Injectivity

A function $f : A \to B$ is called **injective**, iff for any $x_1, x_2 \in A$ there holds

$$f(x_1) = f(x_2) \quad \Leftrightarrow \quad x_1 = x_2$$

The **injectivity** of a function depends on the mapping rule and on the domain of definition.

Surjectivity

A function $f : A \to B$ is called **surjective**,
iff for any $y \in B$ there exists a $x \in A$ such that $f(x) = y$.

Equivalently, iff $f(A) = B$,
$f(A)$ being the range of values, B the target area.

Bijectivity

Functions that are injective and surjective are called **bijective**.

Thus, for a bijective function

$$f : A \to B, \quad x \mapsto f(x)$$

there exists for any $y \in B$ exactly one $x \in A$ with $f(x) = y$.

Inverse function

Hence there exists a mapping

$$f^{-1} : B \to A, \quad y \mapsto f^{-1}(y)$$

with $f^{-1}(f(x)) = x$ for any $x \in A$ and
with $f(f^{-1}(y)) = y$ for any $y \in B$.

f^{-1} is called the **inverse mapping** or **inverse function** or **inverse** of f.

Identity and identical

For any subset A we may define the function

$$Id_A : A \to A, \quad x \mapsto x,$$

called the **identity on** A.

Two mappings

$$f : A \to B, \quad x \mapsto f(x),$$
$$g : C \to D, \quad z \mapsto g(z)$$

are called **identical**, iff $\Leftrightarrow$

e.g. $f(x) = 0$

$f \equiv 0$

$f = 0$

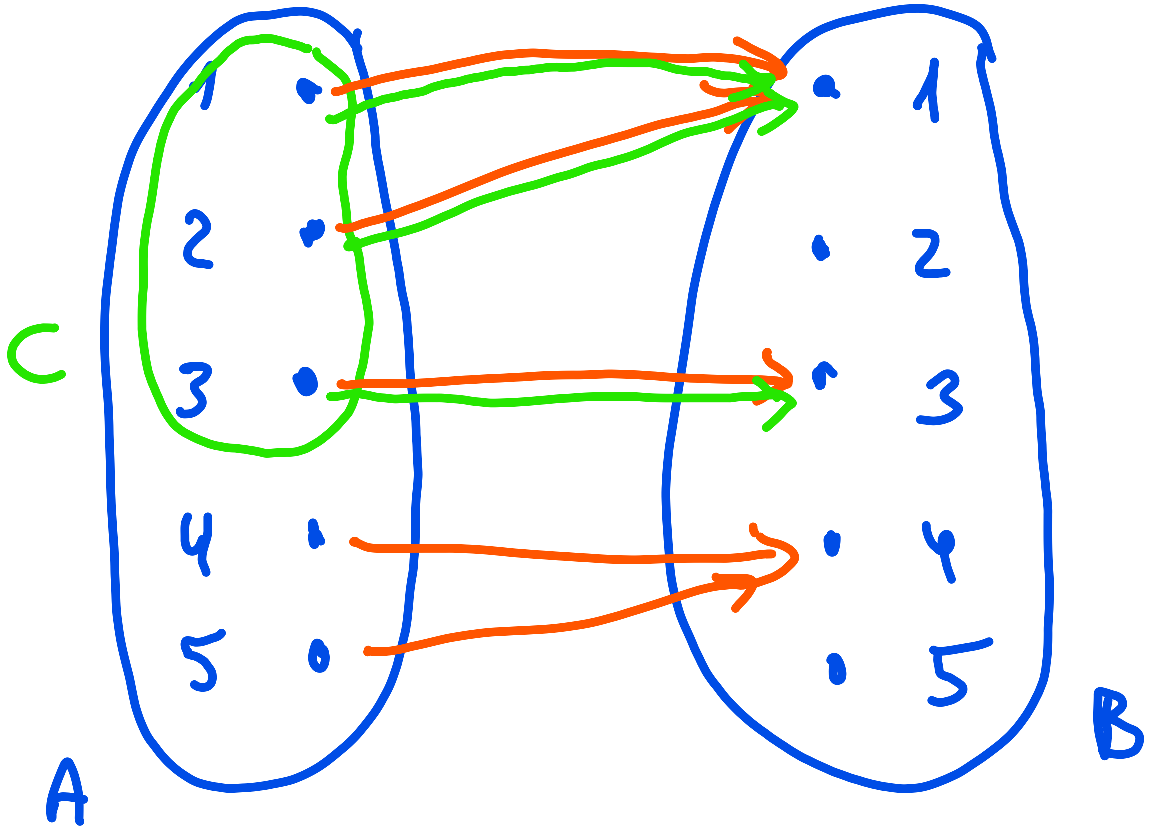
- $A = C$,
- $B = D$, and
- $f(x) = g(x)$ for all $x \in A$.

Then we write $f = g$.   or $f \equiv g$

Restriction

Let $f : A \to B, x \mapsto f(x)$.
By restricting the domain A to a subset $C \subseteq A$, we obtain the **restriction** of f to C:

$$f|_C : C \to B, \quad x \mapsto f(x).$$



Numbers

Special (number) sets:

$\mathbb{N} = \{1, 2, 3, \ldots\}$ $\mathbb{N}$ natural numbers

$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$ $\mathbb{N}_0$ natural numbers incl. zero

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3 \ldots\}$ integers (whole numbers ?)

$\mathbb{Z}$

$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}$ $\mathbb{Q}$ rational numbers

$\mathbb{R} \approx$ “all” numbers in 1d $\mathbb{R}$ real numbers

$\mathbb{C} \approx \mathbb{R}^2$ with “special structure” complex numbers

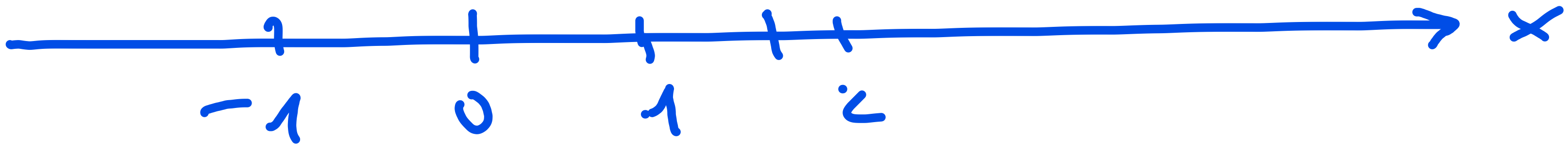
$\mathbb{C}$

Number line

Every point of the number line corresponds to a real number:

$\pi \approx 3{,}14\ldots,\ e = \exp(1) \approx 2{,}71\ldots \in \mathbb{R},\ \notin \mathbb{Q}$

$\sqrt{2} \approx 1{,}41\ldots$



Claim: $\sqrt{2} \notin \mathbb{Q}$ but $\sqrt{2} \in \mathbb{R}$

Proof: suppose $\sqrt{2} \in \mathbb{Q} \Rightarrow \sqrt{2} = \frac{p}{q},\ p,q \in \mathbb{Z},\ q \neq 0$

w.l.o.g. p and q do not have a common divisor (p and q are coprime)

$\sqrt{2} = \frac{p}{q} \overset{(\,)^2}{\Longrightarrow} 2 = \frac{p^2}{q^2} \overset{q\neq 0}{\Longleftrightarrow} 2q^2 = p^2 \Longrightarrow$

p^2 is even $\Rightarrow p$ is even, i.e. $p = 2n,\ n \in \mathbb{Z}$

$\Rightarrow 2q^2 = 4n^2 \overset{:2}{\Longrightarrow} q^2 = 2n^2 \Rightarrow q$ is even $\lightning$

Order of the real numbers

The real numbers are ordered completely by the order relation $\geq$:
For all $x, y \in \mathbb{R}$ there holds:

$$x \geq y :\Leftrightarrow (x - y) \geq 0$$

$\leq$ $\leq$

Analogously:

$x \leq y$

Alternatively, for all $x, y \in \mathbb{R}$ there holds:

$$x > y :\Leftrightarrow (x - y) > 0$$

$<$ $<$

$x < y$

The comparability of real numbers is called the **order** of the real numbers.

Numbers - overview

There holds:

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Field axioms 1

e.g. $\mathbb{F} = \mathbb{R}$
$= \mathbb{C}$
$= \{0; 1\}$

Definition (Field)

A non-empty set $\mathbb{F}$ equipped with 2 operations that are closed **(F0)**

$$\oplus : \overset{a}{\mathbb{F}} \times \overset{b}{\mathbb{F}} \to \mathbb{F} \ni c \qquad c = a + b$$
$$\otimes : \mathbb{F} \times \mathbb{F} \to \mathbb{F} \qquad ab = a * b$$

is called a field, supposed the following conditions **(F1)** – **(F5)** hold:

(F1) For all $a, b \in \mathbb{F}$ we have the **commutative laws**

$$a \oplus b = b \oplus a, \quad a \otimes b = b \otimes a.$$

(F2) For all $a, b, c \in \mathbb{F}$ we have the **associative laws**

$$(a \oplus b) \oplus c = a \oplus (b \oplus c), \quad (a \otimes b) \otimes c = a \otimes (b \otimes c).$$

...

Field axioms 2

Definition (Field (continued))

(F3) For all $a \in \mathbb{F}$ there exists a **neutral element w.r.t.** $\oplus$, $e_\oplus \in \mathbb{F}$, such that

$$a \oplus e_\oplus = a$$

and for all $a \in \mathbb{F} \setminus e_\oplus$ there exists a **neutral element w.r.t.** $\otimes$, $e_\otimes \in \mathbb{F}$, such that

$$a \otimes e_\otimes = a.$$

(F4) For all $a \in \mathbb{F}$ there exists an **inverse element w.r.t.** $\oplus$, $a_\oplus^{-1} \in \mathbb{F}$, such that

$$a \oplus a_\oplus^{-1} = e_\oplus$$

and for all $a \in \mathbb{F} \setminus e_\oplus$ there exists an **inverse element w.r.t.** $\otimes$, $a_\otimes^{-1} \in \mathbb{F}$, such that

$$a \otimes a_\otimes^{-1} = e_\otimes.$$

(F5) For all $a, b, c \in \mathbb{F}$ there holds the **distributive law**

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c).$$

Axioms of ordered fields

Definition (Ordered fields)

If $\mathbb{F}$ is a field and
if some elements of $\mathbb{F}$ are designated as positive,
then $\mathbb{F}$ is called ordered if:

(A1) For all $a \in \mathbb{F}$ there holds exactly one of the following relations

- a is positive,
- $a = e_\oplus$ (is neutral), or
- $a_\oplus^{-1}$ is positive.

(A2) For all $a, b \in \mathbb{F}$ positive this implies $a \oplus b$ is positive.

(A3) For all $a, b \in \mathbb{F}$ positive this implies $a \otimes b$ is positive.

Axiom of Archimedes for ordered fields

Definition (Archimidean ordered fields)

Let $\mathbb{F}$ be an ordered field.
$\mathbb{F}$ is called an Archimidean ordered field, if in addition holds:

(AR) For any 2 positive $a, b \in \mathbb{F}$ there exists an $n \in \mathbb{N}$ such that:

$$\underbrace{a \oplus a \oplus \ldots \oplus a}_{n \text{ times}} \oplus b_{\oplus}^{-1} \text{ is positive.}$$

Remarks to the field axioms

- Both neutral elements, $e_\oplus$ and $e_\otimes$, are unique.
- In (F4) the neutral element w.r.t. $\oplus$ has to be excluded.
  There exists no inverse element for $e_\oplus$ w.r.t. $\otimes$.
- If we set $e_\oplus = e_\otimes$, then there would hold for all $a \in \mathbb{F}$

$$a \otimes e_\oplus = e_\oplus = e_\otimes = a \otimes e_\otimes = a,$$

  i.e. $a = e_\otimes$ and thus $\mathbb{F} = \{e_\otimes\}$.
- Both inverse elements, $a_\oplus^{-1}$ and $a_\otimes^{-1}$, are uniquely defined.