Ana 2 HW 4/1a

$$f(x) = \ln(x^2+1)$$

$$f'(x) = \frac{d}{dx} \ln(v) \cdot \frac{d}{dx}(x^2+1)$$

$$= \frac{1}{x^2+1} \cdot 2x = \frac{2x}{x^2+1}$$

$$f''(x) = \frac{g' \cdot h - g \cdot h'}{h^2} = \frac{2 \cdot (x^2+1) - (2x \cdot 2x)}{x^4+2x^2+1} = \frac{2x^2 - 4x^2 + 2}{x^4+2x^2+1} = \frac{-2x^2+2}{x^4+2x^2+1}$$

$$f'''(x) = \frac{-4x \cdot (x^4+2x^2+1) - (-2x^2+2) \cdot (4x^3+4x)}{(x^4+2x^2+1)^2} = \frac{(-4x^5 - 8x^3 - 4x) + (+8x^5 + 8x^3 - 8x^3 - 8x)}{x^8 + 4x^6 + 6x^4 + 4x^2 + 1}$$

$$(x^4+2x^2+1)^2 = \begin{matrix} x^8 + 2x^6 + x^4 + \\ 2x^6 + 4x^4 + 2x^2 \\ + x^4 + 2x^2 + 1 \end{matrix}$$

$$= \frac{4x^5 - 8x^3 - 12x}{x^8+4x^6+6x^4+4x^2+1} = \frac{4x(x^2-3)}{(x^2+1)^3} = \frac{4x^3 - 12x}{(x^2+1)^3}$$

$$f^{(4)}(x) = \frac{(12x^2-12) \cdot (x^2+1)^3 - (4x^3-12x) \cdot (6x(x^2+1))}{((x^2+1)^3)^2}$$

$$= \frac{12x^2(x^2+1)^3 - 12(x^2+1)^3 - (4x^3(6x(x^2+1)) - 12x(6x(x^2+1)))}{(x^2+1)^6}$$

$$= \frac{-12x^4 + 72x^2 - 12}{(x^2+1)^4}$$

$$f^{(5)}(x) = \frac{(-48x^3 + 144x) \cdot (x^2+1)^4 - (-12x^4 + 72x^2 - 12) \cdot 4x(x^2+1)^3}{(x^2+1)^8}$$

$$= \frac{48x^5 - 480x^3 + 240x}{(x^2+1)^5}$$

$$f^{(6)}(x) = \frac{(240x^4 - 1440x^2 + 240) \cdot (x^2+1)^5 - (48x^5 - 480x^3 + 240x) \cdot 10x(x^2+1)^9}{(x^2+1)^{10}}$$

$$= \frac{-240x^6 + 3600x^4 - 3600x^2 + 240}{(x^2+1)^6}$$

$$f_5(x) = \underbrace{\frac{\ln(0^2+1)}{0!}}_{0} x^0 + \underbrace{\frac{\frac{2\cdot 0}{0^2+1}}{1!}}_{0} x^1 + \frac{\frac{-2\cdot 0^2+2}{0^4+2\cdot 0^2+1}}{2!} x^2$$

$$+ \underbrace{\frac{\frac{4\cdot 0^3 - 12\cdot 0}{(0^2+1)^3}}{3!}}_{0} x^3 + \frac{\frac{-12\cdot 0^4 + 72\cdot 0^2 - 12}{(0^2+1)^4}}{4!} x^4 + \underbrace{\frac{\frac{48\cdot 0^5 - 480\cdot 0^3 + 240\cdot 0}{(0^2+1)^5}}{5!}}_{0} x^5$$

$$+ \frac{\frac{-240(\Theta_x\cdot x)^6 + 3600(\Theta_x\cdot x)^4 - 3600(\Theta_x\cdot x)^2 + 240}{((\Theta_x\cdot x)^2+1)^6}}{6!} x^6$$

$$= x^2 - 0.5x^4 + \ldots$$

b) $$g(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{(1-x^3)^{\frac{1}{2}}} = (1-x^3)^{-\frac{1}{2}}$$

$$g'(x) = \frac{3x^2}{2(1-x^3)^{\frac{3}{2}}}$$

$$g''(x) = \frac{(15x^4 + 12x)}{4(1-x^3)^{\frac{5}{2}}}$$

$$g'''(x) = \frac{(105x^6 + 276x^3 + 24)}{8(1-x^3)^{\frac{7}{2}}}$$

$$g_3(x) = \frac{1}{\sqrt{1-0^3}} x^0 + \frac{3\cdot 0^2}{2(1-0^3)^{\frac{3}{2}}} x^1$$

$$+ \frac{\frac{(15\cdot 0^4 + 12\cdot 0)}{4(1-0^3)^{\frac{5}{2}}}}{2} x^2 + \frac{\frac{(105\cdot 0^6 + 276\cdot 0^3 + 24)}{8(1-0^3)^{\frac{7}{2}}}}{6} x^3$$

$$= 1 + \frac{\frac{24}{8\cdot 1}}{6} x^3 = 1 + \frac{1}{2}x^3$$

$$g_3(0.2) = 1 + \frac{1}{2}(0.2)^3 = 1.004$$

$$g(0.2) = \frac{1}{\sqrt{1-0.2^3}} = 1.004024\ldots$$

$$E_3(0.2) = |g_3(0.2) - g(0.2)| \approx 0.000024$$

/2 $f(x) = \sin(x)$

$f'(x) = \cos(x)$

$f''(x) = -\sin(x)$

$$T_2(f, x, \tfrac{\pi}{2}) = \sin\left(\tfrac{\pi}{2}\right)\cdot\left(x - \tfrac{\pi}{2}\right)^0 + \cos\left(\tfrac{\pi}{2}\right)\cdot\left(x - \tfrac{\pi}{2}\right)^1 - \sin\left(\tfrac{\pi}{2}\right)\cdot\left(x - \tfrac{\pi}{2}\right)^2$$

$$= 1\cdot 1 + 0\cdot\left(x - \tfrac{\pi}{2}\right) - 1\cdot\left(x - \tfrac{\pi}{2}\right)^2$$

$$= 1 - \left(x - \tfrac{\pi}{2}\right)^2$$

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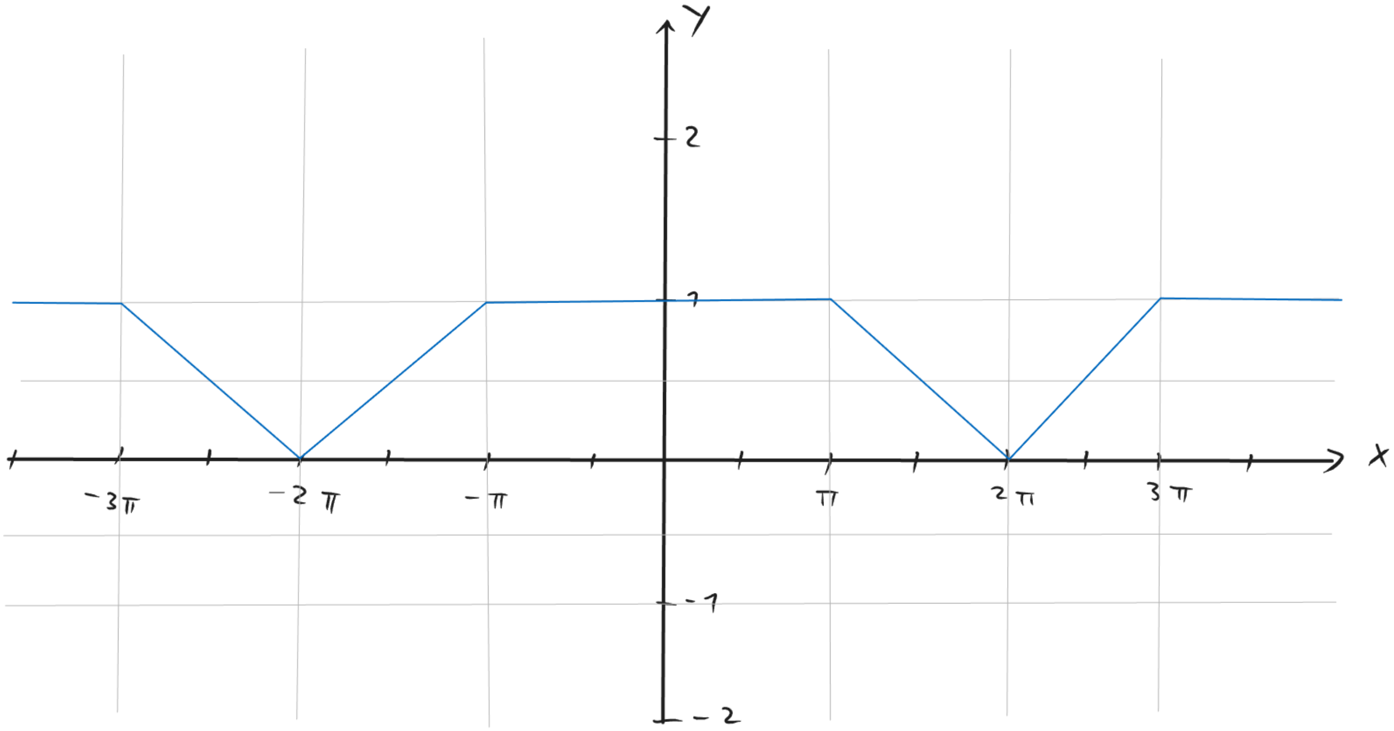


$$\omega = \frac{2\pi}{T} = \frac{2\pi}{4\pi} = 0.5$$

b) the plot is symmetrical to the y-achsis. (= even function)
this means that the fourier coefficient bk is always 0.
for ak the interval of the integral gets halved.

c) $$a_0 = \frac{2}{4\pi}\int_0^{4\pi} f(x)\,dx = \frac{2}{4\pi}\cdot 3\pi = 1.5$$

$$b_k = 0$$

$$a_k = \frac{4}{4\pi}\int_0^{2\pi} f(x)\cdot\cos(k\cdot 0.5x)\,dx$$

$$= \frac{4}{4\pi}\left(\int_0^{\pi} \underbrace{f(x)}_{=1} \cdot \cos(k \cdot 0.5x)\,dx + \int_{\pi}^{2\pi} \underbrace{f(x)}_{=\frac{x+2\pi}{\pi}} \cdot \cos(k \cdot 0.5x)\,dx\right)$$

$$= \frac{4}{4\pi}\left(\left[\frac{2\sin(k \cdot 0.5x)}{k}\right]_0^{\pi} + \left[\frac{x+2\pi}{\pi} \cdot \frac{2\sin(k \cdot 0.5x)}{k}\right]_{\pi}^{2\pi} - \int_{\pi}^{2\pi} \frac{1}{\pi} \cdot \frac{2\sin(k \cdot 0.5x)}{k}\,dx\right)$$

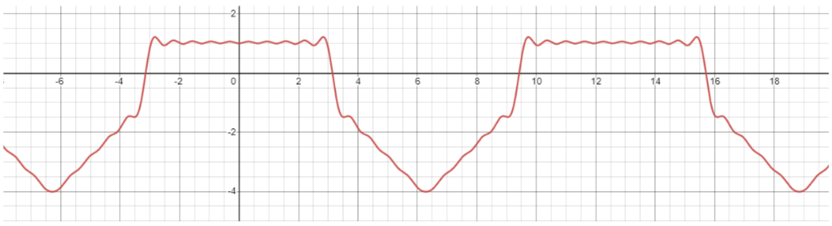
$$= \frac{4}{4\pi}\left(\left[\frac{2\sin(k \cdot 0.5x)}{k}\right]_0^{\pi} + \left[\frac{x+2\pi}{\pi} \cdot \frac{2\sin(k \cdot 0.5x)}{k}\right]_{\pi}^{2\pi} - \frac{1}{\pi}\left[\frac{-2\cos(k \cdot 0.5x)}{k} \cdot \frac{1}{0.5k}\right]_{\pi}^{2\pi}\right)$$

$$= \frac{1}{\pi}\left(\frac{2\sin(k \cdot 0.5\pi)}{k} - 0 + \frac{4\pi}{\pi} \cdot \frac{2\sin(k \cdot \pi)}{k} - \frac{3\pi}{\pi} \cdot \frac{2\sin(k \cdot 0.5\pi)}{k} - \frac{1}{\pi}\left(\frac{-4\cos(k \cdot \pi)}{k^2} + \frac{4\cos(k \cdot 0.5\pi)}{k^2}\right)\right)$$

$$= \frac{2\sin(k \cdot 0.5\pi)}{k\pi} + \frac{8\sin(k \cdot \pi)}{k\pi} - \frac{6\sin(k \cdot 0.5\pi)}{k\pi} + \frac{4\cos(k \cdot \pi)}{k^2\pi} - \frac{4\cos(k \cdot 0.5\pi)}{k^2\pi}$$

$$a_k = \frac{8\sin(k \cdot \pi) - 4\sin(k \cdot 0.5 \cdot \pi)}{k \cdot \pi} + \frac{4\cos(k \cdot \pi) - 4\cos(k \cdot 0.5 \cdot \pi)}{k^2 \cdot \pi}$$

$$F(x) = \frac{1.5}{2} + \sum_{k=1}^{\infty}\left(\frac{8\sin(k \cdot \pi) - 4\sin(k \cdot 0.5 \cdot \pi)}{k \cdot \pi} + \frac{4\cos(k \cdot \pi) - 4\cos(k \cdot 0.5 \cdot \pi)}{k^2 \cdot \pi}\right) \cdot \cos\left(\frac{kx}{2}\right)$$



This shows the graph of F(x) when multiplied by -1 and k up to 20. With my solution the Fourier series does not fully represent u(x) but has obvious similarities to the original graph, especially around y = 1 and the diagonals.

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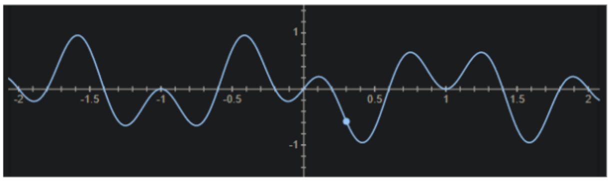
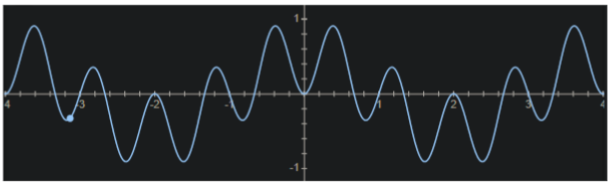
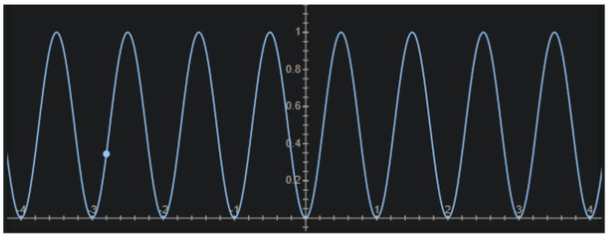


This shows a function of the form sin(mwx) * cos(nwx) with m = 2, n = 5, T = 4 as an example.
The graph is symmetric to the origin (f(x) = -f(-x)).
When taking the integral, all areas above and below the x-achsis cancel out due to the symmetry, which returns the value 0 as a result.



This shows a function of the form sin(mwx) * sin(nwx) with m = 2, n = 3, T = 4 as an example.
The graph is symmetric to the y-achsis (f(x) = f(-x)).
You can see that the integral between x = 0 and x = 2 would result in 0, which is mirrored from x = 2 to x = 4 (= 0+0). Since this pattern repeats for every period, the total integral value is again 0 for $m \neq n$.



This shows a function of the form sin(mwx) * sin(nwx) with m = n = 2, T = 4 as an example.
Contrary to the previous function, the argument for sine is now the same. This basically turns the function into $\sin(mwx)^2$ which results in the graph being one-sided (above x-achsis because it's positive).
As for the result of the integral of T/2, that is obtained from simple geometry calculations. Since one bump (x = 0 -> x = 1) splits the area into 3 parts, of which one is exactly 50% of 1*1 (=T) you get 50% of T.
For different parameters it would be: T*Amplitude*0.5

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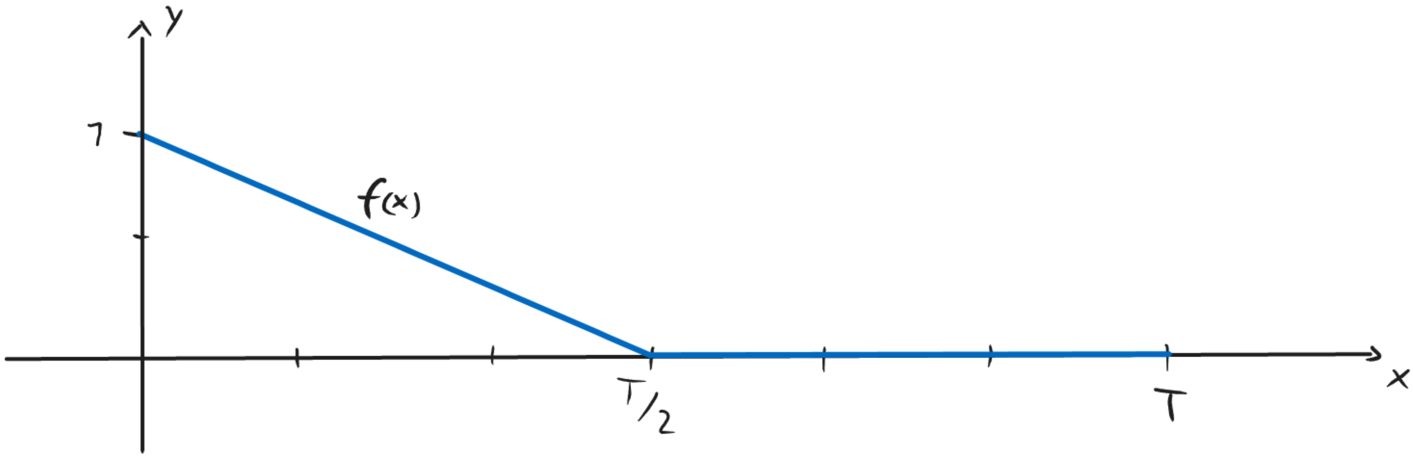

$$F(x) = \sum_{k=-\infty}^{\infty} c_k \cdot e^{ik\omega x} \qquad \omega = \frac{2\pi}{T}$$

$$c_k = \frac{1}{T}\int_0^T f(x)\cdot e^{-ik\omega x}\,dx$$

$$= \frac{1}{T}\left(\int_0^{T/2}\left(1-\frac{2x}{T}\right)\cdot e^{-ik\omega x}\,dx + \underbrace{\int_{T/2}^{T} 0\cdot e^{-ik\omega x}\,dx}_{=0}\right)$$

$$= \frac{1}{T}\left(\int_0^{T/2} 1\, e^{-ik\omega x}\,dx - \int_0^{T/2}\frac{2x}{T}\cdot e^{-ik\omega x}\,dx\right)$$

$$= \frac{1}{T}\int_0^{T/2} 1\, e^{-ik\omega x}\,dx - \frac{1}{T}\int_0^{T/2}\frac{2x}{T}\cdot e^{-ik\omega x}\,dx$$

$$= \frac{1}{T}\int_0^{T/2} e^{-ik\omega x}\,dx - \frac{1}{T^2}\int_0^{T/2} 2x\cdot e^{-ik\omega x}\,dx$$

$$= \frac{1}{T}\left[\frac{-1}{ik\omega}\cdot e^{-ik\omega x}\,dx\right]_0^{T/2} - \frac{1}{T^2}\left(\left[2x\cdot e^{-ik\omega x}\right]_0^{T/2} - 2\int_0^{T/2} e^{-ik\omega x}\,dx\right)$$

$$= \frac{1}{T}\left(-\frac{e^{-ik\omega\cdot\frac{T}{2}}}{ik\omega} + \frac{e^{-ik\omega\cdot 0}}{ik\omega}\right)$$

$$- \frac{1}{T^2}\left(\left[2xe^{-ik\omega x}\right]_0^{T/2} - 2\cdot\left(-\frac{e^{-ik\omega\cdot\frac{T}{2}}}{ik\omega} + \frac{e^{-ik\omega\cdot 0}}{ik\omega}\right)\right.$$

$$= \frac{-e^{-ik\omega\frac{T}{2}}+1}{ik\omega T} - \frac{T\cdot e^{-ik\omega\frac{T}{2}}}{T^2} - 0 - \frac{2}{T^2}\left(\frac{-e^{-ik\omega\frac{T}{2}}+e^{-ik\omega\cdot 0}}{ik\omega}\right)$$

$$= \frac{-e^{-ik\omega\frac{T}{2}}+1}{ik\omega T} - \frac{T\cdot e^{-ik\omega\frac{T}{2}}}{T^2} + \frac{2e^{-ik\omega\frac{T}{2}}-1}{ik\omega T^2}$$

$$F(x) = \sum_{k=-\infty}^{\infty}\left(\frac{-e^{-ik\omega\frac{T}{2}}+1}{ik\omega T} - \frac{T\cdot e^{-ik\omega\frac{T}{2}}}{T^2} + \frac{2e^{-ik\omega\frac{T}{2}}-1}{ik\omega T^2}\right)e^{ik\omega x}$$

$$= \sum_{k=-\infty}^{\infty}\left(\frac{-e^{-ik\pi}+1}{ik2\pi} - \frac{Te^{-ik\pi}}{T^2} + \frac{2e^{-ik\pi}-1}{ik2\pi T}\right)e^{ik\frac{2\pi}{T}x}$$