Exercise 13.1 *(Formulating LP models; graphical solution)***.**
Romeo Winery produces two types of wines, Bordeaux and Romerlot, by blending Merlot and Cabernet Sauvignon grapes. Making one barrel of Bordeaux blend requires 250 pounds of Merlot and 250 pounds of Cabernet Sauvignon, whereas making one barrel of Romerlot requires 450 pounds of Merlot and 50 pounds of Cabernet Sauvignon. The profit received from selling Bordeaux is £800 per barrel, and from selling Romerlot, £600 per barrel. Romeo Winery has 9000 pounds of Merlot and 5000 pounds of Cabernet Sauvignon available. Formulate an LP model aiming to maximize the winery's profit. Solve the LP graphically.

Suggested Solution. We define the decision variables as follows:

$$\begin{aligned} x_1 &= \text{ barrels of Bordeaux produced} \\ x_2 &= \text{ barrels of Romerlot produced} \end{aligned}$$

Then we obtain the following LP:

$$\begin{aligned} \text{maximize} \quad & 800x_1 + 600x_2 & \\ \text{subject to} \quad & 250x_1 + 450x_2 \leq 9000 & \text{(Merlot constraint)} \\ & 250x_1 + \ \ 50x_2 \leq 5000 & \text{(Cabernet Sauvignon constraint)} \\ & x_1, x_2 \geq 0 & \end{aligned}$$

The LP is solved graphically in Figure 1.

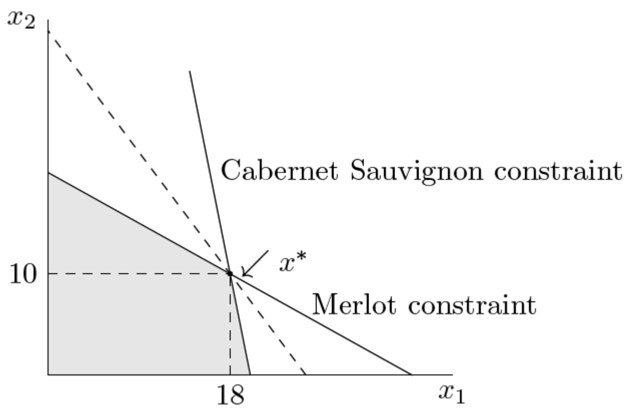

Figure 1: Graphical solution of the LP

Exercise 13.2 *(beginners)***.** Solve the following LP using the simplex method (with Dantzig's rule):

$$\begin{aligned} \text{maximize} \quad & x_1 + \ \ x_2 \\ \text{subject to} \quad & 3x_1 + 7x_2 \le 10 \\ & 2x_1 + \ \ x_2 \le 3 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{1}$$

Suggested Solution. Since all constraints are $\le$-constraints, we already have the (P)-form (cf. the slides). Namely, we have

$$c = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \ A' = \begin{pmatrix} 3 & 7 \\ 2 & 1 \end{pmatrix}; \ b' = \begin{pmatrix} 10 \\ 3 \end{pmatrix}; \ n = 2; \ m' = 2; \ m'' = 0; \ m = 2.$$

Moreover, since $b' \ge 0$ we have $\mathcal{I}^- = \{\}$. To formulate the Problem in standard form (PS), we introduce slack variables s_1, s_2 for each inequality and obtain the LP

$$\begin{aligned} \text{maximize} \quad & x_1 + \ \ x_2 + 0s_1 + 0s_2 \\ \text{subject to} \quad & 3x_1 + 7x_2 + \ \ s_1 \qquad = 10 \\ & 2x_1 + \ \ x_2 \qquad + \ \ s_2 = 3 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Letting

$$A = (A' \ E_2) = \begin{pmatrix} 3 & 7 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{pmatrix}; \ b = b'; \ x = (x_1, x_2, s_1, s_2)^T; \ c := (1, 1, 0, 0)^T$$

we get

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x_1, x_2 \ge 0. \end{aligned}$$

Since $\mathcal{I}^- = \{\}$ (i.e. we do not have $\le$-constraints with neg. rhs) we do not have to introduce artificial variables for the inequality constraints. Further, since there are no equality constraints, we do not need a artificial variables here either. That is $\mathcal{I}^a = \{\}$. Because we already have the basis $BV_0 = \{s_1, s_2\}$ we do not need the two-phase method here. Thus, we can write down the initial tableau

z	x_1	x_2	s_1	s_2	rhs	basis$_0$	ratio
1	-1	-1	0	0	0	z	–
0	3	7	1	0	10	s_1	$10/3$
0	2	1	0	1	3	s_2	$3/2$

(2)

Here there are two variables (x_1, x_2) which have the smallest coefficient. According to Dantzig's rule, we can choose one of the two at random, e.g. x_1:

z	x_1	x_2	s_1	s_2	rhs	basis$_1$	ratio
1	0	$-1/2$	0	$1/2$	$3/2$	z	–
0	0	$11/2$	1	$-3/2$	$11/2$	s_1	1
0	1	$1/2$	0	$1/2$	$3/2$	x_1	3

$\rightsquigarrow$

z	x_1	x_2	s_1	s_2	rhs	basis$_2$	ratio
1	0	0	$1/11$	$4/11$	2	z	–
0	0	1	$2/11$	$-3/11$	1	x_2	–
0	1	0	$-1/11$	$7/11$	1	x_1	–

This tableau is optimal since all coefficients in the z-row are nonnegative. The optimal solution is

$$x_1^* = x_2^* = 1; \ z^* = 2. \tag{3}$$

If we choose x_2 as the entering variable in Equation 2, we obtain

z	x_1	x_2	s_1	s_2	rhs	basis$_1$	ratio
1	-1	-1	0	0	0	z	$-$
0	3	7	1	0	10	s_1	$10/7$
0	2	1	0	1	3	s_2	3

$\rightsquigarrow$

z	x_1	x_2	s_1	s_2	rhs	basis$_2$	ratio
1	$-4/7$	0	$1/7$	0	$10/7$	z	$-$
0	$3/7$	1	$1/7$	0	$10/7$	x_2	$10/3$
0	$11/7$	0	$-1/7$	1	$11/7$	s_2	1

$\rightsquigarrow$

z	x_1	x_2	s_1	s_2	rhs	basis$_2$	ratio
1	0	0	$1/11$	$4/11$	2	z	$-$
0	0	1	$2/11$	$-3/11$	1	x_2	$-$
0	1	0	$-1/11$	$7/11$	1	x_1	$-$

This tableau is again optimal and of course you get the same optimal solution as in (3).

Exercise 13.3. Solve the following LP using the simplex method:

$$\begin{aligned} \text{maximize} \quad & x_1 - x_2 \\ \text{subject to} \quad & -2x_1 + x_2 \leq -1 \\ & -x_1 - 2x_2 \leq -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Suggested Solution. Since all constraints are $\leq$-constraints, we already have the (P)-form. Namely, we have

$$c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}; \; A' = \begin{pmatrix} -2 & 1 \\ -1 & -2 \end{pmatrix}; \; b' = \begin{pmatrix} -1 \\ -2 \end{pmatrix}; \; n = 2; \; m' = 2; \; m'' = 0; \; m = 2.$$

Moreover, since $b' < 0$ we have $\mathcal{I}^- = \{1, 2\}$. To formulate the Problem in standard form (PS), we introduce slack variables s_1, s_2 for each inequality and obtain the LP

$$\begin{aligned} \text{maximize} \quad & x_1 - x_2 + 0s_1 + 0s_2 \\ \text{subject to} \quad & -2x_1 + x_2 + s_1 = -1 \\ & -x_1 - 2x_2 + s_2 = -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Also, we multiply row i by -1 for each $i \in \mathcal{I}^-$ to force nonnegative right hand sides. Letting

$$A = (A' \; E_2) = \begin{pmatrix} 2 & -1 & -1 & 0 \\ 1 & 2 & 0 & -1 \end{pmatrix}; \; b := \begin{pmatrix} 1 \\ 2 \end{pmatrix}; \; x = (x_1, x_2, s_1, s_2)^T; \; c := (1, -1, 0, 0)^T$$

we get

$$\begin{aligned} \text{maximize} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x_1, x_2 \geq 0. \end{aligned} \tag{PS}$$

Since $\mathcal{I}^- = \{1, 2\} = \mathcal{I}^a$ (s_1, s_2 is not a feasible basis), we have to introduce artificial variables for both equations. In this case we use the two-phase simplex method. For this, we formulate the auxiliary problem (A) to be solved in phase I:

$$\begin{aligned} \max \quad & -a_1 - a_2 \\ \text{s.to} \quad & 2x_1 - x_2 - s_1 + a_1 = 1 \\ & x_1 + 2x_2 - s_2 + a_2 = 2 \\ & x_1, x_2, s_1, s_2, a_1, a_2 \geq 0 \end{aligned} \tag{A}$$

The corresponding tableau is

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	basis$^{\mathrm{I}}_{-1}$	ratio
1	0	0	0	0	1	1	0	z	–
0	2	−1	−1	0	1	0	1	a_1	–
0	1	2	0	−1	0	1	2	a_2	–

Since the z-row has to depend on nonbasic variables (NVs) only, we must perform a Gaussian step beforehand to obtain the phase I initial tableau:

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	basis$^{\mathrm{I}}_{0}$	ratio
1	−3	−1	1	1	0	0	−3	z	–
0	2	−1	−1	0	1	0	1	a_1	1/2
0	1	2	0	−1	0	1	2	a_2	2

The coefficient of x_1 has the most negative coefficient. Thus, x_1 enters the basis. Row 1 wins the ratio test so that a_1 leaves the basis. 2 is the pivot element. We now perform a Gauss-step to clear out the 2$^{\text{nd}}$ column above and below the pivot:

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	basis$^{\mathrm{I}}_{1}$	ratio
1	0	−5/2	−1/2	1	3/2	0	−3/2	z	–
0	1	−1/2	−1/2	0	1/2	0	1/2	x_1	–
0	0	5/2	1/2	−1	−1/2	1	3/2	a_2	3/5

Performing the next pivot yields

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	basis$^{\mathrm{I}}_{2}$	ratio
1	0	0	0	0	1	1	0	z	–
0	1	0	−2/5	−1/5	2/5	1/5	4/5	x_1	–
0	0	1	1/5	−2/5	−1/5	2/5	3/5	x_2	–

This tableau is optimal with optimal value $z^* = -a_1 - a_2 = 0$. That is, $a_1 = a_2 = 0$. To start with phase II, we need to substitute the original z-row. Also we can remove the a_1, a_2-columns, since we do not need them any more. We obtain

z	x_1	x_2	s_1	s_2	rhs	basis$^{\mathrm{II}}_{-1}$	ratio
1	−1	1	0	0	0	z	–
0	1	0	−2/5	−1/5	4/5	x_1	–
0	0	1	1/5	−2/5	3/5	x_2	–

To get the initial phase II tableau we have to make sure that the z-row does not depend on the basic variables (here: x_1, x_2). Therefore, we have to perform another Gauss-step:

z	x_1	x_2	s_1	s_2	rhs	basis$^{\mathrm{II}}_{0}$	ratio
1	0	0	−3/5	1/5	1/5	z	–
0	1	0	−2/5	−1/5	4/5	x_1	–
0	0	1	1/5	−2/5	3/5	x_2	–

$\rightsquigarrow$

z	x_1	x_2	s_1	s_2	rhs	basis$^{\mathrm{II}}_{1}$	ratio
1	0	3	0	−1	2	z	–
0	1	2	0	−1	2	x_1	–
0	0	5	1	−2	3	s_1	–

Because there are only nonpositive coefficients in the s_2-column, the corresponding LP is unbounded. (We could let $s_1 = 0$ and $s_2 \to \infty$ without violating the constraints.) To illustrate this, we use the dictionary format:

$$\begin{aligned} z &= 2 + \ \ s_2 \\ \hline x_1 &= 2 + \ \ s_2 \\ x_2 &= 3 + 2s_2. \end{aligned}$$

Exercise 13.4. Solve the following LP using the simplex method:

$$\begin{aligned} \text{maximize} \quad & 4x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \leq 7 \\ & x_1 + 2x_2 \leq 4 \\ & -x_1 + 2x_2 = -2 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Suggested Solution. Since all constraints are $\leq$-constraints, we already have the (P)-form. Here, it is $\mathcal{I}^- = \{\}$, for $b' \geq 0$. Introducing slack variables and multiplying the equations with negative rhs we obtain the problem in standard form:

$$\begin{aligned} \max \quad & 4x_1 - 2x_2 \\ \text{s.to} \quad & 2x_1 + x_2 + s_1 = 7 \\ & x_1 + 2x_2 + s_2 = 4 \\ & x_1 - 2x_2 + a_1 = 2 \\ & x_1, x_2, s_1, s_2, a_1, a_2 \geq 0 \end{aligned} \tag{PS}$$

Because the third constraint is an equation we let $\mathcal{I}^a = \{3\}$, that is, we introduce an artificial variable for the equation constraint and employ the two-phase method. (We want to solve for a variable with the additional requirement that the other constraints do not depend on this variable. For example, if we had to solve the LP

$$\begin{aligned} \text{maximize} \quad & 4x_1 - 2x_2 \\ \text{subject to} \quad & 2x_1 + 0x_2 \leq 7 \\ & x_1 + 0x_2 \leq 4 \\ & -x_1 + x_2 = -2 \\ & x_1, x_2 \geq 0, \end{aligned}$$

then s_1, s_2, x_2 could serve as a basis. We only had to express the objective through nonbasic variables only.)

We obtain the auxiliary problem

$$\begin{aligned} \max \quad & -a_1 \\ \text{s.to} \quad & 2x_1 + x_2 + s_1 = 7 \\ & x_1 + 2x_2 + s_2 = 4 \\ & x_1 - 2x_2 + a_1 = 2 \\ & x_1, x_2, s_1, s_2, a_1, a_2 \geq 0 \end{aligned} \tag{A}$$

Note that we have multiplied the the equation by -1 to ensure a nonnegative rhs. To obtain a initial basic feasible solution (bfs) we express the objective through nonbasic variables only - equivalently: perform a Gauss-step in the following corresponding tableau:

z	x_1	x_2	s_1	s_2	a_1	rhs	basis$^{\mathrm{I}}_{-1}$	ratio
1	0	0	0	0	1	0	z	–
0	2	1	1	0	0	7	s_1	–
0	1	2	0	1	0	4	s_2	–
0	1	-2	0	0	1	2	a_1	–

$\rightsquigarrow$

z	x_1	x_2	s_1	s_2	a_1	rhs	basis$^{\mathrm{I}}_{0}$	ratio
1	-1	2	0	0	0	-2	z	–
0	2	1	1	0	0	7	s_1	$7/2$
0	1	2	0	1	0	4	s_2	4
0	1	-2	0	0	1	2	a_1	2

	z	x_1	x_2	s_1	s_2	a_1	rhs	basis$_1^{\mathrm{I}}$	ratio
	1	0	0	0	0	1	0	z	–
$\rightsquigarrow$	0	0	5	1	0	-2	3	s_1	–
	0	0	4	0	1	-1	2	s_2	–
	0	1	-2	0	0	1	2	x_1	–

The latter tableau is optimal with optimal value $z^* = 0$. Also, there is no artificial basic variables. Thus, (PS) is feasible with feasible solution

$$x_2 = 0,\ x_1 = 2,\ s_1 = 3,\ s_2 = 2.$$

As above, to formulate the phase two initial tableau we have to substitute the original z-row and asure that the objective depends on nonbasic variables only.

z	x_1	x_2	s_1	s_2	rhs	basis$_{-1}^{\mathrm{II}}$	ratio
1	-4	2	0	0	0	z	–
0	0	5	1	0	3	s_1	–
0	0	4	0	1	2	s_2	–
0	1	-2	0	0	2	x_1	–

	z	x_1	x_2	s_1	s_2	rhs	basis$_0^{\mathrm{II}}$	ratio
	1	0	-6	0	0	8	z	–
$\rightsquigarrow$	0	0	5	1	0	3	s_1	$3/5$
	0	0	4	0	1	2	s_2	$1/2$
	0	1	-2	0	0	2	x_1	–

	z	x_1	x_2	s_1	s_2	rhs	basis$_1^{\mathrm{II}}$	ratio
	1	0	0	0	$3/2$	11	z	–
$\rightsquigarrow$	0	0	0	1	$-5/4$	$1/2$	s_1	–
	0	0	1	0	$1/4$	$1/2$	x_2	–
	0	1	0	0	$1/2$	3	x_1	–

This tableau is optimal with optimal value $z^* = 11$ and optimal solution $x_1 = 3,\ x_2 = 1/2,\ s_1 = 1/2$.

Exercise 13.5. Solve the following LP using the simplex method:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 7 \\ & x_1 + 2x_2 \ge 5 \\ & -x_1 + 2x_2 = -2 \\ & x_1, x_2 \ge 0 \end{aligned}$$

Suggested Solution. First, we transform this LP to have $\le$-inequalities only to determine the constraints that need artificial variables:

$$\begin{aligned} \text{maximize} \quad & 2x_1 + x_2 \\ \text{subject to} \quad & 2x_1 + x_2 \le 7 \\ & -x_1 - 2x_2 \le -5 \\ & x_1 - 2x_2 = 2 \\ & x_1, x_2 \ge 0 \end{aligned} \tag{P}$$

Since the 2$^{\text{nd}}$ constraint is the only inequality that has a negative rhs we have $\mathcal{I}^- = \{2\}$. For the equations we always introduce artificial variables. Thus, $\mathcal{I}^a = \{2, 3\}$. Introducing slack variables s_1, s_2 for constraint 1 and 2 leads to the standard form

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.to} \quad & 2x_1 + x_2 + s_1 = 7 \\ & x_1 + 2x_2 - s_2 = 5 \\ & x_1 - 2x_2 = 2 \\ & x_1, x_2, s_1, s_2 \ge 0 \end{aligned} \tag{PS}$$

and to the auxiliary problem

$$\begin{array}{rl} \max & -a_1 - a_2 \\ \text{s.to} & 2x_1 + x_2 + s_1 = 7 \\ & x_1 + 2x_2 - s_2 + a_1 = 5 \\ & x_1 - 2x_2 + a_2 = 2 \\ & x_1, x_2, s_1, s_2, a_1, a_2 \geq 0 \end{array} \tag{A}$$

From this we get the corresponding tableau format

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	$\text{basis}^{\text{I}}_{-1}$	ratio
1	0	0	0	0	1	1	0	z	–
0	2	1	1	0	0	0	7	s_1	–
0	1	2	0	−1	1	0	5	a_1	–
0	1	−2	0	0	0	1	2	a_2	–

As always, we still need a preceding Gauss step to get the initial phase I tableau:

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	$\text{basis}^{\text{I}}_{0}$	ratio
1	−2	0	0	1	0	0	−7	z	–
0	2	1	1	0	0	0	7	s_1	7/2
0	1	2	0	−1	1	0	5	a_1	5
0	1	−2	0	0	0	1	2	a_2	2

$\rightsquigarrow$

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	$\text{basis}^{\text{I}}_{1}$	ratio
1	0	−4	0	1	0	2	−3	z	–
0	0	5	1	0	0	−2	3	s_1	3/5
0	0	4	0	−1	1	−1	3	a_1	3/4
0	1	−2	0	0	0	1	2	x_1	–

$\rightsquigarrow$

z	x_1	x_2	s_1	s_2	a_1	a_2	rhs	$\text{basis}^{\text{I}}_{2}$
1	0	0	4/5	1	0	2/5	−3/5	z
0	0	1	1/5	0	0	−2/5	3/5	x_2
0	0	0	−4/5	−1	1	3/5	3/5	a_1
0	1	0	2/5	0	0	1/5	16/5	x_1

The last tableau is optimal with optimal value $z^* = -\frac{3}{5}$, which is negativ. This means that the original LP is infeasible!

The exercises can be discussed in the Tuesday lecture next week.