Absolute value (modulus)

Theorem (Properties of the absolute value)

For any $w, z \in \mathbb{C}$ there holds:

1)a) $z \geq 0$ *1)b)* $|z| = 0 \Leftrightarrow z = 0$

2) $|z + w| \leq |z| + |w|$ *(triangle inequality)*

3) $|z \cdot w| = |z||w|$ *(multiplicativity)*

Remark: A field, for whose elements a mapping with the properties of an absolute value exists, is called a **valued field**.

Complex exponential function

We extend the exponential function into the complex plane.

Definition (Complex exponential function)

For $z \in \mathbb{C}$ we define

$$\exp(z) := \sum_{k=0}^{\infty} \frac{1}{k!} z^k = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \dots$$

Properties:

1. $\exp(z) \neq 0$ for any $z \in \mathbb{C}$
2. $\exp(z)$ is continuous
3. $\exp(\overline{z}) = \overline{\exp(z)}$
4. Let $z = x + yi$,
   then $|\exp(z)| = |\exp(x)|$.

Trigonometric functions

Definition (Cosine and sine)

For $x \in \mathbb{R}$ we define:

$$\cos(x) := \mathrm{Re}\left(e^{ix}\right)$$

$$\sin(x) := \mathrm{Im}\left(e^{ix}\right)$$

We see that the Euler formula holds:

$$\exp(ix) = \cos(x) + i\sin(x), \quad x \in \mathbb{R}$$

Geometric interpretation of the Euler formula

The range of values is the unit circle:

$$
\begin{aligned}
|\exp(ix)| &= \left(\exp(ix)\overline{\exp(ix)}\right)^{1/2} \\
&= (\exp(ix)\exp(-ix))^{1/2} = (\exp(0))^{1/2} = 1, \quad x \in \mathbb{R}
\end{aligned}
$$

$\sin(x)$ and $\cos(x)$ are the projections on the real and imaginary axis, resp.

sin and cos are extended on complex arguments by $\exp(iz)$, $z \in \mathbb{C}$. For real arguments sin and cos coincide with the known functions.

Geometric interpretation of multiplication in $\mathbb{C}$

We consider the polar complex plane.

We have

$$z = r\exp(i\phi) \in \mathbb{C}$$

with $r = |z| \in \mathbb{R}_0^+$ and $\phi \in (-\pi, \pi]$.

Thus for the multiplication of $z_1 = r_1 \exp(i\phi_1)$, $z_2 = r_2 \exp(i\phi_2) \in \mathbb{C}$ there holds

$$z_1 z_2 = r_1 r_2 \exp(i(\phi_1 + \phi_2)).$$

Properties of (co)sine

Theorem (Properties of cosine and sine)

For all $x \in \mathbb{R}$ there holds:

- $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \quad \sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$
- $\cos(-x) = \cos(x), \quad \sin(-x) = \sin(x)$
- $\cos^2(x) + \sin^2(x) = 1$
- $\cos : \mathbb{R} \to \mathbb{R}$ *and* $\sin : \mathbb{R} \to \mathbb{R}$ *are continuous on* $\mathbb{R}$.

Trigonometric addition theorems

Theorem (Addition theorems of cosine and sine)

For all $x, y \in \mathbb{R}$ there holds:

$$\cos(x+y) = \cos(x)\cos(y) - \sin(x) sin(y),$$
$$\sin(x+y) = \sin(x)\cos(y) + \cos(x) sin(y).$$

Theorem (Duplication formulas of cosine and sine)

For all $x \in \mathbb{R}$ there holds:

$$\cos(2x) = \cos^2(x) - \sin^2(x),$$
$$\sin(2x) = 2\sin(x)\cos(x).$$

Series representation of (co)sine

Theorem (Sine and cosine as series)

For $x \in \mathbb{R}$ we define

$$\cos(x) := \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots$$

$$\sin(x) := \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots$$

Both series converge absolutely.

Series representation of (co)sine - proof

Proof.

The exponential series is absolutely convergent. We consider:

$$\exp(ix) = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!}$$

Note that: $i^0 = 1, i^1 = i, i^2 = -1, i^3 = -i, i^4 = 1$
In a cyclic manner: $i^{4k} = 1, i^{4k+1} = i, i^{4k+2} = -1, \ldots$

We order the terms (why is it allowed ?) with even $(4k, 4k+2)$ and odd indices $(4k+1, 4k+3)$:

$$\exp(ix) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k)!} x^{2k} + i \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!} x^{2k+1}$$

Together with the Euler formula

$$\exp(ix) = \cos(x) + i \sin(x)$$

the statement is verified. □

Estimates for the remainder

Since we may evaluate only a finite part of a series, it is important to have estimates for the remainder:

Theorem (Estimates for the remainder for (co)sine)

For all $x \in \mathbb{R}$:

$$\exp(x) = \sum_{k=0}^{n} \frac{x^k}{k!} + r_n(x),$$

$$\text{with } |r_{n+1}(x)| \leq 2\frac{|x|^{n+1}}{(n+1)!} \quad \text{for } |x| \leq \frac{n+1}{2}$$

$$\cos(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} + r_{2n+2}(x),$$

$$\text{with } |r_{2n+2}(x)| \leq \frac{|x|^{2n+2}}{(2n+2)!} \quad \text{for } |x| \leq 2n+3$$

$$\sin(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + r_{2n+3}(x),$$

$$\text{with } |r_{2n+3}(x)| \leq \frac{|x|^{2n+3}}{(2n+3)!} \quad \text{for } |x| \leq 2n+4$$

Limits and estimates involving (co)sine

Theorem (Some limits sin and cos)

For any $x \in \mathbb{R}$ we have

$$1)\ \lim_{x\to 0, x\neq 0} \frac{\sin(x)}{x} = 1, \qquad 2)\ \lim_{x\to\infty} \frac{\sin(x)}{x} = 0,$$

$$3)\ \lim_{x\to 0, x\neq 0} \frac{\cos x - 1}{x^2} = \frac{1}{2}, \qquad 4)\ \lim_{x\to\infty} \frac{\cos(x)}{x} = 0$$

Lemma (2 estimates for sin and cos)

$$\sin(x) > 0 \qquad \textit{for all } x \in (0, 2]$$

$$\cos(2) \leq -\frac{1}{3}$$

Introducing π

Definition (The number π)

The cosine has exactly one zero p in the interval $[0, 2]$. We define

$$\pi := 2p.$$

The real number π is an infinite, non-periodic decimal fraction, i.e. an irrational number,

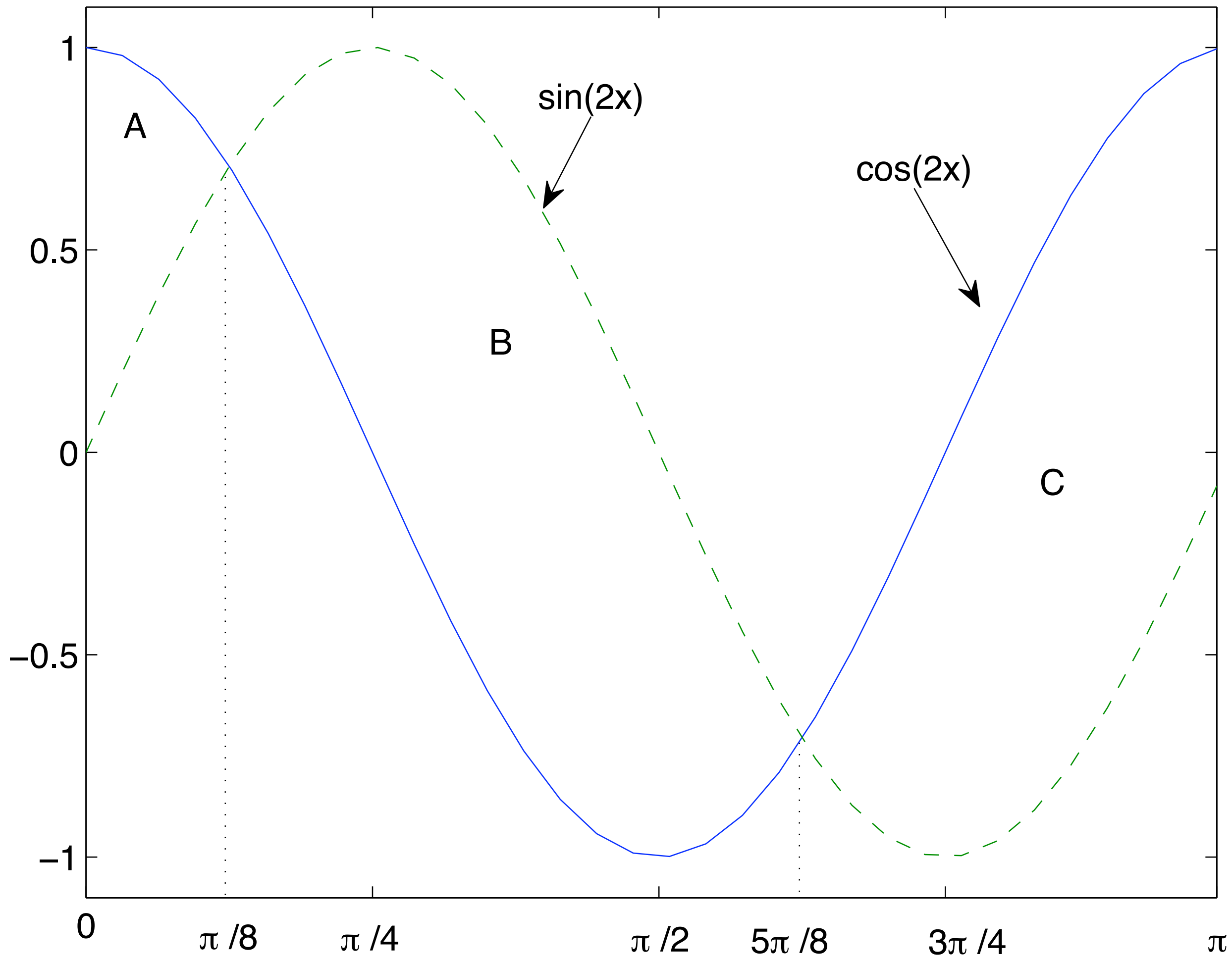
$$\pi = 3,141\,592\,653\,589\ldots.$$

Periodicities of sine and cosine I

$$\cos(x + 2\pi) = \cos(x)$$
$$\sin(x + 2\pi) = \sin(x)$$
$$\cos(x + \pi) = -\cos(x)$$
$$\sin(x + \pi) = -\sin(x)$$

$$\cos(x) = \sin\left(\frac{\pi}{2} - x\right)$$
$$\sin(x) = \cos\left(\frac{\pi}{2} - x\right)$$

Periodicities of sine and cosine II



Sine and cosine with half the periodicity

Tangent

The quotient of sine and cosine has a certain importance. It is called the **tangent**

$$\tan(x) = \frac{\sin(x)}{\cos(x)}.$$

Moreover, we define the **cotangent**

$$\cot(x) = \frac{\cos(x)}{\sin(x)}.$$

Note the singularities at zeros of cos or sin.

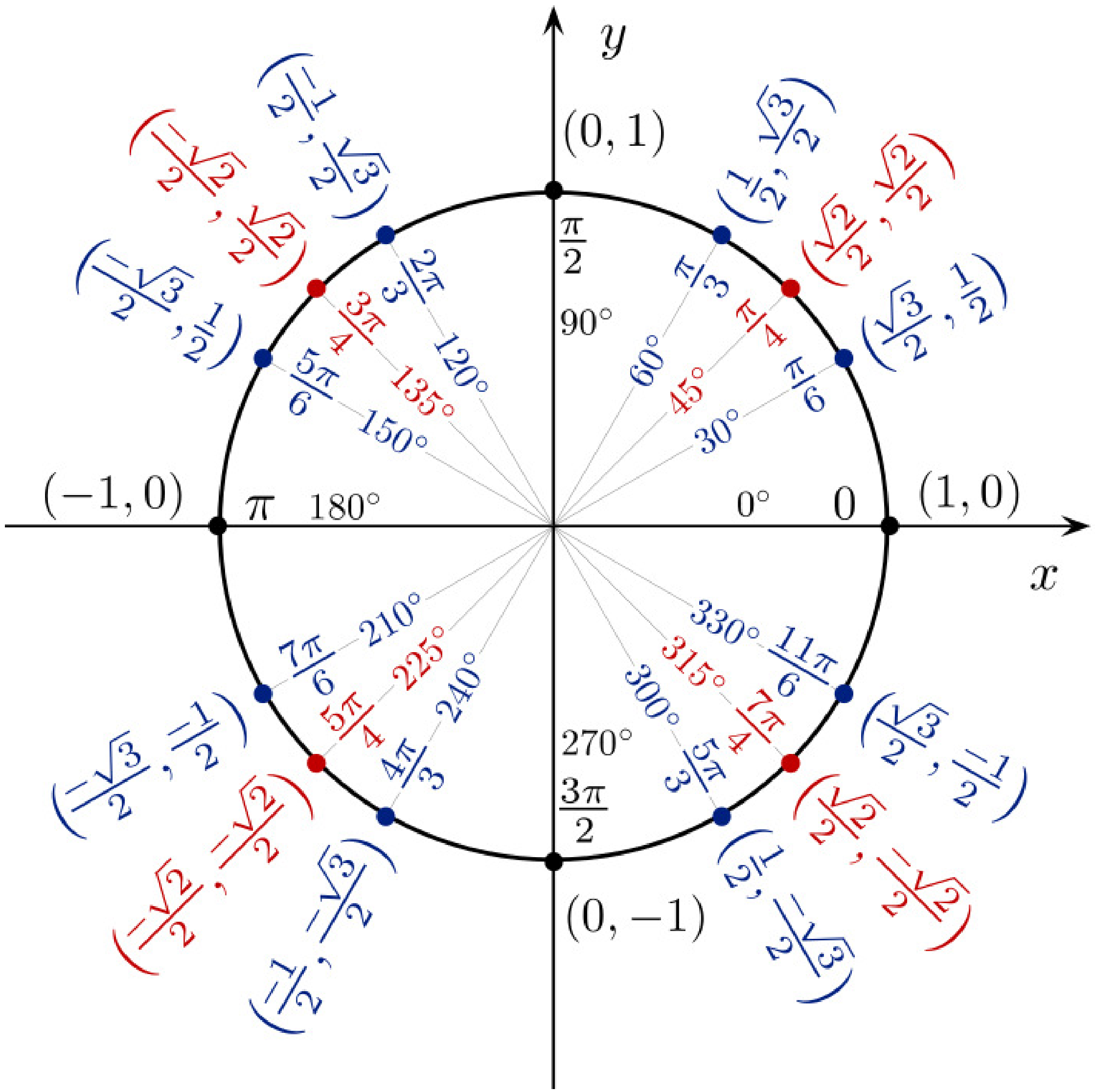
Remark: There exist further trigonometric functions that we do not consider here:

$$\sec(x) = \frac{1}{\cos(x)}, \quad \csc(x) = \frac{1}{\sin(x)}$$

Special values of trigonometric functions

Radian	degree	sin	cos	tan
0	$0°$	0	1	0
$\frac{\pi}{6}$	$30°$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$
$\frac{\pi}{4}$	$45°$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$\frac{\pi}{3}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	$90°$	1	0	∞

Special values of co(sine)



Unit circle with coordinates of important points
Source: Wikipedia

Inverse trigonometric functions I

On the interval $[0, \pi]$ the function cos is strictly monotone decrasing and maps bijectively to $[-1, 1]$.
The inverse function

$$\arccos(x) : [-1, 1] \to \mathbb{R}$$

exists and it is called **arccosine**.

On the interval $[-\pi/2, \pi/2]$ the function sin is strictly monotone incrasing and maps bijectively to $[-1, 1]$.
The inverse function

$$\arcsin(x) : [-1, 1] \to \mathbb{R}$$

exists and it is called **arcsine**.

Inverse trigonometric functions II

On the interval $(-\pi/2, \pi/2)$ the function tan is strictly monotone incrasing and maps bijectively to $\mathbb{R}$.
The inverse function

$$\arctan(x) : \mathbb{R} \to \mathbb{R}$$

exists and it is called **arctangent**.

Note that the inverse functions are restricted to a “principal branch”.

Hyperbolic functions

Let $x \in \mathbb{R}$.

We define the **hyperbolic cosine** as

$$\cosh(x) := \frac{1}{2}\left(\exp(x) + \exp(-x)\right),$$

and the **hyperbolic sine** as

$$\sinh(x) := \frac{1}{2}\left(\exp(x) - \exp(-x)\right).$$

Applications:

- sagging high voltage lines or chains (e.g. of bridges) have the shape of a cosh
- magnetism
- cosmology
- ...