Revision: polynomial division with remainder I

Consider a rational function

$$r : A \to \mathbb{R}, x \mapsto \frac{p(x)}{q(x)} := \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}, \qquad a_n \neq 0, b_m \neq 0$$

where $A := \{x \in \mathbb{R} \mid \sum_{i=0}^{m} b_i x^i \neq 0\}$.

If $n \geq m$, then we set

$$p_1 : \mathbb{R} \to \mathbb{R}, x \mapsto p(x) - \frac{a_n}{b_m} x^{n-m} \cdot q(x)$$

and obtain the following representation

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_n}{b_m} x^{n-m} + \frac{p_1(x)}{q(x)} \qquad \text{for all } x \in A$$

where p_1 is either the zero function or a polynomial of degree smaller than n.

Revision: polynomial division with rem. II

This procedure may be iterated for k steps that produce a polynomial p_k
until the degree of p_k is less than n.

We end up with

$$\frac{p(x)}{q(x)} = g(x) + \frac{p_k(x)}{q(x)},$$

g a polynomial.

Partial fraction expansion I

Before integrating a rational function $r : A \to \mathbb{R}$ as defined above,
we carry out a polynomial division:

$$\int r(x)\,dx = \int g(x)\,dx + \int \frac{p_k(x)}{q(x)}\,dx$$

We know how to integrate the polynomial g.
For the remainder $\frac{p_k(x)}{q(x)}$, we consider the partial fraction expansion, i.e., the rational function is decomposed into a sum of fractions (yielding only a short list of cases with explicit formulas).

Partial fraction expansion II

Let again $r : A \to \mathbb{R}, x \mapsto \frac{p(x)}{q(x)}$ as defined above, but w.l.o.g. let degree $p <$ degree q.
Moreover, let

$$q(x) = x(x-b_1)^{k_1} \cdot (x-b_2)^{k_2} \cdot \ldots \cdot (x-b_r)^{k_r} \cdot q_1(x)^{l_1} \cdot \ldots \cdot q_s(x)^{l_2}$$

with pairwise distinct zeros b_i of multiplicity k_i and pairwise distinct quadratic polynomials q_i that do not have zeros in $\mathbb{R}$.

Then there exists real numbers $A_1^{[1]}, \ldots, A_1^{[k_1]}, \ldots, A_r^{[1]}, \ldots, A_r^{[k_r]}$, $B_1^{[1]}, \ldots, B_1^{[l_1]}, \ldots, B_s^{[1]}, \ldots, B_s^{[l_s]}, C_1^{[1]}, \ldots, C_1^{[l_1]}, \ldots, C_s^{[1]}, \ldots, C_s^{[l_s]}$ s.t.

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{A_i^{[j]}}{(x-b_j)^j} + \sum_{i=1}^{s} \sum_{j=1}^{l_i} \frac{B_i^{[j]} + C_i^{[j]} x}{(q_i(x))^j} \quad \text{for all } x \in A.$$

Integration of rational functions I

Thus we only have to figure out how to integrate functions of the type

$$\frac{\zeta}{(x-x_0)^k}, \quad \frac{\xi+\mu x}{(q_i(x))^{\tilde{k}}}, \quad k, \tilde{k} \in \mathbb{N}:$$

Let $[a,b] \subset A$:

1)

$$\int_a^b \frac{1}{x-x_0}\, dx = [\ln(|x-x_0|)]_a^b, \quad x_0 \notin [a,b]$$

2)

$$\int_a^b \frac{1}{(x-x_0)^k}\, dx = \frac{-1}{k-1}\left[\frac{1}{(x-x_0)^{k-1}}\right]_a^b, \quad k>1, x_0 \notin [a,b]$$

If $4\beta-\alpha^2>0$, then $q(x)=x^2+\alpha x+\beta$ has no real zeros.

Integration of rational functions II

Let $4\beta - \alpha^2 > 0$ and $k > 1$.

3)

$$\int_a^b \frac{1}{x^2 + \alpha x + \beta}\, dx = \left[\frac{2}{\sqrt{4\beta - \alpha^2}} \arctan\left(\frac{2x + \alpha}{\sqrt{4\beta - \alpha^2}} \right) \right]_a^b$$

4)

$$\int_a^b \frac{ax + b}{x^2 + \alpha x + \beta}\, dx = \left[\frac{a}{2} \ln\left(\left| x^2 + \alpha x + \beta \right| \right) \right]_a^b + \left(b - \frac{a\alpha}{2} \right) \int_a^b \frac{1}{x^2 + \alpha x + \beta}\, dx$$

5)

$$\int_a^b \frac{1}{(x^2 + \alpha x + \beta)^k}\, dx = \left[\frac{2x + \alpha}{(k-1)(4\beta - \alpha^2)(x^2 + \alpha x + \beta)^{k-1}} \right]_a^b + \frac{2(2k-3)}{(k-1)(4\beta - \alpha^2)} \int_a^b \frac{1}{(x^2 + \alpha x + \beta)^{k-1}}\, dx$$

6)

$$\int_a^b \frac{ax + b}{(x^2 + \alpha x + \beta)^k}\, dx = \left[\frac{-a}{2(k-1)(x^2 + \alpha x + \beta)^{k-1}} \right]_a^b + \left(b - \frac{a\alpha}{2} \right) \int_a^b \frac{1}{(x^2 + \alpha x + \beta)^k}\, dx$$

Improper integrals I

In many applications we find integrals over infinite intervals. This motivates to consider **improper integrals** in the following.

Definition (Improper integral (infinite interval))

Let $f : [a, \infty) \to \mathbb{R}$ a function,
that is on any interval $[a, R]$, $a < R < \infty$, integrable,

then, assuming that the limit exists, we call

$$\int_a^\infty f(x)\,dx := \lim_{R\to\infty} \int_a^R f(x)\,dx$$

the **improper integral** of f on $[a, \infty)$.

Analoguously we define

$$\int_{-\infty}^a f(x)\,dx := \lim_{R\to\infty} \int_{-R}^a f(x)\,dx.$$

Improper integrals II

Definition (Improper integral (singularity))

Let $f : [a, b) \to \mathbb{R}$ a function,
that is on any interval $[a, b-\varepsilon]$, $0 < \varepsilon < b-a$, integrable (but not on $[a, b]$),

then, assuming that the limit exists,

we call
$$\int_a^b f(x)\,dx := \lim_{\varepsilon\downarrow 0} \int_a^{b-\varepsilon} f(x)\,dx$$
the **improper integral** of f on $[a, b]$.

Analoguously we define
$$\int_a^b f(x)\,dx := \lim_{\varepsilon\downarrow 0} \int_{a+\varepsilon}^{b} f(x)\,dx.$$
Moreover, we may define improper integrals for infinity or singularities, resp., at both ends.

Comparison criterion for series

Improper integrals may allow to decide on the convergence of series:

Theorem (Integral test)

Let $f : [1, \infty) \to \mathbb{R}_0^+$ a monotonically increasing function.

The series $\sum_{n=1}^{\infty} f(n)$ converges, iff the improper integral $\int_1^{\infty} f(x)\, dx$ exists.

Bell curve

An improper integral important for stochastics is the integral over the Gauss bell curve, i.e. the probability density function

$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{(x-\mu)^2}{\sigma}\right), \quad \mu \in \mathbb{R}, \sigma > 0.$$

For a probability we have to check (see Analysis 2)

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

However, for $f(x)$ there exists no closed form for a primitive. We define the **error function** (for the standard normal distribution) as

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{x} \exp\left(-\frac{x^2}{2}\right) dx.$$

Gamma function

Definition (Gamma function)

The function

$$\Gamma : \mathbb{R}^+ \to \mathbb{R}^+,\ x \mapsto \int_0^\infty t^{x-1} \exp(-t)\, dt$$

is called **gamma function**.

The gamma function interpolates the factorial:

$$\begin{aligned} \Gamma(x+1) &= x \cdot \Gamma(x) && \text{for all } x \in \mathbb{R}^+, \\ \Gamma(1) &= 1, && \\ \text{i.e.} \quad \Gamma(n+1) &= n! && \text{for all } n \in \mathbb{N}_0. \end{aligned}$$

Improper integrals of this type are important for the probability of failure (Weibull distribution).

Summary

- Concepts: **integral**, **integrand**, **integration bounds**, **integration variable**
- Approximation by a Riemann sum
- Concepts: **primitive**/indefinite integral, **definite integral**
- Fundamental theorem of differential and integral calculus

  Integration is an “inverse operation” of differentiation
- Practical computation of integrals
- Improper integrals

Further topics in integral calculus

Further topics:

- Volumes of rotational bodies
- Path integrals, surface integrals, volume integrals $\rightsquigarrow$ Analysis 2
- Gauss divergence theorem, Stokes theorem $\rightsquigarrow$ Analysis 2 or later
- ...