Exercise 12.1. Consider Newton's method to approoximate a zero of

$$f(x) = x^3 - x.$$

Give four initial values for which the Newton method does not converge.

Hint: Investigate whether the iterates are well-defined at all or whether they only jump back and forth between two values.

Exercise 12.2. Assume that a function f has an m-fold zero $\hat{x}$ in the interior of the interval $[a, b]$ with $a, b \in \mathbb{R},\ a < b$, and is of the form

$$f(x) = (x - \hat{x})^m g(x),$$

where $g \in C^3([a, b])$ with $g(\hat{x}) \neq 0$.

(a) Determine the corresponding iteration mapping for Newton's method in this case.

(b) Calculate the value $\Phi'(\hat{x})$. What can you conclude from this about the local convergence of the method?

(c) Show that the local quadratic convergence can be recovered using the modified Newton method

$$x^{(k+1)} = x^{(k)} - m\frac{f(x^{(k)})}{f'(x^{(k)})}.$$

Exercise 12.3. Consider the mapping $F\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$F(x) = Ax - b,$$

where $A \in \mathrm{Gl}(n)$ and $b \in \mathbb{R}^n$. Show that the general Newton method in this case yields the exact solution for each initial value already after the first iteration step.

Exercise 12.4. Determine a solution to the following system of nonlinear equations using Newton's method (cf. Exercise 11.5):

$$F(x_1, x_2) = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 6x_1 - 2x_2 + 4x_1\left(x_1^2 - \sqrt{x_2^2 - 1}\right) \\ -2x_1 + 4x_2 - \frac{2x_2(1+x_1^2)}{\sqrt{x_2^2-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Use the initial guess $x^{(0)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(a) Formulate Newton's method for this specific case.

<u>Hint:</u> You can use the following derivatives:

$$\begin{aligned}
\partial_{x_1} F_1(x(1), x(2)) &= \texttt{12*x(1)^2 - 4*(x(2)^2 - 1)^(1/2) + 6} \\
\partial_{x_2} F_1(x(1), x(2)) &= \texttt{-(4*x(1)*x(2))/(x(2)^2-1)^(1/2)-2} \\
\partial_{x_1} F_2(x(1), x(2)) &= \partial_{x_2} F_1(x(1), x(2)) \\
\partial_{x_2} F_2(x(1), x(2)) &= \texttt{(2*x(1)^2+2)/(x(2)^2-1)^(3/2)+4}
\end{aligned}$$

(b) Implement Newton's method for this case in Octave and calculate $x^{(7)}$.

(c) As in (b) but with damping factor $\lambda = 0.8$. Compute in this case also $x^{(20)}$.

The exercises can be discussed on 17.01.2022.