Uniform continuity

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We may sharpen the concept of continuity for a set, starting from the ε-δ formulation.

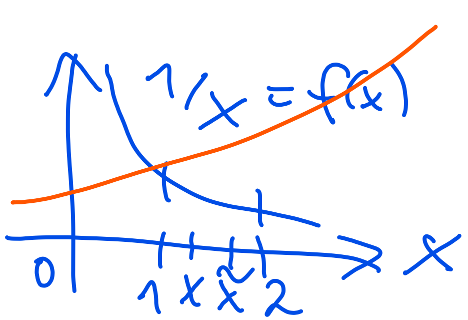
Definition (Uniform continuity)

Let $f : \mathbb{R} \supseteq A \to \mathbb{R}$ be a function.
f is called **uniformly continuous** in A, iff for any $\varepsilon > 0$ there exists a $\delta > 0$, s.t.

the same δ (not depending on ε and on $x, \tilde{x}$)

$$|f(x) - f(\tilde{x})| < \varepsilon \quad \text{for all } x, \tilde{x} \in A \text{ with } |x - \tilde{x}| < \delta.$$

Any uniformly continuous function $f : A \to \mathbb{R}$ is continuous in any point $a \in A$.
The converse does not hold true in general.



$1/x$ is continuous on $(0,\infty)$

$1/x$ is not uniformly cont. on $(0,\infty)$

$1/x$ is uniformy cont. on $(1,2)$

Uniform continuity

If the domain of a continuous function is a compact interval,
then the function is uniformly continuous.

Theorem (Uniform continuity on compact intervals)

Any continuous function $f : [a, b] \to \mathbb{R}$ *(with* $a \leq b$*)*

is uniformly continuous.

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An important application of uniform continuity are integrals. Integrals of uniformly continuous functions may be introduced by approximations by special functions:

Definition (Step functions (or staircase functions))

Let $a < b$ real numbers. A function $\tau : [a, b] \to \mathbb{R}$ is called ~~a~~ **a step function** (or a staircase function), if there exists a **partition**

$n = 11$

$t_0 = a \quad t_1 \quad t_2 \quad t_3 \quad \dots \quad t_{10} \quad b = t_{11}$

$$a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$$

of the interval $[a, b]$ and constants

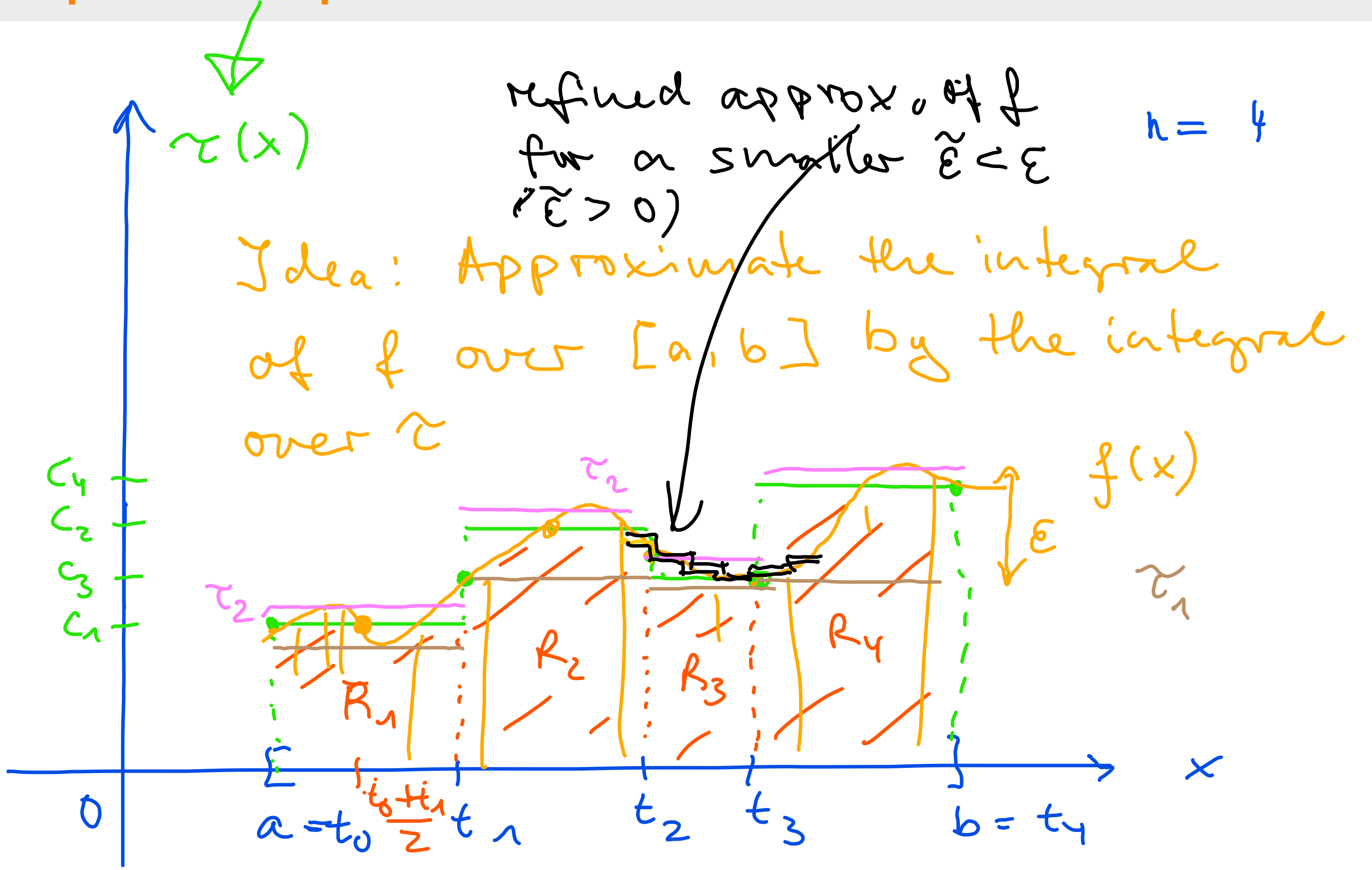
$$c_1, \ldots, c_{n-1}, c_n$$

such that

$$\tau(x) = c_k \quad \text{for all } x \in (t_{k-1}, t_k), k \in \{1, \ldots, n\}.$$

The function values $\tau(t_0), \ldots, \tau(t_n)$ at the "partition points" are arbitrary real numbers.

Example: Step function



We know how to compute the areas of R_1, R_2, R_3, R_4

Integral of τ over $[a,b]$: $|R_1| + |R_2| + |R_3| + |R_4|$

For instance $R_1 = c_1 \cdot (t_1 - t_0) = \tau\left(\frac{t_0+t_1}{2}\right) \cdot (t_1 - t_0)$

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Approximation by step functions

Theorem (Approximation of uniformly continuous functions)

uniformly !

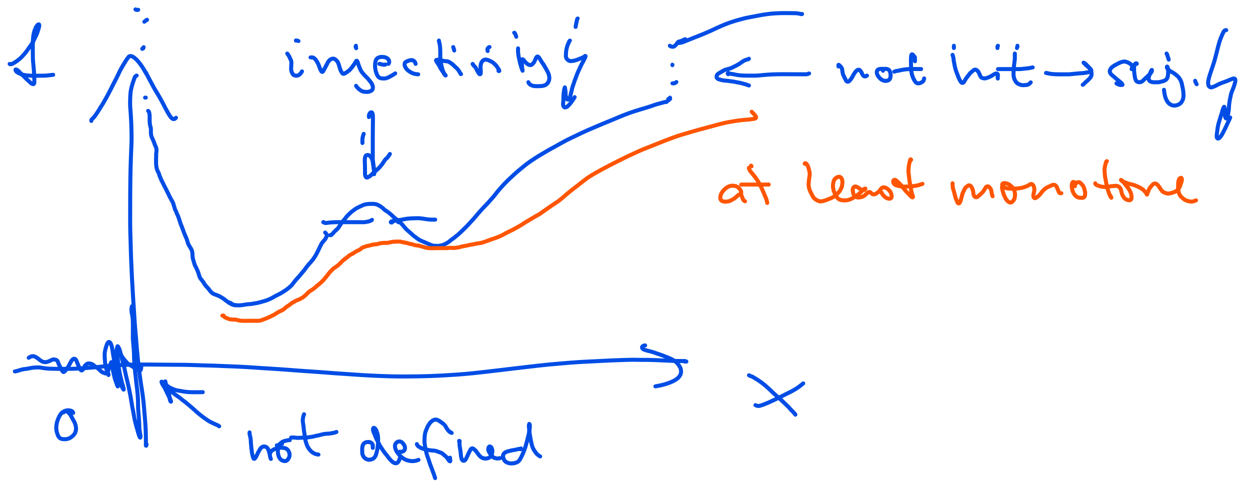
Let $f : [a, b] \to \mathbb{R}$ (with $a \leq b$) a continuous function, then there exist for any given tolerance $\varepsilon > 0$ step functions $\tau_1, \tau_2 : [a, b] \to \mathbb{R}$ with

a) $\tau_1(x) \leq f(x) \leq \tau_2(x)$ *for all* $x \in [a, b]$,

b) $|\tau_2(x) - \tau_1(x)| \leq \varepsilon$ *for all* $x \in [a, b]$.

Inverse functions

In order to invert a function, the bijectivity of the mapping is required.



For considering the typical case of inverse functions, we introduce monotone functions at first.

Monotone functions

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Analogously to monotone sequences we define monotone functions.

Definition (Monotone function)

A function $f : \mathbb{R} \supseteq A \to \mathbb{R}$ is called

- **monotonically increasing**,
  if $f(x) \leq f(\tilde{x})$ for all $x < \tilde{x}$ with $x, \tilde{x} \in A$,
- **strictly monotonically increasing**,
  if $f(x) < f(\tilde{x})$ for all $x < \tilde{x}$ with $x, \tilde{x} \in A$,
- **monotonically decreasing**,
  if $f(x) \geq f(\tilde{x})$ for all $x < \tilde{x}$ with $x, \tilde{x} \in A$
- **strictly monotonically decreasing**,
  if $f(x) > f(\tilde{x})$ for all $x < \tilde{x}$ with $x, \tilde{x} \in A$.

Inverse of continuous, monotone functions

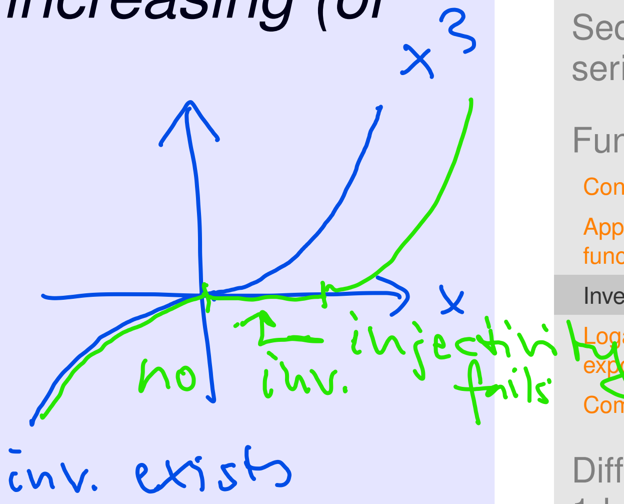
Theorem (Inverse functions of continuous, monotone functions)

Let I be an interval and
$f : I \to \mathbb{R}$ a continuous, strictly monotone increasing (or strictly monotone decreasing) function,

then $f(I)$ is an interval and the inverse function

$$f^{-1} : f(I) \to I$$

is also a continuous, strictly monotone increasing (or strictly monotone decreasing) function.



! **Warning:** $\underbrace{(f^{-1})(x)}_{=:g} = f^{-1}(x) \neq \frac{1}{f(x)} = (f(x))^{-1}$ ⚠

$\sin^2 x = \sin^2(x) = (\sin(x))^2 \neq \sin(x^2)$

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Theorem (k-th root)

Let $k \geq 2$ be a natural number. The function

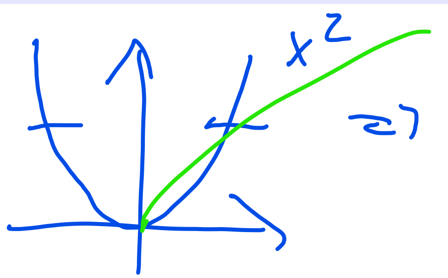
$$f : \mathbb{R}_0^+ \to \mathbb{R}_0^+,\ x \mapsto x^k$$

is continuous and strictly monotone increasing, and bijective.

Thus the inverse function

$$f^{-1} : \mathbb{R}_0^+ \to \mathbb{R}_0^+,\ x \mapsto \sqrt[k]{x}$$

exists and is continuous and strictly monotone increasing. It is called the **k-th root**.



$\sqrt[2]{x} = \sqrt{x}$

$\Rightarrow$ injectivity fails if the domain is $\mathbb{R}$, e.g.

Inverse of the exponential function

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Theorem (Natural logarithm)

The exponential function

$$f : \mathbb{R} \to \mathbb{R}^+,\ x \mapsto \exp(x) = e^x$$

is continuous, strictly monotone increasing, and bijective.

Thus the inverse function

$$f^{-1} : \mathbb{R}^+ \to \mathbb{R},\ x \mapsto \ln(x) = \log_e(x)$$
$$= \log(x)$$

exists and is continuous and strictly monotone increasing. It is called the **(natural) logarithm***.*

Moreover, we have the functional equation

$$\ln(x \cdot y) = \ln(x) + \ln(y) \quad \text{for all } x, y \in \mathbb{R}^+$$

and $\ln(1) = 0$.