Exercise 1. Calculate $\sigma(A), \rho(A), \|A\|_\infty, \|A\|_F$ and $\|A\|_2$ for the following matrices

(a) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & -4 \\ 7 & 8 & -6 \end{pmatrix}$

To calculate the eingenvalues in b) you can use the build-in function `eig( )` in `Octave`. Specify the results on 3 decimal places after the comma.

Exercise 2. Let A be representable as $A = BB^T$ for some invertible matrix $B \in \mathbb{R}^{n\times n}$. Are the following statements true or false? Give an explanation!

(a) $\det(A) \neq 0$

(b) A is symmetric

(c) $A > 0$

(d) The diagonal entries of B are the square roots of the eigenvalues of A.

Exercise 3 *(LU decomposition)*. Determine (by hand calculation) the LU decomposition of the following matrices:

(a) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$ **(b)** $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 8 & 1 \\ 2 & 1 & 2 & 16 \end{pmatrix}$

Check your result and use the decomposition to calculate the determinant.

Exercise 4. Let L_i and L_j for $j > i$ be the Frobenius matrices of the ith and jth step of the LU decomposition of a matrix of dimension n. That is, L_i and L_j are unipotent lower triangular matrices that differ exactly in the ith or jth column from the identity matrix. Show that the matrix L_iL_j arises from the matrix L_j by replacing the ith column there with the ith column of L_i.

Exercise 5. Let $A \in \mathbb{R}^{n\times n}$ be regular. Count the number of multiplications and divisions used in the

(a) LU decomposition

(b) forward and backward substitution (in total)

Hint: To calculate $\sum_{k=1}^{n-1} k^2$ you can use the telescoping sum

$$\sum_{k=1}^{n-1} \Big((k+1)^3 - k^3 \Big).$$

The exercises can be discussed in the Tuesday lecture next week.