



# Computer Science Fundamentals

## Number Systems – Conversions

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- Place-value numeral systems (*Stellenwertsysteme*)
- Conversion of
  - integers (*ganze Zahlen*)
  - fractional numbers (*gebrochene Zahlen*)

- Focus: **efficient** and **reversible** mapping between numbers and bit sequences
- Bit sequences of a fixed length  $N \rightarrow 2^N$  numbers can be represented
- Common choices are  $N = 8, 16, 32$  or  $64$
- Using bit sequences of length  $N$  we can code for example
  - the natural numbers from  $0$  to  $2^N - 1$ , or
  - the integers between  $-2^{N-1}$  und  $2^{N-1} - 1$ , or
  - an interval of the real numbers with limited accuracy.

- Also called positional numeral system or positional/place-value notation
- The value of a number depends on the **position** of a digit
- Advantage: simple calculation rules
- Examples:
  - Dual system (also: Binary system)
  - Octal system
  - Decimal system
  - Hexadecimal system

- a positional system with base  $B$  is a numeral system, in which a number is broken down according to powers of  $B$
- A natural number  $n$  is represented by the following sum:
  - $B$  = Base of the numeral system ( $B \in \mathbb{N}, B \geq 2$ ),
  - $b_i$  = digits ( $b_i \in \mathbb{N}_0, 0 \leq b_i < B$ ),
  - $N$  = number of digits

$$n = \sum_{i=0}^{N-1} b_i B^i$$

## Representation of an **integer** $z$

- Sum of powers of base 10
- $z = a_n 10^n + a_{n-1} 10^{n-1} + \dots + a_2 10^2 + a_1 10^1 + a_0 10^0$ ,  
where  $a_0, a_1, a_2, \dots \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

- Example

$$\begin{aligned} 4711 &= 4 \cdot 10^3 + 7 \cdot 10^2 + 1 \cdot 10^1 + 1 \cdot 10^0 \\ &= 4 \cdot 1000 + 7 \cdot 100 + 1 \cdot 10 + 1 \end{aligned}$$

- Representation of numbers by base 2 using the digits {0, 1}
- For example, the bit sequence 1101 has the numerical value:

$$\begin{aligned} 1101 &= 1 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 \\ &= 1 \cdot 8 + 1 \cdot 4 + 0 \cdot 2 + 1 \\ &= 13 \end{aligned}$$

- Notation:

$$1101_2 = 13_{10}$$

- Disadvantage of **dual** system: **very long** numbers, therefore difficult to read/remember
  - Idea: Combine a certain number of binary digits
- Octal system
  - **3** binary digits are combined into a single octal digit
  - Representation of numbers by base  $2^3 = 8$  using digits  $\{0, 1, 2, 3, 4, 5, 6, 7\}$

• Examples

$$4711_8 = 4 \cdot 8^3 + 7 \cdot 8^2 + 1 \cdot 8^1 + 1 \cdot 8^0 = 2505_{10}$$

$$53_{10} = 110 \ 101_2 = 65_8$$



- Even more compact number representation
  - 4 binary digits are combined into a hexadecimal one
  - Representation of numbers by base  $2^4 = 16$  using digits {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F}

- Examples

$$\begin{aligned} 53_{10} &= 0011 \ 0101_2 \\ &= 35_{16} \end{aligned}$$

$$\begin{aligned} 4711_8 &= 100 \ 111 \ 001 \ 001_2 \\ &= 1001 \ 1100 \ 1001_2 \\ &= 9C9_{16} \end{aligned}$$

- can be done using the sum equation directly:  $n = \sum_{i=0}^{N-1} b_i B^i$
- or, more efficiently, using **Horner's Scheme**:

$$n = \sum_{i=0}^{N-1} b_i B^i = \left( \cdots \left( (b_{N-1} \cdot B + b_{N-2}) \cdot B + b_{N-3} \right) \cdot B + b_{N-4} \right) \cdot B + \cdots + b_1 \right) \cdot B + b_0$$

Examples:

$$1578_{10} = ((1 \cdot 10 + 5) \cdot 10 + 7) \cdot 10 + 8$$

$$\begin{aligned} 754_8 &= (7 \cdot 8 + 5) \cdot 8 + 4 \\ &= 492_{10} \end{aligned}$$

Convert the number  $2375_9$  to decimal using Horner's Scheme.

- Tables
- Direct methods
- Repeated division by base

Can be used for small numbers

Decimal	Dual	Octal	Hexadecimal
0	0	0	0
1	1	1	1
2	10	2	2
3	11	3	3
4	100	4	4
5	101	5	5
6	110	6	6
7	111	7	7
8	1000	10	8
9	1001	11	9
10	1010	12	A
11	1011	13	B
12	1100	14	C
13	1101	15	D
14	1110	16	E
15	1111	17	F

## Conversion from **Decimal to Dual**

- Divide the decimal number to be converted by the **greatest power of 2** less than or equal to that decimal number; write down a 1 for the leftmost (highest) binary digit
- Divide the result by the **next smaller power of 2** (result – 0 or 1 – indicates the next binary digit)
- Proceed until division by  $2^0 = 1$  terminates the procedure

This is basically a decomposition of the decimal into a sum of powers of 2

# Direct Methods: Decomposition Example

Convert  $116_{10}$  to binary

116	:	64	=	1
-64				
52	:	32	=	1
-32				
20	:	16	=	1
-16				
4	:	8	=	0
-0				
4	:	4	=	1
-4				
0	:	2	=	0
-0				
0	:	1	=	0

→  $116_{10} = 111\ 0100_2$

## Conversions between **Dual system and Octal system**

- To convert a number represented in the dual system to the octal system, so-called **dual triads** (groups of three) are formed starting from the right.

Dual

110 111 001 110 010

Octal

6 7 1 6 2

- When converting an octal number into its dual representation, we use the opposite way

Octal

3 2 1 5

Dual

011 010 001 101



## Conversions between **Dual system and Hexadecimal system**

- To convert a number represented in the dual system to the hexadecimal system, so-called **dual tetrades** (groups of four) are formed starting from the right.

Dual	1 0111 0101 1101	
Hex	1 7 5 D	Triad: 1 011 101 011 101 Octal: 1 3 5 3 5

- ... and for Hex  $\rightarrow$  Dual:

Hex	A D A	
Dual	1010 1101 1010	Triad: 101 011 011 010 Octal: 5 3 3 2

- Convert the number  $2E4_{16}$  to dual as well as to decimal
- Convert the number  $753_8$  to dual as well as to hexadecimal
- Convert the number  $1101011111111010_2$  to octal as well as to hexadecimal

## Conversion from decimal system to any base

- Continued division of a decimal number by the desired base
- returns the new digits as a division remainder

Algorithm to convert a decimal number  $x$  to a numeral system with base  $B$ :

1.  $x : B = y$  remainder  $z$
2. Make  $y$  the new  $x$ 
  - if this  $x$  is not equal to 0, proceed with step 1,
  - otherwise proceed with step 3
3. The resulting remainders  $z$  written read bottom to top = left to right are the converted number

# Repeated Division by Base – Example



- $43_{10} = ?_2$

x	B	y	z
43	: 2	= 21	Rem. 1
21	: 2	= 10	Rem. 1
10	: 2	= 5	Rem. 0
5	: 2	= 2	Rem. 1
2	: 2	= 1	Rem. 0
1	: 2	= 0	Rem. 1



The remainders z written from bottom to top provide the dual number you are looking for.

- Result:  $101011_2$

Convert the number  $10172_{10}$  using repeated division by base to the dual system as well as the hexadecimal system

$$\begin{aligned} &10172_{10} \\ &= \boxed{\phantom{00000000}}_2 \\ &= \boxed{\phantom{00}}_{16} \end{aligned}$$

- For fractions, a point separates the integer part of a number from the fractional part.
- A fractional number  $n$  is represented by the following sum:
$$n = \sum_{i=-M}^{N-1} b_i B^i$$
  - $B$  = Base of the numeral system ( $B \in \mathbb{N}, B \geq 2$ ),
  - $b_i$  = digits ( $b_i \in \mathbb{N}_0, 0 \leq b_i < B$ ),
  - $N$  = number of digits to the left of the point
  - $M$  = number of digits to the right of the point

$$17.05_{10} = 1 \cdot 10^1 + 7 \cdot 10^0 + 0 \cdot 10^{-1} + 5 \cdot 10^{-2}$$

$$3758.0_{10} = 3 \cdot 10^3 + 7 \cdot 10^2 + 5 \cdot 10^1 + 8 \cdot 10^0$$

$$9.702_{10} = 9 \cdot 10^0 + 7 \cdot 10^{-1} + 0 \cdot 10^{-2} + 2 \cdot 10^{-3}$$

$$0.503_{10} = 0 \cdot 10^0 + 5 \cdot 10^{-1} + 0 \cdot 10^{-2} + 3 \cdot 10^{-3}$$

- A pure fraction  $n$  ( $n < 1$ ) can be represented using **Horner's Scheme** as

$$n = \sum_{i=-M}^{-1} b_i B^i = \frac{1}{B} \left( b_{-1} + \frac{1}{B} \left( b_{-2} + \frac{1}{B} \left( b_{-3} + \cdots + \frac{1}{B} \left( b_{-M+1} + \frac{1}{B} b_{-M} \right) \right) \right) \right)$$

- Examples

$$0.193_{10} = \frac{1}{10} \cdot \left( 1 + \frac{1}{10} \cdot \left( 9 + \frac{1}{10} \cdot 3 \right) \right)$$

$$0.011_2 = \frac{1}{2} \cdot \left( 0 + \frac{1}{2} \cdot \left( 1 + \frac{1}{2} \cdot 1 \right) \right) = 0.375_{10}$$



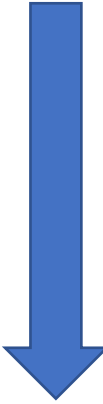
Algorithm for converting the fractional part of a decimal number to a system with base B:

1.  $x \cdot B = y$  overflow  $z$  ( $z$  = integer part of  $y$ , i.e., left of decimal point)
2. Make fractional part of  $y$  the new  $x$ 
  - if this new  $x$  is not equal to 0, and more fractional digits are required, go to step 1,
  - otherwise proceed to step 3
3. Write down the determined overflow-digits from top to bottom on the right side of “0.” to get the corresponding converted number

# Converting Pure Fractions from Decimal – Example

- $0.34375_{10} = ?_2$

<u>x</u>	<u>B</u>	<u>y</u>	<u>z</u>	
0.34375	· 2 =	0.6875	overflow	0
0.6875	· 2 =	1.375	overflow	1
0.375	· 2 =	0.75	overflow	0
0.75	· 2 =	1.5	overflow	1
0.5	· 2 =	1.0	overflow	1
0.0	· 2 =	0.0	overflow	0



The overflows  $z$  written from top to bottom after 0. provide the dual number you are looking for.

- Result:  $0.01011_2$

Some fractional numbers that can be represented exactly in the decimal system cannot be represented exactly as a dual number

- Typical example  
 $0.1_{10} = 0.0\ 0011\ 0011\ \dots_2$
- Periodic sequence of digits, repeating bit pattern 0011
- This leads to a loss of accuracy in the computer

x	B	y	z
0.1	· 2	= 0.2	overf. 0
0.2	· 2	= 0.4	overf. 0
0.4	· 2	= 0.8	overf. 0
0.8	· 2	= 1.6	overf. 1
0.6	· 2	= 1.2	overf. 1
0.2	· 2	= 0.4	overf. 0
0.4	· 2	= 0.8	overf. 0
0.8	· 2	= 1.6	overf. 1
0.6	· 2	= 1.2	overf. 1

- The other way around: **All** pure fractions that can be represented exactly in the dual system can also be represented exactly as a decimal number
- General: Any rational number  $p/q$  with  $\gcd(p, q) = 1$  can be represented exactly to base  $B$ , if all prime factors of  $q$  are also prime factors of  $B$
- Examples
  - $1/3_{10} = 0.33333..._{10}$       3 is not a prime factor of 10
  - $1/3_{10} = 0.010101..._2$       3 is not a prime factor of 2
  - $1/3_{10} = 0.1_3$       3 is a prime factor of 3
  - $1/10_{10} = 0.1_{10}$       2 and 5 are prime factors of 10
  - $1/10_{10} = 0.000110011..._2$       5 is not a prime factor of 2

Convert the following fractional numbers

- $0.375_{10} = \text{dual?}$
- $0.25_{10} = \text{in the base 5 system?}$
- $0.19_{10} = \text{hexadecimal?}$

# Converting Non-pure Fractional Numbers

- Split the number
  - into its integer part
  - and its fractional part,  
and convert separately.

• Example  $12.25_{10} = 1100.01_2$

integer part

12	:	2	=	6	Rem.	0
6	:	2	=	3	Rem.	0
3	:	2	=	1	Rem.	1
1	:	2	=	0	Rem.	1

$12_{10} = 1100_2$

fractional part

0.25	·	2	=	0.5	overf.	0
0.5	·	2	=	1.0	overf.	1
0.0	·	2	=	0.0	overf.	0

$0.25_{10} = 0.01_2$

Convert  **$39.6875_{10}$**  to binary as well as the hexadecimal system.