Properties of a^x

Proof:

① $(a^x)^y = (\exp_a(x))^y = \exp_{a^x}(y)$
$= \exp(y\ln(a^x)) = \exp((y\,x)\ln(a))$
$= \exp_a(xy) = a^{xy}$

② $a^0 = \exp(0\cdot\ln(a)) = \exp(0) = 1$

③ $a^x b^x = \exp(x\ln(a))\cdot\exp(x\ln(b))$

$a,b\in\mathbb{R}^+$ $\overset{\text{funct. eq. for exp}}{=} \exp(x\ln(a)+x\ln(b))$
$= \exp(x(\ln(a)+\ln(b)))$
$\overset{\text{funct. eq. for ln}}{=} \exp(x\ln(ab)) = (ab)^x$

④ $\left(\frac{1}{a}\right)^x = \exp(x\ln(\tfrac{1}{a})) = \exp(x(\underbrace{\ln(1)}_{=0}-\ln(a)))$
$= \exp(\underbrace{x\cdot(-1)}_{=-x}\cdot\ln(a)) = \exp_a(-x) = a^{-x}$

$\square$

To show: $\log_a(x) \overset{!}{=} \frac{\ln(x)}{\ln(a)}$, $a\neq 1$ (*)

$x = \exp_a(\log_a(x)) \overset{\text{def.}}{=} \exp(\log_a(x)\cdot\ln(a))$ $\mid \ln(\ldots)$

$\Leftrightarrow \ln(x) = \log_a(x)\cdot\ln(a)$ $\quad\mid : \ln(a)\neq 0$ (since $a\neq 1$)

$\Leftrightarrow \frac{\ln(x)}{\ln(a)} = \log_a(x)$ $\square$

Corollary: $\log_a(xy) \overset{(*)}{=} \frac{\ln(xy)}{\ln(a)} \overset{\text{funct. eq. ln}}{=} \frac{\ln(x)}{\ln(a)} + \frac{\ln(y)}{\ln(a)} \overset{(*)}{=} \log_a(x) + \log_a(y)$