
Exercise 1. Determine the LU decomposition with row swaps, i.e. L, U and the corresponding permutation vector p for the following matrices. Then solve the system $Ax = e_1$. Here, $e_1 = (1, 0, \ldots, 0)^T$.

a) $A = \begin{pmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix}$ b) $A = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 2 & 2 & 4 \\ 1 & 1 & 2 & 4 \end{pmatrix}$.

Exercise 2. Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ be arbitrary. The Frobenius norm is defined by

$$\|A\|_F = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{\frac{1}{2}}.$$

Proof the following:

(a) For $n \geq 2$ the Frobenius norm is not induced by any norm.
Hint: Exercise sheet 1, Exercise 2(d).

(b) $\|A\|_F^2 = \operatorname{trace}(A^T A)$

(c) The Frobenius norm is compatible with the Euclidean norm $\|\cdot\|_2$, i.e. for all $x \in \mathbb{R}^n$ it is

$$\|Ax\|_2 \leq \|A\|_F \|x\|_2.$$

Hint: Consider the squared inequality and use the (squared) Cauchy-Schwarz inequality to estimate $\|Ax\|_2^2$.

(d) $\|A\|_2 \leq \|A\|_F$.
Hint: Use subtask (c)

Exercise 3. Let $A \in \mathbb{R}^{n \times n}$ be ***nilpotent***, i.e. for some $N \in \mathbb{N}$ it is

$$A^N = 0.$$

Are the following statements true or false (give a reason for each!):

(a) $\det(A) = 0$

(b) $\sigma(A) = \{0\}$

(c) $\operatorname{trace}(A) = 0$

(d) $A = 0$ or A is not diagonalizable

(e) $E - A$ is invertible

Exercise 4. Write a program that will implement the LU decomposition (without permutations) preferably with Octave/Matlab. The program should receive an arbitrary $(n \times n)$-matrix A as input and the matrices L (unipotent lower triangular matrix) and U (upper triangular matrix) as output, such that

$$A = LU.$$

Test your program with the matrices from Sheet 3, Exercise 3.

The exercises can be discussed in the Tuesday lecture next week.