Technische Hochschule Rosenheim WiSe 2021/22
Applied Artificial Intelligence - Bachelor
Analysis 1

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Exercise 8: functions III

Exercise 24

Let

$$z_1 = 1+i, \quad z_2 = 2+i, \quad z_3 = 3+4i, \quad z_4 = 4-3i, \quad z_5 = i.$$

a) Express z_1+z_3, z_1z_2, z_1/z_2, z_1^2, and z_4/z_3 in cartesian coordinates and calculate their modulus.

b) Plot z_1, z_4/z_3, and z_5 and express these complex numbers in polar coordinates.

Solution for exercise 24

a) $z_1+z_3 = 1+i+3+4i = 4+5i \quad \text{and} \quad |z_1+z_3| = \sqrt{4^2+5^2} = \sqrt{41}$

$$z_1z_2 = (1+i)(2+i) = 2+i+2i+i^2 = 1+3i \quad \text{and} \quad |z_1z_2| = \sqrt{1^2+3^2} = \sqrt{10}$$

$$\frac{z_1}{z_2} = \frac{1+i}{2+i} = \frac{(1+i)(2-i)}{(2+i)(2-i)} = \frac{3}{5}+\frac{1}{5}i \quad \text{and} \left|\frac{z_1}{z_2}\right| = \sqrt{\left(\frac{3}{5}\right)^2+\left(\frac{1}{5}\right)^2} = \sqrt{\frac{2}{5}}$$

$$z_1^2 = (1+i)^2 = 1+2i+i^2 = 2i \quad \text{and} \quad |z_1^2| = \sqrt{0^2+2^2} = 2$$

$$z_4/z_3 = \frac{4-3i}{3+4i} = \frac{(4-3i)(3-4i)}{(3+4i)(3-4i)} = \frac{-25i}{25} = -i \text{ and } \left|\frac{z_4}{z_3}\right| = \sqrt{0^2+(-1)^2} = 1$$

b) The polar form is $r\,e^{i\varphi}$ $(= r(\cos(\varphi)+i\sin(\varphi)))$ where r is the modulus and φ is the argument.

We have $|z_1| = \sqrt{1^2+1^2} = \sqrt{2}$. If we plot this point in the complex plane (alternatively by the arccos-formula), then we find $\varphi = \pi/4$.

$$z_1 = \sqrt{2}\,e^{i\pi/4}.$$

We have $|z_4/z_3| = 1$. If we plot this point in the complex plane, then we find $\varphi = -\pi/2$.

$$z_4/z_3 = 1\cdot e^{-i\pi/2} = e^{-i\pi/2}.$$

We have $|z_5| = \sqrt{0^2 + 1^2} = 1$. If we plot this point in the complex plane, then we find $\varphi = \pi/2$.

$$z_5 = 1 \cdot e^{i\pi/2} = e^{i\pi/2}.$$

Exercise 25
Solve for $x \in \mathbb{R}^+$

a) $\ln(\sqrt{x}) + \frac{3}{2}\ln(x) = \ln(21x)$

b) $\exp(x^2 - 2x) = 2$

c) $\ln^2(x) - \ln(x) = 2 + \frac{1}{4}\ln(x^2)$

Solution for exercise 25

a)

$$\begin{aligned}
& \ln(\sqrt{x}) + \frac{3}{2}\ln(x) = \ln(21x) \\
\Leftrightarrow \quad & \ln(x^{1/2}) + \ln(x^{3/2}) = \ln(21x) \\
\overset{\text{funct. eq.}}{\Leftrightarrow} \quad & \ln(x^2) = \ln(21x) \\
\overset{\exp(.)}{\Leftrightarrow} \quad & x^2 = 21x \\
\Leftrightarrow \quad & x = 21 \quad (\text{since } x \neq 0)
\end{aligned}$$

b)

$$\begin{aligned}
& \exp(x^2 - 2x) = 2 \\
\overset{\cdot \exp(1)}{\Leftrightarrow} \quad & \exp(x^2 - 2x + 1) = 2\exp(1) \\
\Leftrightarrow \quad & \exp((x-1)^2) = 2e \\
\overset{\ln(\cdot)}{\Leftrightarrow} \quad & (x-1)^2 = \ln(2e) \\
\overset{\text{funct. eq.}}{\Leftrightarrow} \quad & (x-1)^2 = \ln(2) + 1 \\
\overset{(\cdot)^{1/2}}{\Leftrightarrow} \quad & x - 1 = \pm\sqrt{\ln(2) + 1} \\
\overset{x>0, \ln(2)>0}{\Rightarrow} \quad & x = 1 + \sqrt{1 + \ln(2)}
\end{aligned}$$

c)

$$\begin{aligned}
& \ln^2(x) - \frac{1}{2}\ln(x) = 2 + \frac{1}{4}\ln(x^2) \\
\Leftrightarrow \quad & (\ln(x))^2 - \ln(x) - 2 = 0
\end{aligned}$$

Substitution: $s := \ln(x) \in \mathbb{R}$

$$\begin{aligned}
& s^2 - s - 2 = 0 \\
\Leftrightarrow \quad & s_1 = 2,\ s_2 = -1 \\
\Leftrightarrow \quad & \ln(x_1) = 2,\ \ln(x_2) = -1 \\
\Leftrightarrow \quad & x_1 = e^2 (\approx 7{,}389),\ x_2 = e^{-1} (\approx 0{,}3679)
\end{aligned}$$

Exercise 26
Show for all $x \in \mathbb{R}$

a) $\cosh(-x) = \cosh(x)$, $\sinh(-x) = -\sinh(x)$

b) $\cosh^2(x) - \sinh^2(x) = 1$

c) $\cosh : \mathbb{R} \to \mathbb{R}$ and $\sinh : \mathbb{R} \to \mathbb{R}$ are continuous on $\mathbb{R}$.

d) Addition theorems of cosh and sinh:
For all $x, y \in \mathbb{R}$ there holds:

$$\begin{aligned}\cosh(x+y) &= \cosh(x)\cosh(y) + \sinh(x)\sinh(y),\\ \sinh(x+y) &= \sinh(x)\cosh(y) + \cosh(x)\sinh(y).\end{aligned}$$

Solution for exercise 26

a)

$$\cosh(-x) = \frac{1}{2}(\exp(-x) + \exp(-(-x))) = \frac{1}{2}(\exp(x) + \exp(-x)) = \cosh(x)$$

$$\sinh(-x) = \frac{1}{2}(\exp(-x) - \exp(-(-x))) = -\frac{1}{2}(\exp(x) - \exp(-x)) = -\sinh(x)$$

b)

$$\begin{aligned}\cosh^2(x) - \sinh^2(x) &= \frac{1}{4}(\exp(2x) + 2\exp(x)\exp(-x) + \exp(-2x)) - \frac{1}{4}(\exp(2x) - 2 + \exp(-2x))\\ &= \frac{2-(-2)}{4} = \frac{2+2}{4} = 1\end{aligned}$$

c) $\cosh : \mathbb{R} \to \mathbb{R}$ and $\sinh : \mathbb{R} \to \mathbb{R}$ are an addition of exponential functions that are continuous on $\mathbb{R}$. The addition of continuous functions is again continuous and thus cosh and sinh are continuous on $\mathbb{R}$.

d) Left hand side:

$$\cosh(x+y) = \frac{1}{2}\left(e^{x+y} + e^{-(x+y)}\right)$$

Right hand side:

$$\begin{aligned}\cosh(x)\cosh(y) + \sinh(x)\sinh(y) &= \frac{1}{2}\left(e^x + e^{-x}\right) \cdot \frac{1}{2}\left(e^y + e^{-y}\right) + \frac{1}{2}\left(e^x - e^{-x}\right) \cdot \frac{1}{2}\left(e^y - e^{-y}\right)\\ &= \frac{1}{4}\left(e^x e^y + e^{-x}e^y + e^x e^{-y} + e^{-x}e^{-y}\right) + \frac{1}{4}\left(e^x e^y - e^{-x}e^y - e^x e^{-y} + e^{-x}e^{-y}\right)\\ &= \frac{1}{4}\left(e^x e^y + e^{-x}e^{-y} + e^x e^y + e^{-x}e^{-y}\right)\\ &= \frac{1}{4}\left(e^{x+y} + e^{-(x+y)} + e^{x+y} + e^{-(x+y)}\right)\\ &= \frac{1}{2}\left(e^{x+y} + e^{-(x+y)}\right) \quad \checkmark\end{aligned}$$

Analogously follows the addition theorem for sinh.