Exercise 6: functions: limits and continuity

Exercise 17
We consider a connection of n identical voltage sources in series (e.g., electrical batteries) in a circuit with one consumer.
Each of the voltage sources has an interior ohmic resistance R_i and yields a source voltage U_q. Hence the total voltage is $U_0 = nU_q$.
The consumer has an ohmic resistance R_a.

Compute the resulting current $I(n)$ in the circuit as function of the number of voltage sources. Plot $I(n)$. What is the limit $I(n)$ as n tends to infinity? Remark: The latter is the so-called short-circuit current I_{sc}.

Note that according to the Kirchhoff laws, the total ohmic resistance is

$$R_g = nR_i + R_a$$

and by Ohm's law

$$I = \frac{U_0}{R_g}.$$

Solution for exercise 17
We plug in:

$$\frac{U_0}{R_g} = \frac{nU_q}{nR_i + R_a} = \frac{U_q}{R_i + R_a/n} =: I(n).$$

Note for the plot that n is a discrete variable and $I(n)$ is a discrete function that approaches I_{sc}.

The limit yields

$$I_{sc} = \lim_{n\to\infty} I(n) = \lim_{n\to\infty} I(n) \frac{U_q}{R_i + R_a/n} = \frac{U_q}{R_i + 0} = \frac{U_q}{R_i}.$$

Exercise 18
Compute the limits

a) $\lim_{x\to 2} \frac{x^3 - x^2 - 2x}{x^3 + x^2 - 7x + 2}$

b) $\lim_{x\to 1} \frac{\sqrt{x} - 1}{x - 1}$

c) $\lim_{x\downarrow 1} \frac{x - 3}{x^2 + x - 2}$

Solution for exercise 18

a) By a polynomial division we find

$$(x^3 - x^2 - 2x) : (x-2) = x^2 + x$$

$$(x^3 + x^2 - 7x + 2) : (x-2) = x^2 + 3x - 1$$

$$\lim_{x\to 2} \frac{x^3 - x^2 - 2x}{x^3 + x^2 - 7x + 2} = \lim_{x\to 2} \frac{x^2 + x}{x^2 + 3x - 1} = \frac{6}{9} = \frac{2}{3}$$

b)

$$\frac{\sqrt{x}-1}{x-1} = \frac{\sqrt{x}-1}{(\sqrt{x}-1)(\sqrt{x}+1)} = \frac{1}{\sqrt{x}+1}$$

$$\lim_{x\to 1} \frac{\sqrt{x}-1}{x-1} = \lim_{x\to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$$

c)

$$x^2 + x - 2 = (x+2)(x-1) > 0 \quad \Leftrightarrow \quad x \in (-\infty, -2) \cup (1, \infty)$$

For $x \to 1$ the denominator goes to 0, thus the fraction gets singular.

Since $\lim_{x\downarrow 1} x - 3 = -2$, we find

$$\lim_{x\downarrow 1} \frac{x-3}{x^2 + x - 2} = -\infty.$$

Exercise 19

Let

$$\begin{aligned} f &: [0,1] \to \mathbb{R}, x \mapsto f(x) := x^2 - 2x + 1, \\ g &: [0,1] \to \mathbb{R}, x \mapsto g(x) := -x + 1. \end{aligned}$$

a) Justify that f and g are continuous.
Show that f and g attain its maximum and minimum.
Determine the images of f and g.

b) Compute $h_1(x) = f(g(x))$ and $h_2(x) = g(f(x))$.

c) Show that h_1 and h_2 are strictly monotone increasing.

Solution for exercise 19

a) f and g are polynomials and, thus, continuous.

According to the Weierstrass theorem the continuous functions f and g take its maximum and minimum on the compact interval $[0,1]$.

f is a part of a parabola, since $x^2 - 2x + 1 = (x-1)^2$, and attains its maximum at the boundary $x = 1$, $f(1) = 1$, and its minimum in the vertex $x = 0$, $f(0) = 0$.

g is a straight line and attains its maximum in $x = 0$, $g(0) = 1$, and its minimum in $x = 1$, $g(1) = 0$. Show that f and g take its maximum and minimum.

The image of an interval under a continuous function is again an interval: $f(0,1) = (0,1)$ and $g(0,1) = (0,1)$.

b)

$$h_1(x) = f(g(x)) = (g(x)-1)^2 = (-x+1-1)^2 = x^2$$

$$h_2(x) = g(f(x)) = -f(x)+1 = -(x-1)^2+1 = -x^2+2x$$

c)

$$x_1 < x_2 \overset{x_1,x_2 \geq 0}{\Longrightarrow} h_1(x_1) = x_1^2 < x_2^2 = h_1(x_2) \Rightarrow h_1 \text{ strictly monotone increasing}$$

$$x_1 < x_2 \overset{0 \leq x_1,x_2 \leq 1}{\Longrightarrow} h_2(x_2)-h_2(x_1) = 2x_2-x_2^2-(2x_1-x_1^2) = 2(x_2-x_1)+x_1^2-x_2^2 = (x_2-x_1)(2-(x_1+x_2))$$
$$\overset{x_1+x_2<2}{\Rightarrow} h_2 \text{ strictly monotone increasing}$$