

Exercise Sheet 0

Linear Algebra (AAI)

Exercise 0.1 (H)

Verify that $G = \{0, 1\}$ together with

$*$	0	1
0	0	1
1	1	0

is a commutative group, see Example 3(iii) in Section I.1.

Solution:

By definition, we have $*$: $G \times G \rightarrow G$. Moreover, $(G, *)$ satisfies:

i) Associativity: We have

$$\forall a, b, c \in G: (a * b) * c = a * (b * c)$$

since (there are 2^3 cases)

$$\begin{aligned} (0 * 0) * 0 &= 0 = 0 * (0 * 0), \\ (0 * 0) * 1 &= 1 = 0 * (0 * 1), \\ (0 * 1) * 0 &= 1 = 0 * (1 * 0), \\ (0 * 1) * 1 &= 0 = 0 * (1 * 1), \\ (1 * 0) * 0 &= 1 = 1 * (0 * 0), \\ (1 * 0) * 1 &= 0 = 1 * (0 * 1), \\ (1 * 1) * 0 &= 0 = 1 * (1 * 0), \\ (1 * 1) * 1 &= 1 = 1 * (1 * 1). \end{aligned}$$

ii) The neutral element is given by 0 since $0 * 0 = 0$ and $0 * 1 = 1$.

iii) The inverse elements are given by $-1 = 1$ and $-0 = 0$ since

$$\begin{aligned} 1 * 1 &= 0, \\ 0 * 0 &= 0. \end{aligned}$$

iv) Commutativity: We have

$$\forall a, b \in G: a * b = b * a$$

since $0 * 1 = 1 = 1 * 0$.

Exercise 0.2 (H)

Prove Proposition 10 from Section I.1.

Solution:

See Proposition I.1.10 in the lecture notes.

Exercise 0.3 (H)

Prove Lemma 12 from Section I.1.

Solution:

See Lemma I.1.12 in the lecture notes.

Exercise 0.4 (H)

Let G_1 and G_2 be subgroups of a group G . Show that $G_1 \cup G_2$ is not a subgroup in general.
Hint: Cf. Example 18 from Section I.1.

Solution:

See Example I.1.18 in the lecture notes.

Exercise Sheet 1

Linear Algebra (AAI)

Exercise 1.1 (H)

Prove Lemma I.2.4.

Solution:

See Lemma I.2.4 in the lecture notes.

Exercise 1.2 (H)

a) Verify that $V = \mathbb{R}^n$ is a \mathbb{R} -vector space.

Solution:

By using the field properties of \mathbb{R} in every coordinate we obtain:

1.) (\mathbb{R}^n, \oplus) is a commutative group, i.e., we have $\oplus: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ and the following properties:

i) Associativity: We have

$$\forall x, y, z \in \mathbb{R}^n: (x \oplus y) \oplus z = x \oplus (y \oplus z)$$

since

$$\begin{aligned} (x \oplus y) \oplus z &= ((x_1 + y_1) + z_1, \dots, (x_n + y_n) + z_n), \\ &= (x_1 + (y_1 + z_1), \dots, x_n + (y_n + z_n)) \\ &= x \oplus (y \oplus z). \end{aligned}$$

ii) The neutral element is given by $\mathbf{0} = (0, \dots, 0)$ since

$$\mathbf{0} \oplus x = (0 + x_1, \dots, 0 + x_n) = (x_1, \dots, x_n) = x$$

for all $x \in \mathbb{R}^n$.

iii) The inverse element of $x \in \mathbb{R}^n$ is given by $-x = (-x_1, \dots, -x_n)$ since

$$\begin{aligned} -x \oplus x &= (-x_1, \dots, -x_n) \oplus (x_1, \dots, x_n) \\ &= (-x_1 + x_1, \dots, -x_n + x_n) \\ &= (0, \dots, 0) = \mathbf{0}. \end{aligned}$$

iv) Commutativity: We have

$$\forall x, y \in \mathbb{R}^n: x \oplus y = y \oplus x$$

since

$$\begin{aligned} x \oplus y &= (x_1 + y_1, \dots, x_n + y_n) \\ &= (y_1 + x_1, \dots, y_n + x_n) \\ &= y \oplus x. \end{aligned}$$

2.) For all $\lambda, \mu \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$ we have

$$\begin{aligned}
(\lambda + \mu) \odot x &= ((\lambda + \mu) \cdot x_1, \dots, (\lambda + \mu) \cdot x_n) \\
&= (\lambda x_1 + \mu x_1, \dots, \lambda x_n + \mu x_n) \\
&= (\lambda \cdot x_1, \dots, \lambda \cdot x_n) \oplus (\mu \cdot x_1, \dots, \mu \cdot x_n) \\
&= \lambda \odot x \oplus \mu \odot x, \\
\lambda \odot (x \oplus y) &= (\lambda \cdot (x_1 + y_1), \dots, \lambda \cdot (x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, \dots, \lambda x_n + \lambda y_n) \\
&= (\lambda \cdot x_1, \dots, \lambda \cdot x_n) \oplus (\lambda \cdot y_1, \dots, \lambda \cdot y_n) \\
&= \lambda \odot x \oplus \lambda \odot y, \\
\lambda \odot (\mu \odot x) &= \lambda \odot (\mu \cdot x_1, \dots, \mu \cdot x_n) \\
&= (\lambda \cdot \mu \cdot x_1, \dots, \lambda \cdot \mu \cdot x_n) \\
&= ((\lambda \cdot \mu) \cdot x_1, \dots, (\lambda \cdot \mu) \cdot x_n) \\
&= (\lambda \cdot \mu) \odot x, \\
1 \odot x &= (1 \cdot x_1, \dots, 1 \cdot x_n) \\
&= (x_1, \dots, x_n) \\
&= x.
\end{aligned}$$

b) Is \mathbb{C}^n a \mathbb{R} -vector space (based on (1) from Example II.1.2)?

Solution:

The scalar multiplication

$$\odot: \mathbb{R} \times \mathbb{C}^n \rightarrow \mathbb{C}^n$$

is given by the corresponding operation on $\mathbb{C} \times \mathbb{C}^n$ (\mathbb{C}^n seen as a \mathbb{C} -vector space), where the scalars are restricted to the subset $\mathbb{R} \subseteq \mathbb{C}$. Since \mathbb{C}^n is a \mathbb{C} -vector space, all vector space conditions are also met for this restriction.

Exercise 1.3 (H)

Let $V = \mathbb{R}^{[0,1]}$.

a) Verify that V is a \mathbb{R} -vector space.

Solution:

By using the field properties of \mathbb{R} in every coordinate (i.e., for every evaluation site $x \in [0, 1]$) we obtain the analogous properties as in Exercise 1.2 a). [...]

b) Show that $C([0, 1])$ is a subspace of V .

Solution:

We verify the subspace conditions from Definition II.1.5:

- i) $\mathbf{0} \in \mathbb{R}^{[0,1]}$ given by $\mathbf{0}(x) = 0$ for all $x \in [0, 1]$ (= zero function) is continuous and hence $C([0, 1]) \neq \emptyset$.
- ii) Since the sum of two continuous functions is again a continuous function (see Analysis I), the set $C([0, 1])$ is closed w.r.t. the vector addition.
- iii) Since a continuous function multiplied by a constant is again a continuous function (see Analysis I), the set $C([0, 1])$ is closed w.r.t. the scalar multiplication.

Exercise 1.4 (H)

Consider Example II.1.7.(i) with $a \in \mathbb{R}^n$, $b \in \mathbb{R}$, and

$$U = \left\{ x \in \mathbb{R}^n : \sum_{i=1}^n a_i \cdot x_i = b \right\}.$$

a) Show that U is a subspace of \mathbb{R}^n if and only if $b = 0$.

Solution:

“ \Rightarrow ”: Let U be a subspace and $x \in U$, i.e., we have

$$\sum_{i=1}^n a_i \cdot x_i = b$$

and hence

$$(1) \quad \sum_{i=1}^n a_i \cdot 2x_i = 2 \cdot \sum_{i=1}^n a_i \cdot x_i = 2b.$$

Since U is a subspace, we also have $2x \in U$ (see Definition II.1.5.(iii)) and thus

$$(2) \quad \sum_{i=1}^n a_i \cdot 2x_i = b.$$

Combining (1) and (2) we obtain $2b = b$, which implies $b = 0$.

“ \Leftarrow ”: Let $b = 0$. We verify the subspace conditions from Definition II.1.5:

i) We have

$$\sum_{i=1}^n a_i \cdot 0 = 0$$

and hence $\mathbf{0} = (0, \dots, 0) \in U$ such that $U \neq \emptyset$.

ii) Let $x, y \in U$. Then we have

$$\sum_{i=1}^n a_i \cdot (x_i + y_i) = \sum_{i=1}^n a_i \cdot x_i + \sum_{i=1}^n a_i \cdot y_i = 0 + 0 = 0$$

and hence $x + y \in U$.

iii) Let $x \in U$ and $\lambda \in \mathbb{R}$. Then we have

$$\sum_{i=1}^n a_i \cdot (\lambda \cdot x_i) = \lambda \cdot \sum_{i=1}^n a_i \cdot x_i = \lambda \cdot 0 = 0$$

and hence $\lambda \cdot x \in U$.

b) Let $n = 2$. Determine and illustrate U for all combinations of $a \in \{(1, 1), (-1, 0)\}$ and $b \in \{0, 1, -2\}$.

Solution:

...

c) Show Remark II.1.10 (regarding the union of subspaces).

Solution:

Consider $U_1 = \{x \in \mathbb{R}^2 : x_1 = 0\}$ and $U_2 = \{x \in \mathbb{R}^2 : x_2 = 0\}$. Then U_1 and U_2 are subspaces of \mathbb{R}^2 according to part a). However, $U_1 \cup U_2$ is not a subspace since

$$(0, 1) \in U_1 \subseteq U_1 \cup U_2,$$

$$(1, 0) \in U_2 \subseteq U_1 \cup U_2,$$

and

$$(0, 1) + (1, 0) = (1, 1) \notin U_1 \cup U_2.$$

Exercise Sheet 2

Linear Algebra (AAI)

Exercise 2.1 (H)

Determine whether U is a subspace of \mathbb{R}^3 :

- a) $U = \{(x, y, z) \in \mathbb{R}^3 : z = 3x - y\}$,
- b) $U = \{(x, y, z) \in \mathbb{R}^3 : x \cdot y \cdot z = 0\}$,
- c) $U = \{(x, y, z) \in \mathbb{R}^3 : x \cdot (\exp(y) + z) = 0\}$,
- d) $U = \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0\}$,
- e) $U = \{(\lambda, 2\lambda, 4\lambda) \in \mathbb{R}^3 : \lambda \in \mathbb{R}\}$.

Solution:

The sets from a) and e) are subspaces of \mathbb{R}^3 .

... (Verify the subspace conditions from Definition II.1.5.)

The sets from b), c), and d) are not subspaces of \mathbb{R}^3 . As an example we show that b) is not a subspace: Note that $(1, 1, 0), (0, 0, 1) \in U$, but

$$(1, 1, 0) + (0, 0, 1) = (1, 1, 1) \notin U.$$

Hence condition (ii) from Definition II.1.5 is not satisfied.

Exercise 2.2 (H)

Consider $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^4$ given by

$$v_1 = (1, 2, 1, 2), \quad v_2 = (1, 1, 1, 1), \quad v_3 = (0, 1, 1, 0), \quad v_4 = (0, 1, 0, 1), \quad v_5 = (1, 0, 0, 1).$$

- a) Express each vector v_i as a linear combination of the remaining vectors v_j with $j \neq i$ (if possible).

Solution:

We have $v_1 = v_2 + v_4$ and $v_2 = v_3 + v_5$. Using these two equalities we obtain, e.g.,

$$v_1 = v_2 + v_4,$$

$$v_2 = v_3 + v_5,$$

$$v_3 = v_2 - v_5,$$

$$v_4 = v_1 - v_2,$$

$$v_5 = v_2 - v_3.$$

- b) Prove or disprove:

- i) $\text{span}(\{v_2, v_3, v_5\}) = \text{span}(\{v_3, v_5\})$,
- ii) $\text{span}(\{v_2, v_3, v_5\}) = \text{span}(\{v_1, v_3, v_5\})$,
- iii) $\text{span}(\{v_2, v_5\}) = \text{span}(\{v_3, v_5\})$,
- iv) $\text{span}(\{v_1, v_5\}) = \text{span}(\{v_1, v_2, v_3, v_4, v_5\})$,

v) $\text{span}(\{v_1, v_2, v_4\}) = \text{span}(\{v_2, v_3, v_5\})$.

Solution:

Statements i) and iii) are correct. Statements ii), iv), and v) are false.

ad i): We clearly have

$$\text{span}(\{v_2, v_3, v_5\}) = L(v_2, v_3, v_5) \supseteq L(v_3, v_5) = \text{span}(\{v_3, v_5\}),$$

see Corollary II.1.17. Moreover, every linear combination of v_2, v_3, v_5 can be expressed as a linear combination of v_3, v_5 since $v_2 = v_3 + v_5$ and thus

$$\lambda_2 v_2 + \lambda_3 v_3 + \lambda_5 v_5 = \lambda_2(v_3 + v_5) + \lambda_3 v_3 + \lambda_5 v_5 = (\lambda_2 + \lambda_3)v_3 + (\lambda_2 + \lambda_5)v_5$$

for $\lambda_2, \lambda_3, \lambda_5 \in \mathbb{R}$. Hence we get

$$\text{span}(\{v_2, v_3, v_5\}) = L(v_2, v_3, v_5) \subseteq L(v_3, v_5) = \text{span}(\{v_3, v_5\}).$$

ad ii): Using part (i) we obtain

$$\text{span}(\{v_2, v_3, v_5\}) = \text{span}(\{v_3, v_5\}) \subseteq \text{span}(\{v_1, v_3, v_5\}).$$

Moreover, we have

$$\begin{aligned} \text{span}(\{v_3, v_5\}) &= L(v_3, v_5) = \{\lambda v_3 + \mu v_5 : \lambda, \mu \in \mathbb{R}\} \\ &= \{(\mu, \lambda, \lambda, \mu) : \lambda, \mu \in \mathbb{R}\}. \end{aligned}$$

This shows $v_1 = (1, 2, 1, 2) \notin \text{span}(\{v_3, v_5\})$ (Note that the first and the last entry of v_1 are not equal.) and hence

$$\text{span}(\{v_2, v_3, v_5\}) = \text{span}(\{v_3, v_5\}) \not\supseteq \text{span}(\{v_1, v_3, v_5\}).$$

ad iii): ...

ad iv): ...

ad v): ...

c) Determine all linearly independent families $(v_i)_{i \in I}$ with $I \subseteq \{1, \dots, 5\}$ and $|I| \leq 3$.

Solution:

$|I| = 0$: $(v_i)_{i \in \emptyset}$ is linearly independent per definition.

$|I| = 1$: (v_i) is linearly independent if and only if $v_i \neq 0$. Hence (v_i) is linearly independent for every $i \in \{1, \dots, 5\}$.

$|I| = 2$: By Lemma II.2.3 we have

$$(v_i, v_j) \text{ linearly independent} \Leftrightarrow v_i \notin L(v_j) \wedge v_j \notin L(v_i).$$

Hence (v_i, v_j) is linearly independent for all $i, j \in \{1, \dots, 5\}$ with $i \neq j$.

$|I| = 3$: Use Lemma II.2.3 for each of the 10 combinations: The families (v_1, v_2, v_4) and (v_2, v_3, v_5) are linearly dependent according to part a). E.g., (v_3, v_4, v_5) is linearly independent since for $\lambda_3, \lambda_4, \lambda_5 \in \mathbb{R}$ we have

$$\lambda_3 v_3 + \lambda_4 v_4 + \lambda_5 v_5 = 0 \Leftrightarrow \begin{cases} \lambda_5 = 0 \\ \lambda_3 + \lambda_4 = 0 \\ \lambda_3 = 0 \\ \lambda_4 + \lambda_5 = 0 \end{cases} \Leftrightarrow \lambda_3 = \lambda_4 = \lambda_5 = 0.$$

Exercise 2.3 (H)

Let $U \subseteq V$ be a subspace of V and let $x, y \in V$. Show that

$$y \in \text{span}(U \cup \{x\}) \wedge y \notin U \Rightarrow x \in \text{span}(U \cup \{y\}).$$

Solution:

Let $y \in \text{span}(U \cup \{x\})$. Then there exist $\lambda \in K$ and $u \in U$ such that

$$y = \lambda x + u,$$

see Proposition II.1.16 and note that U is a subspace. If additionally $y \notin U$, we have $\lambda \neq 0$. Hence we get

$$x = \lambda^{-1} \cdot (y - u) = \lambda^{-1}y - \lambda^{-1}u \in \text{span}(U \cup \{y\}).$$

Exercise 2.4 (H)

a) Let $v_1, v_2, v_3 \in \mathbb{R}^{[0, \infty)}$ be given by

$$v_1(x) = 1, \quad v_2(x) = \sqrt{x}, \quad v_3(x) = \sin(x)$$

for $x \in [0, \infty)$. Show that (v_1, v_2, v_3) is linearly independent.

Solution:

... (Cf. Example II.2.4.(ii) and consider, e.g., the evaluation sites $x_1 = 0$, $x_2 = \pi$, and $x_3 = \pi/2$.)

b) Let V be a \mathbb{R} -vector space and let (v_1, v_2) be linearly independent for $v_1, v_2 \in V$. Show that $(v_1 - v_2, v_1 + v_2)$ is linearly independent.

Solution:

Let $\lambda_1, \lambda_2 \in \mathbb{R}$ and

$$\lambda_1(v_1 - v_2) + \lambda_2(v_1 + v_2) = 0.$$

Then we obtain

$$(\lambda_1 + \lambda_2)v_1 + (-\lambda_1 + \lambda_2)v_2 = 0.$$

Since (v_1, v_2) is linearly independent by assumption, we have

$$\begin{aligned} \lambda_1 + \lambda_2 &= 0, \\ -\lambda_1 + \lambda_2 &= 0, \end{aligned}$$

which is equivalent to

$$\begin{aligned} \text{(I)} \quad \lambda_1 + \lambda_2 &= 0, \\ \text{(I)} + \text{(II)} \quad 2\lambda_2 &= 0. \end{aligned}$$

Hence we get $\lambda_1 = \lambda_2 = 0$.

Exercise Sheet 3

Linear Algebra (AAI)

Exercise 3.1 (H)

- a) Show that (v_1, v_2, v_3) given by

$$v_1 = (1, 0, 1), \quad v_2 = (1, 1, 0), \quad v_3 = (0, 1, 1)$$

is a basis of \mathbb{R}^3 .

Solution:

Since $\dim \mathbb{R}^3 = 3$, it suffices to show that (v_1, v_2, v_3) is linearly independent.
... (Cf. Exercise 2.2 c).)

- b) Let V be an \mathbb{R} -vector space with $\dim V = 3$, and let (v_1, v_2, v_3) be a basis of V .
Show that

$$(v_1 - v_3, v_1 + v_2 - v_3, v_1 + v_2 + v_3)$$

is a basis of V , too.

Solution:

... (Cf. Exercises 3.1 a) and 2.4 b).)

Exercise 3.2 (H)

Determine a basis and the dimension of the following subspaces of \mathbb{R}^3 :

- a) $U_1 = \{(x, y, z) \in \mathbb{R}^3 : x = y = z = 0\}$,

Solution:

We have $U_1 = \{(0, 0, 0)\} = \text{span}(\emptyset)$ and $\dim U_1 = 0$.

- b) $U_2 = \{(x, y, z) \in \mathbb{R}^3 : x + y - z = 0\}$.

Solution:

For $(x, y, z) \in \mathbb{R}^3$ we have

$$(x, y, z) \in U_2 \Leftrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z - y \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} + \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$$

This shows

$$\begin{aligned} U_2 &= \{\lambda \cdot (1, 0, 1) + \mu \cdot (-1, 1, 0) : \lambda, \mu \in \mathbb{R}\} \\ &= \text{span}\{(1, 0, 1), (-1, 1, 0)\}. \end{aligned}$$

Moreover, the family $((1, 0, 1), (-1, 1, 0))$ is linearly independent [since ...] and hence a basis of U_2 .

Exercise 3.3 (H)

Let V be a K -vector space, let (v_1, \dots, v_n) be a basis of V , and let (w_1, \dots, w_m) be a generating set of V . Prove or disprove:

- a) (v_1, w_2, \dots, w_m) is a generating set of V .

Solution:

False: Choose $V = \mathbb{R}^2$, $n = m = 2$, and

$$v_1 = (1, 0), \quad v_2 = (0, 1), \quad w_1 = v_2, \quad w_2 = v_1.$$

Then, $(v_1, w_2) = (v_1, v_1)$ is not a generating set of \mathbb{R}^2 .

- b) $(v_1 + w_1, \dots, v_n + w_n)$ is a basis of V .

Solution:

False: ...

- c) There exists $i \in \{1, \dots, n\}$ such that $(v_1, \dots, v_{i-1}, w_1, v_{i+1}, \dots, v_n)$ is a basis of V .

Solution:

False: Choose $m = n + 1$ and $w_1 = 0$ as well as

$$(w_2, \dots, w_{n+1}) = (v_1, \dots, v_n).$$

A family containing the zero vector is linearly dependent and hence not a basis.
(Note that the statement is correct for $w_1 \neq 0$ due to Lemma II.2.10.)

Exercise 3.4 (H)

Let U_1, U_2 be subspaces of V , and let

$$U_1 + U_2 = \{u_1 + u_2 : u_1 \in U_1, u_2 \in U_2\}.$$

- a) Show that $U_1 + U_2$ is a subspace of V .

Solution:

... (Verify the subspace conditions from Definition II.1.5.)

- b) Let $V = \mathbb{R}^n$ and $\dim U_1 = \dim U_2 = n - 1$. Determine all possible cases of $\dim U_1 \cap U_2$ and provide an explicit example for each case with $n = 3$. Cf. Exercise 1.4.

Solution:

By Proposition II.2.17 we have

$$\begin{aligned} \dim(U_1 \cap U_2) &= \dim(U_1) + \dim(U_2) - \dim(U_1 + U_2) \\ &= (n - 1) + (n - 1) - \dim(U_1 + U_2) \end{aligned}$$

Moreover, we have

$$0 \leq \dim(U_1 \cap U_2) \leq \min(\dim(U_1), \dim(U_2)) = n - 1$$

and

$$n - 1 = \max(\dim(U_1), \dim(U_2)) \leq \dim(U_1 + U_2) \leq \dim V = n,$$

see Corollary II.2.15. This yields the following possible cases:

	$\dim(U_1 \cap U_2)$	$\dim(U_1 + U_2)$
case 1	$n - 1$	$n - 1$
case 2	$n - 2$	n

... (examples for $n = 3$)

Exercise Sheet 4

Linear Algebra (AAI)

Exercise 4.1 (H)

- a) Consider $A \in \mathbb{R}^{4 \times 3}$ and $B \in \mathbb{R}^{3 \times 4}$ given by

$$A = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & 0 & 2 \\ -1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

Transform A and B into row echelon form and determine row space and rank.

Solution:

... (Cf. **sheet4.m**.)

- b) Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^5$ be given by

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ -1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ -1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \end{pmatrix}.$$

Determine a basis of $\text{span}(\{v_1, v_2, v_3, v_4\})$.

Solution:

Write $v_1^\top, v_2^\top, v_3^\top, v_4^\top$ as rows into a matrix $C \in \mathbb{R}^{4 \times 5}$ and use Gaussian elimination.
... (Cf. **sheet4.m**.)

Exercise 4.2 (H)

- a) Compute $A \cdot B$ and $B \cdot A$ for A, B given by Exercise 4.1a).

Solution:

... (Cf. **sheet4.m**.)

- b) Compute every possible matrix product $C_i \cdot C_j$ for

$$C_1 = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 3 & 5 \\ 1 & 8 & -7 \end{pmatrix}, \quad C_2 = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1 \\ 0 \\ 8 \\ -7 \end{pmatrix},$$
$$C_4 = \begin{pmatrix} -1 & 2 & 0 & 8 \end{pmatrix}, \quad C_5 = \begin{pmatrix} 1 & 4 \\ 0 & 5 \\ 6 & 8 \end{pmatrix}.$$

Solution:

... (Cf. `sheet4.m`.)

- c) Let $m, n, p, q \in \mathbb{N}$ and $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$, and $C \in \mathbb{R}^{p \times q}$. Verify that

$$(A \cdot B) \cdot C = A \cdot (B \cdot C).$$

Solution:

...

Exercise 4.3 (H)

- a) Let $A \in \mathbb{R}^{4 \times 4}$ and $\lambda \in \mathbb{R} \setminus \{0\}$. Express the following elementary row operations on A as a matrix product $L \cdot A$ with $L \in \mathbb{R}^{4 \times 4}$:

i) multiplication of the third row of A by λ ,

ii) addition of the λ -fold of the first row of A to the fourth row of A ,

iii) switching the second and the fourth row of A .

Solution:

... (Cf. `sheet4.m`.)

- b) Let $A = (a_{i,j})_{i,j} \in \mathbb{R}^{3 \times 3}$ be given by

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{pmatrix}.$$

Transform the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & 1 & 0 & 0 \\ a_{2,1} & a_{2,2} & a_{2,3} & 0 & 1 & 0 \\ a_{3,1} & a_{3,2} & a_{3,3} & 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 6}$$

into

$$\begin{pmatrix} 1 & 0 & 0 & b_{1,1} & b_{1,2} & b_{1,3} \\ 0 & 1 & 0 & b_{2,1} & b_{2,2} & b_{2,3} \\ 0 & 0 & 1 & b_{3,1} & b_{3,2} & b_{3,3} \end{pmatrix} \in \mathbb{R}^{3 \times 6}$$

by using elementary row operations. Compute the matrix products $A \cdot B$ and $B \cdot A$ for $B = (b_{i,j})_{i,j} \in \mathbb{R}^{3 \times 3}$.

Solution:

... (Cf. `sheet4.m`.)

Exercise 4.4 (H)

Let $A, B \in \mathbb{R}^{n \times n}$. Prove or disprove:

- a) $\text{rank}(A + B) = \text{rank } A + \text{rank } B$.

Solution:

False: Choose $n = 2$ and $A = B = E_2$ (identity matrix).

...

- b) $\text{rank}(A \cdot B) = \text{rank } A \cdot \text{rank } B$.

Solution:

False: ...

Exercise Sheet 5

Linear Algebra (AAI)

Exercise 5.1 (H)

Prove Lemma II.4.3.(iv)-(v).

Solution:

See Lemma II.4.3 in the lecture notes.

Exercise 5.2 (H)

- a) Show Remark II.4.4.(i).

Solution:

See Remark II.4.4 in the lecture notes.

- b) Let V and W be K -vector spaces, and let $F: V \rightarrow W$ be linear. Moreover, let $v_1, v_2 \in V$ such that $v_1 \neq v_2$ and $F(v_1) = F(v_2) \neq 0$. Show that (v_1, v_2) is linearly independent.

Hint: Consider $\lambda_1 v_1 + \lambda_2 v_2 = 0$, apply F to both sides of the equation, and use Lemma II.1.4.(iii).

Solution:

Let $\lambda_1, \lambda_2 \in K$ with $\lambda_1 v_1 + \lambda_2 v_2 = 0$. By applying F and using $F(v_1) = F(v_2)$ we have

$$0 = F(0) = F(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 F(v_1) + \lambda_2 F(v_2) = (\lambda_1 + \lambda_2) \cdot F(v_1).$$

Since $F(v_1) \neq 0$, Lemma II.1.4.(iii) yields $\lambda_1 + \lambda_2 = 0$, or equivalently, $\lambda_2 = -\lambda_1$. Using this we obtain

$$0 = \lambda_1 v_1 + \lambda_2 v_2 = \lambda_1 v_1 - \lambda_1 v_2 = \lambda_1 \cdot (v_1 - v_2).$$

Since $v_1 \neq v_2$, Lemma II.1.4.(iii) shows $\lambda_1 = 0$. Hence we get $\lambda_1 = \lambda_2 = 0$.

Exercise 5.3 (H)

- a) Let $F: \mathbb{R}^n \rightarrow \mathbb{R}^m$ be linear. Show that there exists a unique matrix $A \in \mathbb{R}^{m \times n}$ such that $F(x) = Ax$ for all $x \in \mathbb{R}^n$.

Hint: According to Proposition II.4.7 a linear map is uniquely determined by the image of a basis. Use the standard basis (e_1, \dots, e_n) in \mathbb{R}^n to obtain A . Then show that A satisfies $F(x) = Ax$ for all $x \in \mathbb{R}^n$.

Solution:

...

- b) Determine the matrix $A \in \mathbb{R}^{3 \times 3}$ from part a) for the following choices of F :

$$\text{i) } F\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad F\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad F\left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

$$\text{ii) } F\left((1, 0, 0)^\top\right) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad F\left((1, 0, 1)^\top\right) = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad F\left((0, 1, 1)^\top\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Hint: Note that the j -th column $a_j \in \mathbb{R}^3$ of the matrix A satisfies $a_j = Ae_j$.

Solution:

... (Cf. **sheet5.m.**)

Exercise 5.4 (H)

- a) Let U , V , and W be K -vector spaces, and let $F: V \rightarrow W$ and $G: U \rightarrow V$ be linear. Show that $F \circ G$ is linear.

Solution:

Since F and G are linear, we have

$$F(G(\lambda u_1 + u_2)) = F(\lambda G(u_1) + G(u_2)) = \lambda F(G(u_1)) + F(G(u_2))$$

for $\lambda \in K$ and $u_1, u_2 \in U$.

- b) Consider the situation from Exercise 5.3.

- i) Determine the matrix $B \in \mathbb{R}^{3 \times 3}$ that represents the linear map $F: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ given by

$$F\left((1, 0, 0)^\top\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad F\left((0, 1, 0)^\top\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad F\left((0, 0, 1)^\top\right) = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

- ii) Compute B^{-1} using the algorithm from Exercise 4.3 b).

- iii) Compute $B^{-1}v_i$ for $i \in \{1, 2, 3\}$ and

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

- iv) Let $A \in \mathbb{R}^{3 \times 3}$ be the matrix from Exercise 5.3 b).i). Compute AB^{-1} and compare the result with the matrix from Exercise 5.3 b).ii).

Solution:

... (Cf. **sheet5.m.**)