Outline

1. Introduction
2. Power series
   - Sequences of Functions
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6. Summary - Outlook and Review

We know the concepts of

functions,
sequences, and
series.

We are interested in series representing a function $f(x)$ at every x:

$$f(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j$$

For this purpose we need the concept of a sequence of functions at first.

Sequence of partial sums: $\left\{ \sum_{j=0}^{n} a_j (x-x_0)^j \right\}_{n \in \mathbb{N}_0}$

Sequences of Functions

A sequence of elements of $\mathbb{R}$ (cf. Analysis 1):

$$g : \mathbb{N} \to \mathbb{R},\ n \mapsto a_n =: g(n)$$

$$\{a_n\}_{n\in\mathbb{N}} = \{a_n\}_{n\geq 1} = \{a_1, a_2, \ldots, a_k, \ldots\}$$

Definition (Sequences of Functions)

Let D be a set. A mapping

$$g : D \times \mathbb{N} \to \mathbb{R},\ (x, n) \mapsto f_n = f_n(x) =: g(x, n)$$

is called a **sequence** of functions $f_n : D \to \mathbb{R}$, $n \in \mathbb{N}$.

Other notations by writing the functions, e.g., are:

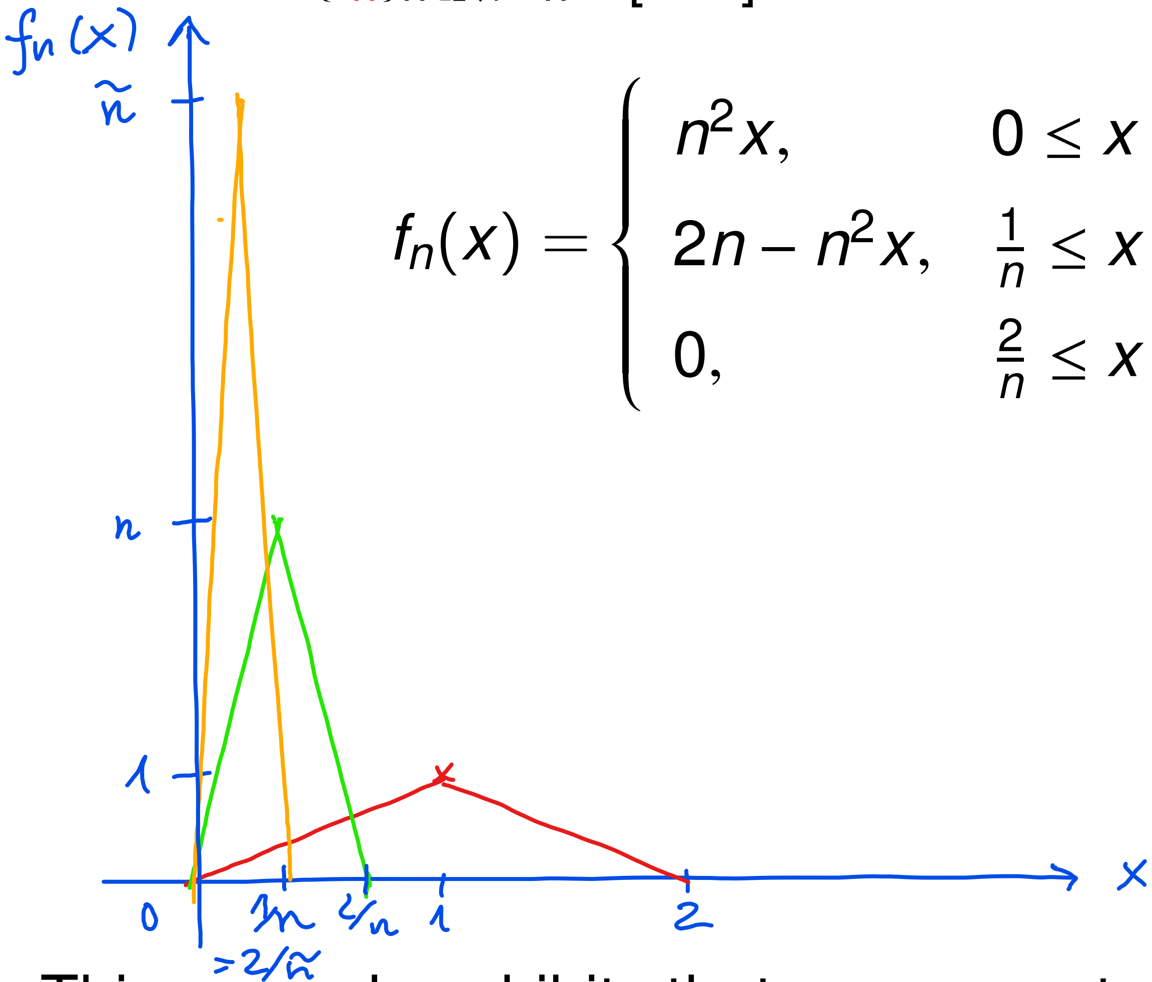
$$\{f_n\}_{n\in\mathbb{N}} = \{f_n\}_{n\geq 1} = \{f_1, f_2, f_3, \ldots, f_n, \ldots\}$$

The domain of definition D and the target area, here $\mathbb{R}$, have to be identical for all functions f_n.

Example (Pointwise Convergence)

Consider $\{f_n\}_{n\in\mathbb{N}}$, $f_n : [0,2] \to \mathbb{R}$ with

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le \frac{1}{n}, \\ 2n - n^2 x, & \frac{1}{n} \le x \le \frac{2}{n}, \\ 0, & \frac{2}{n} \le x \le 2. \end{cases}$$



$\tilde{n} > n > 1$

This example exhibits that we may not swap the limit and the integral (another limit process) in general!

Pointwise Convergence

Definition (Pointwise Convergence)

Let D be a set. A sequence $\{f_n\}_{n \geq n_0}$ of functions $f_n : D \to \mathbb{R}$ is called **pointwise convergent** to a function $f : D \to \mathbb{R}$,
if and only if

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for any } x \text{ in } D.$$

Equivalently,

For any $x \in D$ and $\varepsilon > 0$
there exists a $N = N(x, \varepsilon) \geq n_0$ s.t.:

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for any } x \text{ in } D \text{ and all } n \geq N.$$

Uniform Convergence

Definition (Uniform Convergence)

Let D be a set. A sequence $\{f_n\}_{n \geq n_0}$ of functions $f_n : D \to \mathbb{R}$ is called **uniformly convergent** to a function $f : D \to \mathbb{R}$,
if and only if

For any $x \in D$ and $\varepsilon > 0$
there exists a $N = N(\varepsilon) \geq n_0$ s.t.:

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for any } x \text{ in } D \text{ and all } n \geq N.$$

Notice that N may depend only on ε but not on the point x.

Uniform convergence always implies pointwise convergence, the opposite is not true (see last example).

Example Uniform Convergence

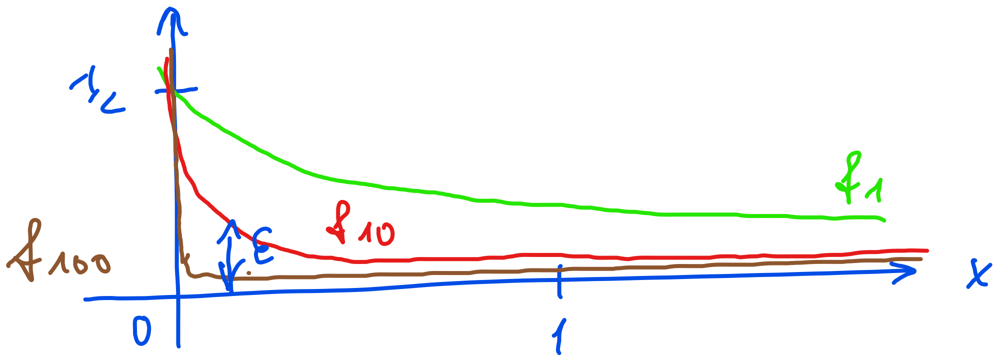
Let $x \in D = [0, 1)$. The sequence

$$\left\{ \frac{1}{2^{x+n}} \right\}_{n \in \mathbb{N}}$$

converges uniformly:

Assume $\varepsilon = \frac{1}{4}$:

$$f_n(x) = \frac{1}{2^{x+n}} = \frac{1}{2^n} \cdot \frac{1}{2^x} < \frac{1}{4}$$ for all $n \geq 2$ for any $x \in D$



$f_n \xrightarrow{n \to \infty} f_\infty \equiv 0$

as $x \in D$ $(x \neq 0!)$

Continuity and Uniform Convergence

Theorem (Uniform convergence preserves continuity)

Let $D \subseteq \mathbb{C}$ and $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$, a sequence of continuous functions,
that uniformly converge to a function $f : D \to \mathbb{C}$,

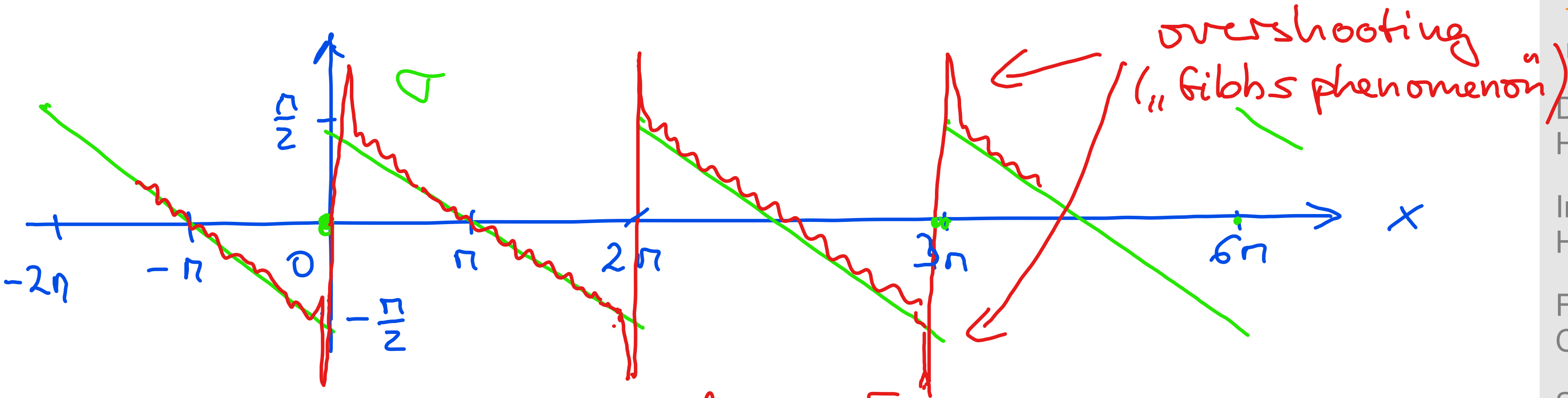
then f is continuous.

The limit of a uniformly convergent sequence of continuous functions,
is again continuous.

Example: Saw-tooth function

Let $\sigma: \mathbb{R} \to \mathbb{R}$,

$$x \mapsto \begin{cases} 0 & ; \ x = 0 \\ \frac{\pi - x}{2} & ; \ 0 < x < 2\pi \\ \ldots & \text{repeat periodically with period } 2\pi \end{cases}$$



σ approx. by a Fourier sum

Spoiler: $\sigma(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{x}$

whatever „equal" means here

σ is discontinuous $\Rightarrow$ this Fourier series is not uniformly convergent

Uniform norm

Definition (Uniform norm or sup norm)

Let D be a set and $f : D \to \mathbb{C}$.

We set

$$\|f\|_D := \sup_{x \in D} |f(x)|.$$

$\| \cdot \|_D$ defines a norm on D.

A function f is bounded iff $\|f\| < \infty$.

When misunderstandings are excluded, we just write $\|f\|$ instead of $\|f\|_D$.

Uniform Norm and Uniform Convergence

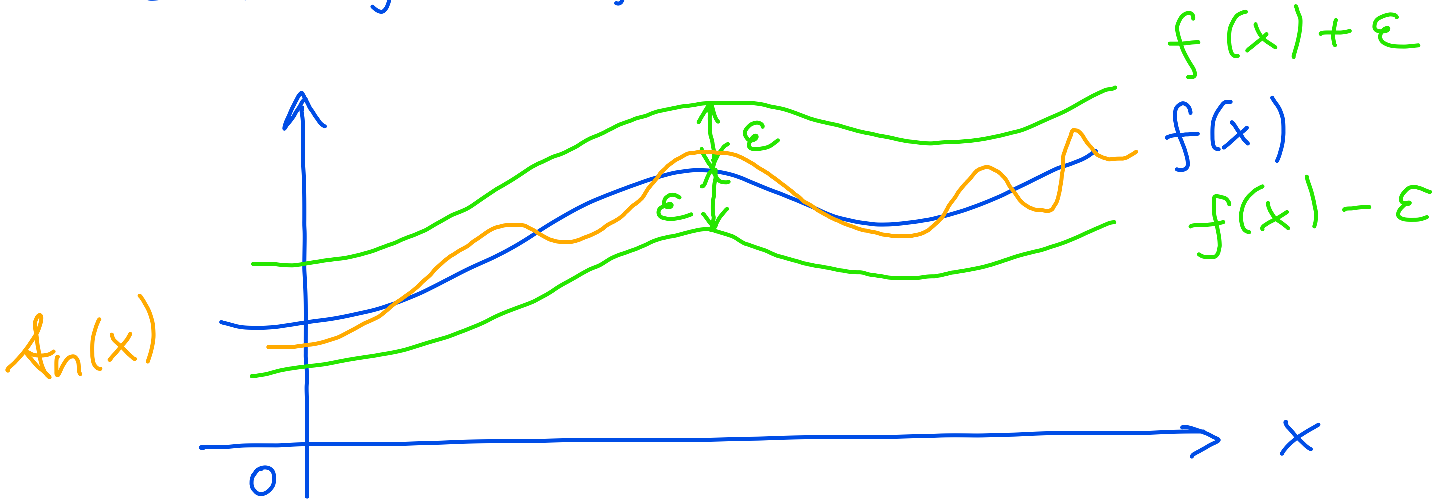
By this notation we may reformulate the uniform convergence:

$$\{f_n\}_{n\in\mathbb{N}} \text{ converges uniformly on } D$$

$$\iff \lim_{n\to\infty} \|f_n - f\|_D = 0$$

$$\|f_n - f\|_D \le \varepsilon$$

$$\iff |f_n(x) - f(x)| \le \varepsilon \quad \text{for all } x \in D$$



Criterion for Uniform Convergence of a Series

Theorem (Weierstrass Convergence Criterion)

Let $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$.
If

$$\sum_{n=0}^{\infty} \|f_n\|_D < \infty$$

then the series

$$\sum_{n=0}^{\infty} f_n = F$$

converges absolutely and uniformly on D to a function $F : D \to \mathbb{C}$.

For an example, see next slide

Example: Convergence of a Power Series

The series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

converges uniformly on $\mathbb{R}$.

$$f_n(x) = \frac{\cos(nx)}{n^2}, \; n \in \mathbb{N} \; ; \quad f_0 = 0$$

We know $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty \implies$ $\left\{ \sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2} \right.$ converges absolutely & uniformly on $D = \mathbb{R}$

$\|\cos(nx)\| \leq 1$