

# Complex Representation

$$c_k := \begin{cases} a_k - ib_k & , k \in \mathbb{N} \\ a_0/2 & , k = 0 \\ a_{-k} + ib_{-k} & , k \in \mathbb{N} \end{cases}$$

Analysis 2

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Introduction

Power series

Sequences of Functions

Uniform Convergence

Continuity and Uniform Convergence

Power Series

Taylor Series

Fourier Series

Differentiation in Higher Dimensions

Integration in Higher Dimensions

Further Topics in Calculus

Summary - Outlook and Review

By the complex exponential function we may rewrite

$$F(x) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega x) \quad \text{with } \omega := \frac{2\pi}{T}$$

where the complex Fourier coefficient  $c_k$ ,  $k \in \mathbb{Z}$ , is given by

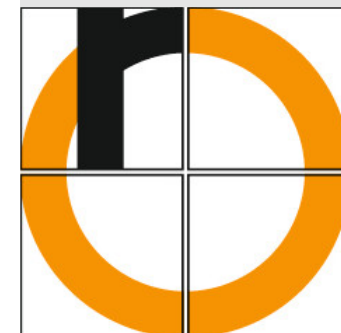
$$c_k = \frac{1}{T} \int_0^T f(x) \exp(-ik\omega x) dx.$$

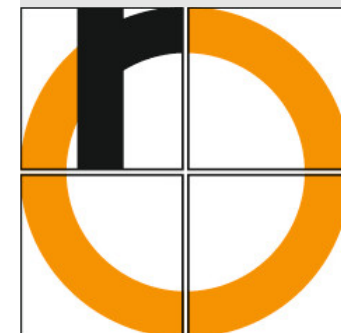
in particular  $c_0 = \frac{1}{T} \int_0^T f(x) dx = \frac{a_0}{2}$

Note that the limit is to be understood symmetrically:

$$\sum_{k=-\infty}^{\infty} c_k \exp(ik\omega x) := \lim_{n \rightarrow \infty} \sum_{k=-n}^n c_k \exp(ik\omega x).$$

$n=2$   
 $\approx c_0 + c_1 \exp(i\omega x) + c_{-1} \exp(-i\omega x) + c_2 \exp(2i\omega x) + c_{-2} \exp(-2i\omega x)$





## Theorem (Orthonormality Relations)

If  $m, n \in \mathbb{N}$ , then:

Kronecker - delta

$$\frac{2}{T} \int_0^T \sin(m\omega x) \sin(n\omega x) dx = \delta_{m,n} = \begin{cases} 1, & m=n \\ 0, & m \neq n \end{cases}$$

$$\frac{2}{T} \int_0^T \cos(m\omega x) \cos(n\omega x) dx = \delta_{m,n},$$

$$\frac{2}{T} \int_0^T \sin(m\omega x) \cos(n\omega x) dx = 0,$$

and if  $m, n \in \mathbb{Z}$ , then:

$$\frac{1}{T} \int_0^T \exp(im\omega x) \exp(-in\omega x) dx = \delta_{m,n}.$$

## Definition (Piecewise Continuously Differentiable Function)

Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  with

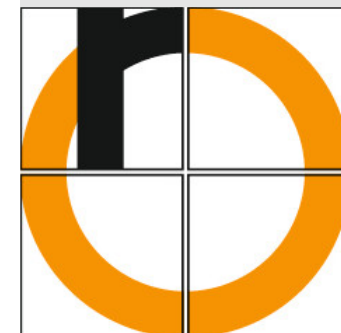
- (i) Only at a finite number of **singularities** the function  $f$  is not continuously differentiable.
- (ii) At any singularity  $x_0$  there exist the following one-sided limits:

$$f(x_0+) := \lim_{x \rightarrow x_0+} f(x) \quad f(x_0-) := \lim_{x \rightarrow x_0-} f(x),$$

$$f'(x_0+) := \lim_{x \rightarrow x_0+} f'(x) \quad f'(x_0-) := \lim_{x \rightarrow x_0-} f'(x).$$

Then  $f$  is called **piecewise continuously differentiable**.

Discontinuities are singularities, but not any singularity is a discontinuity.



## Theorem (Convergence of Fourier Series)

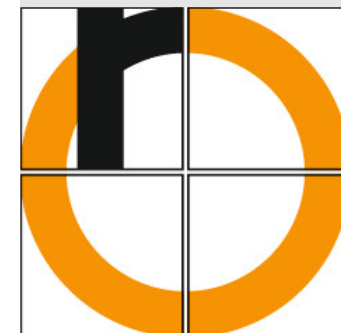
*Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a periodic function that is piecewise continuously differentiable. Then:*

- *The Fourier series  $F$  converges at any  $x$  that is not a singularity to  $f$ .*
- *At any singularity  $x_0$  the Fourier series converges to the “mean value” of the jump*

$$\frac{1}{2} (f(x_0+) + f(x_0-)).$$

- *In any compact interval that does not contain a discontinuity, the convergence of  $F$  to  $f$  is uniform.*

Note that there exist periodic, continuous functions, whose Fourier series does not converge to  $f$ !



# Properties of Fourier Series I

Let  $f$  and  $g$  be piecewise continuous, periodic functions with Fourier series  $F = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega x)$ ,  $G = \sum_{k=-\infty}^{\infty} d_k \exp(ik\omega x)$ , resp. There holds:

- For any  $\alpha, \beta \in \mathbb{R}$ :

$$\alpha F(x) + \beta G(x) = \sum_{k=-\infty}^{\infty} (\alpha c_k + \beta d_k) \exp(ik\omega x)$$

*works for the trigonometric Fourier series as well*

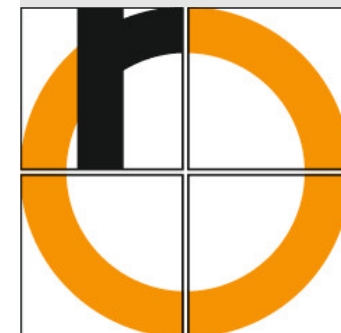
- $F(-x) = \sum_{k=-\infty}^{\infty} c_{-k} \exp(ik\omega x)$
- For any  $\alpha \in \mathbb{R}$

$$F(\alpha x) = \sum_{k=-\infty}^{\infty} c_k \exp(ik\omega \alpha x)$$

- For any  $\alpha \in \mathbb{R}$

$$F(\alpha + x) = \sum_{k=-\infty}^{\infty} (c_k \exp(ik\omega \alpha)) \exp(ik\omega x)$$

*new Fourier coefficient*



# Properties of Fourier Series II

Let  $f$  be a piecewise continuous **differentiable**, periodic function with Fourier series  $F = \sum_{k=-\infty}^{\infty} c_k \exp(\cancel{i}k\omega x)$ . There holds:

- The Fourier series  $F'$  represents  $f'$ :

$$F'(x) = \sum_{k=-\infty}^{\infty} (ik\omega c_k) \exp(ik\omega x)$$

- $F(-x) = \sum_{k=-\infty}^{\infty} c_{-k} \exp(ik\omega x)$
- Suppose  $c_0 = 0$ , then the Fourier series  $\tilde{F} := \int F(\xi) d\xi$  represents  $\tilde{f} := \int f(\xi) d\xi$ :

$$\tilde{F}(x) = \overset{1}{\cancel{2}} \frac{1}{T} \int_0^T \tilde{f}(\xi) d\xi + \sum_{k=-\infty, k \neq 0}^{\infty} \frac{c_k}{ik\omega} \exp(ik\omega x)$$

