Technische Hochschule Rosenheim
Applied Artificial Intelligence - Bachelor
Analysis 1

WiSe 2021/22

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Homework 6: Cauchy product, continuity

To submit: on Thursday, 18.11.2021, 9:30 a.m., online by the learning campus

Exercise 1 (9 pts.)

a) Compute the Cauchy product of the series

$$\sum_{k=0}^{\infty} \frac{1}{9^k}$$

with the series

$$\sum_{k=0}^{\infty} \frac{1}{3^k}.$$

$$c_n = \sum_{k=0}^{n} \frac{1}{9^k} \cdot \frac{1}{3^{n-k}}$$

b) Consider the alternating series $\sum_{k=0}^{\infty} r_k$ with

$$\{r_k\}_{k\in\mathbb{N}_0} = \frac{(-1)^k}{\sqrt{k+1}}.$$

Show that the Cauchy product of $\sum_{k=0}^{\infty} r_k$ with itself is not absolutely convergent.

Please explain why this is no contradiction to the result derived in the lecture!

Exercise 2 (3 pts.)

Consider the fractional rational function

$$f : \mathbb{R} \setminus \{-1\} \to \mathbb{R}, x \mapsto f(x) = \frac{x^2-1}{x+1}.$$

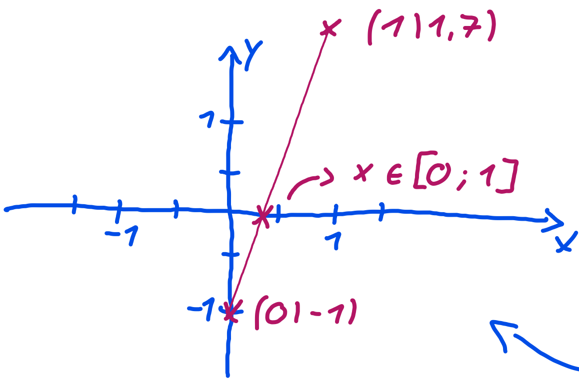
Explain why f is not continuous ! it's not continuous because x = -1 would be a division by zero.
How could you derive from f a new function g by complementing $g(-1)$ such that the function g becomes continuous in $x = -1$, too?

$$\frac{x^2-1}{x+1} \Rightarrow \frac{x-1}{1} = x-1 \qquad g(-1) = -1-1 = -2$$

Exercise 3 (3 pts.)

Demonstrate that there exists a solution $x \in [0,1]$ for the equation

$$\exp(x) = \frac{1}{x}.$$



$$e^x = \frac{1}{x} \quad |\cdot x$$

$$xe^x = 1 \quad |-1$$

$$xe^x - 1 = 0$$

$x = 0$

$\to 0 \cdot e^0 - 1 = -1$

$x = 1$

$\to 1 \cdot e^1 - 1 \approx 1{,}7$

4b

$$c_n = \sum_{k=0}^{n} \frac{(-1)^k}{\sqrt{k+1}} \cdot \frac{(-1)^{n-k}}{\sqrt{(n-k)+1}}$$

$$= (-1)^{k+n-k} \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}} \cdot \frac{1}{\sqrt{n-k+1}}$$

$$= (-1)^n \sum_{k=0}^{n} \frac{1 \cdot 1}{\sqrt{(k+1)\cdot(n-k+1)}}$$

$$(k+1)(n-k+1) = \left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2$$

$$\sqrt{(k+1)(n-k+1)} = \sqrt{\left(\frac{n}{2}+1\right)^2 - \left(\frac{n}{2}-k\right)^2}$$

$$\leq \sqrt{\left(\frac{n}{2}+1\right)^2} = \frac{n}{2}+1$$

$$|c_n| = \sum_{k=0}^{n} \frac{1}{\sqrt{(k+1)(n-k+1)}}$$

$$\geq \sum_{k=0}^{n} \frac{1}{\frac{n}{2}+1}$$

≥ 1 -> c_n doesn't converge to 0

-> the cauchy product diverges

There is no contradiction because the sums aren't abs. convergent themselves.