Exercise 1 *(Computing)***.** Given

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ -3 & -6 & -11 & -15 \\ 0 & 4 & 3 & 2 \\ 2 & 0 & 4 & 9 \end{pmatrix}.$$

(a) Determine one (or the unique - see below -) LU decomposition of A with row permutations. That is, determine permutation matrices P_i as well as matrices L and U, so that

$$PA = LU$$

holds for $P = P_1P_2P_3$. For the sake of uniqueness, apply the partial pivot strategy only in the case of a zero pivot element.

(b) For $b = (-6, 26, 1, -20)^T$ solve the system of linear equations $Ax = b$ using (a) and forward/backward substitution.

Exercise 2 *(Computing; Octave)***.** The so-called Hilbert matrices of the form

$$A = (a_{ij}) \in \mathbb{R}^{n\times n}, \quad a_{ij} = \frac{1}{i+j-1}$$

are ill-conditioned. Consider the system of linear equations $Ax = b$, where $b = (b_j) \in \mathbb{R}^n$ with $b_j = \frac{1}{j+1}$. You can use the solution routine `A\b` included in Matlab/Octave for the solution of this SLE respectively.

(a) Determine a numerical solution to this SLE for matrix dimension $n = 5$: Once the solution x for unperturbed b as above and once $\tilde{x}$ with perturbation of b by assuming $b_1 = 0.51$ (2% perturbation in the first component). Calculate the relative error

$$\frac{\|\tilde{x} - x\|_2}{\|x\|_2}$$

of the solution $\tilde{x}$ of the perturbed system (measured in the Euclidean norm).

(b) Calculate the solution x_{num} of $Ax = b$ with b as above (without perturbation) for matrix dimension $n = 20$. Compare x_{num} with the exact solution $x = (0, 1, 0, \dots, 0)^T$ and calculate $\|x_{\text{num}} - x\|_2$ and $\|x_{\text{num}} - x\|_\infty$. Determine with Matlab/Octave also the condition $\text{cond}_2(A)$ of the matrix A.

Useful Octave-commands: `hilb(n)` produces a Hilbert matrix of order n; `cond(A)` calculates (numerically) $\text{cond}_2(A)$; `norm(x,p)` calculates $\|x\|_p$ ($p =$`inf` for the maximum norm);

Exercise 3 *(Eigenvalues and matrix norms)***.** Given $A \in \mathbb{R}^{n\times n}$. Show that

(a)
$$\sigma(A^2) = \{\lambda^2 \mid \lambda \in \sigma(A)\}$$

(b)
$$\|A\|_F \leq \sqrt{n}\|A\|_2.$$

Is this inequality sharp, i.e. is there a matrix A s.t. the "=" sign is valid?

Exercise 4 *(Eigenvalues, condition numbers; computation in (c))***.** Let $A \in \mathbb{R}^{n \times n}$ be symmetric and $\lambda_{\max}(A)$ be the largest and $\lambda_{\min}(A)$ be the smallest absolute eigenvalue of A.

(a) If A is regular and $\sigma(A) = \{\lambda_1, \dots, \lambda_n\}$ (where the numbers λ_i do not necessarily have to be different) show that the spectrum of A^{-1} is given by

$$\sigma(A^{-1}) = \left\{ \frac{1}{\lambda_1}, \dots, \frac{1}{\lambda_n} \right\}.$$

What is the relationship between the respective eigenvectors of A and A^{-1}?

(b) Let A be regular and symmetric. Show that

$$\operatorname{cond}_2(A) = \frac{|\lambda_{\max}(A)|}{|\lambda_{\min}(A)|}.$$

(c) Using Point (b), determine the value $\operatorname{cond}_2(A)$ of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 2 \end{pmatrix}.$$

The exercises can be discussed in the Tuesday lecture next week.