Exercise for induction

Show $n^n > (n+1)!$ for $n \in \mathbb{N} \setminus \{1, 2\}$
$= \{3, 4, 5, \dots\}$

Hint: You may use the binomial theorem by
using the first / last 2 summands!

Proof: by complete induction

i.c. $n = 3$

l.h.s. $n^n = 3^3 = 27$ r.h.s.: $(3+1)!$
$= 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

$27 > 24$ ✓

i.s. to show $(n+1)^{(n+1)} > (n+1+1)! = (n+2)!$
under the i.h. $n^n > (n+1)!$

$$(n+1)^{n+1} = (n+1)\,(n+1)^n \overset{\text{bin.th.}}{=} (n+1)\Bigg(\overbrace{\sum_{k=0}^{n} \binom{n}{k} n^k \underbrace{1^{n-k}}_{=1}}^{(*)}\Bigg)$$

$$2^{n+1} = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2}_{n+1} = 2 \cdot 2^n$$

We observe that every summand in (*) is non-negative.

$$\geq (n+1)\Bigg(\underbrace{\binom{n}{n}}_{=1} n^n + \underbrace{\underbrace{\binom{n}{n-1}}_{=n} n^{n-1}}_{n^n}\Bigg) \geq (n+1) \cdot 2n^n$$

$2n+2 > n+2$

$$\overset{\text{i.h.}}{>} 2(n+1)\,(n+1)! = (2n+2)(n+1)! > (n+2)(n+1)!$$

$= (n+2)!$ ✓ i.s.

$\square$