Mapping or function

Definition (Mapping / function)

Let A and B be two sets.
A **mapping** or **function** f from A onto b is represented by

$$f : A \to B, x \mapsto f(x).$$

It is a rule, that assigns to every $x \in A$ an element $f(x) \in B$.

A is called **domain (of definition).**
Elements of A are called **arguments** of the mapping f.

$f(x)$ is called the **image of** x **under** f or
the **function value of** f **at the point** x.

Image and preimage

For a mapping f with domain of definition A, we introduce for an arbitrary subset $C \subseteq A$ the set

$$f(C) := \{f(x) \,|\, x \in C\}.$$

$f(C)$ is called the **image of C under f.**

Range and preimage

The image $f(A)$ of the domain of definition A of f is called the **range of values** of f.

In general, for $f(A) \subseteq B$, the set B is called **target area** or **co-domain.**

For $D \subseteq B$ we define the **preimage of D (under f)** as

$$f^{-1}(D) := \{x \in A \,|\, f(x) \in D\}.$$

Injectivity

A function $f : A \to B$ is called **injective**,
iff for any $x_1, x_2 \in A$ there holds

$$f(x_1) = f(x_2) \quad \Leftrightarrow \quad x_1 = x_2$$

The **injectivity** of a function depends
on the mapping rule and
on the domain of definition.

Surjectivity

A function $f : A \to B$ is called **surjective**,
iff for any $y \in B$ there exists a $x \in A$ such that $f(x) = y$.

Equivalently, iff $f(A) = B$,
$f(A)$ being the range of values, B the target area.

Bijectivity

Functions that are injective and surjective are called **bijective**.

Thus, for a bijective function

$$f : A \to B, \quad x \mapsto f(x)$$

there exists for any $y \in B$ exactly one $x \in A$ with $f(x) = y$.

Inverse function

Hence there exists a mapping

$$f^{-1} : B \to A, \quad y \mapsto f^{-1}(y)$$

with $f^{-1}(f(x)) = x$ for any $x \in A$ and
with $f(f^{-1}(y)) = y$ for any $y \in B$.

f^{-1} is called the **inverse mapping** or **inverse function** or **inverse** of f.

Identity and identical

For any subset A we may define the function

$$Id_A : A \to A, \quad x \mapsto x,$$

called the **identity on** A.

Two mappings

$$f : A \to B, \quad x \mapsto f(x),$$
$$g : C \to D, \quad z \mapsto g(z)$$

are called **identical**, iff

- $A = C$,
- $B = D$, and
- $f(x) = g(x)$ for all $x \in A$.

Then we write $f = g$.

Restriction

Let $f : A \to B, x \mapsto f(x)$.
By restricting the domain A to a subset $C \subseteq A$, we obtain the **restriction** of f to C:

$$f|_C : C \to B, \quad x \mapsto f(x).$$

Numbers

Special (number) sets:

$\mathbb{N} = \{1, 2, 3, \ldots\}$	natural numbers
$\mathbb{N}_0 = \{0, 1, 2, 3, \ldots\}$	natural numbers incl. zero
$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, 3 \ldots\}$	integers (whole numbers ?)
$\mathbb{Q} = \left\{ \frac{p}{q} \,\middle|\, p, q \in \mathbb{Z}, q \neq 0 \right\}$	rational numbers
$\mathbb{R} \approx$ "all" numbers in 1d	real numbers
$\mathbb{C} \approx \mathbb{R}^2$ with "special structure"	complex numbers

Number line

Every point of the number line corresponds to a real number:

Order of the real numbers

The real numbers are ordered completely by the order relation $\geq$:
For all $x, y \in \mathbb{R}$ there holds:

$$x \geq y :\Leftrightarrow (x - y) \geq 0$$

Alternatively, for all $x, y \in \mathbb{R}$ there holds:

$$x > y :\Leftrightarrow (x - y) > 0$$

The comparability of real numbers is called the **order** of the real numbers.

Numbers - overview

There holds:

$$\mathbb{N} \subset \mathbb{N}_0 \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Field axioms 1

Definition (Field)

A non-empty set K equipped with 2 operations that are closed **(F0)**

$$\oplus : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$
$$\otimes : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$$

is called a field, supposed the following conditions **(F1)** – **(F5)** hold:

(F1) For all $a, b \in \mathbb{F}$ we have the **commutative laws**

$$a \oplus b = b \oplus a, \quad a \otimes b = b \otimes a.$$

(F2) For all $a, b, c \in \mathbb{F}$ we have the **associative laws**

$$(a \oplus b) \oplus c = a \oplus (b \oplus c), \quad (a \otimes b) \otimes c = a \otimes (b \otimes c).$$

...

Field axioms 2

Definition (Field (continued))

(F3) For all $a \in \mathbb{F}$ there exists a **neutral element w.r.t.** $\oplus$, $e_\oplus \in \mathbb{F}$, such that
$$a \oplus e_\oplus = a$$
and for all $a \in \mathbb{F} \setminus e_\oplus$ there exists a **neutral element w.r.t.** $\otimes$, $e_\otimes \in \mathbb{F}$, such that
$$a \otimes e_\otimes = a.$$

(F4) For all $a \in \mathbb{F}$ there exists an **inverse element w.r.t.** $\oplus$, $a_\oplus^{-1} \in \mathbb{F}$, such that
$$a \oplus a_\oplus^{-1} = e_\oplus$$
and for all $a \in \mathbb{F} \setminus e_\oplus$ there exists an **inverse element w.r.t.** $\otimes$, $a_\otimes^{-1} \in \mathbb{F}$, such that
$$a \otimes a_\otimes^{-1} = e_\otimes.$$

(F5) For all $a, b, c \in \mathbb{F}$ there holds the **distributive law**
$$a \otimes (b \oplus c) = (a \otimes b) \oplus (a \otimes c).$$

Axioms of ordered fields

Definition (Ordered fields)

If $\mathbb{F}$ is a field and
if some elements of $\mathbb{F}$ are designated as positive,
then $\mathbb{F}$ is called ordered if:

(A1) For all $a \in \mathbb{F}$ there holds exactly one of the following relations

- a is positive,
- $a = e_{\oplus}$ (is neutral), or
- $a_{\oplus}^{-1}$ is positive.

(A2) For all $a, b \in \mathbb{F}$ positive this implies $a \oplus b$ is positive.

(A3) For all $a, b \in \mathbb{F}$ positive this implies $a \otimes b$ is positive.

Axiom of Archimedes for ordered fields

Definition (Archimidean ordered fields)

Let $\mathbb{F}$ be an ordered field.
$\mathbb{F}$ is called an Archimidean ordered field, if in addition holds:

(AR) For any 2 positive $a, b \in \mathbb{F}$ there exists an $n \in \mathbb{N}$ such that:

$$\underbrace{a \oplus a \oplus \ldots \oplus a}_{n \text{ times}} \oplus b_{\oplus}^{-1} \text{ is positive.}$$

Remarks to the field axioms

- Both neutral elements, $e_\oplus$ and $e_\otimes$, are unique.
- In (F4) the neutral element w.r.t. $\oplus$ has to be excluded.
  There exists no inverse element for $e_\oplus$ w.r.t. $\otimes$.
- If we set $e_\oplus = e_\otimes$, then there would hold for all $a \in \mathbb{F}$

$$a \otimes e_\oplus = e_\oplus = e_\otimes = a \otimes e_\otimes = a,$$

  i.e. $a = e_\otimes$ and thus $\mathbb{F} = \{e_\otimes\}$.
- Both inverse elements, $a_\oplus^{-1}$ and $a_\otimes^{-1}$, are uniquely defined.

Intervals

By means of the relations $\leq$ or $\geq$, resp., we may introduce

intervals of real numbers:

- closed interval with $a, b \in \mathbb{R}$, $a \leq b$:

$$[a, b] := \{x \in \mathbb{R} \,|\, a \leq x \leq b\}$$

- open interval with $a, b \in \mathbb{R}$, $a < b$:

$$(a, b) := \{x \in \mathbb{R} \,|\, a < x < b\}$$

- semi-open interval with $a, b \in \mathbb{R}$, $a < b$:

$$(a, b] := \{x \in \mathbb{R} \,|\, a < x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} \,|\, a \leq x < b\}$$

Infinite intervals

By introducing the symbols

$$-\infty \text{ (minus infinity) and}$$
$$\infty \text{ (plus infinity)}$$

we may define for $a \in \mathbb{R}$:

$$\begin{aligned}
(-\infty, a] &:= \{x \in \mathbb{R} \mid x \leq a\} \\
(-\infty, a) &:= \{x \in \mathbb{R} \mid x < a\} \\
[a, \infty) &:= \{x \in \mathbb{R} \mid a \leq x\} \\
(a, \infty) &:= \{x \in \mathbb{R} \mid a < x\} \\
(-\infty, \infty) &:= \mathbb{R}
\end{aligned}$$

$(-\infty, a]$ and $[a, \infty)$ are called closed intervals
$(-\infty, a)$ and (a, ∞) are called open intervals,
where $a \in \mathbb{R}$.

Upper and lower bounds

Definition (Upper / lower bounds)

A set S of real numbers is **bounded from above**, if there exists a real number b, such that $S \subseteq (-\infty, b]$.

b is called an **upper bound** of S.

A set S of real numbers is **bounded from below**, if there exists a real number a, such that $S \subseteq [a, \infty)$.

a is called a **lower bound** of S.

Smallest upper/largest lower bound

Definition (Supremum and infimum)

If a set $S \subset \mathbb{R}$ is bounded from above,
then the least upper bound is called the **supremum** of S.

We write $\sup(S)$ or $\sup\{S\}$.

If a set $S \subset \mathbb{R}$ is bounded from below,
then the greatest lower bound is called the **infimum** of S.

We write $\inf(S)$ or $\inf\{S\}$.