Properties of limits IV

Theorem (Limits of inequalities)

Consider two convergent real sequences $\{a_n\}_{n\geq n_0}$ *with* $\lim_{n\to\infty} a_n = a$ *and* $\{b_n\}_{n\geq n_0}$ *with* $\lim_{n\to\infty} b_n = b$.

If $a_n \leq b_n$ *for all* $n \geq n_0$, *then*

$$a = \lim_{n\to\infty} a_n \leq \lim_{n\to\infty} b_n = b.$$

Corollary:

$$\alpha \leq a_n \leq A \qquad \lim_{n\to\infty}$$

$$\alpha \leq \underbrace{\lim_{n\to\infty} a_n}_{=a} \leq A$$

Definite divergence

Definition (Definite divergence of a sequence)

A real sequence $\{a_n\}_{n\le n_0}$ is called

- **definitely (or certainly) divergent to** $+\infty$,
  if there exists for any $k_1 \in \mathbb{R}$ a $N(k_1) \in \mathbb{N}$ s.t.

$$a_n > k_1 \text{ for all } n \ge N(k_1),$$

- **definitely (or certainly) divergent to** $-\infty$,
  if there exists for any $k_2 \in \mathbb{R}$ a $N(k_2) \in \mathbb{N}$ s.t.

$$a_n < k_2 \text{ for all } n \ge N(k_2).$$

Otherwise a divergent sequence is called **indefinitely divergent**.

We write

$$\lim_{n\to\infty} a_n = \pm\infty.$$

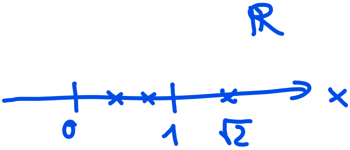
Cauchy sequence

Definition (Cauchy sequence)

A real sequence $\{a_n\}_{n \leq n_0}$ is called a **Cauchy sequence**, if for any $\varepsilon > 0$ there exists a natural number $N(\varepsilon)$ s.t.

$$|a_n - a_m| < \varepsilon \quad \text{for all } n, m \geq N(\varepsilon).$$

Every Cauchy sequence in the real numbers has a limit, this is equivalent to (or also one definition of) the completeness of the real numbers.



for $\mathbb{Q}$: $\sqrt{2} \notin \mathbb{Q}$

Equivalence of convergent and Cauchy sequences

Theorem (Cauchy criterion for the convergence of sequences)

A real sequence $\{a_n\}_{n \geq n_0}$ is convergent, iff $\{a_n\}_{n \geq n_0}$ is a Cauchy sequence.

Proof:

$\{a_n\}_{n \geq n_0}$ is convergent $\Leftarrow$ $\{a_n\}_{n \geq n_0}$ Cauchy sequence

Subsequence

Definition (Subsequence)

Let $\{a_n\}_{n \geq n_0}$ be a sequence of numbers and $n_1 < n_2 < \ldots$ a strictly increasing sequence of natural numbers.
Then $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

not a subsequence: $\{a_1, a_1, a_1, a_3, a_2, a_2, \ldots\}$

$a_n := (-1)^n$ $\quad \{a_1, a_2, a_3, a_4, \ldots\}$

$= \{-1, 1, -1, 1, \ldots\}$

$a_{2n} = (-1)^{2n} = ((-1)^2)^n = 1^n = 1$

subsequence $\rightarrow$ $\{a_2, a_4, a_6, a_8, \ldots\}$

$= \{1, 1, 1, 1, \ldots\}$

$\downarrow$

$a_{2n+1} = (-1)^{2n+1} = ((-1)^2)^n(-1) = -1$

Bolzano-Weierstrass theorem

Weierstraß

Theorem (Bolzano-Weierstrass)

Any bounded real sequence $\{a_n\}_{n\geq n_0}$ has a covergent subsequence.

n

Example:

$a_n = (-1)^n$ bounded ✓

not definitely divergent

convergent subsequences

$\{1, 1, 1, \ldots\}$ with acc. pt. 1

or $\{-1, -1, -1, \ldots\}$ with acc. pt. -1

or $\{\underbrace{-1, -1,}_{2 \text{ exceptions}} 1, 1, 1, \ldots\}$ with acc. pt. 1

Accumulation point

Definition

~~Theorem~~ (Accumulation point)

A real number a is called an **accumulation point (or cluster point** *or limit point) of a real sequence* $\{a_n\}_{n\geq n_0}$*, if there exits a subsequence of* $\{a_n\}_{n\geq n_0}$ *that converges to a.*

s

acc. pt. (it's the only acc. pt. here!)

$$b_n := \frac{1}{n} \longrightarrow 0 \quad \text{as} \quad n \to +\infty$$

limit

$$\left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \right\}$$

* any limit is an accumulation point
* if a sequence converges, the limit is the only accumulation point

Limes superior and inferior

Definition (Limes superior, limes inferior)

Let $\{a_n\}_{n\geq n_0}$ be a sequence of real numbers, then we define

- the **limes superior** of the sequence $\{a_n\}_{n\geq n_0}$ as

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right),$$

- the **limes inferior** of the sequence $\{a_n\}_{n\geq n_0}$ as

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty}\left(\inf_{k\geq n} a_k\right).$$

Remark: This definition is later important for a general approach to integrals required for stochastics.

Monotone sequences

Definition (Monotone real sequences)

A sequence $\{a_n\}_{n \geq n_0}$ of real numbers is called

- **monotonically increasing**, if $a_n \leq a_{n+1}$ for all $n \geq n_0$,
- **strictly monotonically increasing**, if $a_n < a_{n+1}$ for all $n \geq n_0$,
- **monotonically decreasing**, if $a_n \geq a_{n+1}$ for all $n \geq n_0$,
- **strictly monotonically decreasing**, if $a_n > a_{n+1}$ for all $n \geq n_0$.

$a_n := n$

$b_n := \frac{1}{n}$

$c_n := 1 - \frac{1}{n}$ | $e_n := (-1)^n$

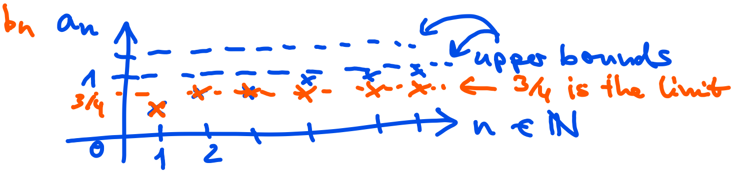
$d_n := \frac{(-1)^n}{2^n}$ | not monotone

$\{-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \dots\}$ not monotone

Properties of monotone sequences

Theorem (Convergence of bounded monotone sequences)

Any bounded monotone real sequence $\{a_n\}_{n \geq n_0}$ *converges.*



An illustration of the statement of the theorem.

(Un)countable

Definition (Countable set)

A non-empty set M is called **countable**, if there exists a sequence $\{a_n\}_{n \geq n_0}$ with

$$S \subseteq M = \{a_n \,|\, n \in \mathbb{N}\}.$$

A non-empty set M is called **uncountable**, if it is not countable.

Evidently, a subset of a countable set is countable.

A finite set is countable. Equivalently, an uncountable set has not a finite number of elements.

A countable set may have a finite or infinite number of elements.

Countability of union of sets

Theorem (Countability of the union of sets)

The union of a countable number of countable sets M_n, $n \in \mathbb{N}$, i.e.

$$\cup_{n\in\mathbb{N}} M_n = M$$

is again countable.

Theorem (Countability of $\mathbb{Q}$)

The set $\mathbb{Q}$ of rational numbers is countable.

Proof: by Cantor's diagonal seq.

$$\mathbb{Q} = \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z},\ q \in \mathbb{Z}\setminus\{0\} \right\}$$

We consider $\left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{N},\ q \in \mathbb{N} \right\}$ w.l.o.g. (see the last theorem)

bijection to $\mathbb{N}$

$$\begin{array}{cccccc}
a_1 = \frac{1}{1} & \to a_2 = \frac{1}{2} & a_5 = \frac{1}{3} & \to \frac{1}{4} & \frac{1}{5} & \to \cdots \\
a_3 = \frac{2}{1} & \frac{2}{2}\,(\cdot) & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \cdots \\
a_4 = \frac{3}{1} & \frac{3}{2} & \frac{3}{3}\,(\cdot) & \frac{3}{4} & \frac{3}{5} & \cdots, \\
\frac{4}{1} & \frac{4}{2}\,(\cdot) & \frac{4}{3} & \frac{4}{4}\,(\cdot) & \frac{4}{5} & \cdots
\end{array}$$