Exercise 7.1 *(The Banach fixed point theorem (BFT))*. For the function

$$f\colon \mathbb{R} \to \mathbb{R},\ f(x) = e^{-x} - x$$

a zero in the interval $I = [\frac{1}{3}, 1]$ is to be calculated approximately. For this, approximate a fixed point of the function

$$\Phi\colon I \to \mathbb{R},\ \Phi(x) := e^{-x}.$$

(a) Show that Φ is a self map on I. (Can you do without a calculator here if you know that $e < 3$?)

Suggested solution:

Since $\Phi'(x) = -e^{-x} < 0$ on $\mathbb{R}$ one knows that Φ is decreasing (and positive) on $\mathbb{R}$ and thus we can infer that

$$1 = \Phi(0) \geq \Phi(1/3) \geq \Phi(x) \geq \Phi(1) = \frac{1}{e} \geq \frac{1}{3},$$

that is

$$\Phi(x) \in [1/3, 1].$$

(b) Show that Φ is contracting with contraction factor $q < 1$ (in other words: show that Φ is Lischitz continuous with Lipschitz constant $q < 1$).

Suggested solution:

Since for all $x \in I$ we have

$$|\Phi'(x)| = e^{-x} \leq e^{-\frac{1}{3}} =: q \approx 0.7165313106 < 1,$$

it follows that

$$\begin{aligned}|\Phi(x) - \Phi(y)| &= |\Phi'(\xi)(x - y)| \\ &\leq |\Phi'(\xi)||x - y| \\ &\leq \max_{x \in I}|\Phi'(x)| \cdot |x - y| \\ &\leq q|x - y|,\end{aligned}$$

where the mean value theorem that for some $\xi \in (1/3, 1)$ was employed.

(c) Using the BFT it follows from 1. and 2. that there exists exactly one fixed point $\hat{x} \in I$. Let $x_0 := \frac{1}{3}$ be the initial guess. (Clearly, you could choose any other $x_0 \in I$. But choosing x_0 as above allows you to directly compare your solution to the suggested solution..) Use the a priori estimates of the BFT to determine the maximum number $N_0 = N_0(\varepsilon, x_0) \in \mathbb{N}$ of iterations needed to determine $\hat{x}$ to an accuracy of (an unspecified) $\varepsilon > 0$. Determine N_0 for $\varepsilon := 10^{-5}$.

Suggested solution:

If we can choose $N_0 \in \mathbb{N}$ s.t. for all $k \geq N_0$

$$\frac{q^k}{1 - q}|x_1 - x_0| < \varepsilon \tag{1}$$

it would then follow by the a priori estimate that

$$|x_k - \hat{x}| \leq \frac{q^k}{1 - q}|x_1 - x_0| < \varepsilon.$$

But

$$
\begin{aligned}
(1) \quad \Leftrightarrow \quad & q^k < \frac{\varepsilon(1-q)}{|x_1 - x_0|} \\
\Leftrightarrow \quad & k \ln q < \ln\left(\frac{\varepsilon(1-q)}{|x_1 - x_0|}\right) \\
\overset{q<1}{\Leftrightarrow} \quad & k > \ln\left(\frac{\varepsilon(1-q)}{|x_1 - x_0|}\right) / \ln(q).
\end{aligned}
$$

(Note that in `Octave` you have three built-in functions to return logarithms: `log(x)`$= \ln(x)$, `log2(x)`$= \log_2(x)$, `log10(x)`$= \log_{10}(x)$.) Thus one can choose $N_0 \in \mathbb{N}$ to be the smallest natural number k with this property.

To calculate N_0 for the ε given, we need to calculate

$$x_1 = \Phi(x_0) = e^{-\frac{1}{3}} (= q).$$

Since

$$\ln\left(\frac{\varepsilon(1-q)}{|x_1 - x_0|}\right) / \ln(q) = 35.44312667497638$$

it follows $N_0 = 36$.

(d) Using `Octave`, calculate the first N_0 iterates of the iteration sequence (x_k) induced by the iteration rule $x_{k+1} = \Phi(x_k)$.

Suggested solution:

Using

```
% sheet 7 ex.1
format long
x0=1/3;
x1=exp(-1/3);
q=x1;
eps=1e-5;
% ceil(x) is the smallest natural number k>=x
% log is the natural logarithm
N0_aux=log(eps*(1-q)/abs(x1-x0))/log(q)
N0=ceil(N0_aux)
x=x0;
for i=1:N0
i
x=exp(-x)
endfor
```

we obtain:

k	$x^{(k)}$	k	$x^{(k)}$	k	$x^{(k)}$
1	0.7165313106	13	0.5673008037	25	0.5671434648
2	0.4884435800	14	0.5670539648	26	0.5671431915
3	0.6135806406	15	0.5671939531	27	0.5671433465
4	0.5414088039	16	0.5671145581	28	0.5671432586
5	0.5819278524	17	0.5671595860	29	0.5671433085
6	0.5588200050	18	0.5671340486	30	0.5671432802
7	0.5718834855	19	0.5671485319	31	0.5671432962
8	0.5644612822	20	0.5671403178	32	0.5671432871
9	0.5686664150	21	0.5671449763	33	0.5671432923
10	0.5662801181	22	0.5671423342	34	0.5671432894
11	0.5676330442	23	0.5671438327	35	0.5671432910
12	0.5668655979	24	0.5671429829	36	0.5671432901

(e) Use the a posteriori estimate of the BFT to obtain an improved estimate for the error $|\hat{x} - x_{N_0}|$.

Suggested solution: From the a posteriori estimate we finally get

$$|x_k - \hat{x}| \leq \frac{q}{1-q} |x_k - x_{k-1}| \approx 1.26e - 07.$$

(f) Use your calculator to compute the best possible iterate (using the "ANS-technique").

Suggested solution: If your calculator has an "ANS-key" then you can proceed as follows:

1. Type: `1÷3=`. Then the value "$\frac{1}{3}$" is stored in the "Answer-Storage"
2. Type: $e^{-\text{ANS}} =$ to obtain $x_1 = e^{-x_0}$
3. Type: $=$ to obtain $x_2 = e^{-x_1}$
4. Type: $=$ to obtain $x_3 = e^{-x_2}$
5. $\vdots$

The exercises can be discussed in the Tuesday lecture next week.