Exercise 9.1. Let $A \in \mathbb{R}^{n\times n}$. In the typical decomposition

$$A = D - L - U,$$

assume that $L = 0$, i.e. A is an upper triangular matrix. Show that in this case

$$\mathcal{L} = \mathcal{J}.$$

That is, the Jacobi method is identical to the Gauss-Seidel method.

Suggested Solution. We have that

$$A = D - L - U \quad \text{with } L = 0$$

and therefore

$$\begin{aligned}\mathcal{J} &= D^{-1}(L+U) \overset{L=0}{=} D^{-1}U\\ \mathcal{L} &= (D-L)^{-1}U \overset{L=0}{=} D^{-1}U.\end{aligned}$$

Exercise 9.2. There are positive definite matrices for which the Jacobi method does not converge. Consider for $\alpha \in \mathbb{R}$ the matrix

$$A = \begin{pmatrix} 1 & \alpha & \alpha\\ \alpha & 1 & \alpha\\ \alpha & \alpha & 1\end{pmatrix}.$$

(a) Determine the Jacobi operator $\mathcal{J}$ corresponding to the matrix A.

Suggested solution:

We have that $A = D - L - U$, where

$$D = E, \quad L = \begin{pmatrix} 0 & 0 & 0\\ -\alpha & 0 & 0\\ -\alpha & -\alpha & 0\end{pmatrix}, \quad U = \begin{pmatrix} 0 & -\alpha & -\alpha\\ 0 & 0 & -\alpha\\ 0 & 0 & 0\end{pmatrix}.$$

Thus

$$\mathcal{J} = D^{-1}(L+U) = \begin{pmatrix} 0 & -\alpha & -\alpha\\ -\alpha & 0 & -\alpha\\ -\alpha & -\alpha & 0\end{pmatrix}.$$

(b) Determine all $\alpha \in \mathbb{R}$ for which the Jacobi method converges.

Suggested solution:

Determine the eigenvalues of $\mathcal{J}$: (To avoid having to find zeros of a third degree polynomial, we first apply elementary row or column transformations to calculate the determinant. In this way, we can split off a linear factor of $\chi_{\mathcal{J}}$. Another possibility is offered by the statement that $\lambda = \alpha$ is an obvious eigenvalue of $\mathcal{J}$.) For the characteristic polynomial we obtain

$$\begin{aligned}-\chi_{\mathcal{J}}(\lambda) &\overset{1)}{=} \begin{vmatrix} \lambda & \alpha & \alpha\\ \alpha & \lambda & \alpha\\ \alpha & \alpha & \lambda\end{vmatrix} \overset{2)}{=} \begin{vmatrix} \lambda-\alpha & \alpha-\lambda & 0\\ \alpha & \lambda & \alpha\\ \alpha & \alpha & \lambda\end{vmatrix} \overset{3)}{=} \begin{vmatrix} \lambda-\alpha & 0 & 0\\ \alpha & \lambda+\alpha & \alpha\\ \alpha & 2\alpha & \lambda\end{vmatrix}\\ &= (\lambda-\alpha)\Big[(\lambda+\alpha)\lambda - 2\alpha^2\Big]\\ &= (\lambda-\alpha)^2(\lambda+2\alpha).\end{aligned}$$

Here, we used in

1) the multilinearity of determinant functions
2) substracting the 2^{nd} row from the 1^{st}
3) adding the 1^{st} column to the second

This means

$$\sigma(\mathcal{J}) = \{\alpha, -2\alpha\}$$

and

$$\rho(\mathcal{J}) < 1 \quad \Leftrightarrow \quad |-2\alpha| < 1 \wedge |\alpha| < 1 \quad \Leftrightarrow \quad |\alpha| < \frac{1}{2}.$$

Consequently, the Jacobi method converges exactly when $|\alpha| < \frac{1}{2}$ by Prop. 3.4.

(c) For which $\alpha \in \mathbb{R}$ is the matrix A positive definite, but the Jacobi method does not converge?

Suggested solution:

Since A is symmetric, we have that: A is pos. definite $\Leftrightarrow$ all eigenvalues are strictly positive. We need to check, for which α all eigenvalues are strictly positive. To compute the characteristic polynomial of A, we substitute λ by $1-\lambda$ in $^{1)}$ to obtain

$$\chi_A(\lambda) = 0 \;\Leftrightarrow\; -\chi_{\mathcal{J}}(1-\lambda) = 0 \;\Leftrightarrow\; (1-\lambda) \in \sigma(\mathcal{J}) \;\Leftrightarrow\; \sigma(A) = \{1-\alpha, 1+2\alpha\}.$$

Hence A is pos. definite exactly when $-\frac{1}{2} < \alpha < 1$. In consequence, we can conclude that A is positive definite and the Jacobi method does not converge in gerneral exactly when $\frac{1}{2} \leq \alpha < 1$.

Exercise 9.3. Consider the iteration method $x^{(k+1)} = \Phi(x^{(k)})$ with $\Phi(x) = Tx + c$, but this time with the additional assumption that T is nilpotent. That is, there exists an $N \in \mathbb{N}$ s.t.

$$T^N = 0.$$

Show that

(a) $\rho(T) = 0$. What does that mean for the convergence of the iteration method?

Suggested solution:

From Exercise 4.3 we know that $\sigma(T) = \{0\}$ and hence $\rho(T) = 0$. According to Prop. 3.4, the iteration method

$$x^{(k+1)} = \Phi(x^{(k)}) = Tx^{(k)} + c$$

therefore converges for all initial vectors $x^{(0)}$ to the unique fixed point $\hat{x}$ of Φ.

(b) The iteration method gives the exact solution for each starting vector $x^{(0)}$ after N iterations at the latest.

Suggested solution:

Let $\hat{x}$ be the exact solution, i.e. the fixed point of Φ. Then by definition of Φ and since $\hat{x}$ is a fixed point we can conclude

$$\begin{aligned}
\|x^{(N)} - \hat{x}\| &= \|T(x^{(N-1)} - \hat{x})\| \\
&= \|T^2(x^{(N-2)} - \hat{x})\| \\
&\;\;\vdots \qquad\qquad \vdots \\
&= \|\underbrace{T^N}_{=0}(x^{(0)} - \hat{x})\| = 0.
\end{aligned}$$

Therefore it is $x^{(N)} = \hat{x}$, i.e. the iteration method attains the solution after N iterations at the latest.

Exercise 9.4. Let the iteration procedure

$$x^{(k+1)} = \Phi(x^{(k)})$$

with any starting vector $x^{(0)} \in \mathbb{R}^n$ be given and let it be defined by the affine-linear mapping

$$\Phi(x) = Tx + c$$

for some matrix $T \in \mathbb{R}^{n\times n}$ and fixed $c \in \mathbb{R}^n$.

(a) Under which conditions is there always exactly one fixed point of Φ?

Suggested solution:

There is exactly one fixed point if and only if $\rho(T) < 1$ (cf. Prop. 3.4). Alternatively one can use Prop. 3.3. For this, one needs a compatible matrix norm $\|\cdot\|_*$ s.t. $\|T\|_* < 1$.

(b) Suppose you know according to the construction of your method that Φ has a fixed point $\hat{x}$. For $\rho(T) \geq 1$ and $\lambda_{\max} \in \mathbb{R}$ ($\lambda_{\max}$ = largest absolute eigenvalue of T), specify an initial guess $x^{(0)}$ for which the iteration method does not converge.

Suggested solution:

Let $\rho(T) \geq 1$ and therefore $|\lambda_{\max}| \geq 1$. Let further v be an eigenvector corresponding to $\lambda_{\max}$. Setting

$$x^{(0)} := v + \hat{x},$$

where $\hat{x}$ denotes a fixed point of Φ, we obtain

$$\begin{aligned}
\|x^{(n)} - \hat{x}\| &= \|T(x^{(n-1)} - \hat{x})\| \\
&= \|T^2(x^{(n-2)} - \hat{x})\| \\
&\vdots \qquad\qquad \vdots \\
&= \|T^n(\underbrace{x^{(0)} - \hat{x}}_{=v})\| \\
&= \|T^n v\| \\
&= \|\lambda_{\max}^n v\| \\
&= |\lambda_{\max}|^n \|v\| \to \begin{cases} \infty & \text{for } |\lambda_{\max}| > 1 \\ \|v\| & \text{for } |\lambda_{\max}| = 1 \end{cases} .
\end{aligned}$$

The error therefore does not tend to zero and thus the method does not converge for the above initial value.

The exercises can be discussed in the Tuesday lecture next week.