Integral for step functions

Integration is the most important application of limits next to the differentiation.

In a 1st step we define an integral for step functions (see Section “Functions”, subsection “Applications for continuous functions”)

With $S[a,b]$ we denote the set of step functions $\tau : [a,b] \to \mathbb{R}$.

Properties of $S[a,b]$:

a) The function being constantly 0 belongs to $S[a,b]$.

b) $\tau \in S[a,b], \lambda \in \mathbb{R} \quad \Longrightarrow \quad \lambda\tau \in S[a,b]$

c) $\tau_1, \tau_2 \in S[a,b] \quad \Longrightarrow \quad \tau_1 + \tau_2 \in S[a,b]$

Proof of property (c)

Linearity of integrals

Theorem (Linearity of an integral (for step functions))

If $\tau_1, \tau_2 \in S[a,b]$ *and* $\lambda \in \mathbb{R}$,

then

$$\int_a^b \left(\tau_1(x) + \tau_2(x)\right)\, dx = \int_a^b \tau_1(x)\, dx + \int_a^b \tau_2(x)\, dx$$

$$\int_a^b \lambda \tau_1(x)\, dx = \lambda \int_a^b \left(\tau_1(x) + \tau_2(x)\right)$$

Monotonicity of integrals

Theorem (Monotonicity of an integral (for step functions))

If $\tau_1, \tau_2 \in S[a,b]$,

then

$$\int_a^b \tau_1(x)\,dx \leq \int_a^b \tau_2(x)\,dx.$$

Generalization of integrals from step functions I

The following defintions prepare the generalization of integrals for step functions to other functions:

Definition (Super-/subintegral)

Let $f : [a, b] \to \mathbb{R}$ a bounded function, then

$$\underline{I}(f) := \inf\left\{\int_a^b \tau(x)\,dx \,|\, \tau \in T[a,b], \tau \geq f\right\}$$

is called a **superintegral** of f and

$$\bar{I}(f) := \sup\left\{\int_a^b \tau(x)\,dx \,|\, \tau \in T[a,b], \tau \leq f\right\}$$

is called a **subintegral** of f.

Evidently, for step functions we have $\underline{I}(f) = \bar{I}(f)$.

Generalization of integrals from step functions II

Theorem (Subaddivity of a superintegral)

Let $f, g : [a, b] \to \mathbb{R}$ bounded functions and let $\lambda \in \mathbb{R}_0^+$, then

$$\bar{I}(f+g) \leq \bar{I}(f) + \bar{I}(g)$$

and

$$\bar{I}(\lambda f) \leq \lambda \bar{I}(f).$$

Analogously we have the superaddivity of a subintegral.

Definition (Riemann integral)

Let $f : [a, b] \to \mathbb{R}$ a bounded function,

then f is called **Riemann integrable** if

$$\underline{I}(f) = \bar{I}(f).$$

We write $I(f) := \bar{I}(f)$.

Evidently, any step function is Riemann integrable.

Characterization of Riemann integrable functions I

Theorem (Criterion for Riemann integrability)

A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable,

iff for any $\varepsilon > 0$ there exist step functions $\tau_1, \tau_2 \in S[a, b]$ with

$$\tau_1 \leq f \leq \tau_2 \quad \textit{and} \quad \int_a^b \tau_2(x) - \int_a^b \tau_1(x) < \varepsilon.$$

Characterization of Riemann integrable functions II

Theorem (Continuity implies Riemann integrability)

Any continuous function

$$f : [a, b] \to \mathbb{R}$$

is Riemann integrable.

Characterization of Riemann integrable functions III

Theorem (Monotonicity implies Riemann integrability)

Any monotone function

$$f : [a, b] \to \mathbb{R}$$

is Riemann integrable.

Riemann integrals by Riemann sums

Theorem (Riemann integrals and sums)

Let $f : [a, b] \to \mathbb{R}$ a Riemann integrable functions,

then there exists for any $\varepsilon > 0$ a $\delta > 0$, such that for any partition

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

of an interval $[a, b]$ with the fineness $\eta \leq \delta$ and any choice of the intermediate points $\xi_k \in [x_{k-1}, x_k], k = 1, \ldots, n$ holds:

$$\left| \int_a^b f(x)\, dx - \sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1}) \right| \leq \varepsilon.$$

In the following we refer with integrable or integrability to Riemann integrable or Riemann integrability, resp.

Properties of integrals I

The following properties transfer from step functions to integrable functions.

Let f, g integrable functions and $a \leq b \leq c$ with a, b, $c \in \mathbb{R}$.

Linearity:

$$\int_a^b f(x) + g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$$

and (factor rule) $\int_a^b c\,f(x)\,dx = c\int_a^b f(x)\,dx$

Additivity: $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

Monotonicity:

$$f \leq g \quad \Longrightarrow \quad \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$$

$$f < g \quad \Longrightarrow \quad \int_a^b f(x)\,dx < \int_a^b g(x)\,dx$$

Properties of integrals II

Inequalities:

"triangle inequality" $\left|\int_a^b f(x)\,dx\right| \leq \int_a^b \left|f(x)\right|\,dx,$

Cauchy-Schwarz

$$\left(\int_a^b f(x)g(x)\,dx\right)^2 \leq \left(\int_a^b f(x)^2\,dx\right)\left(\int_a^b g(x)^2\,dx\right)$$

Change of integration bounds:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

Integration over interval of length zero: $\int_a^a f(x)\,dx = 0$

Computation of integrals: primitive

Definition (Primitive function)

A differentiable[a] A function $F : [a, b] \to \mathbb{R}$ is called a **primitive (function)** of $f : [a, b] \to \mathbb{R}$, if

$$F'(x) = f(x) \quad \text{for all } x \in [a, b]\,.$$

[a] In the boundary points a and b one-sided differentiability is enough.

If there exists a primitive (for f on [a,b]), then it is unique up to a constant C.

Mean value theorem of integration

Theorem (Mean value theorem of integration)

Let $f : [a,b] \to \mathbb{R}$ continuous.

Then there exists a $\xi \in [a,b]$, such that

$$\int_a^b f(x)\,dx = f(\xi)(b-a).$$

Mean value theorem - geometrical interpretation

Fundamental theorem of differentiation and integration

Theorem (Fundamental theorem of differential and integral calculus)

Let $f : [a, b] \to \mathbb{R}$ continuous.

Then

$$I(x) := \int_a^x f(\tilde{x})\, d\tilde{x}$$

is continuously differentiable and there holds $I'(x) = f(x)$.

Thus I is a primitive of f on $[a, b]$.

The fundamental theorem shows,
1.) how to get primitives, and
2.) connects differentiation and integration.

Indefinite integral

Indefinite integral with variable upper bound x:

$$I(x) = \int_a^x f(\tilde{x})\, d\tilde{x} = F(x) + Const = F(x) - F(a)$$

(analogous for variable lower bound),

a **definite integral** is yields a real number

$$\boxed{\int_a^b f(x)\, dx = F(b) - F(a) = [F(x)]_a^b \quad ,}$$

where F is a primitive of f on $[a, b]$.

For a continuous function f its primitive F corresponds to the set of all indefinite integrals:

$$\int f(x)\, dx = F(x) + Const$$

Primitives of common functions

$F(x)$	$F'(x) = f(x)$	$\int f(x)\,dx = F(x) + c$	Bemerkg.
$\frac{1}{n+1}x^{n+1}$	x^n	$\int x^n\,dx = \frac{1}{n+1}x^{n+1} + c$	$n \neq -1$
$\ln\lvert x\rvert$	$\frac{1}{x}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$	$x \neq 0$
$-\cos x$	$\sin x$	$\int \sin x\,dx = -\cos x + c$	
$\sin x$	$\cos x$	$\int \cos x\,dx = \sin x + c$	
$\arctan x$	$\frac{1}{1+x^2}$	$\int \frac{dx}{1+x^2} = \arctan x + c$	
$\frac{1}{2}\ln\frac{1+x}{1-x}$	$\frac{1}{1-x^2}$	$\int \frac{dx}{1-x^2} = \frac{1}{2}\ln\frac{1+x}{1-x} + c$	$\lvert x\rvert < 1$
$\frac{1}{a}e^{ax}$	e^{ax}	$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c$	$a \neq 0$
$\cosh x$	$\sinh x$	$\int \sinh x\,dx = \cosh x + c$	
$\sinh x$	$\cosh x$	$\int \cosh x\,dx = \sinh x + c$	

(Source: [Meyberg, Vachenauer])