
Exercise 1 *(Vector norms)***.**

(a) Let $(V, \|\cdot\|)$ be a normed vector space. Prove that

$$\Big|\|x\| - \|y\|\Big| \leq \|x \pm y\|$$

for all $x, y \in V$.

(b) Prove that all norms in $\mathbb{R}^n$ are equivalent. This means that for arbitrary norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ there exists constants α, β such that for all $x \in \mathbb{R}^n$ it is

$$\alpha\|x\|_* \leq \|x\|_{**} \leq \beta\|x\|_*.$$

Hint: It suffices to show that every norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm $\|\cdot\|_2$. Definition 1 and Theorem 2 below could be helpful.

Definition 1. *A set $X \subset (\mathbb{R}^n, \|\cdot\|_2)$ is called* **bounded** *if there exists $C > 0$ such that $\|x\|_2 \leq C$ for all $x \in X$. X is* **open** *:⇔ any point of X is an interior point. X is* **closed** *:⇔ $\mathbb{R}^n \backslash X$ is open. X is called* **compact** *if it is closed and bounded.* ♢

Theorem 2 (Weierstrass extreme value theorem)**.** *Let $X \subset (\mathbb{R}^n, \|\cdot\|_2)$ be compact and $f\colon X \to \mathbb{R}$ be continuous. Then $f(X)$ is compact and there exist $x_{min}, x_{max} \in X$ such that for all $x \in X$*

$$f(x_{min}) \leq f(x) \leq f(x_{max}).$$ □

Exercise 2 *(matrix norms)***.** Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and $\|\cdot\|_*$ the induced matrix norm. Prove the following

(a) $\|\cdot\|_*$ is a vector norm on (the vector space) $\mathbb{R}^{n\times n}$.

(b) $\|\cdot\|_*$ is compatible with $\|\cdot\|$, i.e. $\|Ax\| \leq \|A\|_*\|x\|$ for all $A \in \mathbb{R}^{n\times n},\ x \in \mathbb{R}^n$.

(c) $\|\cdot\|_*$ is submultiplicative, i.e. for all $A, B \in \mathbb{R}^{n\times n}$ one has

$$\|AB\|_* \leq \|A\|_*\|B\|_*.$$

(d) $\|E_n\|_* = 1$.

(e) If $|||\cdot|||$ is another vector norm on $\mathbb{R}^{n\times n}$ that is compatible with $\|\cdot\|$ then for all $A \in \mathbb{R}^{n\times n}$

$$\|A\|_* \leq |||A|||.$$

Exercise 3 *(Spectral norm and spectral radius)***.** Let $A \in \mathbb{R}^{n\times n}$. Prove that for the spectral norm on $\mathbb{R}^n$ it holds

$$\|A\|_2 = \sqrt{\rho(A^TA)},$$

where for $B \in \mathbb{R}^{n\times n}$

$$\rho(B) := \max_{\lambda\in\sigma(B)} |\lambda|$$

denotes the largest absolute eigenvalue of B and is called the ***spectral radius*** of B .

Hint: Use that $A^T A$ as a symmetric matrix has an orthonormal eigenbasis and a positive spectrum (i.e. only real eigenvalues ≥ 0). Then, represent each vector x in the equation

$$\|A\|_2 = \max_{\|x\|_2=1} \langle Ax, Ax \rangle^{\frac{1}{2}} = \max_{\|x\|_2=1} (x^T A^T A x)^{\frac{1}{2}}$$

with respect to this basis and compute the resulting expression.

The exercises can be discussed in the Tuesday lecture next week.