Exercise 10.1. Find the optimal polynomial p of the second degree, i.e.

$$p(x) = a_0 + a_1 x + a_2 x^2,$$

which minimizes the distance to the following points in the least squares sense:

x	-1	0	1	2
y	0	1	0	2

Sketch the points and the optimal p in a Cartesian coordinate system.

Exercise 10.2. For $n \in \mathbb{N}$, let the following arbitrary pairs of points of $\mathbb{R}^2$ be given:

x	x_1	x_2	$\cdots$	x_n
y	y_1	y_2	$\cdots$	y_n

Here, the x_i should not all be the same. Let the arithmetic mean values of the x_i and y_i be denoted by $\bar{x}$ and $\bar{y}$, i.e.

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \quad \text{and} \quad \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i.$$

Show that

$$\beta = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \quad \text{and} \quad \alpha = \bar{y} - \beta\bar{x}$$

are the optimal parameters of the regression line $y = \alpha + \beta x$ in the least squares sense.

Exercise 10.3. Consider the model function

$$f(x) = a_0 + a_1 x + a_2 e^{x} + a_3 e^{-x}.$$

This is to be optimally fitted to the data points

x	-2	-1	0	1	2
y	e^{-2}	e^{-1}	1	e^{-1}	e^{-2}

in least squares sense.

(a) Determine a matrix A and a vector y, s.t. this is equivalent to minimizing

$$\mathcal{E}(z) = \|Az - y\|_2^2$$

over $\mathbb{R}^2$.

(b) Calculate the optimal model parameters by solving the Gaussian normal equations with `Octave`.

(c) Graph the data points and the best model function (e.g. with `Octave`).

<u>Remark:</u> For part (c) you can use the `Octave` built-in function `plot` (see e.g. Ferreira - MATLAB Codes...)

Exercise 10.4. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and $b \in \mathbb{R}^n$. Consider the following mappings

$$f\colon \mathbb{R}^n \to \mathbb{R},\ \ x \mapsto \langle x, Ax\rangle,$$
$$g\colon \mathbb{R}^n \to \mathbb{R},\ \ x \mapsto \langle x, b\rangle.$$

Show that it then holds for the derivative of f or g:

$$\nabla f = 2Ax \quad \text{and} \quad \nabla g = b.$$

The exercises can be discussed in the Tuesday lecture next week.