Exercise 11.1. In Exercise 7.1 we used Banach's fixed point theorem to approximate a zero of the function

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = e^{-x} - x.$$

In this exercise we use Newton's method as a second method with initial guess $x_0 = 1/3$.

(a) Show that there exists exactly one zero $\hat{x} \in I := (0,1)$ of f.

(b) Show that Newton's method is well-defined on the interval I and that the Newton sequence converges locally quadratically on I.

(c) Determine the iteration function Φ.

(d) Compute the first 4 iterates (up to machine precision).

Exercise 11.2. Consider an iteration method of the form

$$x_{k+1} = \Phi(x_k)$$

with $\Phi\colon \operatorname{dom}(\Phi) \subset \mathbb{R} \to \mathbb{R}$. Suppose Φ has a fixed point $\hat{x}$. Answer the following questions with reasons.

1. Under which conditions is the iteration method locally convergent?
2. What conditions do you need for local quadratic convergence of the iteration method?
3. When does the iteration method have order of convergence 3?

Exercise 11.3. Consider Newton's method for determining a zero of a function.

1. Determine the respective iteration function Φ.
2. Calculate the derivatives Φ' and Φ'' in dependence on f and its derivatives.
3. Specify sufficient conditions on f (and its derivatives) so that the conditions (a) to (c) in Exercise 11.2 are satisfied.

Exercise 11.4. Given are the two curves

$$K_1 = \{(x,y) \in \mathbb{R}^2 \mid y = 1 + x^2\} \quad \text{and} \quad K_2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 = 1 + y^2\}.$$

Sought for is the minimum distance d of these two curves, i.e.

$$d = \inf_{v \in K_1, w \in K_2} \|v - w\|_2.$$

Sketch the two curves and give a non-linear system of equations to determine the minimum distance points on the curves. (We will also solve this system numerically on sheet 12 numerically).

Exercise 11.5. Use Newton's method to determine a zero of each of the functions

$$f(x) = x^4 - 4x^2 + 4$$

and

$$g(x) = x^4 - 6x^2 + 8.$$

Use the initial value $x^{(0)} = 1$ in both cases and determine the first 7 iterates in each case. Why is the observed speed of convergence significantly lower in the case of f?

Merry Christmas and a happy New Year! :-)