Numerical Methods and Optimization (Part I)

6 Linear Optimization

11 January 2023

- Linear programming (LP): methodology for solving "linear optimization problems".
- "Simplex Method" introduced by George Dantzig in the late 1940s
- maximize linear objective function $f\colon \mathbb{R}^n \to \mathbb{R}$

$$f(x) = \sum_{j=1}^{n} c_j x_j = c^T x$$

subject to linear equality and/or inequality constraints

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i \qquad \text{for } i = 1, \dots, m$$

$$\sum_{j=1}^{n} a'_{ij} x_j \ge b'_i \qquad \text{for } i = 1, \dots, m'$$

$$\sum_{j=1}^{n} a''_{ij} x_j = b''_i \qquad \text{for } i = 1, \dots, m''$$

- If $x, y \in \mathbb{R}^n$ we write $x \le y \;\Leftrightarrow\; x_i \le y_i$.
  Constraints: $Ax \le b,\; A'x \ge b',\; A''x = b''$.

- formulating an LP Model:

Example

- Heavenly Pouch Inc. (HP) produces 2 types of baby carriers: non-reversible and reversible

- reversible: sells for $£35$, requires 2 linear yards of a printed fabric, 2 linear yards of solid color fabric $\rightsquigarrow$ costs $£10$ to manufacture
- non-reversible: sells for $£23$, requires 2 linear yards of solid color fabric $\rightsquigarrow$ costs $£8$ to manufacture
- available: 900 lin. yards of solid color fabric, 600 lin. yards of printed fabrics
- budget of $£4000$ is available
- demand is such that all reversible carriers made are projected to sell
- at most 350 non-reversible carriers can be sold
- HP want to maximize profit (difference between revenues and expenses) resulting from manufacturing and selling

[0] Picture source:https://www.youtube.com/watch?v=U59bs1U5aAs&list=PLY9yf2-4yyeTeRnUIiROr_ojkjuQkLx2b&index=4

- defining the **decision variables**:

  $\rightsquigarrow$ variables determining the outcome (we can control):

$$x_1 = \#\text{ non-reversible carriers to manufacture}$$
$$x_2 = \#\text{ reversible carriers to manufacture}$$

- formulating the objective function:
  total revenue: $r = 23x_1 + 35x_2$.
  total manufacturing costs: $c = 8x_1 + 10x_2$.
  Thus, profit (linear function of decision variables):

$$z = r - c = (23x_1 + 35x_2) - (8x_1 + 10x_2) = 15x_1 + 25x_2.$$

- specifying the constraints: (often associated with limited availability of various types of resources (such as materials, time, space, money):
  - company has only 900 linear yards of solid color fabrics available. Each baby carrier uses 2 linear yds of solid color fabrics $\rightsquigarrow 2x_1 + 2x_2 \leq 900$, i.e.

    $$x_1 + x_2 \leq 450 \qquad \text{(solid color fabric constraint)}$$

  - at most 600 linear yards of printed fabrics can be used; printed fabrics are used only for reversible carriers (2 lin. yds/carriers) $\rightsquigarrow\ 2x_2 \leq 600$, i.e.

    $$x_2 \leq 300 \qquad \text{(printed fabric constraint)}$$

  - the manufacturing budget is limited to $\pounds 4000$; HP spends $\pounds 8$/non-reversible and $10\pounds$/reversible carrier $\rightsquigarrow\ 8x_1 + 10x_2 \leq 4000$, i.e.

    $$4x_1 + 5x_2 \leq 2000 \qquad \text{(budget constraint)}$$

  - at most 350 non-reversible carriers can be sold $\rightsquigarrow$

    $$x_1 \leq 350 \qquad \text{(demand constraint)}$$

  - nonnegativity constraints: since x_1 and x_2 represent quantities of physical objects, their values must be nonnegative by definition:

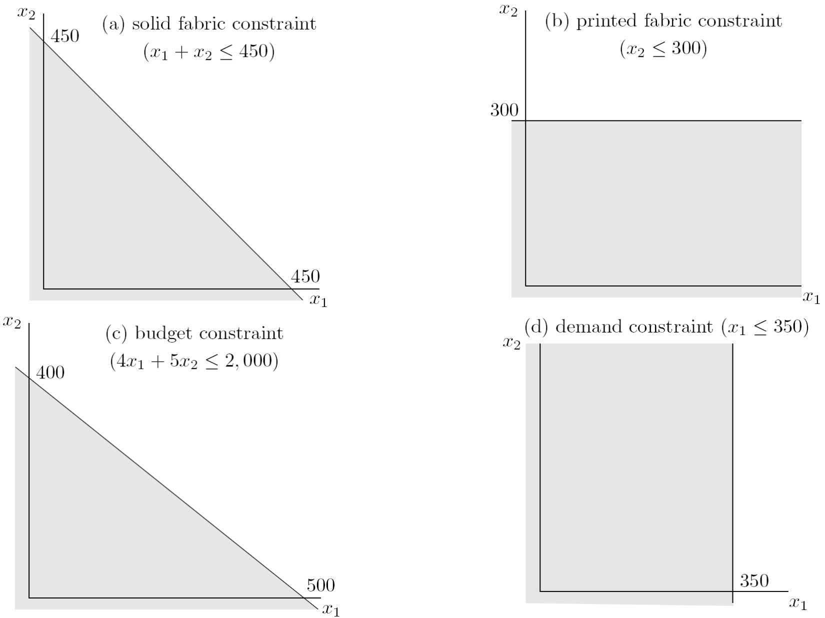
    $$x_1, x_2 \geq 0 \qquad \text{(nonnegativity constraint)}$$

complete linear programming formulation

We obtain the following linear program (LP) that models the considered problem:

$$
\begin{array}{rrcl r}
\text{maximize} & 15x_1 + 25x_2 & & & \text{(profit)} \\
\text{subject to} & x_1 + x_2 & \leq & 450 & \text{(solid color fabric constraint)} \\
& x_2 & \leq & 300 & \text{(printed fabric constraint)} \\
& 4x_1 + 5x_2 & \leq & 2000 & \text{(budget constraint)} \\
& x_1 & \leq & 350 & \text{(demand constraint)} \\
& x_1, x_2 & \geq & 0 & \text{(nonnegativity constraints)}
\end{array}
$$

Solving two-variable LPs graphically: If LP involves only 2 var. $\rightsquigarrow$ can be solved graphically, by plotting lines representing the constraints and level sets of the objective function:



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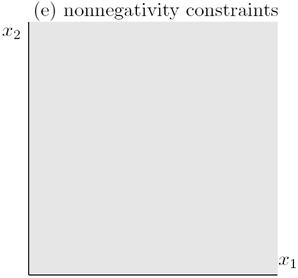
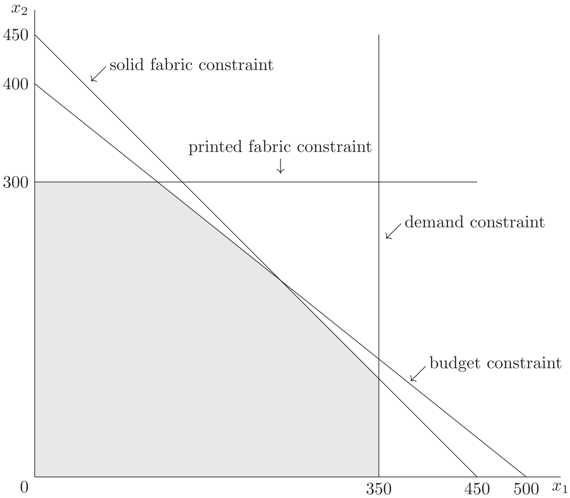


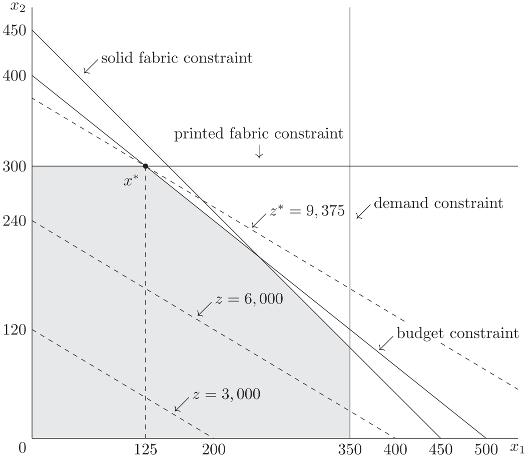


Intersection of all these half-planes with the nonnegative quadrant of the plane will give us the **feasible region** of the problem = set of all points that satisfy all the constraints.

[0] Picture source: ISBN 978-1-4665-7778-7

- to solve LP graphically, we use level sets z_c of the objective function f

$$z_c = f^{-1}(\{c\}) = \{\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{R}^2 \mid f(x_1, x_2) = c\}.$$



- z_c may or may not overlap with the feasible region
- one can start with z_c, where $c := f(x, y)$ and $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$ is a feasible point, e.g. $\left(\begin{smallmatrix} 200 \\ 0 \end{smallmatrix}\right)$ leads to $f(200, 0) = 3000$ or $\left(\begin{smallmatrix} 0 \\ 240 \end{smallmatrix}\right)$ leads to $f(0, 240) = 6000$
- now one can move the lines in parallel until z_c takes on the largest possible value $z^* = f(x^*)$.
Here, $x^* = \begin{pmatrix} x_1^* \\ x_2^* \end{pmatrix} = \left(\begin{smallmatrix} 125 \\ 300 \end{smallmatrix}\right)$.

- <u>This yields the optimal profit</u> $f(x_1^*, x_2^*) = £15 \cdot 125 + £25 \cdot 300 = £9375.$

0 Picture source: ISBN 978-1-4665-7778-7

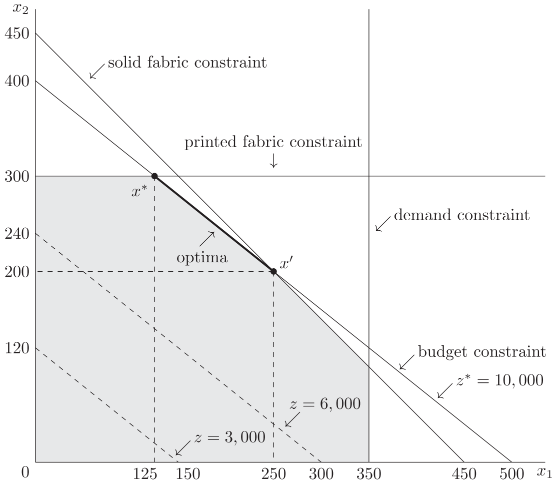
Example (10.8 [Butenko])

Consider again HP ex. above. Assume non-reversible carriers raised by $£5$ so that now $z = 20x_1 + 25x_2$. Solve the LP graphically again.

Here, we have

$$
\begin{array}{llll}
\text{maximize} & 20x_1 + 25x_2 & & \text{(profit)} \\
\text{subject to} & x_1 + x_2 & \leq 450 & \text{(solid color fabric constraint)} \\
 & x_2 & \leq 300 & \text{(printed fabric constraint)} \\
 & 4x_1 + 5x_2 & \leq 2000 & \text{(budget constraint)} \\
 & x_1 & \leq 350 & \text{(demand constraint)} \\
 & x_1, x_2 \geq 0 & & \text{(nonnegativity constraints)}
\end{array}
$$

Note that level sets z_c have same slope $m = \frac{4}{5}$ as budget constraint! Thus these lines are parallel.



- Thus, every $x = \binom{x_1}{x_2}$ between x^* and x' is optimal
- For these we have $z^* = f(x) = 10000$.
- Extreme Point: Given convex set $X \subset \mathbb{R}^n$. $x \in X$ is called **extreme point** of X if there do not exist distinct $x', x'' \in X,\ \alpha \in (0,1)$ s.t. $x = \alpha x' + (1-\alpha)x''$.
- in this context: an extreme point is also called **corner** or **vertex**.

- later: any LP that has an optimal sol. must have a corner optimum

0 Picture source: ISBN 978-1-4665-7778-7

Example (10.9 [Butenko])

- Retail store planning advertising campaign.
- Aim: increase number of customers visiting phy. location and online store
- planned to advertise through local magazine and online social network
- It is estimated:
  - $£1000$ invested in magazine ads will attract 100 new cust. in store/500 on website
  - $£1000$ invested in online ads will attract 50 new cust. in store/1000 on website
- Target:
  - 500 new visitors to phy. store
  - 5000 new visitors to online store

Formulate LP to minimize costs of advertising campain; solve LP graphically.

Solution: Decision variables:

$$x_1 = \text{budget for magazine advertising (in thousands of pounds)}$$

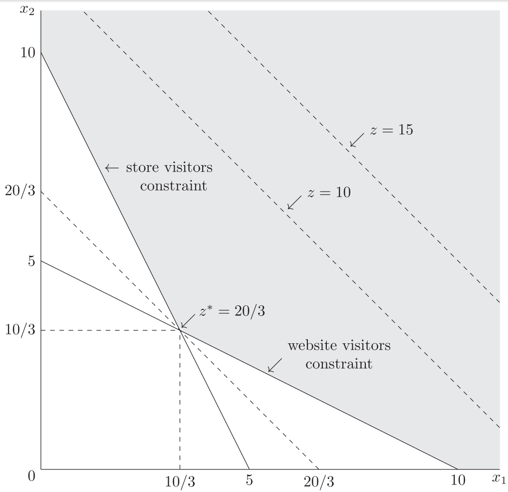
$$x_2 = \text{budget for online advertising (in thousands of pounds)}$$

$$\begin{array}{lrll} \text{minimize} & x_1 + x_2 & & \\ \text{subject to} & 100x_1 + 50x_2 \geq 500 & & \text{(store visitors)} \\ & 500x_1 + 1000x_2 \geq 5000 & & \text{(website visitors)} \\ & x_1, x_2 \geq 0 & & \text{(nonnegativity)} \end{array}$$

- start drawing boundary of each constraint
- obtain feasible region
- draw two level sets, e.g. $z_c = 10$ and $z_c = 15$
- thus z decreases to lower left
- clearly, z^* (see figure) is the global minimum
- z^* is the solution of

$$\begin{aligned} 100x_1 + \quad 50x_2 &= 500 \\ 500x_1 + 1000x_2 &= 5000 \end{aligned}$$

- i.e. $x_1^* = x_2^* = 10/3$
- $\Rightarrow z^* = x_1^* + x_2^* \approx 6.666$



- $\rightsquigarrow$ Store should spend $£6666$ on ads and split budget evenly between magazine and online ads
- Note: although feasible region unbounded $\rightsquigarrow$ optimal solution exists; However, if f was improving (i.e. decreasing) along direction of unboundedness of feasible region: then optimal sol. would not exist

[0] Picture source: ISBN 978-1-4665-7778-7

Example (10.10 [Butenko])

Consider same problem (as above) with only difference that objective function is to be maximized:

$$
\begin{array}{rlll}
\text{maximize} & x_1 + x_2 & & \\
\text{subject to} & 100x_1 + 50x_2 \geq 500 & & \text{(store visitors)} \\
& 500x_1 + 1000x_2 \geq 5000 & & \text{(website visitors)} \\
& x_1, x_2 \geq 0 & & \text{(nonnegativity)}
\end{array}
$$

Sol.: Clearly, objective fct. $\to \infty$ if $x_1 \to \infty$ or $x_2 \to \infty$. //.

- LP s.t. objective is unbounded on feasible region is highly unlikely in practice
- next: LP example in decision-making situation with limited resources
- insufficient resources may lead to
  - nonexistence of optimal sol.
  - a feasible region even being empty

Example (10.11 [Butenko])

Assume retail store (as above) has advertising budget limited to $\pounds 5000$.
Corresponding LP model:

$$
\begin{array}{rrcl}
\text{minimize} & x_1 + x_2 & & \\
\text{subject to} & 100x_1 + 50x_2 \geq 500 & & \text{(store visitors)} \\
& 500x_1 + 1000x_2 \geq 5000 & & \text{(website visitors)} \\
& x_1 + x_2 \leq 5 & & \text{(budget constraint)} \\
& x_1, x_2 \geq 0 & & \text{(nonnegativity)}
\end{array}
$$

Sol.:

- As can be seen from figure above, halfspace determined from budget constraint does not intersect feasible region from before.
- Thus, feasible region does not exist for this LP

Classification of LPs:

LPs can be classified in terms of their feasibility and optimality properties.

Definition

An LP is called

- **feasible** if it has at least one feasible solution and **infeasible**, otherwise;
- **optimal** if it has an optimal solution;
- **unbounded** if it is feasible and its objective function is not bounded (from above for a maximization problem and from below for a minimization problem) in the feasible region.

For example:

- HP ex. above: optimal and ($\Rightarrow$) feasible
- ex. 10.8 (HP with infinitely many optima): optimal and feasible
- ex. 10.9 (advertising campaign with unlimited budget): optimal and feasible (but the LP is **not** unbounded although the feasible region is unbounded)
- ex. 10.10 (advertising campaign with maximization): feasible, not optimal. feasible region and LP unbounded
- ex. 10.11 infeasible, ($\Rightarrow$) not optimal (and the LP is not unbounded)

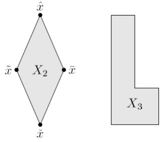
Theorem (will be proven later)

If an LP is not optimal, it is either unbounded or infeasible.

Definition

(Polyhedral Set) A set defined by linear equations and/or inequalities is called a **polyhedral set** or a **polyhedron**.

E.g. X_2: polyhedron; X_3: no polyhedron



Proposition

An optimal LP has either a unique or infinitely many solutions. Moreover, the set of all optimal solutions of an LP is convex.

Proof

Consider, say, a maximization LP $\max_{x \in X} c^T x$, X *a polyhedron and assume we have 2 different optimal sol.* x^*, x' *with optimal objective value* $z^* = c^T x* = c^T x'$*. Then any convex combination of* x^* *and* x', $y = \alpha x^* + (1-\alpha)x' \underbrace{\in X}_{ex!}$, *is also an optimal sol.:* $c^T y = c^T(\alpha x^* + (1-\alpha)x') = \alpha c^T x^* + (1-\alpha)c^T x' = \alpha z^* + (1-\alpha) z^* = z^*$

[0] Picture source: ISBN 978-1-4665-7778-7

- Here, we discuss one of the first and most popular methods for solving LPs: **simplex method**.
- originally proposed by George Dantzig in late 1940s for solving problems arising in military operations.
- To apply this method: first convert LP into **standard form**.

The Standard Form of LP

- LP in **standard form**: only equality and nonnegativity constraints
- An inequality constraint is easily converted into equality constraint:
- e.g. if i^{th} constraint is of $\leq$-type: introduce new variable (**slack variable**) $s_i \geq 0$ in the lhs to turn it into equality constraint
- example: $2x_1 + 3x_2 + 4x_3 \leq 5$ becomes $2x_1 + 3x_2 + 4x_3 + s_i = 5$, where

$$s_i = 5 - 2x_1 - 3x_2 - 4x_3 \geq 0.$$

- similarly: if i^{th} constraint is of $\geq$-type: introduce new variable (**excess variable**) $e_j \geq 0$ in the lhs to turn it into equality constraint
- example: $2x_1 + 3x_2 + 4x_3 \geq 5$ becomes $2x_1 + 3x_2 + 4x_3 - e_j = 5$, where

$$e_j = 2x_1 + 3x_2 + 4x_3 - 5 \geq 0.$$

Example

The standard form of the LP

$$\begin{array}{rrcl} \text{maximize} & 3x_1 - 5x_2 + 7x_3 & & \\ \text{subject to} & 2x_1 + 4x_2 - x_3 & \geq & -3 \\ & 4x_1 - 2x_2 + 8x_3 & \leq & 7 \\ & 9x_1 + x_2 + 3x_3 & = & 11 \\ & x_1, x_2, x_3 & \geq & 0 \end{array}$$

is given by

$$\begin{array}{rrcl} \text{maximize} & 3x_1 - 5x_2 + 7x_3 & & \\ \text{subject to} & 2x_1 + 4x_2 - x_3 - e_1 & = & -3 \\ & 4x_1 - 2x_2 + 8x_3 + s_2 & = & 7 \\ & 9x_1 + x_2 + 3x_3 & = & 11 \\ & x_1, x_2, x_3, e_1, s_2 & \geq & 0 \end{array}$$

- required in standard form: var. nonnegative
- in practice sometimes: $x \in \mathbb{R}$ (free variables)
- represent $x_j \in \mathbb{R}$ as difference of 2 nonnegative var.: $x_j = x'_j - x''_j,\ x'_j, x''_j \geq 0$

Example

The standard form of the LP

$$\begin{array}{rrcl}
\text{maximize} & 3x_1 - 5x_2 + 7x_3 & & \\
\text{subject to} & 2x_1 + 4x_2 - x_3 & \geq & -3 \\
& 4x_1 - 2x_2 + 8x_3 & \leq & 7 \\
& 9x_1 + x_2 + 3x_3 & = & 11 \\
& x_1 \in \mathbb{R};\ x_2, x_3 & \geq & 0
\end{array}$$

is given by

$$\begin{array}{rrcl}
\text{maximize} & 3x_1' - 3x_1'' - 5x_2 + 7x_3 & & \\
\text{subject to} & 2x_1' - 2x_1'' + 4x_2 - x_3 - e_1 & = & -3 \\
& 4x_1' - 4x_1'' - 2x_2 + 8x_3 + s_2 & = & 7 \\
& 9x_1' - 9x_1'' + x_2 + 3x_3 & = & 11 \\
& x_1', x_1'', x_2, x_3, e_1, s_2 & \geq & 0
\end{array}$$

- After multiplying by -1 (if necessary), we can assume that all constraints are in $\leq$-form.
- Consider general LP

$$\begin{aligned} \text{maximize} \quad & c^T x & \\ \text{subject to} \quad & A'x \leq b' & (m' \text{ inequalities}) \\ & A''x = b'' & (m'' \text{ equalities}) \\ & x \geq 0 & (x \in \mathbb{R}^n) \end{aligned}$$

Then the **standard form** of the above LP is

$$\begin{aligned} \text{maximize} \quad & \bar{c}^T \bar{x} & \\ & A\bar{x} = b & (m := m' + m'' \text{ equalities}) \\ & \bar{x} \geq 0 & (\bar{x} \in \mathbb{R}^{n+m'}) \end{aligned}$$

where (s_i slack variables)

$$\bar{x} = (x_1, \dots, x_n, s_1, \dots, s_{m'})^T,$$

$$\bar{c} = (c_1, \dots, c_n, 0, \dots, 0)^T \in \mathbb{R}^{n+m'}$$

$$A = \begin{pmatrix} A' & E_{m'} \\ A'' & 0 \end{pmatrix} \in \mathbb{R}^{m \times (n+m')},$$

$$b = (b'_1, \dots, b'_{m'}, b''_1, \dots, b''_{m''})^T \in \mathbb{R}^m$$

The Simplex Method:

- At first, we restrict to LPs

$$\begin{aligned} \text{maximize} \quad & c^T x \\ & Ax \leq b, \quad (b \geq 0) \\ & x \geq 0 \end{aligned}$$

- reason for restrictive form $Ax \leq b$ and $b \geq 0$:
  - initial feasible solution is $x^{(0)} = 0$ (satisfies all constraints!)
  - simplex method needs "starting feasible solution" $x^{(0)}$
  - needed to generate finite sequence $x^{(0)}, x^{(1)}, x^{(2)}, \ldots, x^{(N)}$ with property
    - $z(x^{(k)}) \leq z(x^{(k+1)})$ for $k = 0, \ldots, N-1$ and
    - $x^* = x^{(N)}$ is optimal sol. to LP
  - if cond. $b \geq 0$ was dropped: finding starting point more challenging ($\rightarrow$ later)
- we introduce basic idea using HP LP

Consider again the LP (see above; HP LP)

$$\begin{array}{rlr} \text{maximize} & 15x_1 + 25x_2 & \text{(profit)} \\ \text{subject to} & x_1 + x_2 \le 450 & \text{(solid color fabric constraint)} \\ & x_2 \le 300 & \text{(printed fabric constraint)} \\ & 4x_1 + 5x_2 \le 2000 & \text{(budget constraint)} \\ & x_1 \le 350 & \text{(demand constraint)} \\ & x_1, x_2 \ge 0 & \text{(nonnegativity constraints)} \end{array}$$

First, convert LP into standard form (using slack var.):

$$\begin{array}{rl} \text{maximize} & 15x_1 + 25x_2 \\ \text{subject to} & x_1 + x_2 + s_1 = 450 \\ & x_2 + s_2 = 300 \\ & 4x_1 + 5x_2 + s_3 = 2000 \\ & x_1 + s_4 = 350 \\ & x_1, x_2, s_1, s_2, s_3, s_4 \ge 0 \end{array}$$

In the sequel we'll represent this standard form LP

$$
\begin{array}{ll}
\text{maximize} & 15x_1 + 25x_2 \\
\text{subject to} & x_1 + x_2 + s_1 = 450 \\
& x_2 + s_2 = 300 \\
& 4x_1 + 5x_2 + s_3 = 2000 \\
& x_1 + s_4 = 350 \\
& x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
\end{array}
$$

in the equivalent **dictionary format**

$$
\begin{array}{rcrcrcr}
z & = & & & 15x_1 & + & 25x_2 \\
\hline
s_1 & = & 450 & - & x_1 & - & x_2 \\
s_2 & = & 300 & & & - & x_2 \\
s_3 & = & 2000 & - & 4x_1 & - & 5x_2 \\
s_4 & = & 350 & - & x_1 & &
\end{array}
$$

- set of var. on the lhs of right-hand box: **basic variables (BV)**; here: $\{s_1, \ldots, s_4\}$ obviously: # basic var. = # (inequality) constraints
- set of remaining var. in rhs box: **non-basic variables (NV)**; here: $\{x_1, x_2\}$ obviously: # non-basic var. = # descision var.
- BVs and NVs will be updated step by step; using operation called **pivot**.
- set of basic and non-basic var. at step k denoted by BV_k and NV_k respectively
- here: $BV_0 = \{s_1, \ldots, s_4\}$ and $NV_0 = \{x_1, x_2\}$
- note: get feasible sol. by setting $x_i = 0 \ \forall x_i \in NV \rightsquigarrow$ uniquely det. BVs and z
- here: $x_1 = x_2 = 0 \ \Rightarrow \ s_1 = 450, s_2 = 300, s_3 = 2000, s_4 = 350$ and $z = 0$
- this sol. is called **basic solution** corresponding to basis BV_0

- this sol. is called: **basic solution** corr. to basis BV_0
- if all var. in basic sol. are nonnegative: we call it **basic feasible solution (bfs)**
- corresponding dictionary is called: **feasible**
- note: here: basic sol. is a bfs and the corr. dictionary is thus feasible

$$\begin{array}{rcrrr}
z = & 0 & +\ 15x_1 & +\ 25x_2 \\
\hline
s_1 = & 450 & -\ x_1 & -\ x_2 \\
s_2 = & 300 & & -\ x_2 \\
s_3 = & 2000 & -\ 4x_1 & -\ 5x_2 \\
s_4 = & 350 & -\ x_1 &
\end{array}$$

- Our LP can also be converted to **tableau format** (to apply el. row op.s):

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	basis
1	-15	-25	0	0	0	0	0	z
0	1	1	1	0	0	0	450	s_1
0	0	1	0	1	0	0	300	s_2
0	4	5	0	0	1	0	2000	s_3
0	1	0	0	0	0	1	350	s_4

- entries of tableau are just coeff. of LP in standard form
- in z-row all var. have moved to lhs, e.g. instead of $z = 15x_1 + 25x_2$ we write $z - 15x_1 - 25x_2 = 0$
- dictionary good for explanation; tableau format handy for computations
- top row: row 0 or z-row
- row i corresponds to i^{th} constraint

Step 1 (of simplex method):

$$\begin{array}{rcrcrcr} z & = & & & 15x_1 & + & 25x_2 \\ \hline s_1 & = & 450 & - & x_1 & - & x_2 \\ s_2 & = & 300 & & & - & x_2 \\ s_3 & = & 2000 & - & 4x_1 & - & 5x_2 \\ s_4 & = & 350 & - & x_1 & & \end{array}$$

- for the bfs we have $x_1 = x_2 = 0 \;\Rightarrow\; z = 0$
- z is expressed in terms of NVs only $\rightsquigarrow$ only way to change value of z is by changing at least one of NVs from 0 to value ≥ 0 (note: all var.≥ 0)
- to increase z value, we can increase any NV with coeff. > 0 in z-row of dictionary
- z fct. is linear, it makes sense to incr. var. with highest coeff. $\to$ **pivot variable**
- here: since $z = 15x_1 + 25x_2$ we pick x_2 as pivot variable
- corr. column is called **pivot column**
- want to increase pivot var. as much as possible while keeping other NVs$= 0$
- amount by which we can increase pivot var. is restricted by nonnegativity constraints of the BVs (to ensure feasibility)!
- note: here, x_1 is ignored since $x_1 = 0$
- largest possible (feasible) value for x_2 is $x_2 = 300$
- more generally: largest possible increase corr. to smallest ratio of free coeff. to the absolute value of coeff. of pivot var. x_2 in case it is negative

$$\begin{array}{rcrcl} s_1 & = & 450 & - & x_2 \geq 0 \\ s_2 & = & 300 & - & x_2 \geq 0 \\ s_3 & = & 2000 & - & 2x_2 \geq 0 \\ s_4 & = & 350 & - & 0x_2 \geq 0 \end{array}$$

- ratios: $\frac{450}{1}, \frac{300}{1}, \frac{2000}{2}$ $\rightsquigarrow$ smallest ratio is $\frac{300}{1}$ $\rightsquigarrow$ corr. row wins **ratio test**
- rows with coeff. ≥ 0 for x_2 can be ingnored $\rightsquigarrow$ always satisfied; e.g. if we had $500 + 5x_2 \geq 0 \rightsquigarrow$ always satisfied (since $x_2 \geq 0$)

$$\begin{array}{rcrcrcr} z & = & & & 15x_1 & + & 25x_2 \\ \hline s_1 & = & 450 & - & x_1 & - & x_2 \\ s_2 & = & 300 & & & - & x_2 \\ s_3 & = & 2000 & - & 4x_1 & - & 5x_2 \\ s_4 & = & 350 & - & x_1 & & \end{array}$$

- row winning ratio test$\rightsquigarrow$ **pivot row**; here: 2^{nd} row
- to "carry out pivot": solve for pivot variable (NV) (here: x_2) in pivot row (here: 2^{nd}): $x_2 = 300 - s_2$
- now substitute this expression in remaining rows of dictionary:

$$\begin{aligned} z &= 15x_1 + 25x_2 &&= 15x_1 + 25(300 - s_2) &&= 7500 + 15x_1 - 25s_2 \\ s_1 &= 450 - x_1 - x_2 &&= 450 - x_1 - (300 - s_2) &&= 150 - x_1 + s_2 \\ s_3 &= 2000 - 4x_1 - 5x_2 &&= 2000 - 4x_1 - 5(300 - s_2) &&= 500 - 4x_1 + 5s_2 \\ s_4 &= 350 - x_1 - 0x_2 && &&= 350 - x_1 \end{aligned}$$

- we obtain step 1 dictionary:

$$\begin{array}{rcrcrcr} z & = & 7500 & + & 15x_1 & - & 25s_2 \\ \hline s_1 & = & 150 & - & x_1 & - & s_2 \\ s_2 & = & 300 & & & - & x_2 \\ s_3 & = & 500 & - & 4x_1 & + & 5s_2 \\ s_4 & = & 350 & - & x_1 & & \end{array}$$

- now: perform same step using tableau

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	basis	ratio
1	-15	-25	0	0	0	0	0	z	
0	1	1	1	0	0	0	450	s_1	450
0	0	1	0	1	0	0	300	s_2	300
0	4	5	0	0	1	0	2000	s_3	400
0	1	0	0	0	0	1	350	s_4	–

- find most negative coeff. in z-row
- $\rightsquigarrow$ corresponds to x_2
- corresponding col.: pivot col.
- perform ratio test:
- divide entries in rhs col. by corresponding pos.(!) entries in pivot col.
- min. ratio (300) corresponds to 2nd row $\rightsquigarrow$ pivot row
- intersection of pivot row and column $\rightsquigarrow$ **pivot element**
- perform pivot:use el. row op.s; goal:clear out the pivot col. above/below pivot el.
- in particular: mult. pivot row by 25,-1 and -5 and add result to rows 0,1,3 resp.
- x_2 is now BV and s_2 is now NV
- result is step 1 tableau:

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	basis
1	-15	0	0	25	0	0	7500	z
0	1	0	1	-1	0	0	150	s_1
0	0	1	0	1	0	0	300	x_2
0	4	0	0	-5	1	0	500	s_3
0	1	0	0	0	0	1	350	s_4

compare tableau to corr. dictionary (see above)

$$
\begin{array}{rcl}
z & = & 7500 + 15x_1 - 25s_2 \\
\hline
s_1 & = & 150 - x_1 + s_2 \\
x_2 & = & 300 \qquad\quad - s_2 \\
s_3 & = & 500 - 4x_1 + 5s_2 \\
s_4 & = & 350 - x_1
\end{array}
$$

- $BV_1 = \{s_1, x_2, s_3, s_4\}$; $NV_1 = \{x_1, s_2\}$; x_2: **entering var.**; s_2: **leaving var.**
- step 1 bfs: $x_1 = 0, x_2 = 300, s_1 = 150, s_2 = 0, s_3 = 500, s_4 = 350;\ z = 7500$

Recall:

$$\begin{array}{rcrrr} z = & 7500 & + 15x_1 & - 25s_2 \\ \hline s_1 = & 150 - & x_1 + & s_2 \\ x_2 = & 300 & - & s_2 \\ s_3 = & 500 - & 4x_1 + & 5s_2 \\ s_4 = & 350 - & x_1 & \end{array}$$

Step 2 (of simplex method):

- step 2 analogously to 1^{st} step
- only x_1 has pos. coeff. in z-row; only entering var. candidate
- row 3 wins ratio test $\rightsquigarrow$ leaving var.
- solving for x_1 in row 3 gives

$$x_1 = 125 + \frac{5}{4}s_2 - \frac{1}{4}s_3;$$

substituting in remaining rows yields

$$z = 9375 - \frac{25}{4}s_2 - \frac{15}{4}s_3$$
$$s_1 = 25 - \frac{1}{4}s_2 + \frac{1}{4}s_3$$
$$s_4 = 225 - \frac{5}{4}s_2 + \frac{1}{4}s_3$$

obtain step 2 dictionary:

$$\begin{array}{rcrrr} z = & 9375 - & \frac{25}{4}s_2 - & \frac{15}{4}s_3 \\ \hline s_1 = & 25 - & \frac{1}{4}s_2 + & \frac{1}{4}s_3 \\ x_2 = & 300 & - & s_2 \\ x_1 = & 125 + & \frac{5}{4}s_2 - & \frac{1}{4}s_3 \\ s_4 = & 225 - & \frac{5}{4}s_2 + & \frac{1}{4}s_3 \end{array}$$

- now we carry out step 2 in the tableau format

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	basis	ratio
1	-15	0	0	25	0	0	7500	z	
0	1	0	1	-1	0	0	150	s_1	150
0	0	1	0	1	0	0	300	x_2	–
0	4	0	0	-5	1	0	500	s_3	125
0	1	0	0	0	0	1	350	s_4	350

- find most negative coeff. in z-row
- $\rightsquigarrow$ corresponds to x_1 (entering var.)
- row 3 wins ratio test: (s_3 is the leaving var.)

- apply el. row op.s to clear out 2nd col., i.e. obtain $(0,0,0,1,0)^T$. obtain:

z	x_1	x_2	s_1	s_2	s_3	s_4	rhs	basis
1	0	0	0	$25/4$	$15/4$	0	9375	z
0	0	0	1	$1/4$	$-1/4$	0	25	s_1
0	0	1	0	1	0	0	300	x_2
0	1	0	0	$-5/4$	$-1/4$	0	125	x_1
0	0	0	0	$5/4$	$-1/4$	1	225	s_4

- tableau is equivalent to corr. dictionary.

Step 2 basic feasible solution	
BV_2 :	x_1, x_2, s_1, s_4
NV_2 :	s_2, s_3
bfs :	$x_1 = 125, x_2 = 300$
	$s_1 = 25, s_2 = 0, s_3 = 0, s_4 = 225$
	$z = 9375$

$$
\begin{array}{rcl}
z & = & 9375 - \frac{25}{4}s_2 - \frac{15}{4}s_3 \\
\hline
s_1 & = & 25 - \frac{1}{4}s_2 + \frac{1}{4}s_3 \\
x_2 & = & 300 \qquad\qquad - s_2 \\
x_1 & = & 125 + \frac{5}{4}s_2 - \frac{1}{4}s_3 \\
s_4 & = & 225 - \frac{5}{4}s_2 + \frac{1}{4}s_3
\end{array}
$$

Recognizing optimality

- note: $s_2, s_3 \geq 0$
- highest possible value for z: iff $s_2 = s_3 = 0$
- thus current bfs is optimal
- can ignore slack var.: were not part of the original LP

- thus, optimal solution is: $x_1^* = 125,\ x_2^* = 300,\ z^* = 9375$
- (same sol. as obtained graphically)

Remark

If in a feasible dictionary, all nonbasic variables have nonpositive coefficients in the z-row, then the corresponding basic feasible solution is an optimal solution of the LP.

If we use the tableau format, then the basic feasible solution is optimal if all nonbasic variables have nonnegative coefficients in row 0 of the corresponding tableau.

Recognizing unbounded LPs:

- step of simplex method:
  1) select entering var., 2) select leaving var., 3) perform pivot (el. row op.s)
- any (NV) var. with pos. coeff. in z-row of dictionary (/neg. coeff. in tableau) is entering variable candidate
- if there is no such var.: current bfs is optimal $\rightsquigarrow$ LP is solved
- leaving var. is basic var. representing a row that wins ratio test
- however, if all coeff. of pivot col. are pos. $\rightsquigarrow$ ratio test produces no result

$$\begin{array}{rcl} z &=& 90 - 25x_1 + \ \ 4x_2 \\ \hline s_1 &=& 25 - 14x_1 + \quad x_2 \\ s_2 &=& 30 - \quad x_1 \\ s_3 &=& 12 + \ \ 5x_1 + 14x_2 \\ s_4 &=& 22 - \ \ 4x_1 + \ 7x_2. \end{array}$$

- e.g., consider dictionary on rhs
- ratio test $\rightsquigarrow$ non of the rows participate!
- since coeff. for x_2 is nonneg. in each row: increasing x_2 does not violate any constraint
- thus, $x_2 \to \infty \Rightarrow z \to \infty \rightsquigarrow$ problem unbounded
- tableau version: all coeff. nonpos. in row $i \geq 1$

Remark

If during the execution of the simplex method we encounter a variable that has all nonnegative coefficients in the dictionary format, then the LP is unbounded.

In tableau format, an LP is proved to be unbounded as soon as a column with no positive entries is detected.

- the simplex alg. applied to HP LP produced sequence of bfs s.t. $z_{\text{step }k+1} > z_{\text{step }k}$
- is not always the case; for example, consider the following LP:

$$\begin{array}{rl} \text{maximize} & x_2 \\ \text{subject to} & -x_1 + x_2 \le 0 \\ & x_1 \quad\quad \le 2 \\ & x_1, x_2 \ge 0 \end{array}$$

dictionary format:

$$\begin{array}{l} z = \qquad\qquad x_2 \\ \hline x_3 = 0 + x_1 - x_2 \\ x_4 = 2 - x_1 \end{array}$$

- here: $BV_0 = \{x_3, x_4\}$ (slack var.); $NV_0 = \{x_1, x_2\}$
- we obtain initial basic feasible solution: $\text{bfs}_0 = \{x_1 = 0,\ x_2 = 0,\ x_3 = 0,\ x_4 = 2\}$

Definition

Basic solutions with one or more basic variables equal to 0 are called degenerate.

- the corr. tableau of above LP is

z	x_1	x_2	x_3	x_4	rhs	basis$_0$
1	0	-1	0	0	0	z
0	-1	1	1	0	0	x_3
0	1	0	0	1	2	x_4

z	x_1	x_2	x_3	x_4	rhs	basis$_0$
1	0	-1	0	0	0	z
0	-1	1	1	0	0	x_3
0	1	0	0	1	2	x_4

$\rightsquigarrow$

z	x_1	x_2	x_3	x_4	rhs	basis$_1$
1	-1	0	1	0	0	z
0	-1	1	1	0	0	x_2
0	1	0	0	1	2	x_4

entering var.: x_2; leaving var.: x_3;
bfs$_0 = \{x_1 = x_2 = x_3 = 0, x_4 = 2\}$

entering var.: x_1; leaving var.: x_4;
bfs$_0 = \{x_1 = x_2 = x_3 = 0, x_4 = 2\}$

z	x_1	x_2	x_3	x_4	rhs	basis$_2$
1	0	0	1	1	2	z
0	0	1	1	1	2	x_2
0	1	0	0	1	2	x_1

$\rightsquigarrow$

dictionary format:

$$\begin{array}{rcl} z &=& 2 - x_3 - x_4 \\ \hline x_1 &=& 2 \phantom{{}- x_3} - x_4 \\ x_2 &=& 2 - x_3 - x_4 \end{array}$$

optimal solution: $x_1 = x_2 = 2$

Definition

An iteration of the simplex method, which results in a new basis with the basic feasible solution that is identical to the previous basic feasible solution ($\Rightarrow z$ remains unchanged) is called a **degenerate iteration** and the corresponding phenomenon is referred to as **degeneracy**. (corr. dictionary/LP is called **degenerate**)

- degenerate LPs can lead to "cycling":

$$\begin{aligned} \text{maximize} \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} \quad & \tfrac{1}{2}x_1 - \tfrac{11}{2}x_2 - \tfrac{5}{2}x_3 + 9x_4 \le 0 \\ & \tfrac{1}{2}x_1 - \tfrac{3}{2}x_2 - \tfrac{1}{2}x_3 + x_4 \le 0 \\ & x_1 + x_2 + x_3 + x_4 \le 1 \\ & x_1, x_2, x_3, x_4 \ge 0 \end{aligned}$$

- we apply simplex method to this LP (using tableau format)
- let x_5, x_6 and x_7 be the slack var. for the respective constraints; obtain:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_0$
1	-10	57	9	24	0	0	0	0	z
0	$\frac{1}{2}$	$-\frac{11}{2}$	$-\frac{5}{2}$	9	1	0	0	0	x_5
0	$\frac{1}{2}$	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	1	0	0	x_6
0	1	1	1	1	0	0	1	1	x_7

- we use the following **pivoting rule**
  - always choose NV with the most neg. coeff. in row 0 ("Dantzig's rule"; tie-breaking rule: choose randomly within candidates)
  - in case of multiple ratio test winners: choose BV with lowest index
- here: pivot col.: 2^{nd}; entering variable: x_1 leaving variable: x_5 (row 2 and 3 both win ratio test; x_5 has smallest index)

- (again) initial tableau

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis_0
1	-10	57	9	24	0	0	0	0	z
0	$1/2$	$-11/2$	$-5/2$	9	1	0	0	0	x_5
0	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0	x_6
0	1	1	1	1	0	0	1	1	x_7

- x_1 enters; x_5 leaves (in a tie for smallest ratio); obtain 1st tableau

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis_1
1	0	-53	-41	204	20	0	0	0	z
0	1	-11	-5	18	2	0	0	0	x_1
0	0	4	2	-8	-1	1	0	0	x_6
0	0	12	6	-17	-2	0	1	1	x_7

- x_2 enters; x_6 leaves; obtain 2nd tableau

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis_2
1	0	0	$-29/2$	98	$27/4$	$53/4$	0	0	z
0	1	0	$1/2$	-4	$-3/4$	$11/4$	0	0	x_1
0	0	1	$1/2$	-2	$-1/4$	$1/4$	0	0	x_2
0	0	0	0	7	1	-3	1	1	x_7

- x_3 enters; x_1 leaves;

- (again) 2^{nd} tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_2$
1	0	0	$-29/2$	98	$27/4$	$53/4$	0	0	z
0	1	0	$1/2$	-4	$-3/4$	$11/4$	0	0	x_1
0	0	1	$1/2$	-2	$-1/4$	$1/4$	0	0	x_2
0	0	0	0	7	1	-3	1	1	x_7

- x_3 enters; x_1 leaves; obtain 3^{rd} tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_3$
1	29	0	0	-18	-15	93	0	0	z
0	2	0	1	-8	$-3/2$	$11/2$	0	0	x_3
0	-1	1	0	2	$1/2$	$-5/2$	0	0	x_2
0	0	0	0	7	1	-3	1	1	x_7

- x_4 enters; x_2 leaves; obtain 4^{th} tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_4$
1	20	9	0	0	$-21/2$	$141/2$	0	0	z
0	-2	4	1	0	$1/2$	$-9/2$	0	0	x_3
0	$-1/3$	$1/2$	0	1	$1/4$	$-5/4$	0	0	x_4
0	$7/2$	$-7/2$	0	0	$-3/4$	$23/4$	1	1	x_7

- x_5 enters; x_3 leaves (another tie for smallest ratio);

- (again) 4^{th} tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_4$
1	20	9	0	0	$-21/2$	$141/2$	0	0	z
0	-2	4	1	0	$1/2$	$-9/2$	0	0	x_3
0	$-1/3$	$1/2$	0	1	$1/4$	$-5/4$	0	0	x_4
0	$7/2$	$-7/2$	0	0	$-3/4$	$23/4$	1	1	x_7

- x_5 enters; x_3 leaves (another tie for smallest ratio); obtain 5^{th} tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_5$
1	-22	93	21	0	0	-24	0	0	z
0	-4	8	2	0	1	-9	0	0	x_5
0	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0	x_4
0	$1/2$	$5/2$	$3/2$	0	0	-1	1	1	x_7

- x_6 enters; x_4 leaves; obtain 6^{th} tableau (identical to initial tableau!):

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_6$
1	-10	57	9	24	0	0	0	0	z
0	$1/2$	$-11/2$	$-5/2$	9	1	0	0	0	x_5
0	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0	x_6
0	1	1	1	1	0	0	1	1	x_7

- a continuation of the algorithm would always lead to the same (initial) tableau!

Definition

A situation when the simplex method goes through a series of degenerate steps, and as a result, revisits a basis it encountered previously is called **cycling**.

- several methods available avoiding cycling
- one of them: **Bland's rule**:
  - order var.s in certain way; e.g. order in the increasing order of their indices e.g. $x_1, x_2, \ldots, x_{n+m}$.
  - whenever multiple candidates for entering or leaving var: preference given to var. appearing earlier in the ordering
  - all NV with pos. coeff. in the dictionary (neg. coeff. in tableau) are candidates for entering var.
  - all BV representing ratio test winning rows are candidates for leaving var.

Theorem

If Bland's rule is used to select the entering and leaving variables in the simplex method, then cycling never occurs.

- if we had used Bland's rule in the last ex., we had obtained an optimal sol.:
- all steps but the last would have remained the same
- penultimate tableau was:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_5$
1	-22	93	21	0	0	-24	0	0	z
0	-4	8	2	0	1	-9	0	0	x_5
0	$1/2$	$-3/2$	$-1/2$	1	0	1	0	0	x_4
0	$1/2$	$5/2$	$3/2$	0	0	-1	1	1	x_7

- instead of x_6 we now choose x_1 as the entering var. and obtain

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_6$
1	0	27	-1	44	0	20	0	0	z
0	0	-4	-2	8	1	-1	0	0	x_5
0	1	-3	-1	2	0	2	0	0	x_1
0	0	4	2	-1	0	-2	1	1	x_7

- x_3 entering var.; x_7 leaving var.; obtain optimal tableau:

z	x_1	x_2	x_3	x_4	x_5	x_6	x_7	rhs	basis$_7$
1	0	29	0	$87/2$	0	19	$1/2$	$1/2$	z
0	0	0	0	7	1	-3	1	1	x_5
0	1	-1	0	$3/2$	0	1	$1/2$	$1/2$	x_1
0	0	2	1	$-1/2$	0	-1	$1/2$	$1/2$	x_3

- optimal sol.: $x_2 = x_4 = x_6 = x_7 = 0$; $x_1 = 1/2, x_3 = 1/2, x_5 = 1,\ z^* = 1/2$

Properties of LP dictionaries and the simplex method

- note: the dictionary format represents a certain SLE;
- z and m BVs are expressed through n NVs (n:# var.s; m:# constraints)
- in the initial example (HP LP): $n = 2$ and $m = 4$; namely, for the LP

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \le b_i \\ & x_j \ge 0, \end{aligned}$$

where $b_i \ge 0,\ i = 1, \ldots, m;\ j = 1, \ldots, n$
we constructed initial (feasible) dictionary$\nearrow$

$$\begin{array}{rcl} z & = & \sum_{j=1}^{n} c_j x_j \\ \hline s_i & = b_i & - \sum_{j=1}^{n} a_{ij} x_j \end{array}$$

where $i = 1, \ldots, m$

- initial dic. is SLE representing the original LP written in standard form
- the only transformations applied: el. row op.s (to express new BVs through NVs)

Thus: Every solution of the set of equations comprising the dictionary obtained at any step of the simplex method is also a solution of the step 0 dictionary, and vice versa.

To show theorem concerning cycling, we first show: any two dic.s (of the same LP) with the same basis must be identical.

- consider 2 dic.s (of the same LP) with the same basis
- they are equivalent in the sense that they have the same solution space (on $\mathbb{R}^n$)
- let $\mathcal{B}$ be the set of indices of BVs
- let $\mathcal{N}$ be the set of indices of NVs

$$\frac{z = \bar{z} + \sum_{j\in\mathcal{N}} \bar{c}_j x_j}{x_i = \bar{b}_i - \sum_{j\in\mathcal{N}} \bar{a}_{ij} x_j \quad (i \in \mathcal{B})} \qquad \frac{z = \tilde{z} + \sum_{j\in\mathcal{N}} \tilde{c}_j x_j}{x_i = \tilde{b}_i - \sum_{j\in\mathcal{N}} \tilde{a}_{ij} x_j \quad (i \in \mathcal{B})}$$

- let $k_0 \in \mathcal{N}$ ($\Rightarrow x_{k_0} \in NV$), set $x_{k_0} = t \in \mathbb{R}$;
- for $j \in \mathcal{N}\setminus\{k_0\}$, set $x_j = 0$
- thus we obtain

$$x_j = \bar{b}_i - \bar{a}_{ik_0} t = \tilde{b}_i - \tilde{a}_{ik_0} t$$

  for all $i \in \mathcal{B}$, and

$$z = \bar{z} + \bar{c}_{k_0} t = \tilde{z} + \tilde{c}_{k_0} t.$$

- setting $t = 0$ at first yields $\bar{z} = \tilde{z}$ and $\bar{b}_i = \tilde{b}_i$.
- using this, setting $t = 1$ then yields $\bar{a}_{ik_0} = \tilde{a}_{ik_0}$ and $\bar{c}_{k_0} = \tilde{c}_{k_0}$ for all $i \in \mathcal{B}$
- hence, $\bar{z} = \tilde{z},\ \bar{b}_i = \tilde{b}_i,\ \bar{c}_{k_0} = \tilde{c}_{k_0}$ for all $i \in \mathcal{B}, k \in \mathcal{N}$

Thus: Any two dictionaries of the same LP with the same basis are identical.

- with ratio test we ensured, that constant terms in rhs are ≥ 0
- this in turn ensures that corr. basic sol. is even a basic feasible sol. (i.e. $x_i \geq 0$)
- (recall: dic.s with feasible basic solutions are called "feasible dic.s")
  thus, if we start with a feasible dic., feasibility is preserved under simplex steps

If step 0 dictionary is feasible, then each consecutive dictionary generated using the simplex method is feasible.

- recall: in each simplex step the entering var. is chosen s.t. $z_{k+1} \geq z_k$
- we've seen however, that degeneracy may occur (i.e. bfs$_{k+1}$ =bfs$_k$ $\Rightarrow z_{k+1} = z_k$)
- this is the only case where algorithm does not terminate:

Theorem

If the simplex method avoids cycling, it must terminate by either finding an optimal solution or by detecting that the LP is unbounded.

Proof

only $\binom{n+m}{m}$ different ways of choosing m BVs from a set of $n+m$ var.s. Seen above: any 2 dic.s corr. to the same basis are identical. Thus, only finitely many simplex steps can be different. Hence, if the simplex method does not terminate, it must eventually revisit a previously visited basis. That is, cycling occurs.

- we can use Bland's rule (or others) to prevent cycling.

Geometry of the Simplex Method

- we illustrate simplex steps geometrically using HP LP:

$$\begin{array}{llrl}
\text{maximize} & 15x_1 + 25x_2 & & \text{(profit)} \\
\text{subject to} & x_1 + \quad x_2 \leq 450 & & \text{(solid color fabric constraint)} \\
& x_2 \leq 300 & & \text{(printed fabric constraint)} \\
& 4x_1 + \ 5x_2 \leq 2000 & & \text{(budget constraint)} \\
& x_1 \qquad \leq 350 & & \text{(demand constraint)} \\
& x_1, x_2 \geq 0 & & \text{(nonnegativity constraints)}
\end{array}$$

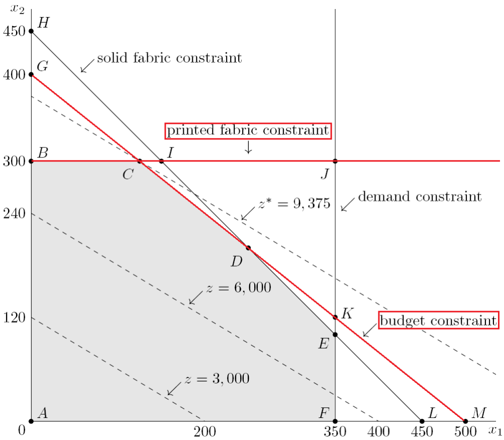
- we've converted this LP into standard form (introducing slack var.s):

$$\begin{array}{ll}
\text{maximize} & 15x_1 + 25x_2 \\
\text{subject to} & x_1 + x_2 + s_1 = 450 \\
& x_2 + s_2 = 300 \\
& 4x_1 + 5x_2 + s_3 = 2000 \\
& x_1 + s_4 = 350 \\
& x_1, x_2, s_1, s_2, s_3, s_4 \geq 0
\end{array}$$

- note: s_2: slack of printed fabric constraint
  (for given x_1, x_2: "amount of unused printed fabric")
- s_3: slack of budget constraint (for given x_1, x_2: "amount of unused budget")

recall what we've obtained after each step in simplex method:

Step	BV	bfs
0	s_1, s_2, s_3, s_4	$x_1 = x_2 = 0, s_1 = 450, s_2 = 300, s_3 = 2,000, s_4 = 350$
1	s_1, x_2, s_3, s_4	$x_1 = 0, x_2 = 300, s_1 = 150, s_2 = 0, s_3 = 500, s_4 = 350$
2	x_1, x_2, s_1, s_4	$x_1 = 125, x_2 = 300, s_1 = 25, s_2 = 0, s_3 = 0, s_4 = 225$



- recall: s_2: slack of printed fabric constraint
- s_3: slack of budget constraint
- bfs are represented by vertices A, B and C
- each pair of consecutive bfs (have all but one BV in common) represents 2 vertices of polyhedron that are connected by an "edge" of feasible region

[0] Picture source: ISBN 978-1-4665-7778-7 (modified)

Definition

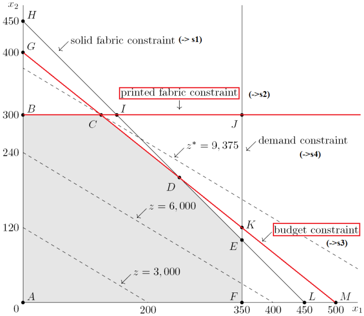
We will call two bfs **adjacent** if their sets of BVs differ by just one element. For a given bfs, another bfs is (also) called adjacent, if both bfs are adjacent.

- i.e.: if the LP has m constraints $\rightsquigarrow$ adjacent bfs have $m-1$ var.s in common
- thus: two consecutive bfs are adjacent
- recall: in HP example above: any bfs represents vertex of feasible region
- this is true in general: "vertex=bfs"; to show this:
- for given LP in standard form we denote by $X = \{x \mid Ax = b,\ x \geq 0\}$ the feasible region

Theorem

A point $\bar{x} \in \mathbb{R}^{m+n}$ is an extrem point of $X = \{x \mid Ax = b,\ A \in \mathbb{R}^{m\times(m+n)},\ x \geq 0\}$ iff it can be represented as a bfs of an LP with feasible region given by X. □

- return to last example:



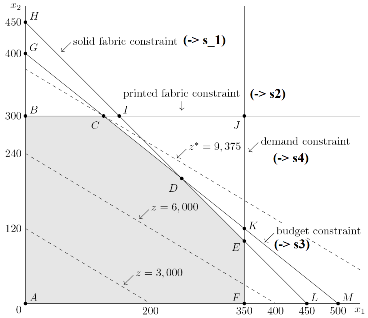
- we've 6 vertices: A, B, C, D, E, F
- correspondence between vertices and bfs:

BVs	Basic feasible solution						Vertex
	x_1	x_2	s_1	s_2	s_3	s_4	
s_1, s_2, s_3, s_4	0	0	450	300	2000	350	A
x_2, s_1, s_3, s_4	0	300	150	0	500	350	B
x_1, x_2, s_1, s_4	125	300	25	0	0	225	C
x_1, x_2, s_2, s_4	250	200	0	100	0	100	D
x_1, x_2, s_2, s_3	350	100	0	200	100	0	E
x_1, s_1, s_2, s_3	350	0	100	300	600	0	F

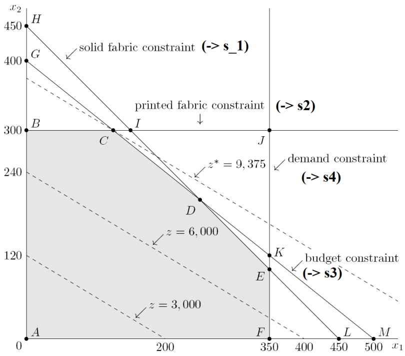
- recall: any basic (not necessarily feasible) sol. is obtained by setting the NVs $= 0$
- in our 2d-case: we've 2 NVs
- if one of original ($=$ decision) var.s is zero then corr. basic sol. lies on corr. coordinate axis (e.g.: A, B, F)
- if slackvar.$= 0$ then constraint it represents is binding for corr.basic sol. (e.g.B-F)
- thus: basic sol. corr. to pairs of lines defining the feasible region (incl. coord.axis)

[0] Picture source: ISBN 978-1-4665-7778-7 (modified)

- total number of <u>potential</u> basic sol. in our ex.: $\binom{2+4}{4} = \frac{6!}{(6-4)!4!} = 15$
- however: not every set of 4 var.s may form basis
- e.g. basis consisting of x_1, s_1, s_3, s_4 is <u>not</u> possible; $\rightsquigarrow$ no corr. basic sol.: $x_2, s_2 = 0$ would imply $0 = x_2 + s_2 \overset{1)}{=} 300$, where [1)] refers to printed fabric constraint; geometrically: this corr. to parallel lines a def. pair of constraints; in our ex.: boundary for printed fabric constraint ($x_2 = 300$) is parallel to boundary of nonneg. constraint x_2 ($x_2 = 0$); i.e. both constraints can't be binding at same time; analogous situation: basis consisting of x_2, s_1, s_2, s_3 not possible; lines corr. to $x_1, s_4 = 0$ parallel.



[0] Picture source: ISBN 978-1-4665-7778-7 (modified)



- excluding these 2 cases: $15 - 2 = 13$ (potential) basic sol. remaining; discussed 6 of them (A-F); A-F are basic feasible solutions (bfs)
- we can see: remaining basic sol. corr. to pairwise intersections of lines def. constraints (incl. nonneg. constraints) are infeasible; $\rightsquigarrow$ points G, H, I, J, K, L, M
- correspondence between these points and basic sol.: exercise

In summary:

- from geom. viewpoint: simplex method starts from vertex of feasible region
- then moves to better (no worse) adjacent vertex (if exists)
- terminates at vertex that has no adjacent vertex with a better objective

[0] Picture source: ISBN 978-1-4665-7778-7 (modified)

The Simplex Method for a General LP

- assume in HP ex. we had additional constraint: $\geq$100 carriers have to be manufactured; i.e. $x_1 + x_2 \geq 100$; we obtain following LP:

$$
\begin{array}{llr}
\text{maximize} & 15x_1 + 25x_2 & \text{(profit)} \\
\text{subject to} & x_1 + x_2 \leq 450 & \text{(solid color fabric constraint)} \\
& x_2 \leq 300 & \text{(printed fabric constraint)} \\
& 4x_1 + 5x_2 \leq 2000 & \text{(budget constraint)} \\
& x_1 \leq 350 & \text{(demand constraint)} \\
& x_1 + x_2 \geq 100 & \text{(manufacturing constraint)} \\
& x_1, x_2 \geq 0 & \text{(nonnegativity constraints)}
\end{array}
$$

- as before: we convert LP to standard form by introducing slack var. $s_1, \ldots, s_4$ and an excess var. e_5 for the new constraint; obtain

$$
\begin{array}{ll}
\text{maximize} & 15x_1 + 25x_2 \\
\text{subject to} & x_1 + x_2 + s_1 = 450 \\
& x_2 + s_2 = 300 \\
& 4x_1 + 5x_2 + s_3 = 2000 \\
& x_1 + s_4 = 350 \\
& x_1 + x_2 - e_5 = 100 \\
& x_1, x_2, s_1, s_2, s_3, s_4, e_5 \geq 0
\end{array}
$$

corr. dictionary:

$$
\begin{array}{rcrrr}
z & = & & 15x_1 & + 25x_2 \\
\hline
s_1 & = & 450 & - x_1 & - x_2 \\
s_2 & = & 300 & & - x_2 \\
s_3 & = & 2000 & - 4x_1 & - 5x_2 \\
s_4 & = & 350 & - x_1 & \\
e_5 & = & -100 & + x_1 & + x_2
\end{array}
$$

is clearly infeasible:
$(x_1, x_2 = 0 \Rightarrow e_5 = -100 < 0)$

- therefore, simplex method can't be initialized with this dic.; e_5 cannot be used as basic var. for 5^{th} constraint

- plan: discuss a variant of simplex method to overcome this obstacle: "two-phase-simplex-method" (other variant: "big-M-method").
- both methods based on similar idea: introduce "artificial variable" in case basic var. is not readily available; e.g. consider

$$x_1 + x_2 - e_5 = 100.$$

Introduce artificial var. $a_5 \geq 0$ as follows

$$x_1 + x_2 - e_5 + a_5 = 100.$$

- now we can use a_5 as starting BV for this constraint; $\rightsquigarrow$ obtain bfs for resulting LP
- to obtain feasible sol. of original LP (if there's one): try to achieve $a_5 = 0$
- both methods try to exclude artificial var.s out of basis; (thus, they eventually vanish, whenever LP is feasible)
- first, we discuss general setup

- Consider general LP (Problem (P))

$$\begin{aligned} \text{maximize} \quad & c^T x & \\ \text{subject to} \quad & A'x \leq b' \in \mathbb{R}^{m'} & (m' \text{ inequalities}) \\ & A''x = b'' \in \mathbb{R}^{m''} & (m'' \text{ equalities}) \\ & x \geq 0 & (x \in \mathbb{R}^n) \end{aligned}$$

- let $\mathcal{I}^- = \{i \mid b'_i < 0\}$;
- intr. slack var $x_{n+1}, \ldots, x_{n+m'}$ and rewrite LP in standard form (Problem (PS)):

$$\begin{aligned} \text{maximize} \quad & \bar{c}^T x & \\ & Ax = b = \begin{pmatrix} b' \\ b'' \end{pmatrix} & (m := m' + m'' \text{ equalities}) \\ & x \geq 0 & (x \in \mathbb{R}^{n+m'}) \end{aligned}$$

- wlog $b \geq 0$ (otherwise mult. resp. eqn. by -1)
- eqn. i ($i \in \mathcal{I}^-$) requires introducing artificial var.s; e.g.

$$x_1 + x_2 \leq -3 \rightsquigarrow x_1 + x_2 + s_1 = -3 \rightsquigarrow -x_1 - x_2 - s_1 = 3 \rightsquigarrow -x_1 - x_2 - s_1 + a_1 = 3$$

- in addition, we also add artificial var.s to last m'' eqn.s; will serve as initial basic var.s in initial dic.
- for convenience: set of indices requiring artificial var.s:

$$\mathcal{I}^a = \mathcal{I}^- \cup \{m' + 1, \ldots, m' + m''\}$$

With (P) and (PS) we associate the LP (which shares the same feasible region):

"Auxiliary Problem" (A)

$$\begin{array}{llll}
\text{maximize} & -\sum_{i\in\mathcal{I}^a} a_i & & \\
\text{subject to} & \sum_{j=1}^{n+m'} a_{ij}x_j = b_i, & i\in\{1,\dots,m\}\setminus\mathcal{I}^a \\
& \sum_{j=1}^{n+m'} a_{ij}x_j + a_i = b_i, & i\in\mathcal{I}^a \\
& x_j, a_i \geq 0 & j=1,\dots,n+m',\ i\in\mathcal{I}^a
\end{array}$$

Example: Consider the following LP:

$$\begin{array}{ll}
\text{maximize} & x_1 - 2x_2 + 3x_3 \\
\text{subject to} & -2x_1 + 3x_2 + 4x_3 \geq 12 \\
& 3x_1 + 2x_2 + x_3 \geq 6 \\
& x_1 + x_2 + x_3 \leq 9 \\
& x_1, x_2, x_3 \geq 0
\end{array}$$

This LP in standard form is given by

$$\begin{array}{ll}
\text{maximize} & x_1 - 2x_2 + 3x_3 \\
\text{subject to} & -2x_1 + 3x_2 + 4x_3 - x_4 = 12 \\
& 3x_1 + 2x_2 + x_3 - x_5 = 6 \\
& x_1 + x_2 + x_3 + x_6 = 9 \\
& x_1, x_2, x_3, x_4, x_5, x_6 \geq 0
\end{array}$$

- clearly: basis $\{x_4, x_5, x_6\}$ infeasible since $x_4, x_5 < 0$
- we introduce artificial var.s for first 2 constraints
- this gives:

$$\begin{array}{ll}
\text{maximize} & -a_1 - a_2 \\
\text{subject to} & -2x_1 + 3x_2 + 4x_3 - x_4 + a_1 = 12 \\
& 3x_1 + 2x_2 + x_3 - x_5 + a_2 = 6 \\
& x_1 + x_2 + x_3 + x_6 = 9 \\
& x_1, x_2, x_3, x_4, x_5, x_6, a_1, a_2 \geq 0
\end{array}$$

- note: bfs for (A) is easily obtained: select basis consisting of the
  - slack var.s x_{n+i} $(i \in \{1,\ldots,m\}\backslash\mathcal{I}^a)$ for rows where art. var.s were not needed
  - artificial var.s a_i $(i \in \mathcal{I}^a)$ for remaining rows
- also, for the objective of (A) we have $z = -a_1 - a_2 \leq 0$;
  thus, any feasible sol. of (A) with $a_i = 0$ $(i \in \mathcal{I}^a)$ is optimal
- The following property holds for LP (P) and the associated auxiliary problem (A):

Theorem

The LP (P) is feasible if and only if (A) is optimal with optimal objective value $z^* = 0$.

Proof

If the LP (P) is feasible, then any feasible sol. x to (P) and $a_i = 0$ $(i \in \mathcal{I}^a)$ provide an optimal sol. for (A).
Conversely, if $z^* = 0$ is optimal for (A) then for the optimal sol. we have that $a_i = 0$ $(i \in \mathcal{I}^a)$ and $x_1^*,\ldots,x_{n+m'}^*$ is a feasible sol. for (P).

The two-phase simplex method

- Phase I: Solve (A); note: (A) is optimal because feasible and bounded
  - case 1: if $z^* = 0$ then x^* is a feasible sol. for (P)
  - case 2: if $z^* \neq 0$ then (P) is infeasible $\rightarrow$ STOP
- Phase II: If (P) is feasible solve (P) using optimal tableau from (A) as initial tableau (as follows:)

- to get feasible tab. for (P) from optimal tab. for (A) in Phase II: get rid of $a_i s$:
  - case 1: ($a_i \in NV \ \forall a_i$). Drop a_i-columns and express objective of of (P) solely through NVs. (Alternatively: perform Gauss-step to obtain zero BV-coeff.s in original z-row). Then the BVs from (A) are now BVs for the initial tableau for (P).
  - case 2: ($\exists a_i \in BV$ obtained in optimal tab. for (A)). corr. sol. must be degenerate since $a_i = 0$ for all i. we try to drive them out of the basis by performing additional degenerate pivots:
    - case 2a: $a_i \in BV$ and at least one x_k-coeff. in corr. row is nonzero. in this case we perform one Gauss step with this nonzero coeff. as pivot; result: pivot was degenerate since $a_i = 0 \Rightarrow z$ unchanged; x_k entered basis and a_i left basis
    - case 2b: $a_i \in BV$ and all the x_k-coeff. in corr. row are zero. in this case the corr. eqn. reads $\sum \lambda_i a_i = 0$ and can be dropped. This can only happen when (PS) has linearly dependent constraints; (see last Example)

Example. Use two-phase simplex method to solve LP (cf. above)
(note: steps could be shortend by intro. excess var.s)

$$\begin{array}{ll} \text{maximize} & x_1 - 2x_2 + 3x_3 \\ \text{subject to} & -2x_1 + 3x_2 + 4x_3 \geq 12 \\ & -3x_1 - 2x_2 - x_3 \leq -6 \\ & x_1 + x_2 + x_3 \leq 9 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

step 1: force $\leq$-inequalities

$$\begin{array}{ll} \text{maximize} & x_1 - 2x_2 + 3x_3 \\ \text{subject to} & 2x_1 - 3x_2 - 4x_3 \leq -12 \\ & -3x_1 - 2x_2 - x_3 \leq -6 \\ & x_1 + x_2 + x_3 \leq 9 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$\mathcal{I}^- = \{1,2\} \overset{\text{no eqn.s}}{=} \mathcal{I}^a$

step 2: introduce slack var.s x_4, x_5, x_6

$$\begin{array}{ll} \text{max} & x_1 - 2x_2 + 3x_3 \\ \text{s.to} & 2x_1 - 3x_2 - 4x_3 + x_4 = -12 \\ & -3x_1 - 2x_2 - x_3 + x_5 = -6 \\ & x_1 + x_2 + x_3 + x_6 = 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

step 3: mult. row i by -1 forall $i \in \mathcal{I}^-$

$$\begin{array}{ll} \text{max} & x_1 - 2x_2 + 3x_3 \\ \text{s.to} & -2x_1 + 3x_2 + 4x_3 - x_4 = 12 \\ & 3x_1 + 2x_2 + x_3 - x_5 = 6 \\ & x_1 + x_2 + x_3 + x_6 = 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

step 4: introduce artificial var.s a_i ($i \in \mathcal{I}^a = \{1,2\}$);
This is (PS); Formulate auxiliary problem (A):

$$\begin{array}{ll} \text{max} & -a_1 - a_2 \\ \text{s.to} & -2x_1 + 3x_2 + 4x_3 - x_4 + a_1 = 12 \\ & 3x_1 + 2x_2 + x_3 - x_5 + a_2 = 6 \\ & x_1 + x_2 + x_3 + x_6 = 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

This is (A); let $e_1 = x_4, e_2 = x_5, s_3 = x_6$
step 5: create initial tableau for (A)

z	x_1	x_2	x_3	e_1	e_2	s_3	a_1	a_2	rhs	basis$_{-1}$
1	0	0	0	0	0	0	1	1	0	z
0	-2	3	4	-1	0	0	1	0	12	a_1
0	3	2	1	0	-1	0	0	1	6	a_2
0	1	1	1	0	0	1	0	0	9	s_3

Perform Gauss-step to obtain 0 coeff.s for a_i in z row

z	x_1	x_2	x_3	e_1	e_2	s_3	a_1	a_2	rhs	basis$_{-1}$
1	0	0	0	0	0	0	1	1	0	z
0	-2	3	4	-1	0	0	1	0	12	a_1
0	3	2	1	0	-1	0	0	1	6	a_2
0	1	1	1	0	0	1	0	0	9	s_3

Perform Gauss-step to obtain 0 coeff.s for a_i in z row

z	x_1	x_2	x_3	e_1	e_2	s_3	a_1	a_2	rhs	basis$_0$
1	-1	-5	-5	1	1	0	0	0	-18	z
0	-2	3	4	-1	0	0	1	0	12	a_1
0	3	2	1	0	-1	0	0	1	6	a_2
0	1	1	1	0	0	1	0	0	9	s_3

Now: simplex steps as usual...

z	x_1	x_2	x_3	e_1	e_2	s_3	a_1	a_2	rhs	basis$_1$
1	$13/2$	0	$-5/2$	1	$-3/2$	0	0	$5/2$	-3	z
0	$-13/2$	0	$5/2$	-1	$3/2$	0	1	$-3/2$	3	a_1
0	$3/2$	1	$1/2$	0	$-1/2$	0	0	$1/2$	3	x_2
0	$-1/2$	0	$1/2$	0	$1/2$	1	0	$-1/2$	6	s_3

z	x_1	x_2	x_3	e_1	e_2	s_3	a_1	a_2	rhs	basis$_2$
1	0	0	0	0	0	0	1	1	0	z
0	$-13/5$	0	1	$-2/5$	$3/5$	0	$2/5$	$-3/5$	$6/5$	x_3
0	$14/5$	1	0	$1/5$	$-4/5$	0	$-1/5$	$4/5$	$12/5$	x_2
0	$4/5$	0	0	$1/5$	$1/5$	1	$-1/5$	$-1/5$	$27/5$	s_3

- The latter tab. is optimal;
- Here, we have case 1 ($a_i \in NV \ \forall a_i$);
- To obtain feasible tab. for ori. LP: drop a_i col.s;
- substitute original z-row

z	x_1	x_2	x_3	e_1	e_2	s_3	rhs	basis$_2$
1	-1	2	-3	0	0	0	0	z
0	$-13/5$	0	1	$-2/5$	$3/5$	0	$6/5$	x_3
0	$14/5$	1	0	$1/5$	$-4/5$	0	$12/5$	x_2
0	$4/5$	0	0	$1/5$	$1/5$	1	$27/5$	s_3

perform Gauss-step to transform red numbers to zero

z	x_1	x_2	x_3	e_1	e_2	s_3	rhs	basis$_2$
1	-1	2	-3	0	0	0	0	z
0	$-13/5$	0	1	$-2/5$	$3/5$	0	$6/5$	x_3
0	$14/5$	1	0	$1/5$	$-4/5$	0	$12/5$	x_2
0	$4/5$	0	0	$1/5$	$1/5$	1	$27/5$	s_3

perform Gauss-step to transform red numbers to zero

z	x_1	x_2	x_3	e_1	e_2	s_3	rhs	basis$_0^{II}$
1	$-72/5$	0	0	$-8/5$	$17/5$	0	$-6/5$	z
0	$-13/5$	0	1	$-2/5$	$3/5$	0	$6/5$	x_3
0	$14/5$	1	0	$1/5$	$-4/5$	0	$12/5$	x_2
0	$4/5$	0	0	$1/5$	$1/5$	1	$27/5$	s_3

Now: simplex steps as usual...

z	x_1	x_2	x_3	e_1	e_2	s_3	rhs	basis$_1^{II}$
1	0	$36/7$	0	$-4/7$	$-5/7$	0	$78/7$	z
0	0	$13/14$	1	$-3/14$	$-1/7$	0	$24/7$	x_3
0	1	$5/14$	0	$1/14$	$-2/7$	0	$6/7$	x_1
0	0	$-2/7$	0	$1/7$	$3/7$	1	$33/7$	s_3

z	x_1	x_2	x_3	e_1	e_2	s_3	rhs	basis$_2^{II}$
1	0	$14/3$	0	$-1/3$	0	$5/3$	19	z
0	0	$5/6$	1	$-1/6$	0	$1/3$	5	x_3
0	1	$1/6$	0	$1/6$	0	$2/3$	4	x_1
0	0	$-2/3$	0	$1/3$	1	$7/3$	11	e_2

z	x_1	x_2	x_3	e_1	e_2	s_3	rhs	basis$_3^{II}$
1	2	5	0	0	0	3	27	z
0	1	1	1	0	0	1	9	x_3
0	6	1	0	1	0	4	24	e_1
0	-2	-1	0	0	1	1	3	e_2

- The latter tab. is optimal;
- thus, optimal sol. to orig. LP (P) is $x_1^* = x_2^* = 0,\ x_3^* = 9;\ z^* = 27.$

Example. Use two-phase simplex method to solve the following LP

$$\begin{aligned} \max \quad & x_1 + x_2 + x_3 \\ \text{s.to} \quad & 2x_1 + 2x_2 + 3x_3 = 6 \\ & x_1 + 3x_2 + 6x_3 = 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$\mathcal{I}^- = \{\}$; and $\mathcal{I}^a = \{1, 2\}$;
intr. artificial var.s. a_1, a_2; obtain phase I initial tab.

z	x_1	x_2	x_3	a_1	a_2	rhs	basis$^{\text{I}}_0$
1	-3	-5	-9	0	0	-18	z
0	2	2	3	1	0	6	a_1
0	1	3	6	0	1	12	a_2

obtain optimal tab. already:

z	x_1	x_2	x_3	a_1	a_2	rhs	basis$^{\text{I}}_1$
1	3	1	0	3	0	0	z
0	$2/3$	$2/3$	1	$1/3$	0	2	x_3
0	-3	-1	0	-2	1	0	a_2

Corr. bfs is degenerate (case 2a from above); optimality will not be altered when we make add. pivot with (e.g.) x_2 entering var. and a_2 as leaving var.

z	x_1	x_2	x_3	a_1	a_2	rhs	basis$^{\text{I}}_2$
1	0	0	0	1	1	0	z
0	$-4/3$	0	1	-1	$2/3$	2	x_3
0	3	1	0	2	-1	0	x_2

This tab. is still optimal with optimal value $z^* = 0$;
All $a_i \in NV$;
Remove all a_i col.s and substitute original z-row

z	x_1	x_2	x_3	rhs	basis$^{\text{II}}_{-1}$
1	-1	-1	-1	0	z
0	$-4/3$	0	1	2	x_3
0	3	1	0	0	x_2

Perform Gauss-step to obtain 0 for BV coeff. in z-row

z	x_1	x_2	x_3	rhs	basis$^{\text{II}}_0$
1	$2/3$	0	0	2	z
0	$-4/3$	0	1	2	x_3
0	3	1	0	0	x_2

- The latter tab. is optimal.
- Thus, optimal sol. to orig. LP (P) is
- $x_1^* = x_2^* = 0, \; x_3^* = 2; \; z^* = 2.$

Example. Use two-phase simplex method to solve the following LP (with linearly dependent constraints)

$$\begin{aligned} \text{max} \quad & x_1 + x_2 + 2x_3 \\ \text{s.to} \quad & x_1 + 2x_2 + 3x_3 = 6 \\ & 2x_1 + 4x_2 + 6x_3 = 12 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

$\mathcal{I}^- = \{\}$; and $\mathcal{I}^a = \{1, 2\}$;
intr. artificial var.s. a_1, a_2; obtain phase I initial tab.

z	x_1	x_2	x_3	a_1	a_2	rhs	basis$^{\text{I}}_0$
1	-3	-6	-9	0	0	-18	z
0	1	2	3	1	0	6	a_1
0	2	4	6	0	1	12	a_2

obtain optimal phase I tab. already:

z	x_1	x_2	x_3	a_1	a_2	rhs	basis$^{\text{I}}_1$
1	0	0	0	3	0	0	z
0	⅓	⅔	1	⅓	0	2	x_3
0	0	0	0	-2	1	0	a_2

Corr. bfs is again degenerate (case 2a from above); but this time a_2 not removable from basis; however, 2nd row does not involve orig. var.s $\rightsquigarrow$ remove this row; remove a_i col.s; substitute z by original z-row;

z	x_1	x_2	x_3	a_1	a_2	rhs	basis$^{\text{II}}_{-1}$
1	-1	-1	-2	0	0	0	z
0	⅓	⅔	1	⅓	0	2	x_3

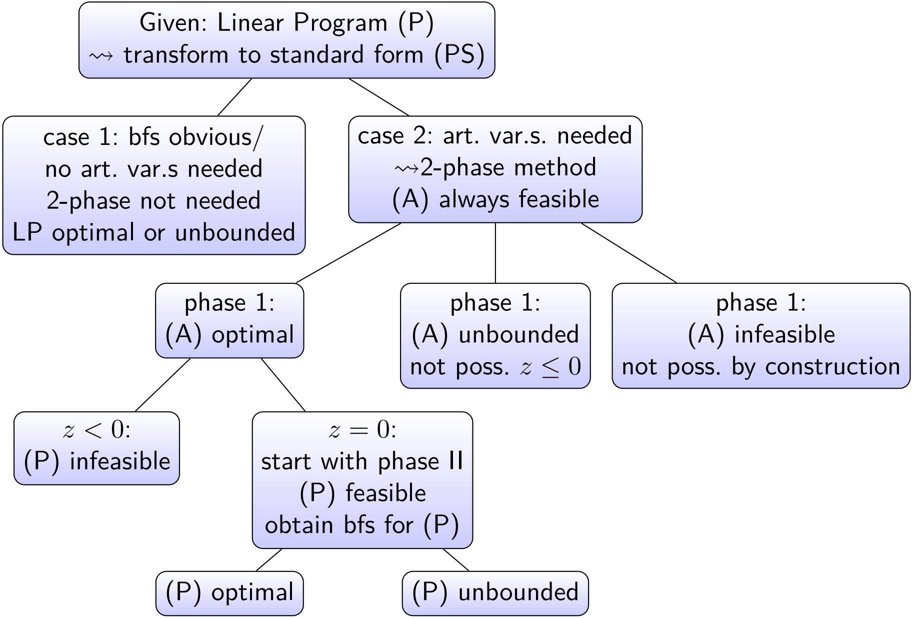
Perform Gauss-step to obtain 0 for BV coeff. in z-row; obtain feasible tab. for phase II:

z	x_1	x_2	x_3	rhs	basis$^{\text{II}}_0$
1	$-$⅓	⅓	0	4	z
0	⅓	⅔	1	2	x_3

carry out pivot:

z	x_1	x_2	x_3	rhs	basis$^{\text{II}}_0$
1	0	1	1	6	z
0	1	2	3	6	x_1

- The latter tab. is optimal.
- Thus, optimal sol. to orig. LP (P) is
- $x_1^* = 6,\ x_2^* = x_3^* = 0;\ z^* = 6.$


Given: Linear Program (P)
⇝ transform to standard form (PS)
case 1: bfs obvious/
no art. var.s needed
2-phase not needed
LP optimal or unbounded
case 2: art. var.s. needed
⇝2-phase method
(A) always feasible
phase 1:
(A) optimal
phase 1:
(A) unbounded
not poss. $z \leq 0$
phase 1:
(A) infeasible
not poss. by construction
$z < 0$:
(P) infeasible
$z = 0$:
start with phase II
(P) feasible
obtain bfs for (P)
(P) optimal
(P) unbounded

Theorem (Fundamental theorem of Linear Programming)

Every LP has the following properties:

1) *If it has no optimal solution, then it is either infeasible or unbounded.*
2) *If it has a feasible solution, then it has a standard-form representation in which it has a basic feasible solution.*
3) *If it has an optimal solution, then it has a standard-form representation in which it has a basic optimal solution.*

Proof

The proof follows from the analysis of the two-phase simplex method above. If an LP has a feasible solution, then Phase I of the two-phase simplex method will find a basic feasible solution. If the LP is optimal, then Phase II of the two-phase simplex method will find a basic optimal solution. If the LP has no optimal solution and was not proved infeasible at Phase I of the twophase simplex method, then we start Phase II. Phase II will always terminate if, e.g., we use Bland's rule to avoid cycling. If the LP is not optimal, Phase II will prove that the problem is unbounded, since this is the only remaining possibility.

List of abbreviations/acronyms

- bfs: basic feasible solution(s) (corresponding to a certain basis)
- bfs$_k$: basic feasible solution (corresponding to a certain basis BV_k)
- BV: set of basic variables
- NV: set of nonbasic variables
- corr.: corresponds/corresponding etc.
- el. row op.s: elementary row operations
- mult.: multiply (or the like)
- resp.: respectively/respective...
- LP: Linear Program/Programming
- SLE: System of Linear Equations
- s.f.a.t.a.: Sorry for all the abbreveations

- neg.: negative
- nonpos.: nonpositive
- nonneg.: nonnegative
- sol.: solution
- var.: variable
- coeff.: coefficient(s)
- col.: column
- HP: Heavenly Pouch Inc.
- dic.: dictionary/dictionaries
- iff: if and only if
- ex.: example (or: exercise)
- eqn.(s): equation(s)
- tab.: tableau