Intervals

By means of the relations $\leq$ or $<$, resp., we may introduce

intervals of real numbers:

- closed interval with $a, b \in \mathbb{R}$, $a \leq b$:

$$[a, b] := \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$$

- open interval with $a, b \in \mathbb{R}$, $a < b$:

$$(a, b) := \{x \in \mathbb{R} \,|\, a < x < b\}$$

- semi-open interval with $a, b \in \mathbb{R}$, $a < b$:

$$(a, b] := \{x \in \mathbb{R} \,|\, a < x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} \,|\, a \leq x < b\}$$

Infinite intervals

By introducing the symbols

$$-\infty \text{ (minus infinity) and}$$
$$\infty \text{ (plus infinity)}$$

we may define for $a \in \mathbb{R}$:

$$\begin{aligned}
(-\infty, a] &:= \{x \in \mathbb{R} \,|\, x \leq a\} \\
(-\infty, a) &:= \{x \in \mathbb{R} \,|\, x < a\} \\
[a, \infty) &:= \{x \in \mathbb{R} \,|\, a \leq x\} \\
(a, \infty) &:= \{x \in \mathbb{R} \,|\, a < x\} \\
(-\infty, \infty) &:= \mathbb{R}
\end{aligned}$$

$(-\infty, a]$ and $[a, \infty)$ are called closed intervals
$(-\infty, a)$ and (a, ∞) are called open intervals

Upper and lower bounds

Definition (Upper / lower bounds)

A set S of real numbers is **bounded from above**,
if there exists a real number b,
such that $S \subseteq (-\infty, b]$.

b is called an **upper bound** of S.

A set S of real numbers is **bounded from below**,
if there exists a real number a,
such that $S \subseteq [a, \infty)$.

a is called a **lower bound** of S.

Smallest upper/largest lower bound

Definition (Supremum and infimum)

If a set $S \subset \mathbb{R}$ is bounded from above,
then the least upper bound is called the **supremum** of S.

We write $\sup(S)$ or $\sup\{S\}$.

If a set $S \subset \mathbb{R}$ is bounded from below,
then the greatest lower bound is called the **infimum** of S.

We write $\inf(S)$ or $\inf\{S\}$.

Completeness of the real numbers

We assume the **axiom of completeness**:
For any non-empty set of real numbers that is bounded from above
there exists a least upper bound.

This property of the real numbers is called **completeness**.

Thus $(\mathbb{R}, +, \cdot, >)$ is a complete, Archimidean ordered field.

$(\mathbb{R}, +, \cdot, >)$ is (essentially) the only complete, Archimidean ordered field.

Real numbers - further properties

Further properties following from the axioms of a complete, Archimidean ordered field:

- No zero divisor:

$$x, y \in \mathbb{R} \text{ and } xy = 0 \Rightarrow x = 0 \text{ or } y = 0$$

- $(-x)y = -(xy)$ for all $x, y \in \mathbb{R}$
- $(-x)(-y) = xy$ for all $x, y \in \mathbb{R}$

Absolute value

Definition (Absolute value (or modulus))

The absolute value (or modulus) $|a|$ of a real number a is defined by

$$|a| := \begin{cases} a & \text{falls } a \geq 0, \\ -a & \text{falls } a < 0. \end{cases}$$

"Trivially" we see that for $a, b \in \mathbb{R}$:

- $-|a| \leq a \leq |a|$
- $|-a| = |a|$
- $|ab| = |a| \cdot |b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \quad b \neq 0$
- $|a| \leq b \quad \Longleftrightarrow \quad -b \leq a \leq b$

Triangular inequality

From

$$-|a| \leq a \leq |a|$$

and

$$-|b| \leq b \leq |b|$$

we obtain:

$$-\left(|a| + |b|\right) \leq a + b \leq |a| + |b|.$$

This yields the **triangular inequality**:

$$|a + b| \leq |a| + |b|.$$

A sharper variant of the triangular inequality

If we set $a = c + d$ and $b = -d$, then:

$$\begin{aligned} & |c| \leq |c + d| + |-d| \\ \iff \quad & |c| - |d| \leq |c + d|. \end{aligned}$$

If we set $a = c + d$ and $b = -c$, then:

$$\begin{aligned} & |d| \leq |c + d| + |-c| \\ \iff \quad & -|c + d| \leq |c| - |d|. \end{aligned}$$

Altogether:

$$\big||c| - |d|\big| \leq |c + d|.$$

Geometrical interpretation of the absolute value

Consider 2 real numbers a, b with $a < b$ on the number line:

Here $|a - b|$ is the distance on the number line between the points belonging to a and b.

Reformulations of the axiom of Archimedes

Let $x \in \mathbb{R}$ is positive, if $x > 0$. (We may abbreviate $x \in \mathbb{R}^+$.)
We rewrite the axiom (AR) as follows:

For any 2 positive numbers x, y there exists an $n \in \mathbb{N}$ with

$$nx > y.$$

Now choose $x = 1$, $y = \frac{1}{\varepsilon}$ for a real $\varepsilon > 0$.
Moreover we write $N(\varepsilon)$ instead of n in order to emphasize the dependence of this natural number on ε.
Then (AR) reads:

For any $\varepsilon > 0$ there exists an $n \in \mathbb{N}$ with

$$N(\varepsilon) > \frac{1}{\varepsilon}$$

The last reformulation will be exploited frequently in Chapter 3.

Outline

Some logic

Mathematical content is expressed by **statements.**

A statement describes a fact that is either clearly true (✓) or false (f), i.e.:

Negation

Logical linking of statements:

The **negation** $\neg A$ of a statement A is the statement "not A":

A	$\neg A$
$\checkmark$	f
f	$\checkmark$

Conjunction

The **conjunction** $A \wedge B$ is the statement "A and B", defined as:

A	B	$A \wedge B$
✓	✓	✓
✓	f	f
f	✓	
f	f	

Disjunction

The **disjunction** $A \vee B$ is the statement "*A* or *B*", defined as:

A	B	$A \vee B$
$\checkmark$	$\checkmark$	$\checkmark$
$\checkmark$	f	
f	$\checkmark$	$\checkmark$
f	f	

Proof strategies

For a theorem, we have to proof that
under the given conditions A
the statement B of the theorem is always true .

For the **implication** $A \Rightarrow B$ we have to proof that the case A true and B false never occurs:

A	B	$A \Rightarrow B$
✓	✓	✓
✓	f	f
f	✓	✓
f	f	✓

Direct proof

If A is false, then $A \Rightarrow B$ holds no matter what is the value of B.

Thus it is sufficient to consider the case A is true. Then we have to show that this implies B is true.

Example (4 is a divisor $\Rightarrow$ 2 is a divisor)

Indirect proof I: contraposition

Since

$$(A \Rightarrow B) \quad \Longleftrightarrow \quad (\neg B \Rightarrow \neg A),$$

we may proof

$$\neg B \Rightarrow \neg A$$

instead of a direct proof.

Example (n^2 is even $\Rightarrow$ n is even)

Indirect proof II: contradiction

We assume that A is true and B is false
(this the only case where $A \Rightarrow B$ is false)
and show that this assumption yields a contradiction.

Example ($\sqrt{2}$ is an irrational number)

Equivalence

If the implication holds in both directions, i.e.

$$A \Leftrightarrow B$$

we say A holds if b holds.

A	B	$A \Leftrightarrow B$
✓	✓	✓
✓	f	f
f	✓	f
f	f	✓

Complete induction: induction principle

Complete induction is a proof scheme allowing to proof statements $A(n)$ that hold for any integer $n \geq n_0$ ($n_0 \in \mathbb{Z}$ being fixed, e.g. $n = 0$)

We have 2 steps:

- **initial case**:
  We show that $A(n_0)$ holds.
- **induction step**:
  For any $n \neq n_0$ we may assume that $A(n)$, $A(n-1)$, ..., $A(n_0)$ holds (**induction hypothesis**).
  Using this we derive that $A(n+1)$ is valid.

If both steps are successful, then $A(n)$ holds for all $n \geq n_0$.

Motivation

$A(n)$ holds for all $n \geq n_0$ ($n, n_0 \in \mathbb{Z}$), since:

- Step 1:
  yields that $A(n_0)$ holds
- Step 2:
  yields that $A(n_0 + 1)$ holds
  again by step 2 now $A(n_0 + 2)$ holds
  again by step 2 now $A(n_0 + 3)$ holds
  ...
  this may be continued for all $n \in \mathbb{Z}$ with $n \geq n_0$

Example