$\begin{aligned} f : D \subseteq \mathbb{R}^n &\to \mathbb{R}^m \\ x = (x_1, \ldots, x_n)^\top &\mapsto (f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n))^\top \end{aligned}$$

Special Case: n = 2

A real-valued function in 2 variables

$$f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$$
$$x = (x_1, x_2)^\top \mapsto f(x_1, x_2)$$

We may plot the function value as 3rd coordinate over the real plane $\mathbb{R}^2$.
The graph of f is a subset of $\mathbb{R}^3$: a "landscape" or "mountains".

Contour Line, Equipotential Surface, ...

Definition (Level set)

We define the **level set** of a function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $(x_1, \ldots, x_n)^\top \mapsto f(x_1, \ldots, x_n)$ for the function value $c \in \mathbb{R}$ as the set

$$N_c := \{x \in D \,|\, f(x) = c\}.$$

The structure of N_c may be "complicated", it might also be the empty set.

For $n = 2$ the level set is called a contour line (though it may be an area, e.g.),
for $n = 3$ the level set is called an equipotential surface (though it may be an solid, e.g.).

Partial Function: Definition

A partial function is a "cross section"-function that is obtained by freezing all but 1 variables, e.g. x_i:

$$\begin{aligned} g_i : D_i \subseteq \mathbb{R} &\to \mathbb{R} \\ x_i &\mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n), \end{aligned}$$

with $(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) \in D$ for all $x_i \in D_i$, $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n$ fixed.

Partial Function: Example

$$f(x_1, x_2) = x_1^2 + x_2^2$$

Partial Derivatives: Definition

Definition (Partial derivative)

Let $f : D \subseteq \mathbb{R}^n \to \mathbb{R}$, $x = (x_1, \ldots, x_n)^\top \mapsto f(x_1, \ldots, x_n)$ and $a \in D$, D an open set.

If the derivative of the partial function

$$x_i \mapsto f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$$

exists at $x_i = a_i$,
then we call it the **partial derivative** of f w.r.t. x_i at a.

We write:

$$\frac{\partial f}{\partial x_i}(a) \quad \text{or} \quad f'_{x_i}(a) \quad \text{or} \quad \ldots$$

We say f is **partially differentiable** in a, if all $\frac{\partial f}{\partial x_i}(a)$ exist.
We say f is partially differentiable in $E \subseteq D$, if f is partially differentiable at any $a \in E$.

Partial Derivatives: Examples