Exercise 14.1 *(LU decomposition)*.

(a) Let $A \in \mathbb{R}^{3\times 3}$ be given with

$$A = \begin{pmatrix} 2 & -1 & 2 \\ -4 & 5 & -2 \\ 2 & -10 & -3 \end{pmatrix}.$$

Compute the LU decomposition of A, i.e. determine a unipotent lower triangular matrix L and an upper triangular matrix U such that $A = LU$ is valid.

(b) Let $A \in \mathbb{R}^{3\times 3}$ be given with

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -4 \end{pmatrix}.$$

Compute the LU decomposition (with permutations) of A, i.e. determine a unipotent lower triangular matrix L, an upper triangular matrix U and a permutation matrix P such that $PA = LU$ is valid. For comparability reasons do not use partial pivoting.

(c) Solve the system of linear equations $Ax = b$, where

$$A = \begin{pmatrix} 2 & 1 \\ -4 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 5 \\ -13 \end{pmatrix},$$

by forward and backward substitution.

Suggested Solution.

(a) We have

$$\begin{aligned} A &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & -9 & -5 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{pmatrix} = LU. \end{aligned}$$

(b) Step 1: Permutation vector $p = (1, 0)$

$$P_1 A = \begin{pmatrix} 2 & 3 & 1 \\ 2 & 3 & 3 \\ -4 & -7 & -4 \end{pmatrix}$$

Step 2:

$$\begin{aligned} P_1 A &= L_1^{-1} \cdot L_1 P_1 A \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & -1 & -2 \end{pmatrix} \end{aligned}$$

Step 3: Permutation vector $p = (1, 3)$

$$P_2 P_1 A = P_2 L_1^{-1} P_2 \cdot P_2 L_1 P_1 A$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{pmatrix} = LU,$$

where

$$P = P_2 P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

(c) With

$$Ax = \underbrace{\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}}_{=:L} \underbrace{\begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix}}_{=:U} x = \begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} y = \begin{pmatrix} 5 \\ -13 \end{pmatrix}$$

it follows using forward substitution

$$y = \begin{pmatrix} 5 \\ -3 \end{pmatrix}.$$

Using backward substitution in

$$Ux = y \quad \Leftrightarrow \quad \begin{pmatrix} 2 & 1 \\ 0 & -3 \end{pmatrix} x = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$$

we finally obtain

$$x = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Exercise 14.2 *(Cholesky decomposition)***.** Let $A \in \mathbb{R}^{3\times 3}$ be given with

$$A = \begin{pmatrix} 4 & -2 & 6 \\ -2 & 17 & -11 \\ 6 & -11 & 17 \end{pmatrix}.$$

Determine the Cholesky decomposition of A, i.e. determine a lower triangular matrix $G \in \mathbb{R}^{n\times n}$ with positive diagonal entries, such that $A = GG^T$.

Suggested Solution. Applying the Cholesky algorithm we obtain

$\underline{k = 1:}$ (i) $g_{11} = \sqrt{a_{11}^{(1)}} = \sqrt{4} = \underline{\underline{2}}$

(ii) $g_{21} = \dfrac{a_{21}^{(1)}}{g_{11}} = \dfrac{-2}{2} = \underline{\underline{-1}}$

$g_{31} = \dfrac{a_{31}^{(1)}}{g_{11}} = \dfrac{6}{2} = \underline{\underline{3}}$

(iii) $$(g_{i1} g_{j1})_{2\le i,j\le 3} = \begin{pmatrix} g_{21}^2 & \star \\ g_{31} g_{21} & g_{31}^2 \end{pmatrix} = \begin{pmatrix} 1 & \star \\ -3 & 9 \end{pmatrix}$$

$$\Rightarrow \; A^{(2)} = \left(a_{ij}^{(2)}\right)_{2\le i,j\le 3} = \begin{pmatrix} 17 & -11 \\ -11 & 17 \end{pmatrix} - \begin{pmatrix} 1 & \star \\ -3 & 9 \end{pmatrix} = \begin{pmatrix} 16 & -8 \\ -8 & 8 \end{pmatrix}$$

$\underline{k=2:}$ (i) $g_{22} = \sqrt{a_{22}^{(2)}} = \sqrt{16} = \underline{\underline{4}}$

(ii) $g_{32} = \dfrac{a_{32}^{(2)}}{g_{22}} = \dfrac{-8}{4} = \underline{\underline{-2}}$

(iii) $(g_{i2}g_{j2})_{3\le i,j\le 3} = g_{32}^2 = (-2)^2 = 4$

$$\Rightarrow A^{(3)} = \left(a_{ij}^{(3)}\right)_{3\le i,j\le 3} = a_{33}^{(2)} - g_{32}^2 = 8-4 = 4$$

$\underline{k=3:}$ (i) $g_{33} = \sqrt{a_{33}^{(3)}} = \sqrt{4} = \underline{\underline{2}}$

From this we finally obtain

$$G = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ 3 & -2 & 2 \end{pmatrix}$$

Exercise 14.3 *(condition number)***.** Compute $\mathrm{cond}_2(A)$ and $\mathrm{cond}_F(A)$ where

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}.$$

Suggested Solution. Since A is symmetric we can compute a largest absolute eigenvalue of A and A^{-1}:

$$\begin{aligned} \chi_A(\lambda) &= (1-\lambda)(5-\lambda) - 4 \\ &= 5 - 6\lambda + \lambda^2 - 4 \\ &= (\lambda-3)^2 - 8 \end{aligned}$$

with solutions $\lambda_{1/2} = 3 \pm \sqrt{8}$. From this we obtain

$$\mathrm{cond}_2(A) = \frac{|\lambda_{\max}|}{|\lambda_{\min}|} = \frac{3+\sqrt{8}}{3-\sqrt{8}}.$$

<u>Computation of $\mathrm{cond}_F(A)$</u>: Since

$$A^{-1} = \frac{1}{\det A}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

we obtain

$$\mathrm{cond}_F(A) = \sqrt{5^2 + 2\cdot 2^2 + 1^2}\sqrt{5^2 + 2(-2)^2 + 1^2} = 34.$$

Exercise 14.4 *(Linear Least-Squares Problem)***.** Determine the optimal least-squares fit of the model function

$$y = f(x; k_0, k_1) = k_0 \sin\left(\frac{\pi}{8}x\right) + k_1\sqrt{|x|}$$

to the data points

x	-4	0	4	16
y	3.9	0.1	7.9	12.1

Suggested Solution. Solution via the normal equations: with

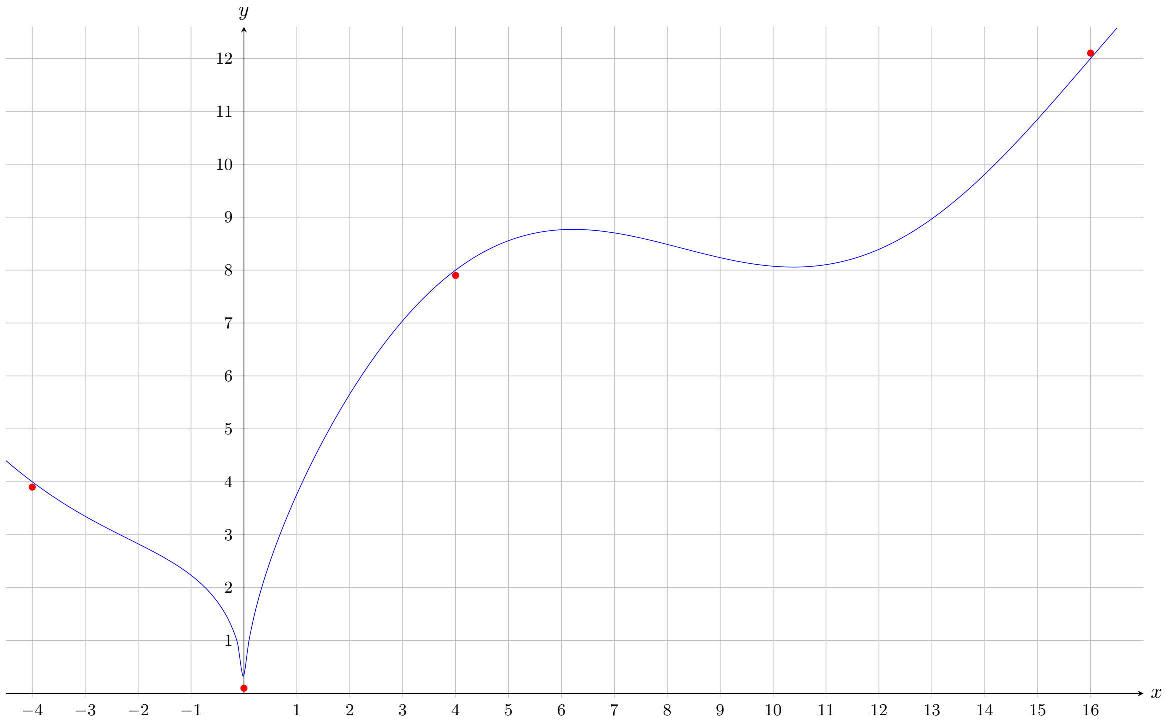
$$A = \begin{pmatrix} -1 & 2 \\ 0 & 0 \\ 1 & 2 \\ 0 & 4 \end{pmatrix}, \quad A^t = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 0 & 2 & 4 \end{pmatrix}, \quad A^tA = \begin{pmatrix} 2 & 0 \\ 0 & 24 \end{pmatrix}, \quad y = \begin{pmatrix} 3.9 \\ 0.1 \\ 7.9 \\ 12.1 \end{pmatrix}, \quad A^ty = \begin{pmatrix} 4 \\ 72 \end{pmatrix}$$

one has to solve the following SLE

$$
\begin{aligned}
& A^t A \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = A^t y \\
\Leftrightarrow\ & \begin{pmatrix} 2 & 0 \\ 0 & 24 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \begin{pmatrix} 4 \\ 72 \end{pmatrix} \\
\Leftrightarrow\ & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} k_0 \\ k_1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\
\Leftrightarrow\ & \begin{cases} k_0 = 2 \\ k_1 = 3 \end{cases}
\end{aligned}
$$

From this we get the optimal model function

$$f(x) = 2\sin\left(\frac{\pi}{8}x\right) + 3\sqrt{|x|}.$$



Exercise 14.5. Consider the system of equations

$$
\begin{aligned}
x^2 - x - y &= 0 \\
x + y - 2 &= 0
\end{aligned}
$$

Apply Newton's method once with the starting point $x^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, i.e. compute $x^{(1)}$.

Suggested Solution. Letting

$$F(x, y) := \begin{pmatrix} x^2 - x - y \\ x + y - 2 \end{pmatrix}$$

we first compute the Jacobian matrix:

$$JF(x, y) = \begin{pmatrix} 2x - 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

With

$$JF(1,0) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad F(1,0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

we need to solve

$$\begin{aligned} JF(x^{(0)})s^{(0)} = -F(x^{(0)}) \quad &\Leftrightarrow \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 1 \end{array}\right) \\ &\Leftrightarrow \quad \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 2 & 1 \end{array}\right) \\ &\Leftrightarrow \quad s^{(0)} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}. \end{aligned}$$

From this we get

$$x^{(1)} = x^{(0)} + s^{(0)} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Exercise 14.6. Let $A \in \mathbb{R}^{n\times n}$ be symmetric an positive definite. Show that

$$\langle\cdot,\cdot\rangle_A \colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R},\ \langle x,y\rangle_A \coloneqq \langle Ax,y\rangle,$$

is a scalar product on $\mathbb{R}^n$. Here, $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product on $\mathbb{R}^n$.

Suggested Solution.

(1) $\forall x,y,z \in \mathbb{R}^n: \quad \langle x+y,z\rangle_A = \langle A(x+y),z\rangle = \langle Ax,z\rangle + \langle Ay,z\rangle = \langle x,z\rangle_A + \langle y,z\rangle_A$

(2) $\forall x,y \in \mathbb{R}^n, \lambda \in \mathbb{R}: \quad \langle \lambda x,y\rangle_A = \langle A(\lambda x),y\rangle = \langle \lambda Ax,y\rangle = \lambda\langle Ax,y\rangle = \lambda\langle x,y\rangle_A$

(3) $\forall x,y \in \mathbb{R}^n: \quad \langle x,y\rangle_A = \langle Ax,y\rangle = \langle x,A^Ty\rangle = \langle x,Ay\rangle = \langle Ay,x\rangle = \langle x,y\rangle_A$

(4) $\forall x \in \mathbb{R}^n\backslash\{0\}: \quad \langle x,x\rangle_A = \langle Ax,x\rangle > 0,$ since A is positive definite.

Exercise 14.7. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite. Show that

$$\max_{1\le i,j\le n} |a_{ij}| = \max_{1\le i\le n} |a_{ii}|.$$

Hint: One of many possible solutions would be to use the result from Exercise 14.6 and apply the Cauchy-Schwarz inequality.

Suggested Solution.

Possibility 1: Let $m \coloneqq \max_{1\le i\le n}|a_{ii}|$ and define the symmetric bilinear form

$$\Psi(x,y) = \langle Ax,y\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean scalar product. By "polarization"we get

$$\Psi(x,y) = \frac{1}{2}\Big(\Psi(x,x) + \Psi(y,y) - \Psi(x-y,x-y)\Big).$$

Letting $x \coloneqq e_i,\ y \coloneqq e_j,\ q_\mp = \Psi(x \mp y, x \mp y)$ we obtain for $1 \le i,j \le n$

$$a_{ij} = \frac{1}{2}\Big(\underbrace{a_{ii}}_{\le m} + \underbrace{a_{jj}}_{\le m} - \underbrace{q_-}_{\ge 0}\Big)$$

and, substituting y by $-y$ we get

$$-a_{ij} = \frac{1}{2}\Big(\underbrace{a_{ii}}_{\leq m} + \underbrace{a_{jj}}_{\leq m} - \underbrace{q_+}_{\geq 0}\Big).$$

This means

$$\max_{1\leq i,j\leq n} |a_{ij}| \leq m.$$

The converse inequality is trivially true.

Possibility 2: Letting $x = e_i, y = e_j$ we get using the Cauchy-Schwarz inequality

$$\begin{aligned} |a_{ij}|^2 &= |\langle e_i, e_j\rangle_A|^2 \\ &\leq |\langle e_i, e_i\rangle_A|\cdot|\langle e_j, e_j\rangle_A| \\ &= |a_{ii}|\cdot|a_{jj}| \end{aligned}$$

First, taking the maximum over all i we get

$$\max_{1\leq i\leq n} |a_{ij}|^2 \leq |a_{jj}| \max_{1\leq i\leq n} |a_{ii}|.$$

Second, we take the maximum over all j to obtain

$$\max_{1\leq i,j\leq n} |a_{ij}|^2 \leq \Big(\max_{1\leq i\leq n} |a_{ii}|\Big)^2.$$

Since

$$\max_{1\leq i,j\leq n} |a_{ij}|^2 = \Big(\max_{1\leq i,j\leq n} |a_{ij}|\Big)^2$$

we can take the square root and the claim follows.

Exercise 14.8. Let $n \in \mathbb{N}$ be arbitrary.

(a) Show that the mapping

$$\rho\colon \mathbb{R}^{n\times n} \to \mathbb{R},\ \rho(A) := \max_{\lambda\in\sigma(A)} |\lambda|$$

defines no matrix norm in $\mathbb{R}^{n\times n}$.

(b) Show that ρ defines a matrix norm on the subset of all symmetric matrices in $\mathbb{R}^{n\times n}$.

Suggested Solution.

(a) For $A := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ we have that $\rho(A) = 0$ contradicting the positive definiteness of a norm.

(b) According to the lecture/exercises we have

$$\rho(A) = |\lambda_{\max}| = \sqrt{\lambda_{\max}^2} = \sqrt{\rho(A^2)} = \sqrt{\rho(A^TA)} = \|A\|_2.$$

Remark: It is also possible to show the norm properties directly (cf. Exercise 14.9).

Exercise 14.9. Show that for all symmetric $A, B \in \mathbb{R}^{n\times n}$ it holds that

$$\rho(A + B) \leq \rho(A) + \rho(B).$$

Suggested Solution. Let v be an eigenvector of (the symmetric matrix) $A+B$ corresponding to a largest eigenvalue $\lambda_{\max}$ with $\|v\|_2 = 1$. Then

$$\begin{aligned}
\rho(A+B) &= |\lambda_{\max}| \\
&= |\lambda_{\max}| \cdot \|v\|_2 \\
&= \|\lambda_{\max} v\|_2 \\
&= \|(A+B)v\|_2 \\
&\leq \|Av\|_2 + \|Bv\|_2 \\
&\leq \|A\|_2 \|v\|_2 + \|B\|_2 \|v\|_2 \\
&= \rho(A) + \rho(B).
\end{aligned}$$