

- 1 Introduction
- 2 Basics (sets, mappings, and numbers)
- 3 Proof techniques
- 4 Sequences and series
- 5 Functions**
 - Continuity
 - Applications for continuous functions
- 6 Differentiation in 1d
- 7 Integration in 1d
- 8 Summary - outlook and review

Introduction

Basics (sets, mappings, and numbers)

Proof techniques

Sequences and series

Functions

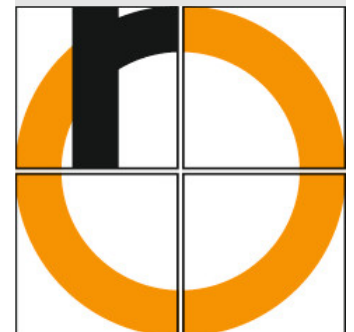
Continuity

Applications for continuous functions

Differentiation in 1d

Integration in 1d

Summary - outlook and review



Functions: Operations for creating new functions

Analysis 1

S.-J. Kimmerle

Introduction

Basics (sets, mappings, and numbers)

Proof techniques

Sequences and series

Functions

Continuity

Applications for continuous functions

Differentiation in 1d

Integration in 1d

Summary - outlook and review

Let $f, g : A \rightarrow \mathbb{R}$ functions with $A \subseteq \mathbb{R}$.

Let $h : B \rightarrow \mathbb{R}$ functions with $B \subseteq \mathbb{R}$.

We obtain new functions by

$$f \pm g : A \rightarrow \mathbb{R}, x \rightarrow f(x) \pm g(x)$$

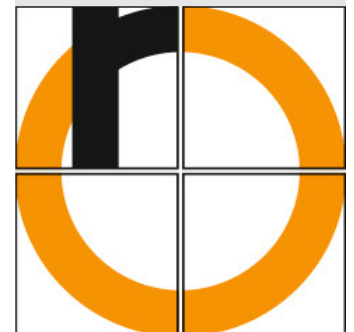
$$f \cdot g : A \rightarrow \mathbb{R}, x \rightarrow f(x) \cdot g(x)$$

$$\lambda g : A \rightarrow \mathbb{R}, x \rightarrow \lambda g(x) \quad \text{for } \lambda \in \mathbb{R}$$

$$\frac{f}{g} : \tilde{A} \rightarrow \mathbb{R}, x \rightarrow \frac{f(x)}{g(x)} \quad \text{for } \tilde{A} := \{x \in A \mid g(x) \neq 0\}$$

$$h \circ f : A \rightarrow \mathbb{R}, x \rightarrow h(f(x)) \quad \text{for } f(A) \subseteq B$$

The latter is the **concatenation of functions**.



Continuity is a central concept in mathematics.

We combine the concepts of limits and functions.

Introduction

Basics (sets,
mappings, and
numbers)

Proof techniques

Sequences and
series

Functions

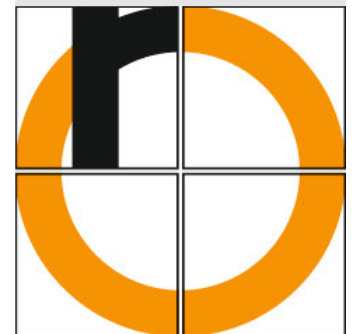
Continuity

Applications for continuous
functions

Differentiation in
1d

Integration in 1d

Summary - outlook
and review



Definition (Limit of a function)

Let $f : A \rightarrow \mathbb{R}$ a function, $A \subset \mathbb{R}$.

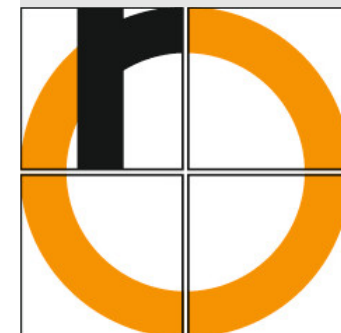
Let $a \in \mathbb{R}$ the limit of at least one sequence $\{a_n\}_{n \geq n_0}$ of real numbers in A .

If for any such sequence $\{a_n\}_{n \geq n_0}$ with $\lim_{n \rightarrow \infty} a_n = a$, we have

$$\lim_{n \rightarrow \infty} f(a_n) = b \text{ with } b \in \mathbb{R},$$

then we define b as the **limit of the function f at the point a** and we write

$$\lim_{x \rightarrow a} f(x) := b.$$



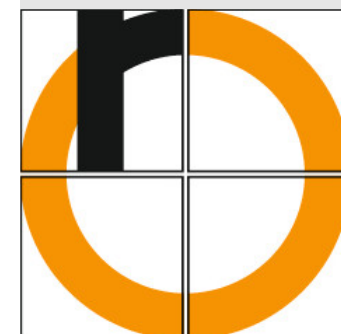
Moreover, we introduce analogously:

$\lim_{x \downarrow a} f(x) = b$ for sequences from the right only

$\lim_{x \uparrow a} f(x) = b$ for sequences from the left only

$\lim_{x \rightarrow \infty} f(x) = b$ for sequences unbounded from above

$\lim_{x \rightarrow -\infty} f(x) = b$ for sequences unbounded from below



Definition (Continuous function)

Let $A \subseteq \mathbb{R}$, $a \in A$ and $f : A \rightarrow \mathbb{R}$ a function.

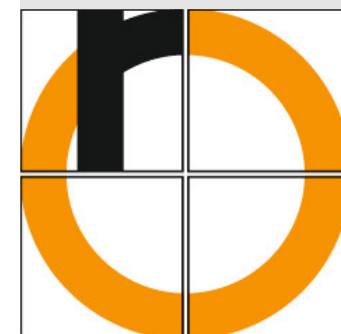
If

$$\lim_{x \rightarrow a} f(x) = f(a) \quad (1)$$

then f is called **continuous in a** .

If f is continuous in any point of A ,
then f is called **continuous on A** .

However, in order to verify the continuity,
it is not always necessary to check (1) for every a in A .



Theorem (Continuity of composed functions)

*Let $f, g : \mathbb{R} \supseteq A \rightarrow \mathbb{R}$ a function,
being continuous in $a \in A$,
let $h : \mathbb{R} \supseteq B \rightarrow \mathbb{R}$ a function,
being continuous in $f(a) \in B$,
then*

$$f \pm g : A \rightarrow \mathbb{R}, \quad x \rightarrow f(x) \pm g(x)$$

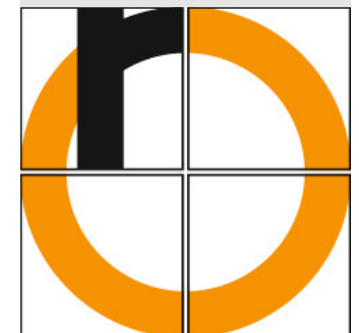
$$f \cdot g : A \rightarrow \mathbb{R}, \quad x \rightarrow f(x) \cdot g(x)$$

$$\lambda g : A \rightarrow \mathbb{R}, \quad x \rightarrow \lambda g(x) \quad \text{for } \lambda \in \mathbb{R}$$

$$\frac{f}{g} : \tilde{A} \rightarrow \mathbb{R}, \quad x \rightarrow \frac{f(x)}{g(x)} \quad \text{for } \tilde{A} := \{x \in A \mid g(x) \neq 0\}$$

$$h \circ f : A \rightarrow \mathbb{R}, \quad x \rightarrow h(f(x)) \quad \text{for } f(A) \subseteq B$$

are continuous in a .



Proof.

Let $\{a_n\}_{n \geq n_0}$ a real sequence in A with $\lim_{n \rightarrow \infty} a_n = a$.

Since f is continuous in a , we have

$$\lim_{n \rightarrow \infty} f(a_n) = f(a).$$

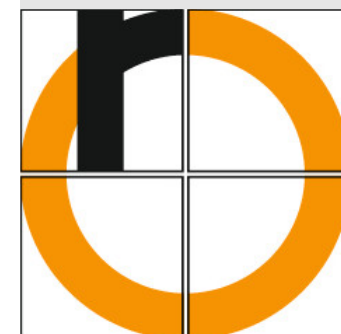
Due to the assumption $\{b_n\}_{n \geq n_0}$ with $b_n = f(a_n)$ is a real sequence in B .

Since h is continuous in $b = f(a)$, we have

$$\lim_{n \rightarrow \infty} h(f(a_n)) = \lim_{n \rightarrow \infty} h(b_n) = h(b) = h(f(a)).$$

Thus there holds

$$\lim_{n \rightarrow \infty} (h \circ f)(a_n) = \lim_{n \rightarrow \infty} h(f(a_n)) = h(f(a)). \quad \square$$



Alternative definition of continuity

Analysis 1

S.-J. Kimmerle

Another, equivalent definition of continuity that does not rely on the concept of limits, can be found in many books (for that reason).

Theorem (ε - δ formulation of continuity)

Let $f : \mathbb{R} \supseteq A \rightarrow \mathbb{R}$ be a function.

f is continuous in $x_0 \in A$,

iff for any $\varepsilon > 0$ there exists a $\delta > 0$, s.t.

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for all } x \in A \text{ with } |x - x_0| < \delta.$$

Introduction

Basics (sets, mappings, and numbers)

Proof techniques

Sequences and series

Functions

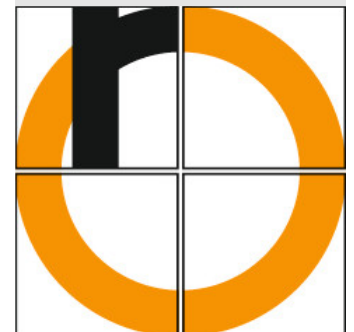
Continuity

Applications for continuous functions

Differentiation in 1d

Integration in 1d

Summary - outlook and review

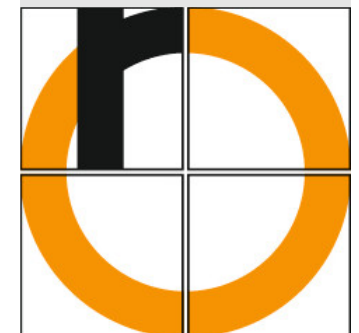


In this subsection we proof important results that hold for continuous functions.

Theorem (Intermediate value theorem (1st version))

*Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function
with $f(a) < 0$ and $f(b) > 0$ or
with $f(a) > 0$ and $f(b) < 0$,
then there exists a $\xi \in (a, b)$ s.t.*

$$f(\xi) = 0.$$



Corollary (Intermediate value theorem)

*Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function
and $c \in \mathbb{R}$*

with $f(a) \leq c \leq f(b)$ or

with $f(a) \geq c \geq f(b)$,

then there exists a $\xi \in [a, b]$ s.t.

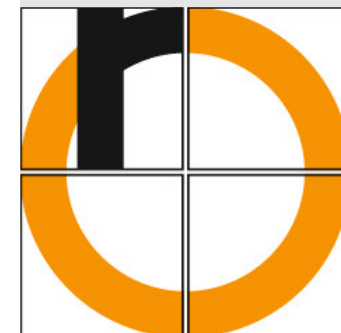
$$f(\xi) = c.$$

Corollary (Image of an interval)

Let $I \subseteq \mathbb{R}$ be an interval and

$f : I \rightarrow \mathbb{R}$ a continuous function,

then $f(I)$ is again an interval.



Definition (Bounded function)

Let $f : A \rightarrow \mathbb{R}$ be a function and $f(A)$ is bounded (from above and below) then f is called a **bounded function**.

Definition (Compact interval)

Let $a, b \in \mathbb{R}$, then a closed and bounded interval $[a, b]$ is called a **compact interval**.

Introduction

Basics (sets, mappings, and numbers)

Proof techniques

Sequences and series

Functions

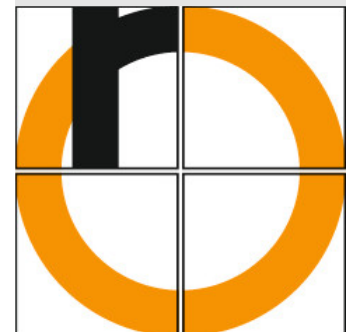
Continuity

Applications for continuous functions

Differentiation in 1d

Integration in 1d

Summary - outlook and review



Theorem (Continuous functions on compact intervals (Weierstrass))

Any continuous function $f : [a, b] \rightarrow \mathbb{R}$ (with $a \leq b$) is bounded and takes its maximum & minimum, i.e. there exists \bar{x} & \underline{x} s.t.

$$f(\bar{x}) = \sup_{x \in [a, b]} f(x),$$

$$f(\underline{x}) = \inf_{x \in [a, b]} f(x).$$

Note this theorem does not hold for open or semi-open (semi-closed) intervals.

This result may be generalized to higher dimensions.

It is of key importance in optimization.

