Exercise 8.1. Consider the SLE $Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{pmatrix} \quad \text{und} \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Show that the Jacobi method converges to the solution of the SLE for each initial vector. Determine the associated Gauss-Seidel operator and calculate the first two iterations to the initial vector $x^{(0)} = (1, 1, 1)^T$.

(b) Show that the Gauss-Seidel method for the iterative solution of the system does not converge in general. Determine the associated Gauss-Seidel operator and calculate the first two iterations to the starting vector $x^{(0)} = (1, 1, 1)^T$.

Exercise 8.2. Given $A = \begin{pmatrix} 1 & \alpha & -\alpha \\ 0 & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix} \in \mathbb{R}^{3\times 3}$ $(\alpha \neq -1)$ and an arbitrary $b \in \mathbb{R}^3$. For which α does the Jacobi and Gauss-Seidel method converge for any initial guess $x^{(0)}$ to the solution of $Ax = b$?:

(a) If you apply the criterion of strict diagonal dominance (Prop. 3.9).

(b) If you apply Prop. 3.4.

Exercise 8.3. For $n \in \mathbb{N}$ let $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^n$ be given as follows:

$$A = \left(\begin{array}{c|ccc} 1 & \frac{1}{n} & \cdots & \frac{1}{n} \\ \hline \frac{1}{2} & & & \\ \vdots & & E & \\ \frac{1}{2} & & & \end{array}\right), \quad b = \begin{pmatrix} 2 \\ \vdots \\ n \end{pmatrix},$$

where E denotes the $(n-1)$-dimensional unit matrix.

(a) Justify why, for the SLE $Ax = b$, both the Jacobi and the Gauss-Seidel method converge to the solution $x = A^{-1}b$ for each initial vector.

(b) Implement the Jacobi method for the SLE $Ax = b$ and the matrix dimension $n = 1000$. To do this, write an `Octave` function that requires as input values the vector b, the initial vector $x^{(0)}$ and the value $N_{\max}$ = number of iterations as input values and returns the approximate solution of $Ax = b$ as return value. Iterate directly, i.e. without saving A or $\mathcal{J}$, or D, L, U. Compare the computing time for 20 iterations with the duration of the calculation of the LU decomposition of A, using the program from Exercise 4.4. (To determine the computing time you can use the command `tic` to start a stop watch and `toc` to stop it.)

(c) Using an effort-only approach and the result of (b), estimate the time difference between calculating the LU decomposition and calculating 20 iterations of the Jacobi method for the matrix dimension $n = 100000$.

Remark: "Effort-only approach" means that you set the (asymptotic) numerical effort for $n = 1000$ and $n = 100000$ in relation to each other and equate this quotient with the ratio of the corresponding runtimes.

The exercises can be discussed in the Tuesday lecture next week.