Exponentiation and logarithm

By combining the exponetial function and the natural logarithm, we may introduce new functions:

Definition

Let $a \in \mathbb{R}^+$.
The function

$$\exp_a : \mathbb{R} \to \mathbb{R}^+, x \mapsto \exp(x \ln(a)) = a^x$$

is called **exponential function to the base** a.

Consistence: $\exp_e = \exp$

Exponential function of the base a

Theorem (Properties of $\exp_a$)

The function

$$\exp_a : \mathbb{R} \to \mathbb{R}^+, x \mapsto \exp(x \ln(a))$$

is continuous and there holds:

1. $\exp_a(x+y) = \exp_a(x) \cdot \exp_a(y)$ *for all* $x, y \in \mathbb{R}$
2. $\exp_a(0) = 1$
3. $\exp_a(n) = a^n$ *for all* $n \in \mathbb{Z}$
4. $\exp_a\left(\frac{p}{q}\right) = \sqrt[q]{a^p}$ *for all* $p \in \mathbb{Z}$ *and* $q \in \mathbb{N}, q \geq 2$

$\exp(1 \cdot \ln(a))$
$= \exp(\ln(a)) = a$

$= a^{p/q}$

Compatibility with exponentiation and (natural) exponential function

As a consequence we may write

$$a^x = \exp_a(x)$$

for all $a \in \mathbb{R}^+$ and $x \in \mathbb{R}$.

Consistency check:

$$e^x = \exp_e(x) = \exp(x)$$

$$= \exp_a\left(\lim_{n\to\infty} \frac{1}{n}\right)$$

Moreover, the continuity of $\exp_a$ yields:

$$\lim_{n\to\infty} \sqrt[n]{a} = \lim_{n\to\infty} \exp_a\left(\frac{1}{n}\right) = \exp_a(0) = 1 \quad \text{for all } a \in \mathbb{R}^+$$

Powers of the base a

Theorem (Properties of a^x)

The function

$$a^{\cdot} : \mathbb{R} \to \mathbb{R}^+, x \mapsto a^x \overset{\text{def}}{=} \exp(x \cdot \ln(a))$$
$$= \exp_a(x)$$

is continuous and there holds:

1. $(a^x)^y = a^{yx}$ *for all* $x, y \in \mathbb{R}$
2. $a^0 = 1$
3. $a^x b^x = (ab)^x$ *for all* $a, b \in \mathbb{R}^+$
4. $\left(\frac{1}{a}\right)^x = a^{-x}$

Uniqueness of exponential functions

Theorem (Functional equation and exponentiation)

If $F : \mathbb{R} \to F(R) \subseteq \mathbb{R}$ *be a continuous function with*

$$F(x+y) = F(x) \cdot F(y) \qquad \text{for all } x, y \in \mathbb{R}$$

then either

$$F(x) = 0 \qquad \text{for all } x \in \mathbb{R}$$

or we have $a := F(1)$ *and*

$$F : \mathbb{R} \to \mathbb{R}^+, x \mapsto a^x.$$

$$\exp(x) = \exp_e(x) \overset{x=1}{=} \exp(1 \cdot \ln(e)) = e$$

$$1^x = 1 \mathrel{\widehat{=}} F(x) \qquad F(x+y) = 1 \qquad \underbrace{F(x)}_{1} \cdot \underbrace{F(y)}_{1} = 1$$

Generalized logarithms

Theorem (Logarithms to the base *a*)

The exponential function to the base $a \in \mathbb{R}^+$ *where* $a \neq 1$

$$f : \mathbb{R} \to \mathbb{R}^+, x \mapsto \exp_a(x) = a^x = \exp(x \cdot \ln(a))$$

is continuous, strictly monotonically (decreasing for $a > 1$*, increasing for* $a < 1$*, resp.), and, thus, bijective.*
Thus we have the existence of the inverse function

$$f^{-1} : \mathbb{R}^+ \to \mathbb{R}, x \mapsto \log_a(x).$$

It fulfills the functional equation

$$\log_a(xy) = \log_a(x) + \log_a(y) \qquad \text{for all } x, y \in \mathbb{R}^+.$$

Moreover, we find

$$\log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Outlook:

$\log'(|x|) = \frac{1}{|x|}$

$x \neq 0$

$\log_a$ *is called* **logarithm to the base** *a*.

Important logarithms

Some logarithms that have a specific application are abbreviated as follows:

- Decadic logarithm

$$ld(x) = lg(x) = \log(x) := \log_{10}(x)$$

- (Natural) logarithm

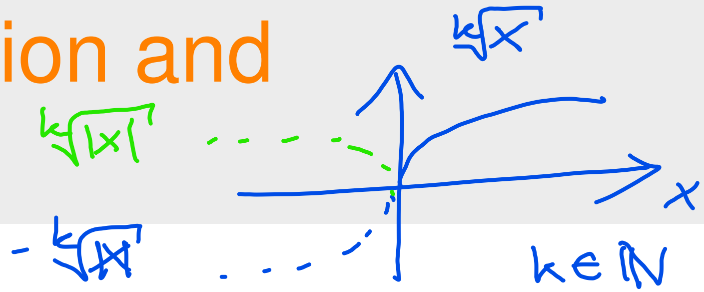
$$\log(x) = \ln(x) := \log_e(x)$$

e Euler number

- Binary logarithm (also called dual logarithm)

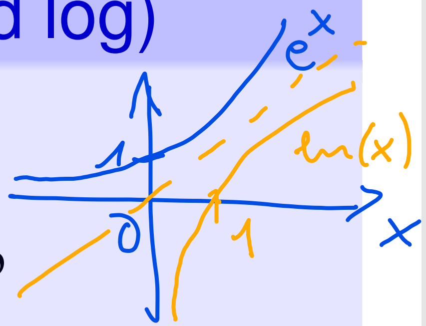
$$\big(ld(x) =\big)\ \mathrm{lb}(x) := \log_2(x)$$

Limits involving exponentiation and logarithms



Theorem (Some limits of powers, exp, and log)

For any real number $\alpha > 0$ we have



1) $\lim_{x \downarrow 0} x^\alpha = 0,$ *2)* $\lim_{x \to \infty} x^\alpha = \infty$

3) $\lim_{x \downarrow 0} \ln(x) = -\infty,$ *4)* $\lim_{x \to \infty} \ln(x) = \infty$

5) $\lim_{x \to \infty} \frac{\exp(x)}{x^\alpha} = \infty,$ *6)* $\lim_{x \to -\infty} x^\alpha \exp(x) = 0$

7) $\lim_{x \to \infty} \frac{\ln(x)}{x^\alpha} = 0,$ *8)* $\lim_{x \downarrow 0} x^\alpha \ln(x) = 0$

Rule of thumb: || $\exp(ix) = \cos(x) + i \sin(x)$

The exponetial grows (i.e. diverges definitely to ∞) faster than any power, $\cos(nx) \to$ express $\cos(x)$, $\sin(x)$

the logarithm grows slower than any power.