Recursive definition

The principle of complete induction allows to define a quantity D for all $n \in \mathbb{Z}$, $n \geq n_0$:

- Define D_{n_0}.
- Assume D_k is known for k with $n_0 \leq k \leq n$.
  Thus we may state D_{n+1} by means of D_k, $n_0 \leq k \leq n$.

This is called a **recursive definition**.

Factorial

Definition (Factorial)

For $n \in \mathbb{N}$ we define "n **factorial**" by

$$\boxed{n! = n \cdot (n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}$$

or recursively by

$$n! = n \cdot (n-1)! \ .$$

Formally we set

$$0! = 1 \ .$$

Remark: The factorial grows faster than any exponential function for sufficiently large n.

Permutation

Definition (Permutation)

A bijective mapping $f : A \to A$ of a set A onto itself is called a **permutation** of the set A.

If A is finite with n distinct elements,
e.g. $A = \{a_1, a_2, \ldots, a_n\}$,
then we interpret a permutation of A as a mapping rule for the a_i to n different, numbered (from 1 to n) places, s.t. a place is occupied by exactly 1 element.

Number of permutations

Theorem (Number of permutations)

For any $n \in \mathbb{N}$ there holds:
Any set with n distinct elements has exactly $n!$ different permutations.

Proof: By complete induction.

Sum and product symbols

For the sum or the product, resp., of the numbers

$$a_m, a_{m+1}, \ldots, a_n \in \mathbb{R}, \qquad n, m \in \mathbb{Z}, n \geq m,$$

we write

$$a_m + a_{m+1} + \ldots + a_n = \sum_{k=m}^{n} a_k,$$

$$a_m \cdot a_{m+1} \cdot \ldots \cdot a_n = \prod_{k=m}^{n} a_k,$$

By a recursive definition we may avoid the dots:

$$\sum_{k=m}^{m} a_k := a_m, \qquad \sum_{k=m}^{n+1} a_k := \left(\sum_{k=m}^{n} a_k\right) + a_{n+1},$$

$$\prod_{k=m}^{m} a_k := a_m, \qquad \prod_{k=m}^{n+1} a_k := \left(\prod_{k=m}^{n} a_k\right) \cdot a_{n+1}.$$

Properties of sums and products

Please note that sums (as well as products) are independent of the summation index, i.e.

$$\sum_{k=m}^{m} a_k = \sum_{i=m}^{n} a_i.$$

Moreover (also for products):

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{l} a_k + \sum_{k=l+1}^{n} a_k, \quad m \leq l \leq n.$$

For $n < m$ $(n, m \in \mathbb{Z})$ we define:

$$\sum_{k=m}^{n} a_k = 0,$$

$$\prod_{k=m}^{n} a_k = 1.$$

Binomial coefficients

Definition (Binomial coefficients)

Let $n, k \in \mathbb{N}_0$. We set

$$\binom{n}{k} := \frac{n!}{k!\,(n-k)!} \qquad \text{for} \quad 0 \le k \le n,$$

$$\binom{n}{k} := 0 \qquad \text{for} \quad k > n.$$

We say: "n choose k".

Moreover, for $1 \le k \le n$ there holds

$$\boxed{\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1}} \qquad (1).$$

Binomial coefficient: examples

Example: 5 gear wheels with 3 large and 2 small gear wheels (identical), how many possibilities to arrange them?

$$\binom{5}{3} =$$

Binomial coefficient: examples

Example: 5 gear wheels with 3 large and 2 small gear wheels (identical), how many possibilities to arrange them?

$$\binom{5}{3} =$$

Example: a German lottery

$$\binom{49}{6} = \qquad = 13\,983\,816$$

Binomial coefficients: properties

Symmetry

$$\binom{n}{k} = \binom{n}{n-k}$$

0th and 1st coefficient

$$\binom{n}{0} = \binom{n}{n} = 1,$$
$$\binom{n}{1} = \binom{n}{n-1} = n$$

Within the representation (1) all numerators and denominators are products with k factors.

The numerator start with n, any neyt factor is smaller by 1. The denominator is the product of the k first natural numbers.

Binomial coefficients: arithmetics

We have:

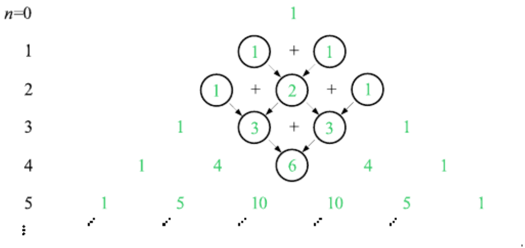
$$\boxed{\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}} \quad \text{für } 1 \leq k \leq n.$$

Since we compute:

$$\begin{aligned}
&\binom{n}{k-1} + \binom{n}{k} \\
&= \frac{k \cdot n \cdot (n-1) \cdot \ldots (n-k+2)}{k \cdot (k-1) \cdot \ldots \cdot 1} + \frac{n \cdot (n-1) \cdot \ldots (n-k+2) \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1} \\
&= \frac{(k+n-k+1) \cdot n \cdot (n-1) \cdot \ldots (n+1-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1} = \binom{n+1}{k}
\end{aligned}$$

(Could be demonstrated by combinatorics as well.)

Pascal’s triangle



Pascal’s triangle (Gerdts: Mathematik I)

$\binom{n}{k}$ appears in the n-th row in the k-th columns (starting with row and column 0)

Binomial

$a + b$ is called a **binomial**
(a special case of a polynomial)

I suppose most of you know

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2,\\ (a-b)^2 &= a^2 - 2ab + b^2,\\ (a+b)(a-b) &= a^2 - b^2\end{aligned}$$

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and maybe also

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
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But what about, e.g., $(a+b)^{100}$?

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\end{aligned}
$$

But what about, e.g., $(a+b)^{100}$?

The coefficients on the r.h.s. are the binomial coefficients.

Binomial theorem: our aim

Wanted:

A general formula for a “hypervolume”

$$(a+b)^n = a^n + \ldots ??? \ldots + b^n$$

without explicitly multiplying out

Binomial theorem

Theorem (Binomial theorem)

For $a, b \in \mathbb{R}$ and $n \in \mathbb{N}_0$ we have

$$\boxed{(a+b)^n} = a^n + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \ldots$$

$$\ldots + \binom{n}{n-2} a^2 b^{n-2} + \binom{n}{n-1} a^1 b^{n-1} + b^n$$

$$\boxed{= \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

Remark: The sum of the exponents in any sum is n.

Since $a + b = b + a$, also $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$.

For $(a-b)^n$ we replace b by $-b$ within the formula.

For odd powers we observe a change of sign.

Binomial theorem: proof

Proof.

By induction ...

Induction principle (Gerdts: Mathematik I)

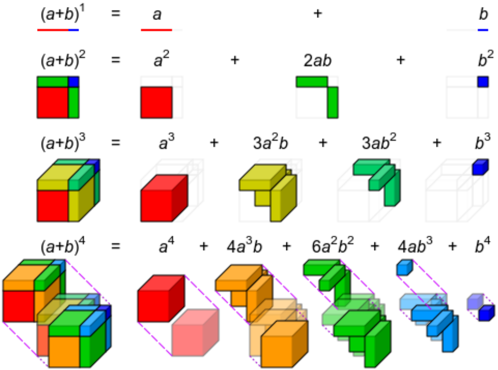
or by combinatorics:
Write

$$(a+b)^n = (a_1+b_1)\cdot(a_2+b_2)\cdot\ldots\cdot(a_n+b_n)$$

with $a_i = a$ and $b_i = b$.

The term $a^k b^{n-k}$ appears in the expansion, iff in k brackets a_i and in $n-k$ brackets b_i is chosen, i.e. in $\binom{n}{k}$ cases. □

Binomial theorem: visualization



Geometrical interpretation of the binomial theorem (for 1d - 4d).
Source: By Cmglee - Own work, CC BY-SA 3.0,
https://commons.wikimedia.org/w/index.php?curid=39642544

Binomial theorem: number of subsets

Theorem (Number of subsets)

Let $n \in \mathbb{N}_0$ and S a set with $|S| = n$ elements.
Then there exist exactly $\binom{n}{k}$ different subsets of S with exactly k elements.
All in all, M has 2^n different subsets.

Binomial theorem: applications

- Sum formulas

$$\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$$

$$\sum_{k=0}^{n} \binom{n}{k}(-1)^k = (1-1)^n = 0$$

- Expansion of the binomial $(a+2x)^4$ w.r.t. (increasing) powers of x

$$\begin{aligned}(a+2x)^4 &= a^4 + 4a^3 \cdot 2x + 6a^2 \cdot 2^2x^2 + 4a \cdot 2^3x^3 + 2^4x^4 \\ &= a^4 + 8a^3x + 24a^2x^2 + 32ax^3 + 16x^4\end{aligned}$$

- Binomial coefficient play a big role in combinatoris, statistics, etc.: binomial distribution, hypergeometric distribution, ...

Binomial theorem etc.: outlook

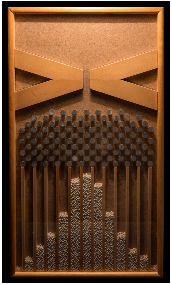
- Bernoulli experiments
- Binomial coefficient, generalized for $n \in \mathbb{R}$, $k \in \mathbb{N}$

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot 1}$$

- Leibniz' rule for derivatives (later!)
- Trigonometric summation formulas for $\sin^n(x)$, $\cos^n(x)$
- Taylor expansions of a binomial with real exponents
  General binomial series

$$(a \pm x)^n = a^n \pm \binom{n}{1} a^{n-1} x^1 \pm \binom{n}{2} a^{n-2} x^2 \pm \binom{n}{3} a^{n-3} x^3 + \ldots$$

- ...



Galton board (Wikipedia)

Binomial theorem etc.: summary

- Terms: **Binomial, factorial, binomial coefficient**
- $\boxed{\text{Properties of binomial coefficients}}$
- $\boxed{\text{Pascal's triangle (with calculation rule)}}$
- $\boxed{\text{Binomial theorem}}$

Binomial theorem etc.: Exercises

1. Calculate

   a) $\binom{13}{11}$, b) $\binom{7}{4}$, c) $\binom{13}{4}$, d) $\binom{67890}{12345}$, e) $\binom{9102}{2019}$, f) $\binom{2019}{9102}$.

   For d) and e) you should use Matlab, Maple, or Mathematica etc.

3. Compute
   $\binom{n+k}{k+1}$?

4. Compute

   a) $101^4 = (100+1)^4$, b) 98^5, c) $1,03^{12}$ (4 digits) .

5. Expand the following powers of binomials:

   a) $(x+4)^5$, b) $(1-5y)^4$, c) $(1-4x)^8$ (up to x^5) .