Technische Hochschule Rosenheim
Applied Artificial Intelligence - Bachelor
Analysis 1

WiSe 2021/22

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Homework 9: complex roots and derivatives

To submit: on Thursday, 09.12.2021, 9:30 a.m., online by the learning campus

Exercise 1 (7 pts.)

a) Determine all $n \in \mathbb{N}$ s.t. $(-8-8\sqrt{3}i)^n$ is a real number. $n = 3k \;,\; k \in \mathbb{N}_0$

b) Find all solutions of the algebraic equation

$$z^4 + 8 + 8\sqrt{3}i = 0,$$

expressed in Cartesian representation.

Exercise 2 (4 pts.)

Let $x > 0$.

a) Compute

$$\lim_{x \to 0} (x^x) .$$

any number to the power of 0 is 1. so lim(x^x) = 1
x -> 0

b) Compute the derivative of

$$f(x) = x^x \quad \text{for } x > 0.$$

$f'(x) = \frac{d}{dx} e^{(x \ln(x))}$

$f'(x) = e^{(x \ln(x))} \cdot \left(1 \ln(x) + x \cdot \frac{1}{x}\right) = e^{x \ln(x)} \cdot (\ln(x) + 1) = x^x \cdot (\ln(x) + 1)$

Exercise 3 (4 pts.)

Show for $x \in \mathbb{R}^+$ and $n \in \mathbb{N}$

$$\left(\frac{d}{dx}\right)^n \ln(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}.$$

Base $n=1$: $\frac{d}{dx} \ln(x) = \frac{(-1)^{1-1} \cdot (1-1)!}{x^1} = \frac{1 \cdot 0!}{x} = \frac{1}{x}$ ✓

$\exists n \in \mathbb{N}$: $\left(\frac{d}{dx}\right)^n \ln(x) = \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$

Proof $n \to n+1$: $\left(\frac{d}{dx}\right)^{n+1} \ln(x) = \frac{(-1)^{n+1-1} \cdot (n+1-1)!}{x^{n+1}}$ ← goal

$= \left(\frac{d}{dx}\right)^1 \left(\frac{d}{dx}\right)^n \ln(x) \overset{IA}{=} \frac{d}{dx} \frac{(-1)^{n-1} \cdot (n-1)!}{x^n}$

/1a $(-8-8\sqrt{3}\,i) = -8(1+\sqrt{3}\,i) = -8\left(\cos(2\pi) + i\cdot\sin\left(\frac{\pi}{3}\right)\right)$

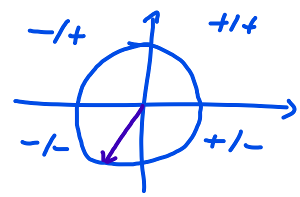
$z := (-8-8\sqrt{3}\,i)^n = (-8)^n\left(\cos(2\pi n) + i\cdot\sin\left(\frac{\pi n}{3}\right)\right)$

$\operatorname{Im}(z) = 0$ if $i\sin\left(\frac{\pi n}{3}\right) = 0$

$\operatorname{Im}(z) = 0$ for $0, 3, 6, 9 \dots$

$\operatorname{Im}(z) = 0$ for $3k$, $k \in \mathbb{N}_0$

b) $z^4 + 8 + 8\sqrt{3}\,i = 0 \qquad z^4 = \underline{-8-8\sqrt{3}\,i}$



$\theta_1 = \arctan\left(\frac{-8\sqrt{3}}{-8}\right)$

$= 60°$

$r = \sqrt{(-8)^2 + (-8\sqrt{3})^2}$

$= \sqrt{256} = 16$

$\theta_2 = 60° + 180° = \underline{240°}$

$\alpha_1 = \theta_2$

$\alpha_2 = \theta_2 + \frac{360°}{4} = \theta_2 + 90° = \underline{330°}$

$\alpha_3 = \alpha_2 + 90° = 420° \mathrel{\hat{=}} \underline{60°}$

$\alpha_4 = \alpha_1 - 90° = \underline{150°}$

$\sqrt[n]{z} = \sqrt[n]{r}\left[\cos\left(\frac{\theta + 360k}{n}\right) + i\sin\left(\frac{\theta+360k}{n}\right)\right]$

$\sqrt[4]{z} = \sqrt[4]{16}$ "

$z_1 = 2\cdot\left(\cos\left(\frac{240°+0°}{4}\right) + i\sin\left(\frac{240°+0°}{4}\right)\right) = 2\cdot\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right) = 1+\sqrt{3}\,i$

$z_2 = 2\cdot(\cos(150°) + i\sin(150°)) = -\sqrt{3} + i$

$z_3 = 2\cdot(\cos(240°) + i\sin(240°)) = -1 - \sqrt{3}\,i$

$z_4 = 2\cdot(\cos(330°) + i\sin(330°)) = \sqrt{3} - i$

13 $$\frac{d}{dx} \frac{(-1)^{n-1} \cdot (n-1)!}{x^n} = \frac{d}{dx}\left((-1)^{n-1} \cdot (n-1)! \cdot x^{-n}\right)$$

$$= (-1)^{n-1} \cdot (n-1)! \cdot \frac{d}{dx} x^{-n}$$

$$= (-1)^{n-1} \cdot (n-1)! \cdot -n \cdot x^{-n-1} = (-1)^{n-1} \cdot (n-1)! \cdot (-1) \cdot n \cdot x^{-n-1}$$

$$= (-1) \cdot (-1)^{n-1} \cdot n \cdot (n-1)! \cdot x^{-n-1}$$

$$= \frac{(-1)^{n+1-1} \cdot (n+1-1)!}{x^{n+1}} \quad \text{qed}$$