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Exercise Sheet 10

Linear Algebra (AAI)

Exercise 10.1 (H)

Let $A_1, A_2, A_3, A_4 \in \mathbb{R}^{3\times 3}$ be given by

$$A_1 = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1/2 & 1/2 & 0 \\ -1/2 & 3/2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 7 & 2 & 4 \\ 1 & 6 & 2 \\ 1 & 1 & 7 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

a) Determine the characteristic polynomials P_{A_i} for all $i = 1, 2, 3, 4$.

b) Determine all eigenvalues and the corresponding eigenspaces of A_i for all $i = 1, 2, 3, 4$.

Exercise 10.2 (H)

Let the basis $\mathcal{A} = (v_1, v_2, v_3, v_4)$ of $\mathbb{R}^4$ be given by

$$v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Construct a matrix $A \in \mathbb{R}^{4\times 4}$ such that

$$Av_1 = v_1, \quad Av_2 = -v_2, \quad Av_3 = 0, \quad Av_4 = 2v_4.$$

Hint: Determine $\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}(\mathcal{F}_A)$ *and note that* $A = \mathcal{M}^{\mathcal{E}}_{\mathcal{E}}(\mathcal{F}_A)$.

Exercise 10.3 (H)

Let $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$.

a) Show that $\mathrm{Eig}(A, \lambda) = \mathrm{Eig}(A^{-1}, \lambda^{-1})$ for $A \in \mathrm{Gl}(n, \mathbb{R})$ and $\lambda \in \sigma(A)$.

b) Let all $x \in \mathbb{R}^n \setminus \{0\}$ be eigenvectors of A. Show that there exists $c \in \mathbb{R}$ such that $A = c \cdot E_n$, i.e., A is a multiple of the identity matrix E_n.
Hint: Note that $Ae_i = \lambda_i e_i$ *and* $Ae_j = \lambda_j e_j$. *Consider* $A(e_i + e_j)$ *to show* $\lambda_i = \lambda_j$.

Exercise 10.4 (H)

Let V be a K-vector space with $\dim V = n \in \mathbb{N}$. Moreover, let $F\colon V \to V$ be linear with $\operatorname{im} F = \ker F$.

a) Show that n is even.

b) Determine all eigenvalues and the corresponding eigenspaces of F.
Hint: Note that an eigenvector $v \in V \setminus \{0\}$ *satisfies* $F(F(v)) = F(\lambda v) = \lambda^2 v$.

c) Construct a linear map $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\operatorname{im} F = \ker F$.