Exercise 14.1 *(LU decomposition)*.

1. Let $A \in \mathbb{R}^{3\times 3}$ be given with
$$A = \begin{pmatrix} 2 & -1 & 2 \\ -4 & 5 & -2 \\ 2 & -10 & -3 \end{pmatrix}.$$
Compute the LU decomposition of A, i.e. determine a unipotent lower triangular matrix L and an upper triangular matrix U such that $A = LU$ is valid.

2. Let $A \in \mathbb{R}^{3\times 3}$ be given with
$$A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -4 \end{pmatrix}.$$
Compute the LU decomposition (with permutations) of A, i.e. determine a unipotent lower triangular matrix L, an upper triangular matrix U and a permutation matrix P such that $PA = LU$ is valid. For comparability reasons do not use partial pivoting.

3. Solve the system of linear equations $Ax = b$, where
$$A = \begin{pmatrix} 2 & 1 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \\ -13 \end{pmatrix},$$
by forward and backward substitution.

Exercise 14.2 *(Cholesky decomposition)*. Let $A \in \mathbb{R}^{3\times 3}$ be given with
$$A = \begin{pmatrix} 4 & -2 & 6 \\ -2 & 17 & -11 \\ 6 & -11 & 17 \end{pmatrix}.$$
Determine the Cholesky decomposition of A, i.e. determine a lower triangular matrix $G \in \mathbb{R}^{n\times n}$ with positive diagonal entries, such that $A = GG^T$.

Exercise 14.3 *(condition number)*. Compute $\mathrm{cond}_2(A)$ and $\mathrm{cond}_F(A)$ where
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Exercise 14.4 *(Linear Least-Squares Problem)*. Determine the optimal least-squares fit of the model function
$$y = f(x; k_0, k_1) = k_0 \sin\left(\frac{\pi}{8} x\right) + k_1 \sqrt{|x|}$$
to the data points

x	-4	0	4	16
y	3.9	0.1	7.9	12.1

Exercise 14.5. Consider the system of equations
$$\begin{aligned} x^2 - x - y &= 0 \\ x + y - 2 &= 0 \end{aligned}$$
Apply Newton's method twice with the starting point $x^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Exercise 14.6. Let $A \in \mathbb{R}^{n\times n}$ be symmetric an positive definite. Show that

$$\langle \cdot, \cdot \rangle_A \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \; \langle x, y \rangle_A \coloneqq \langle Ax, y \rangle,$$

is a scalar product on $\mathbb{R}^n$. Here, $\langle \cdot, \cdot \rangle$ is the Euclidean scalar product on $\mathbb{R}^n$.

Exercise 14.7. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Show that

$$\max_{1\le i,j\le n} |a_{ij}| = \max_{1\le i\le n} |a_{ii}|.$$

Hint: One of many possible solutions would be to use the result from Exercise 14.6 and apply the Cauchy-Schwarz inequality.

Exercise 14.8. Let $n \in \mathbb{N}$ be arbitrary.

1. Show that the mapping
$$\rho \colon \mathbb{R}^{n\times n} \to \mathbb{R}, \; \rho(A) \coloneqq \max_{\lambda\in\sigma(A)} |\lambda|$$
defines no matrix norm in $\mathbb{R}^{n\times n}$.

2. Show that ρ defines a matrix norm on the subset of all symmetric matrices in $\mathbb{R}^{n\times n}$.

Exercise 14.9. Show that for all symmetric $A, B \in \mathbb{R}^{n\times n}$ it holds that

$$\rho(A + B) \le \rho(A) + \rho(B).$$