## Proof

**sum rule:** fix some $x_0 \in A \setminus \{x\}$

$$(f \pm g)'(x_0) \overset{\text{def.}}{=} \lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{(f(x) \pm g(x)) - (f(x_0) \pm g(x_0))}{x - x_0}$$

$$= \lim_{\substack{x \to x_0 \\ x \neq x_0}} \left( \frac{f(x) - f(x_0)}{x - x_0} \pm \frac{g(x) - g(x_0)}{x - x_0} \right)$$

$$\overset{\substack{\text{rules} \\ \text{for limits}}}{=} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \pm \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}$$

$$= f'(x_0) \pm g'(x_0)$$

sum rule implies a „subtraction rule"

**product rule:** fix some $x_0 \in A \setminus \{x\}$

$$(f \cdot g)'(x_0) \overset{\text{def.}}{=} \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)g(x) - f(x)g(x_0) + f(x)g(x_0) - f(x_0)g(x_0)}{x - x_0}$$

$$\overset{\substack{\text{rules} \\ \text{for limits}}}{=} \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} g(x_0) + \underbrace{\lim_{x \to x_0} f(x) \frac{g(x) - g(x_0)}{x - x_0}}_{= \lim_{x \to x_0} f(x_0) \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0}}$$

$$= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0) \quad \square$$