Exercise 12.1. Consider Newton's method to approoximate a zero of

$$f(x) = x^3 - x.$$

Give four initial values for which the Newton method does not converge.

Hint: Investigate whether the iterates are well-defined at all or whether they only jump back and forth between two values.

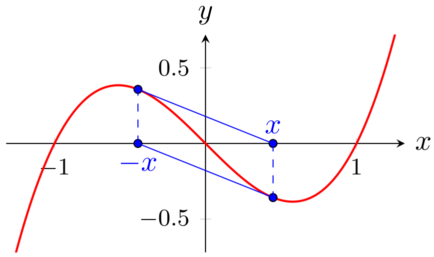
Suggested Solution. Newton's method is defined as follows:

$$x^{(k+1)} = x^{(k)} - \frac{f(x^k)}{f'(x^{(k)})}.$$

Thus, initial values for which Newton's method is not well-defined are obviously given by the requirement $f'(x) = 0$:

$$f'(x) = 0 \quad \Leftrightarrow \quad 3x^2 - 1 = 0 \quad \Leftrightarrow \quad x_{1/2} = \pm\frac{1}{\sqrt{3}}.$$

Thus, in any case, $x_1 = \frac{1}{\sqrt{3}}$ and $x_2 = -\frac{1}{\sqrt{3}}$ are initial values for which Newton's method is not at all is defined. To find two more values for which Newton's method does not converge, consider the sketch of the function $f(x)$:



Due to the symmetry, one expects two values between which Newton's method jumps back and forth, since the tangents meet the same zeros (except for the sign). Since the iteration function $\Phi(x)$ of Newton's method is an odd function in this case, $\Phi(-x) = -\Phi(x)$ is valid. Thus, if one has found an $\tilde{x}$ for which $\Phi(\tilde{x}) = -\tilde{x}$ holds, then, starting from $x(0) := \tilde{x}$ we obtain:

$$x^{(2)} = \Phi(x^{(1)}) = \Phi\Big(\underbrace{\Phi(\tilde{x})}_{=-\tilde{x}}\Big) = -\Phi(\tilde{x}) = \tilde{x} = x^{(0)}.$$

That is, Newton's method continually jumps back and forth between $\tilde{x}$ and $-\tilde{x}$ for all $\tilde{x} \neq 0$. Thus the crucial condition is $\Phi(\tilde{x}) = -\tilde{x} \neq 0$. So, we are looking for solutions of the equation

$$\Phi(x) = -x,$$

i.e.

$$x - \frac{f(x)}{f'(x)} = -x \quad \Leftrightarrow \quad x - \frac{x^3 - x}{3x^2 - 1} = -x.$$

Rearranging leads to

$$\begin{aligned} & -2x(3x^2 - 1) = -(x^3 - x) \\ \Leftrightarrow \quad & -6x^3 + 2x = -x^3 + x \\ \Leftrightarrow \quad & 5x^3 - x = 0 \\ \Leftrightarrow \quad & x(5x^2 - 1) = 0. \end{aligned}$$

Hence, the nonzero solutions are $x_3 = \frac{1}{\sqrt{5}}$ and $x_4 = -\frac{1}{\sqrt{5}}$.

Exercise 12.2. Assume that a function f has an m-fold zero $\hat{x}$ in the interior of the interval $[a, b]$ with $a, b \in \mathbb{R},\ a < b$, and is of the form

$$f(x) = (x - \hat{x})^m g(x),$$

where $g \in C^3([a, b])$ with $g(\hat{x}) \neq 0$.

(a) Determine the corresponding iteration mapping for Newton's method in this case.

(b) Calculate the value $\Phi'(\hat{x})$. What can you conclude from this about the local convergence of the method?

(c) Show that the local quadratic convergence can be recovered using the modified Newton method

$$x^{(k+1)} = x^{(k)} - m\frac{f(x^{(k)})}{f'(x^{(k)})}.$$

Suggested Solution.

(a) In this case we obtain

$$f'(x) = m(x - \hat{x})^{m-1} g(x) + (x - \hat{x})^m g'(x)$$

so that

$$\begin{aligned}\Phi(x) &= x - \frac{(x - \hat{x})^m g(x)}{m(x - \hat{x})^{m-1} g(x) + (x - \hat{x})^m g'(x)}\\ &= x - \frac{(x - \hat{x}) g(x)}{m g(x) + (x - \hat{x}) g'(x)}.\end{aligned}$$

(b) It follows using (a) that

$$\begin{aligned}\Phi'(x) = 1 &- \frac{\Big(g(x) + (x - \hat{x})g'(x)\Big)\Big(mg(x) + (x - \hat{x})g'(x)\Big)}{\Big(mg(x) + (x - \hat{x})g'(x)\Big)^2}\\ &- \frac{(x - \hat{x})g(x)\Big(mg'(x) + g'(x) + (x - \hat{x})g''(x)\Big)}{\Big(mg(x) + (x - \hat{x})g'(x)\Big)^2}\end{aligned}$$

and hence

$$\Phi'(\hat{x}) = 1 - \frac{mg(\hat{x})^2}{m^2 g(\hat{x})^2} = 1 - \frac{1}{m} < 1.$$

Since on the one hand we have $\Phi'(\hat{x}) < 1$, but on the other hand it is $\Phi'(\hat{x}) \neq 0$, Newton's method converges only linearly (and not quadratically). This follows from the local representation

$$x^{(k+1)} - \hat{x} = \Phi(x^{(k)}) - \Phi(\hat{x}) = \underbrace{\Phi'(\hat{x})}_{\neq 0}(x^{(k)} - \hat{x}) + \frac{1}{2}\Phi''(\xi)(x^{(k)} - \hat{x})^2.$$

(c) The modified Newton method leads to the modified iteration mapping

$$\begin{aligned}\Phi(x) &= x - m\frac{f(x)}{f'(x)}\\ &= x - m\frac{(x - \hat{x})^m g(x)}{m(x - \hat{x})^{m-1} g(x) + (x - \hat{x})^m g'(x)}\\ &= x - m\frac{(x - \hat{x}) g(x)}{m g(x) + (x - \hat{x}) g'(x)}.\end{aligned}$$

Analogously to the calculation in (b) it follows that

$$\Phi'(\hat{x}) = 1 - m\frac{mg(\hat{x})^2}{m^2 g(\hat{x})^2} = 1 - m\frac{1}{m} = 0.$$

Thus, the quadratic convergence could be recovered.

Exercise 12.3. Consider the mapping $F\colon \mathbb{R}^n \to \mathbb{R}^n$,

$$F(x) = Ax - b,$$

where $A \in \mathrm{Gl}(n)$ and $b \in \mathbb{R}^n$. Show that the general Newton method in this case yields the exact solution for each initial value already after the first iteration step.

Suggested Solution. Because F is affine linear we obtain for all $x \in \mathbb{R}^n$

$$JF(x) = A.$$

For an arbitrary starting vector we therefore obtain

$$\begin{aligned} x^{(1)} &= x^{(0)} - JF(x^{(0)})^{-1} F(x^{(0)}) \\ &= x^{(0)} - A^{-1}(Ax^{(0)} - b) \\ &= x^{(0)} - x^{(0)} + A^{-1}b \\ &= A^{-1}b. \end{aligned}$$

Hence, the exact solution $\hat{x} = A^{-1}b$ will be obtained after only one iteration.

Exercise 12.4. Determine a solution to the following system of nonlinear equations using Newton's method (cf. Exercise 11.5):

$$F(x_1, x_2) = \begin{pmatrix} F_1(x_1, x_2) \\ F_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} 6x_1 - 2x_2 + 4x_1\left(x_1^2 - \sqrt{x_2^2 - 1}\right) \\ -2x_1 + 4x_2 - \frac{2x_2(1+x_1^2)}{\sqrt{x_2^2-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Use the initial guess $x^{(0)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

(a) Formulate Newton's method for this specific case.

Hint: You can use the following derivatives:

$\partial_{x_1} F_1(x(1), x(2)) =$ `12*x(1)^2 - 4*(x(2)^2 - 1)^(1/2) + 6`
$\partial_{x_2} F_1(x(1), x(2)) =$ `-(4*x(1)*x(2))/(x(2)^2-1)^(1/2)-2`
$\partial_{x_1} F_2(x(1), x(2)) = \partial_{x_2} F_1(x(1), x(2))$
$\partial_{x_2} F_2(x(1), x(2)) =$ `(2*x(1)^2+2)/(x(2)^2-1)^(3/2)+4`

(b) Implement Newton's method for this case in Octave and calculate $x^{(7)}$.

(c) As in (b) but with damping factor $\lambda = 0.8$. Compute in this case also $x^{(20)}$.

Suggested Solution.

(a) For the Jacobian matrix we obtain

$$JF(x_1,x_2) = \begin{pmatrix} 12x_1^2 - 4\sqrt{x_2^2-1}+6 & -4\frac{x_1x_2}{\sqrt{x_2^2-1}} - 2 \\ -4\frac{x_1x_2}{\sqrt{x_2^2-1}} - 2 & \frac{2x_1^2+2}{\sqrt{x_2^2-1}^3} + 4 \end{pmatrix}.$$

With $F(x_1,x_2)$ and $x^{(0)}$ as above, for all $k \geq 0$ one has to solve the following SLE in $s^{(k)}$

$$JF(x^{(k)})s^{(k)} = -F(x^{(k)})$$

and set $x^{(k+1)} := x^{(k)} + s^{(k)}$.

(b) The implementation could be done as follows

```
function [x] = newton_12_4(x_start,Nmax,lambda)
  x = x_start;
  for j = 1:Nmax
    % compute z=F(x),JF(x)
    z(1) = 6*x(1)-2*x(2)+4*x(1)*(x(1)^2-sqrt(x(2)^2-1));
    z(2) = -2*x(1)+4*x(2)-(2*x(2)*(1+x(1)^2))/sqrt(x(2)^2-1);
    JF=[12*x(1)^2-4*(x(2)^2-1)^(1/2)+6,-(4*x(1)*x(2))/(x(2)^2-1)^(1/2)-2;
    -(4*x(1)*x(2))/(x(2)^2-1)^(1/2)-2,(2*x(1)^2+2)/(x(2)^2-1)^(3/2)+4];
    s = JF\[-z(1);-z(2)];
    x = x + lambda*s;
  end
end
```

The function call is then made as follows: `newton_12_4([1;2],7,1)`.

For this initial value we obtain two complex values for x_1 and x_2. It seems that the initial value is too far from a solution.

(c) With `newton_12_4([1;2],20,0.8)`, i.e. damping factor $\lambda = 0.8$, we obtain

$$x_1^{(20)} = 0.707106781186464 \approx \frac{1}{\sqrt{2}} \quad \text{and} \quad x_2^{(20)} = 1.414213562372973 \approx \sqrt{2}.$$

Note that

$$F_1(1/\sqrt{2},\sqrt{2}) = 3\sqrt{2} - 2\sqrt{2} + 2\sqrt{2}\Big(\frac{1}{2} - \sqrt{2-1}\Big) = 0$$

and

$$F_2(1/\sqrt{2},\sqrt{2}) = -\sqrt{2} + 4\sqrt{2} - \frac{2\sqrt{2}(1+\frac{1}{2})}{\sqrt{2-1}} = 0.$$

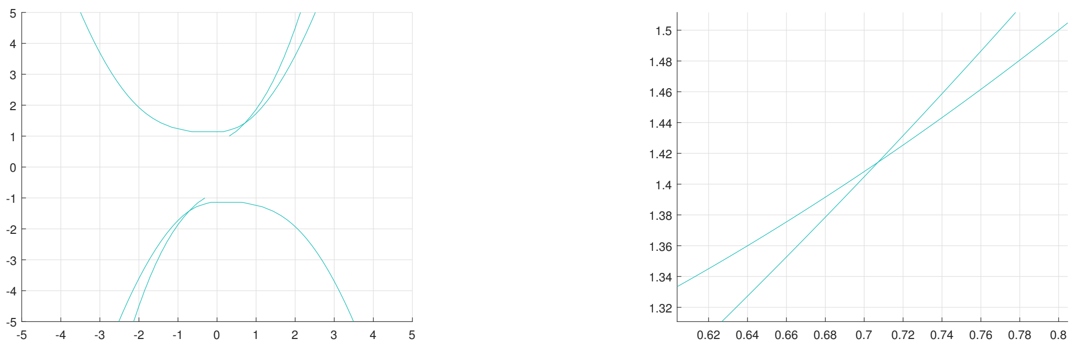


Figure 1: Zero level sets of F_1 and F_2