Series

Series play an important role

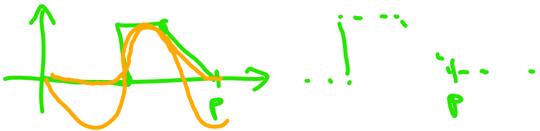
$0,\overline{123} = 0,123123\ldots$

- in analysis: definition of particular functions as exp, sin or cos, digits systems, . . . and
- in applications: Taylor expansions, Fourier series, finite elements . . .).

$e = 2,718\ldots$

$\{a_n\}_{n \geq n_0}$ $\quad$ $\left\{\sum_{k=n_0}^{n} a_k x^k\right\}_{n \geq n_0}$

$$\sum_{k=n_0}^{n} \left(a_k \cos(kx) + b_k \sin(kx)\right)$$



Series as partial sums of a sequence

Definition (Series (or sum of a sequence))

Let $\{a_n\}_{n \geq n_0}$ be a sequence of real numbers, then we define $\{\sum_{k=n_0}^{n} a_k\}_{n \geq n_0}$ as a **sequence of partial sums**. It is called a **series.** Briefly, we write:

$$\sum_{k=n_0}^{\infty} a_k$$

If the sequence of partial sums converges, then we write also for the limit

$$\sum_{k=n_0}^{\infty} a_k := \lim_{n \to \infty} \sum_{k=n_0}^{n} a_k.$$

Examples for series

Exponential series

$$\exp(1) = \sum_{k=0}^{\infty} \underbrace{\frac{1}{k!}}_{=: a_k} = e$$

$$\exp(x) = \sum_{k=0}^{\infty} \underbrace{\frac{1}{k!} x^k}_{= a_k} \quad \text{for fixed } x \in \mathbb{R}$$

Geometric series

$$\sum_{k=0}^{\infty} \underbrace{q^k}_{= a_k} = 1 + q + q^2 + q^3 + \ldots \overset{|q|<1}{=} \frac{1}{1-q}$$

Harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = +\infty$$

$\left\{\frac{1}{k}\right\}_{k \geq 1}$ is a zero sequence

Cauchy criterion for series

Since series are defined as a sequence of partial sums, we may apply the Cauchy criterion for sequences as well to series:

Theorem (Cauchy criterion for the convergence of series)

Let $\{a_n\}_{n\geq n_0}$ be a sequence of real numbers, then the sequence $\{\sum_{k=n_0}^{n} a_k\}_{n\geq n_0}$ of partial sums and thus the series $\sum_{k=n_0}^{\infty} a_k$ converges, iff for any $\varepsilon > 0$ there exists a natural number $N(\varepsilon) \geq n_0$ such that

$\exists$

$\forall$

$\exists_1$ exists one and only one

$$\left|\sum_{k=m+1}^{n} a_k\right| < \varepsilon \text{ for all } n > m \geq N(\varepsilon)$$

$s_n := \sum_{k=n_0}^{n} a_k$ $\quad |s_n - s_m| < \varepsilon$ for all $n > m > N(\varepsilon)$

Necessary criterion for series

We consider further convergence criteria for series.

Theorem (Necessary condition for the convergence of a series) A is sufficient for B

A *If $\{a_n\}_{n\geq n_0}$ is a sequence of real numbers and the series $\sum_{k=n_0}^{\infty} a_k$ is convergent,*
B *then $\{a_n\}_{n\geq n_0}$ is a zero sequence.*

$A \Rightarrow B$

B necessary for A

We write $s_n := \sum_{k=n_0}^{n} a_k$ for the partial sum.

We know $\{s_n\}_{n\geq n_0}$ is convergent $\overset{\text{Cauchy crit.}}{\Longleftrightarrow}$

for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ s.t.

$|s_m - s_n| < \varepsilon$ for all $m, n > N(\varepsilon)$

Let $m = n+1$: $\Rightarrow |s_{n+1} - s_n| = |a_{n+1}| < \varepsilon$

$\Rightarrow \{a_n\}_{n\geq n_0}$ is a zero sequence $\square$

Criterion for non-negative series

Theorem (Necessary and sufficient condition for the convergence of a non-negative series)

If $\{a_n\}_{n \geq n_0}$ is a sequence of real numbers with $a_n \geq 0$, $n \geq n_0$, then series $\sum_{k=n_0}^{\infty} a_k$ is convergent, A
iff $\sum_{k=n_0}^{\infty} a_k$ is bounded. B

A <=> B

Remark:
By this theorem we may decide, whether a series converges or not,
but we do not find the limit by this means, only an upper bound.
This situation is typical for series.

Criterion for alternating series

Now we consider the convergence of series where the sign of the members changes alternatingly.

Theorem (Sufficient condition for the convergence of an alternating series (Leibniz criterion))

If $\{a_n\}_{n \geq n_0}$ is a monotonically decreasing sequence of real numbers ✓ ✓
with $a_n \geq 0$, $n \geq n_0$,
and $\lim_{n\to\infty} a_n = 0$, ✓

then the series $\sum_{k=\underbrace{n_0}_{=1}}^{\infty} (-1)^k |a_k|$ is convergent.

$|a_k| = \frac{1}{k}$

$\sum_{k=1}^{\infty} (-1)^k \frac{1}{k}$ converges

$n_0 = 1$

$\underbrace{\frac{1}{k}}_{= a_k} \xrightarrow{k\to\infty} 0$

$-\sum_{k=1}^{\infty} (-1)^k \frac{1}{k} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$
$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \dots = \ln(2)$

Absolute convergence

Definition (Absolute convergence)

A series $\sum_{k=n_0}^{\infty} a_k$ is called **absolutely convergent**, iff the series $\sum_{k=n_0}^{\infty} |a_k|$ for the absolute values is convergent.

Any absolutely convergent series is also convergent (in the ordinary sense).

$$a_k \leq |a_k|$$

$$\Rightarrow \quad \sum_{k=n_0}^{\infty} a_k \leq \underbrace{\sum_{k=n_0}^{\infty} |a_k|}_{\text{is a bound for } \sum_{k=n_0}^{\infty} a_k}$$

Examples

Alternating harmonic series

$$a = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Rearranging the summands yield

$$a \cancel{\overset{?}{=}} \underbrace{1 - \frac{1}{2}}_{\geq \frac{1}{2}} + \underbrace{\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9}\right) - \frac{1}{4}}_{\geq \frac{1}{2}} + \dots$$

not allowed

$$+ \underbrace{\left(\frac{1}{11} + \frac{1}{13} + \dots + \frac{1}{37}\right) - \frac{1}{6}}_{\geq \frac{1}{2}} + \overbrace{\underbrace{(\dots)}_{\text{as many summands as required}} - \frac{1}{8}}^{\geq \frac{1}{2}} + \dots$$

$$= \lim_{\ell \to \infty} \ell \cdot \frac{1}{2} = +\infty$$

Comparison test

Now we consider the absolute convergence of series.

Theorem (Majorant criterion for the absolute convergence of series)

If $\sum_{k=n_0}^{\infty} c_k$ is a convergent series
with $c_k \geq 0$, $k \geq n_0$,
and if $\{a_n\}_{n \geq n_0}$ is a convergent ~~series~~ sequence
with $c_n \geq |a_n| \, (\geq 0)$ for all $n \geq n_0$
then the series $\sum_{k=n_0}^{\infty} a_k$ converges absolutely.

In this context $\sum_{k=n_0}^{\infty} c_k$ is called a **majorant** of $\sum_{k=n_0}^{\infty} a_k$.

If a series $\sum_{k=n_0}^{\infty} d_k$ that is a **minorant** (defined analogously) of $\sum_{k=n_0}^{\infty} a_k$ does not converge absolutely, then $\sum_{k=n_0}^{\infty} a_k$ does not converge absolutely.

Ratio test

Theorem (Quotient criterion for the absolute convergence of series (by d'Alembert))

If $\sum_{k=n_0}^{\infty} a_k$ is a series with $a_k \neq 0$ for all $k \geq n_1 \geq n_0$, and if there exists a real number Θ (independent of k) with $0 \leq \Theta < 1$ s.t.

< important

$$\left|\frac{a_{k+1}}{a_k}\right| \leq \Theta \text{ for all } k \geq n_1,$$

then the series $\sum_{k=n_0}^{\infty} a_k$ converges absolutely.

The quotient criterion (ratio test) is only a sufficient condition for the absolute convergence.
It is not a necessary condition.

Root test

Theorem (Root criterion for the absolute convergence of series (by Cauchy))

If $\sum_{k=n_0}^{\infty} a_k$ is a series and if there exists a real number t with $0 \leq t < 1$ s.t.

$$\sqrt[k]{|a_k|} \leq t \text{ for all } k \geq n_0,$$

then the series $\sum_{k=n_0}^{\infty} a_k$ converges absolutely.

The root criterion (root test) is only a sufficient condition for the absolute convergence.
It is not a necessary condition.

Rearrangement of series

For brevity, we introduce $\mathbb{N}_{n_0} := \{n_0, n_0 + 1, n_0 + 2 \ldots\}$.

Definition (Rearrangement of a series)

Let $\sum_{k=n_0}^{\infty} a_k$ be a given series and
$r : \mathbb{N}_{n_0} \to \mathbb{N}_{n_0}$ be a bijecitve mapping,
then

$$\sum_{k=n_0}^{\infty} a_{r(k)}$$

is called a **rearrangement** of $\sum_{k=n_0}^{\infty} a_k$.

The rearranged series has the same summands as the original series.

We expect that rearranging a series does not change its limit. However, this is not clear for (infinite) series!

Rearrangement theorem

Theorem (Rearrangement theorem (Lévy-Steinitz))

If $\sum_{k=n_0}^{\infty} a_k$ is an absolutely convergent series, then any rearrangement of the series $\sum_{k=n_0}^{\infty} a_k$ converges to the same limit.

Simple convergent is in general not sufficient to assure the convergence of any rearranged series to the same limit.

see the counter-example with the alternating harmonic series $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$

p-adic numbers

Lemma (p-adic numbers)

Let $p \geq 2$ a natural number and $\{a_n\}_{n\geq 0}$ a sequence of integers with $0 \leq a_n \leq p$, $n \in \mathbb{N}_0$.

Then the series

$$\sum_{n=0}^{\infty} a_n p^{-n}$$

fulfills the Cauchy criterion and is, thus, convergent.

If we start with $n_0 = -k$ instead of $n_0 = 0$ this defines a **p-adic fraction**

$$\pm \sum_{n=-k}^{\infty} a_n b^{-n} := \pm \left(\sum_{n=-k}^{-1} a_n b^{-n} + \sum_{n=0}^{\infty} a_n b^{-n} \right).$$

Real numbers and p-adic fractions

Theorem (Real numbers and p-adic fractions)

Let $b \geq 2$ be a natural number.
Any real number can be represented as a p-adic fraction.

Thus we recover
from the definition of $\mathbb{R}$ as a complete, Archimidean ordered field
the usual representation of $\mathbb{R}$ as an (possibly infinite) decimal fraction (the case $b = 10$).

Uncountability of real numbers

Theorem (Uncountability of $\mathbb{R}$)

The set $\mathbb{R}$ of real numbers is uncountable.

Idea of proof:

- If suffices to show that $(0, 1)$ is uncountable.
- Indirect proof using a Cantor diagonal sequence.
- Assume there exists a $\{a_n\}_{n\in\mathbb{N}}$ s.t. $(0, 1) = \{a_n \,|\, n \in \mathbb{N}\}$.
- Write

$$\begin{aligned} a_1 &= 0.b_{11}b_{12}b_{13}\ldots \\ a_2 &= 0.b_{21}b_{22}b_{23}\ldots \\ a_3 &= 0.b_{31}b_{32}b_{33}\ldots \end{aligned}$$

- We construct a real number $c = 0.c_1c_2c_3\ldots \in (0, 1)$ that is not included in the above enumeration. $\lightning$

Exponential series

Theorem (Convergence of the exponential series)

For any $x \in \mathbb{R}$ the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!}$$

is absolutely convergent.
It is called **exponential series**.

Exponential function

Definition (Exponential function and Euler number)

By means of the exponential series we define the **exponential function**

$$\exp : \mathbb{R} \to \mathbb{R},\ x \mapsto \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and the **Euler number**

$$e := \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!} = 2,71828182846\ldots$$

Properties of the exponential function

Theorem (Properties of the exponential function)

For all $x, y \in \mathbb{R}$ we have

- *(functional equation)*

$$\exp(x + y) = \exp(x) \cdot \exp(y)$$

- $\exp(0) = 1$
- $\exp(-x) = \frac{1}{\exp(x)}$
- $\exp(x) > 0$
- $\exp(n) = e^n$ *for all* $n \in \mathbb{Z}$

The last property motivates the notation

$$e^x = \exp(x)$$

for any real exponent.

Product of series

Theorem (Cauchy product)

Let $\sum_{k=n_0}^{\infty} a_k$ and $\sum_{k=n_0}^{\infty} b_k$ be absolutely convergent series.
We define for $n \geq n_0$

$$c_n := \sum_{i=n_0}^{n} a_i \cdot b_{n+n_0-i}.$$

Then $\sum_{n=n_0}^{\infty} c_n$ is absolutely convergent with the limit

$$\sum_{n=n_0}^{\infty} c_n = \left(\sum_{k=n_0}^{\infty} a_k \right) \cdot \left(\sum_{k=n_0}^{\infty} b_k \right).$$

$\sum_{n=n_0}^{\infty} c_n$ is called the **Cauchy product** *of $\sum_{k=n_0}^{\infty} a_n$ and $\sum_{k=n_0}^{\infty} b_n$.*