Numerical Methods and Optimization

Summary of the basics for chapter 3

10.10.2022

Content

Fixed Point Methods

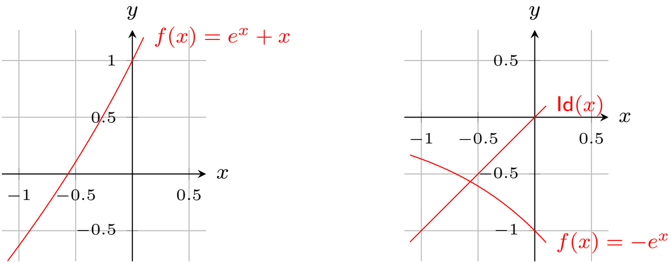
- until now we have tried to solve SLE "exactly"
- $\rightarrow$ "direct methods": LU or Cholesky decomposition with forward/backward substitution; complexity: n^3
- now we want to determine solutions approximatively (with a given precision)
- more generally: approximize zeros of given functions $f(x) = 0$
- for this we use "iterative methods":
- also useful for sparse matrices; LU quickly destroys sparseness
- reformulate: $f(x) = 0 \quad \Leftrightarrow \quad \Phi(x) = x$ (fixed point equation)
- idea: $x^{(k+1)} := \Phi(x^{(k)})$ for given $x^{(0)}$ (fixed point iteration)
- if Φ is continuous and if $x^{(k)} \rightarrow \hat{x}$:

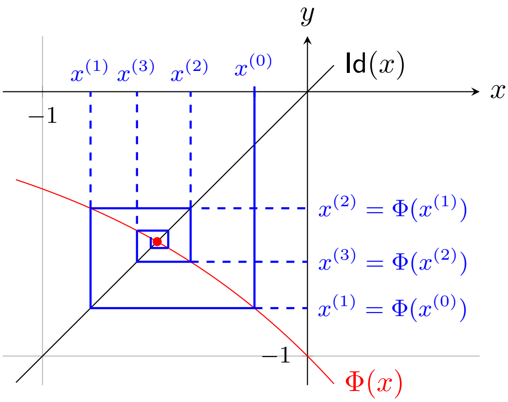
$$\hat{x} = \lim x^{(k+1)} = \lim \Phi(x^{(k)}) = \Phi(\lim x^{(k)}) = \Phi(\hat{x})$$

$\rightsquigarrow \quad \hat{x}$ is a fixed point of Φ and thus a root of f

Example of fixed point iteration

- example: sought: x s.t.
  $f(x) = e^x + x = 0 \quad \Leftrightarrow \quad \Phi(x) := -e^x = x$




y
$\mathsf{Id}(x)$
x
$x^{(1)}$ $x^{(3)}$ $x^{(2)}$ $x^{(0)}$
-1
$x^{(2)} = \Phi(x^{(1)})$
$x^{(3)} = \Phi(x^{(2)})$
$x^{(1)} = \Phi(x^{(0)})$
-1
$\Phi(x)$

Axiom (completeness of the real numbers)

For all nonempty bounded $A \subset \mathbb{R}$ there exists a least upper bound.

Definition (real Cauchy sequences)

A real sequence (a_n) is called a Cauchy sequence if for any $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ s.t. for all $n, m \geq N(\varepsilon)$

$$|a_n - a_m| < \varepsilon.$$

Every Cauchy sequence in the real numbers has a limit. This is equivalent to (or also one definition of) the completeness of the real numbers. This may be generalized to $\mathbb{R}^n$:

Definition (Cauchy sequences in space)

A real sequence (a_n) in $(\mathbb{R}^n, \|\cdot\|)$ is called a Cauchy sequence if for any $\varepsilon > 0$ there exists an $N(\varepsilon) \in \mathbb{N}$ s.t. for all $n, m \geq N(\varepsilon)$

$$\|a_n - a_m\| < \varepsilon.$$

Definition (completeness in space)

A nonempty subset $X \subset (\mathbb{R}^n, \|\cdot\|)$ is called complete, if any Cauchy sequence in X converges in X.

Proposition

Let (a_n) be a convergent sequence in $\mathbb{R}^n$. Then (a_n) is a Cauchy sequence.

PROOF: Let $a_n \to a$. Then

$$\forall \varepsilon > 0 \exists N(\varepsilon) \in \mathbb{N} \forall n \geq N : \|a_n - a\| < \frac{\varepsilon}{2}.$$

Hence, for all $n, m \geq N$

$$\|a_n - a_m\| = \|a_n - a + a - a_m\| \leq \|a_n - a\| + \|a_m - a\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Definition (Lipschitz continuity)

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be norms on $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and $D \subset \mathbb{R}^n$ be nonempty. A map $f\colon D \to \mathbb{R}^m$ is said to be Lipschitz continuous on D (wrt these norms) if there is a constant L (named Lipschitz constant) s.t. for all $x, y \in D$

$$\|f(x) - f(y)\|_2 \leq L\|x - y\|_1.$$

Lipschitz continuity does not depend on the given norms (but L does). Geometric intuition:

- $f\colon \mathbb{R} \to \mathbb{R}$ with $\|\cdot\|_1 = \|\cdot\|_2 = |\cdot|$. Then

$$\left|\frac{f(x) - f(y)}{x - y}\right| \leq L$$

  implies that L is an upper bound of the slope of any chord joining $P = (x, f(x))$ and $Q = (y, f(y))$.

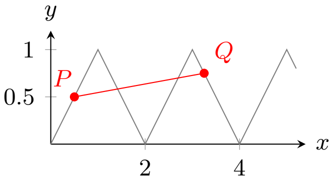


Figure: Function with Lipschitz constant $L = 1$

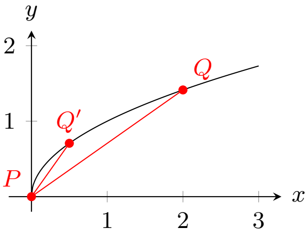


Figure: $\sqrt{\cdot}$ is not Lipschitz continuous on, say, $[0, 3]$

- Geogebra files 015.ggb and 016.ggb (see LC)

Proposition

Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$ and $\|\cdot\|_*$ be any matrix norm that is compatible with $\|\cdot\|$. If $f\colon D \subset \mathbb{R}^n \to \mathbb{R}^n$ is (totally) differentiable on D, where $D \neq \emptyset$ is open and convex, then $\|Jf\|_* \leq L < \infty$ on D implies that f is Lipschitz with constant L.

PROOF: Application of the mean value theorem (of differential calculus). ∎

Examples:

- $\sin\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz with constant $L = 1$:
  Since $|\sin' x| = |\cos x| \leq 1$ it follows

$$|\sin x - \sin y| = |\cos(\xi)(x-y)| \leq 1 \cdot |x-y|.$$

Proposition

If $f\colon D \subset \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous then f is also uniformly continuous on D.

PROOF: Assume that for all $x, y \in D$

$$\|f(x) - f(y)\| \leq L\|x - y\|.$$

Let $\varepsilon > 0$ be arbitrary. Letting $\delta := \frac{\varepsilon}{L}$, it follows that $\forall x, y \in D$ with $\|x - y\| < \delta$

$$\|f(x) - f(y)\| \leq L \underbrace{\|x - y\|}_{<\delta} < L\delta = L\frac{\varepsilon}{L} = \varepsilon.$$