Outline

Sequences and series

Many mathematical problems cannot be solved within a finite number of steps

For instance, the calculation of the area of a circle

Solutions for these problems can be only approximated, but with arbitrary precision

$$\sum_{k=1}^{n} a_k \xrightarrow{n\to\infty} \sum_{k=1}^{\infty} a_k$$

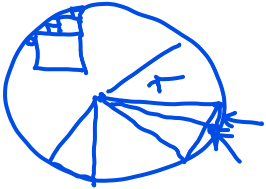
This motivates to consider:

- sequences (and as a special case series),
- limits, and
- the convergence thereof.

$$a_k \xrightarrow{k\to\infty} a_\infty = a$$

Area A ?

$$A = \pi r^2$$



Sequence

Definition (Sequence)

A mapping

$$f : \mathbb{N} \to \mathbb{R},\ n \mapsto a_n =: f(n)$$

or

$$f : \mathbb{N}_0 \to \mathbb{R},\ n \mapsto a_n =: f(n)$$

is called a **sequence** of elements of $\mathbb{R}$.

Other notations by writing the function values, e.g., are:

$$\{a_k\}_{k\in\mathbb{N}} = \{a_k\}_{k\geq 1} = \{\cancel{a_0}, a_1, a_2, \ldots a_k, \ldots\}$$

$$\{a_k\}_{k\in\mathbb{N}} = \{a_k\}_{k\geq 0} = \{a_0, a_1, \ldots, a_k, \ldots\}$$

Since the target area is $\mathbb{R}$, we call f a sequence of real numbers (or a real sequence).

Defining sequences

Examples:

$$f(n): \mathbb{N} \to \mathbb{R}, \quad n \mapsto a_n := \frac{1}{n}$$

$$g(n): \mathbb{N}_0 \to \mathbb{R}, \quad n \longmapsto (-1)^n =: b_n$$

Recursive definition of sequences:

$$a_1 = 1$$

$$a_2 = 1$$

$$a_n = a_{n-1} + a_{n-2}, \quad n \geq 3$$

Fibonacci sequence (originates from population models)

$$\{1, 1, 2, 3, 5, 8, 13, \ldots\}$$

grows exponentially

Bounded sequences

Definition (Bounded sequence)

A sequence $\{a_n\}_{n \geq n_0}$ of real numbers is called

- **bounded from above**,
  if there exists a $B \in \mathbb{R}$,
  s.t. $a_n \leq B$ for all $n \geq n_0$,
- **bounded from below**
  if there exists a $b \in \mathbb{R}$,
  s.t. $a_n \geq b$ for all $n \geq n_0$,
- **bounded**,
  if the sequence is bounded from above & from below.

Evidently, a real sequence $\{a_n\}_{n \leq n_0}$ is bounded,
iff there exists a R,

such that

$$\text{s.t. } |a_n| \leq R \text{ for all } n \geq n_0.$$

Definition (Convergent sequence and limit)

A sequence $\{a_n\}_{n \geq n_0}$ of real numbers is called **convergent**,
if there exists a real number a and
for any real $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ s.t.

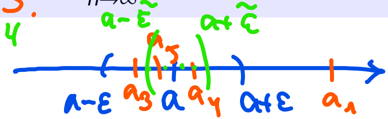
$$|a_n - a| < \varepsilon \text{ for all } n \geq N(\varepsilon).$$

The number a is called the **limit** of the sequence $\{a_n\}_{n \geq n_0}$.

Then we may write

$$\lim_{n\to\infty} a_n = a \quad \text{or} \quad a_n \to a \text{ as } n \to \infty.$$

here $N(\varepsilon) = 3$. $N(\tilde{\varepsilon}) = 4$ $+ \dots$ a_2



I can consider an arbitrary small interval?

Illustration of convergence

We say:
"The sequence $\{a_n\}_{n \geq n_0}$ converges to a". or
"The sequence $\{a_n\}_{n \geq n_0}$ has the limit a".

Descriptively:
the existence of a limit a means for a sequence $\{a_n\}_{n \geq n_0}$ that for any choice of a strictly positive (small) number ε almost all components a_n
are within the intervall $(a - \varepsilon, a + \varepsilon)$.

"Almost all" a_n means for infinitely many components except for a finite number (of exceptions).

Divergence

Definition (Divergent sequence)

A real sequence that is not convergent, is called a **divergent** sequence.

Definition (Zero sequence)

A real sequence that converges to 0, is called a **zero sequence**.

$\{(-1)^n\}_{n\geq 0} = \{1, -1, 1, -1, \ldots\}$ divergent

$\{\frac{1}{n}\}_{n\geq 10} = \{\frac{1}{10}, \frac{1}{11}, \ldots\}$ convergent

and it is a zero sequence

$\{2^n\}_{n\geq 1} = \{2, 4, 8, 16, \ldots\}$ divergent

Examples for sequences

$\lfloor$ here $n \in \mathbb{N}$

1) $a_n := \sin(n \frac{\pi}{2})$? $\rightsquigarrow$ divergent

$\{a_n\}_{n\in\mathbb{N}} = \{1, 0, -1, 0, 1, 0, -1, 0, \dots\}$

2) $b_n := \exp(nx) = e^{nx}$ for a fixed $x \in \mathbb{R}$

$x \leq 0: b_n \leq 1 \rightarrow$ convergence, $x > 0: b_n \rightarrow +\infty$

3) $c_n := \exp(-n) = e^{-n} \xrightarrow{n\to\infty} 0$

4) $d_n := \frac{1}{3^n}$

$\{d_n\}_{n\in\mathbb{N}} = \{\frac{1}{3}, \frac{1}{3^2} = \frac{1}{9}, \frac{1}{3^3} = \frac{1}{27}, \frac{1}{81}, \frac{1}{3^5}, \dots\}$ $\lim_{n\to\infty} d_n = 0$

Boundedness and convergence

An important property of bounded or convergent sequences, resp., is:

Theorem (A convergent sequence is bounded)

Any convergent sequence $\{a_n\}_{n\geq n_0}$ is necessarily bounded.

What about the opposite implication:

bounded sequence $\Rightarrow$ convergent sequence ?

Counter-example: see 1) on the last page

$\{\sin(n\frac{\pi}{2})\}_{n\in\mathbb{N}}$ is bounded,

but not convergent

Properties of limits I

Theorem (Uniqueness of the limit)

The limit of a convergent sequence $\{a_n\}_{n\geq n_0}$ is unique.

Proof.

Suppose the sequence converges to a as well as to $\tilde{a}$.

Thus the sequence $\{a_n - a_n\}_{n\geq 0}$ converges on one hand to $a - \tilde{a}$ and on other hand to 0.

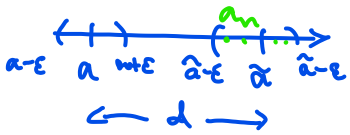
$= \{a_n\}_{n\geq 0} - \{a_n\}_{n\geq 0}$

$\{a_n - a_n\}_{n\geq 0} = \{0\}_{n\geq 0}$

Thus $a - \tilde{a} = 0 \Leftrightarrow a = \tilde{a}$.

$\square$

Alternative idea of proof: (proof by contradiction)



Suppose $|a - \tilde{a}| = d > 0$

We make $\varepsilon < \frac{d}{2}$ $\Longrightarrow$

Intersection is empty

Contradiction to the def.

Properties of limits II

$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = 1$

L'Hôpital's rule

Theorem (Sum and difference of limits)

Consider two convergent real sequences $\{a_n\}_{n\geq n_0}$ with $\lim_{n\to\infty} a_n = a$ and $\{b_n\}_{n\geq n_0}$ with $\lim_{n\to\infty} b_n = b$.

Then the sum $\{a_n + b_n\}_{n\geq n_0}$ of the sequences converges to $a + b$, i.e.:

$$\lim_{n\to\infty}(a_n + b_n) \overset{\not\Leftarrow}{=} \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = a + b$$

Further, the difference $\{a_n - b_n\}_{n\geq n_0}$ of the sequences converges to $a - b$, i.e.:

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n = a - b$$

In general ⚠: $\frac{d}{dx} \lim_{n\to\infty} \sum_{k=1}^{n} a_n(x) \neq \sum_{k=1}^{\infty} a_n'(x)$

$f(\lim_{n\to\infty} x_n) \neq \lim_{n\to\infty} f(x_n)$

Properties of limits III

Theorem (Product of limits)

Consider two convergent real sequences
$\{a_n\}_{n\geq n_0}$ *with* $\lim_{n\to\infty} a_n = a$ *and*
$\{b_n\}_{n\geq n_0}$ *with* $\lim_{n\to\infty} b_n = b$.

Then the product $\{a_n \cdot b_n\}_{n\geq n_0}$ *converges to* $a \cdot b$*, i.e.:*

$$\lim_{n\to\infty} (a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n = ab$$

Moreover, if $b \neq 0$
then there exists a $m_0 \geq n_0$ *s.t.* $b_n \neq 0$ *for all* $n \geq m_0$
and the quotient $\{a_n/b_n\}_{n\geq n_0}$ *converges to* a/b*, i.e.:*

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n} = \frac{a}{b}$$