Local extrema and mean value theorems

We consider

Problem (Optimization problem (OP))

Minimize $f(x)$

subject to the constraint $x \in X \subseteq \mathbb{R}$.

Thereby let $f : X \to \mathbb{R}$ an at least 1x continuously differentiable function.

If $X = \mathbb{R}$, then we call this an unrestricted optimization problem.

Extrema: minima and maxima

Definition (Minima)

- $\hat{x} \in X$ is called a **global minimum** of (OP), if

$$f(\hat{x}) \leq f(x) \quad \text{for all } x \in X \subseteq \mathbb{R}. \tag{2}$$

- $\hat{x} \in X$ is called a **strict global minimum** of (OP), if in (2) “$<$” for all $x \in X$ holds except for $x = \hat{x}$.

- $\hat{x} \in X$ is called a **local minimum** of (OP), if there exists for a $\varepsilon > 0$ a neighbourhood

$$U_\varepsilon(\hat{x}) := \{x \in \mathbb{R}^n \,|\, |x - \hat{x}| < \varepsilon\}$$

with

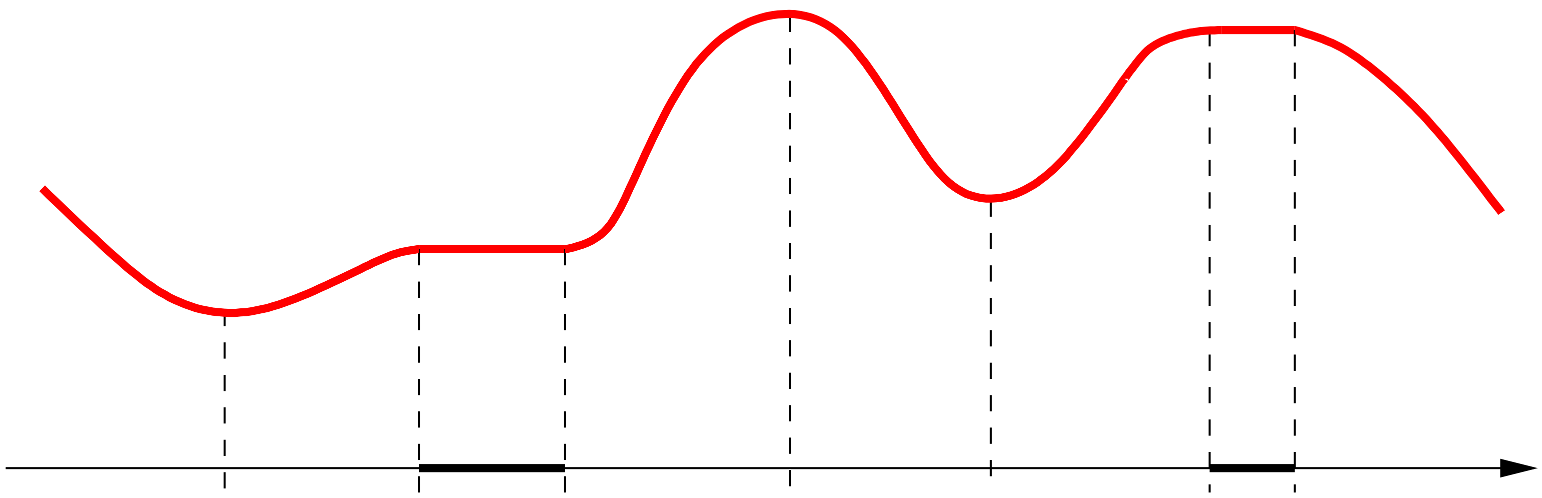
$$f(\hat{x}) \leq f(x) \quad \text{für alle } x \in X \cap U_\varepsilon(\hat{x}). \tag{3}$$

- $\hat{x} \in X$ is called a **strict local minimum** of (OP), if in (3) “$<$” for all $x \in X \cap U_\varepsilon(\hat{x})$ holds except for $x = \hat{x}$.

A strict minimum is also called an isolated minimum.

Analogously corresponding maxima are defined.

Extrema: example - to be completed in the lecture



Theory in 1d

A twice continuously differentiable function $f : \mathbb{R} \to \mathbb{R}$ exhibits a local minimum at $\hat{x}$,

then:

- $f'(\hat{x}) = 0$ (necessary condition of 1st order)
- $f''(\hat{x}) \geq 0$ (necessary condition of 2nd order)

Vice versa let $f'(\hat{x}) = 0$ and $f''(\overline{x}) > 0$,

then $\hat{x}$ is a local minimum of f.

(sufficient condition)

A point $\hat{x}$ where $f'(\hat{x}) = 0$, is called a **stationary point**.

Examples

$$f : (-1, 1) \to \mathbb{R}, x \mapsto x^3$$

$$f : [-1, 1] \to \mathbb{R}, x \mapsto x$$

Rolle theorem

Theorem (Rolle)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ a continuous function that is continuous differentiable in (a, b) with $f(a) = f(b)$,

then there exists a $\xi \in (a, b)$ such that

$$f'(\xi) = 0.$$

Mean value theorem

Corollary (Mean value theorem)

Let $a < b$ and $f : [a, b] \to \mathbb{R}$ a continuous function that is continuously differentiable in (a, b),

then there exists a $\xi \in (a, b)$ such that

$$f'(\xi) = \frac{f(b) - f(a)}{b - a}.$$

Mean value theorem - geometrical interpretation

Geometrically the mean value theorem means that the slope of the secant at the graph of f at the points $(a, f(a))$ and $(b, f(b))$ is equal
to the slope of the tangent line to the graph of f at a point $(\xi, f(\xi))$.

Mean value theorem - generalized version

Theorem (Generalized mean value theorem)

Let $a < b$ and $f, g : [a, b] \to \mathbb{R}$ continuous functions that are continuously differentiable in (a, b), and $g'(x) \neq 0$ for all $x \in (a, b)$,

then there exists a $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$