

Revision: polynomial division with remainder I

Analysis 1

S.-J. Kimmerle

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Summary - outlook and review

Consider a rational function

$$r : A \rightarrow \mathbb{R}, x \mapsto \frac{p(x)}{q(x)} := \frac{\sum_{i=0}^n a_i x^i}{\sum_{i=0}^m b_i x^i}, \quad a_n \neq 0, b_m \neq 0$$

where $A := \{x \in \mathbb{R} \mid \sum_{i=0}^m b_i x^i \neq 0\}$.

If $n \geq m$, then we set

$$p_1 : \mathbb{R} \rightarrow \mathbb{R}, x \mapsto p(x) - \frac{a_n}{b_m} x^{n-m} \cdot q(x)$$

and obtain the following representation

$$r(x) = \frac{p(x)}{q(x)} = \frac{a_n}{b_m} x^{n-m} + \frac{p_1(x)}{q(x)} \quad \text{for all } x \in A$$

where p_1 is either the zero function or a polynomial of degree smaller than n .



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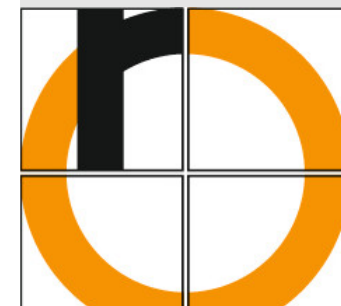
Summary - outlook and review

This procedure may be iterated for k steps that produce a polynomial p_k until the degree of p_k is less than n .

We end up with

$$\frac{p(x)}{q(x)} = g(x) + \frac{p_k(x)}{q(x)},$$

g a polynomial.



Before integrating a rational function $r : A \rightarrow \mathbb{R}$ as defined above,
we carry out a polynomial division:

$$\int r(x) dx = \int g(x) dx + \int \frac{p_k(x)}{q(x)} dx$$

We know how to integrate the polynomial g .
For the remainder $\frac{p_k(x)}{q(x)}$, we consider the partial fraction expansion, i.e., the rational function is decomposed into a sum of fractions (yielding only a short list of cases with explicit formulas).

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Partial fraction expansion II

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Let again $r : A \rightarrow \mathbb{R}, x \mapsto \frac{p(x)}{q(x)}$ as defined above,
but w.l.o.g. let $\text{degree } p < \text{degree } q$.
Moreover, let

$$q(x) = x(x - b_1)^{k_1} \cdot (x - b_2)^{k_2} \cdot \dots \cdot (x - b_r)^{k_r} \cdot q_1(x)^{l_1} \cdot \dots \cdot q_s(x)^{l_s}$$

with pairwise distinct zeros b_i of multiplicity k_i and
pairwise distinct quadratic polynomials q_i that do not have
zeros in \mathbb{R} .

Then there exists real numbers $A_1^{[1]}, \dots, A_1^{[k_1]}, \dots, A_r^{[1]}, \dots, A_r^{[k_r]},$
 $B_1^{[1]}, \dots, B_1^{[l_1]}, \dots, B_s^{[1]}, \dots, B_s^{[l_s]}, C_1^{[1]}, \dots, C_1^{[l_1]}, \dots, C_s^{[1]}, \dots, C_s^{[l_s]}$ s.t.

$$\frac{p(x)}{q(x)} = \sum_{i=1}^r \sum_{j=1}^{k_i} \frac{A_i^{[j]}}{(x - b_j)^j} + \sum_{i=1}^s \sum_{j=1}^{l_i} \frac{B_i^{[j]} + C_i^{[j]}x}{(q_i(x))^j} \quad \text{for all } x \in A.$$



Thus we only have to figure out how to integrate functions of the type

$$\frac{\zeta}{(x - x_0)^k}, \quad \frac{\xi + \mu x}{(q_i(x))^{\tilde{k}}}, \quad k, \tilde{k} \in \mathbb{N} :$$

Let $[a, b] \subset A$:

1)

$$\int_a^b \frac{1}{x - x_0} dx = [\ln(|x - x_0|)]_a^b, \quad x_0 \notin [a, b]$$

2)

$$\int_a^b \frac{1}{(x - x_0)^k} dx = \frac{-1}{k-1} \left[\frac{1}{(x - x_0)^{k-1}} \right]_a^b, \quad k > 1, x_0 \notin [a, b]$$

If $4\beta - \alpha^2 > 0$, then $q(x) = x^2 + \alpha x + \beta$ has no real zeros.

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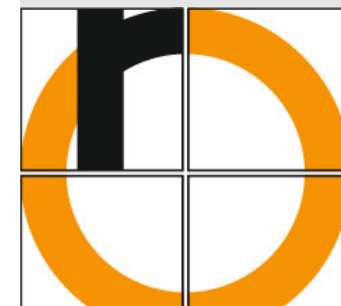
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Integration of rational functions II

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Let $4\beta - \alpha^2 > 0$ and $k > 1$.

3)

$$\int_a^b \frac{1}{x^2 + \alpha x + \beta} dx = \left[\frac{2}{\sqrt{4\beta - \alpha^2}} \arctan \left(\frac{2x + \alpha}{\sqrt{4\beta - \alpha^2}} \right) \right]_a^b$$

4)

$$\int_a^b \frac{ax + b}{x^2 + \alpha x + \beta} dx = \left[\frac{a}{2} \ln(|x^2 + \alpha x + \beta|) \right]_a^b + \left(b - \frac{a\alpha}{2} \right) \int_a^b \frac{1}{x^2 + \alpha x + \beta} dx$$

5)

$$\begin{aligned} \int_a^b \frac{1}{(x^2 + \alpha x + \beta)^k} dx &= \left[\frac{2x + \alpha}{(k-1)(4\beta - \alpha^2)(x^2 + \alpha x + \beta)^{k-1}} \right]_a^b \\ &\quad + \frac{2(2k-3)}{(k-1)(4\beta - \alpha^2)} \int_a^b \frac{1}{(x^2 + \alpha x + \beta)^{k-1}} dx \end{aligned}$$

6)

$$\begin{aligned} \int_a^b \frac{ax + b}{(x^2 + \alpha x + \beta)^k} dx &= \left[\frac{-a}{2(k-1)(x^2 + \alpha x + \beta)^{k-1}} \right]_a^b \\ &\quad + \left(b - \frac{a\alpha}{2} \right) \int_a^b \frac{1}{(x^2 + \alpha x + \beta)^k} dx \end{aligned}$$

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In many applications we find integrals over infinite intervals. This motivates to consider **improper integrals** in the following.

Definition (Improper integral (infinite interval))

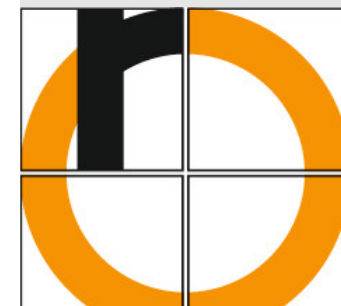
Let $f : [a, \infty) \rightarrow \mathbb{R}$ a function, that is on any interval $[a, R]$, $a < R < \infty$, integrable, then, assuming that the limit exists, we call

$$\int_a^\infty f(x) dx := \lim_{R \rightarrow \infty} \int_a^R f(x) dx$$

the **improper integral** of f on $[a, \infty)$.

Analogously we define

$$\int_{-\infty}^a f(x) dx := \lim_{R \rightarrow \infty} \int_{-R}^a f(x) dx.$$



Definition (Improper integral (singularity))

Let $f : [a, b) \rightarrow \mathbb{R}$ a function,
that is on any interval $[a, b - \varepsilon]$, $0 < \varepsilon < b - a$, integrable
(but not on $[a, b]$),

then, assuming that the limit exists,

we call

$$\int_a^b f(x) dx := \lim_{\varepsilon \downarrow 0} \int_a^{b-\varepsilon} f(x) dx$$

the **improper integral** of f on $[a, b]$.

Analogously we define

$$\int_a^b f(x) dx := \lim_{\varepsilon \downarrow 0} \int_{a+\varepsilon}^b f(x) dx.$$

Moreover, we may define improper integrals for infinity or singularities, resp., at both ends.



Improper integrals may allow to decide on the convergence of series:

Theorem (Integral test)

Let $f : [1, \infty) \rightarrow \mathbb{R}_0^+$ a monotonically increasing function.

*The series $\sum_{n=1}^{\infty} f(n)$ converges,
iff the improper integral $\int_1^{\infty} f(x) dx$ exists.*

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An improper integral important for stochastics is the integral over the Gauss bell curve, i.e. the probability density function

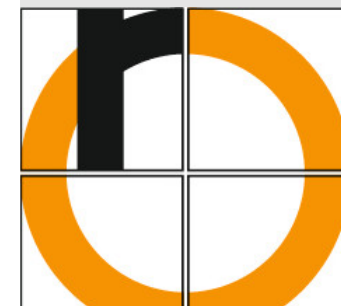
$$f(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} \exp\left(-\frac{(x - \mu)^2}{\sigma^2}\right), \quad \mu \in \mathbb{R}, \sigma > 0.$$

For a probability we have to check (see Analysis 2)

$$\int_{-\infty}^{\infty} f(x) dx = 1.$$

However, for $f(x)$ there exists no closed form for a primitive. We define the **error function** (for the standard normal distribution) as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{-\infty}^x \exp\left(-\frac{x^2}{2}\right) dx.$$



Definition (Gamma function)

The function

$$\Gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+, x \mapsto \int_0^\infty t^{x-1} \exp(-t) dt$$

is called **gamma function**.

The gamma function interpolates the factorial:

$$\Gamma(x+1) = x \cdot \Gamma(x) \quad \text{for all } x \in \mathbb{R}^+,$$

$$\Gamma(1) = 1,$$

$$\text{i.e. } \Gamma(n+1) = n! \quad \text{for all } n \in \mathbb{N}_0.$$

Improper integrals of this type are important for the probability of failure (Weibull distribution).

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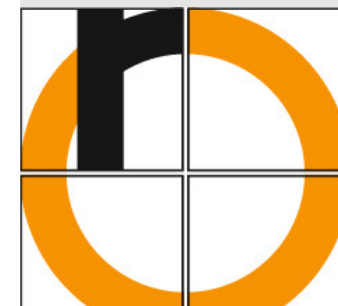
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- Concepts: **integral**, **integrand**, **integration bounds**, **integration variable**
 - Approximation by a Riemann sum
 - Concepts: **primitive**/indefinite integral, **definite integral**
 - Fundamental theorem of differential and integral calculus
- Integration is an “inverse operation” of differentiation
- Practical computation of integrals
 - Improper integrals

Further topics:

- Volumes of rotational bodies
- Path integrals, surface integrals, volume integrals \rightsquigarrow Analysis 2
- Gauss divergence theorem, Stokes theorem \rightsquigarrow Analysis 2 or later
- ...

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