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Exercise Sheet 7

Stochastics (AAI)

Exercise 7.1

Compute the following probabilities approximately using the central limit theorem:

a) $P(\{\sum_{i=1}^{400} X_i > 210\})$ for $X_1, \dots, X_{400}$ i.i.d. with $X_1 \sim \mathrm{Exp}(2)$,

b) $P(\{X > 600\})$ and $P(\{X \in [100, 450[\})$ for $X \sim \mathrm{B}(1000, 0.6)$,

c) $P(\{|\sum_{i=1}^{50} X_i - 150| > 100\})$ for $X_1, \dots, X_{50}$ i.i.d. with $X_1 \sim \mathrm{Poi}(3)$.

Exercise 7.2

Let $X_1, \dots, X_5$ be i.i.d. with $X_1 \sim \mathrm{N}(\mu, 4)$ and unknown $\mu \in \mathbb{R}$. For the estimation of the expected value μ we consider the estimators $g_5^{(i)} : \mathbb{R}^5 \to \mathbb{R}$ for $i = 1, \dots, 4$ given by

$$\begin{aligned}
g_5^{(1)}(x_1, \dots, x_5) &= \frac{1}{3}(x_1 + x_3 + x_5), \\
g_5^{(2)}(x_1, \dots, x_5) &= \frac{1}{5}\sum_{i=1}^{5} x_i, \\
g_5^{(3)}(x_1, \dots, x_5) &= x_1 + x_4, \\
g_5^{(4)}(x_1, \dots, x_5) &= x_3.
\end{aligned}$$

a) Determine the bias of $g_5^{(i)}$ for $i = 1, \dots, 4$.

b) Determine the mean squared error of $g_5^{(i)}$ for $i = 1, \dots, 4$.

Exercise 7.3

Let $n \in \mathbb{N}$ and $X_1, \dots, X_n$ be i.i.d. with $\mathrm{Var}(X_1) > 0$. Let $g_n : \mathbb{R}^n \to \mathbb{R}$ be given by

$$g_n(x_1, \dots, x_n) = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)^2.$$

Prove or disprove:

a) g_n is an unbiased estimator for $\mathrm{Var}(X_1)$, i.e., $\mathrm{E}(g_n(X_1, \dots, X_n)) = \mathrm{Var}(X_1)$.

b) $g_n(X_1, \dots, X_n)$ converges almost surely to $\mathrm{Var}(X_1)$, i.e.,

$$P\Big(\Big\{\lim_{n\to\infty} g_n(X_1, \dots, X_n) = \mathrm{Var}(X_1)\Big\}\Big) = 1.$$

Exercise 7.4

Let $n \in \mathbb{N}$ and $X_1, \ldots, X_n$ be i.i.d. with $X_1 \sim \mathrm{U}(0, b)$ and $b \in \,]0, \infty[$. For the estimation of b we consider the estimators $g_n^{(1)}, g_n^{(2)} \colon \mathbb{R}^n \to \mathbb{R}$ given by

$$g_n^{(1)}(x_1, \ldots, x_n) = \frac{2}{n} \cdot \sum_{i=1}^{n} x_i, \qquad g_n^{(2)}(x_1, \ldots, x_n) = \frac{n+1}{n} \cdot \max(x_1, \ldots, x_n).$$

a) Show that $\max(X_1, \ldots, X_n)$ is absolutely continuous with density $f \colon \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} n \cdot x^{n-1}/b^n, & \text{if } x \in [0, b], \\ 0, & \text{else.} \end{cases}$$

Hint: Determine $P(\{\max(X_1, \ldots, X_n) \leq x\})$ *for* $x \in \mathbb{R}$. *Cf. Exercise 6.1 a).*

b) Show that $g_n^{(1)}$ and $g_n^{(2)}$ are unbiased estimators.

c) Compute the mean squared error of $g_n^{(1)}$ and $g_n^{(2)}$.
Hint: For $n \in \mathbb{N}$ *we have*

$$\left(\frac{n+1}{n}\right)^2 \cdot \left(\frac{n}{n+2} - \left(\frac{n}{n+1}\right)^2\right) = \frac{1}{n(n+2)}.$$

Exercise 7.5

Let $P_X^{\vartheta} \sim \mathrm{Exp}(\lambda)$ with $\lambda = \vartheta \in \Theta = \,]0, \infty[$. Determine the maximum likelihood estimator of λ.

Exercise 7.6* (P)

Let $X_1, X_2, \ldots$ be i.i.d. with $\mu = \mathrm{E}(X_1) \in \mathbb{R}$ and $\sigma = \sqrt{\mathrm{Var}(X_1)} > 0$, and define

$$\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i \qquad \text{and} \qquad \overline{X}_n^* = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma}$$

for $n \in \mathbb{N}$. In the sequel consider the two cases

i) $X_1 \sim \mathrm{Exp}(1)$ with $\mu = \sigma = 1$,

ii) P_{X_1} discrete uniform distribution on $\{1, \ldots, 6\}$ with $\mu = 7/2$ and $\sigma = \sqrt{35/12}$,

separately. Proceed as follows to illustrate the strong law of large numbers (SLLN) and the central limit theorem (CLT).

a) Generate realizations $X_1(\omega_j), \ldots, X_n(\omega_j)$ for $j = 1, \ldots, m$ (i.e., generate m samples of size n).
Hint: `exprnd`, `randi` (Matlab/Octave)

b) Illustration of SLLN: Put $m = 10$ and $n = 10^3$. For each sample $j = 1, \ldots, m$ plot $k \mapsto \overline{X}_k(\omega_j)$ for $k = 1, \ldots, n$.

c) Illustration of CLT: Put $m = 10^3$ and $n = 10$. For each $k = 1, \ldots, n$ plot a histogram of $\overline{X}_k^*$ based on the data $\overline{X}_k^*(\omega_1), \ldots, \overline{X}_k^*(\omega_m)$.