Technische Hochschule Rosenheim
Applied Artificial Intelligence - Bachelor
Analysis 1

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Exercise 11: applications of differentiation, Riemann sum

Exercise 34

Show that among the ellipses, prescribed by

$$\frac{1}{a^2}x^2 + \frac{1}{b^2}y^2 = 1, \quad a, b > 0, \quad x, y \in \mathbb{R},$$

a circle maximizes the enclosed area

$$\pi ab$$

subject to the constraint that the perimeter $p > 0$ of the ellipse is given approximately by:

$$\pi(a+b) = p.$$

a) Formulate the corresponding **minimization** problem in 1d, where the variable b has been eliminated by the constraint.

b) Compute the minimizer of this optimization problem.

c) Check that for the minimizer $a = b =: r$ holds. Show that r corresponds to the radius of a circle.

Solution for exercise 34

a) Maximizing the area $A = \pi ab$ is equivalent to minimizing $-A$.

We resolve the constraint for b, i.e.

$$b = \frac{p}{\pi} - a.$$

The minimization problem reads:

Minimize $-\pi ab = \pi a \left(a - \frac{p}{\pi}\right)$ over $a \in \mathbb{R}^+$.

This problem has no constraint anymore and is a 1d problem in the variable a.

b) In order to solve the optimization problem we consider

$$f(a) := \pi a^2 - pa.$$

(Remark: we could scale f by a non-zero constant, e.g., by $1/\pi$.)

The necessary 1st order condition yields

$$f'(a) = 2\pi a - p \stackrel{!}{=} 0,$$

thus the only stationary point (i.e. candidate for a minimizer) is

$$\hat{a} = \frac{p}{2\pi}.$$

The sufficient 2nd order condition yields

$$f''(a) = 2\pi > 0 \quad \text{for any admissible } a,$$

thus $\hat{a}$ is a minimizer of the problem and a maximizer of the area.

Remark: The maximal area is

$$\pi\hat{a}\left(\frac{p}{\pi} - \hat{a}\right) (= \pi\hat{a}^2) = \frac{p^2}{4\pi}.$$

c) We observe

$$\hat{b} = \frac{p}{\pi} - \hat{a} = \frac{p}{2a},$$

thus $\hat{b} = \hat{a}$ and the equation of the ellipsis simplifies (by multiplying with $\hat{a}$) to the equation of a circle with radius $r = \hat{a}$:

$$x^2 + y^2 = r^2, \quad r > 0.$$

Exercise 35

Calculate the square root of a positive real number a by the Newton method, i.e. find a zero of

$$f(x) = 1 - \frac{a}{x^2}.$$

As initial value we consider $x_0 = \frac{1+a}{2}$

a) Derive the abstract steps of the Newton method for arbitrary a.

b) Solve two steps of the Newton method for $a = 2$.

c) Solve two steps of the Newton method for $a = 5$.

In b) and c) use a calculator or a mathematical software.

Solution for exercise 35

a) The derivative reads

$$f'(x) = 2\frac{a}{x^3},$$

thus the rule of the Newton iteration is here

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{1 - a/x^2}{2a/x_n^3} = \frac{x_n}{2}\left(3 - \frac{x_n^2}{a}\right), \quad x_0 = \frac{1+a}{2}$$

Remark: In the case $x = 0$ to be excluded, we find directly $a = 0$ and no Newton method is required.

b)

$$\begin{aligned} x_0 &= 1.5 \\ x_1 &= 1.40\ldots \\ x_2 &= 1.414\ldots \end{aligned}$$

Remark: A check turns out that all digits after $n = 2$ steps are valid.

c)

$$x_0 = 3$$
$$x_1 = 1.8\dots$$
$$x_2 = 2.1\dots$$

Remark: A check turns out only the 1st digit "2" after $n = 2$ steps is valid.

Exercise 36

a) Let $t \in \mathbb{R}$ and t is not an integer multiple of 2π.
Show that for any $n \in \mathbb{N}$ we have

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kt) = \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{2\sin\left(\frac{1}{2}t\right)}.$$

Hint: Use the definition of the cosine by the complex exponential function and then exploit a geometric series.

b) Compute the integral

$$\int_0^a \cos(x)\,dx, \quad a > 0,$$

by means of a Riemann sum.
Consider an equidistant partition of $[0,a]$ with fineness a/n. We choose as intermediate points $t_k = x_k = ka/n$, $k = 0,1,\dots,n$.

Solution for exercise 36

a) Acording to the hint

$$\frac{1}{2} + \sum_{k=1}^{n} \cos(kt) = \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{n} (\exp(ikt) + \exp(-ikt)) = \frac{1}{2}\sum_{k=-n}^{n} \exp(ikt).$$

This can be rewritten as

$$\frac{1}{2}\sum_{k=-n}^{n} \exp(ikt) = \frac{1}{2}\exp(-int)\sum_{k=0}^{2n} exp(ikt)$$

in order to apply the summation formula for the geometric series ($\exp(it) \neq 1$ since $t \neq 2\pi m$ for some $m \in \mathbb{Z}$)

$$\frac{1}{2}\exp(-int)\sum_{k=0}^{2n} \exp(ikt) = \frac{1}{2}\exp(-int)\frac{1-\exp((2n+1)it)}{1-\exp(it)}.$$

This can be transformed as

$$= \frac{1}{2}\frac{\exp(-int) - \exp((n+1)it)}{1-\exp(it)}\frac{\exp(-i\frac{t}{2})}{\exp(-i\frac{t}{2})} = \frac{1}{2}\frac{\exp(-i(n+\frac{1}{2})t) - \exp((n+\frac{1}{2})it)}{\exp(-i\frac{t}{2}) - \exp(i\frac{t}{2})}\frac{-1/(2i)}{-1/(2i)} = \frac{1}{2}\frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin\left(\frac{1}{2}t\right)}. \quad \square$$

b) The Riemann sum reads for the given partition

$$S_n := \sum_{k=1}^{n} \cos\left(\frac{ka}{n}\right)\frac{a}{n} = \frac{a}{n}\left(\frac{\sin\left(\left(n+\frac{1}{2}\right)\frac{a}{n}\right)}{2\sin\left(\frac{1}{2}\frac{a}{n}\right)} - \frac{1}{2}\right) = \frac{\frac{a}{2n}}{\sin\left(\frac{a}{2n}\right)}\sin\left(\left(1+\frac{1}{2n}\right)a\right) - \frac{a}{2n}$$

where we have applied the result from a) for $t = a/n$ (being not an integer multiple of 2π) in the last step.

Letting $n \to \infty$ we obtain with $\lim_{x\to 0} x/\sin(x) = 1$

$$\int_0^a \cos(x)\,dx = \lim_{n\to\infty} S_n = \sin(a).$$

Remark: According to the exercise the integral has to be computed only, thus we may assume that the integral exists.