Outline

Power Series

We know the concepts of

functions,
sequences, and
series.

We are interested in series representing a function $f(x)$ at every x:

$$f(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^j$$

For this purpose we need the concept of a sequence of functions at first.

Sequences of Functions

A sequence of elements of $\mathbb{R}$ (cf. Analysis 1):

$$f : \mathbb{N} \to \mathbb{R},\ n \mapsto a_n =: f(n)$$

Definition (Sequences of Functions)

Let D be a set. A mapping

$$f : D \times \mathbb{N} \to \mathbb{R},\ (x, n) \mapsto f_n =: f(n)$$

is called a **sequence** of functions $f_k : D \to \mathbb{R}$, $k \in \mathbb{N}$.

Other notations by writing the functions, e.g., are:

$$\{f_k\}_{k\in\mathbb{N}} = \{f_k\}_{k\geq 1} = \{f_1, f_2, f_3, \ldots, f_k, \ldots\}$$

The domain of definition D and the target area, here $\mathbb{R}$, have to be identical for all functions f_k.

Example (Pointwise Convergence)

Consider $\{f_k\}_{k\in\mathbb{N}}$, $f_k : [0,2] \to \mathbb{R}$ with

$$f_k(x) = \begin{cases} n^2 x, & 0 \le x \le \frac{1}{n}, \\ 2n - n^2 x, & \frac{1}{n} \le x \le \frac{2}{n}, \\ 0, & \frac{2}{n} \le x \le 2. \end{cases}$$

This example exhibits that we may not swap the limit and the integral (another limit process) in general!

Pointwise Convergence

Definition (Pointwise Convergence)

Let D be a set. A sequence $\{f_n\}_{k\geq k_0}$ of functions $f_k : D \to \mathbb{R}$ is called **pointwise convergent** to a function $f : D \to \mathbb{R}$,
if and only if

$$\lim_{k\to\infty} f_k(x) = f(x) \quad \text{for any } x \text{ in } D.$$

Equivalently,

For any $x \in D$ and $\varepsilon > 0$
there exists a $N = N(x, \varepsilon)$ s.t.:

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for any } x \text{ in } D \text{ and all } n \geq N.$$

Uniform Convergence

Definition (Uniform Convergence)

Let D be a set. A sequence $\{f_n\}_{k \geq k_0}$ of functions $f_k : D \to \mathbb{R}$ is called **uniformly convergent** to a function $f : D \to \mathbb{R}$,
if and only if

For any $x \in D$ and $\varepsilon > 0$
there exists a $N = N(\varepsilon)$ s.t.:

$$|f_n(x) - f(x)| < \varepsilon \quad \text{for any } x \text{ in } D \text{ and all } n \geq N.$$

Notice that N may depend only on ε but not on the point x.

Pointwise convergence always implies uniform convergence, the opposite is not true (see last example).

Example Uniform Convergence

Let $x \in D = [0, 1)$. The sequence

$$\left\{ \frac{1}{2^{x+n}} \right\}_{n \in \mathbb{N}}$$

converges uniformly:

Continuity and Uniform Convergence

Theorem (Uniform convergence preserves continuity)

Let $D \subseteq \mathbb{C}$ and $f_n : D \to \mathbb{C}$, $n \in \mathbb{C}$, a sequence of continuous functions,
that uniformly converge to a function $f : D \to \mathbb{C}$,

then f is continuous.

The limit of a uniformly convergent sequence of continuous functions,
is again continuous.

Example: Saw-tooth function

Uniform norm

Definition (Uniform norm or sup norm)

Let D be a set and $f : D \to \mathbb{C}$.

We set

$$\|f\|_D := \sup_{x \in D} |f(x)|.$$

$\|\cdot\|_D$ defines a norm on D.

A function f is bounded iff $\|f\| < \infty$.

When misunderstandings are excluded, we just write $\|f\|$ instead of $\|f\|_D$.

Uniform Norm and Uniform Convergence

By this notation we may reformulate the uniform convergence:

$$\{f_n\}_{n\in\mathbb{N}} \text{ converges uniformly on } D$$

$$\iff \lim_{n\to\infty} \|f_n - f\|_D = 0$$

Criterion for Uniform Convergence of a Series

Theorem (Weierstrass Convergence Criterion)

Let $f_n : D \to \mathbb{C}$, $n \in \mathbb{N}$.
If

$$\sum_{n=0}^{\infty} \|f_n\|_D < \infty$$

then the series

$$\sum_{n=0}^{\infty} f_n$$

converges absolutely and uniformly on D *to a function* $F : D \to \mathbb{C}$.

Example: Convergence of a Power Series

The series

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2}$$

converges uniformly on $\mathbb{R}$.

Power Series

Polynomials are among the functions that are easy to handle (e.g. for a machine).

Many other functions become manageable, when they are approximated by polynomials, i.e. as power series.

As for polynomials it is helpful to consider power series on $\mathbb{C}$ from the start.

Definition (Power Series)

Let $\{a_j\}_{j\in\mathbb{N}}$ a sequence of complex numbers and $z_0 \in \mathbb{C}$.

Then

$$f(z) = \sum_{j=0}^{\infty} a_j(z - z_0)^j = a_0 + a_1(z - z_0) + a_2(z - z_0)^2 + \dots$$

is called a **(complex) power series** with the **center** of the series z_0.

Properties of Power Series I

The set of points z where the power series converges form a set $M \subseteq \mathbb{C}$.
Note that $z_0 \in M$.

By this the power series defines a function $f : M \to \mathbb{C}$.

The partial sums of power series are polynomials (multiply out!).

Power series have excellent properties of convergence.

Convergence of Power Series

Theorem (Radius of Convergence)

Let

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

be a complex power series.

Then exactly one of the following 2 cases holds:

- *There exists a* $\rho \in \mathbb{R}_0^+$ *s.t. the series converges absolutely for all* $z \in O_\rho(z_0) = \{z \in \mathbb{C} \,|\, |z - z_0| < \rho$ *and diverges for all* z *with* $|z - z_0| > \rho$.
- *The series converges absolutely for all* $z \in \mathbb{C}$.

$\rho \in \mathbb{R}_0^+ \cup \{+\infty\}$ is called **radius of convergence**,
$O_\rho(z_0)$ is called **circle of convergence**.

For $|z| = \rho$ no general statement on convergence/divergence is possible.

Computing a Radius of Convergence

Assume for a power series holds

$$\overline{\lim}_{j\to\infty}\sqrt[j]{|a_j|} = b \quad \text{or} \quad \lim_{j\to\infty}\left|\frac{a_{j+1}}{a_j}\right| = b,$$

where $b \in \mathbb{R}_0^+ \cup \infty$, then:

- If $b = 0$, then $\rho = +\infty$.
- If $b = +\infty$, then $\rho = 0$.
- If $0 < b$, then $\rho = \frac{1}{b}$.

The limes superior (or inferior) of the quotient is not helpful in general.

Example: Geometric Series

Geometric series:

$$\sum_{j=0}^{\infty} z^n$$

Set $a_j = 1$ for all $j \in \mathbb{N}$ (and $z_0 = 0$).

Since $b = \lim_{j\to\infty} \sqrt[j]{a_j]} = 1$, we find $\rho = 1/b = 1$.

Thus we have (as expected) convergence for $|z| < 1$ and divergence for $|z| > 1$.
What happens for $|z| = 1$? Divergence, since we do not encounter a zero sequence z^n.

Moreover $\sum_{j=0}^{\infty} z^n = \frac{1}{1-z}$ for $|z| < 1$.

Properties of Power Series II

Theorem (Addition & scaling of Power Series)

Consider two power series
$f(z) = \sum_{n=0}^{\infty} a_n z^n$ *with radius of convergence* ρ_f *and*
$g(z) = \sum_{n=0}^{\infty} b_n z^n$ *with radius of convergence* ρ_g.

Then the sum/difference is given for all z *with* $|z| < \min(\rho_f, \rho_g)$ *by:*

$$\sum_{n=0}^{\infty} (a_n \pm b_n)\, z^n = \sum_{n=0}^{\infty} a_n z^n \pm \sum_{n=0}^{\infty} b_n z^n = f(z) \pm g(z)$$

Further, the scaling i.e. multiplication with a factor $c \in \mathbb{C}$ *is given for all* z *with* $|z| < \rho_f$ *by:*

$$\sum_{n=0}^{\infty} c a_n z^n = c \sum_{n=0}^{\infty} a_n z^n = c f(z).$$

Properties of Power Series III

Theorem ((Cauchy) Product of Power Series)

Consider two power series
$f(z) = \sum_{n=0}^{\infty} a_n z^n$ *with radius of convergence* ρ_f *and*
$g(z) = \sum_{n=0}^{\infty} b_n z^n$ *with radius of convergence* ρ_g.

Then the product $f(z_1)g(z_2)$ *is given for all* z_1, z_2 *with* $|z_1|, |z_2| < \min(\rho_f, \rho_g)$ *by:*

$$f(z_1)g(z_2) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k} z_1^k z_2^{n-k}.$$

In particular, if $z = z_1 = z_2$*:*

$$f(z)g(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) z^n.$$

The latter 2 theorems also hold for $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ by a shift.

Continuity of Power Series

Theorem (Series of polynomials is continuous)

Let

$$f(z) = \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

be a complex power series with radius of convergence ρ.

Then the function

$$f : O_\rho(z_0) \to \mathbb{C} : z \mapsto \sum_{j=0}^{\infty} a_j (z - z_0)^j$$

is continuous.

More Examples: Exponential Series etc.

The series
for the exponential function and
for (co)sine, resp.,
have the radius of convergence $\rho = \infty$.

The latter 3 series yield a continuous function.