说 明

有了力的定律和运动定律，动力学的根本任务，即在一定环境下求物体的运动问题，似乎就成为求解运动方程的数学问题了。其实，并非完全如此。

如果我们在动力学定律的基础上引进一些新的概念和新的物理量，如动量、能量和角动量等，就可进而得到关于这些量的新的规律（包括所谓运动定理以及由此引出的守恒定律），而直接用这些规律去分析质点的运动问题，往往比从运动定律出发更为方便。

在力作为位置（或速度、时间）函数的具体形式不十分清楚的情况下（约束力和碰撞中的力就是例子），利用关于动量，能量和角动量的规律，也能为我们求解问题提供一定的信息，使我们获得关于质点系运动的相关知识。

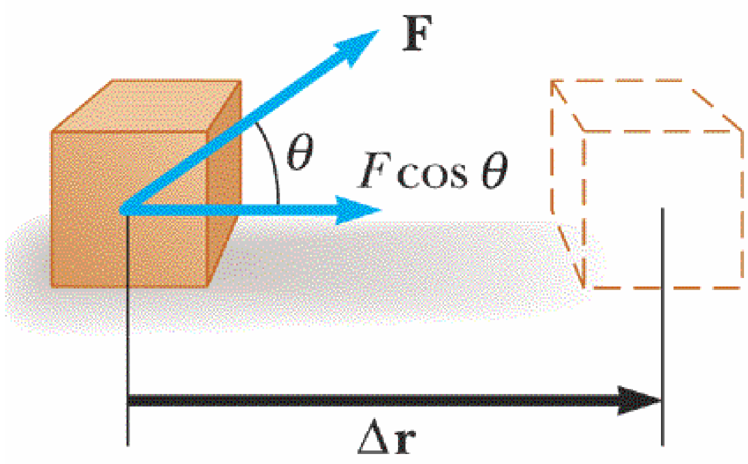
即使在牛顿定律不一定适用的许多场合，包括微观领域，守恒定律仍然有效。这样，原来仅仅作为牛顿定律辅助工具而引入的运动定理的推论——守恒定律，却成为比牛顿定律更为基本的规律。

Chapter 7-8 Work and Energy

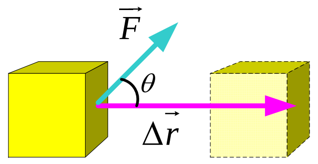
§ 1 Work and Power

- Work done by a constant force

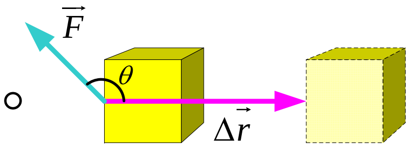
$$W = \vec{F} \cdot \Delta\vec{r} = F\,|\Delta\vec{r}|\cos\theta$$



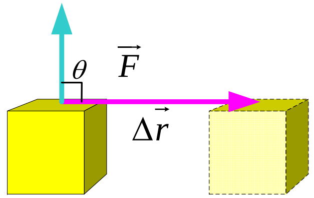
W is positive when $\theta < 90°$



W is negative when $\theta > 90°$



W is zero when $\theta = 90°$

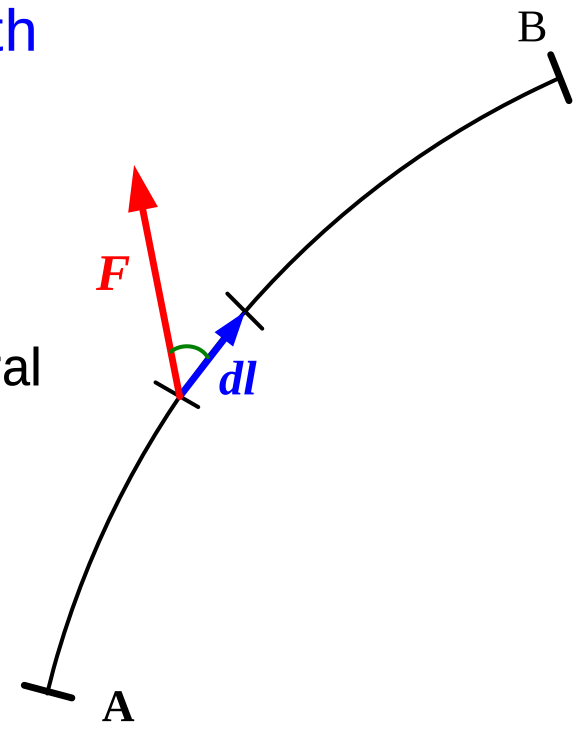


Work

- Work done by a varying force along a curve path
  - Divide the path into a large number of small displacement $d\vec{l}$

$$\boxed{W = \int_A^B \vec{F} \cdot d\vec{l}}$$

Line integral or path integral



The SI unit of work: Newton•meter or Joule

- Work is a process quantity.
- Calculation of work relates to the reference frame.

Work and Power

- **Work done by multiple forces.**

Total work done is the scalar addition of the work done by each force.

$$W_{net} = \int_A^B \vec{F}_{net} \cdot d\vec{l} = \int_A^B \left(\sum_i \vec{F}_i \right) \cdot d\vec{l} = \sum_i \int_A^B \vec{F}_i \cdot d\vec{l} = \sum_i W_i$$

- **The power: The rate at which work is done (P186)**

  - **Average power:**

$$\overline{P} = \frac{\Delta W}{\Delta t}$$

  - **Instantaneous power:**

$$P = \frac{dW}{dt} = \frac{\vec{F} \cdot d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

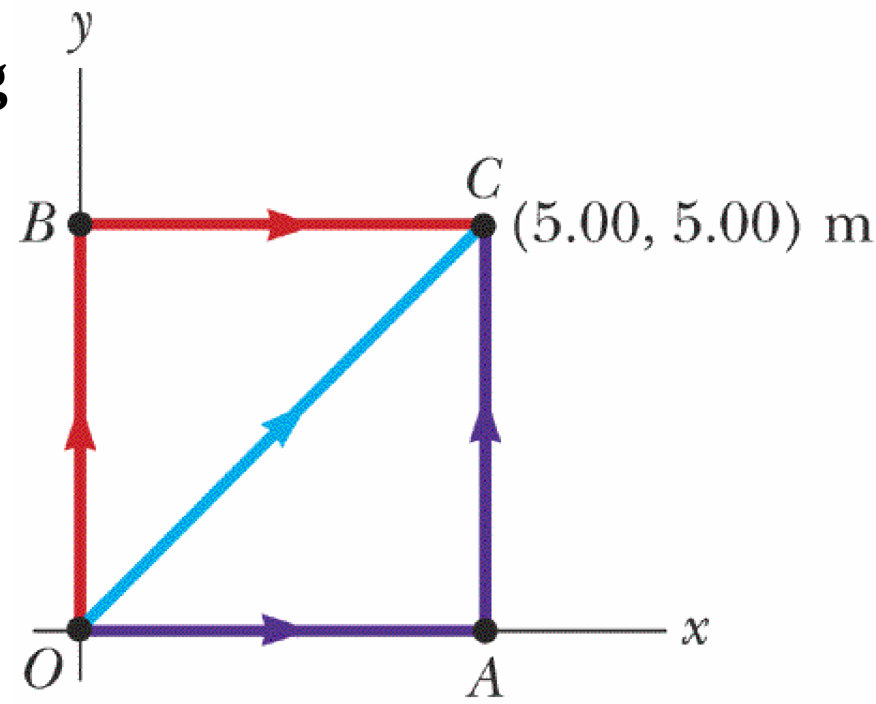
  - **SI unit: watt.**

Example

A force acting on a particle moving in the *xy* plane is given by

$$\vec{F} = 2y\hat{i} + x^2\hat{j} \qquad \textbf{(SI)}$$

The particle moves from the origin to a final position *C* (5.00m, 5.00m). Calculate the work done by $\vec{F}$ along (1) *OC*, (2) *OAC*, (3) *OBC*.

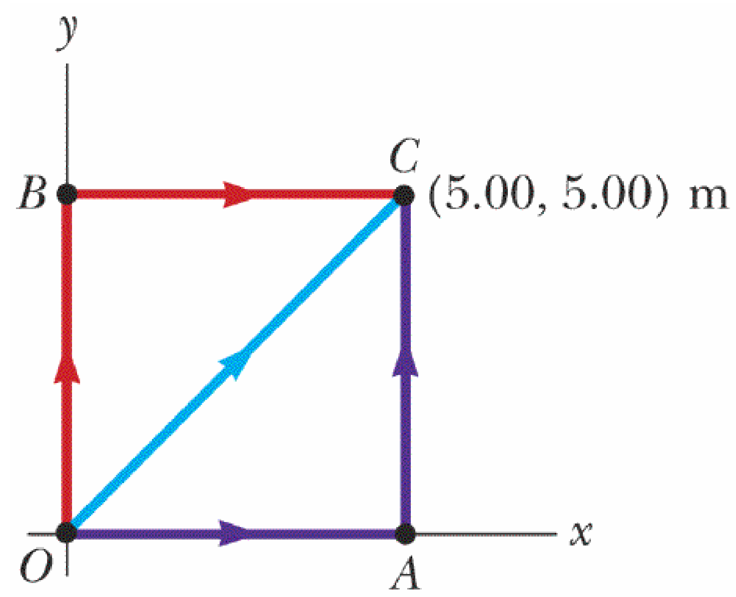


Example (continued)

$$\vec{F} = 2y\hat{i} + x^2\hat{j}$$

Solution:

(1) Along path *OC*:



$$\vec{F}\cdot d\vec{l} = (F_x\hat{i} + F_y\hat{j})\cdot(dx\hat{i} + dy\hat{j})$$

$$= F_x dx + F_y dy$$

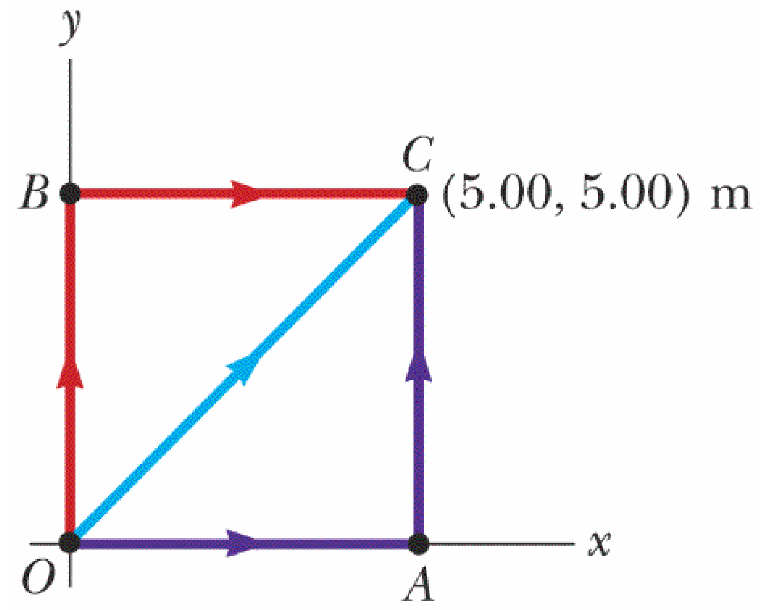
$$= 2y dx + x^2 dy$$

$$\int_{OC} \vec{F}\cdot d\vec{l} = \int_{OC} (2y dx + x^2 dy) = \int_0^5 2x dx + \int_0^5 y^2 dy = \ 66.7\ \text{J}$$

$$OC: y = x$$

Example

$$\vec{F} = 2y\hat{i} + x^2\hat{j}$$



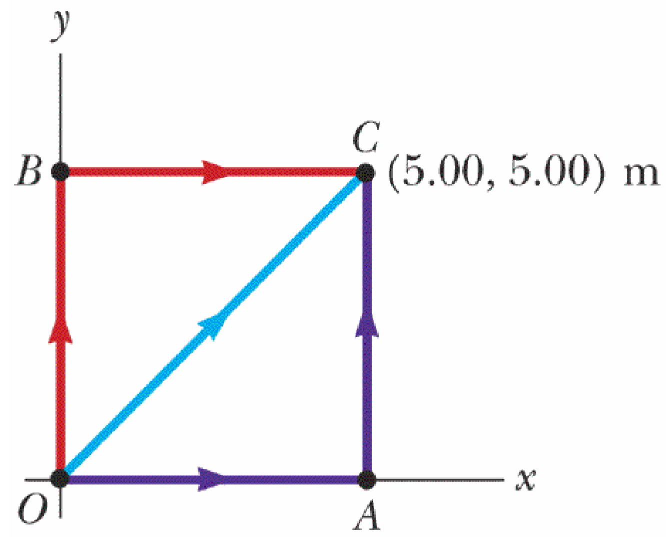
(2) Along path *OAC*:

$$\int_{OAC} \vec{F}\cdot d\vec{l} = \int_{OA} \vec{F}\cdot d\vec{l} + \int_{AC} \vec{F}\cdot d\vec{l}$$

$$= \int_0^5 (2y\hat{i} + x^2\hat{j})\cdot(dx\,\hat{i}) + \int_0^5 (2y\hat{i} + x^2\hat{j})\cdot(dy\,\hat{j})$$

$$= \underbrace{\int_0^5 2y(=0)dx}_{\text{zero}} + \int_0^5 \left(x(=5)\right)^2 dy = \int_0^5 25dy = 125\ \text{J}$$

$$\vec{F} = 2y\hat{i} + x^2\hat{j}$$



(3) Along path *OBC*:

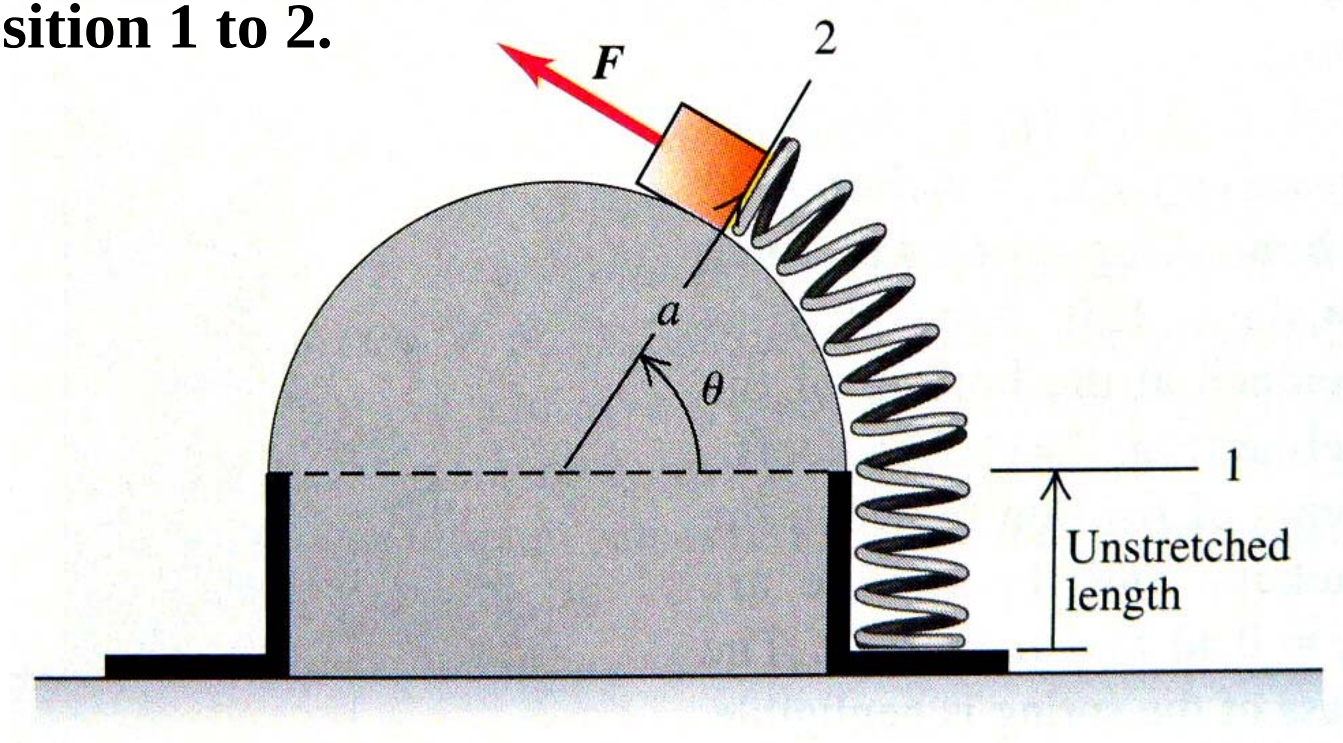
$$\int_{OBC} \vec{F}\cdot d\vec{l} = \int_{OB} \vec{F}\cdot d\vec{l} + \int_{BC} \vec{F}\cdot d\vec{l}$$

$$= \int_0^5 (2y\hat{i} + x^2\hat{j})\cdot dy\,\hat{j} + \int_0^5 (2y\hat{i} + x^2\hat{j})\cdot dx\,\hat{i}$$

$$= \int_0^5 \underbrace{\left(x(=0)\right)^2 dy}_{\text{zero}} + \int_0^5 2y(=5)dx = \int_0^5 2\times 5dx = 50\ \text{J}$$

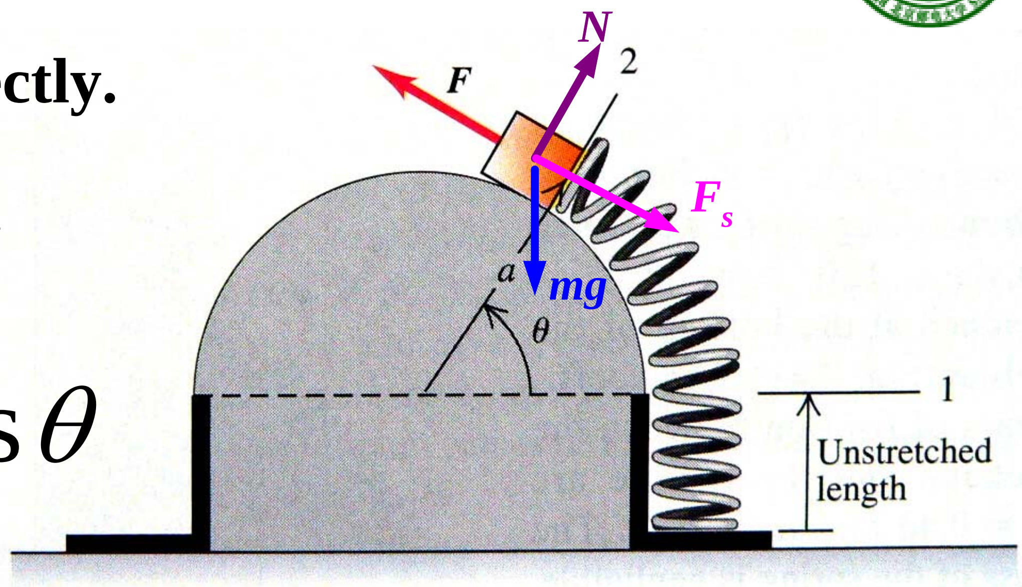
Example

Variable force ***F*** is maintained tangent to a frictionless semicircular surface. By a slowly varying force ***F***, a block with mass of ***m*** is moved, and spring to which it is attached is stretched from position 1 (unstretched length) to position 2 (θ). The spring has negligible mass and force constant ***k***. The end of the spring moves in an arc of radius ***a***. Calculate the work done by the force ***F*** from position 1 to 2.



Solution I: by integration directly.

The block is in equilibrium in tangential direction:



$$F = k(a\theta) + mg\cos\theta$$

$$W_F = \int_1^2 \vec{F}\cdot d\vec{l} = \int_1^2 F ds$$

$$= \int_0^\theta [k(a\theta) + mg\cos\theta] d(a\theta)$$

$$= ka^2\int_0^\theta \theta d\theta + mga\int_0^\theta \cos\theta d\theta = \frac{1}{2}ka^2\theta^2 + mga\sin\theta$$

§ 2 Work – kinetic energy theorem (P156)

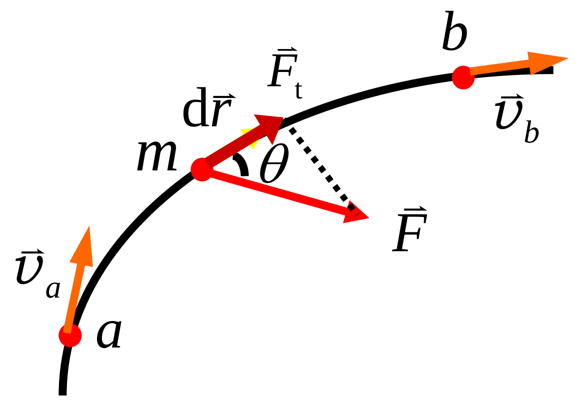
$$W_{net} = \int_A^B \sum_i \vec{F}_i \cdot d\vec{r} = \int_A^B \sum_i F_{it} ds = \int_A^B m\frac{dv}{dt} ds = \int_{v_A}^{v_B} mvdv = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

- Kinetic energy: $K = \frac{1}{2}mv^2$

Process quantity

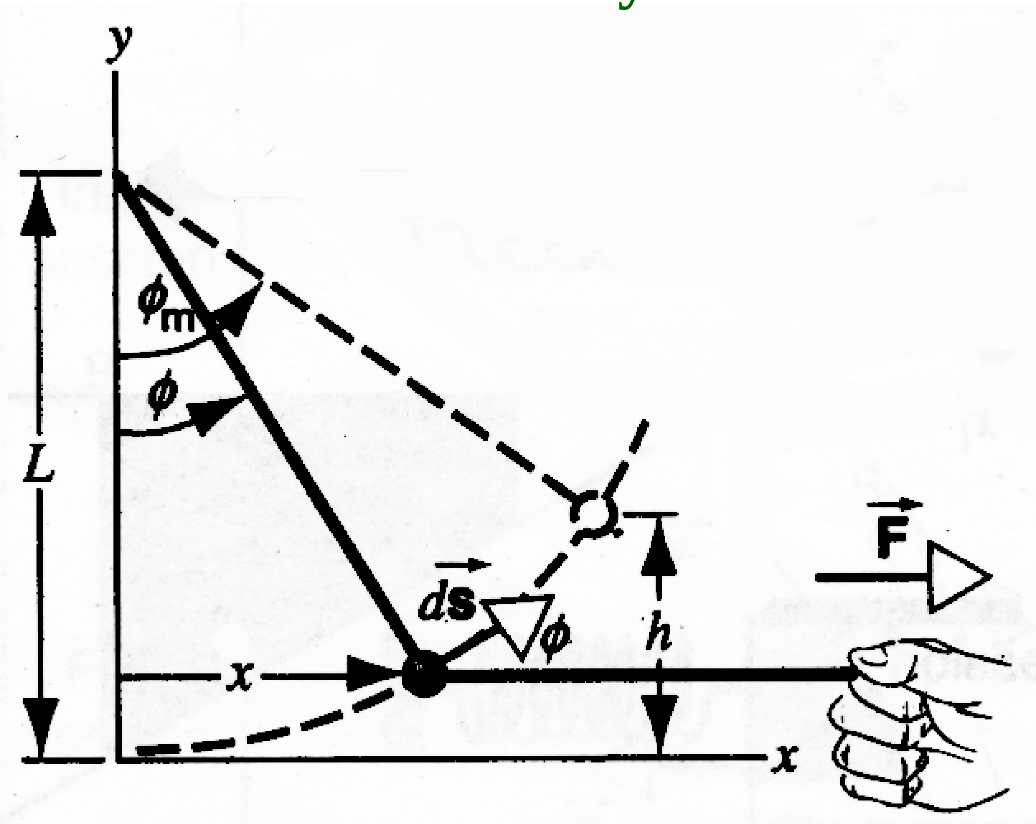
The change of state quantity

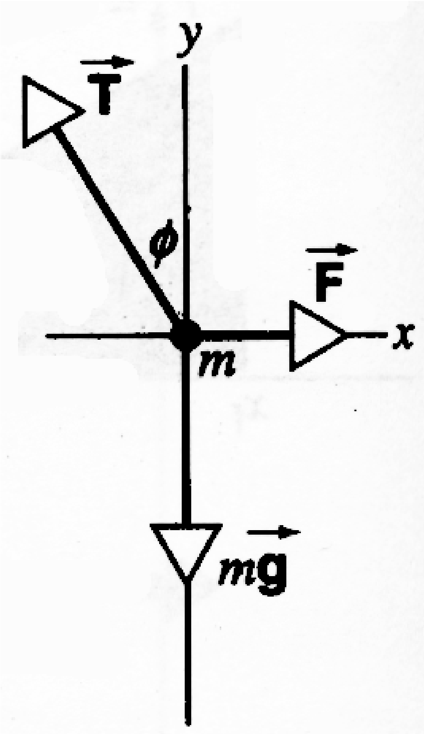
- Work – kinetic energy theorem: $\boxed{W_{net} = K_f - K_i}$
  - The work done by the net force on a particle equals the change in kinetic energy (valid in the inertial frame of reference).



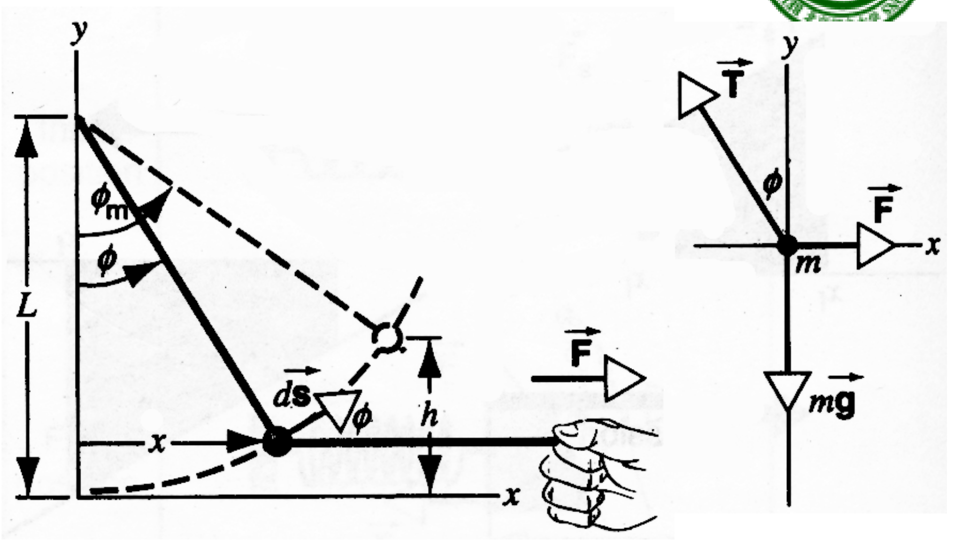
Example

A small object of mass m is suspended from a string of length of L. The object is pulled sideways by a force F that is always horizontal, until the string finally makes an angle ϕ_m. The displacement is accomplished at a very small constant speed. Find the work done by all the forces that act on the object.





Solution



x component: $F - T\sin\phi = 0$

y component: $T\cos\phi - mg = 0$

$$F = mg\tan\phi$$

$$W_F = \int_i^f \vec{F}\cdot d\vec{s} = \int_i^f F ds\cos\phi = \int_i^f F dx$$

$$x = L\sin\phi, \quad dx = L\cos\phi d\phi$$

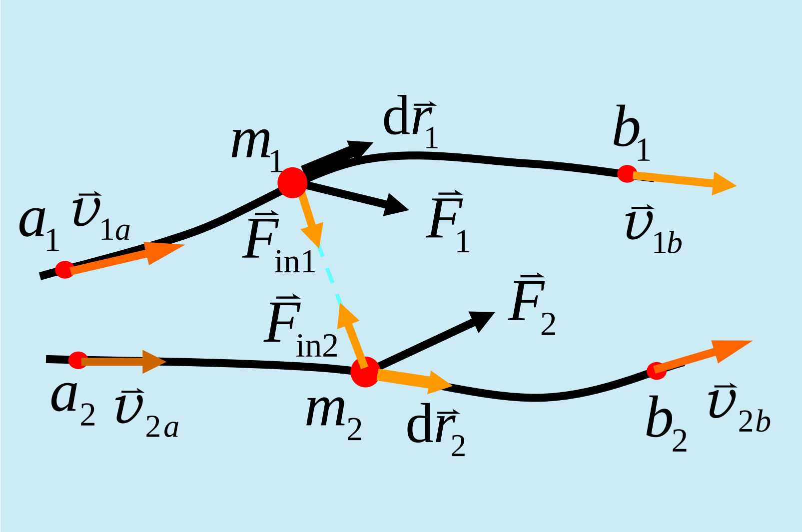
$$W_F = \int_0^{\phi_m} mg\tan\phi L\cos\phi d\phi = mgL\int_0^{\phi_m}\sin\phi d\phi = mgL(1-\cos\phi_m) = mgh$$

$$W_g = \int_i^f (-mg\hat{j})\cdot(dx\hat{i} + dy\hat{j}) = \int_0^h -mgdy = -mgh$$

$\vec{T}$ is perpendicular to the displacement $d\vec{s}$ at every point of the motion.

$$W_{net} = W_F + W_g + W_T = mgh - mgh + 0 = 0$$

- Work – kinetic energy theorem for the system of particles



For $\boldsymbol{m_1}$

$$W_1 = \int_{a_1}^{b_1} \vec{F}_1 \cdot \mathrm{d}\vec{r}_1 + \int_{a_1}^{b_1} \vec{F}_{\text{in}1} \cdot \mathrm{d}\vec{r}_1$$

$$= \frac{1}{2} m_1 v_{1b}^2 - \frac{1}{2} m_1 v_{1a}^2$$

For $\boldsymbol{m_2}$

$$W_2 = \int_{a_2}^{b_2} \vec{F}_2 \cdot \mathrm{d}\vec{r}_2 + \int_{a_2}^{b_2} \vec{F}_{\text{in}2} \cdot \mathrm{d}\vec{r}_2$$

$$= \frac{1}{2} m_2 v_{2b}^2 - \frac{1}{2} m_2 v_{2a}^2$$

$$\int_{a_1}^{b_1} \vec{F}_1 \cdot \mathrm{d}\vec{r}_1 + \int_{a_2}^{b_2} \vec{F}_2 \cdot \mathrm{d}\vec{r}_2 + \int_{a_1}^{b_1} \vec{F}_{\text{in}1} \cdot \mathrm{d}\vec{r}_1 + \int_{a_2}^{b_2} \vec{F}_{\text{in}2} \cdot \mathrm{d}\vec{r}_2$$

$$= \left(\frac{1}{2} m_1 v_{1b}^2 + \frac{1}{2} m_2 v_{2b}^2\right) - \left(\frac{1}{2} m_1 v_{1a}^2 + \frac{1}{2} m_2 v_{2a}^2\right)$$

Work – kinetic energy theorem

For a particle

$$\boxed{W_{net} = K_f - K_i}$$

- Work – kinetic energy theorem for the system of particles

$$\boxed{\sum W_{i-\text{external}} + \sum W_{i-\text{internal}} = \sum K_f - \sum K_i}$$

  - Generally, the works done by internal forces between particles cannot be canceled (the displacements of particles are different).

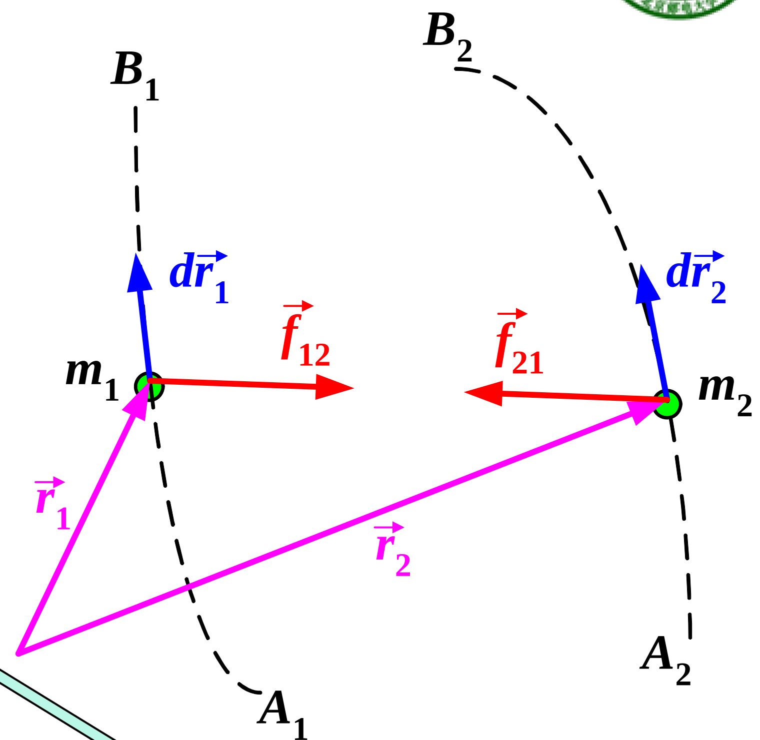
The work done by a pair of internal forces

- The work done by a pair of internal forces

$$\vec{f}_{12} = -\vec{f}_{21}$$

For a infinitesimal process

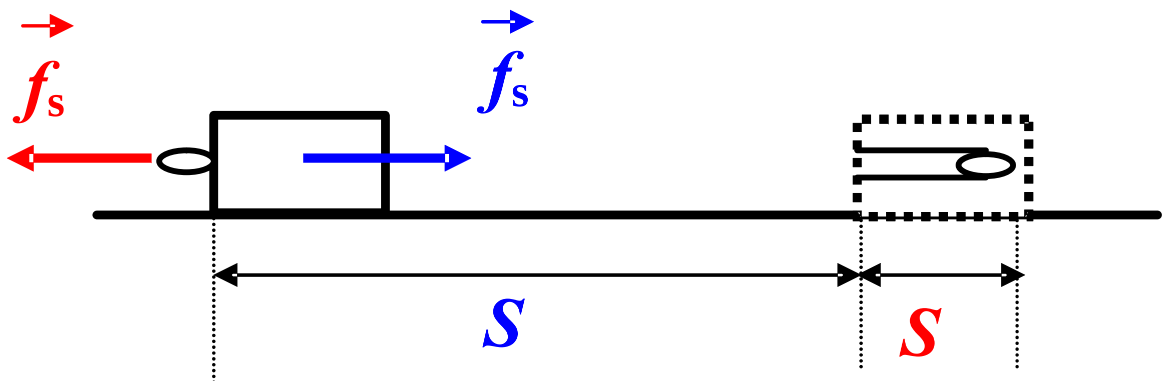
$$\begin{aligned} dW &= \vec{f}_{12}\cdot d\vec{r}_1 + \vec{f}_{21}\cdot d\vec{r}_2 \\ &= \vec{f}_{21}\cdot(d\vec{r}_2 - d\vec{r}_1) = \vec{f}_{21}\cdot d\underbrace{(\vec{r}_2 - \vec{r}_1)} \\ &= \vec{f}_{21}\cdot d\vec{r}_{21} \end{aligned}$$



The displacement of 2 relative to 1

- **The calculation of net work done by a pair of internal forces on two particles is equivalent to —— in the reference frame of particle 1, the calculation of work done by one force acting on particle 2.**

A bullet coming from left is shot into a wooden block and passes through a length of S' in the block. The system of bullet-block comes to a halt after sliding a distance of S. Calculate the net work done by a pair of friction forces f_s and f_s' between the bullet and the block.



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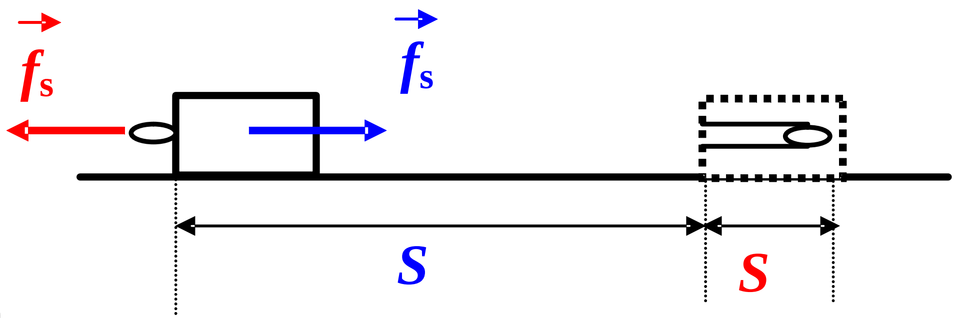
Example

Solution: $\vec{f}_s = -\vec{f}_s', \qquad |\vec{f}_s| = |\vec{f}_s'| = f_s$

For the block:



$$W_s = f_s S$$

For the bullet:

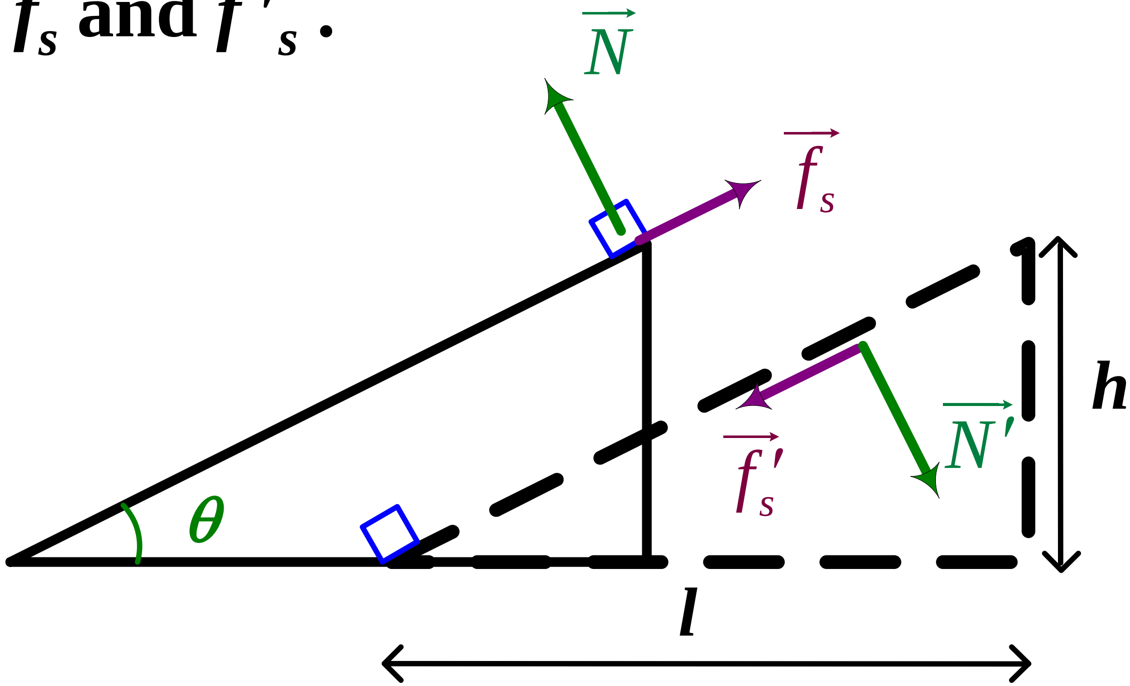
$$W_{s'} = -f_s(S + S')$$

The net work:

$$W_s^{\text{net}} = W_s + W_{s'} = f_s S - f_s(S + S') = -f_s S'$$

Example

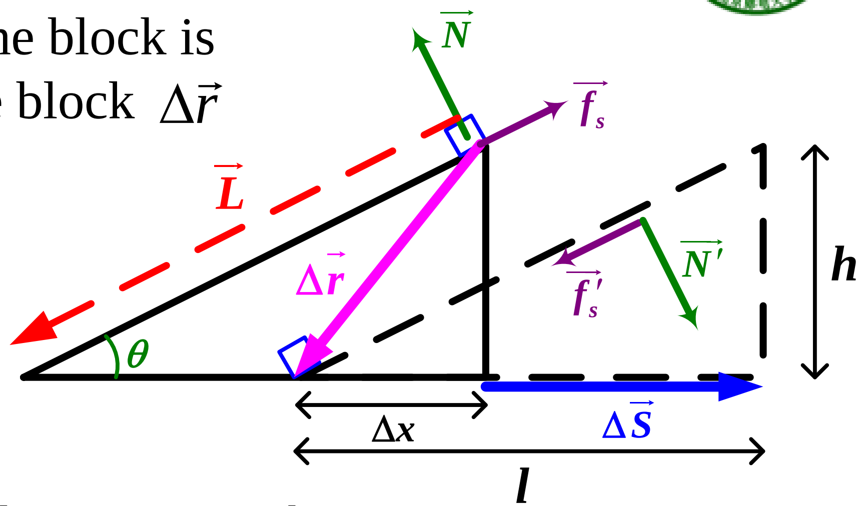
Calculate the net works done respectively by a pair of normal forces N and N', f_s and f'_s between the block and the wedge. Neglecting the frictions except f_s and f'_s.



Example

Solution: The normal force $\vec{N}$ acting on the block is not perpendicular to the displacement of the block $\Delta\vec{r}$

Therefore: $W_N \neq 0$



The normal force $\vec{N}'$ acting on the wedge is not perpendicular to the displacement of the wedge $\Delta\vec{S}$. Therefore:

$$W_{N'} \neq 0$$

$$\Delta\vec{r} = -\Delta x\,\hat{i} - h\,\hat{j}, \qquad \vec{N} = -N\sin\theta\,\hat{i} + N\cos\theta\,\hat{j}$$

$$\Delta\vec{S} = (l - \Delta x)\,\hat{i}, \qquad \vec{N}' = N\sin\theta\,\hat{i} - N\cos\theta\,\hat{j}$$

$$W_N = \vec{N}\cdot\Delta\vec{r} = \Delta x N\sin\theta - hN\cos\theta, \qquad \tan\theta = \frac{h}{l}, \quad h\cos\theta = l\sin\theta$$

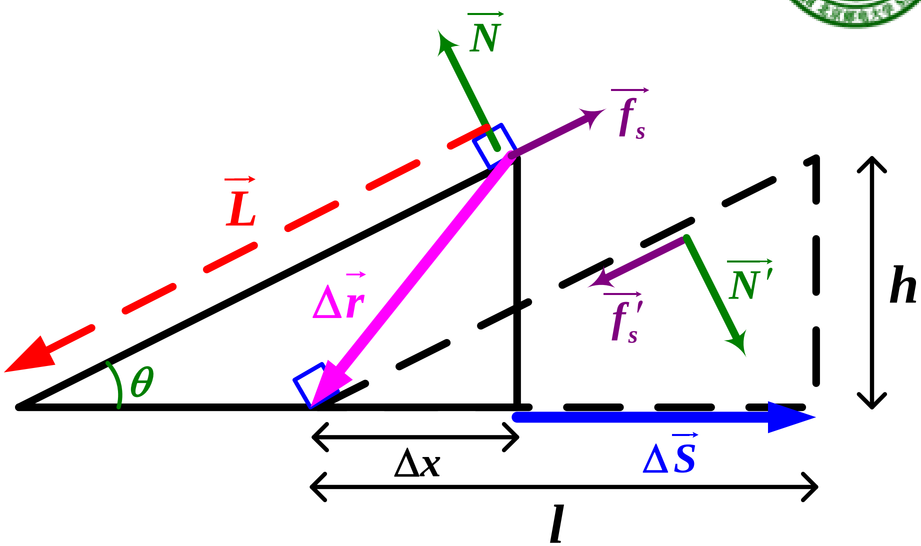
$$W_{N'} = \vec{N}'\cdot\Delta\vec{S} = lN\sin\theta - \Delta x N\sin\theta,$$

$$\boxed{W_N^{\text{net}} = W_N + W_{N'} = 0}$$

Example (continued)

$$\Delta\vec{r} = -\Delta x\,\hat{i} - h\,\hat{j}$$

$$\Delta\vec{S} = (l - \Delta x)\,\hat{i}$$



$$\vec{f}_s = f_s\cos\theta\,\hat{i} + f_s\sin\theta\,\hat{j}$$

$$\vec{f}' = -f_s\cos\theta\,\hat{i} - f_s\sin\theta\,\hat{j}$$

$$W_{f_s} = \vec{f}_s\cdot\Delta\vec{r} = -\Delta x\, f_s\cos\theta - h\, f_s\sin\theta$$

$$W_{f_s'} = \vec{f}_s'\cdot\Delta\vec{S} = -l\, f_s\cos\theta + \Delta x\, f_s\cos\theta$$

$$W_{f_s}^{\text{net}} = W_{f_s} + W_{f_s'} = -(l\cos\theta + h\sin\theta)\, f_s$$

$$= -(L\cos^2\theta + L\sin^2\theta)\, f_s = -f_s L$$