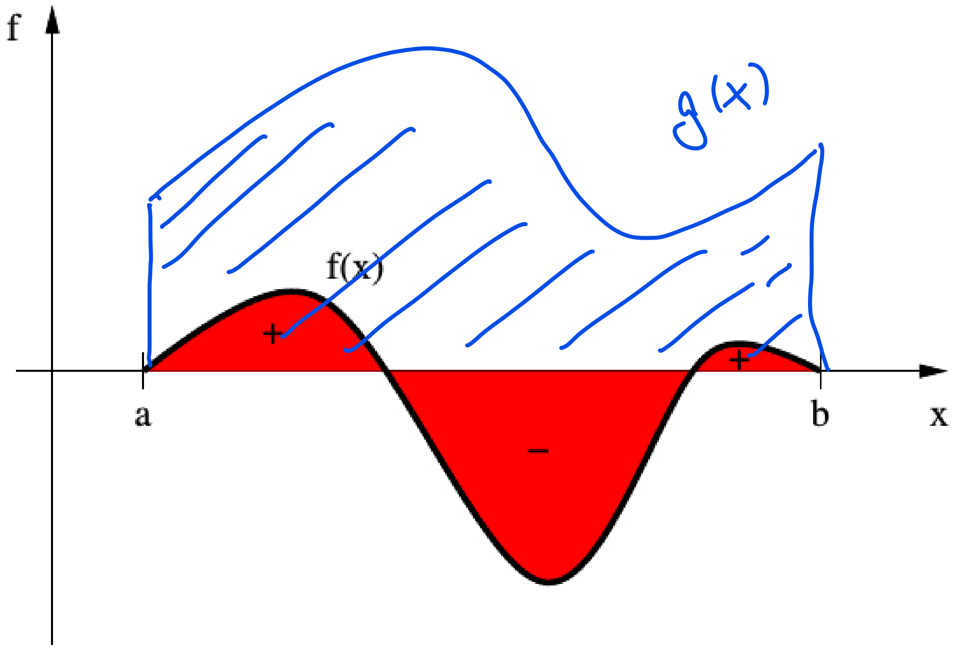
Outline

Integration in 1d

How to compute an area is known for triangles, rectangles, etc.

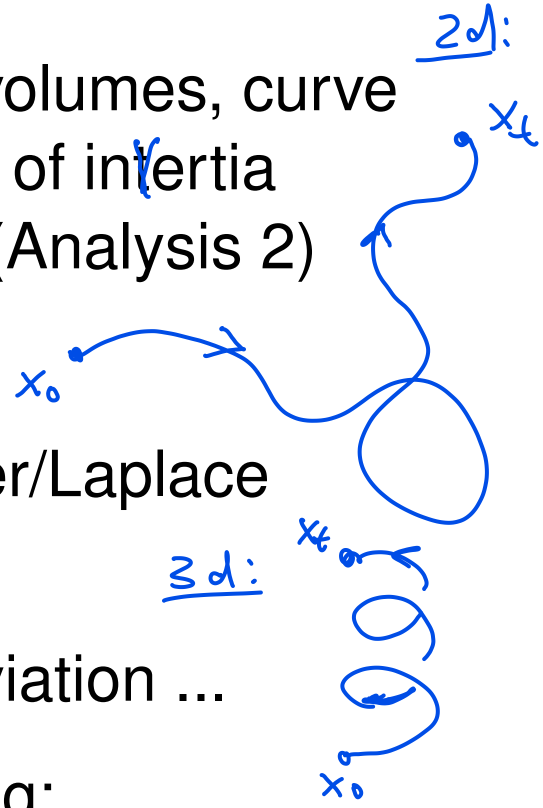
Goal: compute an area, e.g. as the red zone,



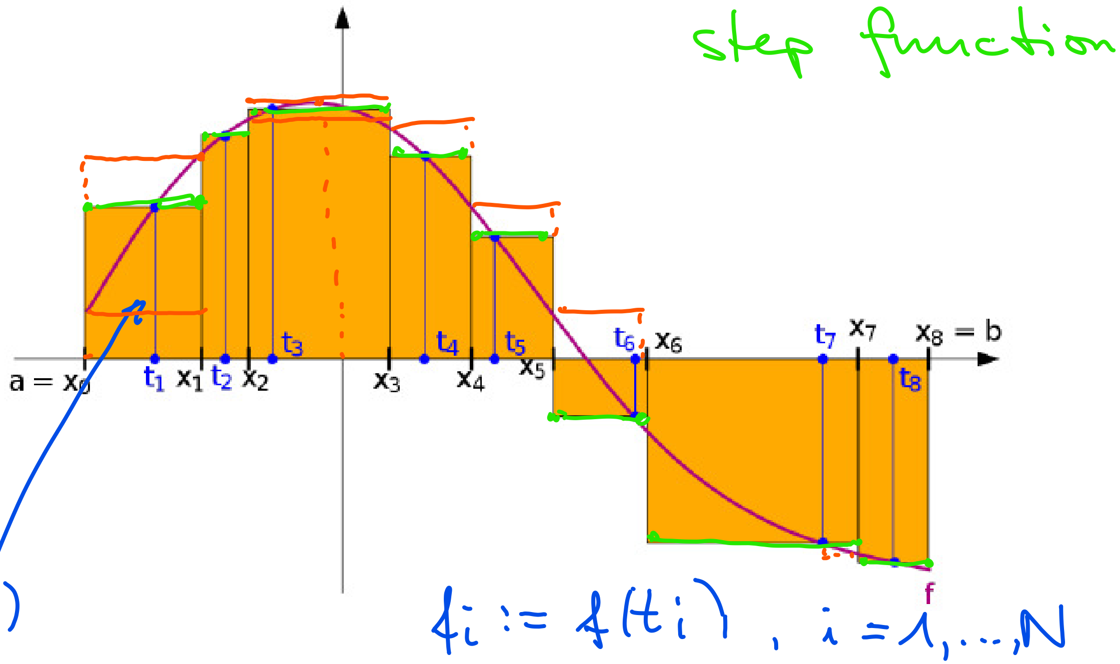
more preciesly the area enclosed by the graph of the function f and the x-axis between $x = a$ and $x = b$, where area below the x-axis are counted as negative and above as positive

Integration in 1d: applications

- Besides areas, also surface areas, volumes, curve lengths, centers of gravity, moments of in~~t~~ertia $\rightsquigarrow$ integration in higher dimensions (Analysis 2)
- Kind of inversion of differentation
- Solving differential equations, Fourier/Laplace transformation $\rightsquigarrow$ Analysis 2 or later
- Statistics: mean value, standard deviation ...
- Applications in science & engineering:
  - Velocity as an integral of acceleration over time, distance as an integral of velocity over time
  - Energy as an integral of power over time
  - Electric charge as an integral of the current over time
  - Work of a space-dependent force as an integral over force along path
  - Applied work for the isothermal compression of an ideal gas



Riemann sums



Partition $\mathcal{Z}$ of an interval $[a, b]$:
$\mathcal{Z} = \{a = x_0 < x_1 < \ldots < x_N = b\}$

$\Delta x_i = x_i - x_{i-1}$, $i = 1, \ldots, N$ } length of i-th sub-interval

Riemann sums:

$$S(f; \mathcal{Z}) := \sum_{i=1}^{N} f_i \, \Delta x_i := \sum_{i=1}^{N} f(t_i)(x_i - x_{i-1}),$$

where $t_i \in [x_{i-1}, x_i]$ are arbitrary intermediate points

Riemann integrable

f is called **(Riemann) integrable**, over $[a, b]$,
if $S(f; \mathcal{Z})$ for arbitrarily fine partitions & arbitrary intermediate points

$$I =: \int_a^b f(x)\,dx \;=\; \int_a^b f(t)\,dt$$

converges against the same real number.

$$= \int_a^b f(\xi)\,d\xi$$

Terminology:

- f is called **integrand**
- a or b is called lower or upper **bound of integration**, resp.
- x is called **integration variable**

the integration variable is arbitrary

Lower and upper sums

If we choose $f_i = m_i := \inf_{x \in [x_{i-1}, x_i]} f(x_i)$, then we call this a lower (Darboux) sum,
if we choose $f_i = M_i := \sup_{x \in [x_{i-1}, x_i]} f(x_i)$~~m~~, then we call this an upper (Darboux) sum.

(Riemann) integrability may be demonstrated, e.g, by convergence of lower and upper sum against the same value:

$$m_i \leq \quad f(x) \quad \leq M_i \qquad \text{for all } x \in [x_{i-1}, x_i]$$

$$\text{for all } i = 1, \ldots, N$$

$$\Longrightarrow \quad \sum_{i=1}^{N} m_i \, \Delta x_i \leq S(f; \mathcal{Z}) \leq \sum_{i=1}^{N} M_i \, \Delta x_i$$

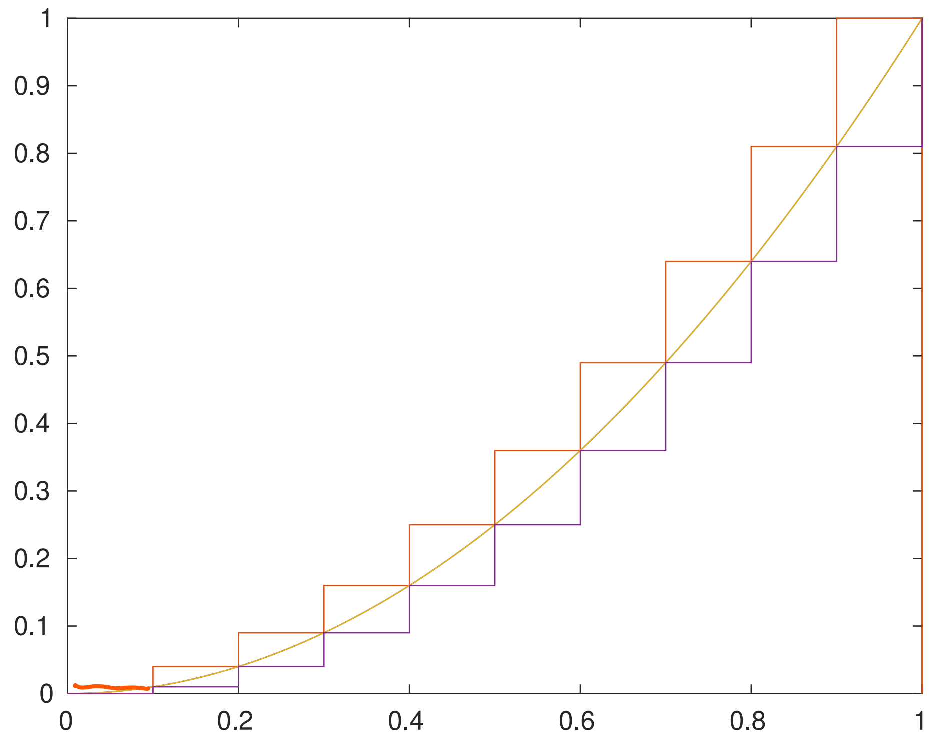
Approximation: example using Matlab

$$S(x^2;\mathcal{Z}) \overset{N\to\infty}{\longrightarrow} I = \int_a^b x^2\,dx = ?$$

Consider equidistant partitions of $[a, b]$

Lower and upper sum for $\Delta x = \frac{b-a}{N} = \frac{1-0}{10} = 0{,}1$

For instance, $a = 0$, $b = 1$, $N = 10$



Approximation: example by feet

$$S(x^2; \mathcal{Z}) \stackrel{N\to\infty}{\longrightarrow} I = \int_0^b x^2\, dx = ?$$

Consider equidistant~~e~~ partition $\mathcal{Z}$ by $x_i = i\,\Delta x$, $i = 1, \ldots, N$, with $\Delta x = \frac{b}{N}$

$$f_i = x_i^2 = (\Delta x)^2 i^2 = (b/N)^2\, i^2$$

proved by induction over N

The summation formula $\sum_{i=1}^N i^2 = \frac{N(N+1)(2N+1)}{6}$ yields for the upper sum:

$$S(x^2; \mathcal{Z}) = \sum_{i=1}^N x_i^2 \Delta x \quad = \quad \left(\frac{b^2}{N^2} \sum_{i=1}^N i^2 \right) \frac{b}{N}$$

Approximation: example by feet

$$S(x^2;\mathcal{Z}) \overset{N\to\infty}{\longrightarrow} I = \int_{0}^{b} x^2\,dx = ?$$

Consider equidistante partition $\mathcal{Z}$ by $x_i = i\,\Delta x$, $i = 1,\dots,N$, with $\Delta x = \frac{b}{N}$

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$$= b^3 \frac{\cancel{N}(N+1)(2N+1)}{6N^2\cancel{N}}$$

Approximation: example by feet

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$$S(x^2;\mathcal{Z}) = \sum_{i=1}^N x_i^2 \Delta x \quad = \left(\frac{b^2}{N^2}\sum_{i=1}^N i^2\right)\frac{b}{N}$$

$$= b^3 \frac{\cancel{N}(N+1)(2N+1)}{6N^2\cancel{N}} = b^3\frac{2N^2+3N+1}{6N^2} \xrightarrow{N\to\infty} \frac{b^3}{3} = I$$

(Analogously consider the lower sum $\sum_{i=1}^N x_{i-1}^2 \Delta x$.)