§ 1 Kinematics of Rigid Bodies

- **What Rigid Body**

The body that has a perfectly definite and unchanging shape and size.

$$\left|\vec{r}_i - \vec{r}_j\right| = d_{ij} = \text{constant}$$

The distance between any two arbitrary points in the body is a constant.

- **Idealized model: the external forces that act on the real-world bodies can deform them —— stretching, twisting, and squeezing.**
- **If these deformations are so little that can be ignored, such bodies can be treated as rigid bodies.**

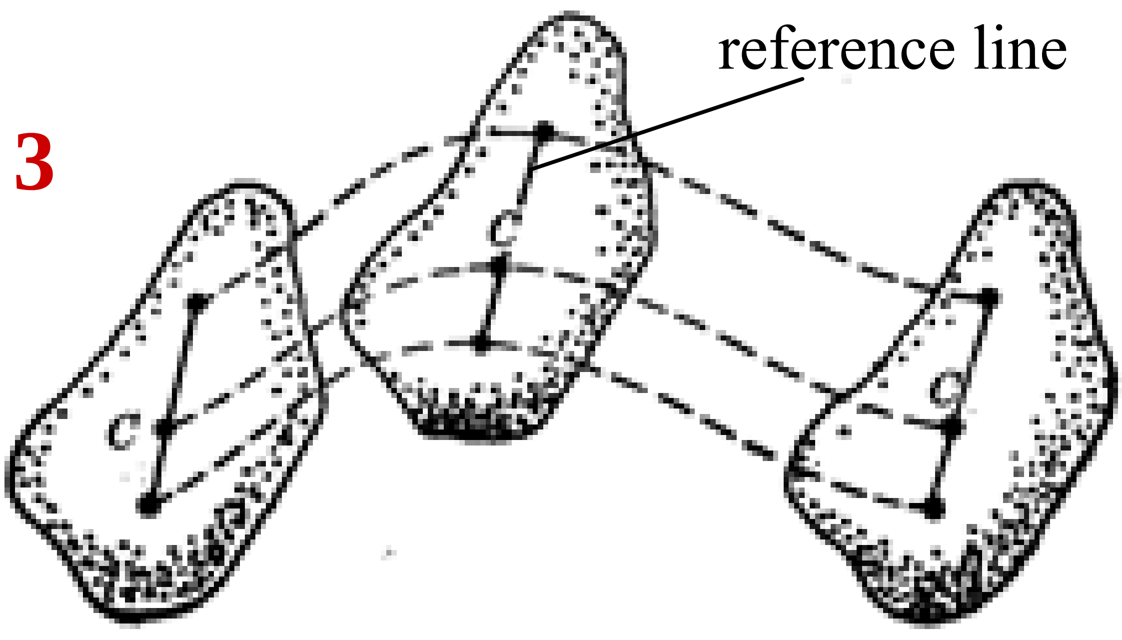
Why Rigid Body

- **Any body can be viewed as a system of *N* numbers of particles.**
- **Generally need *3N* motional equations to describe its motion.**
- **The rigid body model simplifies the description of body's motion.**

$$\left|\vec{r}_i - \vec{r}_j\right| = d_{ij} = \text{constant}$$

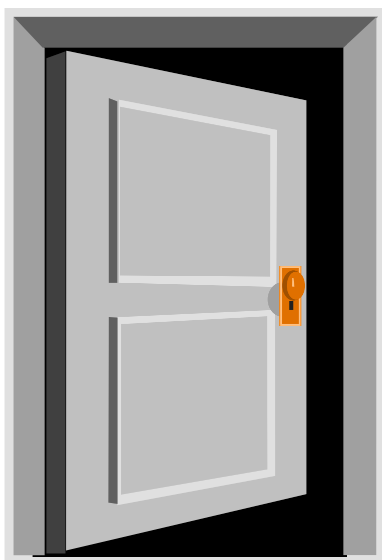
- **Translational motion of a rigid body**
  - **The trajectories of all the points of a rigid body are the same, or the line between any two points of a rigid body keeps its orientation unchanged all the time.**

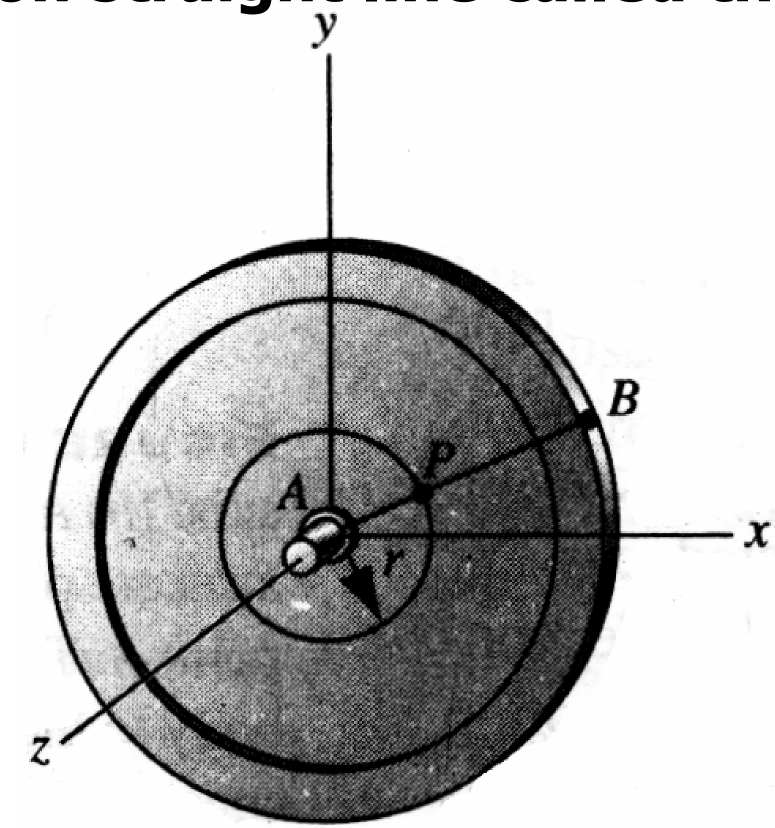
$i = 3$



- **Rotational motion of a rigid body**
  - **Rotation about a fixed axis: every point of the body moves in a circular path. The centers of these circles must lie on a common straight line called the axis of rotation.**

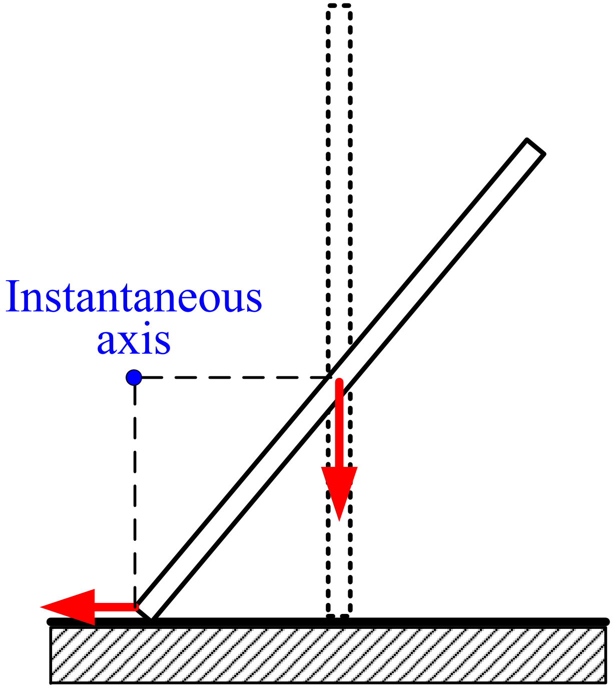
$$i = 1$$

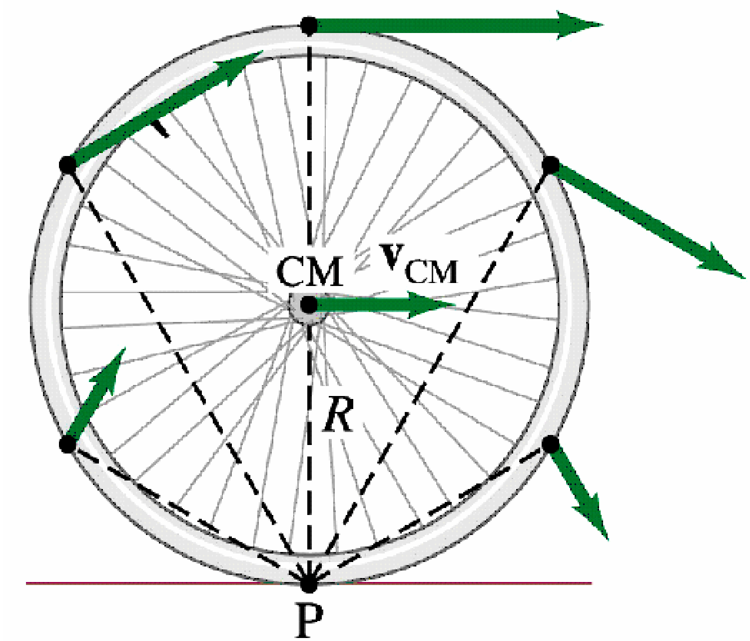




- **Rotation about a non-fixed axis:**

  the position or the orientation of the rotational axis varied with time. An instantaneous rotational axis must exist that the instantaneous velocity of any point in the body is perpendicular to the axis.



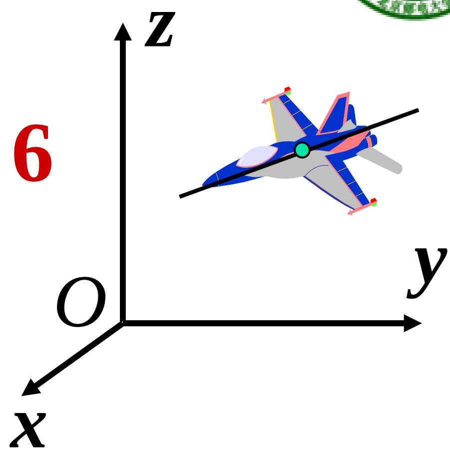


Instantaneous axis

Translational and Rotational Motion of Rigid Bodies

- The **general** motion of a rigid body will include both rotational and translational components.

$$i = 6$$

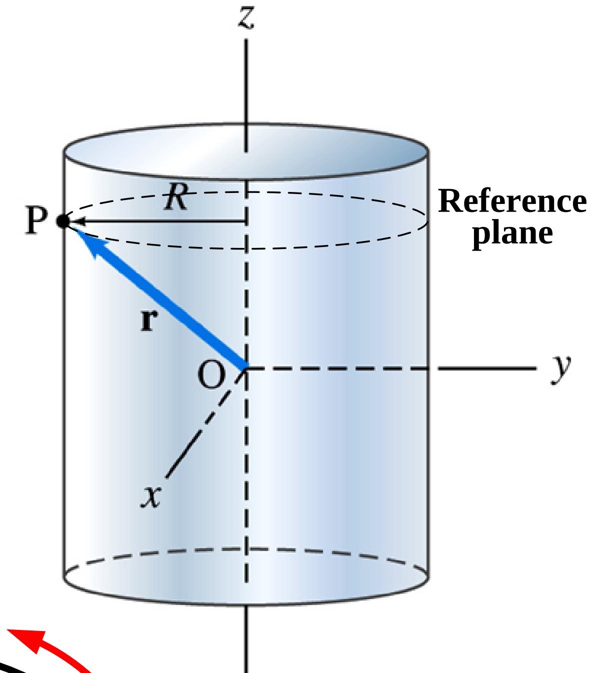


- ✓ **Three** to locate the center of mass.
- ✓ **Two** angles to orient the axis of rotation.
- ✓ **One** angle to describe rotation about the axis.

- The rigid body model **simplifies** the description of body's motion.
  - For a rigid body, we only need **6** coordinates.

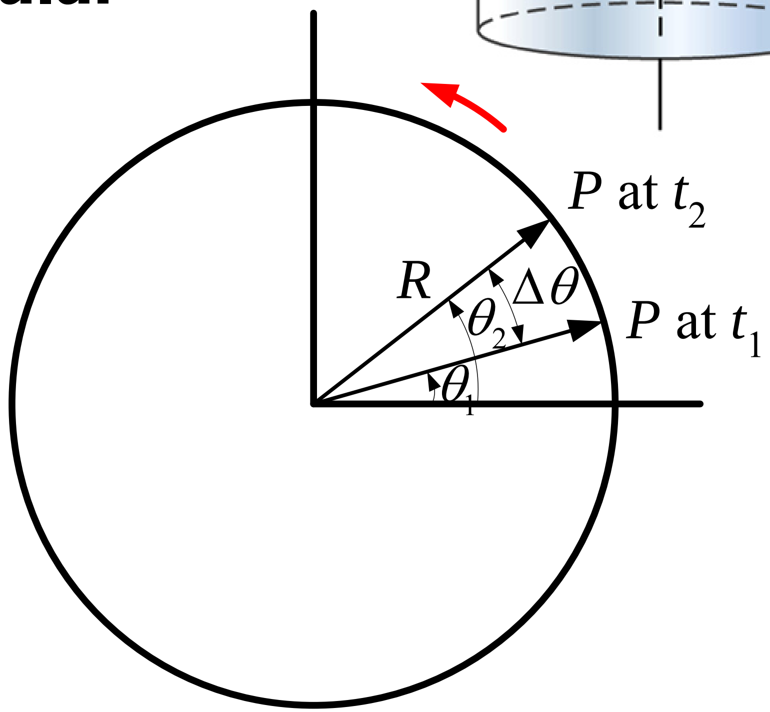
§ 2 Angular Quantities for rigid bodies

- **Rotational radius** R
  - **The perpendicular distance of point P in the reference plane from the axis of rotation.**



- **Angular position and angular displacement**
  - **Angular position:** θ_1, θ_2
  - **Angular displacement:**

    $\Delta\theta = \theta_2 - \theta_1$.



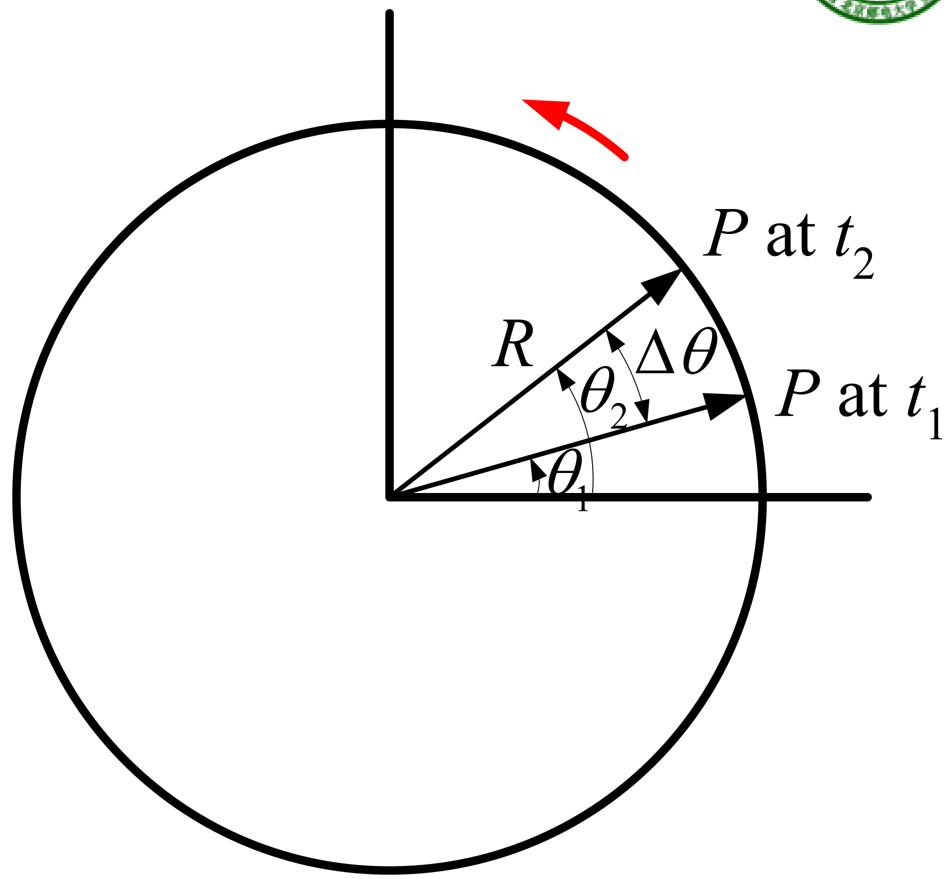
Angular velocity

- **Average angular velocity:**

$$\bar{\omega} = \frac{\theta_2 - \theta_1}{t_2 - t_1} = \frac{\Delta\theta}{\Delta t}$$

- **Instantaneous angular velocity:**

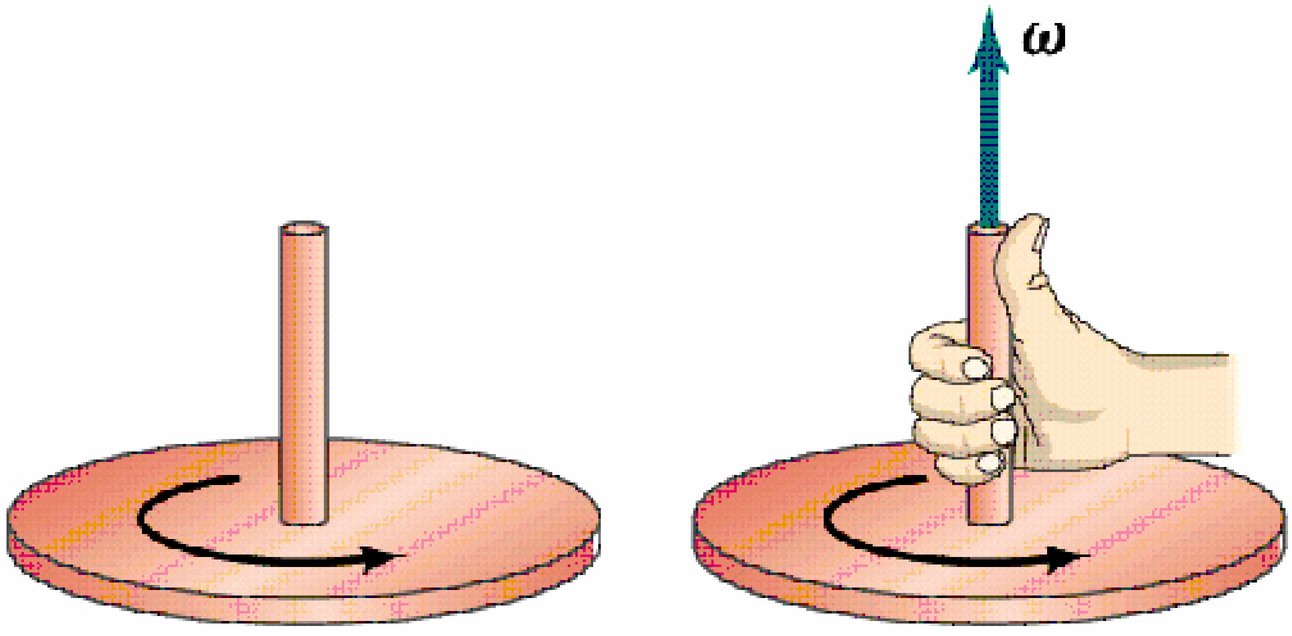
$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$



- **Choose the positive sense of the rotation to be counter-clockwise.**

Angular velocity

- **Angular velocity as a vector**
  - **The direction of angular velocity vector —— right-hand rule**
    - **The right-hand rule: when the fingers of right hand curl in direction of rotation, the thumb position is the direction of** $\vec{\omega}$

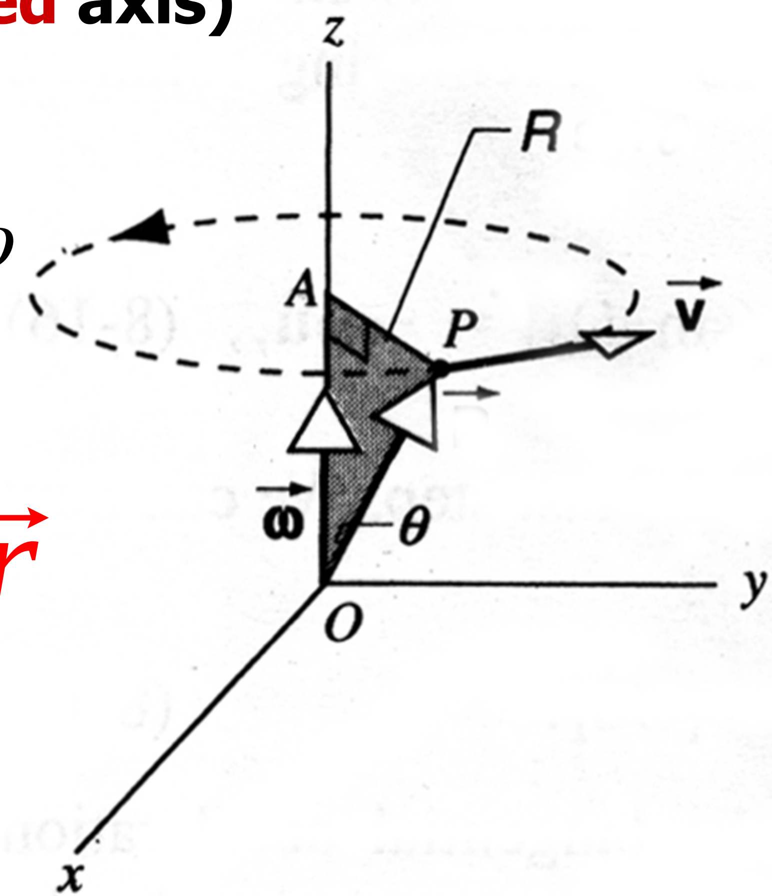


- **Relationship between linear and angular velocities (only for rotation about a fixed axis)**
  - **Magnitude:**

$$v = \frac{ds}{dt} = \frac{d(R\theta')}{dt} = R\frac{d\theta'}{dt} = R\omega$$

  - **Considering the direction:**

$$\vec{v} = \vec{\omega}\times\vec{R} = \vec{\omega}\times\vec{r}$$



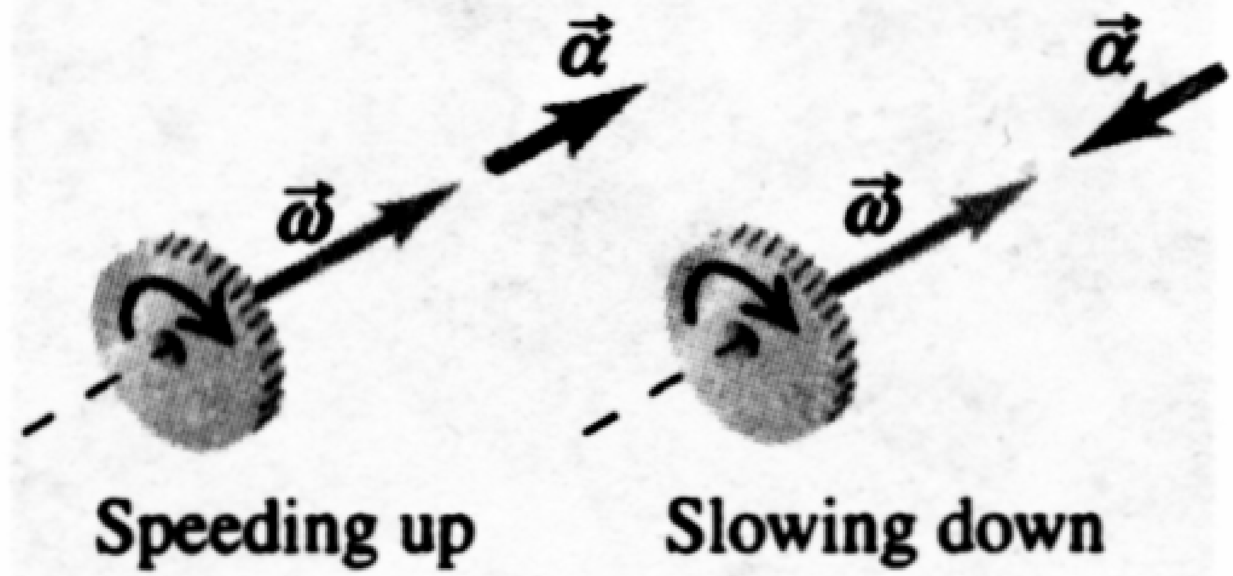
Angular acceleration

- **Average angular acceleration:** $\bar{\alpha} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$

- **Instantaneous angular acceleration:** $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt}$

- **Angular acceleration as a vector**

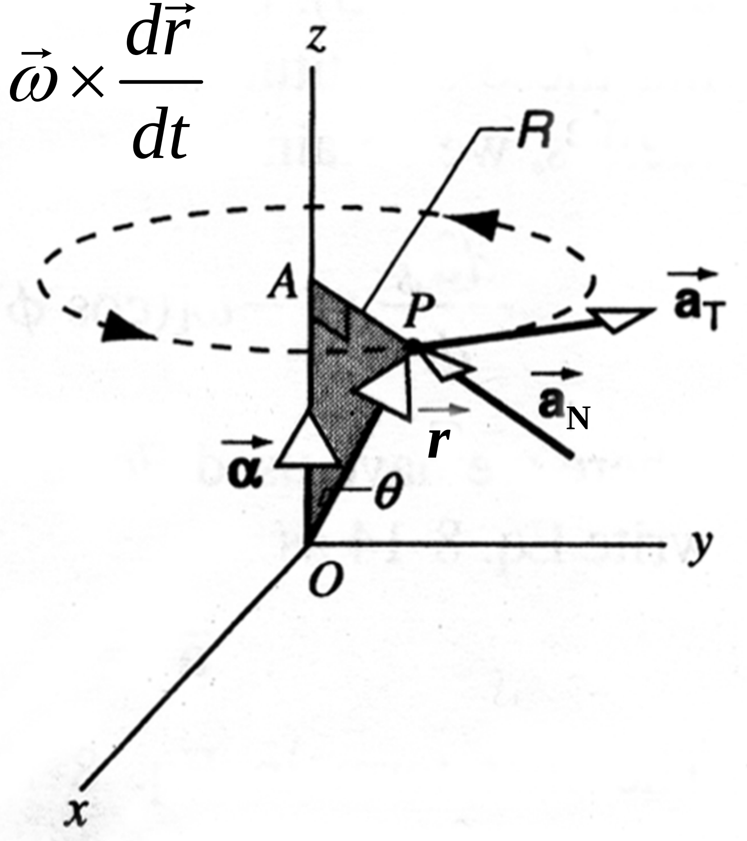
$$\vec{\alpha} = \frac{d\vec{\omega}}{dt}$$



- Relationship between linear and angular accelerations (only for rotation about a fixed axis)

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}(\vec{\omega}\times\vec{r}) = \frac{d\vec{\omega}}{dt}\times\vec{r} + \vec{\omega}\times\frac{d\vec{r}}{dt}$$

$$= \vec{\alpha}\times\vec{r} + \vec{\omega}\times\vec{v} = \vec{\alpha}\times\vec{R} + \vec{\omega}\times\vec{v}$$



- **Tangential acceleration:**
  - **Magnitude:**

$$a_t = R\alpha$$

- **Normal acceleration:**
  - **Magnitude:**

$$a_n = \omega v = \omega^2 R$$

Uniformly accelerated rotational motion

$$\omega = \omega_0 + \alpha t$$

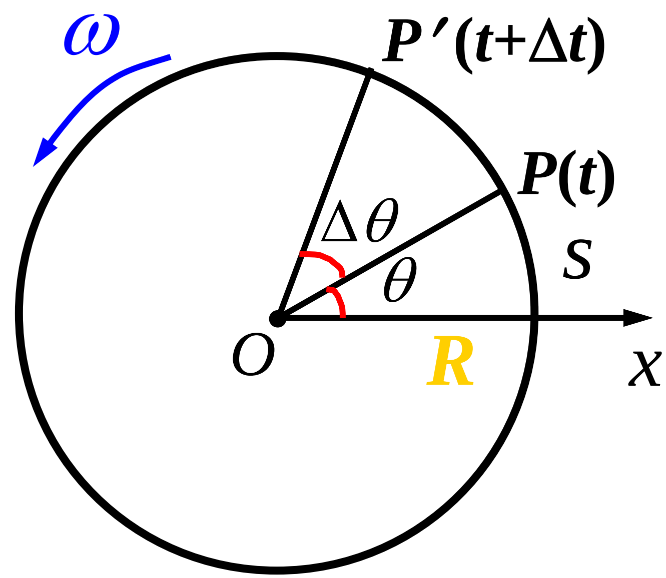
$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 = \omega_0^2 + 2\alpha\left(\theta - \theta_0\right)$$

$$v = v_0 + at$$

$$S = S_0 + v_0 t + \frac{1}{2}at^2$$

$$v^2 = v_0^2 + 2a\left(S - S_0\right)$$

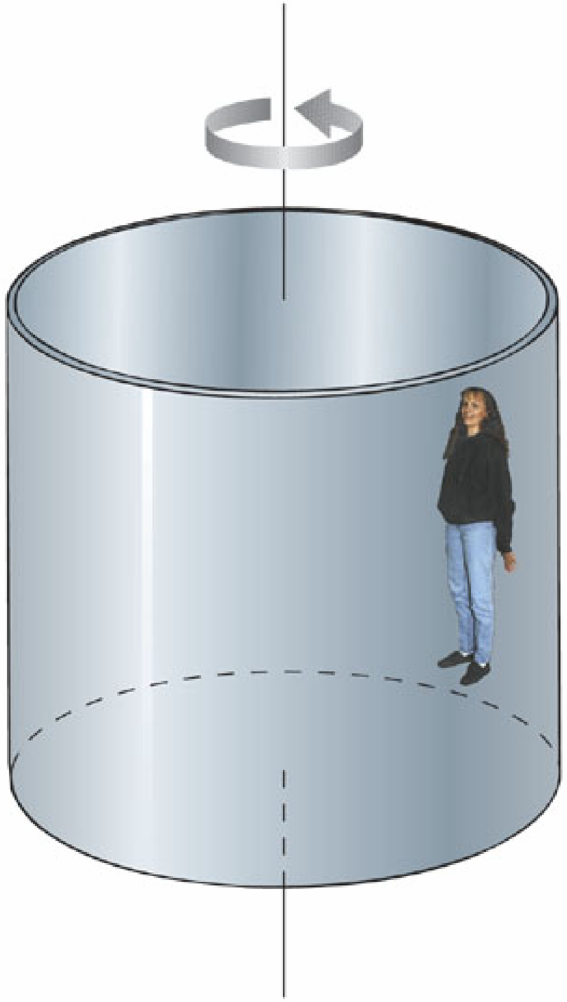


Example

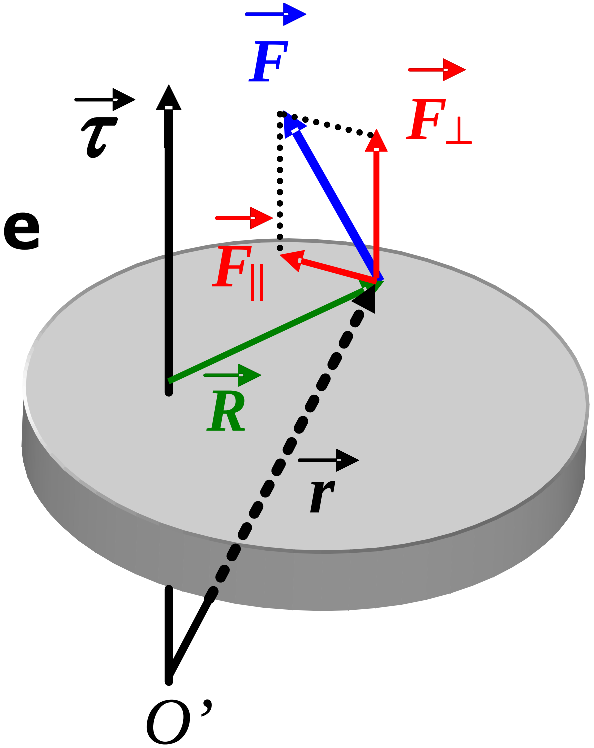
While you are operating a Rotor, you spot a passenger in acute distress and decrease the angular speed of the cylinder from 3.40 rad/s to 2.00 rad/s in 20.0 rev, at constant angular acceleration.

(a) What is the constant angular acceleration during this decrease in angular speed?

(b) How much time did the speed decrease take?



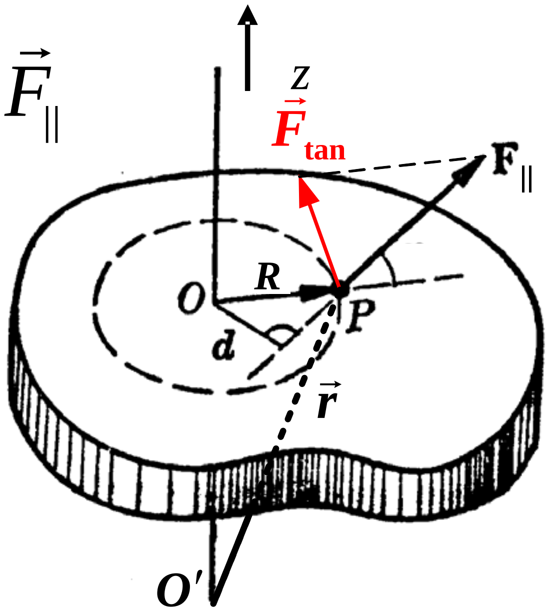
- **The torque about a fixed axis —— torque component along the axis of rotation**
  - **The force $\vec{F}$ can be resolved into the parallel component $\vec{F}_{\parallel}$ lying in the reference plane, and the perpendicular component $\vec{F}_{\perp}$.**
  - **The perpendicular component $\vec{F}_{\perp}$ does not contribute to the torque about the rotation axis, since it can not tend to change the body's rotation about that axis. (or there must be an opposite torque exerted on the axis to balance it)**



The torque about the fixed rotation axis

- **So the torque about the fixed rotation axis:**

$$\vec{\tau} = \vec{r} \times \vec{F}_{\parallel} = (\overrightarrow{O'O} + \vec{R}) \times \vec{F}_{\parallel} = \overrightarrow{O'O} \times \vec{F}_{\parallel} + \vec{R} \times \vec{F}_{\parallel}$$



Perpendicular to the rotation axis $O'O$, and will be balanced by another torque acting on the axis.

$$\tau_z = \tau_{axis} = RF_{\parallel} \sin\theta = F_{\parallel} d = F_{tan} R$$

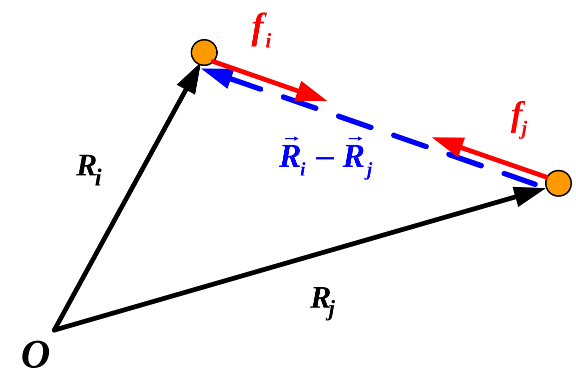
- **The torque about the axis *O'O* is actually the projection of the torque about the point *O'* on the axis *O'O*.**

The Rotational Form of Newton's Second Law (转动定律)

- **Imagine the body as being made up of a large number of particles.**
  - **For *i*-th particle Δm_i external force: $\vec{F}_i$ internal force: $\vec{f}_i$**

$$\vec{F}_i + \vec{f}_i = \Delta m_i \vec{a}_i$$

$$\sum_i \vec{R}_i \times \vec{F}_i + \sum_i \vec{R}_i \times \vec{f}_i = \sum_i \vec{R}_i \times (\Delta m_i \vec{a}_i)$$



The torques of each pair of internal forces are vanished.

$$\vec{R}_i \times \vec{f}_{ij} + \vec{R}_j \times \vec{f}_{ji} = (\vec{R}_i - \vec{R}_j) \times \vec{f}_{ij} = 0 \quad \Rightarrow \quad \sum_i \vec{R}_i \times \vec{f}_i = 0$$

The external torque:

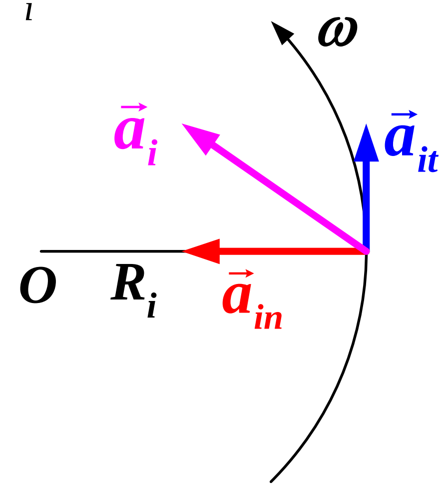
zero

$$\vec{R}_i \times \vec{F}_i = \vec{R}_i \times \vec{F}_{it} + \vec{R}_i \times \vec{F}_{in} = R_i F_{it} \hat{k}$$

The net torque about rotation axis that acts on the body: $\vec{\tau}_{\text{net}} = \sum_i R_i F_{it} \hat{k}$

The Rotational Form of Newton's Second Law (cont'd)

- **It is followed that:** $\vec{\tau}_{\text{net}} = \sum_i R_i F_{it} \hat{k} = \sum_i \vec{R}_i \times \left(\Delta m_i \vec{a}_i\right) = \sum_i \Delta m_i (\vec{R}_i \times \vec{a}_i)$



- **The right side of the equation:**

$$\vec{R}_i \times \vec{a}_i = \vec{R}_i \times \vec{a}_{it} + \vec{R}_i \times \vec{a}_{in} = R_i a_{it} \hat{k} = R_i^2 \alpha \hat{k}$$

zero

$$\vec{\tau}_{\text{net}} = \sum_i R_i F_{it} \hat{k} = \left(\sum_i \Delta m_i R_i^2\right) \alpha \hat{k} = I \alpha \hat{k}$$

- **the moment of inertia of the body (转动惯量)** $I = \sum_i \Delta m_i R_i^2$

$$\sum \tau_{\text{net-axis}} = I\alpha$$

- **The rotational form of Newton's II Law**

Some Comments

$$\boxed{\sum \tau_{\text{net-axis}} = I\alpha}$$

$$\sum F_{z-\text{ext}} = ma_z$$

- **It relates the net external torque about a particular fixed axis to the angular acceleration about that axis. The moment of inertia I must be calculated about that same axis.**
- **The moment of inertia reflects the tendency of a rigid body to resist angular acceleration, just like the mass reflecting the tendency of a object to resist linear acceleration.**
- **Generally, this equation is valid for the rotation of a rigid body about a fixed axis in an inertial reference frame.**
- **It is also valid for the rotation about an axis fixed in the center of mass of the body, although the CM is not an inertial reference frame.**

$$\boxed{\sum \tau_{\text{ext-CM}} = I_{\text{CM}}\alpha}$$

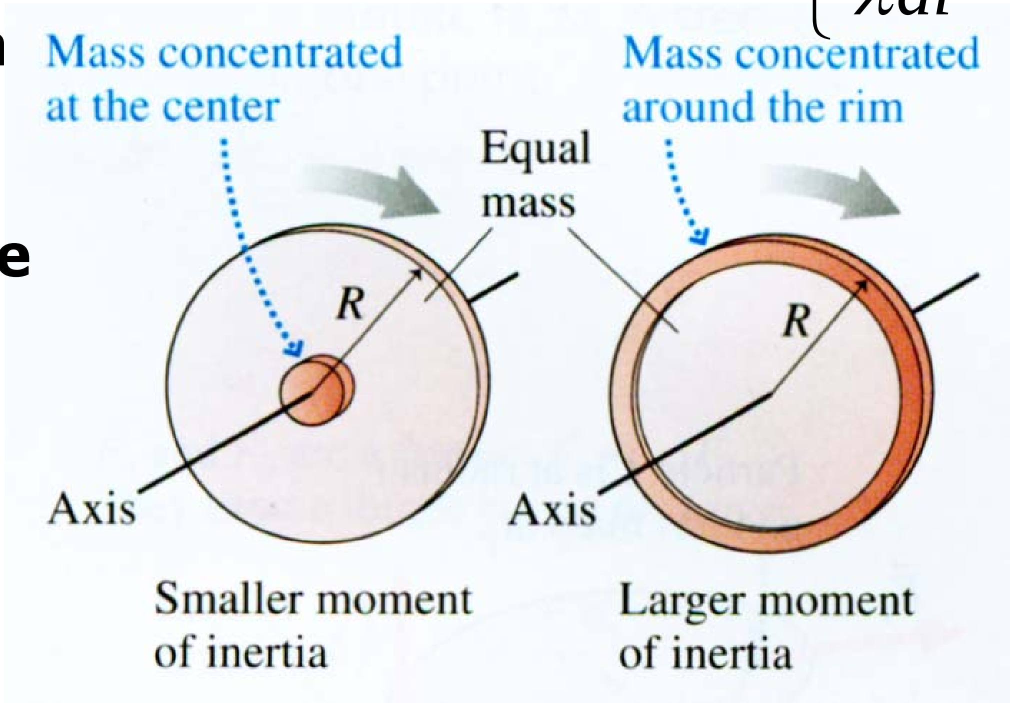
§ 4 The Moment of Inertia

- **The definition:** $\boxed{I = \sum_i \Delta m_i R_i^2}$

It plays the same role in $\alpha = \tau_{\text{net}} / I$ as mass in $\vec{a} = \vec{F}_{\text{net}} / m$. The larger the moment of inertia, the more effort it takes and the slower its angular acceleration.

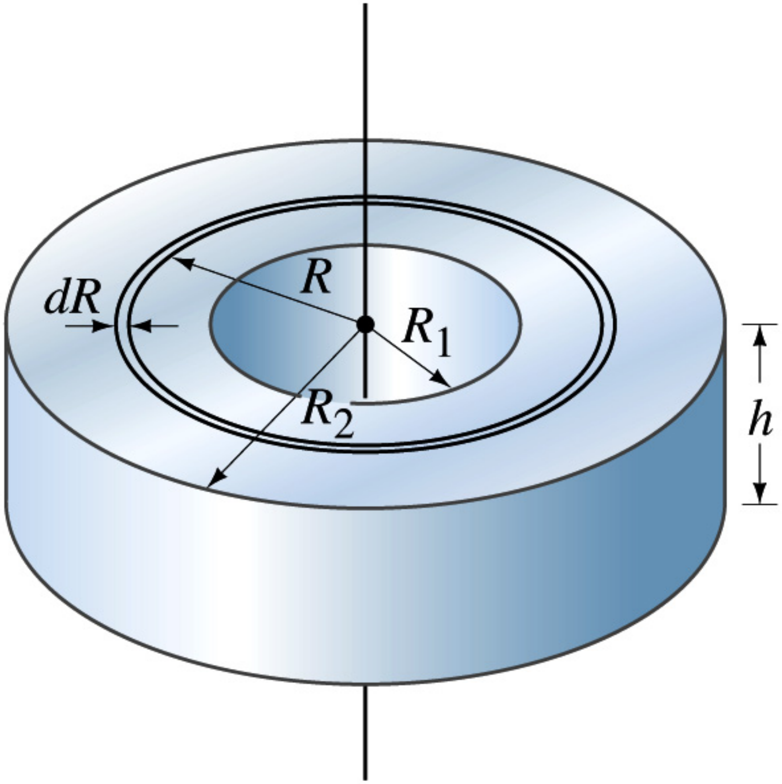
For continuous distribution bodies: $\boxed{I = \int R^2 dm}$ $\quad dm = \begin{cases} \rho dV \\ \sigma dS \\ \lambda dl \end{cases}$

An object's moment of inertia depends not only on the object's mass but on how the mass is distributed around the axis.



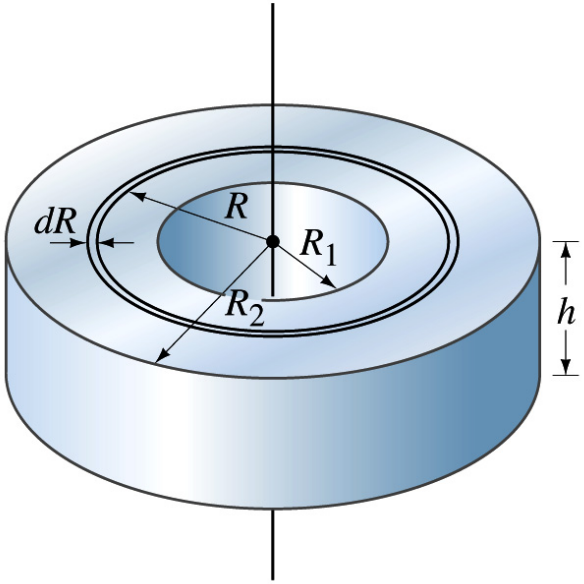
Example (P249 Ex.10-10)

The moment of inertia of a uniform hollow cylinder of inner radius R_1, outer radius R_2, and mass M, if the rotation axis is though the center along the axis of symmetry.



Example

Solution: Divided the cylinder into thin concentric cylindrical rings or hoops of thickness *dR*



$$dI = R^2 dm$$

$$dm = \rho dV$$

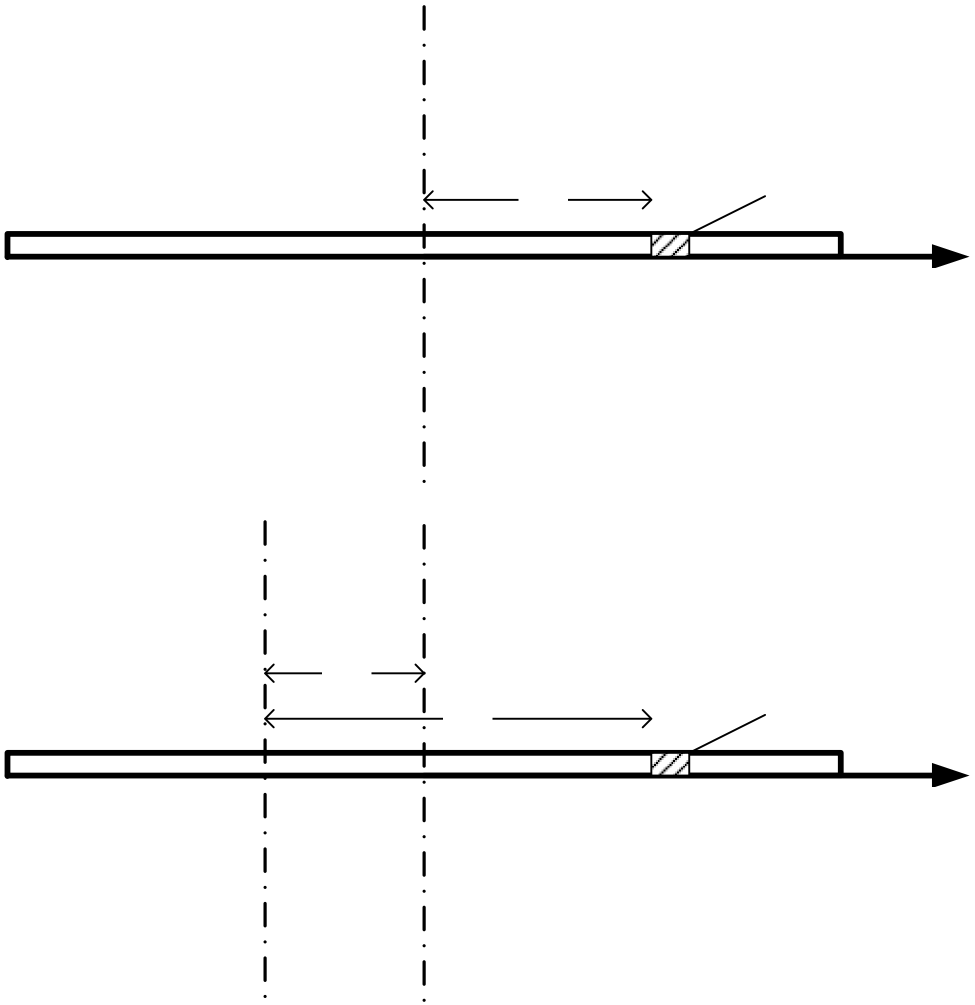
$$= \frac{M}{\pi(R_2^2 - R_1^2)h}(2\pi R)h dR$$

$$= \frac{2M}{R_2^2 - R_1^2} R dR$$

$$I = \int R^2 dm = \frac{2M}{R_2^2 - R_1^2}\int_{R_1}^{R_2} R^3 dR = \frac{1}{2}M(R_1^2 + R_2^2)$$

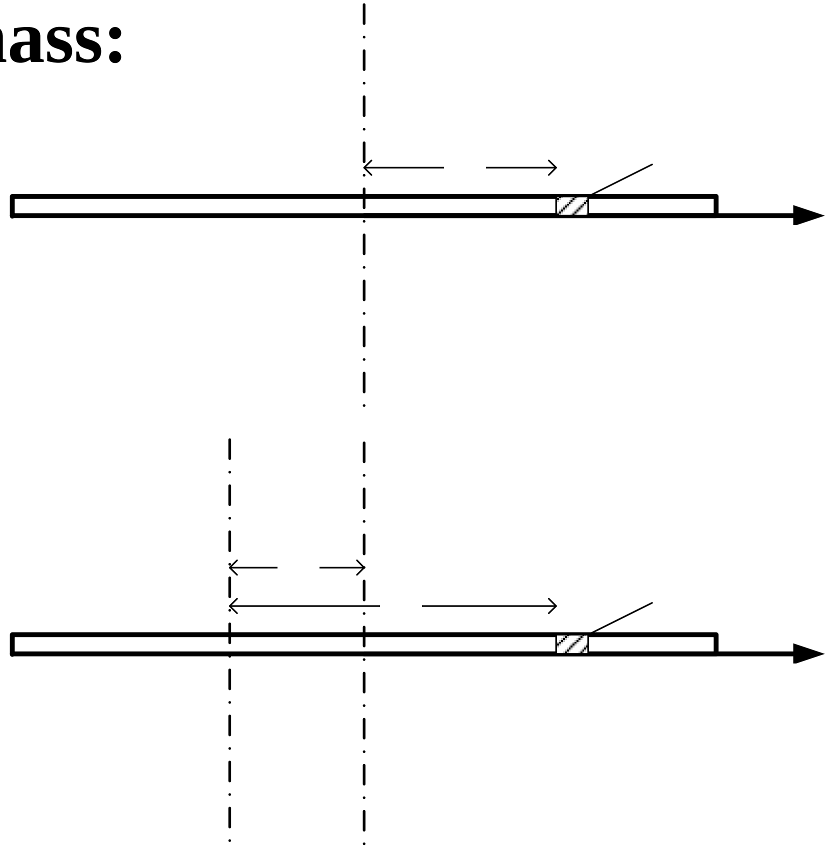
Example

Uniform thin rod with mass *M* and length *l*. Calculate the moment of inertia about the axis located (1) at the CM, (2) at an arbitrary distance *h* from the CM.



Solution: (1) The axis locates at the CM

Take a small element of mass:



$$dm = \lambda dx = \frac{M}{l} dx$$

$$dI = x^2 dm = \lambda x^2 dx$$

$$I = \int dI = \int_{-l/2}^{l/2} \lambda x^2 dx$$

$$= \frac{1}{3} \lambda x^3 \Big|_{-l/2}^{l/2} = \frac{1}{12} M l^2$$

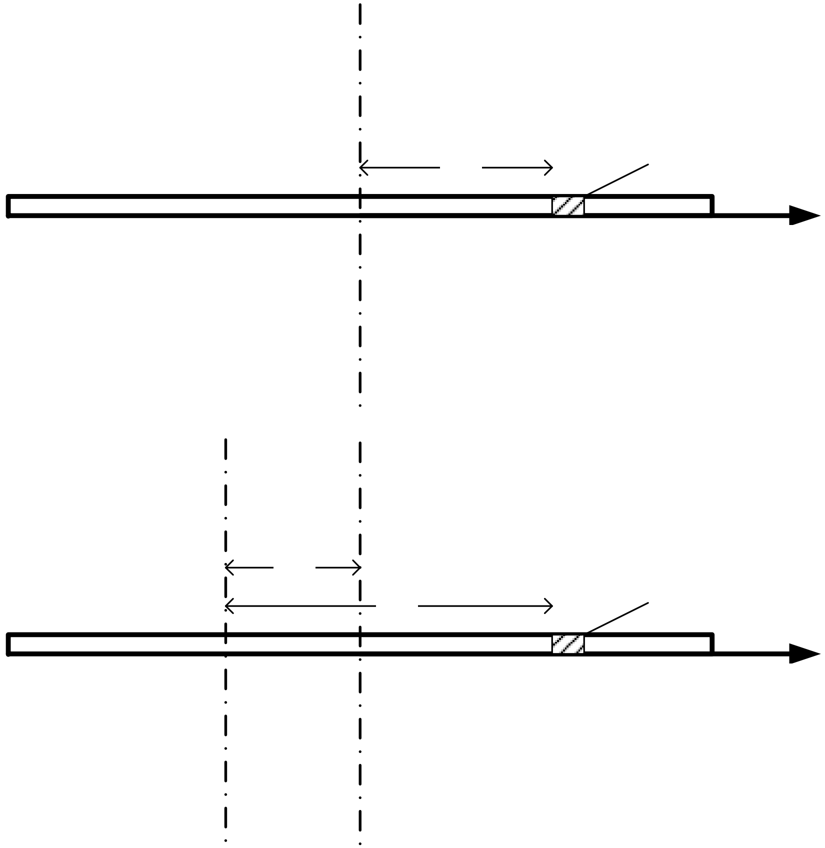
Example

(2) The axis locates at arbitrary distance *h* from the CM.

$$I = \int_{-(l/2-h)}^{l/2+h} \lambda x^2 dx$$

$$= \frac{1}{3}\lambda x^3 \Big|_{-l/2+h}^{l/2+h}$$

$$= \frac{1}{12}Ml^2 + Mh^2$$



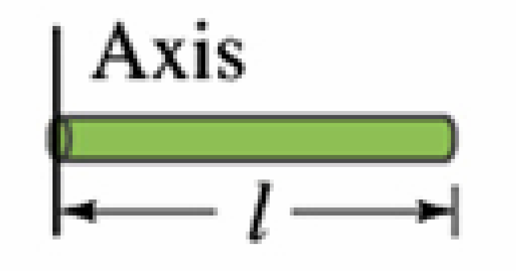
The parallel-axis theorem

The Parallel-axis and Perpendicular-axis Theorems (P249,250)

- **The Parallel-axis Theorem**

$$\boxed{I = I_{\mathrm{CM}} + Mh^2}$$

Long uniform rod of length *l*, axis through one end:



$$I_{\mathrm{end}} = I_{\mathrm{CM}} + M\left(\frac{l}{2}\right)^2 = \frac{1}{12}Ml^2 + \frac{1}{4}Ml^2 = \frac{1}{3}Ml^2$$

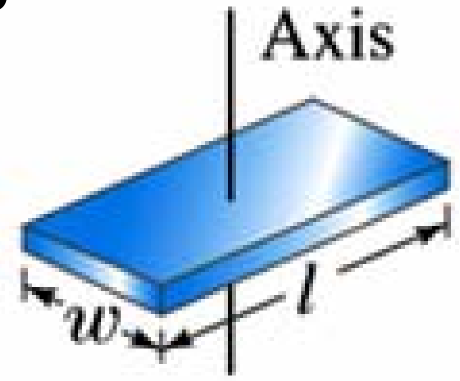
The Perpendicular-axis Theorem

- The sum of the moment of inertia of a **plane** body about any two perpendicular axes in the plane of the body is equal to the moment of inertia about an axis through their point of intersection perpendicular to the plane of the object.

$$\boxed{I_z = I_x + I_y}$$

  - Rectangular thin plate, of length l and width w.

$$I_z = \frac{1}{12} M (l^2 + w^2)$$

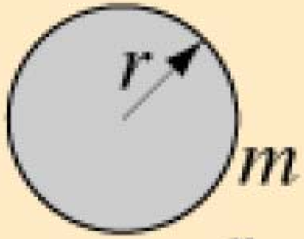


  - Circular thin plate?

The moment of inertia

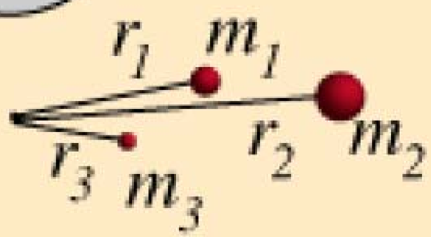
$$I = mr^2$$

For a point mass the moment of inertia is just the mass times the radius from the axis squared. For a collection of point masses (below) the moment of inertia is just the sum for the masses.



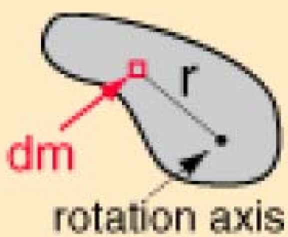
$$I = kmr^2$$

For an object with an axis of symmetry, the moment of inertia is some fraction of that which it would have if all the mass were at the radius r.



$$I = \sum_i m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots$$

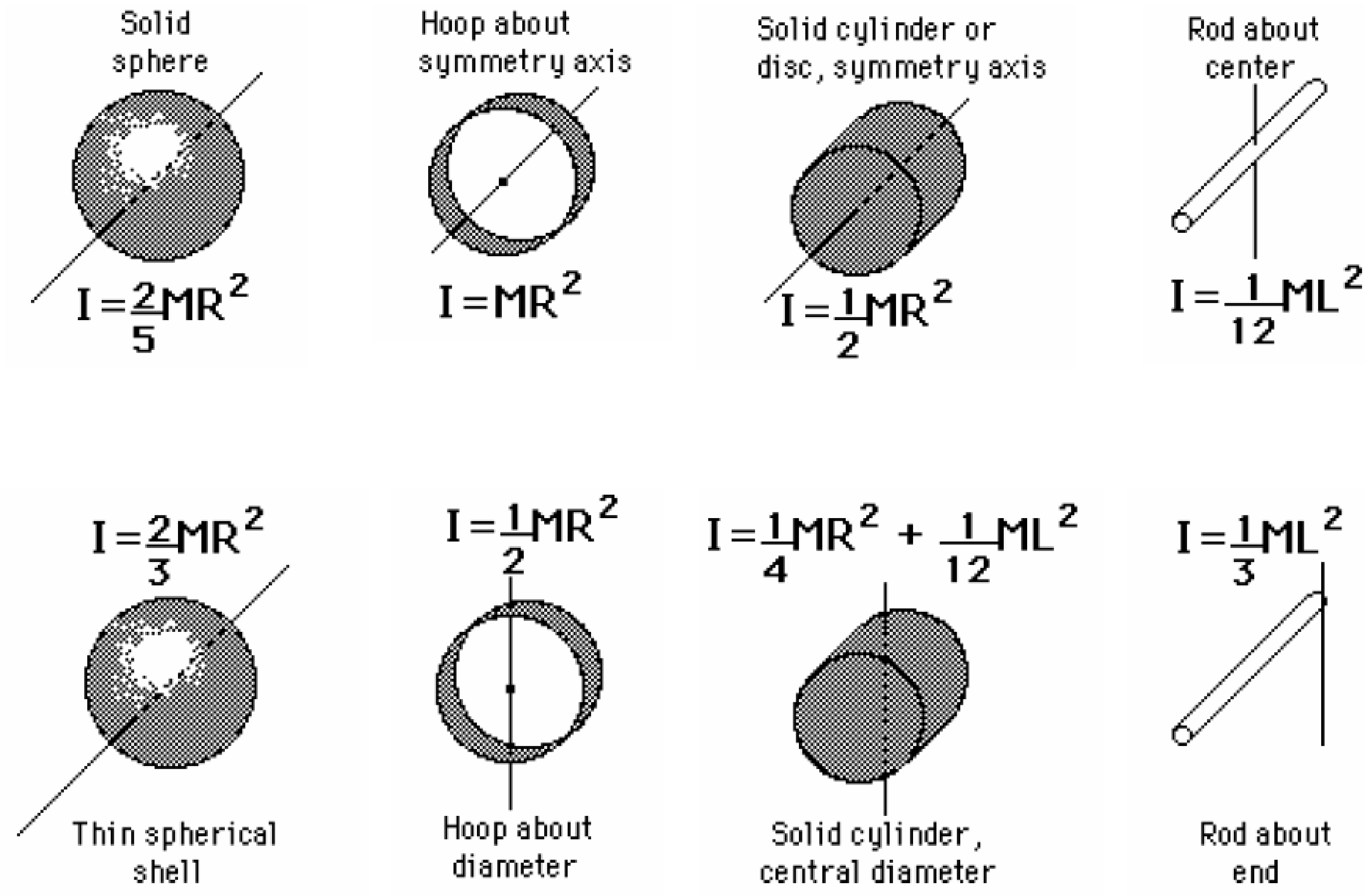
Sum of the point mass moments of inertia.



$$I = \int_0^M r^2 \, dm$$

Continuous mass distributions require an infinite sum of all the point mass moments which make up the whole. This is accomplished by an integration over all the mass.

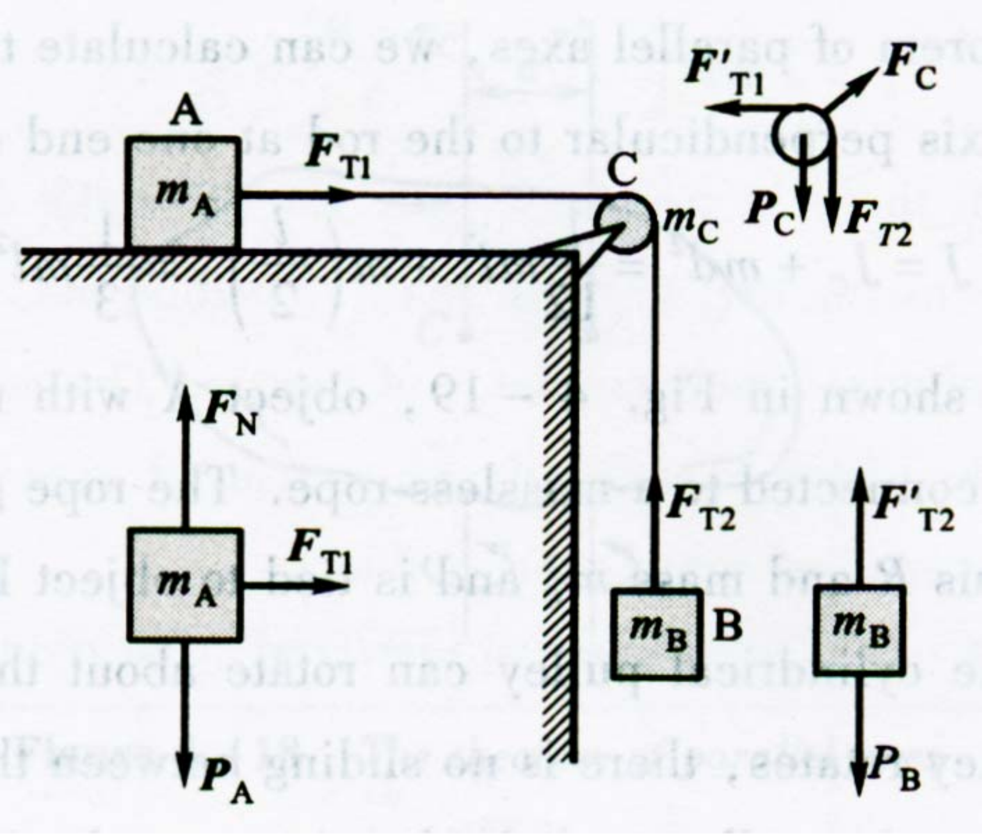
The moment of inertia



Example

Two blocks and a pulley:

Two blocks of masses m_A and m_B are connected by a light cord running over a pulley. The pulley are considered as a uniform cylindrical disk of mass m_C and radius R. There is no sliding between the pulley and the cord. Find the acceleration of two blocks.

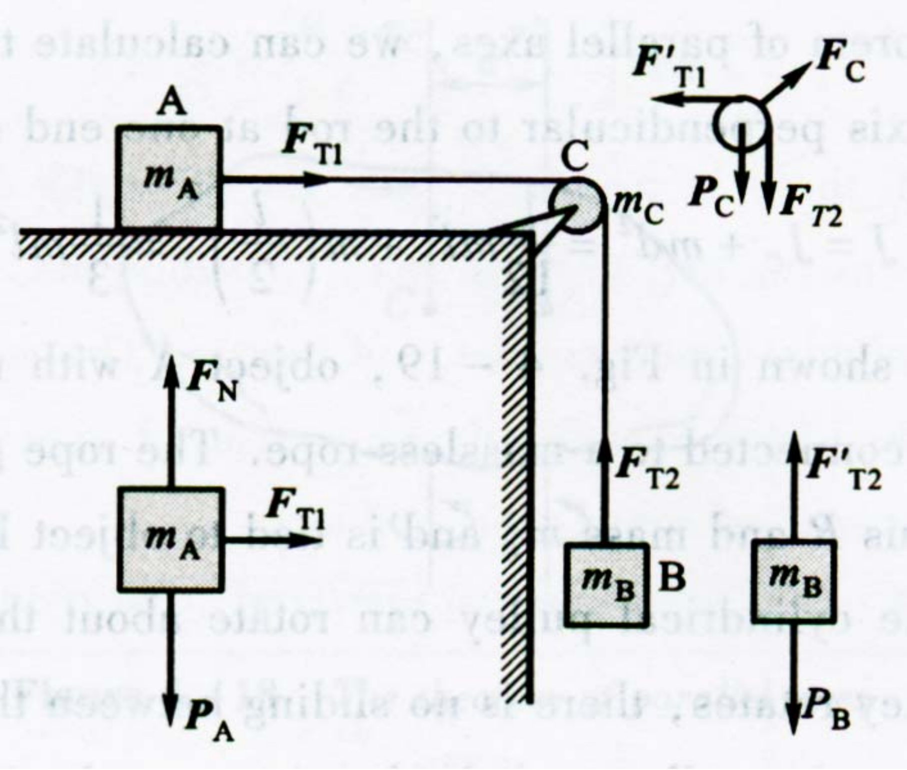


Solution

(1) Draw free-body diagrams.

(2)Newton's II law for every object:

The positive direction of rotation is clockwise.



$$F_{T1} = m_A a$$

$$m_B g - F_{T2} = m_B a$$

$$R(F_{T2} - F_{T1}) = \left(\frac{1}{2} m_c R^2\right)\alpha$$

4 unknowns.

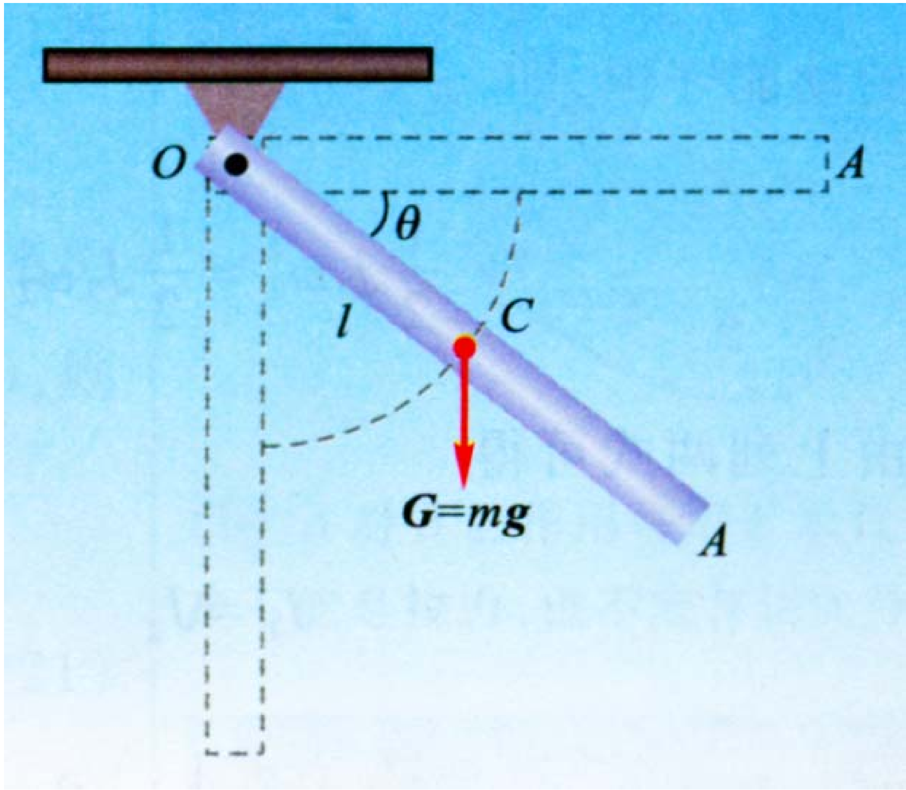
The restriction condition: no sliding between the pulley and the cord.

$$a = R\alpha$$

$$\boxed{a = \frac{m_B g}{m_A + m_B + \frac{1}{2} m_C}}$$

Example
(vs. P248 Ex. 10-9)

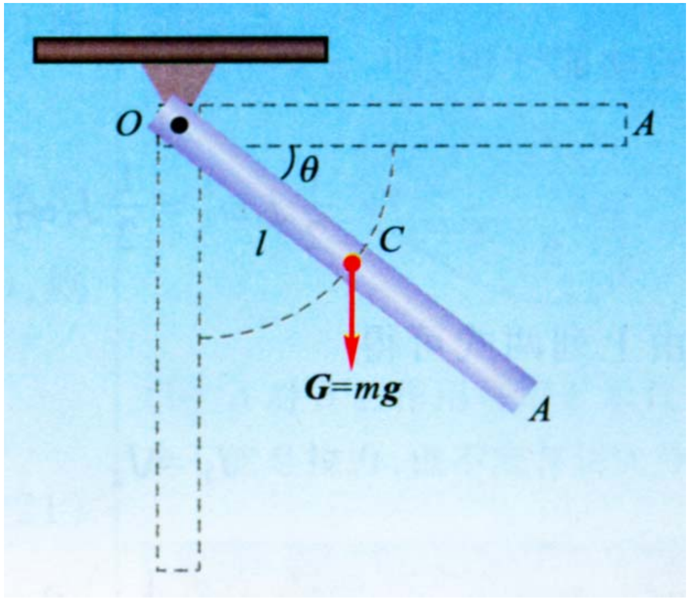
A uniform rod of mass m and length l can pivot freely (no friction on the pivot) about a hinge to the ceiling. The rod is held horizontally and released. Determine: (1) The angular acceleration and angular velocity of the rod as the function of θ. (2) The force on the hinge exerted by the rod.



Example

Solution: (1) Newton's II law for the rotation of rod.

$$mg\frac{l}{2}\cos\theta = I\alpha, \qquad I = \frac{1}{3}ml^2$$



$$\boxed{\alpha = \frac{3}{2}\frac{g}{l}\cos\theta}$$

$$\alpha = \frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta} = \frac{3}{2}\frac{g}{l}\cos\theta$$

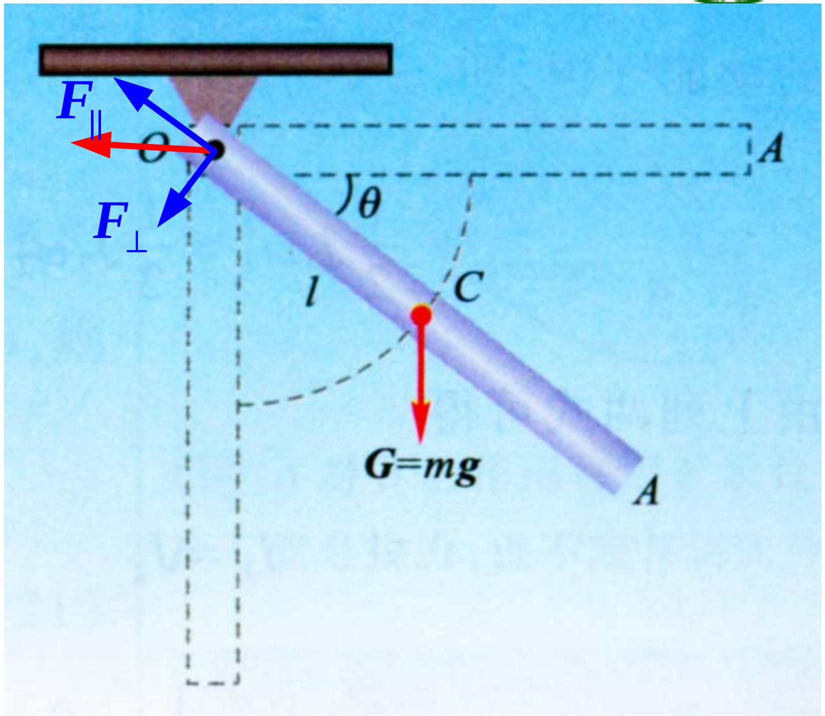
$$\int_0^{\omega}\omega\, d\omega = \frac{3}{2}\frac{g}{l}\int_0^{\theta}\cos\theta\, d\theta$$

$$\boxed{\omega = \sqrt{\frac{3g}{l}\sin\theta}}$$

Example cont'd

$$\alpha = \frac{3}{2}\frac{g}{l}\cos\theta$$

$$\omega = \sqrt{\frac{3g}{l}\sin\theta}$$



Solution: (2) Newton's II law for the CM of the rod.

Normal:

$$F_{\parallel} - mg\sin\theta = ma_{\text{n-CM}}$$

$$= m\frac{l}{2}\omega^2$$

Tangential:

$$F_{\perp} + mg\cos\theta = ma_{\text{t-CM}}$$

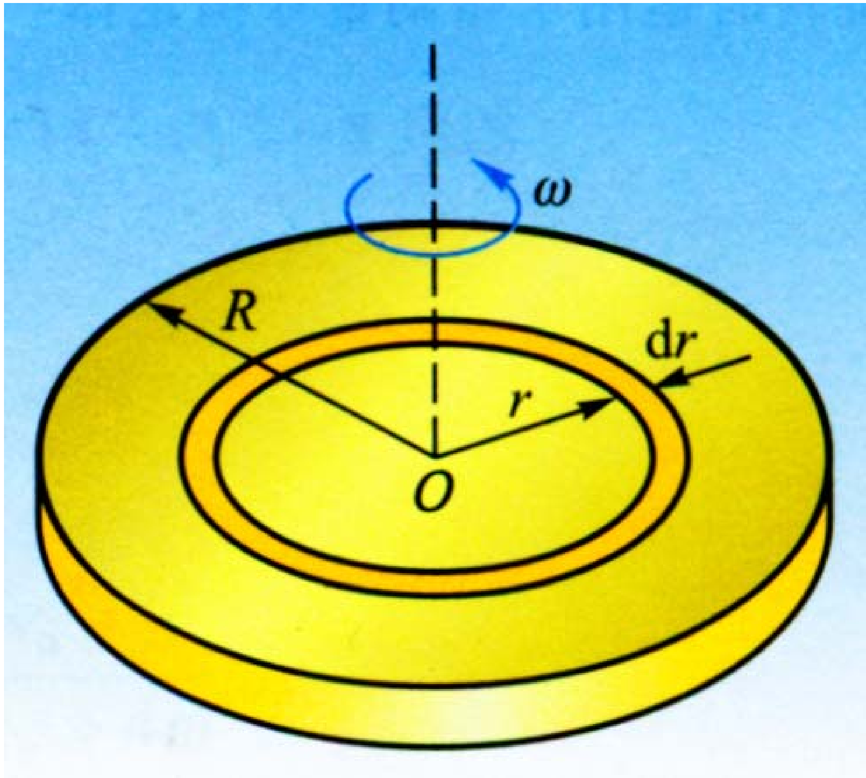
$$= m\frac{l}{2}\alpha$$

$$F_{\parallel} = \frac{5}{2}mg\sin\theta$$

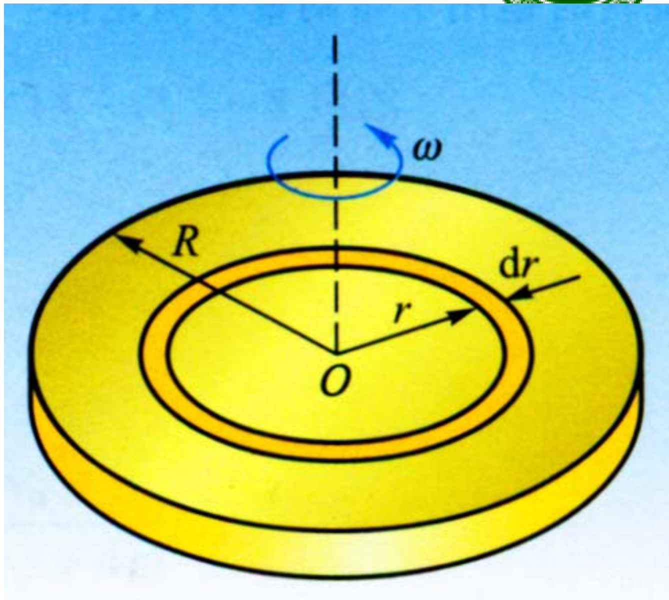
$$F_{\perp} = -\frac{1}{4}mg\cos\theta$$

Example

A circular platform of mass m and radius R rotates initially at an angular velocity ω_0 about its central axis. Then the platform is placed on a rough horizontal surface. Determine (1) the torque acting on the platform by the friction force; (2) the time before the platform comes to a halt. The coefficient of friction between the platform and the surface is μ.



Solution



(1) **The friction force is distributed in the whole area of the platform. Divide the whole platform into many circular rings with a radius of *r* and width *dr*:**

$$dm = \sigma dS = \sigma(2\pi r\, dr), \qquad \sigma = \frac{m}{\pi R^2}$$

$$dF_f = \mu(dm)g, \qquad d\tau_f = -r dF_f = -\mu r g dm$$

$$\tau_f = -\int_m \mu r g dm = -\int_0^R \mu g r \sigma 2\pi r dr = -\frac{2}{3}\pi\mu g R^3 \sigma = -\frac{2}{3}\mu m g R$$

(2) **The Newton's II law for rotation:** $\tau_f = I\alpha$

$$-\frac{2}{3}\mu m g R = \frac{1}{2} m R^2 \frac{d\omega}{dt}, \qquad t = \int_0^t dt = -\frac{3R}{4\mu g}\int_{\omega_0}^0 d\omega = \frac{3R}{4\mu g}\omega_0$$

§ 3 The Rotational Form of Newton's Second Law

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