Exercise 1 *(Vector norms)***.**

(a) Let $(V, \|\cdot\|)$ be a normed vector space. Prove that

$$\Big|\|x\| - \|y\|\Big| \leq \|x \pm y\|$$

for all $x, y \in V$.

(b) Prove that all norms in $\mathbb{R}^n$ are equivalent. This means that for arbitrary norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ there exists constants α, β such that for all $x \in \mathbb{R}^n$ it is

$$\alpha\|x\|_* \leq \|x\|_{**} \leq \beta\|x\|_*.$$

Hint: It suffices to show that every norm on $\mathbb{R}^n$ is equivalent to the Euclidean norm $\|\cdot\|_2$. Definition 1 and Theorem 2 below could be helpful.

Definition 1. *A set $X \subset (\mathbb{R}^n, \|\cdot\|_2)$ is called* **bounded** *if there exists $C > 0$ such that $\|x\|_2 \leq C$ for all $x \in X$. X is* **open** *$:\Leftrightarrow$ any point of X is an interior point. X is* **closed** *$:\Leftrightarrow$ $\mathbb{R}^n \backslash X$ is open. X is called* **compact** *if it is closed and bounded.* ♢

Theorem 2 (Weierstrass extreme value theorem)**.** *Let $X \subset (\mathbb{R}^n, \|\cdot\|_2)$ be compact and $f\colon X \to \mathbb{R}$ be continuous. Then $f(X)$ is compact and there exist $x_{min}, x_{max} \in X$ such that for all $x \in X$*

$$f(x_{min}) \leq f(x) \leq f(x_{max}). \qquad \square$$

Suggested Solution. **(a)** From the triangle inequality it follows that for all $x, y \in \mathbb{R}^n$

$$\|x\| = \|x + y - y\| \leq \|x + y\| + \|y\|$$

i.e.

$$\|x\| - \|y\| \leq \|x + y\|.$$

Swapping x and y yields

$$\Big|\|x\| - \|y\|\Big| \leq \|x + y\|.$$

Replacing y by $-y$ finally proves the assertion.

(b) We show that any norm $\|\cdot\|$ is equivalent to the Euclidean norm $\|\cdot\|_2$. First, for any $x \in \mathbb{R}^n$ we let $\sum_k x_k e_k := x$. We can estimate

$$\|x\| = \|\sum_k x_k e_k\| \leq \sum_k |x_k| \|e_k\| \leq \|x\|_2 \sum_k \|e_k\| =: \beta\|x\|_2.$$

Next, we show that

$$f\colon \mathbb{R}^n \to \mathbb{R}, \quad f(x) = \|x\|$$

is (Lipschitz) continuous: For any $x, y \in \mathbb{R}^n$ we have

$$|f(x) - f(y)| = \Big|\|x\| - \|y\|\Big| \leq \|x - y\| \leq \beta\|x - y\|_2.$$

Thus for any given $\varepsilon > 0$ and $a \in \mathbb{R}^n$ there exists a $\delta > 0$ (e.g. one can choose $\delta := \frac{\varepsilon}{\beta}$) s.t.

$$|f(x) - f(a)| < \varepsilon \quad \text{for all } x \in U_\delta(a).$$

Now we show that the (Euclidean) unit sphere $\mathbb{S} := \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}$ is compact:

(1) <u>$\mathbb{S}$ is closed:</u> $\Leftrightarrow$ $\mathbb{R}^n \backslash \mathbb{S}$ is open: Let $y \in \mathbb{R}^n \backslash \mathbb{S}$ and $r := |\|y\| - 1|$. We claim that $B_r(y) \subset \mathbb{R}^n \backslash \mathbb{S}$: Assume the contrary. Then $\exists p \in B_r(y) \cap \mathbb{S}$. But then

$$\Big| \|y\|_2 - \underbrace{\|p\|_2}_{=1} \Big| \leq \|y - p\|_2 < r = \Big| \|y\|_2 - 1 \Big|$$

yields a contradicion.

(2) <u>$\mathbb{S}$ is bounded:</u> for all $x \in \mathbb{S}:\ \|x\|_2 \leq C = 1$.

Now we can apply the Weierstrass extreme value theorem to ensure the existence of a $x_{\min} \in \mathbb{S}$ ($\Rightarrow x_{\min} \neq 0$) s.t.

$$\|x\| = f(x) \geq f(x_{\min}) = \|x_{\min}\| =: \alpha > 0 \quad \forall x \in \mathbb{S}.$$

For arbitrary $x \in \mathbb{R}^n$ we note that $\frac{x}{\|x\|_2} \in \mathbb{S}$ and therefore

$$\left\| \frac{x}{\|x\|_2} \right\| \geq \alpha \quad \Leftrightarrow \quad \alpha \|x\|_2 \leq \|x\|.$$

Exercise 2 *(matrix norms)***.** Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and $\|\cdot\|_*$ the induced matrix norm. Prove the following

(a) $\|\cdot\|_*$ is a vector norm on (the vector space) $\mathbb{R}^{n\times n}$.

(b) $\|\cdot\|_*$ is compatible with $\|\cdot\|$, i.e. $\|Ax\| \leq \|A\|_* \|x\|$ for all $A \in \mathbb{R}^{n\times n},\ x \in \mathbb{R}^n$.

(c) $\|\cdot\|_*$ is submultiplicative, i.e. for all $A, B \in \mathbb{R}^{n\times n}$ one has

$$\|AB\|_* \leq \|A\|_* \|B\|_*.$$

(d) $\|E_n\|_* = 1$.

(e) If $|||\cdot|||$ is another vector norm on $\mathbb{R}^{n\times n}$ that is compatible with $\|\cdot\|$ then

$$\|A\|_* \leq |||A|||$$

for all $A \in \mathbb{R}^{n\times n}$.

Suggested Solution.

(a) <u>definiteness:</u> We show: $\|A\|_* = 0 \ \Rightarrow\ A = 0$. If

$$0 = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

then $Ax = 0$ for all $x \in \mathbb{R}^n$, i.e. $A = 0$.

<u>positive homogeneity:</u>

$$\|\lambda A\|_* = \sup_{x\neq 0} \frac{\|\lambda A x\|}{\|x\|} = |\lambda| \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} = |\lambda| \|A\|_*.$$

<u>triangle inequality:</u>

$$\|A + B\|_* = \sup_{x\neq 0} \frac{\|(A+B)x\|}{\|x\|}$$

$$\begin{aligned}
&\leq \sup_{x\neq 0} \frac{\|Ax\| + \|Bx\|}{\|x\|} \\
&\leq \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} + \sup_{x\neq 0} \frac{\|Bx\|}{\|x\|} \\
&= \|A\|_* + \|B\|_*
\end{aligned}$$

(b) compatibility: We conclude

$$\begin{aligned}
\|A\|_* = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} &\geq \frac{\|Ax\|}{\|x\|} \\
\Leftrightarrow \quad \|Ax\| &\leq \|A\|_* \|x\|.
\end{aligned}$$

(c) submultiplicativity: Let $A, B \in \mathbb{R}^{n\times n}$. Then

$$\|AB\|_* = \max_{\|x\|=1} \|A(Bx)\| \overset{(b)}{\leq} \max_{\|x\|=1} \|A\|_* \|Bx\| = \|A\|_* \max_{\|x\|=1} \|Bx\| = \|A\|_* \|B\|_*.$$

(d) $\|E\|_* = 1$: We have

$$\|E\|_* = \sup_{x\neq 0} \frac{\|Ex\|}{\|x\|} = \sup_{x\neq 0} \frac{\|x\|}{\|x\|} = 1.$$

Alternatively:

$$\|E\|_* = \max_{\|x\|=1} \|Ex\| = \max_{\|x\|=1} \|x\| = 1.$$

(e) $\|A\|_* \leq |||A|||$: Simply apply sup over $\mathbb{R}^n \backslash \{0\}$ to

$$\frac{\|Ax\|}{\|x\|} \leq \frac{|||A||| \|x\|}{\|x\|} = |||A|||.$$

Exercise 3 *(Spectral norm and spectral radius)***.** Let $A \in \mathbb{R}^{n\times n}$. Prove that for the spectral norm on $\mathbb{R}^n$ it holds

$$\|A\|_2 = \sqrt{\rho(A^T A)},$$

where for $B \in \mathbb{R}^{n\times n}$

$$\rho(B) \coloneqq \max_{\lambda \in \sigma(B)} |\lambda|$$

denotes the largest absolute eigenvalue of B and is called the ***spectral radius*** of B .

Hint: Use that $A^T A$ as a symmetric matrix has an orthonormal eigenbasis and a positive spectrum (i.e. only real eigenvalues ≥ 0). Then, represent each vector x in the equation

$$\|A\|_2 = \max_{\|x\|_2=1} \langle Ax, Ax \rangle^{\frac{1}{2}} = \max_{\|x\|_2=1} (x^T A^T A x)^{\frac{1}{2}}$$

with respect to this basis and compute the resulting expression.

Suggested Solution. Since A^TA is symmetric there exists an orthonormal eigenbasis $\{v_1, \ldots, v_n\}$ with $Av_i = \lambda_i v_i$ and the positive semi-definiteness of A^TA yields $\lambda_i \geq 0$. For arbitrary $x \in \mathbb{R}^n$ we let $\sum_j \mu_j v_j \coloneqq x$. Then

$$\begin{aligned}
\langle Ax, Ax\rangle &= \langle x, A^TAx\rangle \\
&= \langle \sum_i \mu_i v_i, A^TA \sum_j \mu_j v_j\rangle \\
&= \sum_{i,j} \mu_i\mu_j \langle v_i, \underbrace{A^TAv_j}_{=\lambda_j v_j}\rangle \\
&= \sum_{i,j} \mu_i\mu_j\lambda_j \underbrace{\langle v_i, v_j\rangle}_{=\delta_{ij}} \\
&= \sum_i \mu_i^2\lambda_i && (1) \\
&\leq \lambda_{i_0} \sum_i \mu_i, && (2)
\end{aligned}$$

where $\lambda_{i_0} = \rho(A^TA)$ is the largest eigenvalue of A^TA. Specializing to $A \coloneqq E$ and $\|x\|_2 = 1$ in (1) yields $\sum_i \mu_i^2 = 1$ (and further $\mu_i^2 = 1$ when substituting $x = v_i$). From this and (2) we can, on the one hand, estimate for all normalized x

$$\|Ax\|_2^2 \leq \lambda_{i_0}$$

and therefore

$$\|A\|_2 \leq \sqrt{\lambda_{i_0}}.$$

Letting $x \coloneqq v_i$ in (1) we obtain on the other hand

$$\max_{\|x\|_2=1} \|Ax\|_2 \geq \|Av_{i_0}\|_2 = \sqrt{\mu_{i_0}^2\lambda_{i_0}} = \sqrt{\lambda_{i_0}}.$$

The exercises can be discussed in the Tuesday lecture next week.