Exercise 9.1. Let $A \in \mathbb{R}^{n\times n}$. In the typical decomposition

$$A = D - L - U,$$

assume that $L = 0$, i.e. A is an upper triangular matrix. Show that in this case

$$\mathcal{L} = \mathcal{J}.$$

That is, the Jacobi method is identical to the Gauss-Seidel method.

Exercise 9.2. There are positive definite matrices for which the Jacobi method does not converge. Consider for $\alpha \in \mathbb{R}$ the matrix

$$A = \begin{pmatrix} 1 & \alpha & \alpha \\ \alpha & 1 & \alpha \\ \alpha & \alpha & 1 \end{pmatrix}.$$

(a) Determine the Jacobi operator $\mathcal{J}$ corresponding to the matrix A.

(b) Determine all $\alpha \in \mathbb{R}$ for which the Jacobi method converges.

(c) For which $\alpha \in \mathbb{R}$ is the matrix A positive definite, but the Jacobi method does not converge?

Exercise 9.3. Consider the iteration method $x^{(k+1)} = \Phi(x^{(k)})$ with $\Phi(x) = Tx + c$, but this time with the additional assumption that T is nilpotent. That is, there exists an $N \in \mathbb{N}$ s.t.

$$T^N = 0.$$

Show that

(a) $\rho(T) = 0$. What does that mean for the convergence of the iteration method?

(b) The iteration method gives the exact solution for each starting vector $x^{(0)}$ after N iterations at the latest.

Exercise 9.4. Let the iteration procedure

$$x^{(k+1)} = \Phi(x^{(k)})$$

with any starting vector $x^{(0)} \in \mathbb{R}^n$ be given and let it be defined by the affine-linear mapping

$$\Phi(x) = Tx + c$$

for some matrix $T \in \mathbb{R}^{n\times n}$ and fixed $c \in \mathbb{R}^n$.

(a) Under which conditions is there always exactly one fixed point of Φ?

(b) Suppose you know according to the construction of your method that Φ has a fixed point $\hat{x}$. For $\rho(T) \geq 1$ and $\lambda_{\max} \in \mathbb{R}$ ($\lambda_{\max}$ = largest absolute eigenvalue of T), specify an initial guess $x^{(0)}$ for which the iteration method does not converge.