Complex numbers: Motivation

Introduction

Basics (sets, mappings, and numbers)

Proof techniques

Sequences and series

Functions

Continuity

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Complex numbers

Trigonometric functions

Differentiation in 1d

Integration in 1d

Summary - outlook and review

We are going to introduce the trigonometric functions sin and cos by the **Euler formula**

$$\exp(ix) = \cos(x) + i\sin(x), \quad x \in \mathbb{R},\ i \text{ the imaginary unity.}$$

This requires complex numbers $\mathbb{C}$ and the exponential function for complex arguments.

Another important motivation is the solution of equations of the type:

$$x^2 = -a, \quad a \in \mathbb{R}^+.$$

The fundamental theorem of algebra says any polynomial of degree n has n complex roots (if counted properly) $\rightsquigarrow$ Linear Algebra.

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Definition (Complex numbers)

The set $\mathbb{R} \times \mathbb{R}$ of (ordered) pairs together with the operations

$$(x_1, y_1) + (x_2, y_2) := (x_1 + x_2, y_1 + y_2),$$
$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 x_2 - y_1 y_2, x_1 y_2 + y_1 x_2)$$

is called the **complex numbers**, abbreviated with $\mathbb{C}$.

The neutral element of the addition is $(0, 0)$,
of the multiplication $(1, 0)$.
The inverse of (x, y) w.r.t. the addition is $(-x, -y)$,
w.r.t. the multiplication (where $(x, y) \neq (0, 0)$) is

$$\left(\frac{x}{x^2 + y^2}, \frac{-y}{x^2 + y^2}\right).$$

The field of complex numbers

Theorem

Field $\mathbb{C}$ The set of complex numbers $\mathbb{C}$ is a field.

But $\mathbb{C}$ is no ordered field.

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Representation of $\mathbb{C}$

We identify

- $(1, 0)$ with the unity 1 (of real numbers),
- $(0, 1)$ with the **imaginary unity** i

yielding the standard notation

$$z = a + bi \quad \text{for} \quad (a, b)$$

We call $\mathrm{Re}(z) := a$ the **real part of** z,
and $\mathrm{Im}(z) := b$ the **imaginary part of** z.

Cartesian complex plane,
similar to $\mathbb{R}^2$,
but with a different structure

Properties of $\mathbb{C}$

Let $x_1, x_2, y_1, y_2, \lambda \in \mathbb{R}$.
Then

- $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2) i$
- $(x_1 + y_1 i) \cdot (x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1) i$

In particular:

- $i^2 = -1$
- $\lambda(x_1 + y_1 i) = \lambda x_1 + \lambda y_1 i$
- $(x_1 + y_1 i) \cdot (x_1 - y_1 i) = x_1^2 + y_1^2$

  We call $\overline{z} := x_1 - y_1 i$ the **complex conjugate** of $z = x_1 + y_1 i \in \mathbb{C}$.

Analysis 1

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Geometric interpretation of addition in $\mathbb{C}$

Another representation of $\mathbb{C}$

Polar coordinate system for a point $P = (x|y)$:

uses distance $r = \sqrt{x^2 + y^2}$ to the origin 0
and angle ϕ between the positive x-axis and $0P$

For complex numbers:

radius $r = |z| = |x + yi| = \sqrt{z\overline{z}}$, the absolute value in $\mathbb{C}$

$$\phi = \arg(x + yi) := \begin{cases} \arccos\left(\frac{a}{r}\right) & ; b \geq 0 \\ -\arccos\left(\frac{a}{r}\right) & ; b < 0 \end{cases}$$

Polar complex plane
ϕ is not uniquely defined,
here $\phi \in (-\pi, \pi]$

Convergence in $\mathbb{C}$

We consider convergence of the real and imaginary part.

Concepts as (Cauchy) sequences, series, limits . . . may be transferred directly from real to complex numbers!