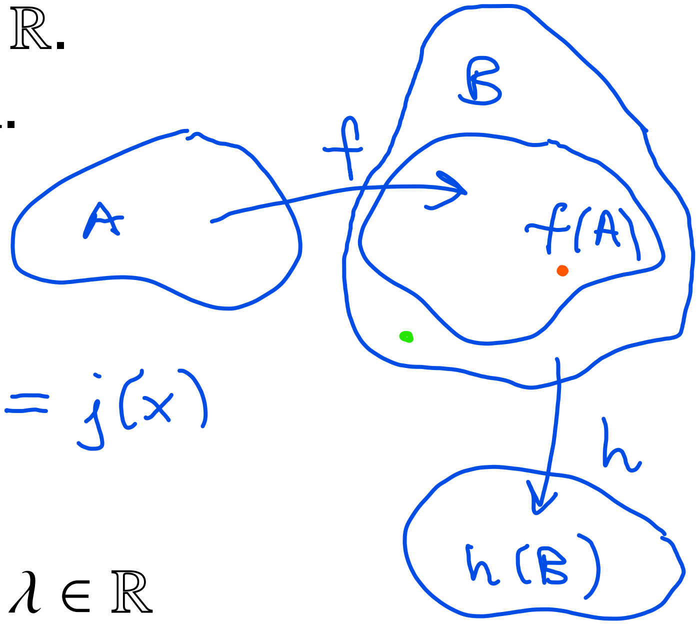
Outline

Functions: Operations for creating new functions

Let $f, g : A \to \mathbb{R}$ functions with $A \subseteq \mathbb{R}$.
Let $h : B \to \mathbb{R}$ functions with $B \subseteq \mathbb{R}$.



We obtain new functions by

$$f \pm g : A \to \mathbb{R}, \quad x \to f(x) \pm g(x) = j(x)$$

$$f \cdot g : A \to \mathbb{R}, \quad x \to f(x) \cdot g(x)$$

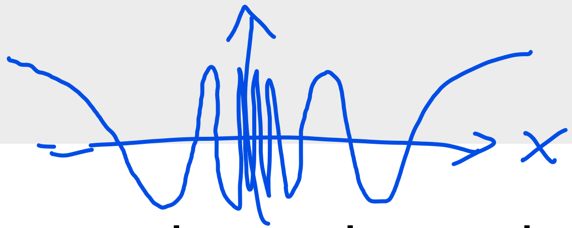
$$\lambda g : A \to \mathbb{R}, \quad x \to \lambda g(x) \quad \text{for } \lambda \in \mathbb{R}$$

$$\frac{f}{g} : \tilde{A} \to \mathbb{R}, \quad x \to \frac{f(x)}{g(x)} \quad \text{for } \tilde{A} := \{x \in A \,|\, g(x) \neq 0\}$$

$$j := h \circ f : A \to \mathbb{R}, \quad x \to h(f(x)) \quad \text{for } f(A) \subseteq B$$

The latter is the **concatenation of functions.**

Continuity

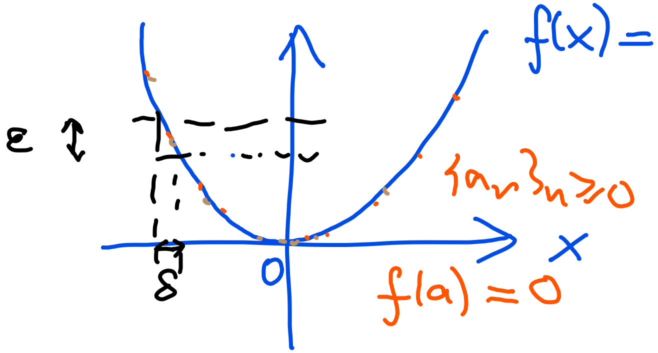


$\cos\left(\frac{1}{x}\right)$ $\downarrow$ $x \to 0$?

Continuity is a central concept in mathematics.

We combine the concepts of limits and functions.

$f: A \to \mathbb{R}$, $A \subset \mathbb{R}$

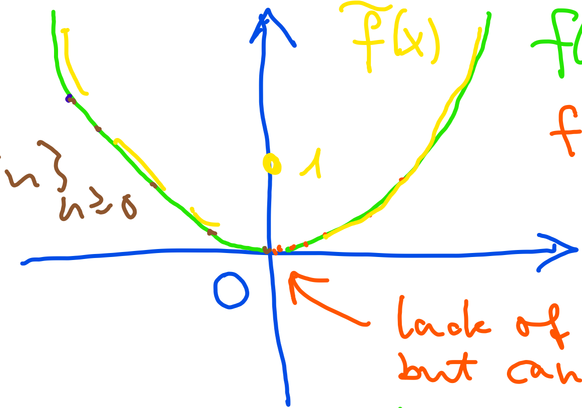


$f(x) = x^2$

$f(a) = 0$

$A = \mathbb{R}$

ε arb. large for $x \to 0$



$f(x) = \frac{x^3}{x} := x^2$

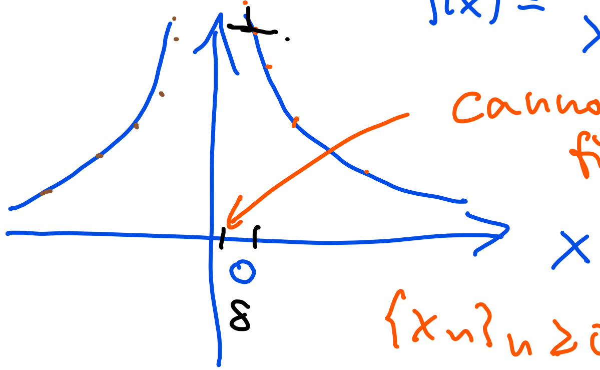
$f(0) := 0$

lack of continuity, but can be fixed

$A = \mathbb{R}\setminus\{0\}$

$f(x) \xrightarrow{x \to 0} +\infty$

$f(x) = \frac{1}{x^2}$



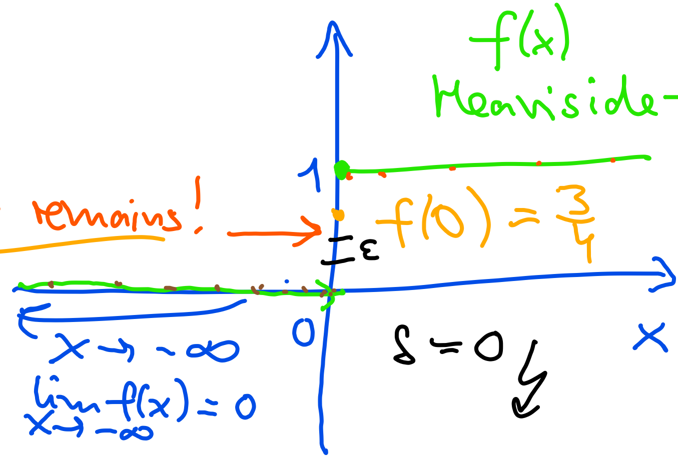
cannot be fixed

$\{x_n\}_{n\geq 0}$, $\lim_{n\to\infty} x_n = 0$

$A = \mathbb{R}\setminus\{0\}$

f(x)

Heaviside-fct.



gap remains!

$f(0) = \frac{3}{4}$

$\lim_{x\to -\infty} f(x) = 0$

$\delta = 0$ ↯

by def. $f(0) = 1$

Limits of functions I

Definition (Limit of a function)

or $f : A (\subseteq \mathbb{R}) \to \mathbb{R}$

or $f : \mathbb{R} \supseteq A \to \mathbb{R}$

Let $f : A \to \mathbb{R}$ a function, $A \subseteq \mathbb{R}$.
Let $a \in \mathbb{R}$ the limit of at least one sequence $\{a_n\}_{n \geq n_0}$ of real numbers in A.

If for any such sequence $\{a_n\}_{n \geq n_0}$ with $\lim_{n\to\infty} a_n = a$, we have

$$\lim_{n\to\infty} f(a_n) = b \text{ with } b \in \mathbb{R},$$

then we define b as the **limit of the function** f **at the point** a and we write

$$\lim_{x\to a} f(x) := b. \overset{?}{=} f(a)$$

might not correspond to the definition of f

Limits of functions II

Moreover, we introduce analogously:

$$\lim_{x\to a+} f(x) := \lim_{x\downarrow a} f(x) = b \quad \text{for sequences from the right only}$$

$$\lim_{x\to a-} f(x) := \lim_{x\uparrow a} f(x) = b \quad \text{for sequences from the left only}$$

$$\lim_{x\to\infty} f(x) = b \quad \text{for sequences unbounded from above}$$

$$\lim_{x\to-\infty} f(x) = b \quad \text{for sequences unbounded from below}$$

Continuity

Definition (Continuous function)

Let $A \subseteq \mathbb{R}$, $a \in A$ and $f : A \to \mathbb{R}$ a function.

If

$$\lim_{x \to a} f(x) = f(a) \tag{1}$$

then f is called **continuous in** a.

If f is continuous in any point of A,
then f is called **continuous on** A.

However, in order to verify the continuity,
it is not always necessary to check (1) for every a in A.

We proof that all polynomial functions are continuous on their domain of definitions.

The exponential function is continuous on $\mathbb{R}$ (w/o proof).

Continuity by operations on functions

Theorem (Continuity of composed functions)

Let $f, g : \mathbb{R} \supseteq A \to \mathbb{R}$ *a function, being continuous in* $a \in A$*, let* $h : \mathbb{R} \supseteq A \to \mathbb{R}$ *a function, being continuous in* $f(a) \in B$*, then*

$$\begin{aligned} f \pm g &: A \to \mathbb{R}, \quad x \to f(x) \pm g(x) \\ f \cdot g &: A \to \mathbb{R}, \quad x \to f(x) \cdot g(x) \\ \lambda g &: A \to \mathbb{R}, \quad x \to \lambda g(x) \qquad \text{for } \lambda \in \mathbb{R} \\ \frac{f}{g} &: \tilde{A} \to \mathbb{R}, \quad x \to \frac{f(x)}{g(x)} \qquad \text{for } \tilde{A} := \{x \in A \,|\, g(x) \neq 0\} \\ h \circ f &: A \to \mathbb{R}, \quad x \to h(f(x)) \qquad \text{for } f(A) \subseteq B \end{aligned}$$

are continuous in a.

Continuity of concatenated functions

Proof.

Let $\{a_n\}_{n\geq n_0}$ a real sequence in A with $\lim_{n\to\infty} a_n = a$.
Since f is continuous in a, we have

$$\lim_{n\to\infty} f(a_n) = f(a).$$

Due to the assumption $\{b_n\}_{n\geq n_0}$ with $b_n = f(a_n)$ is a real sequence in B.
Since h is continuous in $b = f(a)$, we have

$$\lim_{n\to\infty} h(f(a_n)) = \lim_{n\to\infty} h(b_n) = h(b) = h(f(a)).$$

Thus there holds

$$\lim_{n\to\infty} (h \circ f)(a_n) = \lim_{n\to\infty} h(f(a_n)) = h(f(a)). \quad \square$$

$\square$

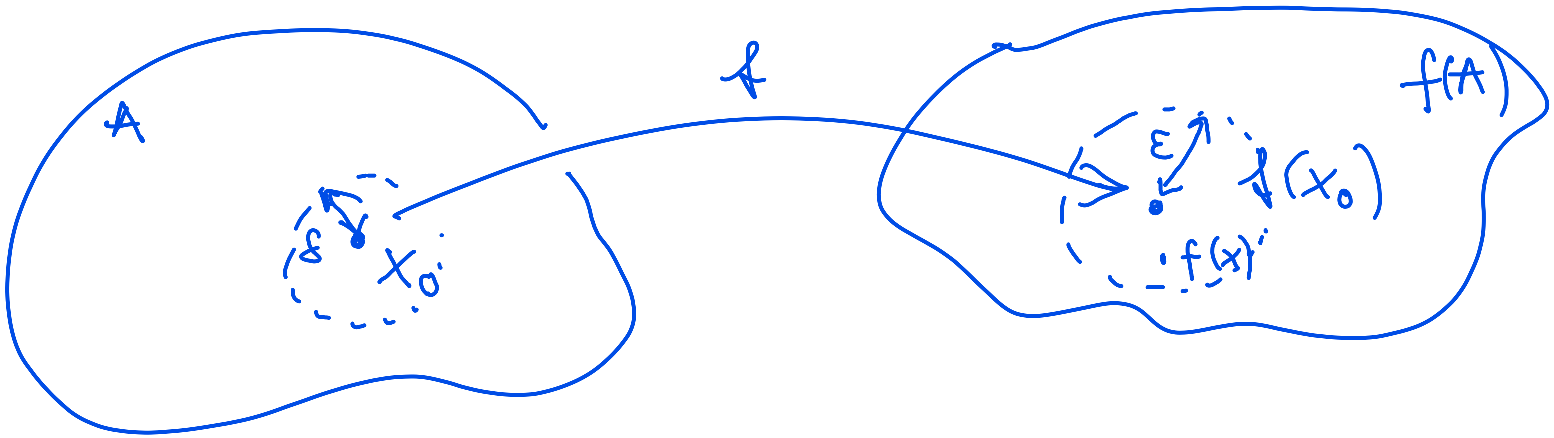
Alternative definition of continuity

Another, equivalent definition of continuity that does not rely on the concept of limits, can be found in many books (for that reason).

Theorem (ε-δ formulation of continuity)

Let $f : \mathbb{R} \supseteq A \to \mathbb{R}$ be a function.
f is continuous in $x_0 \in A$,
iff for any $\varepsilon > 0$ there exists a $\delta > 0$, s.t.

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for all } x \in A \text{ with } |x - x_0| < \delta(\varepsilon).$$



Applications of continuity

In this subsection we proof important results that hold for continuous functions.

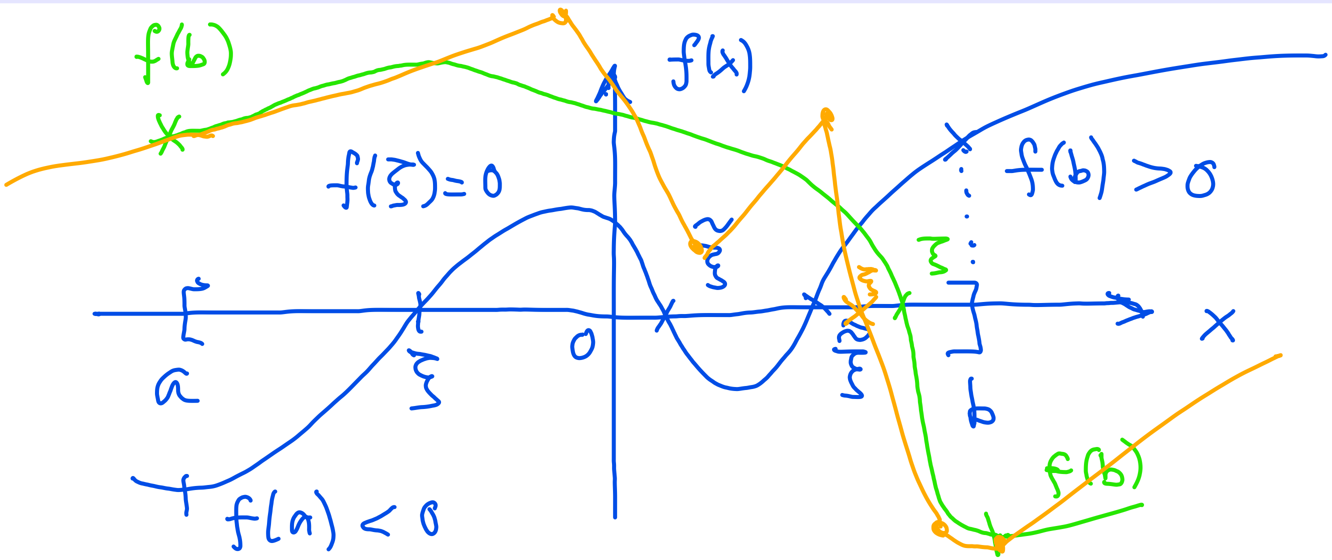
Bolzano's theorem on zeros

Theorem (Intermediate value theorem (1st version))

Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0$ and $f(b) > 0$ or with $f(a) > 0$ and $f(b) < 0$, then there exists a $\xi \in (a, b)$ s.t.

$$f(\xi) = 0.$$

$$\tilde{a} x = \tilde{b} e^x$$
$$\iff \tilde{a} x - \tilde{b} e^x = 0$$
$$\iff f(x) = 0$$



Intermediate value theorem

If we apply the last theorem (with $\tilde{f}$) to $\tilde{f}(x) - c + \varepsilon$:

Corollary (Intermediate value theorem)

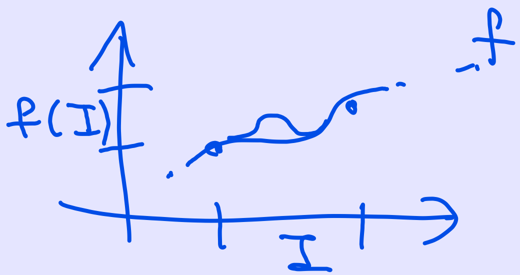
Let $f : [a, b] \to \mathbb{R}$ be a continuous function and $c \in \mathbb{R}$ with $f(a) \leq c \leq f(b)$ or with $f(a) \geq c \geq f(b)$, then there exists a $\xi \in [a, b]$ s.t.

$$f(\xi) = c.$$

Corollary (Image of an interval)

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ a continuous function,

then $f(I)$ is again an interval.



Compactness

Definition (Bounded function)

Let $f : A \to \mathbb{R}$ be a function and
$f(A)$ is bounded (from above and below)

then f is called a **bounded function**.

Definition (Compact interval)

Let $a, b \in \mathbb{R}$,
then a closed and bounded interval $[a, b]$
is called a **compact interval**.

Weierstrass theorem

Theorem (Continuous functions on compact intervals (Weierstrass))

Any continuous function $f : [a, b] \to \mathbb{R}$ (with $a \leq b$)

is bounded and
takes its maximum & minimum, i.e. there exists $\overline{x}$ & $\underline{x}$ s.t.

$$f(\overline{x}) = \sup_{x \in [a,b]} f(x),$$

$$f(\underline{x}) = \inf_{x \in [a,b]} f(x).$$

Note this theorem does not hold for open or semi-open (semi-closed) intervals.
This result may be generalized to higher dimensions.

It is of key importance in optimization.