Exercise 8.1. Consider the SLE $Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{und} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Show that the Jacobi method converges to the solution of the SLE for each initial vector. Determine the associated Jacobi operator and calculate the first two iterations to the initial vector $x^{(0)} = (1, 1, 1)^T$.

(b) Show that the Gauss-Seidel method for the iterative solution of the system does not converge in general. Determine the associated Gauss-Seidel operator and calculate the first two iterations to the starting vector $x^{(0)} = (1, 1, 1)^T$.

Suggested Solution.

(a) To validate the convergence, calculate the Jacobi operator $\mathcal{J}$ and its eigenvalues:

$$A = D - L - U \quad , \qquad D = E \quad , \qquad L = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{pmatrix} \quad , \qquad U = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{J} = D^{-1}(L + U) = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix}$$

Determination of the eigenvalues of $\mathcal{J}$:

$$\chi_{\mathcal{J}}(\lambda) = \det \begin{pmatrix} -\lambda & -2 & 2 \\ -1 & -\lambda & -1 \\ -2 & -2 & -\lambda \end{pmatrix} = -\lambda^3 - 4 + 4 - 4\lambda + 2\lambda + 2\lambda = -\lambda^3$$

$$\Rightarrow \quad \sigma(\mathcal{J}) = \{0\}$$

$$\Rightarrow \quad \rho(\mathcal{J}) = 0 < 1$$

$\overset{\text{Prop. 3.4}}{\Rightarrow}$ Jacobi method converges for any initial value $x^{(0)}$ to the solution of $Ax = b$.

Remains to calculate the first two iterates with initial value $x^{(0)} = (1, 1, 1)^T$:

$$\Rightarrow \quad x^{(1)} = \mathcal{J}x^{(0)} + \underbrace{D^{-1}}_{E} b = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$$

$$\Rightarrow \quad x^{(2)} = \mathcal{J}x^{(1)} + \underbrace{D^{-1}}_{E} b = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 3 \\ 1 \end{pmatrix}$$

($x^{(2)}$ is the unique solution of the SLE!)

(b) Compute the Gauss-Seidel operator $\mathcal{L} = (D - L)^{-1}U$:

Recall: $D = E$, $L = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow (D - L) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}$

$$\Rightarrow (D - L)^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{L} = (D - L)^{-1}U = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

Determination of the eigenvalues of $\mathcal{L}$:

$$\chi_{\mathcal{L}}(\lambda) = \det \begin{pmatrix} -\lambda & -2 & 2 \\ 0 & 2-\lambda & -3 \\ 0 & 0 & -\lambda \end{pmatrix} = -\lambda(2-\lambda)^2$$

$$\Rightarrow \quad \sigma(\mathcal{L}) = \{0, 2\}$$

$$\Rightarrow \quad \rho(\mathcal{L}) = 2 > 1$$

$\overset{\text{Prop. 3.4}}{\Rightarrow}$ The Gauss-Seidel method does not converge in general (i.e. for any $x^{(0)}$).

Remains to calculate the first two iterates with initial value $x^{(0)} = (1,1,1)^T$:

$x^{(1)} = \mathcal{L}x^{(0)} + (D-L)^{-1}b$

Here, it is $(D-L)^{-1}b = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$\Rightarrow \quad x^{(1)} = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad x^{(2)} = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 1 \end{pmatrix}$$

Exercise 8.2. Given $A = \begin{pmatrix} 1 & \alpha & -\alpha \\ 0 & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$ ($\alpha \neq -1$) and an arbitrary $b \in \mathbb{R}^3$. For which α does the Jacobi and Gauss-Seidel method converge for any initial guess $x^{(0)}$ to the solution of $Ax = b$?:

(a) If you apply the criterion of strict diagonal dominance (Prop. 3.9).

(b) If you apply Prop. 3.4.

Suggested Solution.

(a) A strictly diagonally dominant $\quad \Leftrightarrow \quad \begin{cases} \text{I.} & 1 > |\alpha| + |-\alpha| \\ \text{II.} & 1 > |\alpha| \\ \text{III.} & 1 > |\alpha| + |\alpha| \end{cases} \quad \Rightarrow \quad |\alpha| < \frac{1}{2}.$

Thus: $|\alpha| < \frac{1}{2} \quad \Leftrightarrow \quad A$ is strictly diagonally dominant $\Rightarrow$ both methods (J/GS) converge for any initial value to the unique solution of $Ax = b$.

(b) To be able to apply Prop. 3.4 we first have to check whether A is invertible:

$$\det(A) = \begin{vmatrix} 1 & \alpha & -\alpha \\ 0 & 1 & \alpha \\ \alpha & \alpha & 1 \end{vmatrix} = 1 - \alpha^2 + \alpha(\alpha^2 + \alpha) = 1 + \alpha^3 \overset{\alpha \neq 1}{\neq} 0.$$

Secondly, we need to compute the eigenvalues of $\mathcal{J}$ and $\mathcal{L}$:

(1) $\mathcal{J} = \underbrace{D^{-1}}_{E}(L+U) = \begin{pmatrix} 0 & -\alpha & \alpha \\ 0 & 0 & -\alpha \\ -\alpha & -\alpha & 0 \end{pmatrix}$

$$\Rightarrow \quad \chi_{\mathcal{J}}(\lambda) = \det \begin{pmatrix} -\lambda & -\alpha & \alpha \\ 0 & -\lambda & -\alpha \\ -\alpha & -\alpha & -\lambda \end{pmatrix} = -\lambda^3 - \alpha^3 - \alpha^2\lambda + \alpha^2\lambda = -\lambda^3 - \alpha^3$$

$$\Rightarrow \quad \lambda^3 = -\alpha^3 \qquad \Rightarrow \quad \lambda = -\alpha$$

$$\Rightarrow \quad \rho(\mathcal{J}) = |\alpha| \quad \text{, i.e.} \quad \rho(\mathcal{J}) < 1 \quad \text{for} \quad |\alpha| < 1$$

$\Rightarrow$ Thus the Jacobi method converges for each initial value if and only if $|\alpha| < 1$.

(2) Eigenvalues of $\mathcal{L}$: recall $A = \begin{pmatrix} 1 & \alpha & -\alpha \\ 0 & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}$

$$\begin{aligned} \mathcal{L} &= (D-L)^{-1}U \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \alpha & \alpha & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -\alpha & \alpha \\ 0 & 0 & -\alpha \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\alpha & -\alpha & 1 \end{pmatrix} \begin{pmatrix} 0 & -\alpha & \alpha \\ 0 & 0 & -\alpha \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\alpha & \alpha \\ 0 & 0 & -\alpha \\ 0 & \alpha^2 & 0 \end{pmatrix} \end{aligned}$$

$$\Rightarrow \quad \chi_{\mathcal{L}}(\lambda) = \det \begin{pmatrix} -\lambda & -\alpha & \alpha \\ 0 & -\lambda & -\alpha \\ 0 & \alpha^2 & -\lambda \end{pmatrix} = -\lambda(\lambda^2 + \alpha^3)$$

$$\Rightarrow \quad \lambda = 0 \qquad \text{or} \qquad \lambda^2 = -\alpha^3$$

$$\Rightarrow \quad |\lambda| = 0 \qquad \text{or} \qquad |\lambda| = |\alpha|^{\frac{3}{2}}$$

$$\Rightarrow \quad \rho(\mathcal{L}) < 1 \quad \Leftrightarrow \quad |\alpha| < 1$$

$\Rightarrow$ Gauss-Seidel converges for all initial values if $|\alpha| < 1$.

Exercise 8.3. For $n \in \mathbb{N}$ let $A \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$ be given as follows:

$$A = \left(\begin{array}{c|ccc} 1 & \frac{1}{n} & \cdots & \frac{1}{n} \\ \hline \frac{1}{2} & & & \\ \vdots & & E & \\ \frac{1}{2} & & & \end{array}\right), \quad b = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix},$$

where E denotes the $(n-1)$-dimensional unit matrix.

(a) Justify why, for the SLE $Ax = b$, both the Jacobi and the Gauss-Seidel method converge to the solution $x = A^{-1}b$ for each initial vector.

Suggested solution:

A is strictly diagonally dominant, because for the first row of A the condition is:

$$1 > \sum_{j=2}^{n} \frac{1}{n} = \frac{(n-1)}{n} \quad \checkmark$$

and in the following rows

$$1 > \frac{1}{2} \quad \checkmark$$

$$\vdots$$

$$1 > \frac{1}{2} \quad \checkmark$$

Thus the assertion follows directly with Prop. 3.9.

(b) Implement the Jacobi method for the SLE $Ax = b$ and the matrix dimension $n = 1000$. To do this, write an `Octave` function that requires as input values the vector b, the initial vector $x^{(0)}$ and the value $N_{\max}$ = number of iterations as input values and returns the approximate solution of $Ax = b$ as return value. Iterate directly, i.e. without saving A or $\mathcal{J}$, or D, L, U. Compare the computation time for 20 iterations with the duration of the calculation of the LU decomposition of A, using the program from Exercise 4.4. (To determine the computation time you can use the command `tic` to start a stop watch and `toc` to stop it.)

Suggested solution:

The calculation of 20 iterates for $n = 1000$ takes about 0.6 msec. The LU decomposition takes considerably longer, approx. 10.4 sec. The implementation could look as follows: (on the rhs you can see the implementation of the jacobi method.)

```
% jm_script.m
n=1000;
b=(1:n)';
x0=zeros(n,1);
Nmax=20;
tic
approx_sol=jm(b,x0,Nmax);
jacobi_time=toc
A=eye(n);
A(2:n,1)=0.5;
A(1,2:n)=1/n;
tic
[L,U]=LU_wo_permutations(A);
LU_decomp_time=toc
est_time_days=LU_decomp_time*1e6/(60*60*24)
```

```
% run "jm_script.m"
function y = jm(b,x0,Nmax)
  n = length(b);
  % or [n,dummy] = size(b)
  % comp. aux sum:
  for j= 1:Nmax   % j: iteration index
    auxsum=sum(x0);
    % y=x^(k+1)
    y(1)=b(1)-(1/n)*auxsum+(1/n)*x0(1);
    % compute the rest of the components
    y(2:n)=b(2:n)-0.5*x0(1);
    x0=y;
  end
end
```

(c) Using an effort-only approach and the result of (b), estimate the time difference between calculating the LU decomposition and calculating 20 iterations of the Jacobi method for the matrix dimension $n = 100000$.

Remark: "Effort-only approach" means that you set the (asymptotic) numerical effort for $n = 1000$ and $n = 100000$ in relation to each other and equate this quotient with the ratio of the corresponding computation times.

Suggested solution:

If $n = 100000$, i.e. by a factor of 100 greater than in b), we obtain for the LU decomposition due to the effort $\sim \frac{1}{3}n^3$:

$$\frac{\text{computation time LU decomp. for } n = 100000}{\text{computation time LU decomp. for } n = 1000} = \frac{c \cdot 100000^3/3}{c \cdot 1000^3/3} = \frac{10^{15}}{10^9} = 10^6.$$

Thus the computation time for $n = 100000$ amounts to roughly $10^6 \cdot 10.4$ sec, i.e. 120 days.

The effort to calculate 20 iterates of the Jacobi method is $\sim 2n$ (matrix is extremely sparse). Therefore

$$\frac{\text{comp. time for } n = 100000}{\text{comp. time for } n = 1000} = \frac{c \cdot 2 \cdot 100000}{c \cdot 2 \cdot 1000} = 100$$

Thus, regarding the Jacobi method, the computation time for $n = 100000$ amounts to roughly $100 \cdot 0.0006 = 0.06$ sec.

The exercises can be discussed in the Tuesday lecture next week.