Priv.-Doz. Dr. S.-J. Kimmerle Thursday, 02.12.2021

Exercise 9: differential calculus I

Exercise 27

Compute for $x \in \mathbb{R}$ the derivatives of

a) $f(x) = a^x$ where $a \in \mathbb{R}^+$,

b) $g(x) = \cot(x)$ restricted to $(0, \pi)$,

c) $h(x) = \sinh(x)$.

d) $j(x) = \cosh(x)$.

e) $k(x) = \ln\left(1 + (1+x^2)^4\right)$

Solution for exercise 27

a)

$$\begin{aligned} f'(x) &= \frac{d}{dx} \exp(x \ln(a)) \\ &= \exp(x\ln(a)) \cdot \frac{d}{dx}(x \ln(a)) \\ &= \exp(x\ln(a)) \cdot \ln(a) \\ &= a^x \cdot \ln(a) \end{aligned}$$

by using the chain rule and the derivatives for exp and ln.

b)

$$\begin{aligned} g'(x) &= \frac{\cos(x)}{\sin(x)} \\ &= \frac{-\sin(x)\sin(x) - \cos(x)\cos(x)}{\sin^2(x)} \\ &= -\frac{1}{\sin^2(x)} \\ &\left(= -1 - \cot^2(x)\right) \end{aligned}$$

by using the quotient rule and $\sin^2(x) + \cos^2(x) = 1$.

c)

$$\begin{aligned} h'(x) &= \frac{d}{dx}\sinh(x) \\ &= \frac{d}{dx}\frac{1}{2}\left(\exp(x)-\exp(-x)\right) \\ &= \frac{1}{2}\left(\frac{d}{dx}\exp(x)-\frac{d}{dx}\exp(-x)\right) \\ &= \frac{1}{2}\left(\exp(x)+\exp(-x)\right) \\ &= \cosh(x) \end{aligned}$$

by using the definition of sinh and cosh, and the derivative of exp.

d)

$$\begin{aligned} h'(x) &= \frac{d}{dx}\cosh(x) \\ &= \frac{d}{dx}\frac{1}{2}\left(\exp(x)+\exp(-x)\right) \\ &= \frac{1}{2}\left(\frac{d}{dx}\exp(x)+\frac{d}{dx}\exp(-x)\right) \\ &= \frac{1}{2}\left(\exp(x)-\exp(-x)\right) \\ &= \sinh(x) \end{aligned}$$

by using the definition of sinh and cosh, and the derivative of exp.

e)

$$\begin{aligned} k'(x) &= \frac{1}{1+(1+x^2)^4}\cdot 4\left(1+x^2\right)^3\cdot 2x \\ &= \frac{1}{1+(1+x^2)^4}\cdot 8x\left(1+x^2\right)^3 \end{aligned}$$

by using the chain rule twice.

Exercise 28
Show for $x \in \mathbb{R}$ and $n \in \mathbb{N}$

$$\left(\frac{d}{dx}\right)^n \left(x^2 e^x\right) = \left(x^2+2nx+n(n-1)\right)e^x.$$

Solution for exercise 28
We solve by induction over n.

Initial case $n=1$:

$$\begin{aligned} \frac{d}{dx}\left(x^2e^x\right) &= 2xe^x + x^2e^x \\ &= \left(x^2+2x+0\right)e^x \quad \checkmark \end{aligned}$$

We assume the formula holds for n (IA) and show the induction step $n \mapsto n+1$:

$$\begin{aligned}
\left(\frac{d}{dx}\right)^{n+1}\left(x^2 e^x\right) &= \frac{d}{dx}\left(\frac{d}{dx}\right)^{n}\left(x^2 e\right) \\
&\overset{\text{(IA)}}{=} \frac{d}{dx}\left(\left(x^2+2nx+n(n-1)\right)e^x\right) \\
&= \frac{d}{dx}\left(x^2+2nx+n(n-1)\right)e^x + \left(x^2+2nx+n(n-1)\right)e^x \\
&= (2x+2n)\,e^x + \left(x^2+2nx+n^2-n\right)e^x \\
&= \left(x^2+2(n+1)x+n^2+n\right)e^x \\
&= \left(x^2+2(n+1)x+(n+1)n\right)e^x \quad \checkmark
\end{aligned}$$

$\square$

Exercise 29

a) Determine the equation of the tangent line to the graph of the function

$$f(x) = \sqrt{16-x^2}, \quad x \in (-4,4)$$

at the point $x_0 = 1$.

b) Let a curve

$$y = \frac{1}{3}x^3 - x$$

in the real plane be given.

At which point(s) is the tangent line of this curve parallel to the straight line with the equation

$$y = \frac{1}{4}x - 2 \quad ?$$

Solution for exercise 29

a) For a tangent line holds

$$y_T = f(x_0) + f'(x_0)\cdot(x-x_0)$$

$$y = f(x) = \sqrt{16-x^2} \quad \Rightarrow \quad f(x_0) = f(1) = \sqrt{15}$$

$$f'(x) = \frac{-2x}{2\sqrt{16-x^2}} = \frac{-x}{\sqrt{16-x^2}} \quad \Rightarrow \quad f'(x_0) = f'(1) = \frac{-1}{\sqrt{15}}$$

Thus, the equation of the tangent line reads

$$y_T = \sqrt{15} - \frac{1}{\sqrt{15}}(x-1).$$

b) The function $y = \frac{1}{3}x^3 - x$ has the derivative

$$y'(x) = \frac{1}{3}\cdot 3x^2 - 1 = x^2 - 1.$$

The straight line $y = \frac{1}{4}x - 2$ has the slope $c = \frac{1}{4}$.

At which point(s) x has the function the slope $c = \frac{1}{4}$?

$$\begin{aligned} y'(x) = x^2 - 1 &= \frac{1}{4} \\ \Leftrightarrow \qquad x^2 &= \frac{5}{4} \\ \Leftrightarrow \qquad x &= \pm\frac{\sqrt{5}}{2}. \end{aligned}$$