

## Definition (Mapping / function)

Let  $A$  and  $B$  be two sets.

A **mapping** or **function**  $f$  from  $A$  onto  $B$  is represented by

$$f : A \rightarrow B, x \mapsto f(x).$$

It is a rule, that assigns to every  $x \in A$  an element  $f(x) \in B$ .

$A$  is called **domain** (of definition).

Elements of  $A$  are called **arguments** of the mapping  $f$ .

$f(x)$  is called the **image** of  $x$  under  $f$  or the **function value** of  $f$  at the point  $x$ .

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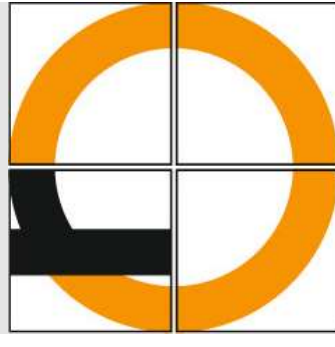
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For a mapping  $f$  with domain of definition  $A$ , we introduce for an arbitrary subset  $C \subseteq A$  the set

$$f(C) := \{f(x) \mid x \in C\}.$$

$f(C)$  is called the **image of  $C$  under  $f$** .

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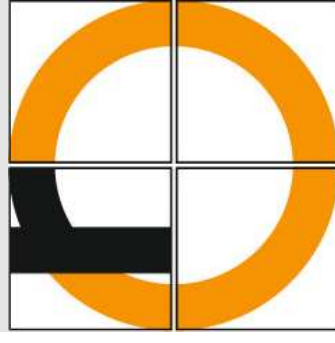
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The image  $f(A)$  of the domain of definition  $A$  of  $f$  is called the **range of values** of  $f$ .

In general, for  $f(A) \subseteq B$ , the set  $B$  is called **target area** or **co-domain**.

For  $D \subseteq B$  we define the **preimage of  $D$  (under  $f$ )** as

$$f^{-1}(D) := \{x \in A \mid f(x) \in D\}.$$

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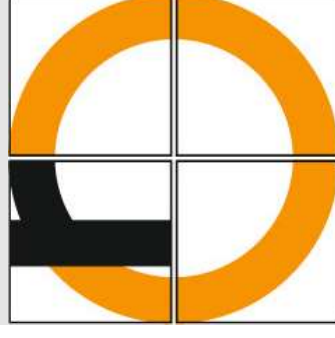
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A function  $f : A \rightarrow B$  is called **injective**,  
iff for any  $x_1, x_2 \in A$  there holds

$$f(x_1) = f(x_2) \quad \Leftrightarrow \quad x_1 = x_2$$

The **injectivity** of a function depends  
on the mapping rule and  
on the domain of definition.

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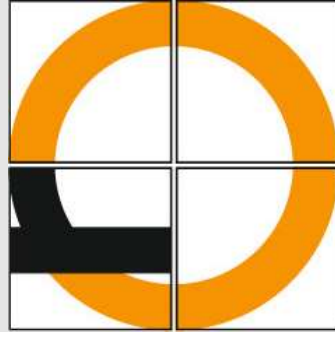
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A function  $f : A \rightarrow B$  is called **surjective**,  
iff for any  $y \in B$  there exists a  $x \in A$  such that  $f(x) = y$ .  
Equivalently, iff  $f(A) = B$ ,  
 $f(A)$  being the range of values,  $B$  the target area.

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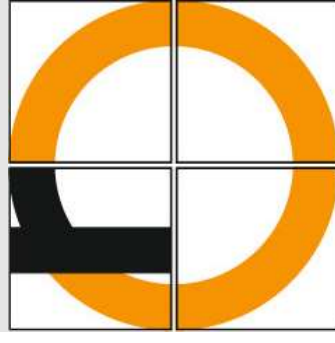
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Functions that are injective and surjective are called **bijective**.

Thus, for a bijective function

$$f : A \rightarrow B, \quad x \mapsto f(x)$$

there exists for any  $y \in B$  exactly one  $x \in A$  with  $f(x) = y$ .

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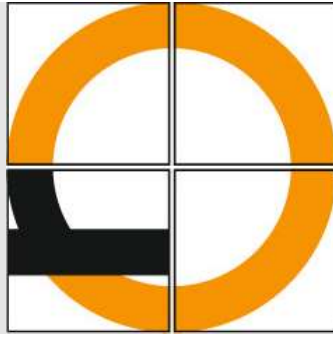
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Hence there exists a mapping

$$f^{-1} : B \rightarrow A, \quad y \mapsto f^{-1}(y)$$

with  $f^{-1}(f(x)) = x$  for any  $x \in A$  and

with  $f(f^{-1}(y)) = y$  for any  $y \in B$ .

$f^{-1}$  is called the **inverse mapping** or **inverse function** or **inverse** of  $f$ .

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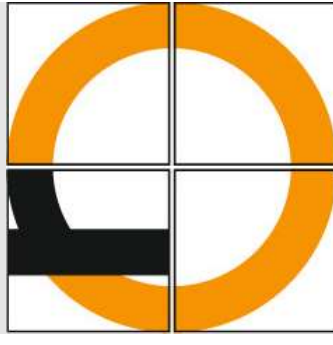
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For any subset  $A$  we may define the function

$$\text{Id}_A : A \rightarrow A, \quad x \mapsto x,$$

called the **identity** on  $A$ .

Two mappings

$$\begin{aligned} f : A &\rightarrow B, & x &\mapsto f(x), \\ g : C &\rightarrow D, & z &\mapsto g(z) \end{aligned}$$

are called **identical**, iff

- $A = C$ ,
- $B = D$ , and
- $f(x) = g(x)$  for all  $x \in A$ .

Then we write  $f = g$ .

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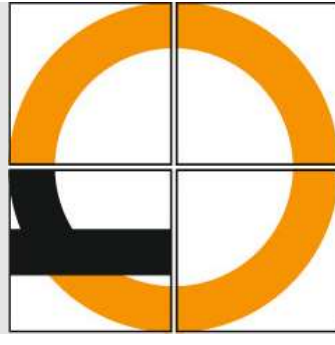
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Let  $f : A \rightarrow B, x \mapsto f(x)$ .

By restricting the domain  $A$  to a subset  $C \subseteq A$ , we obtain the **restriction** of  $f$  to  $C$ :

$$f|_C : C \rightarrow B, \quad x \mapsto f(x).$$

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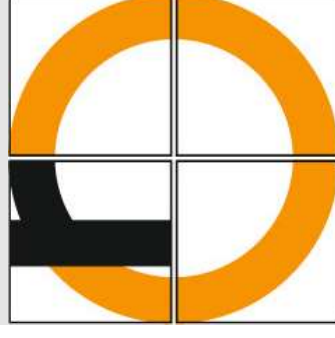
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Special (number) sets:

$$\mathbb{N} = \{1, 2, 3, \dots\}$$

natural numbers

$$\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$$

natural numbers incl. zero

$$\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, 3, \dots\}$$

integers (**whole numbers** ?)

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

rational numbers

$\mathbb{R} \approx$  “all” numbers in 1d

real numbers

$\mathbb{C} \approx \mathbb{R}^2$  with “special structure”

complex numbers

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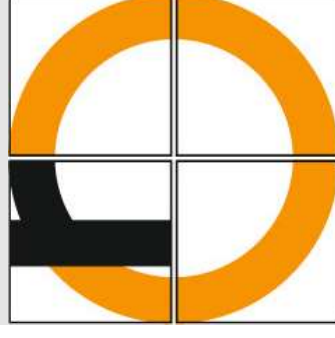
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S.-J. Kimmerle

Every point of the number line corresponds to a real number:

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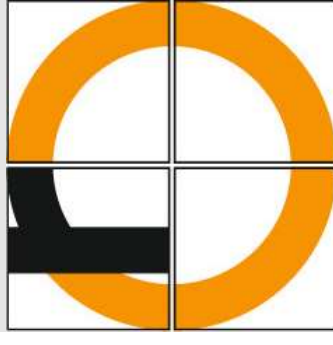
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The real numbers are ordered completely by the order

relation  $\geq$ :

For all  $x, y \in \mathbb{R}$  there holds:

$$x \geq y : \Leftrightarrow (x - y) \geq 0$$

Alternatively, for all  $x, y \in \mathbb{R}$  there holds:

$$x > y : \Leftrightarrow (x - y) > 0$$

The comparability of real numbers is called the **order** of the real numbers.

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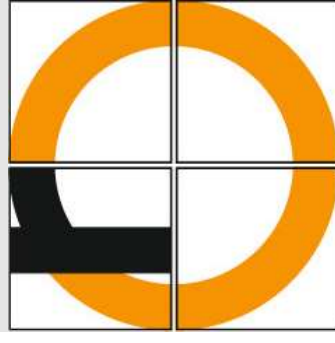
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S.-J. Kimmerle

There holds:

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

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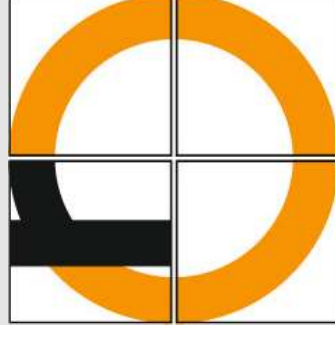
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## Definition (Field)

A non-empty set  $K$  equipped with 2 operations that are closed **(F0)**

$$\oplus : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$$

$$\otimes : \mathbb{F} \times \mathbb{F} \rightarrow \mathbb{F}$$

is called a field, supposed the following conditions **(F1)** – **(F5)** hold:

**(F1)** For all  $a, b \in \mathbb{F}$  we have the **commutative laws**

$$a \oplus b = b \oplus a, \quad a \otimes b = b \otimes a.$$

**(F2)** For all  $a, b, c \in \mathbb{F}$  we have the **associative laws**

$$(a \oplus b) \oplus c = a \oplus (b \oplus c), \quad (a \otimes b) \otimes c = a \otimes (b \otimes c).$$

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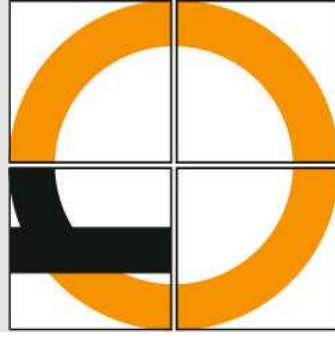
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## Definition (Field (continued))

**(F3)** For all  $a \in \mathbb{F}$  there exists a **neutral element w.r.t.  $\oplus$** ,  $e_{\oplus} \in \mathbb{F}$ , such that

$$a \oplus e_{\oplus} = a$$

and for all  $a \in \mathbb{F} \setminus e_{\oplus}$  there exists a **neutral element w.r.t.  $\otimes$** ,  $e_{\otimes} \in \mathbb{F}$ , such that

$$a \otimes e_{\otimes} = a.$$

**(F4)** For all  $a \in \mathbb{F}$  there exists an **inverse element w.r.t.  $\oplus$** ,  $a_{\oplus}^{-1} \in \mathbb{F}$ , such that

$$a \oplus a_{\oplus}^{-1} = e_{\oplus}$$

and for all  $a \in \mathbb{F} \setminus e_{\oplus}$  there exists an **inverse element w.r.t.  $\otimes$** ,  $a_{\otimes}^{-1} \in \mathbb{F}$ , such that

$$a \otimes a_{\otimes}^{-1} = e_{\otimes}.$$

**(F5)** For all  $a, b, c \in \mathbb{F}$  there holds the **distributive law**

$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c).$$

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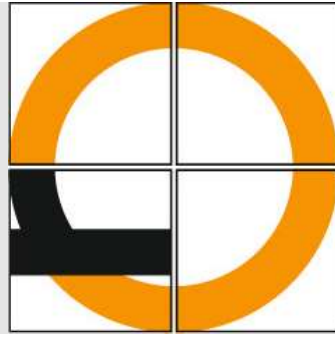
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## Definition (Ordered fields)

If  $\mathbb{F}$  is a field and if some elements of  $\mathbb{F}$  are designated as positive, then  $\mathbb{F}$  is called ordered if:

**(A1)** For all  $a \in \mathbb{F}$  there holds exactly one of the following relations

- $a$  is positive,
- $a = e_{\oplus}$  (is neutral), or
- $a_{\oplus}^{-1}$  is positive.

**(A2)** For all  $a, b \in \mathbb{F}$  positive this implies  $a \oplus b$  is positive.

**(A3)** For all  $a, b \in \mathbb{F}$  positive this implies  $a \otimes b$  is positive.

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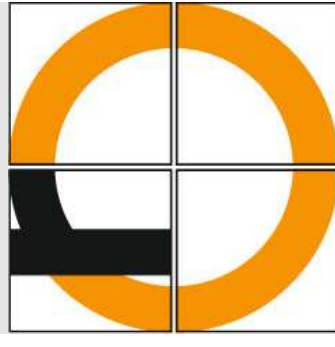
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## Definition (Archimidean ordered fields)

Let  $\mathbb{F}$  be an ordered field.

$\mathbb{F}$  is called an Archimidean ordered field, if in addition holds:

**(AR)** For any 2 positive  $a, b \in \mathbb{F}$  there exists an  $n \in \mathbb{N}$  such that:

$$\underbrace{a \oplus a \oplus \dots \oplus a \oplus b_{\oplus}^{-1}}_{n \text{ times}} \text{ is positive.}$$

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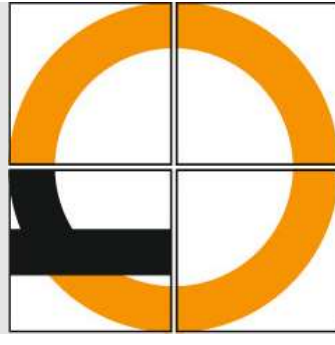
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- Both neutral elements,  $e_{\oplus}$  and  $e_{\otimes}$ , are unique.

- In (F4) the neutral element w.r.t.  $\oplus$  has to be excluded.

There exists no inverse element for  $e_{\oplus}$  w.r.t.  $\otimes$ .

- If we set  $e_{\oplus} = e_{\otimes}$ , then there would hold for all  $a \in \mathbb{F}$

$$a \otimes e_{\oplus} = e_{\oplus} = e_{\otimes} = a \otimes e_{\otimes} = a,$$

i.e.  $a = e_{\otimes}$  and thus  $\mathbb{F} = \{e_{\otimes}\}$ .

- Both inverse elements,  $a_{\oplus}^{-1}$  and  $a_{\otimes}^{-1}$ , are uniquely defined.

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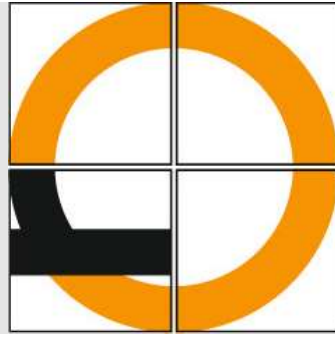
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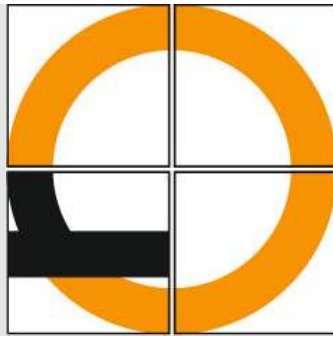
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By means of the relations  $\leq$  or  $\geq$ , resp., we may introduce

**intervals** of real numbers:

- closed interval with  $a, b \in \mathbb{R}$ ,  $a \leq b$ :

$$[a, b] := \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

- open interval with  $a, b \in \mathbb{R}$ ,  $a < b$ :

$$(a, b) := \{x \in \mathbb{R} \mid a < x < b\}$$

- semi-open interval with  $a, b \in \mathbb{R}$ ,  $a < b$ :

$$(a, b] := \{x \in \mathbb{R} \mid a < x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} \mid a \leq x < b\}$$

By introducing the symbols

$-\infty$  (minus infinity) and  
 $\infty$  (plus infinity)

we may define for  $a \in \mathbb{R}$ :

$$(-\infty, a] := \{x \in \mathbb{R} \mid x \leq a\}$$

$$(-\infty, a) := \{x \in \mathbb{R} \mid x < a\}$$

$$[a, \infty) := \{x \in \mathbb{R} \mid a \leq x\}$$

$$(a, \infty) := \{x \in \mathbb{R} \mid a < x\}$$

$$(-\infty, \infty) := \mathbb{R}$$

$(-\infty, a]$  and  $[a, \infty)$  are called closed intervals  
 $(-\infty, a)$  and  $(a, \infty)$  are called open intervals,  
where  $a \in \mathbb{R}$ .

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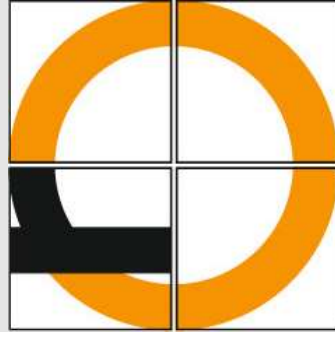
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## Definition (Upper / lower bounds)

A set  $S$  of real numbers is **bounded from above**, if there exists a real number  $b$ , such that  $S \subseteq (-\infty, b]$ .

$b$  is called an **upper bound** of  $S$ .

A set  $S$  of real numbers is **bounded from below**, if there exists a real number  $a$ , such that  $S \subseteq [a, \infty)$ .

$a$  is called a **lower bound** of  $S$ .

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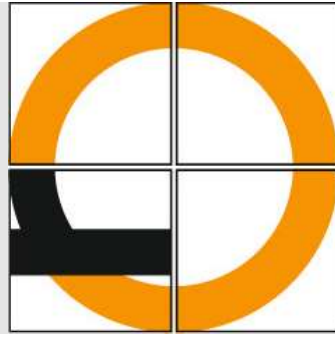
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## Definition (Supremum and infimum)

If a set  $S \subset \mathbb{R}$  is bounded from above, then the least upper bound is called the **supremum** of  $S$ .

We write  $\sup(S)$  or  $\sup\{S\}$ .

If a set  $S \subset \mathbb{R}$  is bounded from below, then the greatest lower bound is called the **infimum** of  $S$ .

We write  $\inf(S)$  or  $\inf\{S\}$ .

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