Exercise 6.1 *(Computing)***.** Determine the Cholesky decompositions of the following matrices. That is, a lower triangular matrix G with $A = GG^T$.

a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 8 & 12 \\ 3 & 12 & 27 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 8 & -8 & 14 \\ 3 & -8 & 11 & -14 \\ -4 & 14 & -14 & 35 \end{pmatrix}$

Suggested Solution.

> `for` $k = 1, \ldots, n$:
> (i) $g_{kk} := \sqrt{a_{kk}^{(k)}}$
>
> (ii) $g_{ik} := \dfrac{a_{ik}^{(k)}}{g_{kk}}$ für $i \geq k+1$
>
> (iii) $a_{ij}^{(k+1)} := a_{ij}^{(k)} - g_{ik}g_{jk}$ für $i, j \geq k+1$
>
> `endfor`

(a) Step $k = 1$:

(i) $g_{11} = \sqrt{a_{11}^{(1)}} = \sqrt{1} = 1$

(ii) $g_{21} = \dfrac{a_{ik}^{(k)}}{g_{kk}} = \dfrac{a_{21}^{(1)}}{g_{11}} = \dfrac{2}{1} = 2,$

$g_{31} = \dfrac{a_{31}^{(1)}}{g_{11}} = \dfrac{3}{1} = 3$

(iii) $(g_{i1}g_{j1})_{2\leq i,j\leq 3} = \left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right)\left(\, 2;\, 3\,\right) = \begin{pmatrix} g_{21}^2 & \star \\ g_{31}g_{21} & g_{31}^2 \end{pmatrix} = \left(\begin{smallmatrix} 4 & 6 \\ 6 & 9 \end{smallmatrix}\right)$

$\Rightarrow\ (a_{ij}^{(2)})_{2\leq i,j\leq 3} = (a_{ij}^{(1)})_{2\leq i,j\leq 3} - (g_{i1}g_{j1})_{2\leq i,j\leq 3} = \left(\begin{smallmatrix} 8 & 12 \\ 12 & 27 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 4 & 6 \\ 6 & 9 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 4 & 6 \\ 6 & 18 \end{smallmatrix}\right)$

Step $k = 2$:

(i) $g_{22} = \sqrt{a_{22}^{(2)}} = \sqrt{4} = 2$

(ii) $g_{32} = \frac{a_{32}^{(2)}}{g_{22}} = \frac{6}{2} = 3$

(iii) $(g_{i2}g_{j2})_{3\leq i,j\leq 3} = (g_{32}^2) = (9)$ and thus

$(a_{ij}^{(3)})_{3\leq i,j\leq 3} = (a_{ij}^{(2)})_{3\leq i,j\leq 3} - (g_{i1}g_{j1})_{3\leq i,j\leq 3} = (18) - (9) = (9)$

Step $k = 3$:

(i) $g_{33} = \sqrt{a_{33}^{(3)}} = \sqrt{9} = 3$

Finally we obtain $A = GG^t$ with $G = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{pmatrix}$.

(b) From

$$A = \begin{pmatrix} 1 & -2 & 3 & -4 \\ -2 & 8 & -8 & 14 \\ 3 & -8 & 11 & -14 \\ -4 & 14 & -14 & 35 \end{pmatrix}$$

we obtain

Step $k=1$:

for $k = 1, \dots, n$:
(i) $g_{kk} := \sqrt{a_{kk}^{(k)}}$
(ii) $g_{ik} := \frac{a_{ik}^{(k)}}{g_{kk}}$ für $i \geq k+1$
(iii) $a_{ij}^{(k+1)} := a_{ij}^{(k)} - g_{ik}g_{jk}$ für $i, j \geq k+1$
endfor

(i) $g_{11} = \sqrt{a_{11}^{(1)}} = \sqrt{1} = 1$

(ii) $g_{21} = \frac{a_{ik}^{(k)}}{g_{kk}} = \frac{a_{21}^{(1)}}{g_{11}} = \frac{-2}{1} = -2,$

$g_{31} = \frac{a_{31}^{(1)}}{g_{11}} = \frac{3}{1} = 3$

$g_{41} = \frac{a_{41}^{(1)}}{g_{11}} = \frac{-4}{1} = -4$

(iii) $(g_{i1}g_{j1})_{2\leq i,j\leq 4} = \begin{pmatrix} -2 \\ 3 \\ -4 \end{pmatrix} \begin{pmatrix} -2; & 3; & -4 \end{pmatrix} = \begin{pmatrix} g_{21}^2 & \star & \star \\ g_{31}g_{21} & g_{31}^2 & \star \\ g_{41}g_{21} & g_{41}g_{31} & g_{41}^2 \end{pmatrix} = \begin{pmatrix} 4 & -6 & 8 \\ -6 & 9 & -12 \\ 8 & -12 & 16 \end{pmatrix}$

$\Rightarrow (a_{ij}^{(2)})_{2\leq i,j\leq 4} = (a_{ij}^{(1)})_{2\leq i,j\leq 4} - (g_{i1}g_{j1})_{2\leq i,j\leq 4} = \begin{pmatrix} 8 & -8 & 14 \\ -8 & 11 & -14 \\ 14 & -14 & 35 \end{pmatrix} - \begin{pmatrix} 4 & -6 & 8 \\ -6 & 9 & -12 \\ 8 & -12 & 16 \end{pmatrix} = \begin{pmatrix} 4 & \star & \star \\ -2 & 2 & -2 \\ 6 & -2 & 19 \end{pmatrix}$

Step $k=2$:

(i) $g_{22} = \sqrt{a_{22}^{(2)}} = \sqrt{4} = 2$

(ii) $g_{32} = \frac{a_{32}^{(2)}}{g_{22}} = \frac{-2}{2} = -1$

$g_{42} = \frac{a_{42}^{(2)}}{g_{22}} = \frac{6}{2} = 3$

(iii) $(g_{i2}g_{j2})_{3\leq i,j\leq 4} = \begin{pmatrix} g_{32}^2 & \star \\ g_{42}g_{32} & g_{42}^2 \end{pmatrix} = \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$ and thus

$(a_{ij}^{(3)})_{3\leq i,j\leq 4} = (a_{ij}^{(2)})_{3\leq i,j\leq 4} - (g_{i1}g_{j1})_{3\leq i,j\leq 4} = \begin{pmatrix} 2 & -2 \\ -2 & 19 \end{pmatrix} - \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 10 \end{pmatrix}$

Step $k=3$:

(i) $g_{33} = \sqrt{a_{33}^{(3)}} = \sqrt{1} = 1$

(ii) $g_{43} = \frac{a_{43}^{(3)}}{g_{33}} = \frac{1}{1} = 1$

(iii) $(g_{i3}g_{j3})_{4\leq i,j\leq 4} = \begin{pmatrix} g_{43}^2 \end{pmatrix} = (1)$ and thus

$A^{(4)} = (a_{ij}^{(4)})_{4\leq i,j\leq 4} = (a_{ij}^{(3)})_{4\leq i,j\leq 4} - (g_{i1}g_{j1})_{4\leq i,j\leq 4} = (10) - (1) = (9)$

Step $k=4$:

(i) $g_{44} = \sqrt{a_{44}^{(4)}} = \sqrt{9} = 3$

Finally we obtain $A = GG^t$ with $G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -2 & 2 & 0 & 0 \\ 3 & -1 & 1 & 0 \\ -4 & 3 & 1 & 3 \end{pmatrix}$.

The `Octave` command `chol(A,'lower')=chol(A)'` computes G.

Exercise 6.2 *(Octave)*. Implement the Cholesky method in `Octave`, i.e. write a program that requires a matrix A as input and - if A is spd - returns the decomposition matrix G as output, so that $A = GG^T$ holds. Test the program with the matrices from Exercise 6.1.

Suggested Solution.

```
function G = ex6_2_cholesky(A)
  % Initialize G as a zero matrix.
  [n,n] = size(A);
  G = zeros(n);
  for k=1:n
    % Step k part (i)
    G(k,k) = sqrt(A(k,k));
    for i=k+1:n
      G(i,k)=A(i,k)/G(k,k);
    end
    % Step k part (ii)
    % Compute auxiliary matrix A(k)
    % Overwrite A thereby
    A(k+1:n,k+1:n) = A(k+1:n,k+1:n) - G(k+1:n,k)*transpose(G(k+1:n,k));
  end
endfunction
```

Exercise 6.3. Show that for all regular $A \in \mathbb{R}^{n\times n}$

$$\operatorname{cond}_2(A) \leq \operatorname{cond}_F(A) \leq n\operatorname{cond}_2(A).$$

Show that the last inequality is sharp.

Suggested Solution. For all regular A we know that

$$\|A\|_2 \leq \|A\|_F$$

from Exercise 4.2 (d) and

$$\|A\|_F \leq \sqrt{n}\|A\|_2$$

from Exercise 5.3 (b). From this we obtain for all regular A

$$\operatorname{cond}_2(A) = \|A\|_2\|A^{-1}\|_2 \leq \underbrace{\|A\|_F\|A^{-1}\|_F}_{=\operatorname{cond}_F(A)} \leq \sqrt{n}\|A\|_2\sqrt{n}\|A^{-1}\|_2 = n\operatorname{cond}_2(A).$$

The last inequality is sharp for $A := E$:

$$\underbrace{\|E\|_F}_{=\sqrt{n}} \leq \sqrt{n}\underbrace{\|E\|_2}_{=1}.$$

Exercise 6.4. Let $A \in \mathbb{R}^{2\times 2}$ be symmetric and assume that

$$\det(A) = 1 \quad \text{and} \quad \frac{\operatorname{trace}(A)}{2} = N,$$

where $N > 0$. Show the following:

(a) It is necessarily true that $N \geq 1$.

(b) It is

$$\operatorname{cond}_2(A) \geq 4N^2 - 2N - 1 \quad \text{for } N \geq 1. \tag{1}$$

Suggested Solution. A sym. $\Rightarrow$ $V^TAV = D = \operatorname{diag}(\lambda_1, \lambda_2)$ for some $V \in \mathcal{O}_n$ and $\lambda_{1/2} \in \mathbb{R}$. Since $|A| = \lambda_1\lambda_2 = 1$ and $\operatorname{trace}(A) = \lambda_1 + \lambda_2 > 0$ it follows (without loss of generality) that $\lambda_1 \geq \lambda_2 > 0$. First, we transform the quadratic characteristic polynomial into normal form

$$\begin{aligned}\chi_A(\lambda) &= \chi_D(\lambda)\\ &= (\lambda_1 - \lambda)(\lambda_2 - \lambda)\\ &= \lambda^2 - (\underbrace{\lambda_1 + \lambda_2}_{=2N})\lambda + \underbrace{\lambda_1\lambda_2}_{=1}\\ &= (\lambda - N)^2 - (N^2 - 1). \end{aligned} \tag{2}$$

(a) From (2) we deduce that

$$\lambda_{1/2} \in \mathbb{R} \;\Leftrightarrow\; N^2 - 1 \geq 0 \;\overset{N\geq 0}{\Leftrightarrow}\; N \geq 1.$$

(b) Additionally, from Equation 2 we can write the eigenvalues of A in terms of N:

$$\lambda_{1/2} = N \pm \sqrt{N^2 - 1}.$$

From this we can finally deduce

$$\operatorname{cond}_2(A) = \frac{\lambda_1}{\lambda_2} = \frac{N + \sqrt{N^2-1}}{N - \sqrt{N^2-1}} = N^2 + 2N\sqrt{N^2-1} + N^2 - 1. \tag{3}$$

The claim follows when rhs of (3) $\geq$ rhs of (1):

$$\begin{aligned} & & 2N^2 + 2N\sqrt{N^2-1} - 1 &\geq 4N^2 - 2N - 1\\ &\Leftrightarrow & \sqrt{N^2-1} &\geq N - 1\\ &\Leftrightarrow & N^2 - 1 &\geq N^2 - 2N + 1\\ &\Leftrightarrow & N &\geq 1, \end{aligned}$$

which is true by (a).

Exercise 6.5 *(Octave)***.** Write an Octave program that calculates the largest and smallest eigenvalues of a matrix $A \in \mathrm{Gl}(n)$. Avoid the calculation of the inverse matrix A^{-1} and use the LU decomposition of A instead (without row permutations).

Suggested Solution.

```
function [lambdamin,lambdamax] = ex6_5_powermethod(A,nmax)
  % nmax = maximum number of iterations
  % Computation of the largest and smallest absolute value
  [n,n] = size(A);
  % Initialize start vector
  v = ones(n,1);
  % Computation of lambdamax
  for j=1:nmax
    v = A*v;
    v = v/norm(v);
    lambdamax = v'*A*v;
  end
  % Computation of lambdamin
  [L,U] = LU_wo_permutations(A); v = ones(n,1);
  for j=1:nmax
    % Solve SLE Ax = v; x is then the next iterate
    y = L\v; x = U\y; v = x;
    v = v/norm(v);
    lambdamin = v'*A*v;
  end
endfunction
```