Exercise 5 (live tutorial): series

Exercise 15

Decide whether the series

$$\sum_{k=1}^{\infty} a_k$$

converges, converges absolutely, or diverges for the following sequences $\{a_k\}_{k\in\mathbb{N}}$:

a) $a_k = \frac{1}{k^m}, \quad m > 2$

b) $a_k = \frac{1}{\sqrt{k}}$

c) $a_k = \frac{1}{k!}$

d) $a_k = \frac{x^k}{k!}, \quad$ for fixed $x \in \mathbb{R}$

e) $a_k = \frac{x^k}{k}, \quad$ for fixed $x \in \mathbb{R}$

f) $a_k = \begin{cases} q_1^k & ; k \text{ even}, \\ q_2^k & ; \text{otherwise}, \end{cases} \qquad \text{for } 0 < q_1 < q_2 < 1.$

Hint: in e) you may use that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Solution for exercise 15

a) For $m > 2$ we have

$$\frac{1}{k^m} < \frac{1}{k^2}$$

and the majorant $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges absolutely.

By the comparison test $\sum_{k=1}^{\infty} \frac{1}{k^m}$, $m > 2$ converges absolutely.

b) We have $\frac{1}{\sqrt{k}} > \frac{1}{k}$ and the minorant $\sum_{k=1}^{\infty} \frac{1}{k^2}$ diverges (definitely to ∞).

By the comparison test $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges (definitely to ∞).

c) We choose the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \frac{k!}{(k+1)!} = \frac{1}{k+1} \overset{k\to\infty}{\to} 0 < 1.$$

Thus the series converges absolutely.

d) We consider again the ratio test:

$$\left|\frac{a_{k+1}}{a_k}\right| = \left|\frac{x^{k+1}}{x^k}\right| \frac{k!}{(k+1)!} = \frac{|x|}{k+1} \overset{k\to\infty}{\to} 0 < 1.$$

This holds for any fixed $x \in \mathbb{R}$. The exponential series converges absolutely.

e) By the root test:

$$\sqrt[k]{\left|\frac{x^k}{k}\right|} = \frac{|x|}{\sqrt[k]{k}} \overset{k\to\infty}{\to} |x| \overset{!}{<} 1.$$

Thus the series converges for $|x| < 1$ absolutely and diverges for $|x| > 1$.
For $|x| = 1$ we have the "critical" case: for $x = 1$ we recover the harmonic series that diverges definitively to ∞,
for $x = -1$ this is the alternating harmonic series that converges, but does not converge absolutely.

f) Again by the root test:

$$\sqrt[k]{|a_k|} = \begin{cases} q_1 & ; k \text{ even}, \\ q_2 & ; \text{otherwise}. \end{cases}$$

Since $q_1 < 1$ and $q_2 < 1$ the series converges absolutely, though the sequence $\{\sqrt[k]{|a_k|}\}_{k\in\mathbb{N}}$ does not converge. (It has 2 accumulation points, namely q_1 and q_2.)

Exercise 16
Prove:

Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are converging series and c a real constant.
Then the series $\sum_{k=1}^{\infty} (a_k + b_k)$ and $\sum_{k=1}^{\infty} c\, a_k$ converge and there holds for the limits:

$$\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k,$$
$$\sum_{k=1}^{\infty} (a_k - b_k) = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty} b_k,$$
$$\sum_{k=1}^{\infty} c\, a_k + b_k = c \sum_{k=1}^{\infty} a_k.$$

Solution for exercise 16
We use the rules for converging sequences applied to the sequences of partial sums. Moreover, a limit is unique.

Remark: The statement for $\sum_{k=1}^{\infty} c\, a_k$ is also a particular case of the Cauchy product (next Monday in the lecture) of two converging series, but can be proved here directly.