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Functions: Operations for creating new functions

Let $f, g : A \to \mathbb{R}$ functions with $A \subseteq \mathbb{R}$.
Let $h : B \to \mathbb{R}$ functions with $B \subseteq \mathbb{R}$.

We obtain new functions by

$$\begin{aligned}
f \pm g &: A \to \mathbb{R},\ x \to f(x) \pm g(x) \\
f \cdot g &: A \to \mathbb{R},\ x \to f(x) \cdot g(x) \\
\lambda g &: A \to \mathbb{R},\ x \to \lambda g(x) \quad \text{for } \lambda \in \mathbb{R} \\
\frac{f}{g} &: \tilde{A} \to \mathbb{R},\ x \to \frac{f(x)}{g(x)} \quad \text{for } \tilde{A} := \{x \in A \,|\, g(x) \neq 0\} \\
h \circ f &: A \to \mathbb{R},\ x \to h(f(x)) \quad \text{for } f(A) \subseteq B
\end{aligned}$$

The latter is the **concatenation of functions.**

Continuity

Continuity is a central concept in mathematics.

We combine the concepts of limits and functions.

Limits of functions I

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Definition (Limit of a function)

Let $f : A \to \mathbb{R}$ a function, $A \subset \mathbb{R}$.
Let $a \in \mathbb{R}$ the limit of at least one sequence $\{a_n\}_{n \geq n_0}$ of real numbers in A.

If for any such sequence $\{a_n\}_{n \geq n_0}$ with $\lim_{n\to\infty} a_n = a$, we have

$$\lim_{n\to\infty} f(a_n) = b \text{ with } b \in \mathbb{R},$$

then we define b as the **limit of the function f at the point** a and
we write

$$\lim_{x\to a} f(x) := b.$$

Limits of functions II

Moreover, we introduce analogously:

$$\lim_{x \downarrow a} f(x) = b \quad \text{for sequences from the right only}$$

$$\lim_{x \uparrow a} f(x) = b \quad \text{for sequences from the left only}$$

$$\lim_{x \to \infty} f(x) = b \quad \text{for sequences unbounded from above}$$

$$\lim_{x \to -\infty} f(x) = b \quad \text{for sequences unbounded from below}$$

Continuity

Definition (Continuous function)

Let $A \subseteq \mathbb{R}$, $a \in A$ and $f : A \to \mathbb{R}$ a function.

If

$$\lim_{x \to a} f(x) = f(a) \tag{1}$$

then f is called **continuous in** a.

If f is continuous in any point of A,
then f is called **continuous on** A.

However, in order to verify the continuity,
it is not always necessary to check (1) for every a in A.

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Continuity by operations on functions

Theorem (Continuity of composed functions)

Let $f, g : \mathbb{R} \supseteq A \to \mathbb{R}$ a function, being continuous in $a \in A$, let $h : \mathbb{R} \supseteq A \to \mathbb{R}$ a function, being continuous in $f(a) \in B$, then

$$\begin{aligned} f \pm g &: A \to \mathbb{R}, \; x \to f(x) \pm g(x) \\ f \cdot g &: A \to \mathbb{R}, \; x \to f(x) \cdot g(x) \\ \lambda g &: A \to \mathbb{R}, \; x \to \lambda g(x) \quad \text{for } \lambda \in \mathbb{R} \\ \frac{f}{g} &: \tilde{A} \to \mathbb{R}, \; x \to \frac{f(x)}{g(x)} \quad \text{for } \tilde{A} := \{x \in A \,|\, g(x) \neq 0\} \\ h \circ f &: A \to \mathbb{R}, \; x \to h(f(x)) \quad \text{for } f(A) \subseteq B \end{aligned}$$

are continuous in a.

Continuity of concatenated functions

Proof.

Let $\{a_n\}_{n\geq n_0}$ a real sequence in A with $\lim_{n\to\infty} a_n = a$.
Since f is continuous in a, we have

$$\lim_{n\to\infty} f(a_n) = f(a).$$

Due to the assumption $\{b_n\}_{n\geq n_0}$ with $b_n = f(a_n)$ is a real sequence in B.
Since h is continuous in $b = f(a)$, we have

$$\lim_{n\to\infty} h(f(a_n)) = \lim_{n\to\infty} h(b_n) = h(b) = h(f(a)).$$

Thus there holds

$$\lim_{n\to\infty} (h \circ f)(a_n) = \lim_{n\to\infty} h(f(a_n)) = h(f(a)). \quad \square$$

$\square$

Alternative definition of continuity

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Another, equivalent definition of continuity that does not rely on the concept of limits, can be found in many books (for that reason).

Theorem (ε-δ formulation of continuity)

Let $f : \mathbb{R} \supseteq A \to \mathbb{R}$ be a function.
f is continuous in $x_0 \in A$,
iff for any $\varepsilon > 0$ there exists a $\delta > 0$, s.t.

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for all } x \in A \text{ with } |x - x_0| < \delta.$$

Applications of continuity

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In this subsection we proof important results that hold for continuous functions.

Theorem (Intermediate value theorem (1st version))

Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0$ and $f(b) > 0$ or with $f(a) > 0$ and $f(b) < 0$, then there exists a $\xi \in (a, b)$ s.t.

$$f(\xi) = 0.$$

Intermediate value theorem

Corollary (Intermediate value theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$ with $f(a) \leq c \leq f(b)$ or with $f(a) \geq c \geq f(b)$, then there exists a $\xi \in [a, b]$ s.t.

$$f(\xi) = c.$$

Corollary (Image of an interval)

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a continuous function,

then $f(I)$ is again an interval.

Compactness

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Definition (Bounded function)

Let $f : A \to \mathbb{R}$ be a function and
$f(A)$ is bounded (from above and below)

then f is called a **bounded function**.

Definition (Compact interval)

Let $a, b \in \mathbb{R}$,
then a closed and bounded interval $[a, b]$
is called a **compact interval**.

Weierstrass theorem

Theorem (Continuous functions on compact intervals (Weierstrass))

Any continuous function $f : [a, b] \to \mathbb{R}$ (with $a \leq b$)

is bounded and
takes its maximum & minimum, i.e. there exists $\overline{x}$ & $\underline{x}$ s.t.

$$f(\overline{x}) = \sup_{x \in [a,b]} f(x),$$

$$f(\underline{x}) = \inf_{x \in [a,b]} f(x).$$

Note this theorem does not hold for open or semi-open (semi-closed) intervals.
This result may be generalized to higher dimensions.

It is of key importance in optimization.