Exercise 1. Determine the LU decomposition with row swaps, i.e. L, U and the corresponding permutation vector p for the following matrices. Then solve the system $Ax = e_1$. Here, $e_1 = (1, 0, \dots, 0)^T$.

a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix}$ b) $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 4 \\ 1 & 1 & 2 & 4 \end{pmatrix}$.

Suggested Solution.

(a) Step 1: Swap rows 1 and 2, i.e. $p = (2)$ and eliminate the 1st column.

$$\Rightarrow \quad P_1 A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}}_{L_1^{-1}} \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix}}_{A^{(1)}}$$

Step 2: Swap rows 2 and 3, i.e. $p = (2, 3)$.

$$\Rightarrow \quad P_2 P_1 A = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{P_2 L_1^{-1} P_2 = L} \underbrace{\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix}}_{P_2 A^{(1)} = U} = LU$$

(b) Step 1: Swap rows 1 and 2, i.e. $p = (2)$ and eliminate the 1st column.

$$P_1 A = \begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & 2 & 4 \\ 1 & 1 & 2 & 4 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & 0 & 1 \end{pmatrix}}_{L_1^{-1}} \underbrace{\begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 3 \\ 0 & 1 & 1 & \frac{7}{2} \end{pmatrix}}_{A^{(1)}}$$

Step 2: Swap rows 2 and 4, i.e. $p = (2, 4)$.

$$\Rightarrow \quad P_2 P_1 A = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{P_2 L_1^{-1} P_2} \underbrace{\begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & \frac{7}{2} \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{P_2 A^{(1)}}$$

Eliminate the 2nd column:

$$\Rightarrow \quad P_2 P_1 A = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{P_2 L_1^{-1} P_2 L_2^{-1} = L} \underbrace{\begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & \frac{7}{2} \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{L_2 P_2 A^{(1)} = A^{(2)} = U} = LU$$

Now we respectively solve $Ax = e_1$: Note that $Ax = b \quad \Leftrightarrow \quad PAx = LUx = Pb$.

(a) $p = (2, 3)$, i.e.

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{I \leftrightarrow II} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \xrightarrow{II \leftrightarrow III} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = Pb$$

(1) Solve $Ly = Pb$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} y_1 = 0 \\ y_2 = -y_1 = 0 \\ y_3 = 1 \end{cases}$$

(2) Solve $Rx = y$:

$$\Leftrightarrow \quad \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = -\frac{1}{2} x_3 = -\frac{1}{2} \\ x_2 = -\frac{3}{2} x_3 = -\frac{3}{2} \\ x_3 = 1 \end{cases}$$

(b) $p = (2, 4)$, i.e.

$$b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{I \leftrightarrow II} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \xrightarrow{II \leftrightarrow IV} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = Pb$$

(1) Solve $Ly = Pb$

$$\Leftrightarrow \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} y = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} y_1 = 0 \\ y_2 = 0 \\ y_3 = 0 \\ y_4 = 1 \end{cases}$$

(2) Solve $Rx = y$:

$$\Leftrightarrow \quad \begin{pmatrix} 2 & 0 & 2 & 1 \\ 0 & 1 & 1 & \frac{7}{2} \\ 0 & 0 & -2 & -4 \\ 0 & 0 & 0 & 1 \end{pmatrix} x = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} x_1 = -x_3 - \frac{1}{2} x_4 = \frac{3}{2} \\ x_2 = -\frac{7}{2} x_4 - x_3 = -\frac{3}{2} \\ x_3 = -2x_4 = -2 \\ x_4 = 1 \end{cases}$$

Exercise 2. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be arbitrary. The Frobenius norm is defined by

$$\|A\|_F = \Big(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \Big)^{\frac{1}{2}}.$$

Proof the following:

(a) For $n \geq 2$ the Frobenius norm is not induced by any norm.
Hint: Exercise Sheet 2, Exercise 2(d).

(b) $\|A\|_F^2 = \operatorname{trace}(A^T A)$

(c) The Frobenius norm is compatible with the Euclidean norm $\|\cdot\|_2$, i.e. for all $x \in \mathbb{R}^n$ it is

$$\|Ax\|_2 \leq \|A\|_F \|x\|_2.$$

Hint: Consider the squared inequality and use the (squared) Cauchy-Schwarz inequality to estimate $\|Ax\|_2^2$.

(d) $\|A\|_2 \leq \|A\|_F$.

Hint: Use subtask (c)

Suggested Solution. **(a)** Assume the contrary. From Sheet 2, Ex. 2(d) we conclude that $\|E\|_F = 1$. But this contradicts

$$\|E\|_F^2 = \sum_{ij} \delta_{ij}^2 = \sum_{i=1}^{n} 1 = n$$

at least for $n \geq 2$.

(b) With $A = [a_1, \ldots, a_n]$ we obtain

$$\operatorname{trace}(A^T A) = \sum_j \langle e_j, A^T A e_j \rangle = \sum_j \langle A e_j, A e_j \rangle = \sum_j \langle a_j, a_j \rangle = \sum_{ij} a_{ij}^2.$$

(c) Let a_i^T be the ith row of A and $x \in \mathbb{R}^n$ be a unit vector. Then

$$\begin{aligned} \|Ax\|_2^2 &= \sum_i \langle a_i, x \rangle \\ &\leq \sum_i \|a_i\|_2^2 \|x\|_2^2 \\ &= \sum_{ij} a_{ij}^2. \end{aligned}$$

(d) Take the supremum over all $x \neq 0$ in

$$\frac{\|Ax\|_2}{\|x\|_2} \leq \|A\|_F$$

yields the result.

Exercise 3. Let $A \in \mathbb{R}^{n \times n}$ be ***nilpotent***, i.e. for some $N \in \mathbb{N}$ it is

$$A^N = 0.$$

Are the following statements true or false (give a reason for each!):

(a) $\det(A) = 0$

(b) $\sigma(A) = \{0\}$

(c) $\operatorname{trace}(A) = 0$

(d) $A = 0$ or A is not diagonalizable

(e) $E - A$ is invertible

Suggested Solution. (a) true, since

$$0 = \det(A^N) = (\det A)^N.$$

(b) $\sigma(A) = \{0\}$ is true: Let $\lambda \in \sigma(A)$ be any eigenvalue and v be a corresponding eigenvector. Then by induction we get $A^N v = \lambda^N v$. Since $v \neq 0$, $0 = A^N v = \lambda^N v$ implies that $\lambda^N = 0$, hence $\lambda = 0$.

(c) $\operatorname{trace}(A) = 0$ is true because $\sigma(A) = \{0\}$ implies

$$\operatorname{trace}(A) = \sum_{\lambda \in \sigma(A)} \mu(\chi_A, \lambda)\lambda = 0.$$

(d) true: Assume the contrary, i.e. $A \neq 0$ is diagonalizable. This means that

$$0 \neq A = SDS^{-1}$$

for some $D = \operatorname{diag}(d_1, \ldots, d_n) \neq 0$ (otherwise $SDS^{-1} = 0$) and $S \in \mathrm{Gl}(n, \mathbb{R})$. By induction we can deduce

$$0 = A^N = (SDS^{-1})^N = SD^N S^{-1} = S \begin{pmatrix} d_1^N & & 0 \\ & \ddots & \\ 0 & & d_n^N \end{pmatrix} S^{-1}.$$

This implies that $D^N = 0 \;\Rightarrow\; D = 0$, contradicting $D \neq 0$.

(e) $E - A$ is invertible since $E - A$ is injective:

$$(E - A)v = 0 \;\Rightarrow\; Av = 1 \cdot v \overset{\sigma(A)=\{0\}}{\Rightarrow} v = 0.$$

Another possibility:

$$(E - A) \sum_{j=0}^{N-1} A^j = E - A^N = E,$$

i.e. $(E - A)$ is invertible with $(E - A)^{-1} = \sum_{j=0}^{N-1} A^j$.

Exercise 4. Write a program that will implement the LU decomposition (without permutations) preferably with Octave/Matlab. The program should receive an arbitrary $(n \times n)$-matrix A as input and the matrices L (unipotent lower triangular matrix) and U (upper triangular matrix) as output, such that

$$A = LU.$$

Test your program with the matrices from Sheet 3, Ex. 3.

Suggested Solution.

```
function [L,U] = LU_wo_permutations(A)
  % Determine the matrix dimension
  [n,n] = size(A);
  % initialize L with a unit matrix
  L = eye(n);
  % Outer for loop: Iterates through all columns to be eliminated
  for j=1:(n-1)
    % Inner for loop: Eliminates the jth column below the diagonal
    for i=(j+1):n
      L(i,j) = A(i,j)/A(j,j);
    end
```

```
    % Carry out row transformations in A
    for i=(j+1):n
      A(i,:) = A(i,:) - L(i,j)*A(j,:);
    end
  end
  U=A;
end
```

The exercises can be discussed in the Tuesday lecture next week.