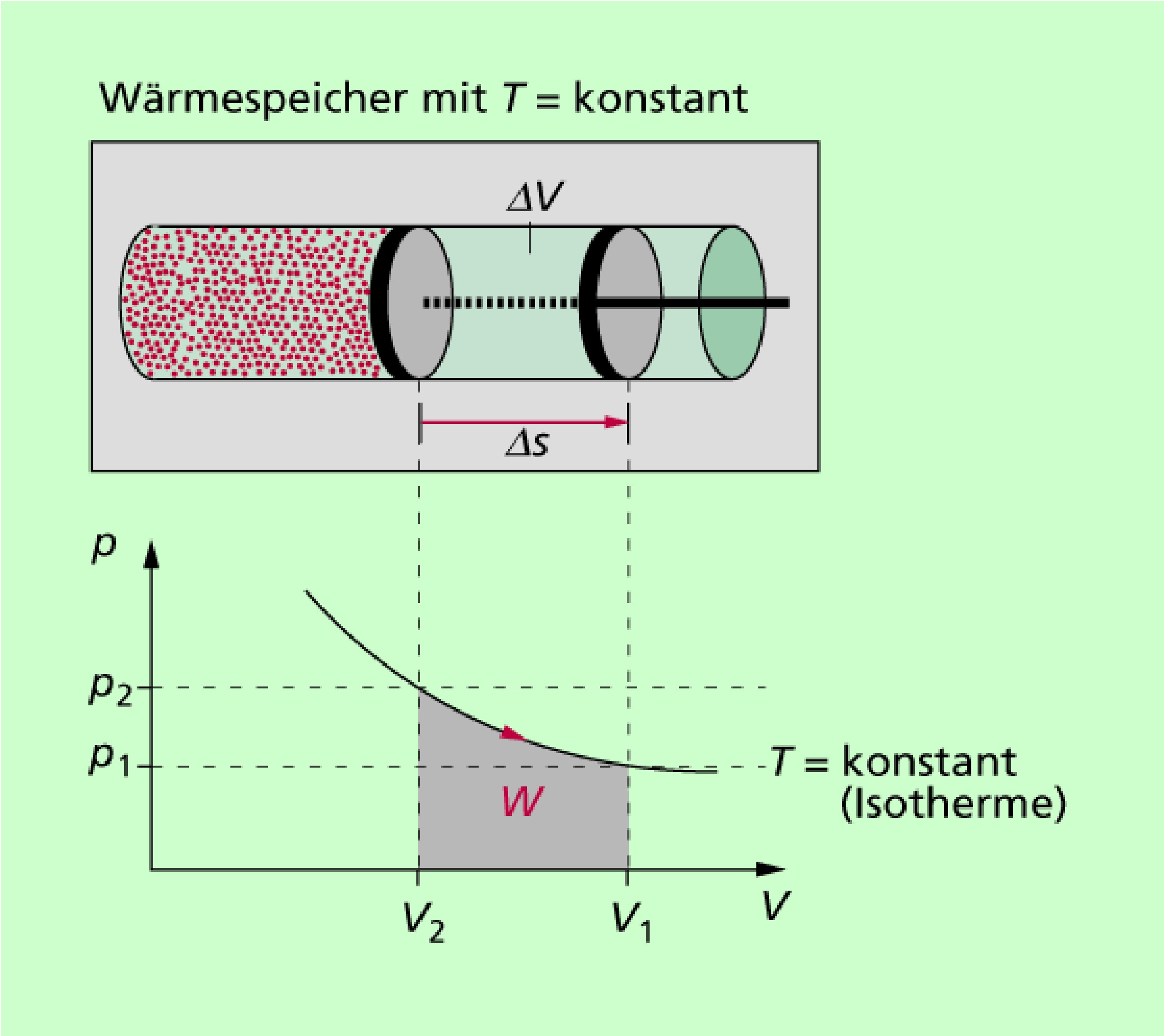
Practical example: isothermal compression of an ideal gas



(Source: lernhelfer.de)

Practical example: isothermal compression of an ideal gas

Example (Isothermal compression of an ideal gas)

The work W carried out on a closed (but non isolated) system for the isothermal compression of an ideal gas from $V_1 = s_1 A$ to $V_2 = s_2 A$ is

$$W = -\int_{s_1}^{s_2} p\,A\,ds.$$

For an ideal gas we have $pV = nRT = const$. Here $V = s\,A$.

Thus we may compute:

$$W = -\int_{s_1}^{s_2} p\,A\,ds = -\int_{s_1}^{s_2} \frac{nRT}{s}\,ds = -nRT\,(\ln(s_2) - \ln(s_1)) = nRT\ln\left(\frac{s_1}{s_2}\right).$$

Let $s_2 = 0,9 s_1$, $R = 8,31$ J/mol/K, $n = 0,22$ mol (5l oxygen) and $T = 300$ K, then

$$W = 0,22 \cdot 8,31 \cdot 300 \cdot \ln(10/9)\text{ J} \approx 57,8\text{ J}.$$

(In general: $W = -\int_{V_1}^{V_2} \frac{nRT}{V}\,dV = -(nRT\ln(V_2) - \ln(V_1)) = RT\ln\left(\frac{V_1}{V_2}\right)$)

Some exercises

1. Compute a primitive for:

   a) $\int (2x + 1)\, dx$, b) $\int \exp(x)\, dx$, c) $\int \frac{3}{1+x^2}\, dx$,
   d) $\int 2(\cos(x) + ax)\, dx$, e) $\int (3x - 2)^2\, dx$, f) $\int (1 + t^2)\, dx$,
   g) $\int (11 + \sqrt{17})\, \sqrt{x}\, dx$.

2. Compute <u>all</u> primitives for:

   a) $f(t) = 2e^t - \frac{5}{t} + 1$, b) $f(x) = 3\exp(x) - \cos(x)$,
   c) $f(u) = 3\sin(u) - \frac{6}{u} + 7u^2$.

3. Which values have the following definite integrals?

   a) $\int_1^e \frac{1}{t}\, dt$, b) $\int_\pi^2 \cos(\psi)\, d\psi$, c) $\int_1^2 5x^{1/4}$,
   d) $\int_0^4 (4s^5 - 6s^3 + 8x^2 + 5)\, ds$.

4. Based on the velocity-time law

$$v(t) = gt + v_0, \quad t \geq 0,$$

   compute a time law for the falling path $s(t)$ of a free falling body. Use $v(t) = s'(t)$.

Integrable functions

Integrand f has to be necessarily bounded on $[a, b]$

For instance, all continuous functions are (Riemann) integrable

Other constructions of integrals

- Regulated integral:

  f bounded, f limit of step functions w.r.t. sup norm

- **Riemann integral:**

  generalizes the regulated integral by considering sequences of uniformly convergent integrands

  Partition only of the domain of definition (“vertical stripes”)

- Lebesgue integral:
  - More arbitrary partitions are possible
  - Any regulated function is also Lebesgue integrable
  - Characteristic functions of bounded sets are Lebesgue integrable, other measures as the geometrical length (are, . . .) are possible

- Stieltjes, Bochner, and Birkhoff integral ...

Here integrable means Riemann integrable.

Small differences that are, e.g., important in probability theory

Practical computation of integrals

Analytical:

- By the fundamental theorem, tables, calculation rules
- (Directly by Riemann sums)
- By an expansion of the integrand into a power series
- ...

Numerical (so-called quadrature):

- Midpoint rule (like Riemann sum with t_i in the midpoint of the subinterval)
- Simpson's rule (Kepler's barrel rule)
- Romberg method
- Newton-Cotes formulas
- ...

or by computer algebra systems (Maple, Matlab Symbolic Toolbox, Mathematica . . .)

Calculation rules for integration

Since integration and differentiatin are coupled, we consider how differentiation rules transfer to integration rules.

- Substitution rule (follows from the chain rule)
- Integration by parts (follows from the product rule)

Moreover, we consider

- Integration of rational functions: Partial fraction expansion
- Improper integrals

Chain rule for integration

Theorem (Substitution rule)

Let $I \subseteq \mathbb{R}$ an interval,
$h : I \to \mathbb{R}$ a continuous function and
$f : [a, b] \to \mathbb{R}$ a continuously differentiable function with $f([a, b]) \subseteq I$,

then

$$\int_a^b h(f(t))f'(t)\,dt = \int_{f(a)}^{f(b)} h(x)\,dx.$$

Important special cases of the substitution rule

Product rule for integration

Theorem (Integration by parts)

Let $a < b$ and $f, g : [a, b] \to \mathbb{R}$ continuously differentiable functions,

then

$$\int_a^b f(x)g'(x)\,dx = [f(x)g(x)]_a^b - \int_a^b f'(x)g(x)\,dx$$

Revision: polynomial division with remainder I

Consider a rational function

$$r : A \to \mathbb{R}, x \mapsto \frac{p(x)}{q(x)} := \frac{\sum_{i=0}^{n} a_i x^i}{\sum_{i=0}^{m} b_i x^i}, \qquad a_n \neq 0, b_m \neq 0$$

where $A := \{x \in \mathbb{R} \mid \sum_{i=0}^{m} b_i x^i \neq 0\}$.

If $n \geq m$, then we set

$$p_1 : \mathbb{R} \to \mathbb{R}, x \mapsto p(x) - \frac{a_n}{b_m} x^{n-m} \cdot q(x)$$

and obtain the following representation

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_n}{b_m} x^{n-m} + \frac{p_1(x)}{q(x)} \qquad \text{for all } x \in A$$

where p_1 is either the zero function or a polynomial of degree smaller than n.

Revision: polynomial division with rem. II

This procedure may be iterated for k steps that produce a polynomial p_k
until the degree of p_k is less than n.

We end up with

$$\frac{p(x)}{q(x)} = g(x) + \frac{p_k(x)}{q(x)},$$

g a polynomial.

Partial fraction expansion I

Before integrating a rational function $r : A \to \mathbb{R}$ as defined above,
we carry out a polynomial division:

$$\int r(x)\,dx = \int g(x)\,dx + \int \frac{p_k(x)}{q(x)}\,dx$$

We know how to integrate the polynomial g.
For the remainder $\frac{p_k(x)}{q(x)}$, we consider the partial fraction expansion, i.e., the rational function is decomposed into a sum of fractions (yielding only a short list of cases with explicit formulas).

Partial fraction expansion II

Let again $r : A \to \mathbb{R}, x \mapsto \frac{p(x)}{q(x)}$ as defined above,
but w.l.o.g. let degree $p <$ degree q.
Moreover, let

$$q(x) = x(x-b_1)^{k_1} \cdot (x-b_2)^{k_2} \cdot \ldots \cdot (x-b_r)^{k_r} \cdot q_1(x)^{l_1} \cdot \ldots \cdot q_2(x)^{l_2}$$

with pairwise distinct zeros b_i of multiplicity k_i and pairwise distinct quadratic polynomials q_i that do not have zeros in $\mathbb{R}$.

Then there exists real numbers $A_1^{[1]}, \ldots, A_1^{[k_1]}, \ldots, A_r^{[1]}, \ldots, A_r^{[k_r]}$, $B_1^{[1]}, \ldots, B_1^{[l_1]}, \ldots, B_s^{[1]}, \ldots, B_s^{[l_s]}, C_1^{[1]}, \ldots, C_1^{[l_1]}, \ldots, C_s^{[1]}, \ldots, C_s^{[l_s]}$ s.t.

$$\frac{p(x)}{q(x)} = \sum_{i=1}^{r} \sum_{j=1}^{k_i} \frac{A_i^{[j]}}{(x-b_j)^j} + \sum_{i=1}^{s} \sum_{j=1}^{l_i} \frac{B_l^{[j]} + C_i^{[j]} x}{(q_i(x))^j} \quad \text{for all } x \in A.$$

Integration of rational functions I

Thus we only have to figure out how to integrate functions of the type

$$\frac{\zeta}{(x-x_0)^k}, \quad \frac{\xi + \mu x}{(q_i(x))^{\tilde{k}}}, \quad k, \tilde{k} \in \mathbb{N}:$$

Let $[a,b] \subset A$:

1)

$$\int_a^b \frac{1}{x-x_0}\, dx = [\ln(|x-x_0|)]_a^b, \quad x_0 \notin [a,b]$$

2)

$$\int_a^b \frac{1}{(x-x_0)^k}\, dx = \frac{-1}{k-1}\left[\frac{1}{(x-x_0)^{k-1}}\right]_a^b, \quad k > 1, x_0 \notin [a,b]$$

If $4\beta - \alpha < 0$, then $q(x) = x^2 + \alpha x + b$ has no real zeros.

Integration of rational functions II

Let $4\beta - \alpha^2 < 0$ and $k > 1$.

3)

$$\int_a^b \frac{1}{x^2 + \alpha x + \beta}\, dx = \left[\frac{2}{\sqrt{4\beta - \alpha^2}} \arctan\left(\frac{2x + \alpha}{\sqrt{4\beta - \alpha^2}} \right) \right]_a^b$$

4)

$$\int_a^b \frac{ax + b}{x^2 + \alpha x + \beta}\, dx = \left[\frac{a}{2} \ln\left(\left| x^2 + \alpha x + \beta \right| \right) \right]_a^b + \left(b - \frac{a\alpha}{2} \right) \int_a^b \frac{1}{x^2 + \alpha x + \beta}\, dx$$

5)

$$\int_a^b \frac{1}{(x^2 + \alpha x + \beta)^k}\, dx = \left[\frac{2x + \alpha}{(k-1)(4\beta - \alpha^2)(x^2 + \alpha x + \beta)^{k-1}} \right]_a^b + \frac{2(2k-3)}{(k-1)(4\beta - \alpha^2)} \int_a^b \frac{1}{(x^2 + \alpha x + \beta)^{k-1}}\, dx$$

6)

$$\int_a^b \frac{ax + b}{(x^2 + \alpha x + \beta)^k}\, dx = \left[\frac{-a}{2(k-1)(x^2 + \alpha x + \beta)^{k-1}} \right]_a^b + \left(b - \frac{a\alpha}{2} \right) \int_a^b \frac{1}{(x^2 + \alpha x + \beta)^k}\, dx$$