Technische Hochschule Rosenheim — WiSe 2021/22
Applied Artificial Intelligence - Bachelor
Analysis 1
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Exercise 4 (live tutorial): sequences and convergence

Exercise 10
Compute
$$\lim_{n\to\infty} \frac{2n}{n+1}$$
by only using the definition of a convergent sequence.

Solution for exercise 10
We guess that
$$\lim_{n\to\infty} \frac{2n}{n+1} = 2$$
by the asymptotic behaviour.
For any given $\varepsilon > 0$, we have to demonstrate there exists a $N(\varepsilon) \in \mathbb{N}$ s.t.
$$\left| \frac{2n}{n+1} - 2 \right| < \varepsilon$$
for all $n > N(\varepsilon)$.
We rewrite
$$\frac{2n}{2n+1} - 2 = -\frac{2}{n+1}.$$
Thus from the inequality
$$\frac{2}{n+1} < \varepsilon \quad \Longleftrightarrow \quad \frac{2}{\varepsilon} - 1 < n.$$
Thus there exists a natural number $N(\varepsilon) > 2/\varepsilon - 1$ s.t.:

For any given $\varepsilon > 0$,
$$\left| \frac{2n}{n+1} - 2 \right| = \frac{2}{n+1} < \frac{2}{N(\varepsilon)+1} < \varepsilon \quad \text{for all } n > N(\varepsilon),$$
what justifies the computed limit.

Exercise 11
Prove for a zero sequence $\{a_n\}_{n\geq n_0}$
$$\lim_{n\to\infty} a_n = 0 \quad \Longrightarrow \quad \frac{1}{|a_n|} \overset{n\to\infty}{\to} \infty.$$

Solution for exercise 11
$\{a_n\}_{n\geq n_0}$ is a zero sequence, thus for any $\varepsilon > 0$ there exists a $N(\varepsilon)$ with $|a_n - 0| < \varepsilon$ for all $n \geq N(\varepsilon)$.

Hence for any $k := 1/\varepsilon > 0$ and there exists a $N(1/k)$ with $|a_n - 0| < 1/K$ for all $n \geq N(1/k)$ or:

For any $k > 0$ there exists a $\tilde{N}(k) := N(1/k)$ s.t.
$|1/a_n| > k$ for all $n \geq \tilde{N}(k)$.

Thus $\{\frac{1}{a_n}\}_{n\in\mathbb{N}}$ is definitely divergent to $+\infty$.

Exercise 12
Compute at least the first 5 elements of the following sequences $\{a_n\}_{n\in\mathbb{N}}$ and compute the limit mathematically.

a) $a_n = \frac{1-n^2}{n^3-3}$

b) $a_n = \frac{(3+\sqrt{n})^2}{n+1}$

c) $a_n = \sqrt{n+1} - \sqrt{n}$

d) $a_n = \frac{(-0,8)^n}{2n-5}$

Solution for exercise 12

a) $a_1 = 0, a_2 = -\frac{3}{5}, a_3 = -\frac{8}{24}, a_4 = -\frac{15}{61}, a_5 = -\frac{12}{61}$
$a_n = \frac{1-n^2}{n^3-3} = \frac{\frac{1}{n^3}-\frac{1}{n}}{1-\frac{3}{n^3}} \xrightarrow{n\to\infty} \frac{0-0}{1-0} = 0$

b) $a_1 = 8, a_2 = \frac{11+6\sqrt{2}}{3}, a_3 = \frac{12+6\sqrt{3}}{4} = 3\cdot\left(1+\frac{\sqrt{3}}{2}\right), a_4 = \frac{25}{5} = 5, a_5 = \frac{7}{3}+\sqrt{5}$
$a_n = \frac{(3+\sqrt{n})^2}{n+1} = \frac{n+6\sqrt{n}+9}{n+1} = \frac{1+\frac{6}{\sqrt{n}}+\frac{9}{n}}{1+\frac{1}{n}} \xrightarrow{n\to\infty} 1$

c) $a_1 = \sqrt{2}-1, a_2 = \sqrt{3}-\sqrt{2}, a_3 = 2-\sqrt{3}, a_4 = \sqrt{5}-2, a_5 = \sqrt{6}-\sqrt{5}$
$a_n = \sqrt{n+1}-\sqrt{n} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \xrightarrow{n\to\infty} 0$

d) $a_1 = \frac{-4/5}{-3} = \frac{4}{15}, a_2 = -\frac{16}{25}, a_3 = -\frac{64}{125}, a_4 = \frac{256}{625\cdot 3}, a_5 = -\frac{1024}{15625}$
$a_n = \frac{\left(-\frac{4}{5}\right)^n}{2n-5} = (-1)^n\left(\frac{4}{5}\right)^n\cdot\frac{1}{2n-5} \xrightarrow{n\to\infty} 0\cdot 0 = 0$

Exercise 13
Prove that every real sequence that converges is a Cauchy sequence (opposite direction of the theorem “Cauchy criterion for the convergence of sequences” from the lecture).

Solution for exercise 13
We consider a convergent sequence $\{a_n\}_{n\geq n_0}$ with the limit a.

For all $\varepsilon > 0$ there exists a $N(\varepsilon)$ s.t.

$$|a_n - a| < \varepsilon \quad \text{for all } n \geq N(\varepsilon).$$

Since

$$|a_n - a_m| = |a_n - a + a - a_m| \leq |a_n - a| + |a_m - a| < 2\varepsilon$$

we have

for all $\tilde{\varepsilon} = 2\varepsilon > 0$ there exists a $\tilde{N}(\tilde{\varepsilon})$ s.t.

$$|a_n - a_m| < \tilde{\varepsilon} \quad \text{for all } m,n \geq \tilde{N}(\tilde{\varepsilon}).$$

Thus the convergent sequence $\{a_n\}_{n \geq n_0}$ is a Cauchy sequence.

Exercise 14

Give all subsets of the set $S = \{1,2,3,4\}$.

Please check that the number of subsets with k elements is $\binom{4}{k}$.

Solution for exercise 14

The order of the elements within the subsets is not important here:

$$\begin{aligned}
&\{\emptyset,\\
&\{1\},\{2\},\{3\},\{4\},\\
&\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\},\\
&\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\\
&\{1,2,3,4\}\}
\end{aligned}$$

The result corresponds to the binomial coefficients: 1, 4, 6, 4, 1.