Stochastics

Bachelor Applied Artificial Intelligence (AAI-B3)

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Literature

H.-O. Georgii, *Stochastics*, DeGruyter, 2nd Edition, 2013.

A. Steland, *Basiswissen Statistik*, Springer, 4. Auflage, 2016. (German)

S.M. Ross, *Introduction to Probability and Statistics for Engineers and Scientists*, Academic Press, 2014.

Contents

Chapter I

Introduction

We consider births in Germany. Question (two variants):

(i) What is the probability of having a girl?

(ii) Is it more likely to have a boy than a girl?

Empirical data from statistical sample yields the gender of N births:

$$k \text{ girls } (1), \quad N-k \text{ boys } (0).$$

The relative frequencies are given by

$$\widehat{p}(1) = \frac{k}{N}, \quad \widehat{p}(0) = \frac{N-k}{N} = 1 - \widehat{p}(1)$$

and serve as a metric of the data (descriptive statistics).
Naive answers to the above questions read:

(i) The sought probability is given by $\widehat{p}(1)$.

(ii) Yes if and only if $\widehat{p}(0) > \widehat{p}(1)$.

Criticism:

- The answers are based on only one sample or data set.
- The sample size and the variability of the data are not taken into account.

Therefore we will consider

inferential statistics (see Chapters **??** and **??**)

to infer properties of an underlying probability distribution. This, in turn, requires a

mathematical model (see Chapters II and **??**)

of the underlying random mechanism.

Chapter II

Probability Theory – Discrete Case

In this section we aim at modelling and analyzing random experiments with at most countably infinite outcomes (discrete case).

In the sequel let Ω be a finite or a countably infinite set. The set Ω is a model for the possible results of a random experiment.

1 Discrete Probability Spaces

Definition 1. The set Ω is called *sample space*. Its elements $\omega \in \Omega$ are called *outcomes*. Any subset $A \subseteq \Omega$ of Ω is called an *event*.

Example 2. Consider a fair coin with head (1) and tail (0) that is tossed twice. The corresponding sample space is given by

$$\Omega = \{(1,1),(1,0),(0,1),(0,0)\} = \{1,0\}^2.$$

The event $A = \{(1,1),(1,0)\}$ contains all outcomes with 1 in the first toss.

Clearly, we may also use this sample space as a model for two-stage processes having two outcomes (e.g., yes/no) per stage, e.g., a two-stage production process where in each stage some tolerance level is satisfied or not.

Terminology: We say that the event $A \subseteq \Omega$ occurs if $\omega \in A$.

Notation: For sets A, B we write $A \subseteq B$ if every element of A is also an element of B, and we write $A = B$ if $A \subseteq B$ and $B \subseteq A$. The empty set containing no elements is denoted by $\varnothing$. The cardinality of a set A is denoted by $|A|$.

Definition 3. Let $A, B \subseteq \Omega$ be events. The *union*, the *intersection*, and the *difference* of A and B as well as the *complement* of A are defined by

$$\begin{aligned} A \cup B &= \{\omega \in \Omega \colon \omega \in A \text{ or } \omega \in B\}, \\ A \cap B &= \{\omega \in \Omega \colon \omega \in A \text{ and } \omega \in B\}, \\ A \setminus B &= \{\omega \in \Omega \colon \omega \in A \text{ and } \omega \notin B\}, \\ A^{\mathsf{c}} &= \Omega \setminus A = \{\omega \in \Omega \colon \omega \notin A\}, \end{aligned}$$

respectively.

Remark 4. Foe events $A, B, C \subseteq \Omega$ we have

$$\begin{aligned} A \cap (B \cup C) &= (A \cap B) \cup (A \cap C), \\ A \cup (B \cap C) &= (A \cup B) \cap (A \cup C), \\ (A \cap B)^{\mathsf{c}} &= A^{\mathsf{c}} \cup B^{\mathsf{c}}, \\ (A \cup B)^{\mathsf{c}} &= A^{\mathsf{c}} \cap B^{\mathsf{c}}, \\ A \setminus B &= A \cap B^{\mathsf{c}}. \end{aligned}$$

Definition 5. The *(countable) union* of events $A_1, A_2, A_3, \ldots \subseteq \Omega$ is defined by

$$\bigcup_{i=1}^{\infty} A_i = \big\{\omega \in \Omega \colon (\exists i \in \mathbb{N} \colon \omega \in A_i)\big\}.$$

The events $A_1, A_2, A_3, \ldots \subseteq \Omega$ are *pairwise disjoint* if $A_i \cap A_j = \varnothing$ for all $i, j \in \mathbb{N}$ with $i \neq j$.

Remark 6. Clearly, we can express the union of finitely many events as a countable union, i.e., for $n \in \mathbb{N}$ and events $A_1, \ldots, A_n \subseteq \Omega$ we have

$$\bigcup_{i=1}^{n} A_i = A_1 \cup \ldots \cup A_n = \bigcup_{i=1}^{\infty} A_i$$

if we put $A_i = \varnothing$ for $i > n$.

We are now going to assign probabilities $P(A)$ to events $A \subseteq \Omega$.

Definition 7. The set

$$\mathcal{P}(\Omega) = \{A \colon A \subseteq \Omega\}$$

of all subsets of Ω is called *power set* of Ω.

Example 8. The power set of $\Omega = \{0, 1\}$ is given by

$$\mathcal{P}(\Omega) = \{\varnothing, \{0\}, \{1\}, \Omega\}.$$

Moreover, we have $|\mathcal{P}(\Omega)| = 2^{|\Omega|}$.

Definition 9. A function $P \colon \mathcal{P}(\Omega) \to \mathbb{R}$ is a *probability measure (on Ω)* (or *probability distribution (on Ω)*) if

(i) $0 \leq P(A) \leq 1$ for all $A \subseteq \Omega$,

(ii) $P(\Omega) = 1$,

(iii) for all pairwise disjoint events $A_1, A_2, \ldots \subseteq \Omega$ we have

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i). \qquad \textit{(σ-additivity)}$$

In such a case the triple $(\Omega, \mathcal{P}(\Omega), P)$ is called a *discrete probability space*.

Proposition 10. Let P be a probability measure on Ω and let $A, B \in \mathcal{P}(\Omega)$. Then we have

(i) $A \cap B = \varnothing \;\Rightarrow\; P(A \cup B) = P(A) + P(B)$ (*additivity*),

(ii) $A \subseteq B \;\Rightarrow\; P(B \setminus A) = P(B) - P(A)$,

(iii) $A \subseteq B \;\Rightarrow\; P(A) \leq P(B)$ (*monotonicity*),

(iv) $P(A^{\mathrm{c}}) = 1 - P(A)$,

(v) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Proof. ad (i): Since $A \cap B = \varnothing$, we have

$$A \cup B = A \cup B \cup \varnothing \cup \varnothing \cup \dots$$

for pairwise disjoint events and thus the σ-additivity of P shows

$$P(A \cup B) = P(A) + P(B) + \sum_{i=3}^{\infty} P(\varnothing).$$

Since $P(A \cup B) \in [0, 1]$, we have $P(\varnothing) = 0$.
ad (ii)-(v): See `Exercise` 1.1. □

2 Probability Mass Functions

Definition 1. Let P be a probability measure on Ω. The function $p\colon \Omega \to \mathbb{R}$ defined by

$$p(\omega) = P(\{\omega\})$$

is called *probability mass function (associated to P)*.

Proposition 2.

(i) For P and p according to Definition 1 and for every $A \subseteq \Omega$ we have

$$P(A) = \sum_{\omega \in A} p(\omega).$$

In particular, P is uniquely determined by p.

(ii) Every probability mass function $p\colon \Omega \to \mathbb{R}$ satisfies

$$\forall\, \omega \in \Omega\colon 0 \leq p(\omega) \leq 1 \quad \wedge \quad \sum_{\omega \in \Omega} p(\omega) = 1. \tag{1}$$

(iii) Every function $p\colon \Omega \to \mathbb{R}$ satisfying (1) defines a probability measure P on Ω by

$$P(A) = \sum_{\omega \in A} p(\omega)$$

for $A \subseteq \Omega$.

Proof. Since Ω is countable, every $A \subseteq \Omega$ is countable and can thus be expressed as a countable union $A = \bigcup_{\omega \in A} \{\omega\}$ of pairwise disjoint sets.
ad (i): The σ-additivity of P yields for $A \subseteq \Omega$

$$P(A) = P\left(\bigcup_{\omega \in A} \{\omega\}\right) = \sum_{\omega \in A} P(\{\omega\}) = \sum_{\omega \in A} p(\omega).$$

ad (ii): Use part (i) with $A = \Omega$ and note that $P(\Omega) = 1$.
ad (iii): Verify the conditions of Definition 1.9. □

Example 3. Let Ω be finite with $|\Omega| = n \in \mathbb{N}$. For $\omega \in \Omega$ we put

$$p(\omega) = \frac{1}{n}.$$

Then p satisfies (1) and the associated probability measure P is given by

$$P(A) = \frac{|A|}{|\Omega|}$$

for every $A \subseteq \Omega$. Hence the calculation of probabilities is based on counting elements.

Definition 4. The probability measure P according to Example 3 is called the *discrete uniform distribution (on the finite set Ω).*

Example 5. Consider a fair coin that is tossed twice, cf. Example 1.2. We model this random experiment by using the discrete uniform distribution P on

$$\Omega = \{(1,1), (1,0), (0,1), (0,0)\}.$$

The events

$$A = \{(1,1), (1,0)\} = \text{“1 in the first toss”},$$
$$B = \{(1,1), (0,1)\} = \text{“1 in the second toss”}$$

satisfy

$$P(A) = \frac{2}{4} = \frac{1}{2} = P(B),$$
$$P(A \cap B) = P(\{(1,1)\}) = \frac{1}{4} = P(A) \cdot P(B),$$
$$P(A \cup B) = P(\{(1,1), (1,0), (0,1)\}) = \frac{3}{4}.$$

Example 6. Consider a two-stage production process where in each process a tolerance level is satisfied (1) or not (0). We model this by

$$\Omega = \{(1,1), (1,0), (0,1), (0,0)\}$$

and the (fictional) probability mass function

ω	$(1,1)$	$(1,0)$	$(0,1)$	$(0,0)$
$p(\omega)$	0.8	0.09	0.01	0.1

The events

$$\begin{aligned} A &= \{(1,1),(1,0)\} = \text{“tolerance level satisfied in first stage”}, \\ B &= \{(1,1),(0,1)\} = \text{“tolerance level satisfied in second stage”} \end{aligned}$$

satisfy

$$\begin{aligned} P(A) &= 0.89, \\ P(B) &= 0.81, \\ P(A \cap B) &= 0.8 \neq 0.7209 = P(A) \cdot P(B). \end{aligned}$$

3 Conditional Probability and Independence

In the sequel let $(\Omega, \mathcal{P}(\Omega), P)$ be a discrete probability space.

Question: How do we change the probability measure P if we know that a certain event $B \subseteq \Omega$ has occured.

Definition 1. For $A, B \subseteq \Omega$ with $P(B) > 0$ the *conditional probability of A given B* is defined by

$$P(A \,|\, B) = \frac{P(A \cap B)}{P(B)}.$$

Example 2. Let P be the discrete uniform distribution on a finite set Ω. For $A, B \subseteq \Omega$ with $B \neq \varnothing$ we have $P(B) > 0$ and

$$P(A \,|\, B) = \frac{|A \cap B|}{|\Omega|} \cdot \frac{|\Omega|}{|B|} = \frac{|A \cap B|}{|B|}.$$

In Example 2.5 (fair coin tossed twice “independently”) we thus have

$$P(A \,|\, B) = P(B \,|\, A) = \frac{1}{2}$$

for $A = \{(1,1),(1,0)\}$ and $B = \{(1,1),(0,1)\}$.

Example 3. In Example 2.6 (two-stage production process) we have

$$P(A \,|\, B) = \frac{P(\{(1,1)\})}{P(\{(1,1),(0,1)\})} = \frac{0.8}{0.81} \approx 0.9876$$

and

$$P(B \,|\, A) = \frac{P(\{(1,1)\})}{P(\{(1,1),(1,0)\})} = \frac{0.8}{0.89} \approx 0.8988$$

for $A = \{(1,1),(1,0)\}$ and $B = \{(1,1),(0,1)\}$.

Remark 4. Let p be the probability mass function associated to P, and let $B \subseteq \Omega$ with $P(B) > 0$. For $A \subseteq \Omega$ we have

$$P(A \mid B) = \frac{1}{P(B)} \cdot \sum_{\omega \in A \cap B} p(\omega) = \sum_{\omega \in A} q(\omega),$$

where

$$q(\omega) = \begin{cases} \dfrac{p(\omega)}{P(B)}, & \text{if } \omega \in B, \\ 0, & \text{else.} \end{cases}$$

Then $Q \colon \mathcal{P}(\Omega) \to \mathbb{R}$ given by

$$Q(A) = P(A \mid B)$$

defines a probability measure on Ω with probability mass function q.

Definition 5. Two events $A, B \subseteq \Omega$ are *independent* if

$$P(A \cap B) = P(A) \cdot P(B).$$

Remark 6. Let $P(B) > 0$. Then we have

$$A, B \text{ independent} \;\Leftrightarrow\; P(A \mid B) = P(A).$$

Example 7. In Example 2.5 (fair coin tossed twice "independently") the events A and B are independent. In Example 2.6 (two-stage production process) the events A and B are dependent.

Proposition 8 (Bayes's law)**.** Let $n \in \mathbb{N}$ and $\{B_1, \ldots, B_n\} \subseteq \mathcal{P}(\Omega)$ be a partition[1] of Ω with $P(B_i) > 0$ for all $i = 1, \ldots, n$.

(i) For all $A \subseteq \Omega$ we have

$$P(A) = \sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i).$$

(ii) For all $A \subseteq \Omega$ with $P(A) > 0$ and for all $k = 1, \ldots, n$ we have

$$P(B_k \mid A) = \frac{P(A \mid B_k) \cdot P(B_k)}{\sum_{i=1}^{n} P(A \mid B_i) \cdot P(B_i)}.$$

Proof. ad (i): Since $\{B_1, \ldots, B_n\}$ is a partition, we obtain for every $A \subseteq \Omega$

$$A = A \cap \left(\bigcup_{i=1}^{n} B_i \right) = \bigcup_{i=1}^{n} (A \cap B_i)$$

with pairwise disjoint sets $A \cap B_1, \ldots, A \cap B_n \subseteq \Omega$. Hence we get

$$P(A) = \sum_{i=1}^{n} P(A \cap B_i).$$

[1] A partition of Ω is a family of non-empty pairwise disjoint sets $B_1, B_2, \ldots \subseteq \Omega$ with $\bigcup_{i=1}^{n} B_i = \Omega$.

Since $P(B_i) > 0$ for all $i = 1, \dots, n$, the conditional probabilities $P(A \,|\, B_i)$ are well-defined such that $P(A \cap B_i) = P(A \,|\, B_i) \cdot P(B_i)$.

ad (ii): If $P(A) > 0$, the conditional probability $P(B_k \,|\, A)$ is well-defined and we have

$$P(B_k \,|\, A) \cdot P(A) = P(B_k \cap A) = P(A \cap B_k) = P(A \,|\, B_k) \cdot P(B_k)$$

for all $k = 1, \dots, n$. Use part (i). $\square$

4 Random Variables

In the sequel let $(\Omega, \mathcal{P}(\Omega), P)$ be a discrete probability space.

Question: How can we describe particular aspects of a random experiment (rather than the full experiment)?

Example 1. Consider a fair die that is rolled twice independently, i.e., we consider a discrete uniform distribution P on $\Omega = \{1, \dots, 6\}^2$. Particular aspects could be:

(i) "number of pips in first die roll",

(ii) "number of pips in second die roll",

(iii) "sum of pips".

In the sequel let $\mathfrak{X} \subseteq \mathbb{R}$ be finite or countably infinite. We will typically consider $\mathfrak{X} \subseteq \mathbb{N}_0$.

Definition 2. A function $X \colon \Omega \to \mathfrak{X}$ is called a *random variable (with values in $\mathfrak{X}$)*. Its function values $x = X(\omega) \in \mathfrak{X}$ are called *realizations* of X.

Example 3 (Continuation of Example 1)**.** The first two aspects are described by $\mathfrak{X} = \{1, \dots, 6\}$ and the random variables $X_1, X_2 \colon \Omega \to \mathfrak{X}$ given by

$$X_1(\omega) = \omega_1, \quad X_2(\omega) = \omega_2$$

for $\omega = (\omega_1, \omega_2) \in \Omega$. The third aspect is described by $\mathfrak{S} = \{2, \dots, 12\}$ and the random variable $S \colon \Omega \to \mathfrak{S}$ given by

$$S(\omega) = X_1(\omega) + X_2(\omega) = \omega_1 + \omega_2.$$

In the sequel let $X \colon \Omega \to \mathfrak{X}$ be a random variable. In many cases one is just interested in the probabilities

$$P_X(A) = P(\{\omega \in \Omega \colon X(\omega) \in A\})$$

for $A \subseteq \mathfrak{X}$ and in particular in

$$p_X(x) = P_X(\{x\}) = P(\{\omega \in \Omega : X(\omega) = x\})$$

for $x \in \mathfrak{X}$.

Example 4 (Continuation of Example 3)**.** For $x \in \{1, \ldots, 6\}$ we have

$$p_{X_1}(x) = p_{X_2}(x) = \frac{1}{6},$$

and p_S is given by

s	2	3	4	5	6	7	8	9	10	11	12
$p_S(s)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Proposition 5. The following holds true:

(i) $p_X \colon \mathfrak{X} \to \mathbb{R}$ is a probability mass function,

(ii) P_X is the associated probability measure on $\mathfrak{X}$, i.e., for all $A \subseteq \mathfrak{X}$ we have

$$P_X(A) = \sum_{x \in A} p_X(x).$$

Definition 6. P_X and p_X are called *distribution* and *probability mass function of* X, respectively.

Example 7 (Continuation of Example 4)**.** A *rod graph* can be used to illustrate the probability mass functions p_{X_1} and p_S, see Figure 4.1.

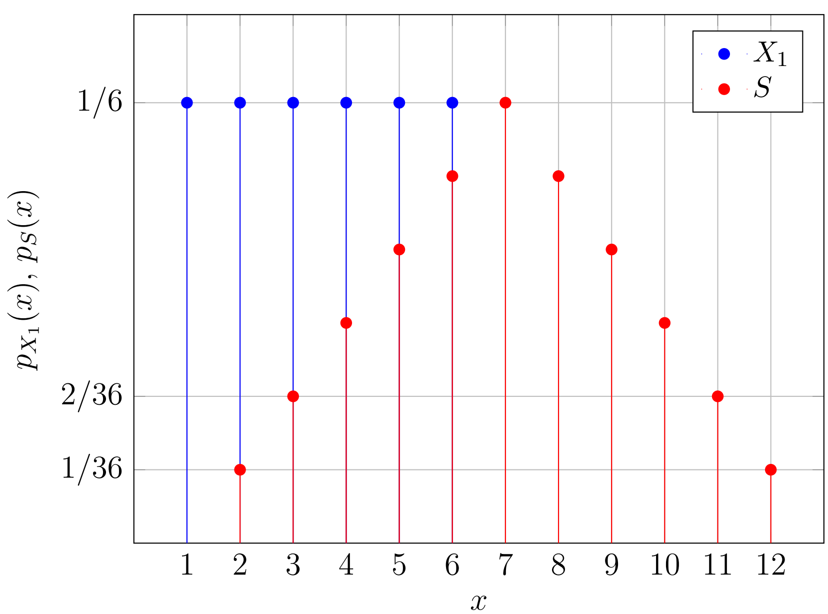


Figure 4.1: Probability mass functions p_{X_1} and p_S from Example 4.

Definition 8. Two random variables $X_1, X_2 \colon \Omega \to \mathfrak{X}$ are *identically distributed* if

$$P_{X_1}(A) = P_{X_2}(A)$$

for all $A \subseteq \mathfrak{X}$.

Example 9 (Continuation of Example 4)**.** P_{X_1} and P_{X_2} are both the discrete uniform distribution on $\{1,\dots,6\}$. Hence X_1 and X_2 are identically distributed.

Proposition 10 (Criterion for identical distributions)**.** $X_1, X_2 \colon \Omega \to \mathfrak{X}$ are identically distributed if and only if

$$p_{X_1}(x) = p_{X_2}(x)$$

for all $x \in \mathfrak{X}$.

Proof. Use Proposition 5. □

Remark 11. The case of a multivariate random variable (random vector)

$$X = (X_1, \dots, X_n) \colon \Omega \to \mathfrak{X}$$

with an at most countably infinite set $\mathfrak{X} \subseteq \mathbb{R}^n$ is treated analogously. The components X_i of X are random variables.

Example 12 (Continuation of Example 3)**.**

(i) For $\mathfrak{X} = \{1,\dots,6\}^2$ and $x = (x_1, x_2) \in \mathfrak{X}$ we have

$$p_{(X_1,X_2)}(x) = P(\{\omega \in \Omega \colon \omega = x\}) = P(\{x\}) = \frac{1}{36}.$$

Hence $P_{(X_1,X_2)}$ is the discrete uniform distribution on $\mathfrak{X}$.

(ii) For $\mathfrak{X} = \{(x_1, s) \in \mathbb{N}^2 \colon 1 \le x_1 \le 6 \wedge x_1 + 1 \le s \le x_1 + 6\}$ and $(x_1, s) \in \mathfrak{X}$ we have

$$p_{(X_1,S)}(x_1, s) = P(\{\omega \in \Omega \colon \omega_1 = x_1 \wedge \omega_1 + \omega_2 = s\}) = \frac{1}{36}.$$

Hence $P_{(X_1,S)}$ is the discrete uniform distribution on $\mathfrak{X}$.

In the sequel we use

$$\{X = x\} = \{\omega \in \Omega \colon X(\omega) = x\},$$
$$\{X \in A\} = \{\omega \in \Omega \colon X(\omega) \in A\},$$

and we consider random variables $X_1, \dots, X_n$ on $(\Omega, \mathcal{P}(\Omega), P)$ taking values in an at most countably infinite set $\mathfrak{X}$.

Definition 13. $X_1, \dots, X_n$ are *independent* if for all $A_1, \dots, A_n \subseteq \mathfrak{X}$ we have

$$P\left(\bigcap_{i=1}^{n} \{X_i \in A_i\}\right) = \prod_{i=1}^{n} P(\{X_i \in A_i\}).$$

Example 14 (Continuation of Example 3)**.**

(i) For $\mathfrak{X} = \{1, \dots, 6\}$ and $A_1, A_2 \subseteq \mathfrak{X}$ we have

$$\begin{aligned} P(\{X_1 \in A_1\} \cap \{X_2 \in A_2\}) &= P(\{\omega \in \Omega \colon \omega_1 \in A_1 \wedge \omega_2 \in A_2\}) \\ &= \frac{|A_1| \cdot |A_2|}{|\Omega|} = \frac{|A_1| \cdot |A_2|}{36} \end{aligned}$$

and according to Example 9 for $i = 1, 2$

$$P(\{X_i \in A_i\}) = \frac{|A_i|}{6}.$$

Hence X_1 and X_2 are independent.

(ii) For $\mathfrak{X} = \{1, \dots, 12\}$, $A_1 = \{6\}$ and $B = \{2\}$ we have

$$\begin{aligned} P(\{X_1 \in A_1\} \cap \{S \in B\}) &= P(\{\omega \in \Omega \colon \omega_1 = 6 \wedge \omega_1 + \omega_2 = 2\}) \\ &= P(\varnothing) = 0 \end{aligned}$$

as well as $P(\{X_1 \in A_1\}) > 0$ and $P(\{S \in B\}) > 0$. Hence X_1 and S are not independent.

Proposition 15 (Criterion for independence)**.** $X_1, \dots, X_n$ are independent if and only if for all $x_1, \dots, x_n \in \mathfrak{X}$ we have

$$P\left(\bigcap_{i=1}^{n} \{X_i = x_i\}\right) = \prod_{i=1}^{n} p_{X_i}(x_i). \tag{2}$$

Proof. "$\Rightarrow$": Take $A_i = \{x_i\}$.
"$\Leftarrow$": Since $\mathfrak{X}$ is countable, every $A_i \subseteq \mathfrak{X}$ can be expressed as a countable union $A_i = \bigcup_{x_i \in A_i} \{x_i\}$ of pairwise disjoint sets. Use the σ-additivity of P and (2). [...] $\square$

Remark 16. Consider the special case of $n = 2$ and $\mathfrak{X} = \{0, \dots, k\}$ for some $k \in \mathbb{N}$. Put

$$p_{i,j} = P(\{X_1 = i\} \cap \{X_2 = j\})$$

and

$$p_{i,\bullet} = P(\{X_1 = i\}) \quad \text{and} \quad p_{\bullet,j} = P(\{X_2 = j\})$$

for $i, j \in \{0, \dots, k\}$. Clearly, we have

$$p_{i,\bullet} = \sum_{j=0}^{k} p_{i,j} \quad \text{and} \quad p_{\bullet,j} = \sum_{i=0}^{k} p_{i,j}$$

for all $i, j \in \{0, \dots, k\}$. The corresponding *contingency table* is given by Table II.1. Proposition 15 shows that

$$X_1, X_2 \text{ independent } \Leftrightarrow \forall\, i, j \in \{0, \dots, k\} \colon p_{i,j} = p_{i,\bullet} \cdot p_{\bullet,j}.$$

X_1 \ X_2	0	$\cdots$	k	$\sum$
0	$p_{0,0}$	$\cdots$	$p_{0,k}$	$p_{0,\bullet}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
k	$p_{k,0}$	$\cdots$	$p_{k,k}$	$p_{k,\bullet}$
$\sum$	$p_{\bullet,0}$	$\cdots$	$p_{\bullet,k}$	1

Table II.1: Contingency table from Remark 16.

5 Empirical Distribution

In the sequel let $(\Omega, \mathcal{P}(\Omega), P)$ be a discrete probability space and let $X\colon \Omega \to \mathfrak{X}$ be a random variable with a countable set $\mathfrak{X} \subseteq \mathbb{R}$.

We consider the case where the set $\mathfrak{X}$ of possible realizations is known but the distribution P_X is unknown. Instead, a *sample* (data set)

$$x_1, \ldots, x_N \in \mathfrak{X} \tag{3}$$

consisting of realizations of N independent repetitions of the random experiments is available.

Question: Can we approximately determine the probability mass function p_X of X?

Definition 1. Given the sample (3) the *relative frequency* of $x \in \mathfrak{X}$ is given by

$$\begin{aligned} \widehat{p}(x) &= \frac{|\{\ell \in \{1, \ldots, N\} \colon x_\ell = x\}|}{N} \\ &= \frac{\text{number of elements of the sample equal to } x}{N}. \end{aligned}$$

Remark 2. For $p = \widehat{p}$ we have (1) in Proposition 2.2(ii). According to Proposition 2.2(iii), see also Proposition 4.5, we obtain a probability distribution $\widehat{P}$ on $\mathfrak{X}$ satisfying

$$\begin{aligned} \widehat{P}(A) = \sum_{x \in A} \widehat{p}(x) &= \frac{|\{\ell \in \{1, \ldots, N\} \colon x_\ell \in A\}|}{N} \\ &= \frac{\text{number of elements of the sample with values in } A}{N} \end{aligned}$$

for $A \subseteq \mathfrak{X}$.

Definition 3. $\widehat{P}$ is called *empirical distribution* of the sample (3).

Remark 4. A rod graph can be used to illustrate empirical distributions in terms of relative frequencies. Based on the scale of measure we distinguish the following types:

(i) Nominal scale: Elements of $\mathfrak{X}$ indicate a name (operations $=, \neq$). Example: gender (male (0), female (1)).

$X_1 \backslash X_2$	0	$\cdots$	k	$\sum$
0	$\widehat{p}_{0,0}$	$\cdots$	$\widehat{p}_{0,k}$	$\widehat{p}_{0,\bullet}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
k	$\widehat{p}_{k,0}$	$\cdots$	$\widehat{p}_{k,k}$	$\widehat{p}_{k,\bullet}$
$\sum$	$\widehat{p}_{\bullet,0}$	$\cdots$	$\widehat{p}_{\bullet,k}$	1

Table II.2: Contingency table for empirical distribution from Remark 6.

(ii) Ordinal scale: Elements of $\mathfrak{X}$ allow for a ranking (operations $<, >, =$). Example: rank order.

(iii) Metric (cardinal) scale: Elements of $\mathfrak{X}$ are numeric and allow for arithmetic operations. Examples: time, temperature, sales.

Remark 5. The case of a multivariate random variable (random vector)

$$X = (X_1, \dots, X_n) \colon \Omega \to \mathfrak{X}$$

with an at most countably infinite set $\mathfrak{X} \subseteq \mathbb{R}^n$ is treated analogously.

Remark 6 (Counterpart of Remark 4.16)**.** Consider the special case of two random variables $(X_1, X_2) \colon \Omega \to \mathfrak{X}$ and $\mathfrak{X} = \{0, \dots, k\}^2$ with $k \in \mathbb{N}$. Put

$$\widehat{p}_{i,j} = \frac{|\{\ell \in \{1, \dots, N\} \colon x_\ell = (i,j)\}|}{N}$$

and

$$\widehat{p}_{i,\bullet} = \frac{|\{\ell \in \{1, \dots, N\} \colon x_{\ell,1} = i\}|}{N}, \quad \widehat{p}_{\bullet,j} = \frac{|\{\ell \in \{1, \dots, N\} \colon x_{\ell,2} = j\}|}{N}$$

for $i, j \in \{0, \dots, k\}$. We clearly have

$$\widehat{p}_{i,\bullet} = \sum_{j=0}^{k} \widehat{p}_{i,j} \quad \text{and} \quad \widehat{p}_{\bullet,j} = \sum_{i=0}^{k} \widehat{p}_{i,j}$$

for all $i, j \in \{0, \dots, k\}$. The corresponding contingency table is given by Table II.2. The random variables X_1 and X_2 are assumed to be independent if and only if

$$\widehat{p}_{i,j} \approx \widehat{p}_{i\,\bullet} \cdot \widehat{p}_{\bullet,j}$$

for all $i, j \in \{0, \dots, k\}$.

Example 7. The relative frequencies of tossing two fair coins, see Example 2.5 (one coin tossed twice independently), resulting from a computer simulation with sample size $N = 10^3$ is shown in Figure 5.1. The corresponding contingency table for

$$X_1(\omega) = \omega_1, \quad X_2(\omega) = \omega_2$$

with $\omega = (\omega_1, \omega_2) \in \Omega$ is shown in Table II.3.

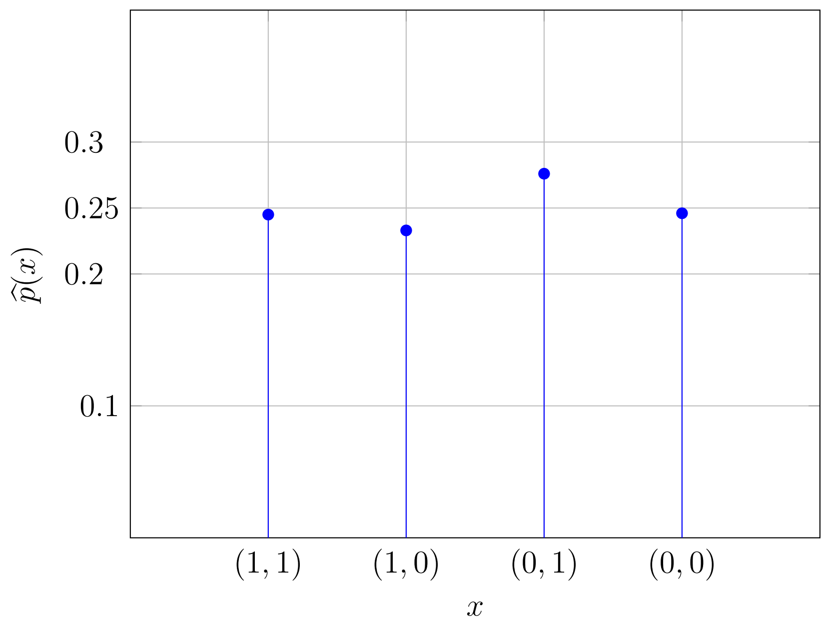


Figure 5.1: Relative frequencies of tossing two fair coins based on 10^3 repetitions.

X_1 \ X_2	0	1	$\sum$
0	0.246	0.276	0.522
1	0.233	0.245	0.478
$\sum$	0.479	0.521	1

Table II.3: Contingency table for empirical distribution in Figure 5.1.

6 Special Discrete Distributions

In this section we discuss several discrete probability distributions that serve as standard models for special random experiments.

Binomial Distribution

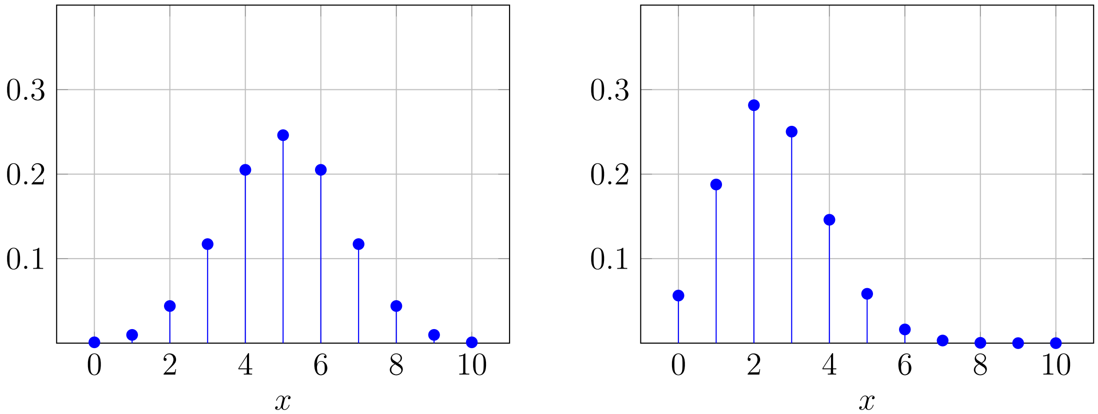
Consider a random experiment with the two outcomes 1 (success) and 0 (failure) that is repeated n times independently. This can be modeled by

(i) parameters $n \in \mathbb{N}$ (number of repetitions) and $p \in [0,1]$ (probability of success),

(ii) independent and identically distributed (*i.i.d.*) random variables $X_1, \ldots, X_n$ satisfying

$$p = P(\{X_i = 1\}) = 1 - P(\{X_i = 0\})$$

for all $i = 1, \ldots, n$.



(a) $p = 1/2$. (b) $p = 1/4$.

Figure 6.1: Probability mass functions of $X \sim \mathrm{B}(10, p)$ with $p = 1/2$ and $p = 1/4$.

Put

$$S_n = \sum_{i=1}^{n} X_i$$

and note that for $\omega \in \Omega$ we have

$$S_n(\omega) = |\{i \in \{1, \ldots, n\} \colon X_i(\omega) = 1\}|,$$

i.e., the number of successes is given by S_n.

Proposition 1. For $k \in \{0, \ldots, n\}$ we have

$$P(\{S_n = k\}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}.$$

Definition 2. A random variable X is *binomially distributed* with parameters $n \in \mathbb{N}$ und $p \in [0, 1]$ if

$$P(\{X = k\}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

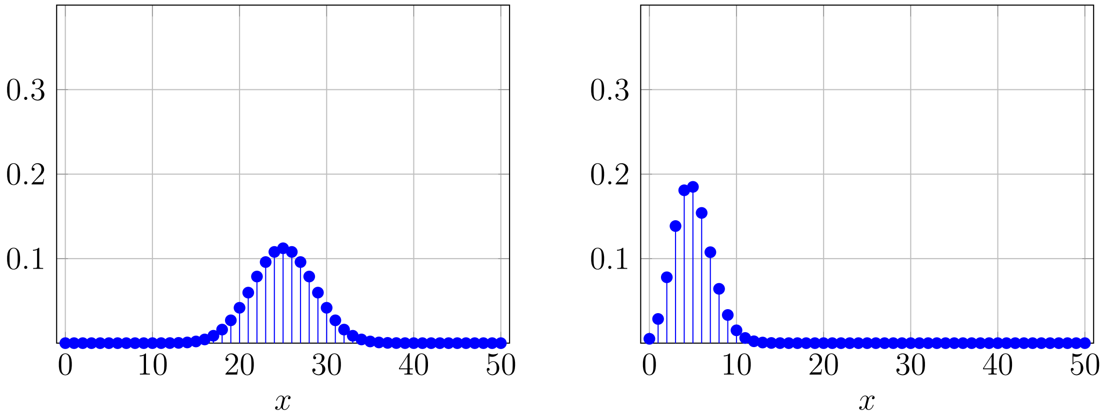
for all $k \in \{0, \ldots, n\}$.
Notation: $X \sim \mathrm{B}(n, p)$.

Statistical Problem. Given n and k (by a sample), estimate p.

Example 3. The probability mass functions of $X \sim \mathrm{B}(n, p)$ for $n \in \{10, 50\}$ and different values of $p \in [0, 1]$ are illustrated in Figure 6.1 and Figure 6.2.

Proposition 4. Let X and Y be independent with $X \sim \mathrm{B}(n, p)$ and $Y \sim \mathrm{B}(m, p)$ for $m, n \in \mathbb{N}$ and $p \in [0, 1]$. Then we have

$$X + Y \sim \mathrm{B}(n + m, p).$$



(a) $p = 1/2$. (b) $p = 1/10$.

Figure 6.2: Probability mass functions of $X \sim \mathrm{B}(50, p)$ with $p = 1/2$ and $p = 1/10$.

Multinomial Distribution

Consider a random experiment with outcomes $0, \dots, m-1$ that is repeated n times independently. This can be modeled by

(i) parameters $n \in \mathbb{N}$ (number of repetitions), $m \in \mathbb{N} \setminus \{1\}$ (number of outcomes), and $p_0, \dots, p_{m-1} \in [0, 1]$ such that $\sum_{j=0}^{m-1} p_j = 1$ (probabilities of m outcomes),

(ii) i.i.d. random variables $X_1, \dots, X_n$ satisfying

$$P(\{X_i = j\}) = p_j$$

for all $i = 1, \dots, n$ and $j = 0, \dots, m-1$.

Put

$$S_j(\omega) = |\{i \in \{1, \dots, n\} \colon X_i(\omega) = j\}|$$

for $\omega \in \Omega$ and $j \in \{0, \dots, m-1\}$, i.e., S_j is the number of random experiments with outcome j, and put

$$S = (S_0, \dots, S_{m-1}).$$

Remark 5. (i) For $j = 0, \dots, m-1$ we have $S_j \sim \mathrm{B}(n, p_j)$.

(ii) The random variables $S_0, \dots, S_{m-1}$ are not independent in general. We have

$$\sum_{j=0}^{m-1} S_j(\omega) = n$$

for all $\omega \in \Omega$.

Proposition 6. For $k = (k_0, \dots, k_{m-1}) \in \mathbb{N}_0^m$ with $\sum_{j=0}^{m-1} k_j = n$ we have

$$P(\{S = k\}) = \frac{n!}{k_0! \cdots k_{m-1}!} \cdot p_0^{k_0} \cdots p_{m-1}^{k_{m-1}}.$$

Definition 7. A random vector X follows a *multinomial distribution* with parameters $n \in \mathbb{N}$ and $p_0, \ldots, p_{m-1} \in [0,1]$ with $\sum_{j=0}^{m-1} p_j = 1$ if

$$P(\{X = k\}) = \frac{n!}{k_0! \cdots k_{m-1}!} \cdot p_0^{k_0} \cdots p_{m-1}^{k_{m-1}}$$

for all $k = (k_0, \ldots, k_{m-1}) \in \mathbb{N}_0^m$ with $\sum_{j=0}^{m-1} k_j = n$.
Notation: $X \sim \mathrm{M}(n, p_0, \ldots, p_{m-1})$.

Hypergeometric Distribution

Consider a sample of size $n \in \mathbb{N}$ drawn without replacement from a set with $N \in \mathbb{N}$ elements consisting of K elements of type "success" and $N - K$ elements of type "failure". This can be modeled by

(i) parameters $N, K, n \in \mathbb{N}$ with $n \leq N$ and $K \leq N$,

(ii) the uniform distribution P on

$$\Omega = \{\omega \subseteq \{1, \ldots, N\} \colon |\omega| = n\}.$$

Put

$$X(\omega) = |\omega \cap \{1, \ldots, K\}|$$

for $\omega \in \Omega$, i.e., $\omega \subseteq \{1, \ldots, N\}$ with $|\omega| = n$, to count the number of successes contained in the subset ω.
A typical application is given by quality control.

Remark 8. Proposition $\boxed{\text{??.??}}$ shows

$$|\Omega| = \binom{N}{n}.$$

Proposition 9. For $k \in \mathbb{N}_0$ with

$$n - (N - K) \leq k \leq \min(n, K) \tag{4}$$

we have

$$P(\{X = k\}) = \frac{\binom{K}{k} \cdot \binom{N-K}{n-k}}{\binom{N}{n}}. \tag{5}$$

Definition 10. A random variable X is *hypergeometrically distributed* with parameters $N, K, n \in \mathbb{N}$ with $n \leq N$ and $K \leq N$ if (5) holds for all $k \in \mathbb{N}_0$ with (4).
Notation: $X \sim \mathrm{H}(N, K, n)$.

Statistical Problem. (i) Given N, n, and k (by a sample), estimate K.

(ii) Given K, n, and k (by a sample), estimate N.

Proposition 11. Let $X_N \sim \mathrm{H}(N, K_N, n)$ for $N \in \mathbb{N}$ such that

$$\lim_{N\to\infty} \frac{K_N}{N} \in \left]0,1\right[.$$

Put $p = \lim_{N\to\infty} \frac{K_N}{N}$. Then we have

$$\lim_{N\to\infty} P(\{X_N = k\}) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k}$$

for all $k \in \{0,\dots,n\}$.

Example 12. The probability mass functions of $X \sim \mathrm{H}(100, 20, 10)$ and $Y \sim \mathrm{B}(10, 1/5)$ are illustrated in Figure 6.3. As indicated by Proposition 11, the distance

$$\max_{k\in\{0,\dots,n\}} |p_X(k) - p_Y(k)|$$

with $n = 10$ is rather small.

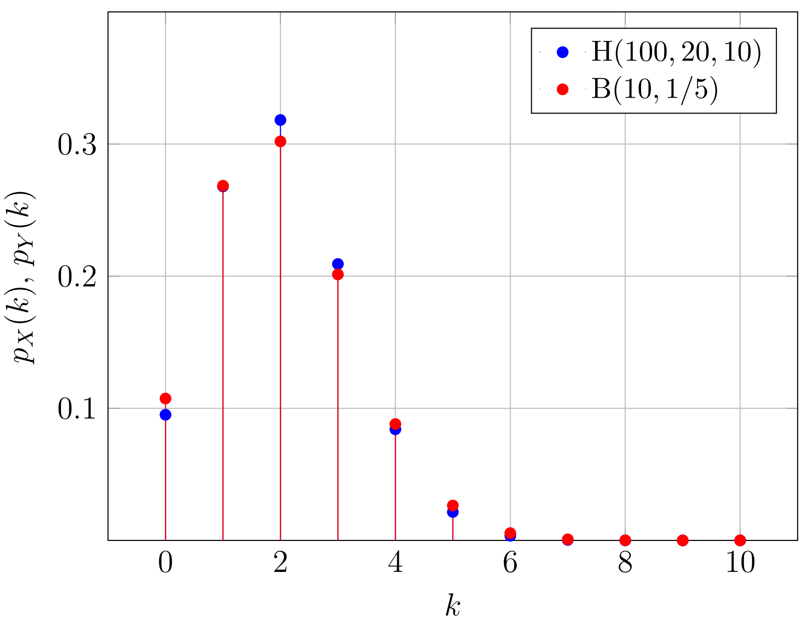


Figure 6.3: Probability mass functions of $X \sim \mathrm{H}(100, 20, 10)$ and $Y \sim \mathrm{B}(10, 1/5)$.

Poisson Distribution

Proposition 13 (Poisson limit theorem)**.** Let $X_n \sim \mathrm{B}(n, p_n)$ for $n \in \mathbb{N}$ such that

$$\lim_{n\to\infty} n \cdot p_n \in \left]0,\infty\right[.$$

Put $\lambda = \lim_{n\to\infty} n \cdot p_n$. Then we have

$$\lim_{n\to\infty} P(\{X_n = k\}) = \exp(-\lambda) \cdot \frac{\lambda^k}{k!}$$

for all $k \in \mathbb{N}_0$.

Remark 14. For all $\lambda \in \mathbb{R}$ we have $\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = \exp(\lambda)$.

Definition 15. A random variable X follows a *Poisson distribution* with parameter $\lambda \in \,]0, \infty[$ if

$$P(\{X = k\}) = \exp(-\lambda) \cdot \frac{\lambda^k}{k!}$$

for all $k \in \mathbb{N}_0$.

Notation: $X \sim \mathrm{Poi}(\lambda)$.

Example 16. The probability mass functions of $X \sim \mathrm{Poi}(\lambda)$ with $\lambda \in \{1/2, 3\}$ are illustrated in Figure 6.4. Moreover, the probability mass functions of $Y \sim \mathrm{B}(50, 1/10)$ and $Z \sim \mathrm{Poi}(5)$ are illustrated in Figure 6.5. As indicated by Proposition 13, the distance

$$\max_{k \in \{0,\dots,n\}} |p_Y(k) - p_Z(k)|$$

with $n = 50$ is rather small.

Typical examples where a Poisson distribution serves as the stochastic model include

(i) number of decay events from a radioactive source within a certain time interval,

(ii) incoming calls in a call center per hour.

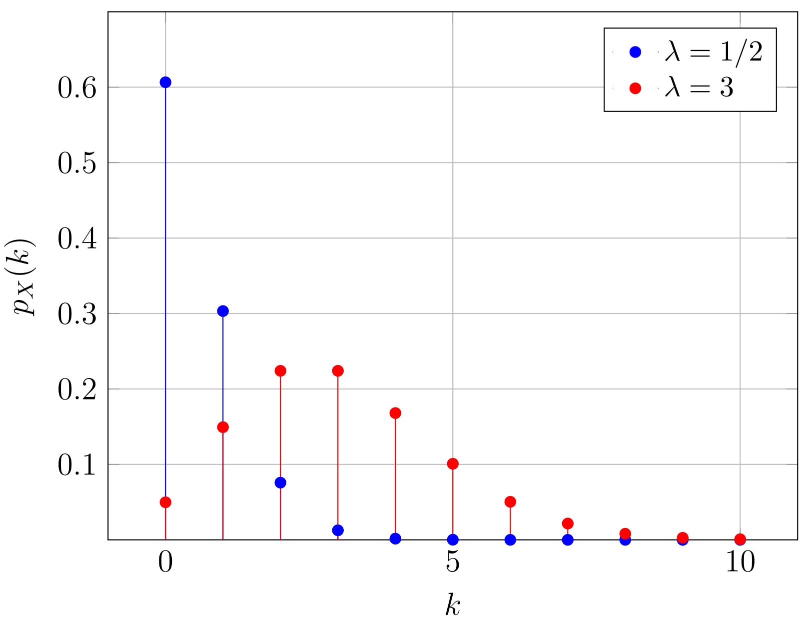


Figure 6.4: Probability mass functions of $X \sim \mathrm{Poi}(\lambda)$ with $\lambda = 1/2$ and $\lambda = 3$.

Geometric Distribution

Consider a random experiment with the two outcomes 1 (success) and 0 (failure) that is repeated n times independently. This can be modeled by

(i) parameters $n \in \mathbb{N}$ (number of repetitions) and $p \in [0, 1]$ (probability of success),

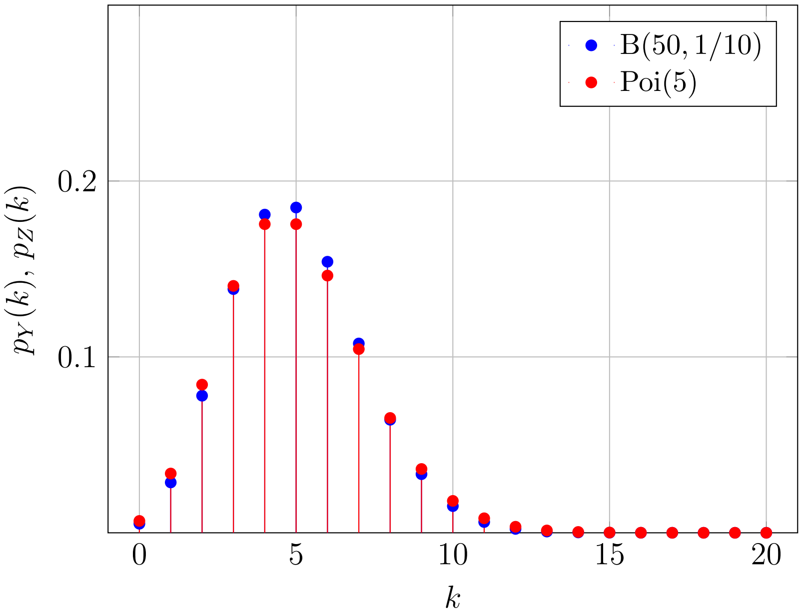


Figure 6.5: Probability mass functions of $Y \sim \mathrm{B}(50, 1/10)$ and $Z \sim \mathrm{Poi}(5)$.

(ii) i.i.d. random variables $X_1, \dots, X_n$ satisfying

$$p = P(\{X_i = 1\}) = 1 - P(\{X_i = 0\})$$

for all $i = 1, \dots, n$.

For $\omega \in \Omega$ we put

$$T_n(\omega) = 0$$

if

$$X_1(\omega) = \ldots = X_n(\omega) = 0,$$

and we put

$$T_n(\omega) = \min\{k \in \{1, \dots, n\} \colon X_k(\omega) = 1\}$$

otherwise, i.e., if $\{k \in \{1, \dots, n\} \colon X_k(\omega) = 1\} \neq \varnothing$. Note that T_n describes the (discrete) waiting time until the first success occurs within n trials.

Remark 17. (i) For $n \in \mathbb{N}$ and $k \in \{0, \dots, n\}$ we have

$$P(\{T_n = k\}) = \begin{cases} (1-p)^{k-1} \cdot p, & \text{if } k \in \{1, \dots, n\}, \\ (1-p)^n, & \text{if } k = 0. \end{cases}$$

In particular, we have $\lim_{n \to \infty} P(\{T_n = 0\}) = 0$.

(ii) For all $p \in \,]0, 1]$ we have $\sum_{k=1}^{\infty} (1-p)^{k-1} = 1/p$.

Definition 18. A random variable X is *geometrically distributed* with parameter $p \in \,]0, 1]$ if

$$P(\{X = k\}) = (1-p)^{k-1} \cdot p$$

for all $k \in \mathbb{N}$.

Notation: $X \sim \mathrm{Geo}(p)$.