Taylor Series

In many applications “higher order” terms are neglected in order to find simpler or manageable formulas.

The base for this and for an estimate of the resulting error is the following theorem:

Theorem (Taylor theorem)

Let $f : [a, b] \to \mathbb{R}$ be an $n+1$-times continuously differentiable function.
Then for any two points x and x_0 in (a, b) there exists a number $\Theta_{x,x_0} \in (0, 1)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + \underbrace{\frac{f^{(n+1)}(x_0 + \Theta_{x,x_0} \cdot (x - x_0))}{(n+1)!}(x - x_0)^{n+1}}_{o(|x-x_0|^n)}.$$

Notation:

* $(x-x_0)^0 = 1$
* $0! = 1$
* $f^{(0)} = f$

Taylor Polynomial & Remainder

Definition (Taylor Polynomial, Lagrange Form of the Remainder)

We call

$$T_n(f, x, x_0) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

the **Taylor polynomial of n-th order** (not of n-th degree!) around the center x_0.

The term

$$\frac{f^{(n+1)}(x_0 + \Theta_{x,x_0} \cdot (x - x_0))}{(n+1)!}(x - x_0)^{n+1}$$

is called the **Lagrange form of the remainder.**

There are other forms of the ~~R~~remainder: Peano form, Cauchy form, ...

Examples for the Taylor Theorem

$$f(x) = \sqrt{1+x} = (1+x)^{1/2}$$

Taylor polynomial of order 3 around $x_0 = 0$

$$f'(x) = \frac{1}{2}(1+x)^{-1/2} \qquad f''(x) = -\frac{1}{4}(1+x)^{-3/2}$$

$$f'''(x) = \frac{3}{8}(1+x)^{-5/2} \qquad f''''(x) = \frac{-15}{16}(1+x)^{-7/2}$$

Thus $$T_3(f,x,0) = f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3$$

and $$R_3(f,x,0) = \frac{-\cancel{1}5}{16\cdot 2\cancel{4}\,8}\,\frac{x^4}{(1+\vartheta_{x,x_0}x)^{7/2}} = \frac{-5x^4}{128(1+\underbrace{\vartheta_{x,x_0}}_{\in(0,1)}x)^{7/2}}$$

$4! = 4\cdot 3\cdot 2\cdot 1 = 24$

Taylor Series

Definition (Taylor Series)

Let $f : [a, b] \to \mathbb{R}$ be arbitrarily many times differentiable.

Then for $x \in [a, b]$ the power series

$$T(f, x, x_0) = \sum_{k=0}^{\infty} \frac{f^{(k)}(\overset{x_0}{\cancel{0}})}{k!}(x - x_0)^k.$$

is called the **Taylor series** of the function f with the center x_0.

Taylor Series - Convergence

Theorem (Convergence Criterion for Taylor Series)

The Taylor series $T(f, x, x_0)$ converges for a given $x \in [a, b]$ to $f(x)$

iff

$$\lim_{n\to\infty} R_n(f, x, x_0) = 0.$$

We are interested in an interval $[a, b]$ such that the remainder converges to 0 and this should happen “fast”.

Warning: There exist functions whose Taylor series represent the function in a single point only!

Maclaurin Series

An important special case: $x_0 = 0$.

Let f be as in the Taylor theorem and $0 \in (a, b)$. For any point $x \in (a, b)$ there exists a number $\Theta_x \in (0, 1)$ such that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(\Theta_x x)}{(n+1)!} x^{n+1}.$$

The corresponding series is called **Maclaurin series** of the function f:

$$M(f, x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k.$$

Its center is $x_0 = 0$.

A Taylor Series for the Logarithm

- $\underbrace{\ln(1+x)}_{=f(x)} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$, $x \in [-\frac{1}{2}, 1]$, $\underline{\underline{x_0 = 0}}$

By induction: $f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k}$, $k \in \mathbb{N}$

$$M(f,x) \overset{\shortparallel}{} \; T(f,x,0) = \underbrace{\ln(1)}_{=0} + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\cancel{(n-1)!}}{(1+0)^n}\frac{1}{n\cancel{!}}(x-0)^n$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$R_n(f,x,0) = \frac{f^{(n+1)}(\vartheta_x x)}{(n+1)!}x^{n+1} = \frac{(-1)^n \cancel{n!}}{(n+1)\cancel{!}}\frac{x^{n+1}}{(1+\vartheta_x x)^{n+1}} = \frac{(-1)^n}{n+1}\left(\frac{x}{1+\vartheta_x x}\right)^{n+1}$$

For $0 \le x \le 1$: Clearly $\left|\frac{x}{1+\vartheta_x x}\right| \le 1$ also holds

For $-\frac{1}{2} \le x < 0$: $|x| = -x \le \frac{1}{2} \le 1 + x < 1 + \vartheta_x x$

$\Rightarrow |R_n(f,x,0)| \le \frac{1}{n+1} \xrightarrow{n\to\infty} 0$

$\Rightarrow$ the series converges

- $\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

$$= \underbrace{-\sum_{n=1}^{\infty}(-1)^{n\cancel{-1}}\frac{1}{n}}_{\text{alternating harmonic series}}$$