Exercise 7.1 *(The Banach fixed point theorem (BFT))***.** For the function

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = e^{-x} - x$$

a zero in the interval $I = [\frac{1}{3}, 1]$ is to be calculated approximately. For this, approximate a fixed point of the function

$$\Phi\colon I \to \mathbb{R},\ \Phi(x) := e^{-x}.$$

(a) Show that Φ is a self map on I. (Can you do without a calculator here if you know that $e < 3$?)

(b) Show that Φ is contracting with contraction factor $q < 1$ (in other words: show that Φ is Lischitz continuous with Lipschitz constant $q < 1$).

(c) Using the BFT it follows from 1. and 2. that there exists exactly one fixed point $\hat{x} \in I$. Let $x_0 := \frac{1}{3}$ be the initial guess. (Clearly, you could choose any other $x_0 \in I$. But choosing x_0 as above allows you to directly compare your solution to the suggested solution..) Use the a priori estimates of the BFT to determine the maximum number $N_0 = N_0(\varepsilon, x_0) \in \mathbb{N}$ of iterations needed to determine $\hat{x}$ to an accuracy of (an unspecified) $\varepsilon > 0$. Determine N_0 for $\varepsilon := 10^{-5}$.

(d) Using `Octave`, calculate the first N_0 iterates of the iteration sequence (x_k) induced by the iteration rule

$$x_{k+1} = \Phi(x_k).$$

(e) Use the a posteriori estimate of the BFT to obtain an improved estimate for the error $|\hat{x} - x_{N_0}|$.

(f) Use your calculator to compute the best possible iterate (using the "ANS-technique").

The exercises can be discussed in the Tuesday lecture next week.