Exercise 1. Calculate $\sigma(A), \rho(A), \|A\|_\infty, \|A\|_F$ and $\|A\|_2$ for the following matrices

(a) $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ **(b)** $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & -4 \\ 7 & 8 & -6 \end{pmatrix}$

To calculate the eingenvalues in b) you can use the build-in function `eig( )` in `Octave`. Specify the results on 3 decimal places after the comma.

Suggested Solution.

(a)
- $\chi_A(\lambda) = \det(A - \lambda E) = (1-\lambda)^3 \quad \Rightarrow \quad \sigma(A) = \{1\}$ and $\rho(A) = 1$
- $\|A\|_\infty = \max\limits_i \sum\limits_j |a_{ij}| = 2$
- $\|A\|_F = \Big(\sum\limits_{i,j} a_{ij}^2\Big) = 4$

For the calculation of $\|A\|_2$ we determine A^TA at first:

$$A^TA = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

Now we calculate $\sigma(A^TA)$: From

$$\begin{aligned} \chi_{A^TA}(\lambda) = \det \begin{pmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{pmatrix} &= \Big((1-\lambda)^2(2-\lambda) - (1-\lambda)\Big) \\ &= (1-\lambda)\Big((1-\lambda)(2-\lambda) - 1\Big) \\ &= (1-\lambda)(\lambda^2 - 3\lambda + 1) \end{aligned}$$

we obtain

$$\lambda_1 = 1, \quad \lambda_{2/3} = \frac{3 \pm \sqrt{5}}{2}$$

and thus

$$\|A\|_2 = \sqrt{\rho(A^TA)} = \sqrt{\frac{3+\sqrt{5}}{2}} \approx 1.618$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & -4 \\ 7 & 8 & -6 \end{pmatrix}$$

Using `eig(A)` in Octave we obtain the eigenvalues

$$\sigma(A) = \{-5.5414; 5.0000; 0.5414\}$$

and therefore

$$\rho(A) = 5.5414.$$

Moreover, we can calculate

$$\|A\|_\infty = \max_i \sum_j |a_{ij}| = \max_i \{6, 13, 21\} = 21$$

$$\|A\|_F = \Bigg(\sum_{i,j} a_{ij}^2\Bigg)^{\frac{1}{2}} = (1+4+9+16+25+16+49+64+39)^{\frac{1}{2}} = \sqrt{220} \approx 14.832$$

Exercise 2. Let A be representable as $A = BB^T$ for some invertible matrix $B \in \mathbb{R}^{n\times n}$. Are the following statements true or false? Give an explanation!

(a) $\det(A) \neq 0$

(b) A is symmetric

(c) $A > 0$

(d) The diagonal entries of B are the square roots of the eigenvalues of A.

Suggested Solution. Let $A = BB^T$ with invertible B.

(a) $\det(A) \neq 0$ is true:

$$\det(A) = \det(BB^T) = \det(B)\det(B^T) = \underbrace{\det(B)}_{\neq 0}{}^2 \neq 0.$$

(b) A is sym.:

$$A^T = (BB^T)^T = (B^T)^T B^T = A.$$

(c) A is positive definite: for all $x \neq 0$ we have

$$\langle x, Ax\rangle = \langle x, BB^T x\rangle = \langle B^T x, B^T x\rangle = \|B^T x\|_2^2 > 0$$

since B^T is invertible ($B^T(B^{-1})^T = E$).

(d) The statement is false due to the following counter example:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Exercise 3 *(LU decomposition)***.** Determine (by hand calculation) the LU decomposition of the following matrices:

(a) $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix}$ **(b)** $B = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 8 & 1 \\ 2 & 1 & 2 & 16 \end{pmatrix}$

Check your result and use the decomposition to calculate the determinant.

Suggested Solution.

(a)

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}}_{L_1^{-1}}\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 2 & 5 & 9 \\ 0 & 3 & 9 & 19 \end{pmatrix}}_{A^{(1)}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix}}_{L_1^{-1}L_2^{-1}}\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 10 \end{pmatrix}}_{A^{(2)}}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{pmatrix}}_{L_1^{-1}L_2^{-1}L_3^{-1}=L}\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{A^{(3)}=R}$$

$\Rightarrow \det(A) = \det(L)\det(R) = 1.$

(b)
$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 1 & 8 & 1 \\ 2 & 4 & 2 & 16 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{pmatrix}}_{L_1^{-1}} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & -1 & 0 & 14 \end{pmatrix}}_{A^{(1)}} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -\frac{1}{3} & 0 & 1 \end{pmatrix}}_{L_1^{-1}L_2^{-1}=L} \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 14 \end{pmatrix}}_{A^{(2)}=R}$$

$\Rightarrow \det(A) = \det(L)\det(R) = 3 \cdot 7 \cdot 14 = 294.$

Exercise 4. Let L_i and L_j for $j > i$ be the Frobenius matrices of the ith and jth step of the LU decomposition of a matrix of dimension n. That is, L_i and L_j are unipotent lower triangular matrices that differ exactly in the ith or jth column from the identity matrix. Show that the matrix L_iL_j arises from the matrix L_j by replacing the ith column there with the ith column of L_i.

Suggested Solution. First, it is

$$L_i = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & l_{i+1,i} & \ddots & \\ & & \vdots & & 1 & \\ & & & & & \ddots \\ & & l_{n,i} & & & & 1 \end{pmatrix} \quad \text{and} \quad L_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \ddots & & \\ & & & & 1 & \\ & & & & l_{j+1,j} & \ddots \\ & & & & \vdots & \\ & & & & l_{n,j} & & 1 \end{pmatrix}.$$

Since $j > i$ we obtain

$$L_iL_j = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & l_{i+1,i} & \ddots & & \\ & & \vdots & & 1 & \\ & & \vdots & & l_{j+1,j} & \ddots \\ & & \vdots & & \vdots & \\ & & l_{n,i} & & l_{n,j} & & 1 \end{pmatrix}.$$

Exercise 5. Let $A \in \mathbb{R}^{n\times n}$ be regular. Count the number of multiplications and divisions used in the LU decomposition and in the forward and backward substitution.

Hint: To calculate $\sum_{k=1}^{n-1} k^2$ you can use the telescoping sum

$$\sum_{k=1}^{n-1} \Big((k+1)^3 - k^3\Big).$$

Suggested Solution.
(a) LU decomposition: Number of mult./div in the kth step:

Recall that

$$L_k = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & -l_{k+1,k} & 1 & & \\ & & \vdots & & \ddots & \\ & & -l_{n,k} & & & 1 \end{pmatrix}.$$

To calculate the l_{ij} we need $(n-k)$ divisions.

$$A^{(k-1)} = \left(\begin{array}{c|cccc} \begin{matrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{matrix} & & & * & \\ \hline & a_{kk}^{(k-1)} & a_{k,k+1}^{(k-1)} & \cdots & a_{kn}^{(k-1)} \\ & a_{k+1,k}^{(k-1)} & * & \cdots & * \\ 0 & \vdots & \vdots & & \vdots \\ & a_{n,k}^{(k-1)} & * & \cdots & * \end{array}\right)$$

To determine the stars in the $(k+1)$th row we have to multiply the gray shaded entries by the respective l_{ij} and add them to the $(k+1)$th row. That needs $(n-k)$ multiplications. Because this has to be done for $(n-k)$ rows, we have in summary $(n-k)^2$ multiplications. Therefore, we have

$$\mathrm{Ops}_k = (n-k)(n-k+1)$$

mult./div. in the kth step of the LU decomposition.

Using a telescoping sum we first note that

$$\begin{aligned} n^3 - 1 &= \sum_{k=1}^{n-1} \Big((k+1)^3 - k^3\Big) \\ &= \sum_{k=1}^{n-1} (3k^2 + 3k + 1) \end{aligned}$$

This is equivalent to

$$\frac{n^3 - n}{3} = \sum_{k=1}^{n-1} k(k+1).$$

Thus the total number of mult./div. equals

$$\sum_{k=1}^{n-1} (n-k)(n-k+1) = \sum_{k=1}^{n} k(k+1) = \frac{n^3 - n}{3}.$$

(b) backward substitution: Number of mult./div in the kth step:

We have to isolate x_k in

$$\begin{aligned} & u_{kk}x_k + x_{k+1}u_{k,k+1} + \ldots + x_n u_{k,n} = b_k \\ \Leftrightarrow \quad & x_k = \Big(b_k - \underbrace{x_{k+1}u_{k,k+1} + \ldots + x_n u_{k,n}}_{(n-k)\text{ mult}}\Big) \underbrace{/u_{kk}}_{1\text{ div.}} \end{aligned}$$

Thus for the total number of mult./div. we obtain

$$\sum_{k=1}^{n}(n-k+1) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

backward/forward substitution: Since for the forward substitution we obtain the same effort, we obtain in total for the backward and forward substitution:

$$\#\text{Ops} = n(n+1) \sim n.$$

The exercises can be discussed in the Tuesday lecture next week.