

## Homework 8: logarithms, trigonometric and hyperbolic functions

To submit: on Thursday, 02.12.2021, 9:30 a.m., online by the learning campus

$$\begin{array}{l} \log_a(x) = n \quad x = a^n \\ \log_a(y) = m \quad y = a^m \end{array} \quad \left| \quad \frac{x}{y} = \frac{a^n}{a^m} = a^{n-m} \right| \log_a$$

$$\log_a\left(\frac{x}{y}\right) = n - m$$

### Exercise 1 (4 pts.)

a) Show for  $a > 0, a \neq 1$  and  $x, y > 0$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y).$$

b) Simplify

$$\log_{a^p} b = \frac{1}{p} \log_a b \quad \log_{a^2}(t) + \log_a(t) \quad a \in \mathbb{R}^+ / \{1\}$$

and determine for which  $a$  and  $t$  the last term is defined.

$$= \frac{1}{2} \log_a(t) + \log_a(t) = \frac{3}{2} \log_a(t) \quad t \in \mathbb{R}^+$$

### Exercise 2 (4 pts.)

Show for all  $x, y \in \mathbb{R}$ :

$$\cos(x) - \cos(y) = -2 \sin\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right),$$

$$\sin(x) - \sin(y) = 2 \cos\left(\frac{x+y}{2}\right) \sin\left(\frac{x-y}{2}\right).$$

$$\begin{array}{ll} x = a + b & a = \frac{x+y}{2} \\ y = a - b & b = \frac{x-y}{2} \end{array}$$

$$x = \frac{x+y+x-y}{2} = \frac{2x}{2} = x$$

$$y = \frac{x+y-x+y}{2} = \frac{2y}{2} = y$$

Hint: consider  $x = u + v$  and  $y = u - v$ .

### Exercise 3 (7 pt.)

Show that for any  $x \in \mathbb{R}$  and  $n \in \mathbb{N}$ ,  $n$  odd,

$$\cos(nx) = \sum_{j=0}^{(n-1)/2} (-1)^j \binom{n}{2j} (\cos(x))^{n-2j} (\sin(x))^{2j},$$

$$\sin(nx) = \sum_{j=0}^{(n-1)/2} (-1)^j \binom{n}{2j+1} (\cos(x))^{n-2j-1} (\sin(x))^{2j+1}.$$

$$\cos(x) = \cos(a+b)$$

$$= \cos(a)\cos(b) - \sin(a)\sin(b)$$

$$\cos(y) = \cos(a-b)$$

$$\parallel \quad + \quad \parallel \quad *$$

Hint: consider  $(\exp(ix))^n = \exp(inx)$  and apply the binomial theorem.

Calculate by the given formulas:

$$\cos(3x), \sin(5x).$$

**Exercise 4 (5 pt.)**

Show for all  $x \in \mathbb{R}$

$$\cosh(x) = \sum_{j=0}^{\infty} \frac{1}{(2j)!} x^{2j},$$

Justify that the series converges absolutely.

Just as a remark (not to show here!): For all  $x \in \mathbb{R}$

$$\sinh(x) = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!} x^{2j+1}$$

converges absolutely.

$$\begin{aligned} * \sin(x) &= \sin(a+b) \\ &= \sin(a)\cos(b) + \cos(a)\sin(b) \end{aligned}$$

$$\begin{aligned} \sin(y) &= \sin(a-b) \\ &= \sin(a)\cos(b) - \cos(a)\sin(b) \end{aligned}$$

$$\begin{aligned} \cos(x) - \cos(y) &= \cos(a)\cos(b) - \sin(a)\sin(b) - \cos(a)\cos(b) - \sin(a)\sin(b) \\ &= -2\sin(a)\sin(b) \\ &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \end{aligned}$$

$$\begin{aligned} \sin(x) - \sin(y) &= \sin(a+b) - \sin(a-b) \\ &= \sin(a)\cos(b) + \cos(a)\sin(b) - \sin(a)\cos(b) + \cos(a)\sin(b) \\ &= 2\cos(a)\sin(b) \\ &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \end{aligned}$$

