Exercise 1 *(Reading - Getting started with Octave/Matlab)***.** -

Exercise 2 *(Programming)***.** It is (without proof)

$$\sum_{j=1}^{\infty} \frac{1}{j^2} = \frac{\pi^2}{6}.$$

How large must N be chosen in

$$\sum_{j=1}^{N} \frac{1}{j^2} \tag{1}$$

in order to calculate π with an error $\leq 10^{-6}$? To determine such a number N, write a Matlab/Octave script that can calculate the sum in (1) above. Find the solution by trying.

Suggested Solution. The associated program can look like this, for example:

```
function y = sheet1_ex2(N)
  j = 1:N;
  aux = 1./(j.*j);
  y = sqrt(sum(aux)*6);
  err=abs(y-pi)
end
```

This script has to be stored in a separate file with filename (equal to the function name) `sheet1_ex2.m`. The distance from π to the number `sheet1_ex2(954935)` is smaller than 10^{-6}, i.e. one can choose $N \geq 954935$.

Exercise 3 *(Math)***.** A symmetric matrix $A \in \mathbb{R}^{n\times n}$ is called positive definite if

$$\langle x, Ax\rangle = x^T Ax > 0 \quad \forall x \in \mathbb{R}^n,\ x \neq 0 \tag{2}$$

holds. Proof the following

(a) For symmetric $A \in \mathbb{R}^{n\times n}$ we have

$$A \text{ is positive definite} \Leftrightarrow \text{all eigenvalues of } A \text{ are strictly positive}$$

Hint: For the reverse direction, use that the symmetric matrix A has an orthonormal basis of eigenvectors, and then represent x in Ax with respect to this particular basis.

(b) If $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is symmetric and positive definite, then all the diagonal elements of A are strictly positive, i.e.

$$a_{ii} > 0 \quad i = 1, \dots, n.$$

Suggested Solution.

(a) <u>"$\Rightarrow$":</u> Let $\lambda \in \mathbb{R}$ be any eigenvalue of A with corresponding eigenvector $v \neq 0$. Then

$$0 < \langle v, Av\rangle = \langle v, \lambda v\rangle = \lambda \underbrace{\|v\|^2}_{>0} \quad \Leftrightarrow \quad \lambda > 0.$$

<u>"$\Leftarrow$":</u> By the symmetry of A there exists an orthonormal eigenbasis $\{v_1, \dots, v_n\}$ with $Av_i = \lambda_i v_i$ and $\lambda_i > 0$ for all i. Let $x \in \mathbb{R}^n$ be arbitrary and $\sum_j \mu_j v_j \coloneqq x$. Then

$$\begin{aligned}
\langle x, Ax\rangle &= \langle \sum_i \mu_i v_i, A \sum_j \mu_j v_j\rangle \\
&= \langle \sum_i \mu_i v_i, \sum_j \mu_j \underbrace{Av_j}_{=\lambda_j v_j}\rangle \\
&= \langle \sum_i \mu_i v_i, \sum_j \mu_j \lambda_j v_j\rangle \\
&= \sum_{i,j} \mu_i \mu_j \lambda_j \underbrace{\langle v_i, v_j\rangle}_{=\delta_{ij}} \\
&= \sum_i \mu_i^2 \underbrace{\lambda_i}_{>0}.
\end{aligned}$$

Since $x \neq 0$ there exists a least one index i_0 s.t. $\mu_{i_0}^2 > 0$ and thus we have that $\langle x, Ax\rangle > 0$ for all $x \neq 0$, yielding the positive definiteness of A.

<u>Remark</u>: Writing

$$\begin{Bmatrix} A > 0 \\ A \geq 0 \\ A < 0 \\ A \leq 0 \end{Bmatrix} \text{ when } A \text{ is symmetric and } \begin{Bmatrix} \text{positive definite} \\ \text{positive semidefinite} \\ \text{negative definite} \\ \text{negative semidefinite} \end{Bmatrix}$$

we analogously obtain

(1) $A > 0 \;\Leftrightarrow$ all $\lambda_i > 0$
(2) $A \geq 0 \;\Leftrightarrow$ all $\lambda_i \geq 0$
(3) $A < 0 \;\Leftrightarrow$ all $\lambda_i < 0$
(4) $A \leq 0 \;\Leftrightarrow$ all $\lambda_i \leq 0$
(5) A is indefinite $\Leftrightarrow\; \exists i, j$ s.t. $\lambda_i < 0$ and $\lambda_j > 0$

(b) Letting $x \coloneqq e_i$ $(i = 1, \dots, n)$ we obtain

$$0 < \langle x, Ax\rangle = \langle e_i, Ae_i\rangle = \langle e_i, a_i\rangle = a_{ii},$$

where a_i denotes the i-th column of A.

The exercises can be discussed in the Tuesday lecture next week.