Technische Hochschule Rosenheim
Applied Artificial Intelligence - Bachelor
Analysis 1
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Exercise 3 (live tutorial): induction

Exercise 8
Proof by complete induction:

$$5^n + 7 \qquad \text{is divisible by } 4 \qquad\qquad \text{for} \quad n \in \{0, 1, 2, \ldots\}$$

Solution for exercise 8
Initial case (i.c.) $\underline{n = 0:}$ $\quad 5^0 + 7 = 1 + 7 = 8$ is divisible by 4 ✓

Induction step (i.s.) To show $\underbrace{5^n + 7 \quad \text{is divisible by } 4}_{\text{induction hypothesis (i.h.)}} \qquad \Longrightarrow \qquad 5^{n+1} + 7 \quad \text{is divisible by } 4$

$$5^{n+1} + 7 = 5 \cdot 5^n + 7 = (4+1) \cdot 5^n + 7 = \underbrace{4 \cdot 5^n}_{\text{is divisible by } 4} + \underbrace{5^n + 7}_{\substack{\text{is divisible by } 4 \\ \text{according to (i.h.)}}} \qquad \text{is divisible by } 4$$

$\square$

Exercise 9
Proof by using the induction principle:

$$\text{for all } n \in \mathbb{N}: \quad \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$

a) Check the initial case,

b) state the induction hypothesis, and

c) demonstrate that the induction step holds.

Solution for exercise 9
Claim for all $n \in \mathbb{N}: \quad \sum_{k=0}^{n} 2^k = 2^{n+1} - 1$

(i.c.) $n = 0$

$$\left.\begin{aligned} \text{l.h.s.} &= \sum_{i=0}^{0} 2^i &&= 2^0 = 1 \\ \text{r.h.s.} &= 2^{0+1} - 1 &&= 2 - 1 = 1 \end{aligned}\right\} \quad \text{l.h.s.=r.h.s. ✓}$$

or also possible $n = 1$

$$\left.\begin{array}{ll} \text{l.h.s.} = \sum\limits_{i=0}^{1} 2^i & = 2^0 + 2^1 = 1 + 2 = 3 \\ \text{r.h.s.} = 2^{1+1} - 1 & = 4 - 1 = 3 \end{array}\right\} \quad \text{l.h.s.=r.h.s.} \checkmark$$

(i.s.) To show $\underbrace{\sum_{k=0}^{n} 2^k = 2^{n+1} - 1}_{\text{(i.h.)=induction hypothesis}} \implies \sum_{k=0}^{n+1} 2^k = 2^{n+2} - 1$

<u>Proof (i.s.):</u>

$$\begin{aligned} \sum_{k=0}^{n+1} 2^k &= \left(\sum_{k=0}^{n} 2^k \right) + 2^{n+1} \\ &\overset{\text{(IV)}}{=} \left(2^{n+1} - 1\right) + 2^{n+1} \\ &= \left(2^{n+1} + 2^{n+1}\right) - 1 \\ &= \left(2 \cdot 2^{n+1}\right) - 1 \\ &= 2^{n+2} - 1 \end{aligned}$$

$\square$

Remark:

The statement can be illustrated as follows:
$\sum\limits_{i=0}^{n} 2^i = 2^{n+1}$ corresponds to a binary number with $n+1$ digits, that consist of ones only (since $2^0 = (1)_2, 2^1 = (10)_2, 2^2 = (100)_2, \ldots$). This binary number is smaller by 1 than a digit 1 followed by $n+1$ zeros, corresponding to 2^{n+1}.