Computer Science Fundamentals

Number Systems – Conversions

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Overview

- Place-value numeral systems (*Stellenwertsysteme*)
- Conversion of
  - integers (*ganze Zahlen*)
  - fractional numbers (*gebrochene Zahlen*)

Number Representation

- Focus: **efficient** and **reversible** mapping between numbers and bit sequences
- Bit sequences of a fixed length N $\longrightarrow$ 2^N numbers can be represented
- Common choices are $N = 8, 16, 32$ or 64

- Using bit sequences of length N we can code for example
  - the natural numbers from 0 to $2^N - 1$, or
  - the integers between -2^{N-1} und $2^{N-1} - 1$, or
  - an interval of the real numbers with limited accuracy.

Place-value Numeral Systems

- Also called positional numeral system or positional/place-value notation

- The value of a number depends on the **position** of a digit
- Advantage: simple calculation rules

- Examples:
  - Dual system (also: Binary system)
  - Octal system
  - Decimal system
  - Hexadecimal system

Place-value Notation – Natural Numbers

- a positional system with base B is a numeral system, in which a number is broken down according to powers of B
- A natural number n is represented by the following sum: $n = \sum_{i=0}^{N-1} b_i B^i$
  - $B =$ Base of the numeral system $(B \in \mathbb{N}, B \geq 2)$,
  - $b_i =$ digits $(b_i \in \mathbb{N}_0, 0 \leq b_i < B)$,
  - $N =$ number of digits

Decimal System

Representation of an integer z

- Sum of powers of base 10
- $z = a_n 10^n + a_{n-1} 10^{n-1} + \ldots + a_2 10^2 + a_1 10^1 + a_0 10^0$,
  where $a_0, a_1, a_2, \ldots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$
- Example

$$\begin{aligned} 4711 &= 4 \cdot 10^3 + 7 \cdot 10^2 + 1 \cdot 10^1 + 1 \cdot 10^0 \\ &= 4 \cdot 1000 + 7 \cdot 100 + 1 \cdot 10 + 1 \end{aligned}$$

Dual System

- Representation of numbers by base 2 using the digits {0, 1}
- For example, the bit sequence 1101 has the numerical value:

$$\begin{aligned} 1101 &= 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\ &= 1 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \\ &= 13 \end{aligned}$$

- Notation: $1101_2 = 13_{10}$

Octal System

- Disadvantage of **dual** system: **very long** numbers, therefore difficult to read/remember
  - Idea: Combine a certain number of binary digits

- Octal system
  - 3 binary digits are combined into a single octal digit
  - Representation of numbers by base $2^3 = 8$ using digits {0, 1, 2, 3, 4, 5, 6, 7}
  - Examples

$$4711_8 = 4 \cdot 8^3 + 7 \cdot 8^2 + 1 \cdot 8^1 + 1 \cdot 8^0 = 2505_{10}$$

$$53_{10} = 110\ 101_2 = 65_8$$

Hexadecimal System

- Even more compact number representation
  - **4** binary digits are combined into a hexadecimal one
  - Representation of numbers by base $2^4 = 16$ using digits {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}

- Examples

$$53_{10} = 0011\ 0101_2 = 35_{16}$$

$$4711_8 = 100\ 111\ 001\ 001_2 = 1001\ 1100\ 1001_2 = 9C9_{16}$$

Conversion from any Base to Decimal

- can be done using the sum equation directly: $n = \sum_{i=0}^{N-1} b_i B^i$
- or, more efficiently, using Horner's Scheme:

$$n = \sum_{i=0}^{N-1} b_i B^i = \left(\cdots\left(\left(\left(b_{N-1} \cdot B + b_{N-2}\right) \cdot B + b_{N-3}\right) \cdot B + b_{N-4}\right) \cdot B + \cdots + b_1\right) \cdot B + b_0$$

Examples:

$1578_{10} = ((1 \cdot 10 + 5) \cdot 10 + 7) \cdot 10 + 8$

$754_8 = (7 \cdot 8 + 5) \cdot 8 + 4$
$= 492_{10}$

Exercise

Convert the number 2375_9 to decimal using Horner's Scheme.

Conversion from Decimal to any Base

- Tables
- Direct methods
- Repeated division by base

Direct Methods: Tables

Can be used for small numbers

Decimal	Dual	Octal	Hexadecimal
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

Direct Methods: Decomposition (by Integer Division)

Conversion from Decimal to Dual

- Divide the decimal number to be converted by the greatest power of 2 less than or equal to that decimal number; write down a 1 for the leftmost (highest) binary digit

- Divide the result by the next smaller power of 2
(result – 0 or 1 – indicates the next binary digit)

- Proceed until division by $2^0 = 1$ terminates the procedure

This is basically a decomposition of the decimal into a sum of powers of 2

Direct Methods: Decomposition Example

Convert 116_{10} to binary

```
116 : 64 = 1
-64
 52 : 32 = 1
-32
 20 : 16 = 1
-16
  4 :  8 = 0
 -0
  4 :  4 = 1
 -4
  0 :  2 = 0
 -0
  0 :  1 = 0
```

$\longrightarrow 116_{10} = 111\ 0100_2$

Direct Methods: Dual ⟷ Octal

Conversions between **Dual system and Octal system**

- To convert a number represented in the dual system to the octal system, so-called dual triads (groups of three) are formed starting from the right.

Dual	110	111	001	110	010
Octal	6	7	1	6	2

- When converting an octal number into its dual representation, we use the opposite way

Octal	3	2	1	5
Dual	011	010	001	101

Direct Methods: Dual ⟷ Hexadecimal

Conversions between **Dual system and Hexadecimal system**

- To convert a number represented in the dual system to the hexadecimal system, so-called dual tetrades (groups of four) are formed starting from the right.

Dual	1	0111	0101	1101
Hex	1	7	5	D

Triad: 1 011 101 011 101
Octal: 1 3 5 3 5

- … and for Hex ⟶ Dual:

Hex	A	D	A
Dual	1010	1101	1010

Triad: 101 011 011 010
Octal: 5 3 3 2

Exercises

- Convert the number $2E4_{16}$ to dual as well as to decimal
- Convert the number 753_8 to dual as well as to hexadecimal
- Convert the number 1101011111111010_2 to octal as well as to hexadecimal

Repeated Division by Base

Conversion from decimal system to any base

- Continued division of a decimal number by the desired base
- returns the new digits as a division remainder

Algorithm to convert a decimal number x to a numeral system with base B:

1. x : B = y remainder z
2. Make y the new x
   - if this x is not equal to 0, proceed with step 1,
   - otherwise proceed with step 3
3. The resulting remainders z written read bottom to top = left to right are the converted number

Repeated Division by Base – Example

- $43_{10} = ?_2$

x		B		y		z
43	:	2	=	21	Rem.	1
21	:	2	=	10	Rem.	1
10	:	2	=	5	Rem.	0
5	:	2	=	2	Rem.	1
2	:	2	=	1	Rem.	0
1	:	2	=	0	Rem.	1

The remainders z written from bottom to top provide the dual number you are looking for.

- Result: 101011_2

Exercise

Convert the number **10172_{10}** using repeated division by base to the dual system as well as the hexadecimal system

10172_{10}
$= \boxed{}_2$
$= \boxed{}_{16}$

Place-value Notation – Fractions

- For fractions, a point separates the integer part of a number from the fractional part.

- A fractional number n is represented by the following sum: $n = \sum_{i=-M}^{N-1} b_i B^i$
  - $B =$ Base of the numeral system $(B \in \mathbb{N}, B \geq 2)$,
  - $b_i =$ digits $(b_i \in \mathbb{N}_0, 0 \leq b_i < B)$,
  - $N =$ number of digits to the left of the point
  - $M =$ number of digits to the right of the point

Decimal Fractions – Examples

$$17{,}05_{10} = 1 \cdot 10^1 + 7 \cdot 10^0 + 0 \cdot 10^{-1} + 5 \cdot 10^{-2}$$

$$3758{,}0_{10} = 3 \cdot 10^3 + 7 \cdot 10^2 + 5 \cdot 10^1 + 8 \cdot 10^0$$

$$9{,}702_{10} = 9 \cdot 10^0 + 7 \cdot 10^{-1} + 0 \cdot 10^{-2} + 2 \cdot 10^{-3}$$

$$0{,}503_{10} = 0 \cdot 10^0 + 5 \cdot 10^{-1} + 0 \cdot 10^{-2} + 3 \cdot 10^{-3}$$

Converting Pure Fractions to Decimal

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- A pure fraction n $(n < 1)$ can be represented using Horner's Scheme as

$$n = \sum_{i=-M}^{-1} b_i B^i = \frac{1}{B}\left(b_{-1} + \frac{1}{B}\left(b_{-2} + \frac{1}{B}\left(b_{-3} + \cdots + \frac{1}{B}\left(b_{-M+1} + \frac{1}{B} b_{-M}\right)\right)\right)\right)$$

- Examples

$$0{,}193_{10} = \frac{1}{10} \cdot \left(1 + \frac{1}{10} \cdot \left(9 + \frac{1}{10} \cdot 3\right)\right)$$

$$0{,}011_2 = \frac{1}{2} \cdot \left(0 + \frac{1}{2} \cdot \left(1 + \frac{1}{2} \cdot 1\right)\right) = 0.375_{10}$$

Converting Pure Fractions from Decimal

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Algorithm for converting the fractional part of a decimal number to a system with base B:

1. $x \cdot B = y$ overflow z (z = integer part of y, i.e., left of decimal point)
2. Make fractional part of y the new x
   - if this new x is not equal to 0, and more fractional digits are required, go to step 1,
   - otherwise proceed to step 3
3. Write down the determined overflow-digits from top to bottom on the right side of “0.” to get the corresponding converted number

Converting Pure Fractions from Decimal – Example

- $0.34375_{10} = ?_2$

x	B		y	z	
0.34375	· 2	=	0.6875	overflow	0
0.6875	· 2	=	1.375	overflow	1
0.375	· 2	=	0.75	overflow	0
0.75	· 2	=	1.5	overflow	1
0.5	· 2	=	1.0	overflow	1
0.0	· 2	=	0.0	overflow	0

The overflows z written from top to bottom after 0. provide the dual number you are looking for.

- Result: 0.01011_2

Converting Pure Fractions from Decimal

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Some fractional numbers that can be represented exactly in the decimal system cannot be represented exactly as a dual number

- Typical example
  $0.1_{10} = 0.0\ 0011\ 0011\ \ldots_2$
- Periodic sequence of digits, repeating bit pattern 0011
- This leads to a loss of accuracy in the computer

x		B		y		z
0.1	·	2	=	0.2	overf.	0
0.2	·	2	=	0.4	overf.	0
0.4	·	2	=	0.8	overf.	0
0.8	·	2	=	1.6	overf.	1
0.6	·	2	=	1.2	overf.	1
0.2	·	2	=	0.4	overf.	0
0.4	·	2	=	0.8	overf.	0
0.8	·	2	=	1.6	overf.	1
0.6	·	2	=	1.2	overf.	1

Converting Pure Fractions from Decimal

- The other way around: **All** pure fractions that can be represented exactly in the dual system can also be represented exactly as a decimal number

- General: Any rational number *p/q* with gcd(*p*, *q*) = 1 can be represented exactly to base *B*, if all prime factors of *q* are also prime factors of *B*

- Examples
  - $1/3_{10}$ $= 0.33333..._{10}$ 3 is not a prime factor of 10
  - $1/3_{10}$ $= 0.010101..._{2}$ 3 is not a prime factor of 2
  - $1/3_{10}$ $= 0.1_{3}$ 3 is a prime factor of 3
  - $1/10_{10}$ $= 0.1_{10}$ 2 and 5 are prime factors of 10
  - $1/10_{10}$ $= 0.000110011..._{2}$ 5 is not a prime factor of 2

Exercises

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Convert the following fractional numbers

- 0.375_{10} = dual?

- 0.25_{10} = in the base 5 system?

- 0.19_{10} = hexadecimal?

Converting Non-pure Fractional Numbers

- Split the number
  - into its integer part
  - and its fractional part,

  and convert separately.

- Example $12.25_{10} = 1100.01_2$

integer part

```
12 : 2 = 6   Rem.   0
 6 : 2 = 3   Rem.   0
 3 : 2 = 1   Rem.   1
 1 : 2 = 0   Rem.   1
```

$12_{10} = 1100_2$

fractional part

```
0.25 · 2 = 0.5   overf.   0
0.5  · 2 = 1.0   overf.   1
0.0  · 2 = 0.0   overf.   0
```

$0.25_{10} = 0.01_2$

Exercise

Convert **39.6875$_{10}$** to binary as well as the hexadecimal system.