Exercise 7.1 *(The Banach fixed point theorem (BFT))*. For the function

$$f : \mathbb{R} \to \mathbb{R},\ f(x) = e^{-x} - x$$

a zero in the interval $I = [\frac{1}{3}, 1]$ is to be calculated approximately. For this, approximate a fixed point of the function

$$\Phi : I \to \mathbb{R},\ \Phi(x) := e^{-x}.$$

(a) Show that Φ is a self map on I. (Can you do without a calculator here if you know that $e < 3$?)

(b) Show that Φ is contracting with contraction factor $q < 1$ (in other words: show that Φ is Lischitz continuous with Lipschitz constant $q < 1$).

(c) Using the BFT it follows from 1. and 2. that there exists exactly one fixed point $\hat{x} \in I$. Let $x_0 := \frac{1}{3}$ be the initial guess. (Clearly, you could choose any other $x_0 \in I$. But choosing x_0 as above allows you to directly compare your solution to the suggested solution..) Use the a priori estimates of the BFT to determine the maximum number $N_0 = N_0(\varepsilon, x_0) \in \mathbb{N}$ of iterations needed to determine $\hat{x}$ to an accuracy of (an unspecified) $\varepsilon > 0$. Determine N_0 for $\varepsilon := 10^{-5}$.

(d) Using `Octave`, calculate the first N_0 iterates of the iteration sequence (x_k) induced by the iteration rule

$$x_{k+1} = \Phi(x_k).$$

(e) Use the a posteriori estimate of the BFT to obtain an improved estimate for the error $|\hat{x} - x_{N_0}|$.

(f) Use your calculator to compute the best possible iterate (using the "ANS-technique").

Suggested Solution.

(a) Since $\Phi'(x) = -e^{-x} < 0$ on $\mathbb{R}$ one knows that Φ is decreasing on $\mathbb{R}$ and thus we can infer that

$$1 = \Phi(0) \geq \Phi(0.1) \geq \Phi(x) \geq \Phi(1) = \frac{1}{e} \geq \frac{1}{3},$$

that is

$$\Phi(x) \in [1/3, 1].$$

(b) Since for all $x \in I$ we have

$$|\Phi'(x)| = e^{-x} \leq e^{-\frac{1}{3}} =: q \approx 0.7165313106 < 1,$$

it follows that

$$\begin{aligned}
|\Phi(x) - \Phi(y)| &= |\Phi'(\xi)(x - y)| \\
&\leq |\Phi'(\xi)||x - y| \\
&\leq \max_{x \in I} |\Phi'(x)| \cdot |x - y| \\
&\leq q|x - y|,
\end{aligned}$$

where the mean value theorem that for some $\xi \in (1/3, 1)$ was employed.

(c) If we can choose $N_0 \in \mathbb{N}$ s.t. for all $k \geq N_0$

$$\frac{q^k}{1-q}|x_1 - x_0| < \varepsilon \tag{1}$$

it would then follow by the a priori estimate that

$$|x_k - \hat{x}| \leq \frac{q^k}{1-q}|x_1 - x_0| < \varepsilon.$$

But

$$\begin{aligned} (1) \quad \Leftrightarrow \quad q^k &< \frac{\varepsilon(1-q)}{|x_1 - x_0|} \\ \Leftrightarrow \quad k \ln q &< \ln\left(\frac{\varepsilon(1-q)}{|x_1 - x_0|}\right) \\ \overset{q<1}{\Leftrightarrow} \quad k &> \ln\left(\frac{\varepsilon(1-q)}{|x_1 - x_0|}\right) / \ln(q). \end{aligned}$$

Thus one can choose $N_0 \in \mathbb{N}$ to be the smallest natural number k with this property.

To calculate N_0 for the ε given, we need to calculate

$$x_1 = \Phi(x_0) = e^{-\frac{1}{3}} (= q).$$

Since

$$\ln\left(\frac{\varepsilon(1-q)}{|x_1 - x_0|}\right) / \ln(q) = 28.53537139599424$$

it follows $N_0 = 29$.

(d) Using

```
% sheet 7 ex.1
format long
x0=1/3;
x1=exp(-1/3);
q=x1;
eps=10e-5;
% ceil(x) is the smallest natural number k>=x
% log is the natural logarithm
N0_aux=log(eps*(1-q)/abs(x1-x0))/log(q)
N0=ceil(N0_aux)
x=x0;
for i=1:N0
  i
  x=exp(-x)
endfor
```

we obtain:

k	$x^{(k)}$	k	$x^{(k)}$
0	0.33333334	15	0.56719395
1	0.71653131	16	0.56711456
2	0.48844358	17	0.56715959
3	0.61358064	18	0.56713405
4	0.54140880	19	0.56714853
5	0.58192785	20	0.56714032
6	0.55882000	21	0.56714498
7	0.57188349	22	0.56714233
8	0.56446128	23	0.56714383
9	0.56866641	24	0.56714298
10	0.56628012	25	0.56714346
11	0.56763304	26	0.56714319
12	0.56686560	27	0.56714335
13	0.56730080	28	0.56714326
14	0.56705396	29	0.56714331

(e) From the a posteriori estimate we finally get

$$|x_k - \hat{x}| \leq \frac{q}{1-q}|x_k - x_{k-1}| \approx 1.26e - 07.$$

(f) If your calculator has an "ANS-key" then you can proceed as follows:

1. Type: `1÷3=`. Then the value "$\frac{1}{3}$" is stored in the "Answer-Storage"
2. Type: $e^{-\text{ANS}} =$ to obtain $x_1 = e^{-x_0}$
3. Type: $=$ to obtain $x_2 = e^{-x_1}$
4. Type: $=$ to obtain $x_3 = e^{-x_2}$
5. $\vdots$

The exercises can be discussed in the Tuesday lecture next week.