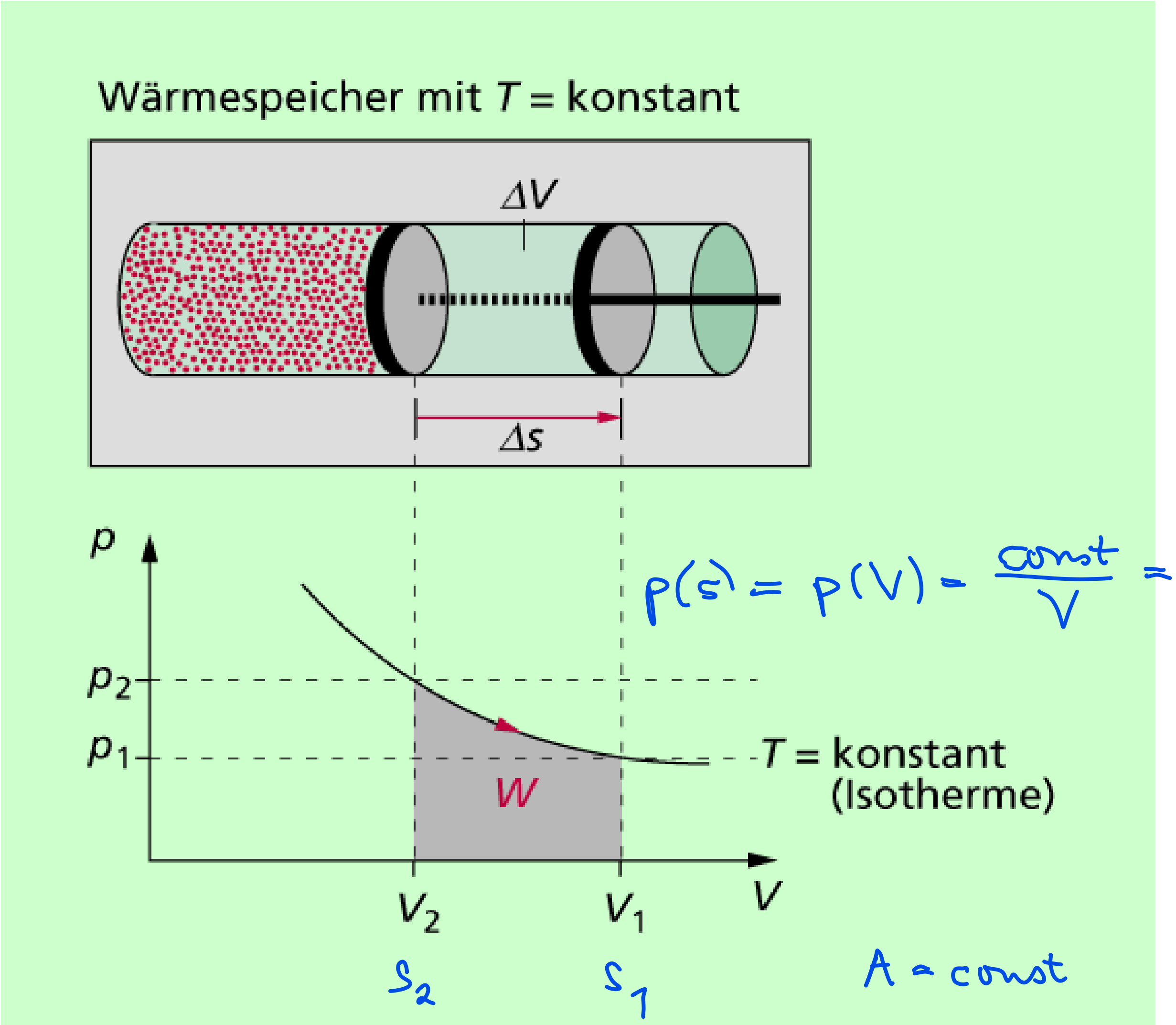
Practical example: isothermal compression of an ideal gas



$$p(s) = p(V) = \frac{\text{const}}{V} = \frac{\text{const}}{A} \frac{1}{s}$$

s_2 s_1

$A = \text{const}$

(Source: lernhelfer.de)

Practical example: isothermal compression of an ideal gas

Example (Isothermal compression of an ideal gas)

The work W carried out on a closed (but non isolated) system for the isothermal compression of an ideal gas from $V_1 = s_1 A$ to $V_2 = s_2 A$ is

$$W = -\int_{s_1}^{s_2} p\,A\,ds. \qquad (dV = A\,ds)$$

For an ideal gas we have $pV = \underbrace{nR}_{k_B}T = const$. Here $V = s\,A$.

Boltzmann constant

Thus we may compute:

$$W = -\int_{s_1}^{s_2} p\,A\,ds = -\int_{s_1}^{s_2} \frac{nRT}{s}\,ds = -nRT\,\underbrace{(\ln(s_2) - \ln(s_1))}_{= \ln\left(\frac{s_2}{s_1}\right)} = nRT\ln\left(\frac{s_1}{s_2}\right).$$

Let $s_2 = 0{,}9 s_1$, $R = 8{,}31$ J/mol/K, $n = 0{,}22$ mol (5l oxygen) and $T = 300$ K, then

$$W = 0{,}22 \cdot 8{,}31 \cdot 300 \cdot \ln(10/9)\ \mathrm{J} \approx 57{,}8\ \mathrm{J}\,.$$

(In general: $W = -\int_{V_1}^{V_2} \frac{nRT}{V}\,dV = -(nRT\ln(V_2) - \ln(V_1)) = RT\ln\left(\frac{V_1}{V_2}\right)$)

Some exercises

1. Compute a primitive for: $a) \int (2x+1)\,dx = x^2 + x + 313$

   $= e^x$

   a) $\int (2x+1)\,dx$, b) $\int \exp(x)\,dx$, c) $\int \frac{3}{1+x^2}\,dx$, $= 3\arctan(x)$
   d) $\int 2(\cos(x)+ax)\,dx$, e) $\int (3x-2)^2\,dx$, f) $\int (1+t^2)\,dx$,
   g) $\int (11+\sqrt{17})\sqrt{x}\,dx$. $= (11+\sqrt{17})\,\frac{2}{3}x^{3/2}$ $= (1+t^2)\,x \quad (t \in \mathbb{R})$

   $d) = +2\sin(x) + ax^2 \quad (a \in \mathbb{R})$
   $e) = \frac{1}{9}(3x-3)^3$

2. Compute <u>all</u> primitives for:

   $t \neq 0$ a) $f(t) = 2e^t - \frac{5}{t} + 1$, b) $f(x) = 3\exp(x) - \cos(x)$, $3e^x - \sin(x) + C,\ C \in \mathbb{R}$
   c) $f(u) = 3\sin(u) - \frac{6}{u} + 7u^2$. $= -3\cos(u) - 6\ln|u| + \frac{7}{3}u^3 + C,\ C \in \mathbb{R}$

   $a) \int f(t) = 2e^t - 5\ln|t| + t + C,\ c \in \mathbb{R}$

3. Which values have the following definite integrals?

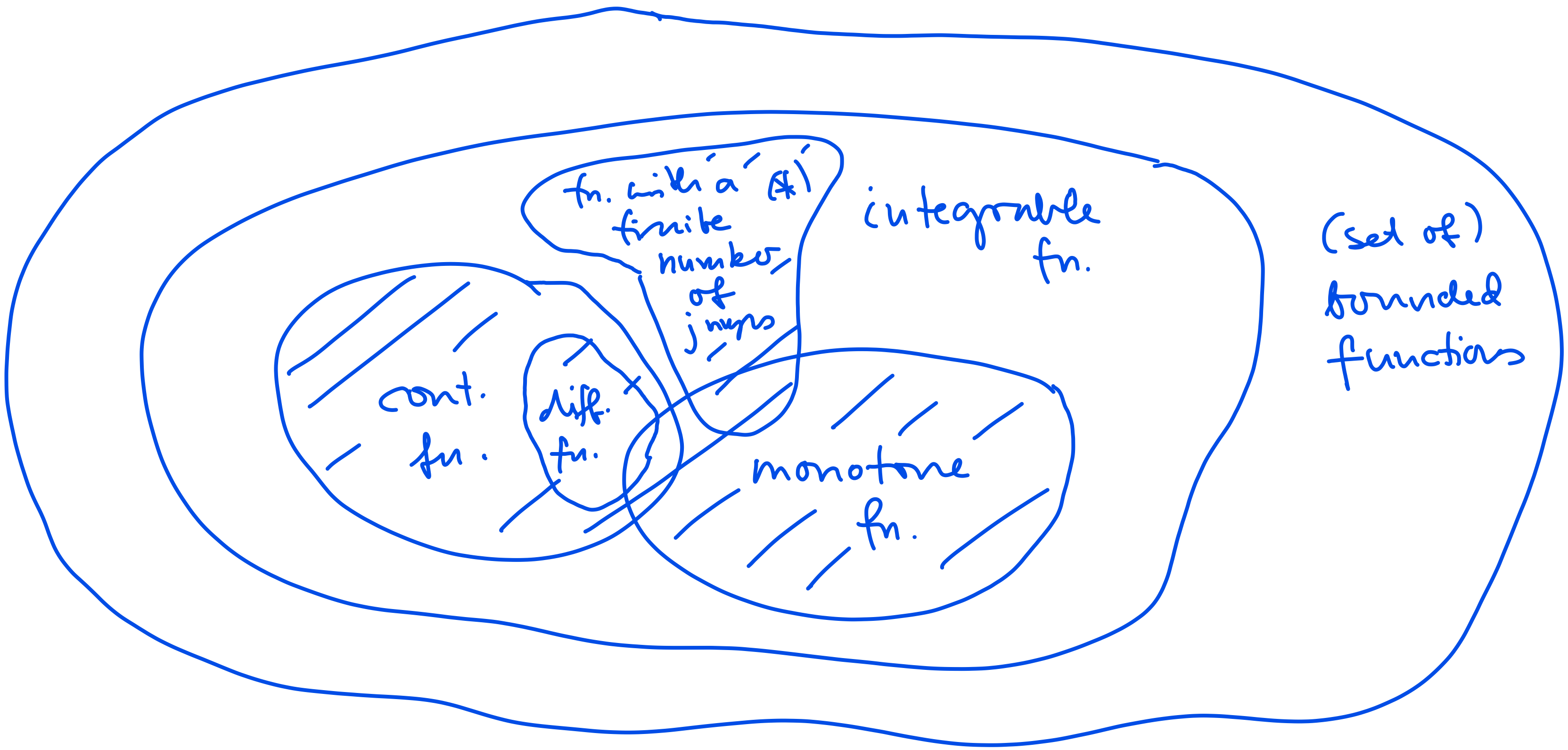
   a) $\int_1^e \frac{1}{t}\,dt$, b) $\int_\pi^2 \cos(\psi)\,d\psi$, c) $\int_1^2 5x^{1/4}$ dx,
   d) $\int_0^4 (4s^5 - 6s^3 + 8x^2 + 5)\,ds$.

4. Based on the velocity-time law

$$v(t) = gt + v_0, \quad t \geq 0,$$

   compute a time law for the falling path $s(t)$ of a free falling body.
   Use $v(t) = s'(t)$.

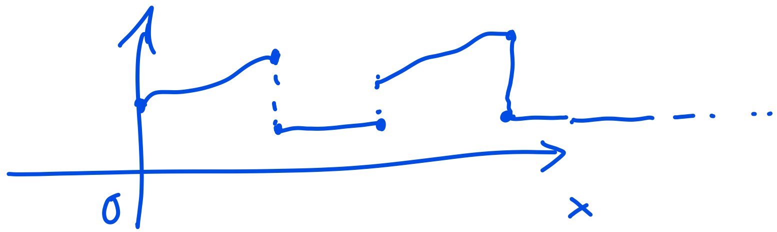
Integrable functions



Integrand f has to be necessarily bounded on $[a, b]$

For instance, all continuous functions are (Riemann) integrable

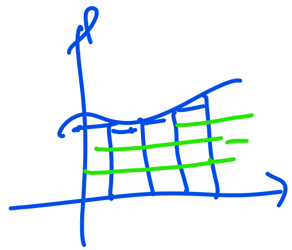
Example for the set (*) :



Other constructions of integrals

- Regulated integral:

  f bounded, f limit of step functions w.r.t. sup norm



- **Riemann integral:**

  generalizes the regulated integral by considering sequences of uniformly convergent integrands

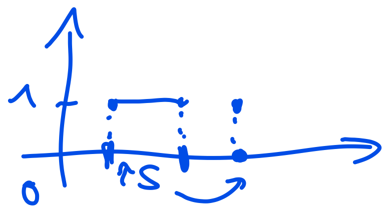
  Partition only of the domain of definition ("vertical stripes")

- Lebesgue integral:
  - More arbitrary partitions are possible
  - Any regulated function is also Lebesgue integrable
  - Characteristic functions of bounded sets are Lebesgue integrable, other measures as the geometrical length (are, . . .) are possible

Characteristic fn.:

$\chi_S(x) = \begin{cases} 1 & ; x \in S \\ 0 & ; \text{otherw.} \end{cases}$

- Stieltjes, Bochner, and Birkhoff integral ...



Here integrable means Riemann integrable.

Small differences that are, e.g., important in probability theory

Practical computation of integrals

Analytical:

- By the fundamental theorem, tables, calculation rules
- (Directly by Riemann sums) ☹
- By an expansion of the integrand into a power series
- ...

⇝ Analysis 2 or later

Numerical (so-called quadrature):

- Midpoint rule (like Riemann sum with t_i in the midpoint of the subinterval)
- Simpson's rule (Kepler's barrel rule)
- Romberg method
- Newton-Cotes formulas
- ...

or by computer algebra systems (Maple, Matlab Symbolic Toolbox, Mathematica . . .)

Calculation rules for integration

Since integration and differentiatin are coupled, we consider how differentiation rules transfer to integration rules.

- Substitution rule (follows from the chain rule)
- Integration by parts (follows from the product rule)

Moreover, we consider

- Integration of rational functions: Partial fraction expansion
- Improper integrals

Chain rule for integration

Theorem (Substitution rule)

Let $I \subseteq \mathbb{R}$ an interval,
$h : I \to \mathbb{R}$ a continuous function and
$f : [a, b] \to \mathbb{R}$ a continuously differentiable function with
$f([a, b]) \subseteq I$,

then

$$\int_a^b h(f(t)) \overbrace{f'(t)\, dt}^{\text{symbolically } = df} = \int_{f(a)}^{f(b)} h(x)\, dx.$$

Proof: Let $H : I \to \mathbb{R}$ a primitive of h, i.e. $H' = h$ (*)

Define $H \circ f : [a,b] \to \mathbb{R}$

By the chain rule of diff. $(H \circ f)'(t) = H'(f(t)) f'(t)$

$= h(f(t)) f'(t)$

$$\int_a^b h(f(t)) f'(t)\, dt = [H \circ f]_{t=a}^{b} = H(f(a)) - H(f(b)) \stackrel{(*)}{=} \int_{f(a)}^{f(b)} h(x)\, dx \quad \square$$

Important special cases of the substitution rule

Let H be a primitive of h.

1) $f: [a,b] \to \mathbb{R},\ t \to ct + d$ with $c, d \in \mathbb{R},\ c \neq 0$

$\Rightarrow f': [a,b] \to \mathbb{R},\ t \longrightarrow c + 0 = c$

$$\int_a^b h(\underbrace{ct+d}_{=f(t)})\,dt = \frac{1}{c}\int_a^b h(ct+d)\underbrace{c}_{f'(t)}\,dt = \frac{1}{c}\int_{ca+d}^{cb+d} h(x)\,dx$$

$$= \frac{1}{c}\Big[H(ct+d)\Big]_{t=a}^{b}$$

2) $h: \mathbb{R} \to \mathbb{R},\ x \mapsto x^n$ with $H(x) = \frac{1}{n+1}x^{n+1},\ n \neq -1$

$$\int_a^b (f(t))^n f'(t)\,dt = \frac{1}{n+1}\Big[(f(t))^{n+1}\Big]_{t=a}^{b}$$

3) $h: \mathbb{R}\setminus\{0\} \longrightarrow \mathbb{R},\ x \longmapsto \frac{1}{x}$ with $H(x) = \ln|x|$
$= \ln(|x|)$

$$\int_a^b h(f(t))\,f'(t)\,dt = \int_a^b \frac{f'(t)}{f(t)}\,dt = \Big[\ln|f(t)|\Big]_{t=a}^{b}$$

Product rule for integration

Theorem (Integration by parts)

Let $a < b$ and $f, g : [a,b] \to \mathbb{R}$ continuously differentiable functions,

then

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

Proof: $\underbrace{\int (f(x)g(x))'\,dx}_{= f(x)g(x) + \text{Const}} = \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$

Example: $\int_a^b \underbrace{x}_{f(x)} \underbrace{e^x}_{g'(x)}\,dx = [x \underbrace{e^x}_{=g(x)}]_a^b - \int_a^b \underbrace{1}_{f'(x)} e^x = [(x-1)e^x]_a^b$, where $\int_a^b 1\, e^x = [e^x]_a^b$