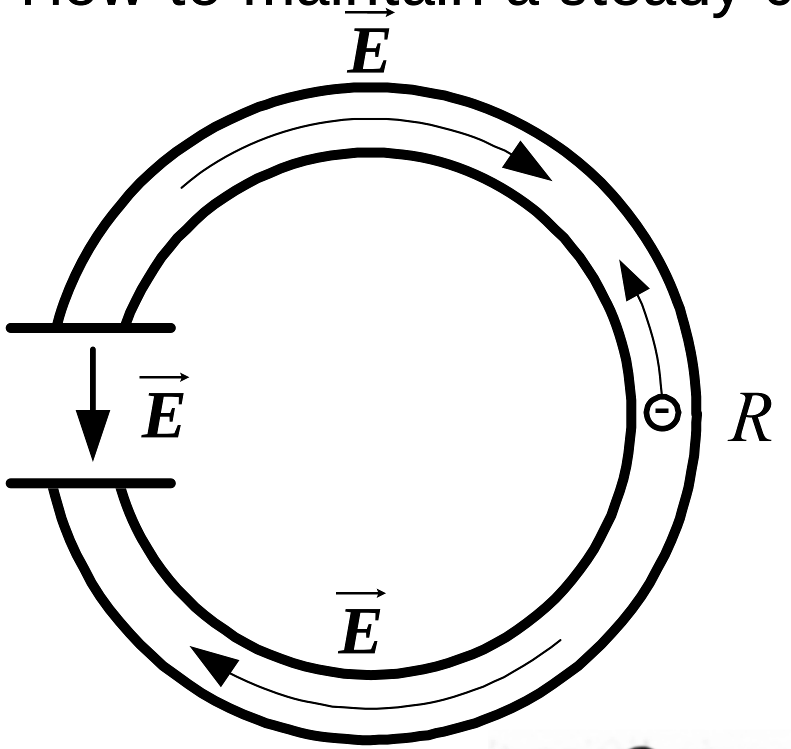
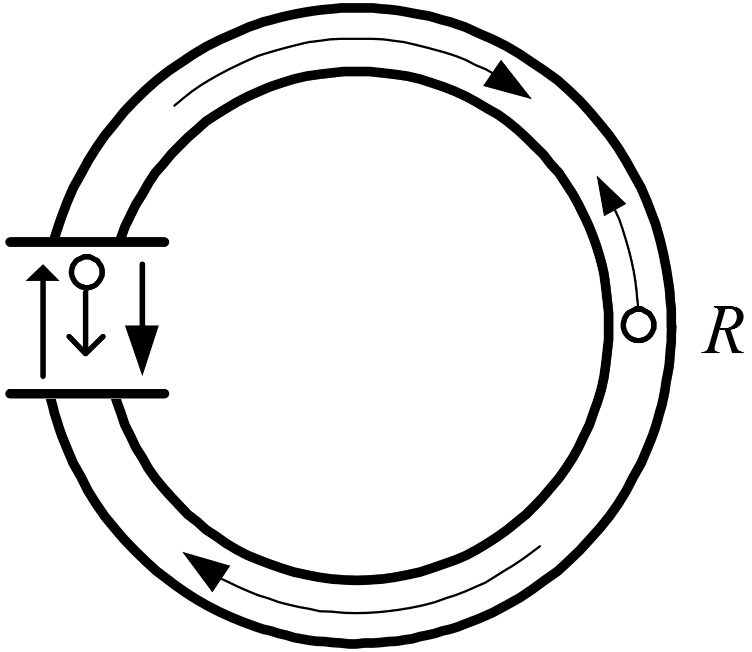
The Electromotive Force (*emf*) (P566 § 24-1)

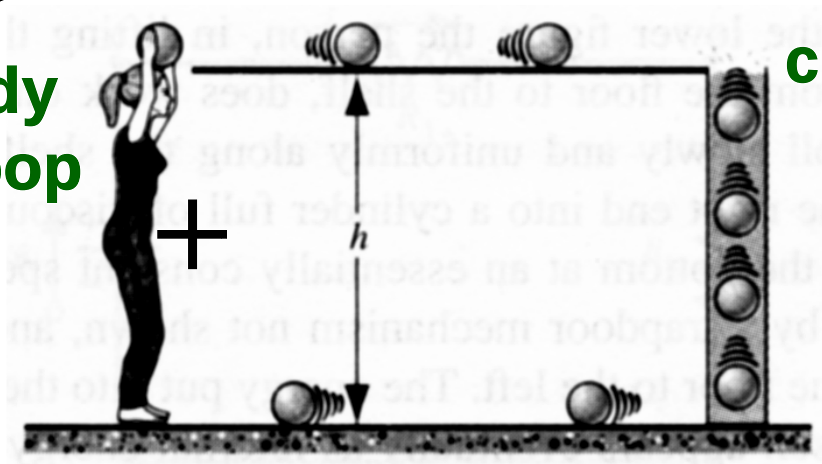
- How to maintain a steady current?



Nonsteady current loop

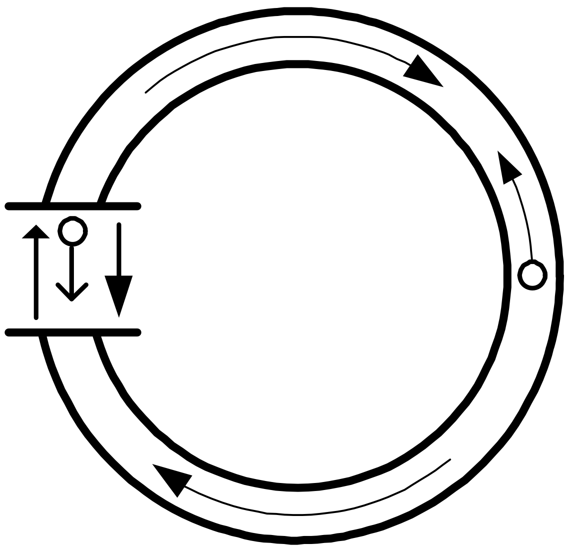


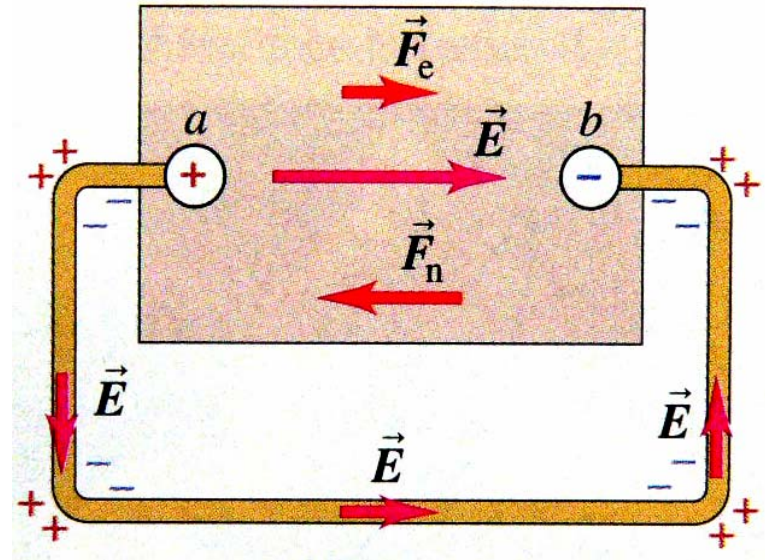
Steady current loop



Electromotive force —— emf

Steady current loop





$$W_n = \int_{-}^{+} \vec{F}_n \cdot d\vec{s} = \int_{-}^{+} q\vec{E}_n \cdot d\vec{s}$$

$$\boxed{\mathcal{E} = \frac{W_n}{q} = \int_{-}^{+} \vec{E}_n \cdot d\vec{s}}$$

$$\boxed{\mathcal{E} = \oint \vec{E}_n \cdot d\vec{s}}$$

ideal source: $q\mathcal{E} = qV_{ab}$ ⇒ $V_{ab} = \mathcal{E}$

§ 1 Faraday's Law of Induction and Lenz's Law

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\mathrm{encl}}$$

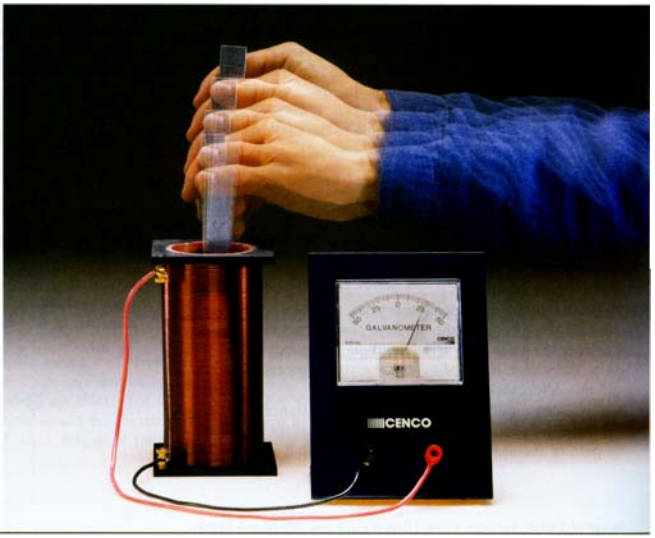
electric current ⇄ magnetic field

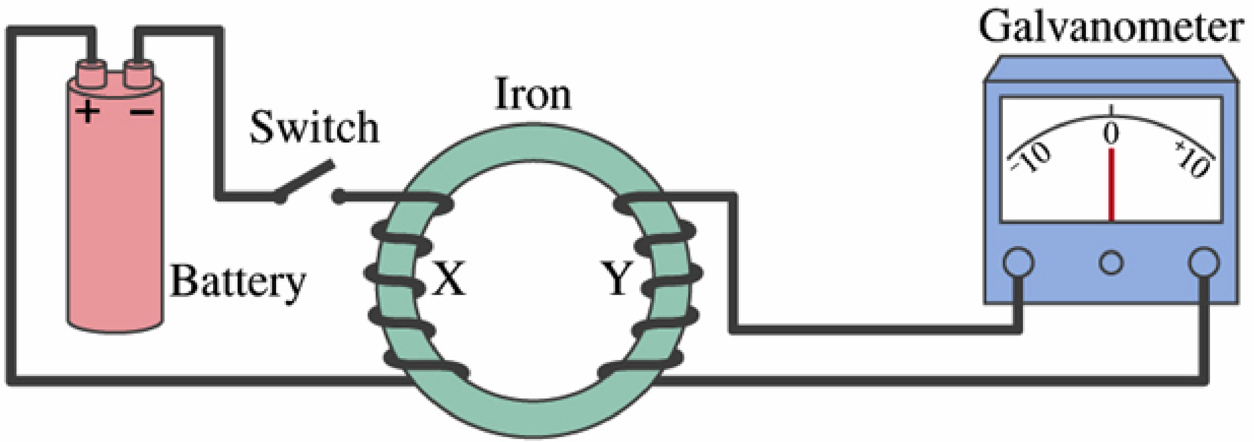
?

- **Question: Can an electric current be produced by a magnetic field?**
  - **M. Faraday (1791-1867) answered this question in 1831.**

The Experiment of Induction

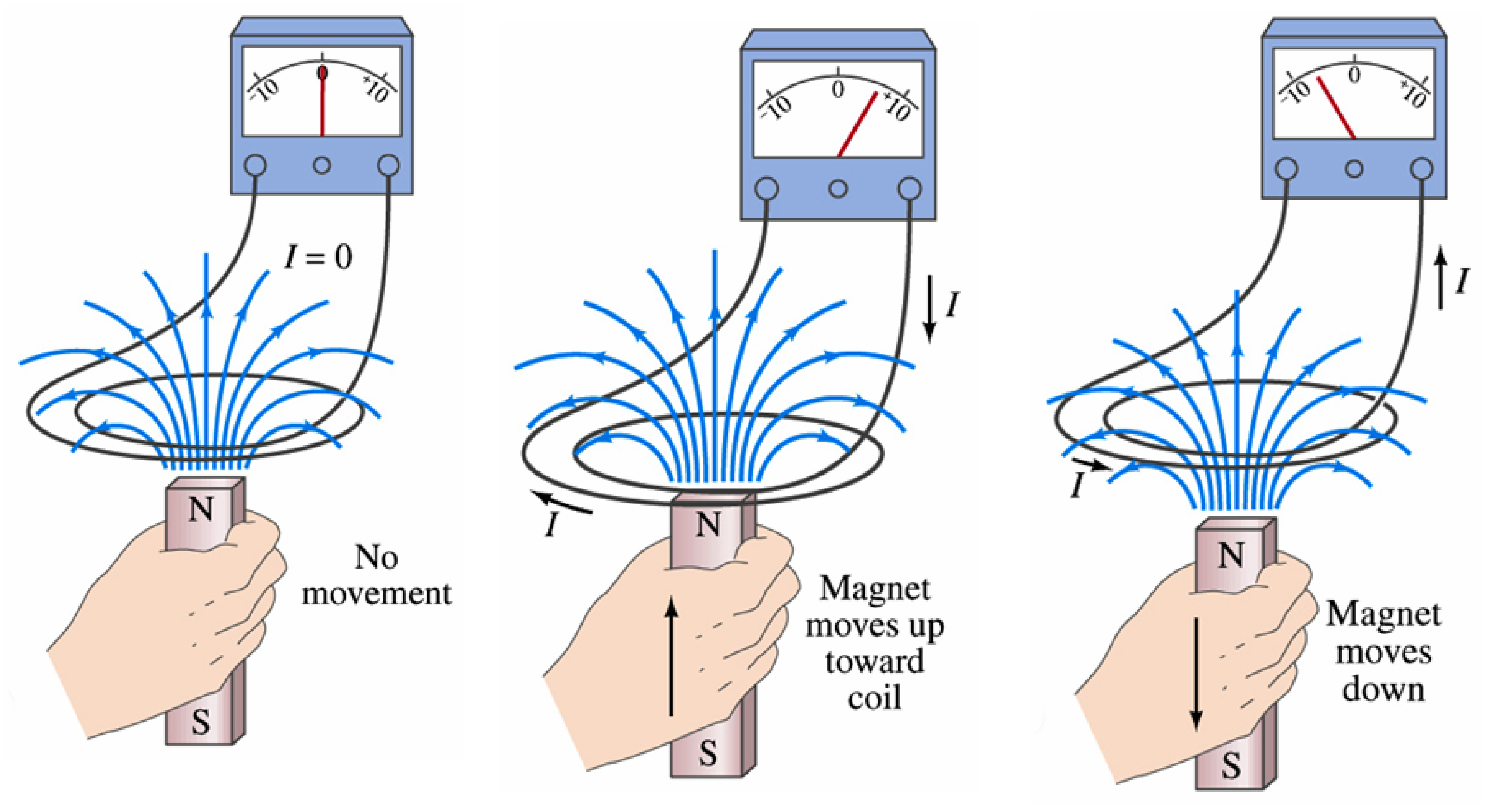
- From the experiment:
  - Steady magnetic field can not produce any current.
  - A time-varying magnetic field can induce an electric current.
  - The galvanometer shows a larger induced current when the relative motion of the magnet is faster.
  - It is the rate of change in the number of the magnetic field lines passing through the loop that determine the induced emf in the loop.





The Experiment of Induction

- It is the **rate of change** in the number of the magnetic field lines passing through the loop that determine the induced **emf** in the loop.



- **Faraday's law:**
  - **The emf induced in a circuit is equal to the time rate of change of magnetic flux through the circuit.** ⇨ $$|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right|$$
  - **If the circuit is a coil consists of *N* turns.** ⇨ $$|\mathcal{E}| = N\left|\frac{d\Phi_B}{dt}\right|$$

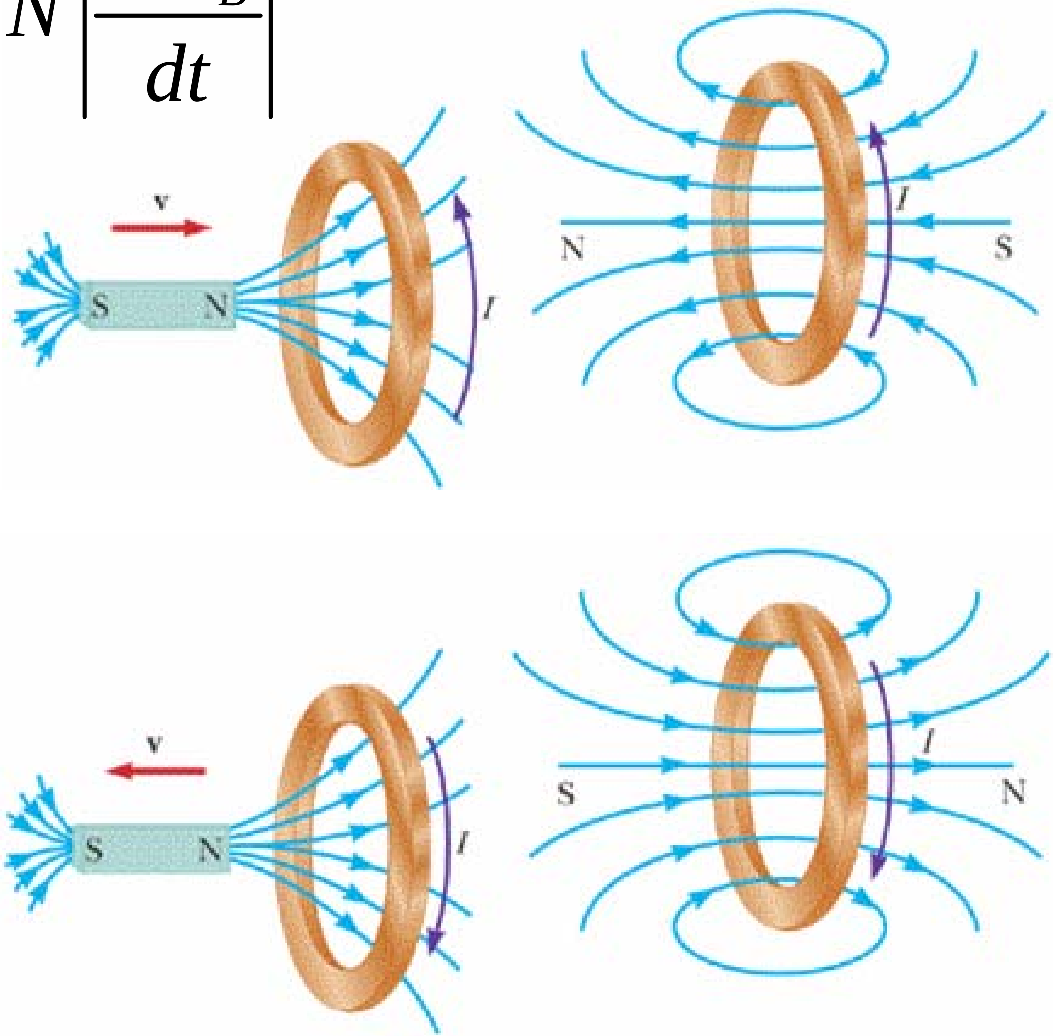
  - **How about the direction of the induced emf?**

—— Lenz's law

$$|\mathcal{E}| = N\left|\frac{d\Phi_B}{dt}\right|$$

- **Lenz's law**
  - **The polarity of the induced emf in a loop is such that it produces a current whose magnetic field opposes the change in magnetic flux through the loop.**

    Another statement:
  - **The induced current is in a direction such that the induced magnetic field attempts to maintain the original flux through the loop.**



- **Complete Faraday's law:**

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt} \iint\limits_{\substack{\text{surrounding}\\ \text{surface}}} \vec{B} \cdot d\vec{A}$$

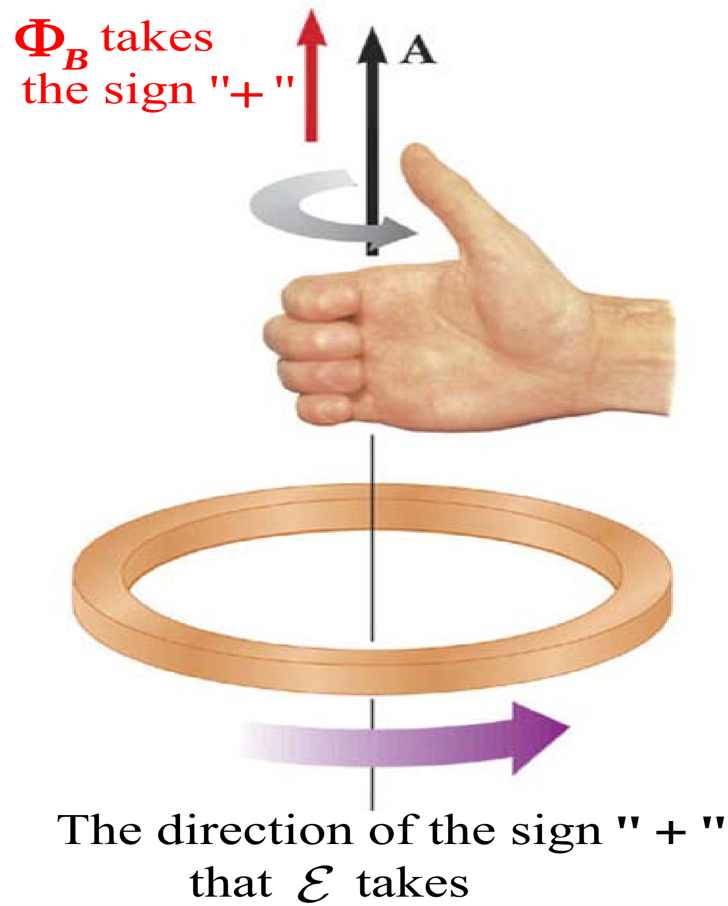
  - **A coil consists of N turns:**

$$\mathcal{E} = -N\frac{d\Phi_B}{dt}$$

How to Determine the Sign of Induced emf

- The relationship between the direction of emf $\mathcal{E}$ and the sign of Φ_B
  - Using the **right-hand rule** to determine the sign of Φ_B and the sign of emf $\mathcal{E}$.

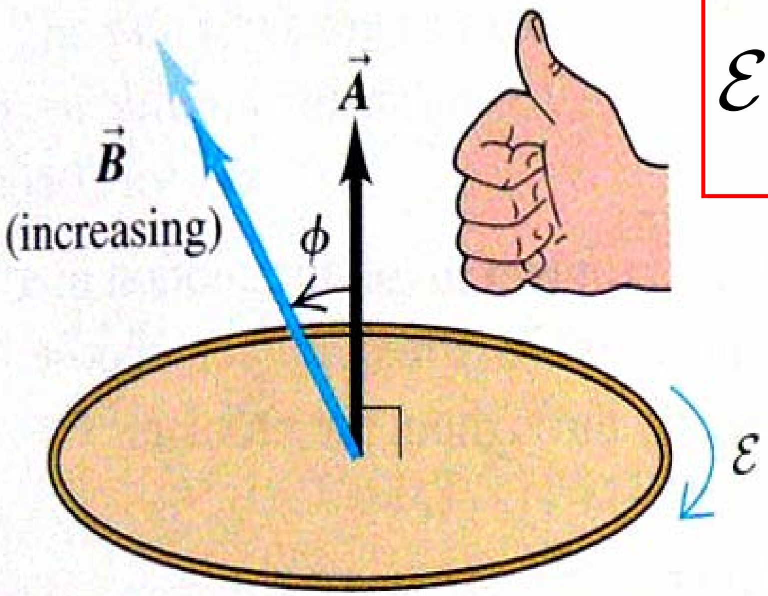
$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\substack{\text{surrounding}\\ \text{surface}}} \vec{B}\cdot d\vec{A}$$



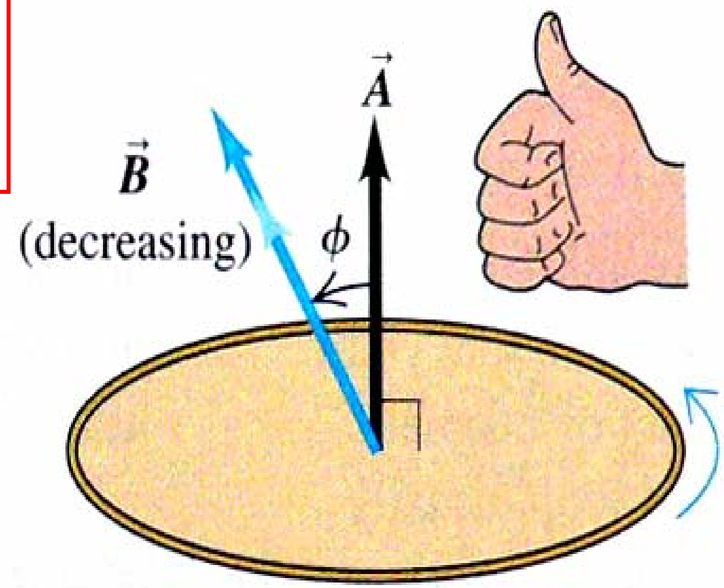
How to Determine the Sign of Induced emf

- Using the **right-hand rule** to determine the sign of Φ_B and the sign of emf $\mathcal{E}$.

$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}$$



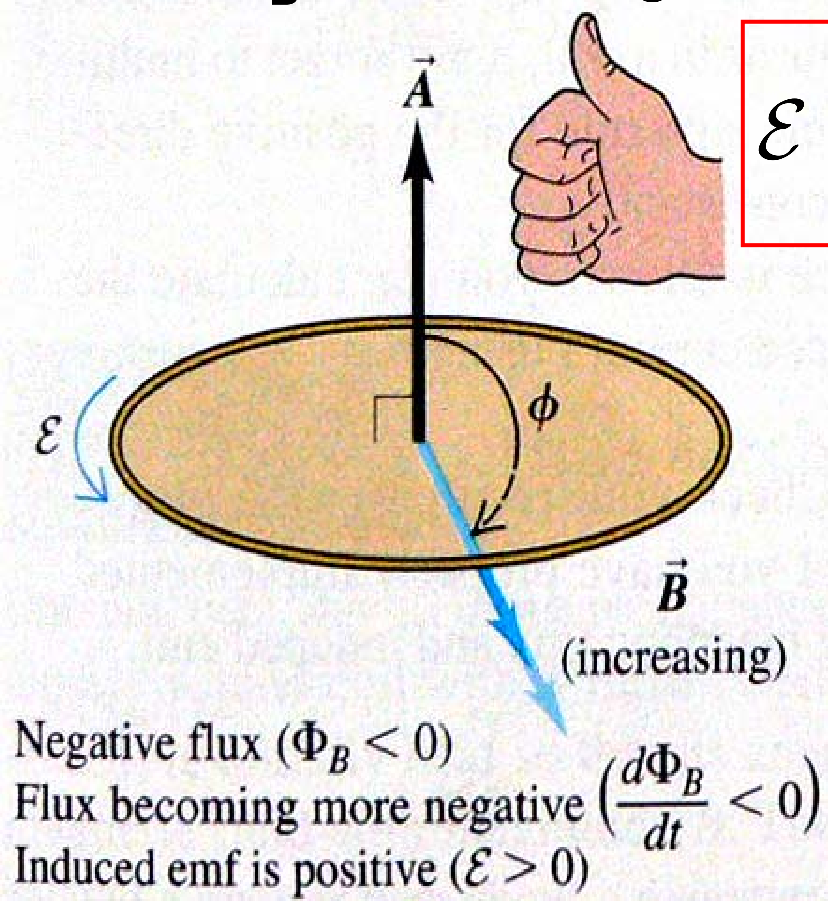
Positive flux ($\Phi_B > 0$)
Flux becoming more positive $\left(\frac{d\Phi_B}{dt} > 0\right)$
Induced emf is negative ($\mathcal{E} < 0$)



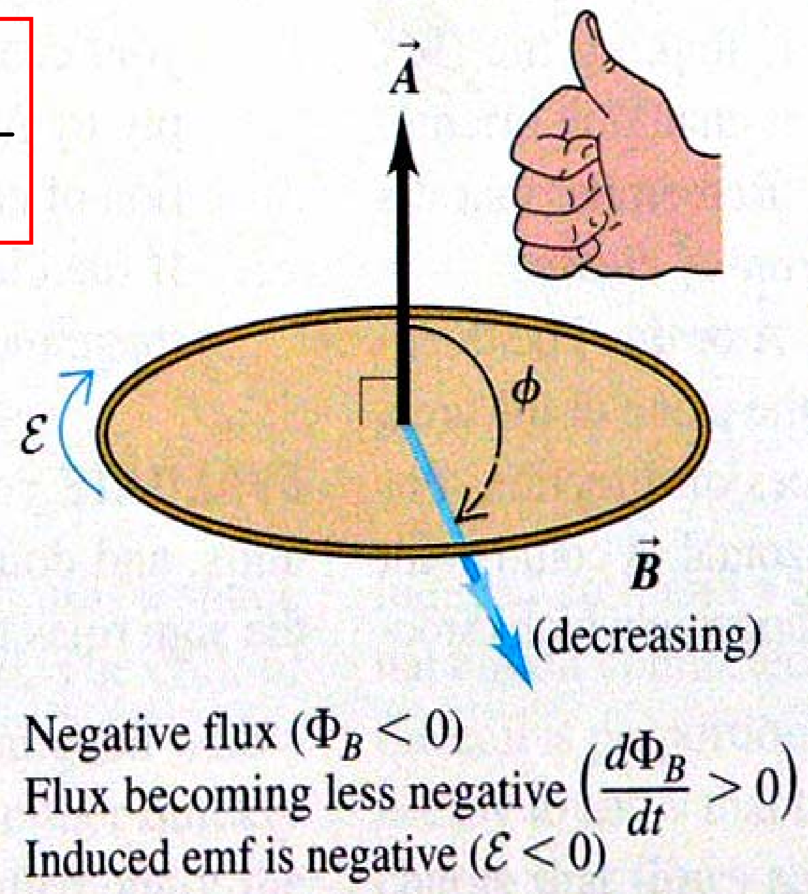
Positive flux ($\Phi_B > 0$)
Flux becoming less positive $\left(\frac{d\Phi_B}{dt} < 0\right)$
Induced emf is positive ($\mathcal{E} > 0$)

How to Determine the Sign of Induced emf

- **Using the right-hand rule to determine the sign of Φ_B and the sign of emf $\mathcal{E}$.**

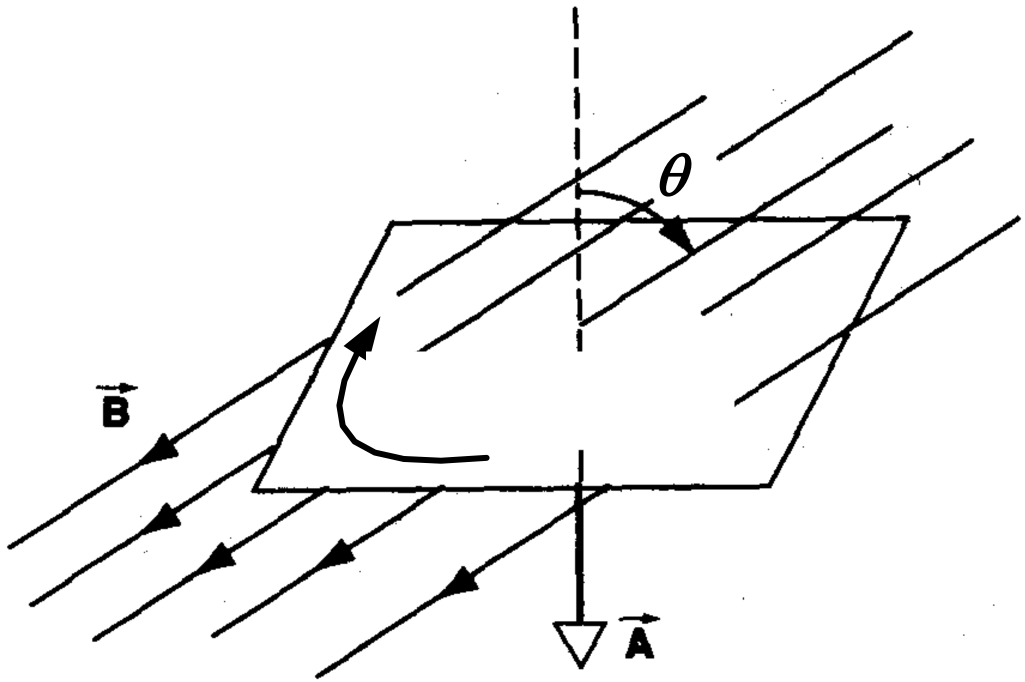


$$\boxed{\mathcal{E} = -\frac{d\Phi_B}{dt}}$$



Example

A plane loop of area A is placed in a region where a uniform magnetic field is an angle θ to the normal to the plane. The magnitude of the magnetic field varies with time according to the expression $B = B_{max} e^{-\alpha t}$. Find the induced emf in the loop as a function of time.

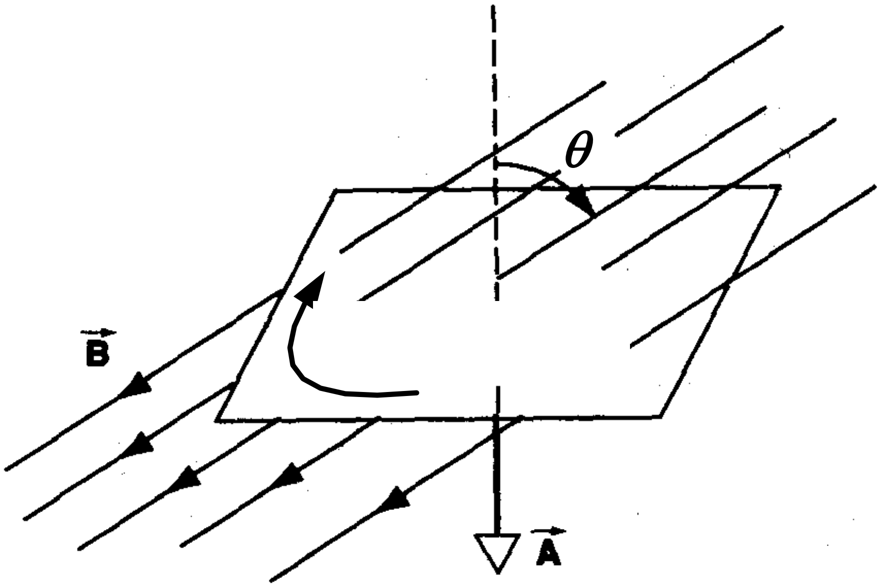


Example

Solution: **Choose the direction of area vector point to downward.**

$$\Phi_B = \vec{B}\cdot\vec{A} = BA\cos\theta$$

$$= AB_{\max}e^{-\alpha t}\cos\theta$$



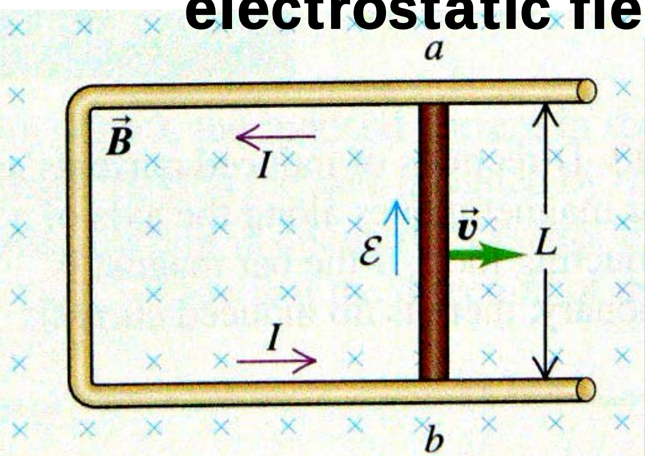
$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

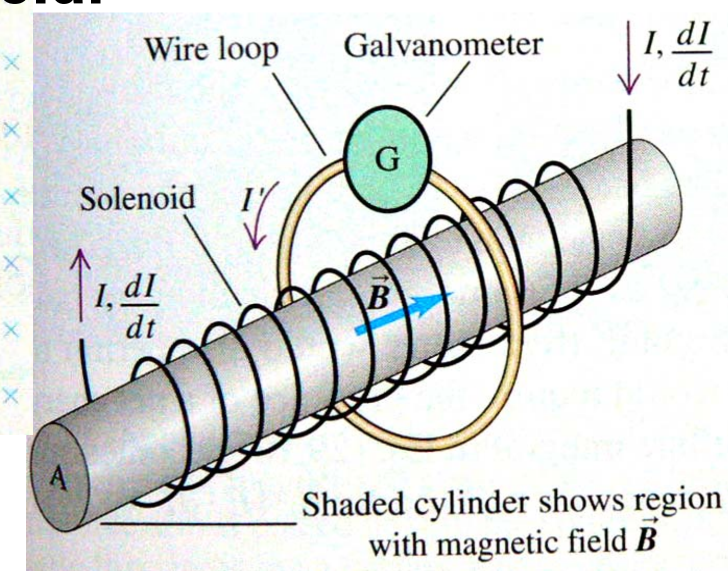
$$= -\left(-\alpha AB_{\max}e^{-\alpha t}\cos\theta\right)$$

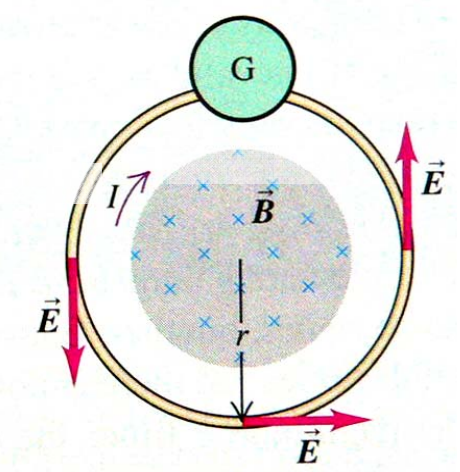
$$= \alpha AB_{\max}\cos\theta e^{-\alpha t}$$

- **What makes the magnetic flux change?**
  - **Is the loop or coil changing orientation or part of the loop moving? —— Motional emf.**
  - **Is the magnetic field changing? —— Induced electric field as the non-electrostatic field.**

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\substack{\text{surrounding}\\ \text{surface}}} \vec{B}\cdot d\vec{A}$$







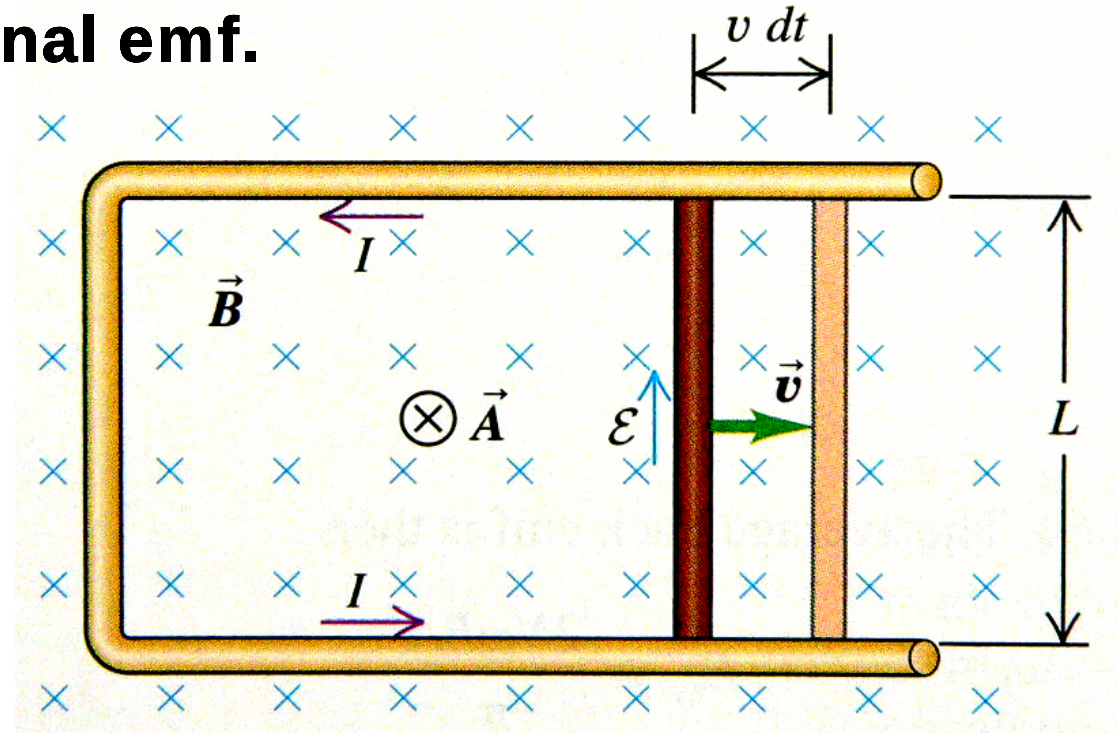
Motional emf

Induced emf

§ 2 Motional emf

- **Staring with the slide-wire generator**

A U-shaped conductor in a uniform magnetic field $\vec{B}$ perpendicular to the plane, directed into page. A metal rod with length L across the two arms of the conductor, forming a circuit. The metal rod slides to the right with a constant velocity $\vec{v}$. Find the motional emf.



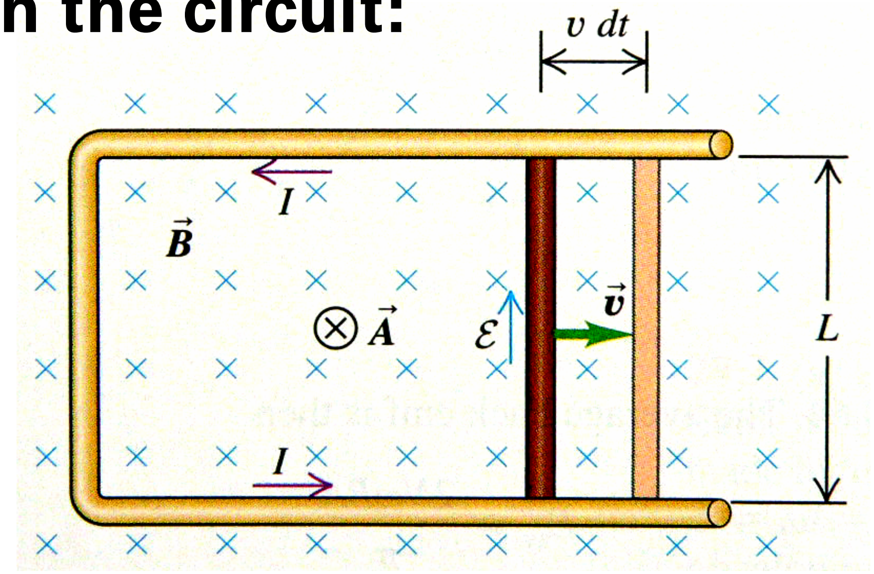
Choose the direction of area $\vec{A}$ as directing into the page.

- The magnetic flux through the circuit:

$$\Phi_B = \vec{B} \bullet \vec{A} = B(Lvt)$$

- The induced emf:

$$\mathcal{E} = -\frac{d\Phi_B}{dt} = -BLv$$



- The negative sign means that direction of emf is counterclockwise.

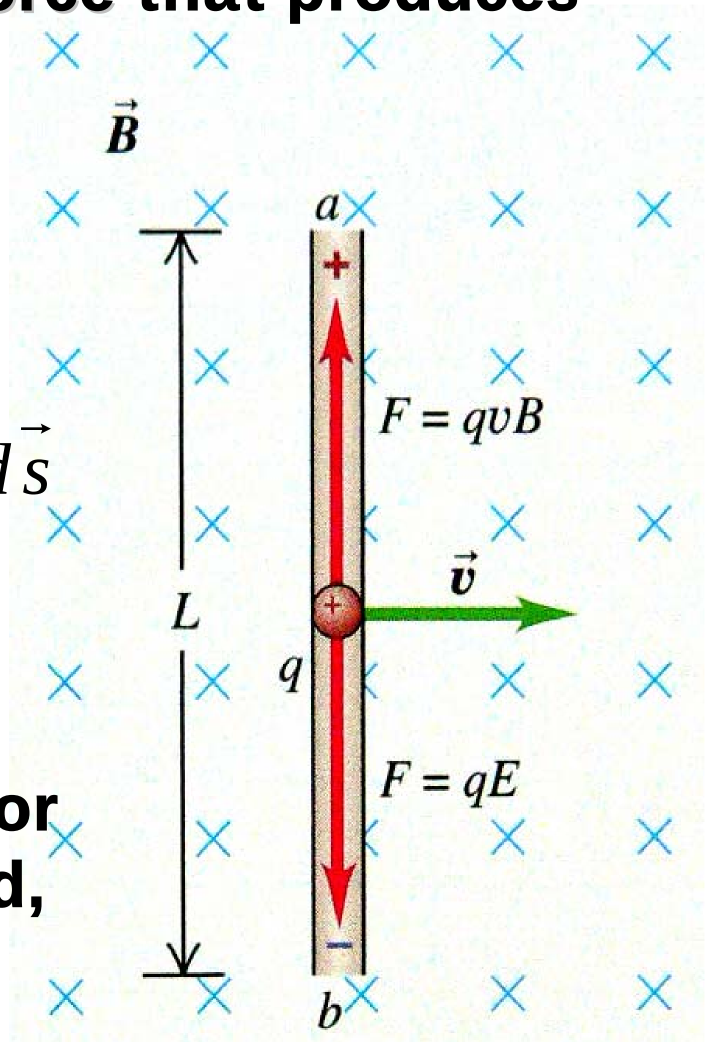
The Origin of the Motional emf

- **The magnetic force exerting on the moving charge in rod acts as the non-electric force that produces the emf.**
- **The magnetic force:** $\vec{F} = q\vec{v} \times \vec{B}$
- **The emf along the rod:**

$$\mathcal{E} = \int_a^b \vec{E}_n \cdot d\vec{s} = \int_a^b \frac{\vec{F}}{q} \cdot d\vec{s} = \int_a^b (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

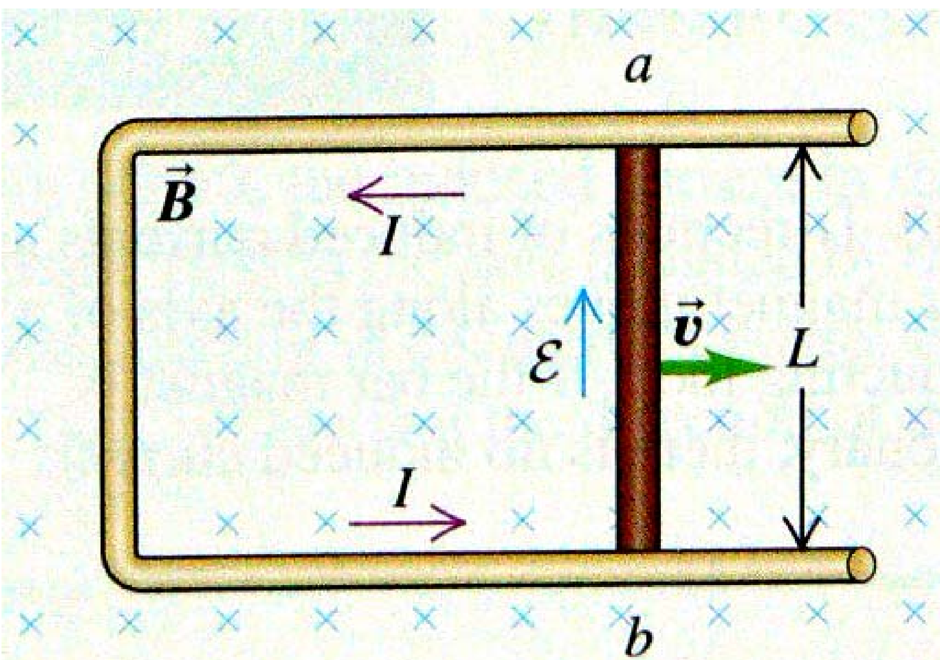
$$= -\int_0^L vBds = -vBL$$

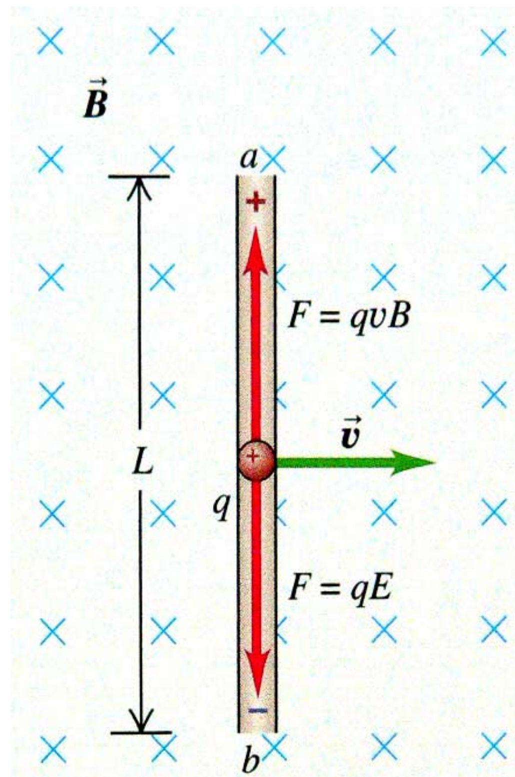
- **The emf is induced in a conductor moving through a magnetic field, called motional emf.**



The Origin of the Motional emf

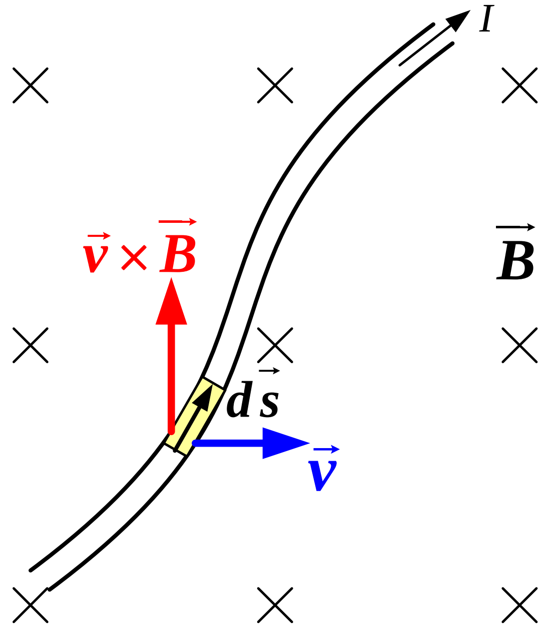
- With Faraday's law, we cannot know which part of the circuit is the source of the emf. Here we know that the moving rod is the source of emf; within it, positive charge moves from lower to higher potential, and in the remainder of the circuit, charge moves from higher to lower potential.





- **Definition of motional emf:**
  - **For moving current-carrying wire of any shape in a magnetic field**

$$d\mathcal{E} = (\vec{v} \times \vec{B}) \cdot d\vec{s}, \quad \boxed{\mathcal{E} = \int_L (\vec{v} \times \vec{B}) \cdot d\vec{s}}$$



  - **For any closed conducting loop:**

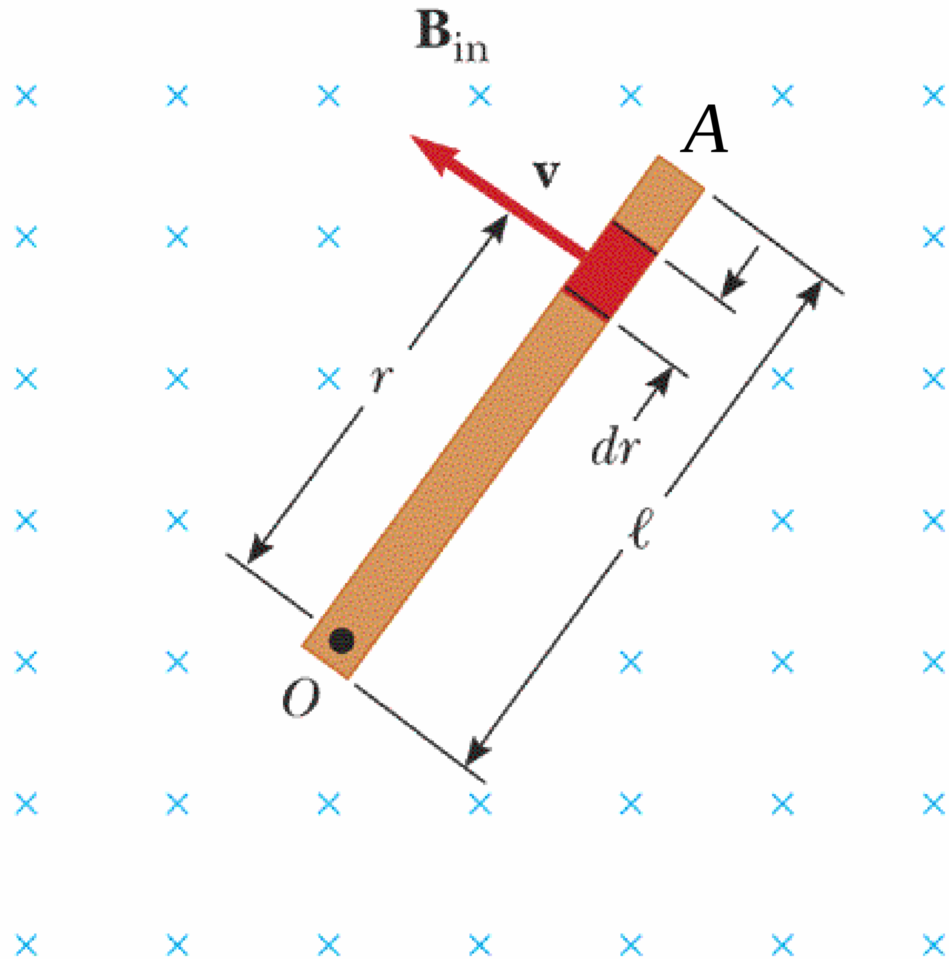
$$\mathcal{E} = \oint_L (\vec{v} \times \vec{B}) \cdot d\vec{s}$$

  - **The direction of motional emf: determined by the projection direction of** $\vec{v} \times \vec{B}$

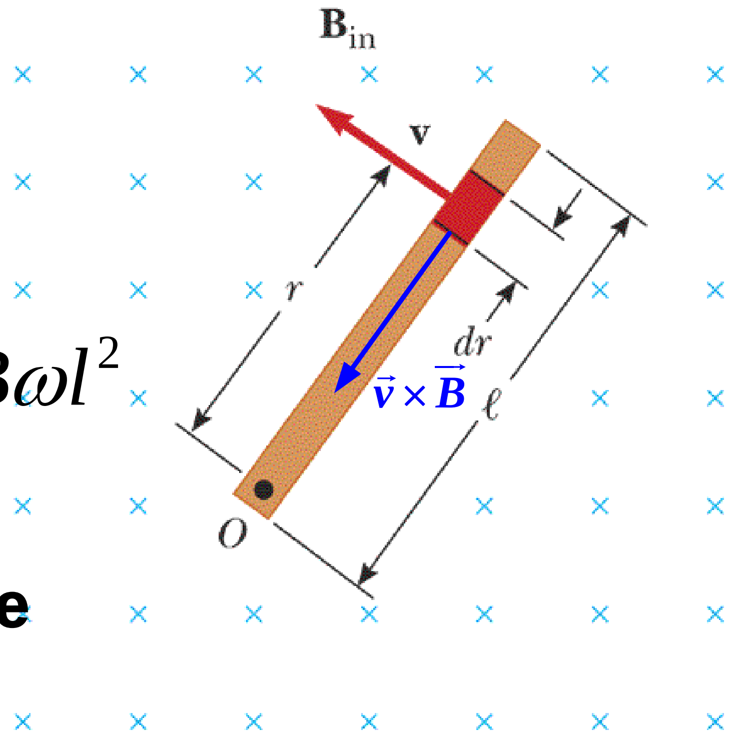
Example

Motional emf induced in a rotating bar

A conducting bar of length l rotates with a angular speed ω about a pivot at one end. B is uniform and perpendicular to the plane of rotation. Find the emf induced between the ends of the bar.



Solution: Choose the direction of integration to be from end O to end A.



$$\mathcal{E}=\int_O^A(\vec{v}\times\vec{B})\cdot d\vec{s}$$

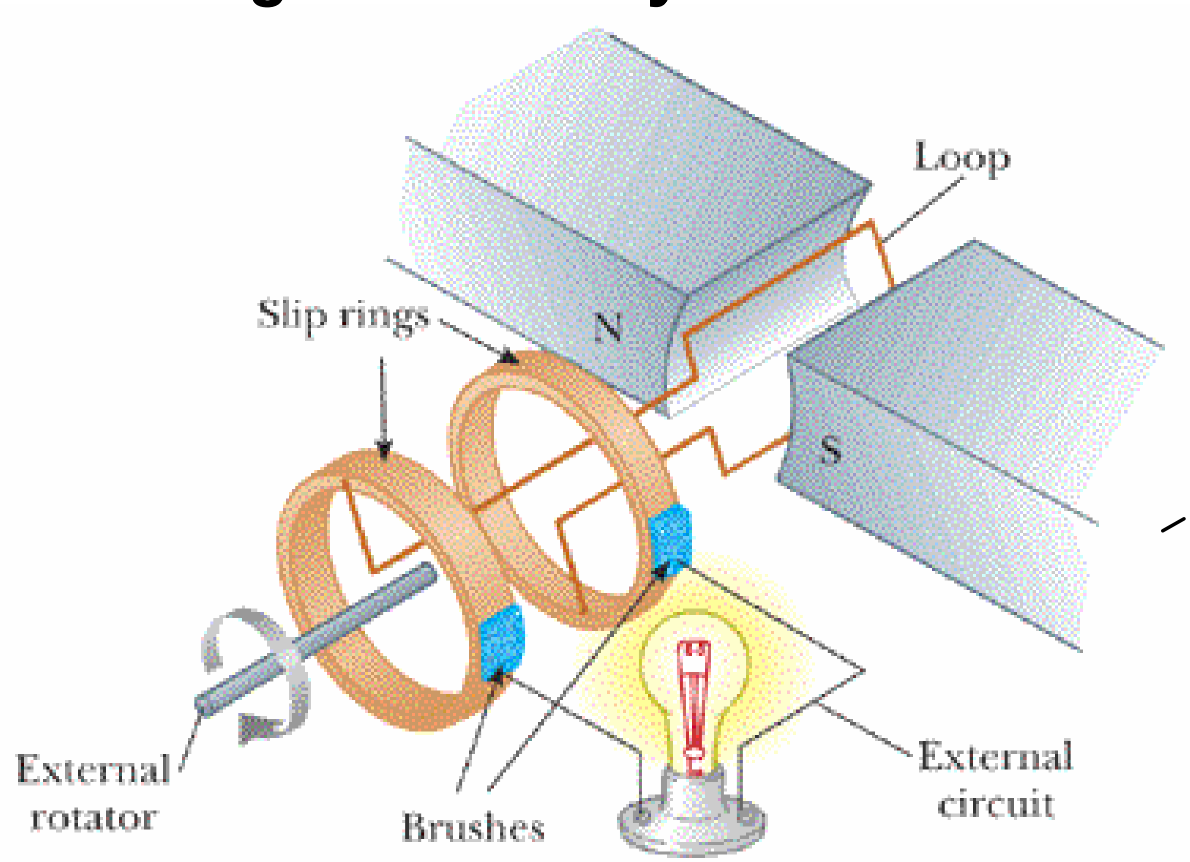
$$=\int_0^l(-Bv)dr=-\int_0^l B\omega r dr=-\frac{1}{2}B\omega l^2$$

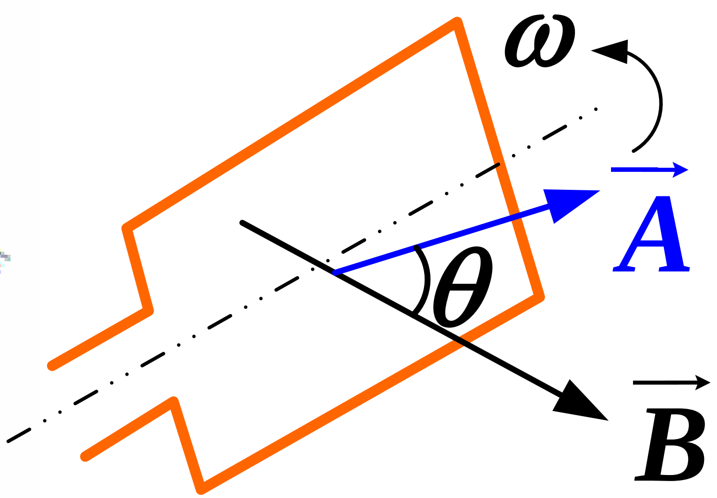
The negative sign means that the real direction of emf is opposite to the direction of integration, and potential at end *A* is lower than end *O*.

Example

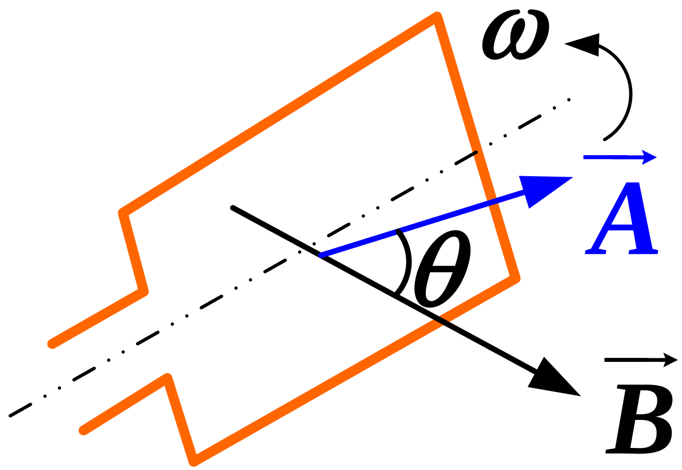
The alternating-current generator

A *N* -turn rectangular loop of area *A* is made to rotate in an external uniform magnetic field, with a angular velocity ω about the axis. Find the emf.





Solution: Assume at time t=0, the direction of area $\vec{A}$ is in alignment with $\vec{B}$.



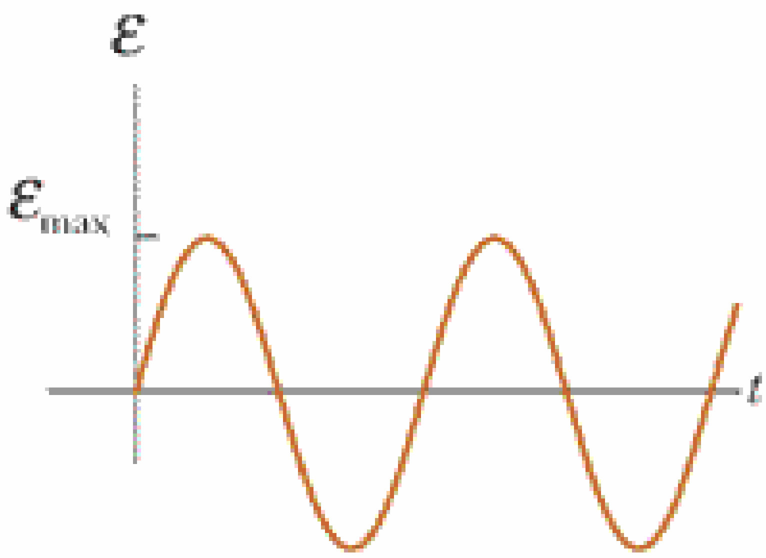
The flux through the loop

$$\Phi_B = \vec{B}\cdot\vec{A} = BA\cos\theta = BA\cos\omega t$$

By Faraday's law,

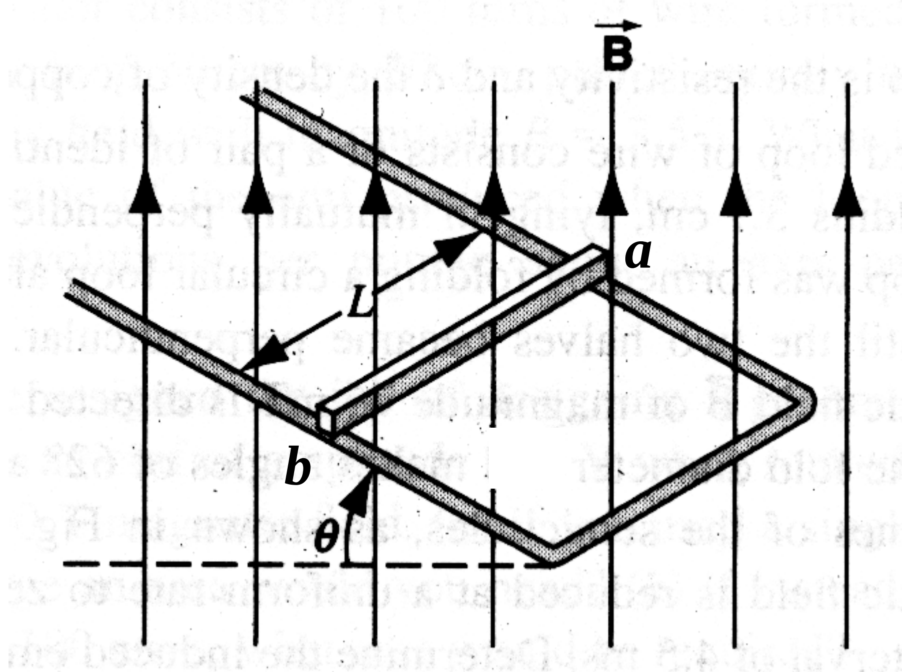
$$\mathcal{E} = -N\frac{d\Phi_B}{dt} = \omega NAB\sin\omega t$$

$$= \mathcal{E}_{\max}\sin\omega t$$



Example

A rod with length l, mass m, and resistance R slides without friction down parallel conducting rails of negligible resistance. The rails are connected together at the bottom, forming a conducting loop with the rod as the top member. The plane of the rails makes an angle θ with the horizontal, and a uniform vertical magnetic field B exists throughout the region. (1) What is the terminal speed of the rod? (2) What is the induced current in the rod when the terminal speed has been reached?



Solution: **(1) Newton's law for the rod**

$$m\frac{dv}{dt} = mg\sin\theta - F_B\cos\theta$$

The motional emf: $\vec{L}:\ a \to b$

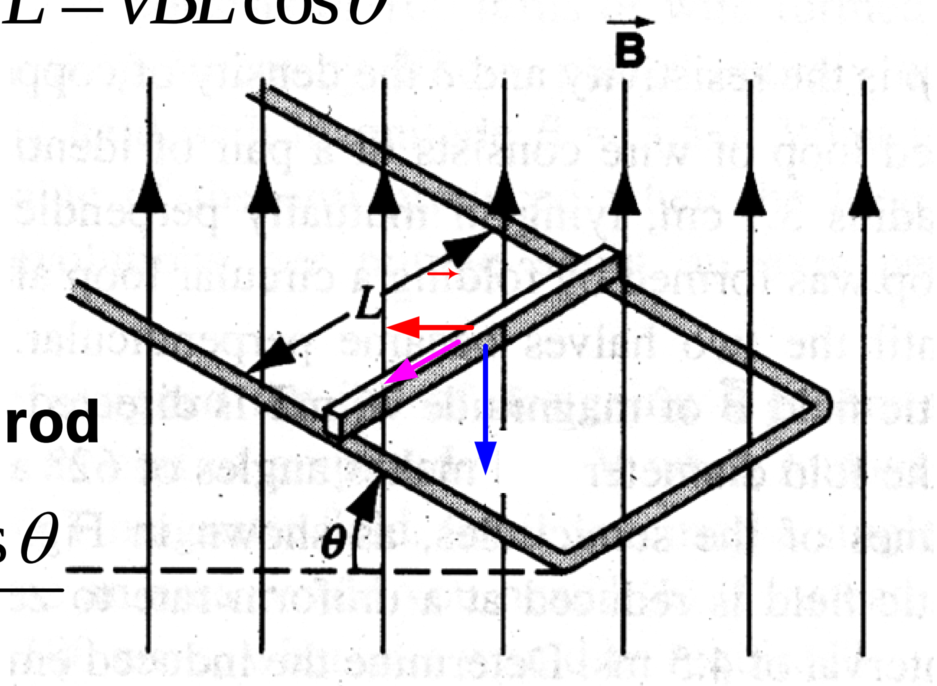
$$\mathcal{E} = (\vec{v}\times\vec{B})\cdot\vec{L} = vB\sin(90^\circ+\theta)L = vBL\cos\theta$$

The current in the loop:

$$I = \frac{\mathcal{E}}{R} = \frac{vBL\cos\theta}{R}$$

The magnetic force acts on the rod

$$F_B = I\,|\vec{L}\times\vec{B}| = ILB = \frac{vB^2L^2\cos\theta}{R}$$



Example Cont'd

Newton's law for the rod becomes:

$$m\frac{dv}{dt} = mg\sin\theta - \frac{vB^2L^2\cos^2\theta}{R}$$

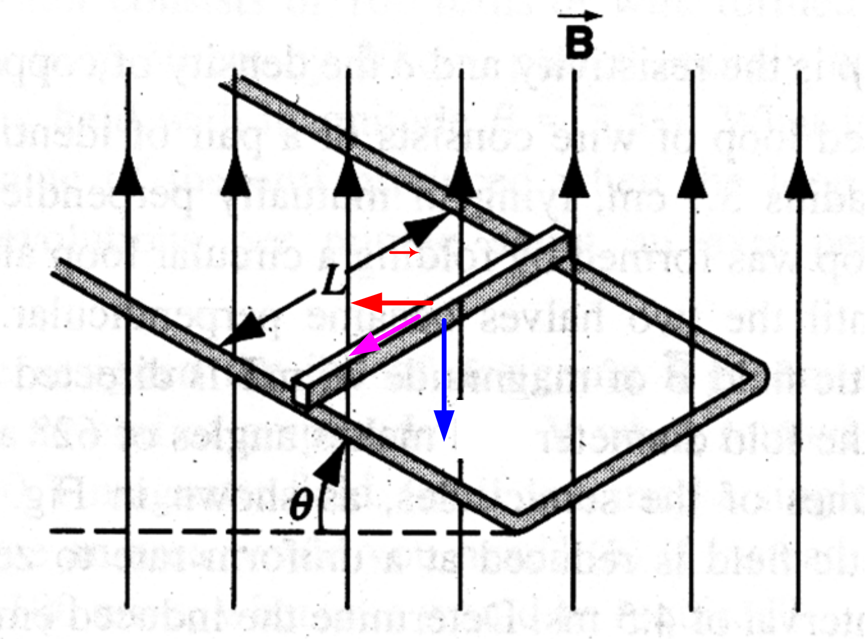
When the rod reaches its terminal speed: $\dfrac{dv}{dt} = 0$

The terminal speed:

$$v = \frac{mgR}{B^2L^2}\frac{\sin\theta}{\cos^2\theta}$$

(2) When the rod reaches the terminal speed, the induced current is:

$$I = \frac{vBL\cos\theta}{R} = \frac{mg}{BL}\tan\theta$$



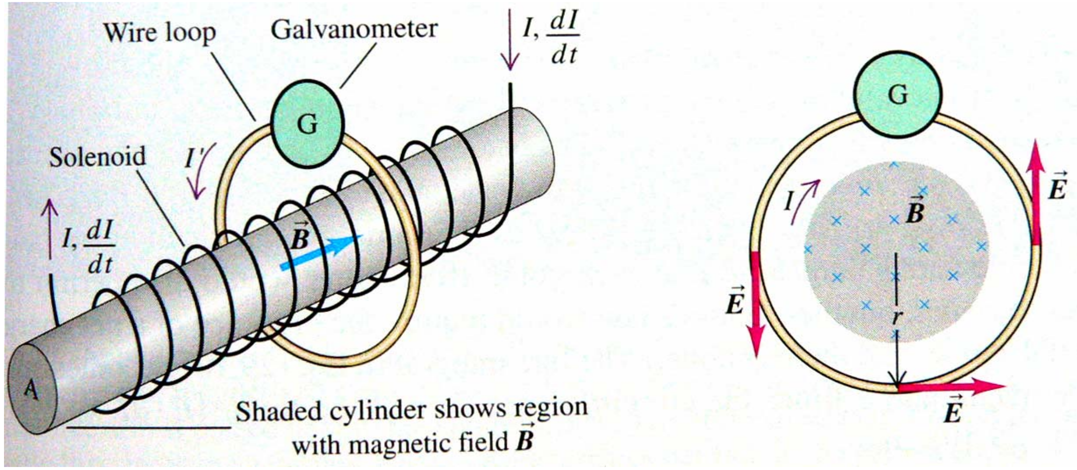
Problems

Ch27 Prob. 11, 26, 27 (P640)

§ 3 Induced Electric Field

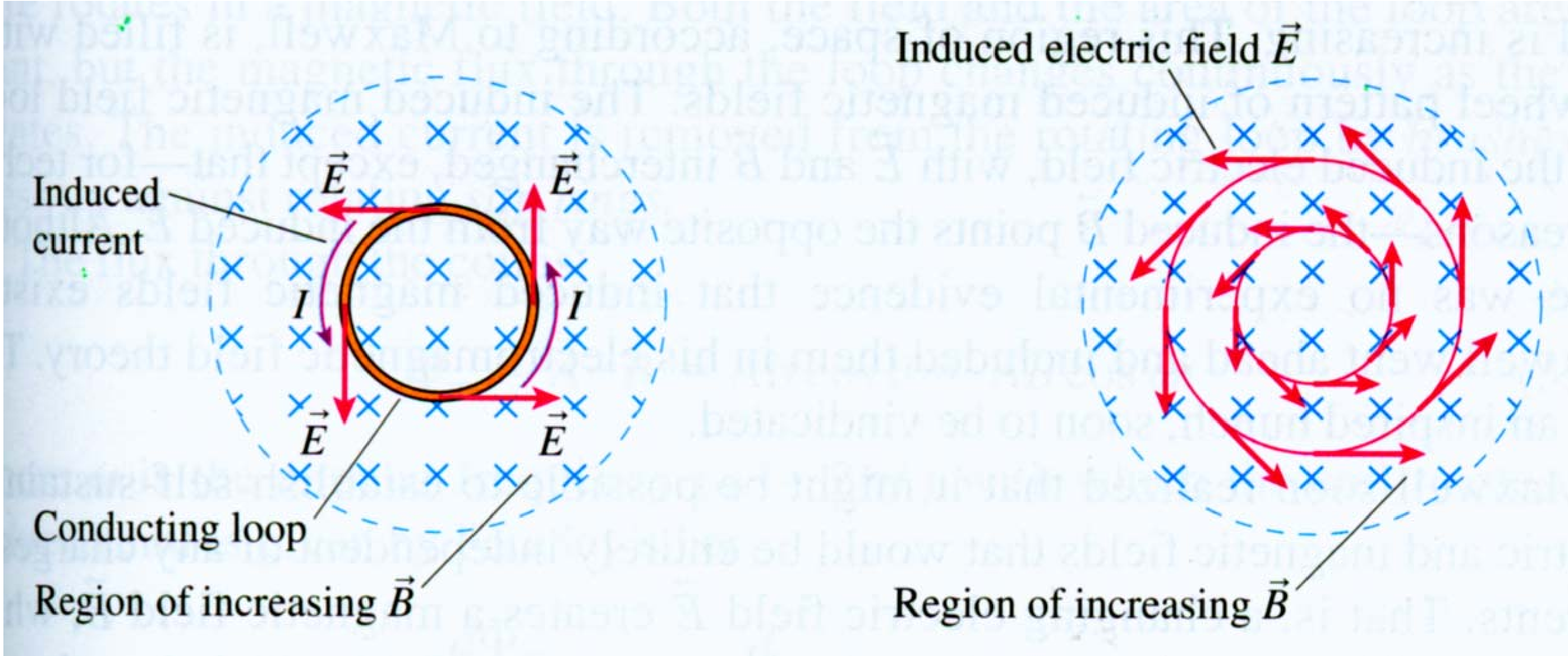
- **What is the basis of induced emf when there is a changing flux through a stationary conducting loop?**
  - **Now we can understand that magnetic force is the reason of the induced emf in a moving conductor.**
  - **By Faraday's law, we only know the result that an induced emf also occurs when there is a changing flux through a stationary conducting loop.**
  - **But up to now, we don't know what *force* makes the charges moving around the loop. It can't be a magnetic force because the conductor is not moving in the magnetic field.**

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$



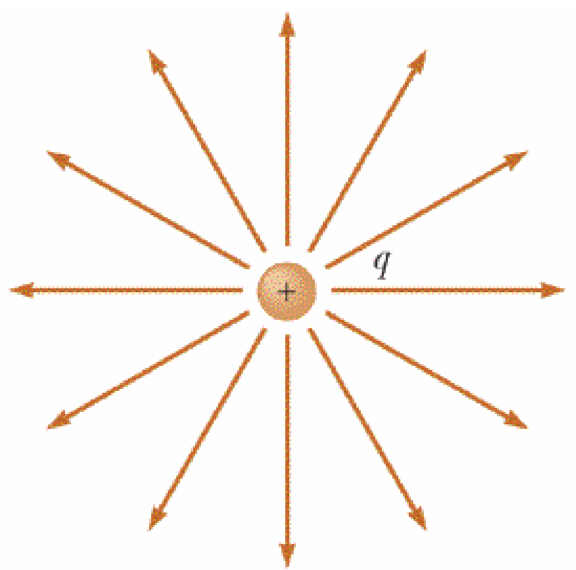
The Induced Electric Field as the Source of Induced emf

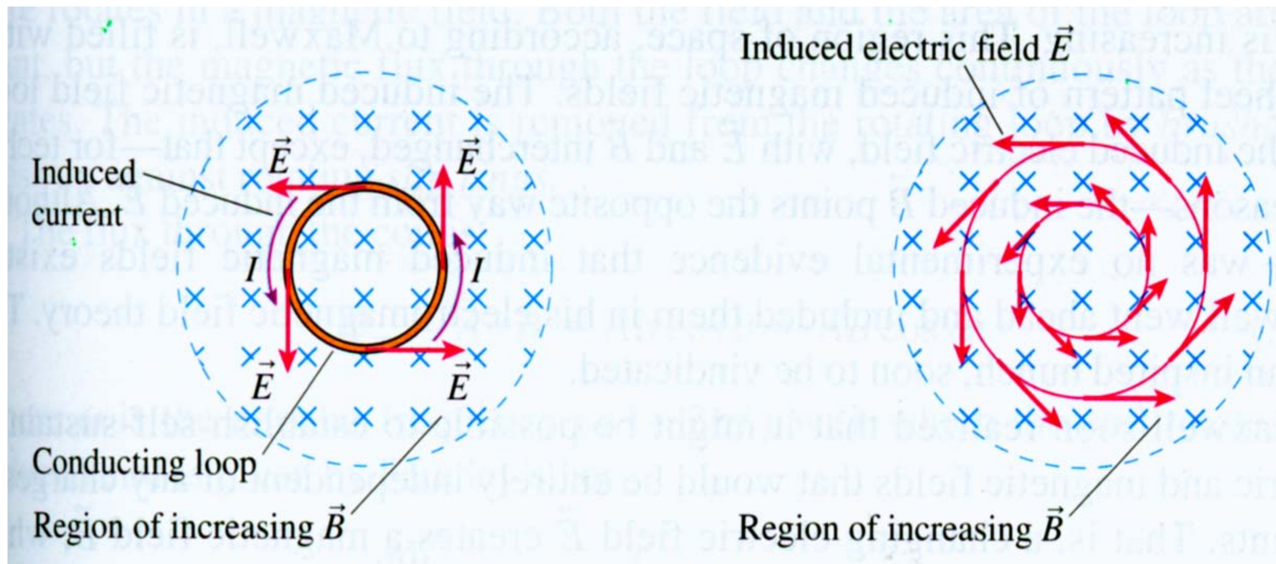
- **Maxwell's suggestion: induced electric field**
  - **There must be an induced electric field (non-electrostatic field) created in the conductor as a result of changing magnetic flux.**
  - **This kind of electric field is induced even when no conductor is present.**



The Confused Points for Induced emf

- **Confused points**
  - **We were accustomed to thinking about electric field as being caused by electric charges. Now we know that a changing magnetic field can also act as a source of electric field.**





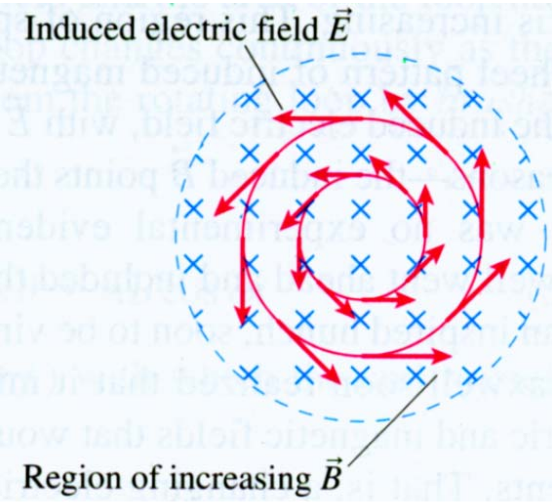
Electrostatic field

Induced electric field

The Confused Points for Induced emf

- **By the definition of emf, $\mathcal{E}$ is equal to the work done by a non-electrostatic field, induced electric field $\vec{E}_i$, per unit charge.**

$$\mathcal{E} = \oint \vec{E}_i \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint_{\substack{\text{the surface}\\ \text{suround the loop}}} \vec{B}\cdot d\vec{A}$$



$$= \iint_{\substack{\text{the surface}\\ \text{suround the loop}}} -\frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

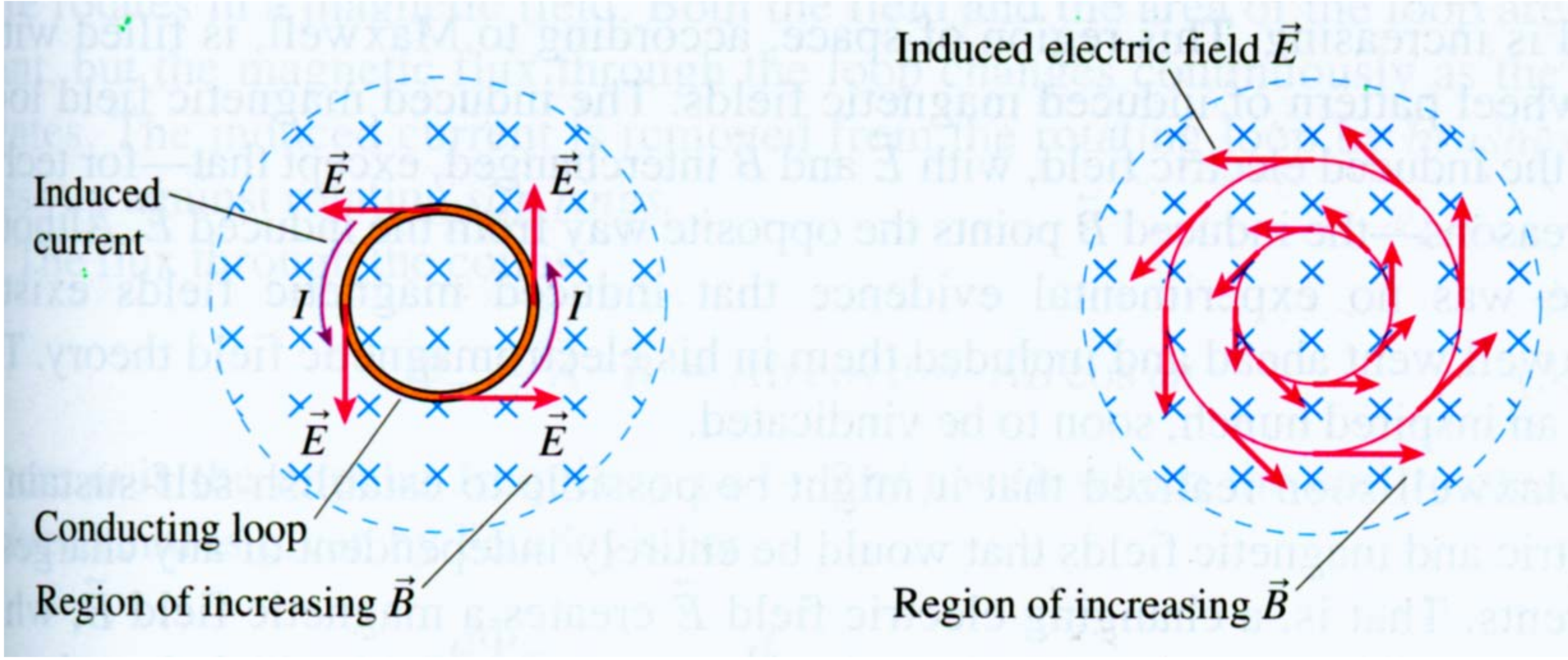
- **The line integral around a closed path is not zero. So the induced electric field is not conservative.**

General Form of Faraday's Law

- **The relationship between the induced electric field and the changing magnetic field**

$$\oint_L \vec{E}_i \cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$$

Valid not only in conductors, but in any region of space.



Electrostatic field vs. Induced electric field

	Electrostatic field $\vec{E}_s$	Induced electric field $\vec{E}_i$
The source of the field	The charges	The changing magnetic field
Line integral around a closed path	$\oint_L \vec{E}_s \cdot d\vec{s} = 0$ Conservative	$\oint_L \vec{E}_i \cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{A}$ Non-conservative
Gauss's law	$\oiint_S \vec{E}_s \cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$ Field lines begin and end on charge	$\oiint_S \vec{E}_i \cdot d\vec{A} = 0$ Field lines form closed loops

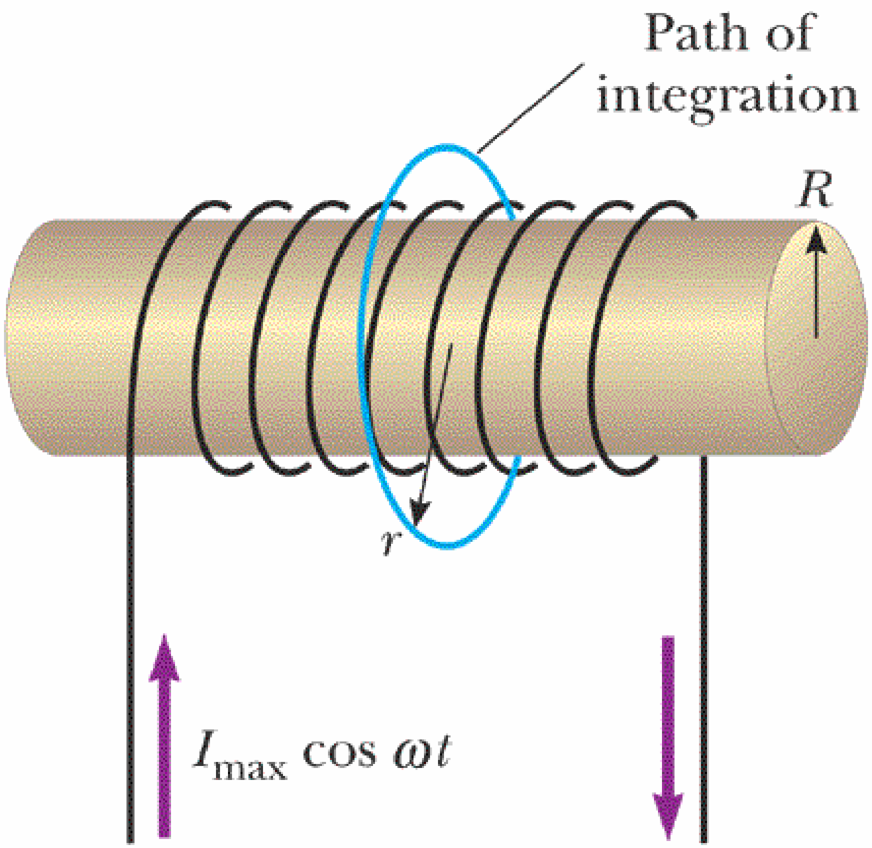
Example

Electric field induced by a changing magnetic field in a solenoid

A long solenoid of radius R has n turns of wire per unit length and carries a time-varying current that varies sinusoidally as $I=I_{max}\cos\omega t$.

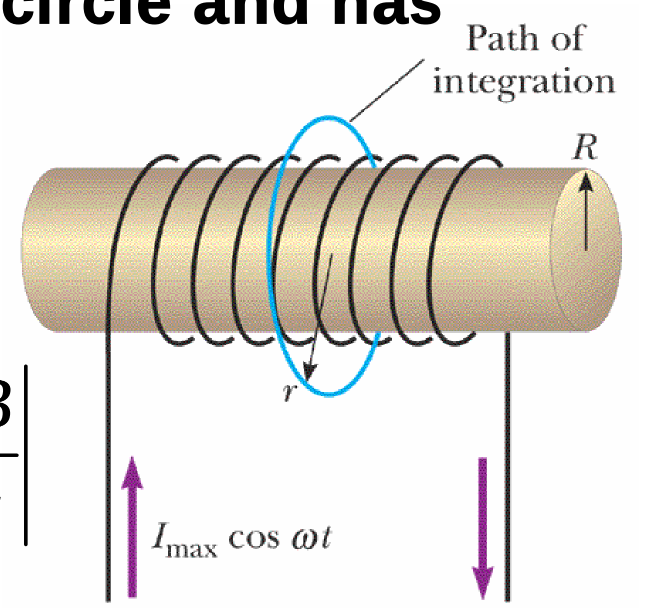
(1) Determine the magnitude of the induced electric field outside the solenoid, a distance $r>R$ from its long central axis.

(2) Find the induced electric filed inside the solenoid, a distance $r<R$ from its axis.



Electric field induced by a changing magnetic field in a solenoid

Solution: **Choose a path for the line integral to be a circle of radius r centered on the solenoid. By symmetry, the $\vec{E}$ is tangent to the circle and has constant magnitude on it.**



$$\left|\oint_L \vec{E}\cdot d\vec{s}\right| = \left|E\oint_L ds\right| = |E|(2\pi r)$$

$$= \left|-\frac{d\Phi_B}{dt}\right| = \left|\frac{d}{dt}\left(B\pi R^2\right)\right| = \pi R^2\left|\frac{dB}{dt}\right|$$

$$\left|\vec{E}\right| = \frac{R^2}{2r}\left|\frac{dB}{dt}\right| = \frac{R^2}{2r}\left|\frac{d}{dt}(\mu_0 n I_{\max}\cos\omega t)\right|$$

$$= \frac{\mu_0 n I_{\max}\omega R^2}{2r}\left|\sin\omega t\right|$$

(for $r > R$)

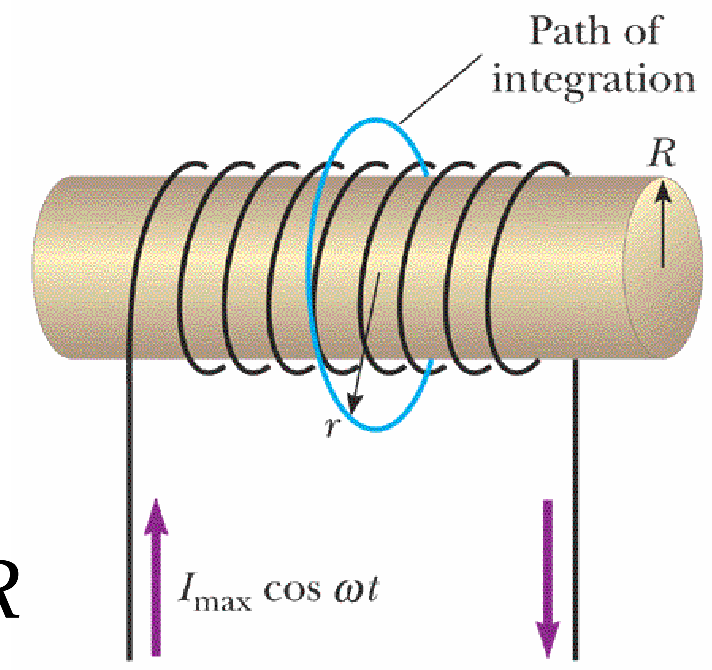
For an interior point ($r < R$)

$$|E|(2\pi r) = \left|-\frac{d\Phi_B}{dt}\right| = \left|\frac{d}{dt}\left(B\pi r^2\right)\right| = \pi r^2\left|\frac{dB}{dt}\right|$$

$$|\vec{E}| = \frac{r}{2}\left|\frac{dB}{dt}\right|$$

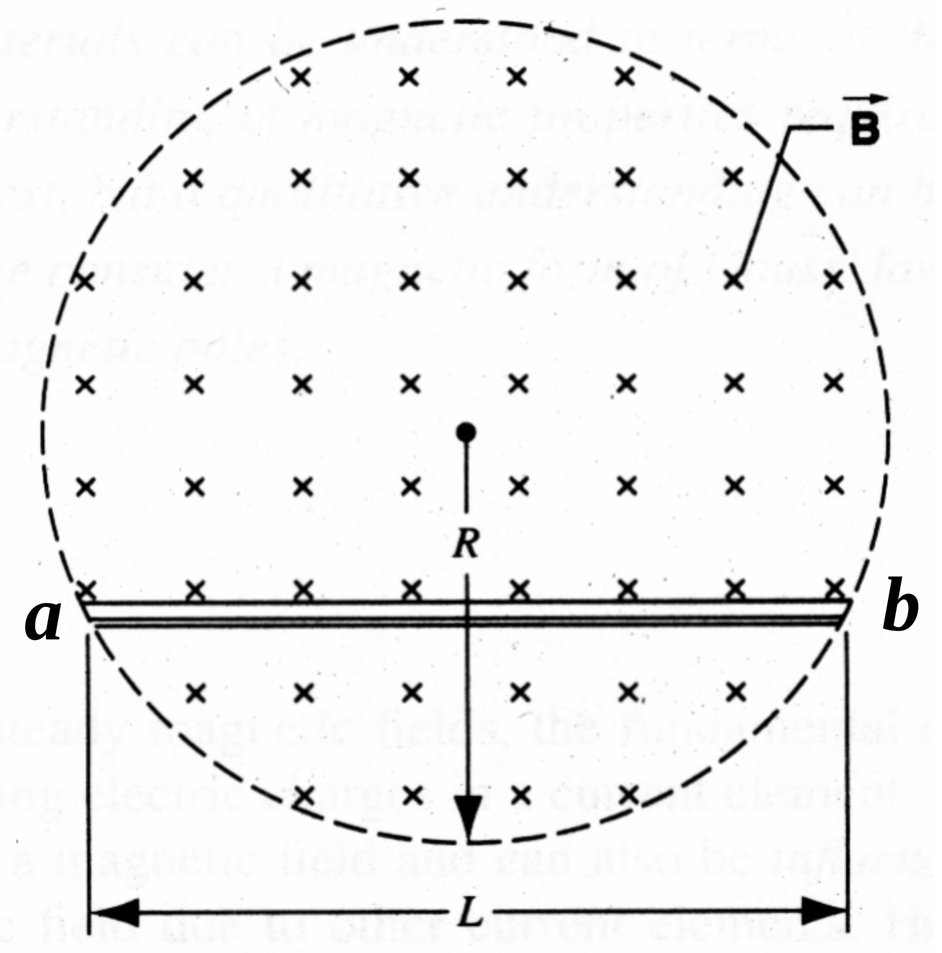
$$= \frac{r}{2}\left|\frac{d}{dt}\left(\mu_0 n I_{\max}\cos\omega t\right)\right|$$

$$= \frac{\mu_0 n I_{\max}\omega}{2} r|\sin\omega t| \quad \text{for } r < R$$



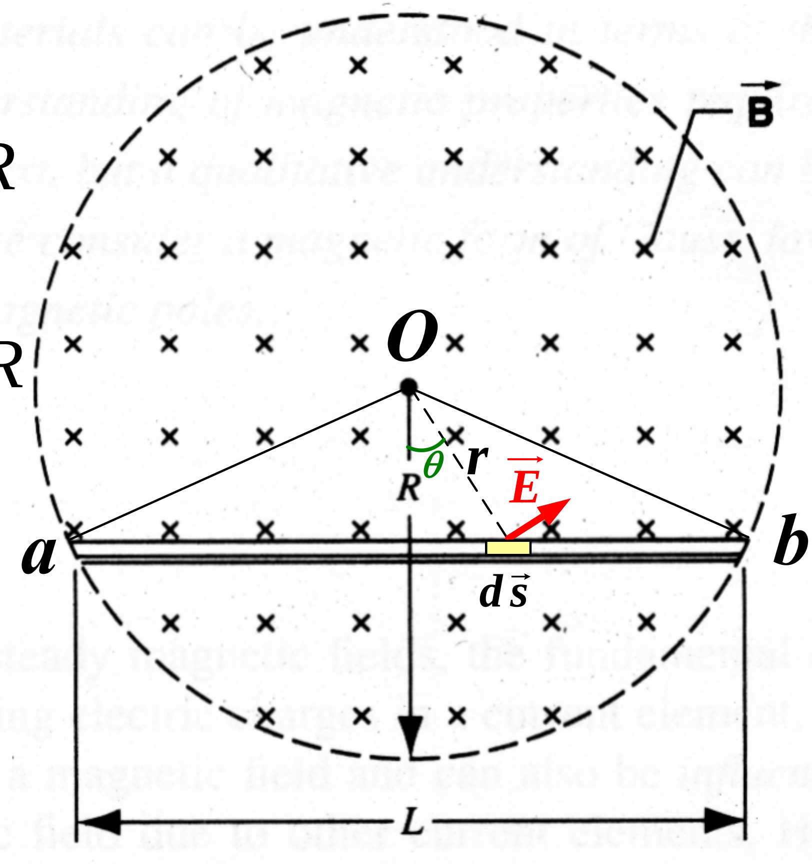
Example

A uniform magnetic field ***B*** fill with cylinrical volume of radius ***R***. A metal rod *ab* of length ***L*** is placed as shown in the figure. If *B* is changing at the constant rate $(dB/dt) > 0$, find the emf acting between the end *a* and *b* of the rod.



Solution I: By line integration of induced electric field.

For $(dB/dt) > 0$, we have know that : $E = \begin{cases} \dfrac{r}{2}\dfrac{dB}{dt} & \text{for } r < R \\ \dfrac{R^2}{2r}\dfrac{dB}{dt} & \text{for } r > R \end{cases}$



$$\mathcal{E}_{ab} = \int_a^b \vec{E}\cdot d\vec{s} = \int_{-L/2}^{L/2} E\cos\theta ds$$

$$= \int_{-L/2}^{L/2} \frac{r}{2}\frac{dB}{dt}\cos\theta ds$$

$$= \frac{1}{2}\frac{dB}{dt}\int_{-L/2}^{L/2} r\cos\theta ds$$

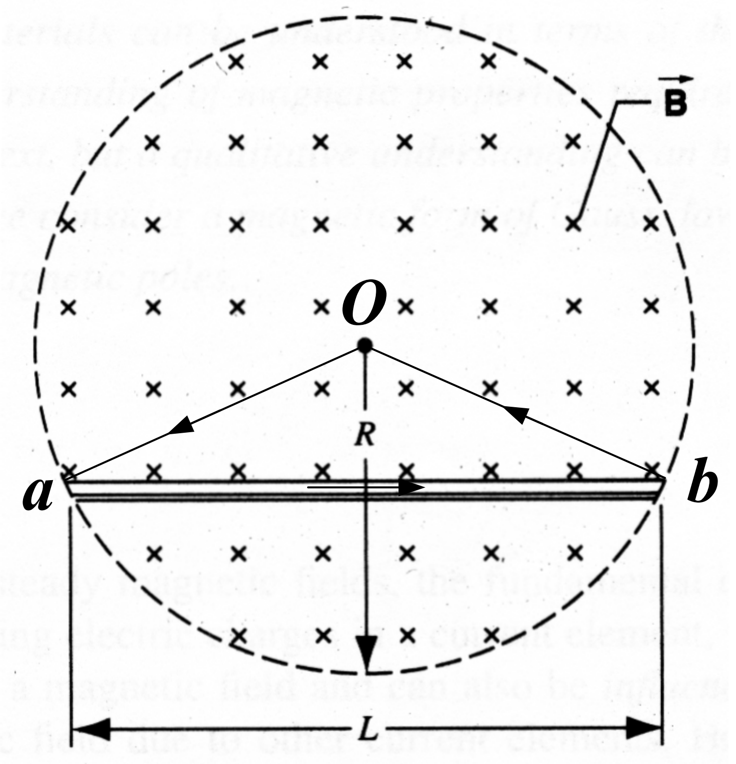
$$r\cos\theta = \sqrt{R^2 - \frac{L^2}{4}}, \qquad \mathcal{E}_{ab} = \frac{1}{2}\sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}\int_{-L/2}^{L/2} ds = \frac{L}{2}\sqrt{R^2 - \frac{L^2}{4}}\frac{dB}{dt}$$

Example

Solution II: Using Faraday's law

Choose the loop *abO*.

$$\Phi_B = \vec{B}\cdot\vec{A}_{abO} = -BA_{abO} = -B\frac{L}{2}\sqrt{R^2-\frac{L^2}{4}}$$



$$\mathcal{E}_{OabO} = \mathcal{E}_{Oa} + \mathcal{E}_{ab} + \mathcal{E}_{bO} = -\frac{d\Phi_B}{dt}$$

$$= A_{abO}\frac{dB}{dt}$$

$$\mathcal{E}_{Oa} = \int_O^a \vec{E}_n \cdot d\vec{s} = 0,$$

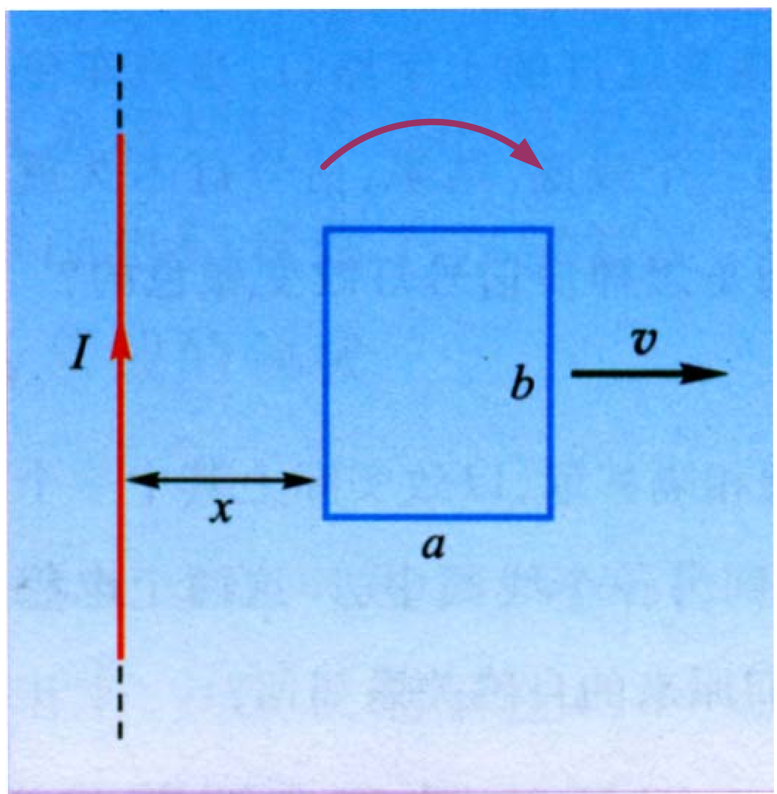
$$\mathcal{E}_{bO} = 0, \qquad \mathcal{E}_{ab} = \frac{L}{2}\sqrt{R^2-\frac{L^2}{4}}\frac{dB}{dt}$$

When $\frac{dB}{dt} > 0, \ \varepsilon_{ab} > 0$

The potential at end *b* is higher than end *a*.

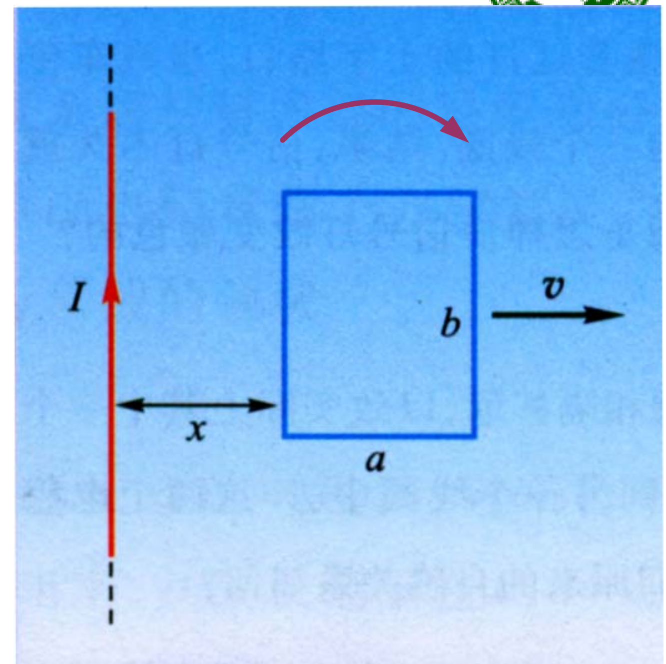
Example

A long, straight wire carries a time-varying current $I = I_0 \sin\omega t$. A rectangular wire loop of sides a and b is placed in the same plane as the straight current is, and a distance x_0 from the straight current. The wire loop starts to move to the right at the speed of v at $t = 0$. Determine the induced emf in the wire loop at time t.



Solution I: Using Faraday's law

Choose the loop direction as shown in the figure



$$x = x_0 + vt$$

$$\Phi_B = \int_x^{x+a} \frac{\mu_0 I}{2\pi x} b dx = \frac{b\mu_0 I_0 \sin\omega t}{2\pi} \ln\frac{x+a}{x}$$

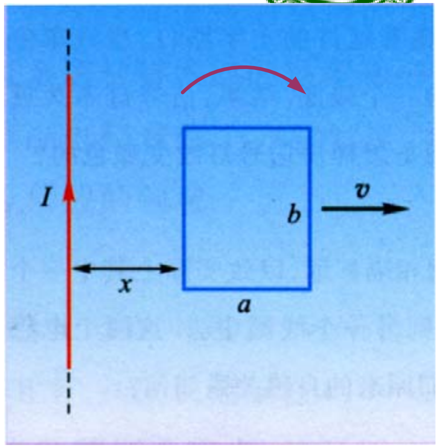
$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

$$= -\frac{b\mu_0 I_0}{2\pi}\left[\sin\omega t \frac{x}{x+a}\frac{x-(x+a)}{x^2}\frac{dx}{dt} + \ln\frac{x+a}{x}\omega\cos\omega t\right]$$

$$= \frac{b\mu_0 I_0}{2\pi}\left[\frac{av}{x(x+a)}\sin\omega t - \ln\frac{x+a}{x}\omega\cos\omega t\right]$$

Example Cont'd

$$\mathcal{E} = \frac{b\mu_0 I_0}{2\pi}\left[\frac{av}{x(x+a)}\sin\omega t - \ln\frac{x+a}{x}\omega\cos\omega t\right]$$



Solution II: By calculation of motional emf and induced electric field.

$$\mathcal{E} = \mathcal{E}_m + \mathcal{E}_i$$

$$\mathcal{E}_m = vbB_x - vbB_{x+a} = vb\frac{\mu_0 I}{2\pi}\left(\frac{1}{x} - \frac{1}{x+a}\right) = \frac{vb\mu_0 I}{2\pi}\frac{a}{x(x+a)}$$

$$= \frac{b\mu_0 I_0}{2\pi}\frac{av}{x(x+a)}\sin\omega t$$

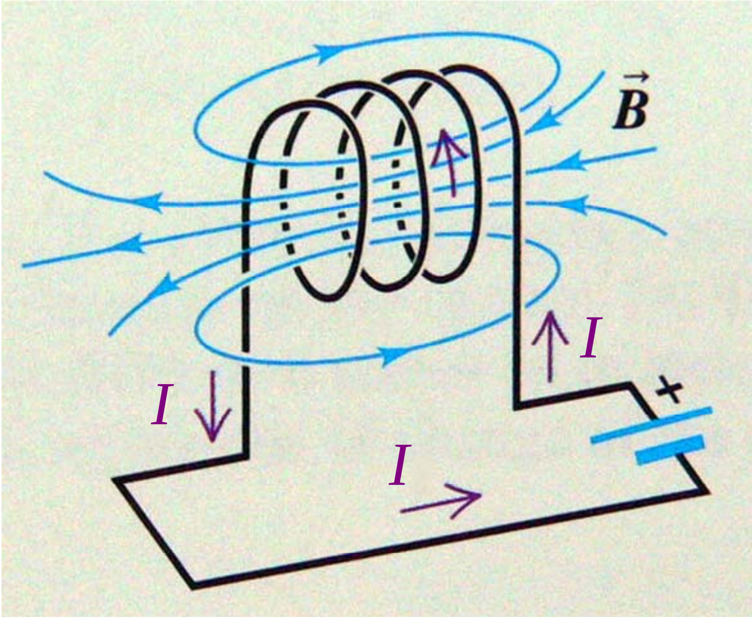
$$\mathcal{E}_i = -\frac{d\Phi_B}{dt}\bigg|_{x=\text{const}} = -\frac{b\mu_0 I_0}{2\pi}\ln\frac{x+a}{x}\omega\cos\omega t$$

§ 4 Self-Inductance

- **Inductor and self-induced emf:**
  - **An inductor is a circuit element such as solenoid that stores energy in the magnetic field surrounding its current-carrying wires, just as a capacitor store energy in the electric field between its charged plates.**
  - **For a circuit including a solenoid**

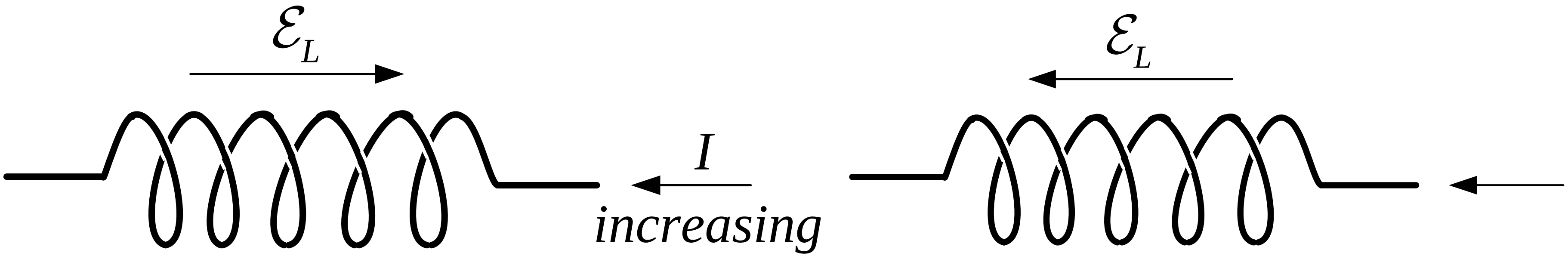
Current I in wire of solenoid → Flux Φ_B through the solenoid

I changes → Φ_B changes → Induced $\mathcal{E}_L$



Inductor and self-induced emf

- **The emf set up by changing self-current is called self-induced emf $\mathcal{E}_L$**
- **By Lenz's law a self-induced emf always opposes the change in the current that caused the emf, and then tends to make it more difficult for variation in current to occur.**

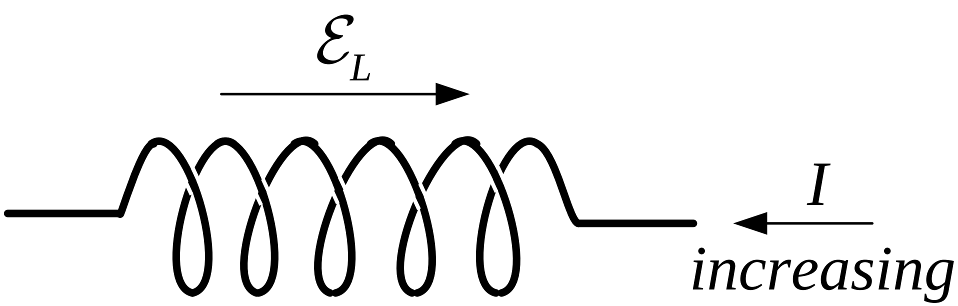


Definition of the Self-inductance

- **Self-induced emf:**

$$\boxed{\mathcal{E}_L = -L\frac{dI}{dt}}$$

- ***L*** **> 0**
- **The negative sign reflects Lenz's law.**



Definition of the Self-inductance

- The self-inductance
  - The proportionality constant *L* is called the **self-inductance**.
  - From Faraday's law

$$\mathcal{E}_L = -\frac{d(N\Phi_B)}{dt} \quad \Rightarrow \quad L\frac{dI}{dt} = \frac{d(N\Phi_B)}{dt}$$

  - Integrating with respect to the time, and assuming that $\Phi_B = 0$ when $I = 0$

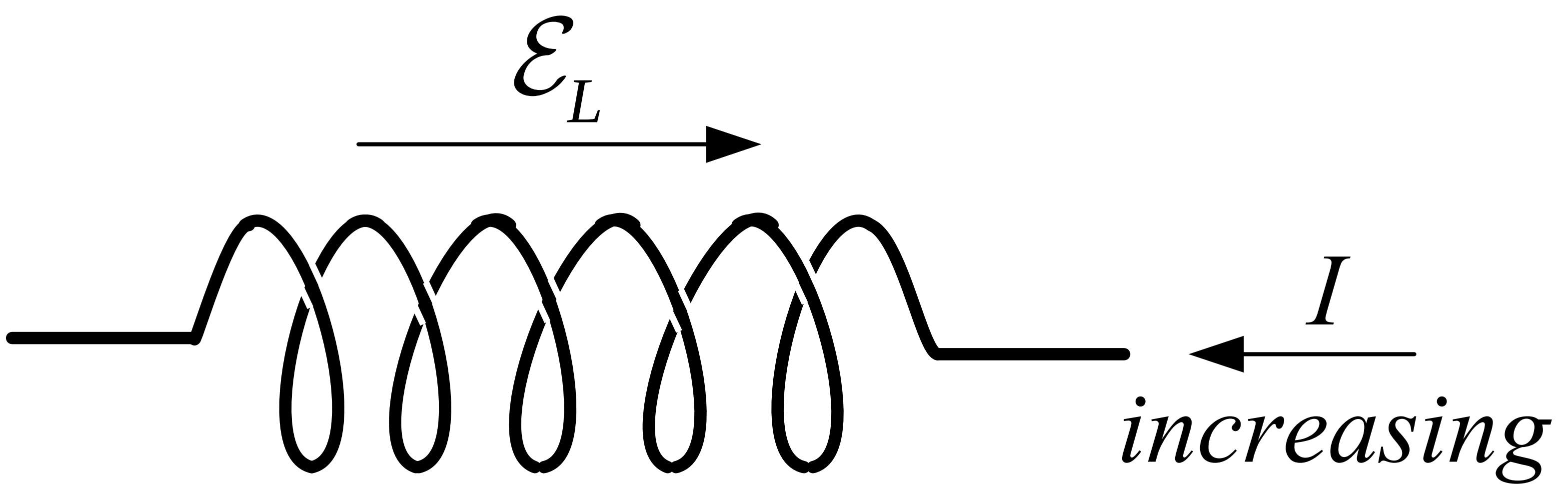
$$\boxed{L = \frac{N\Phi_B}{I}}$$

SI unit: H (Henry)

  - Note that, since Φ_B is proportional to the current, the self-inductance is **independent** of *I*. Just as the capacitance, the self-inductance depends only on the geometry of the device.

Inductance of a solenoid

Find the inductance of a uniformly round solenoid having *N* turns and length *l*. Assume that *l* is long compared with the radius and the core of the solenoid.



Inductance of a solenoid

Solution: **For an ideal solenoid, the interior magnetic field is uniform.**

$$B = \mu_0 nI = \mu_0 \frac{N}{l} I$$

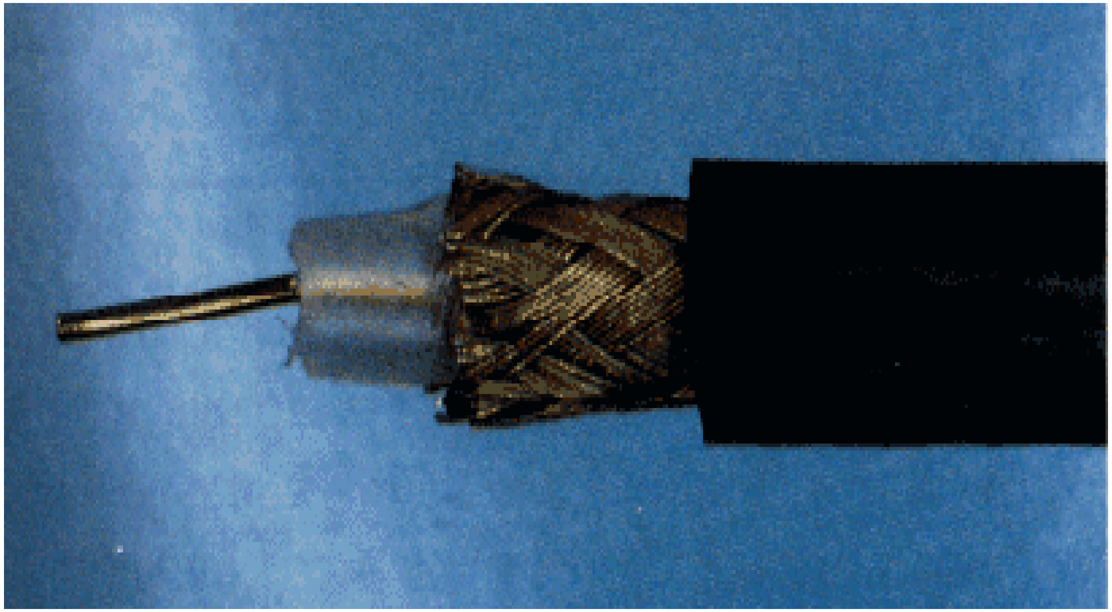
The magnetic flux through each turn is

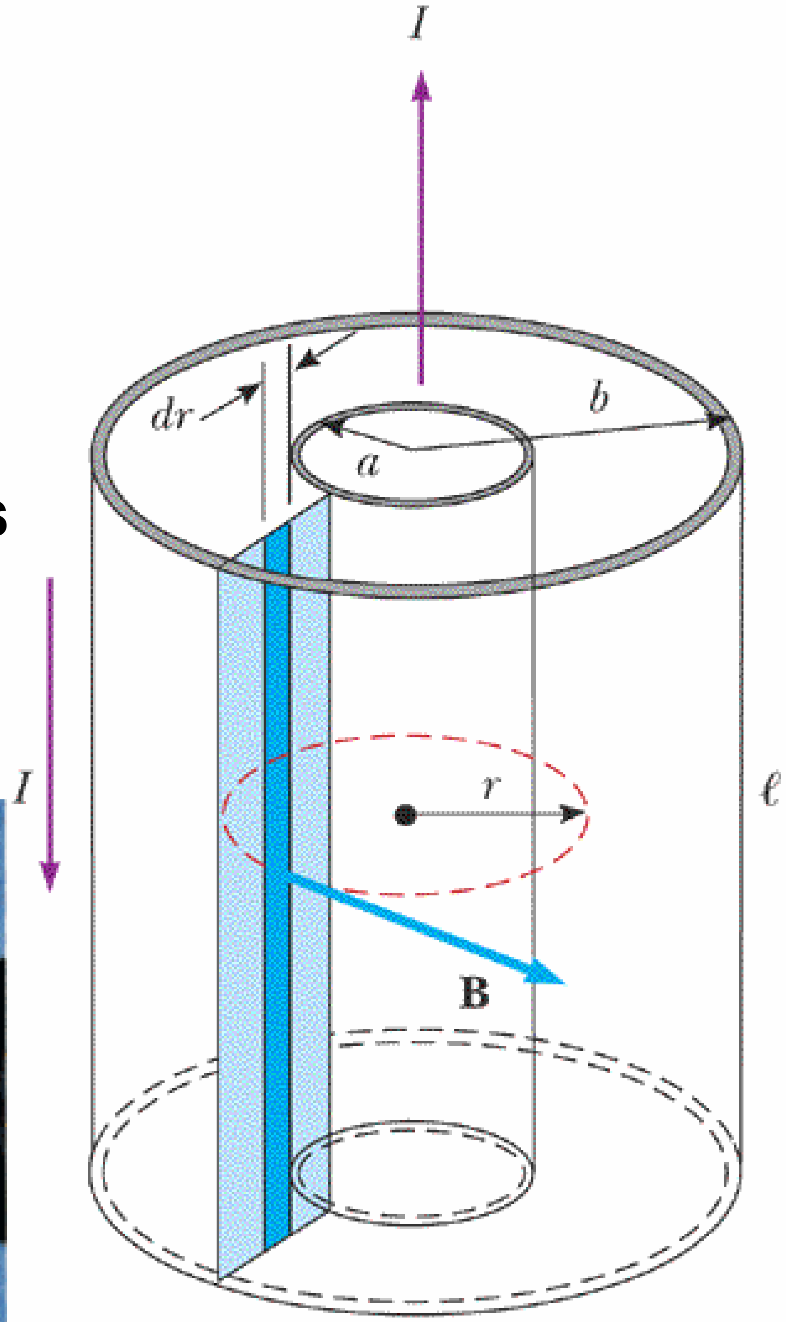
$$\Phi_B = BA = \mu_0 \frac{NA}{l} I$$

$$L = \frac{N\Phi_B}{I} = \frac{\mu_0 N^2 A}{l} = \mu_0 \frac{N^2}{l^2}(Al) = \mu_0 n^2 V$$

Inductance of a coaxial cable

A long coaxial cable consists of two concentric cylindrical conductors of radii *a* and *b* and length *l*. The conductors carry current *I* in opposite directions. Find the self-inductance of this cable.





Solution: Firstly, we find the magnetic flux through cross-section between the two conductors.

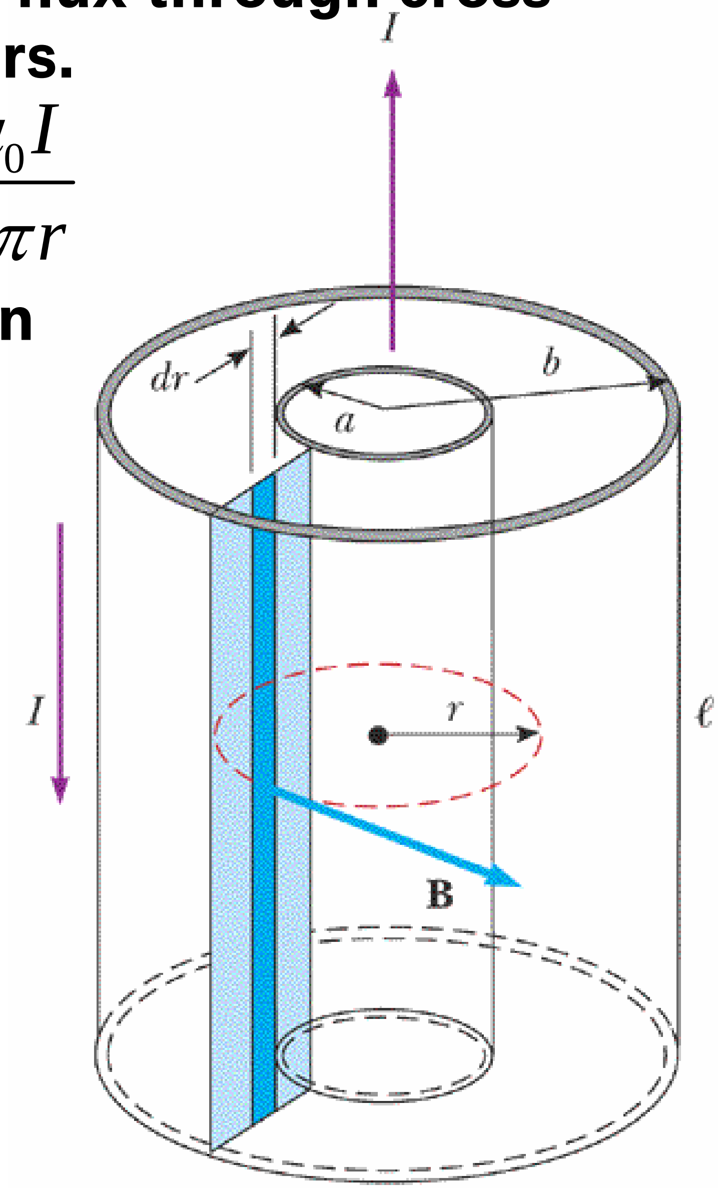
The magnetic field between the conductors: $$B=\frac{\mu_0 I}{2\pi r}$$

Divide the rectangular cross section into strips of width d*r*.

$$\Phi_B=\iint \vec{B}\cdot d\vec{A}=\int_a^b\left(\frac{\mu_0 I}{2\pi r}\right)(l dr)$$

$$=\frac{\mu_0 I l}{2\pi}\int_a^b\frac{dr}{r}=\frac{\mu_0 I l}{2\pi}\ln\left(\frac{b}{a}\right)$$

$$L=\frac{\Phi_B}{I}=\frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)$$



Problems

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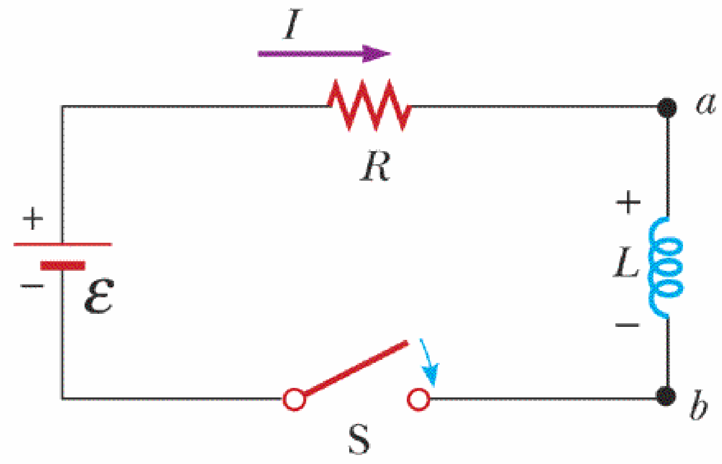
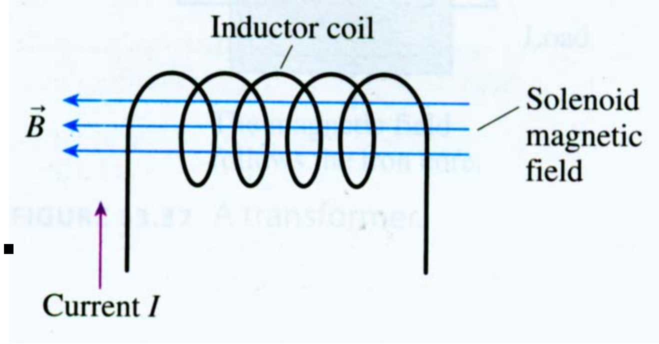
§ 6 Energy Stored in a Magnetic Field



- **Starting with a RL circuit:**
  - **The switch jumps to 1 from 2.**

$$\mathcal{E} = IR + L\frac{dI}{dt}$$

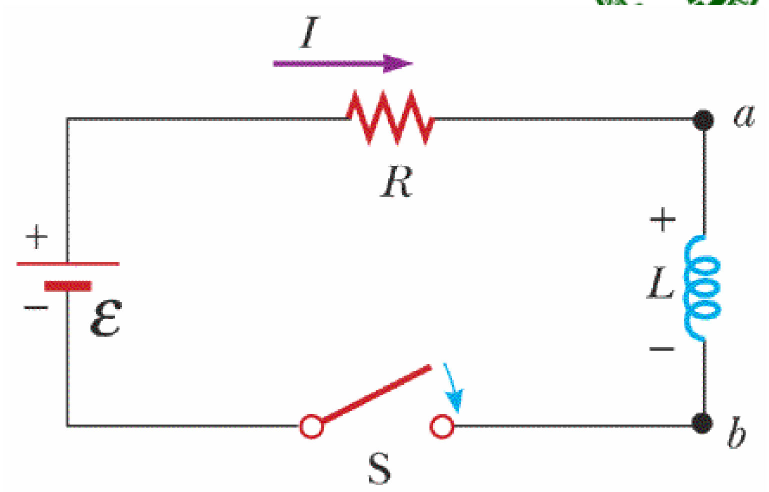
$$\int_0^t \mathcal{E} I dt = \int_0^t I^2 R dt + \int_0^t LI\frac{dI}{dt}dt$$



  - **The term on left side:**
    The energy is supplied by the source.
  - **The first term on right side:**
    The energy is dissipated in the resistor.
  - **The second term on right side:**
    The energy that is delivered to the inductor and is stored in the magnetic field through the coil.

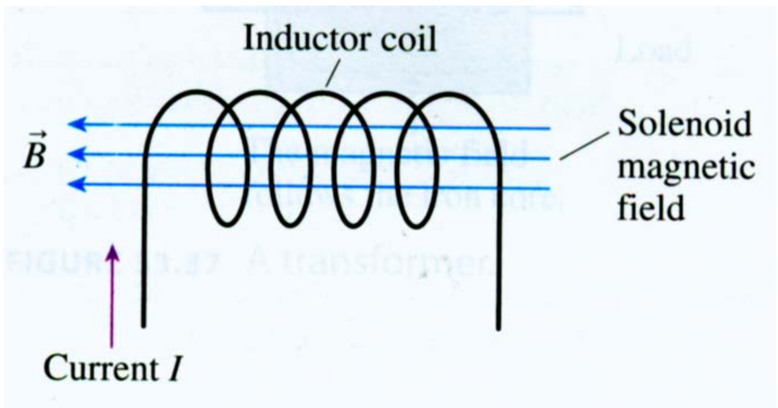
Energy stored in an inductor

$$\int_0^t \mathcal{E} I dt = \int_0^t I^2 R dt + \int_0^t LI \frac{dI}{dt} dt$$



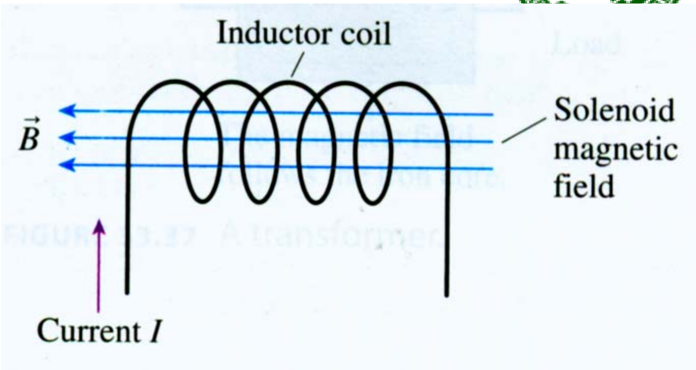
- **Energy stored in the inductor**

$$U_B = \int_0^t LI \frac{dI}{dt} dt = \int_0^I LI dI = \frac{1}{2} LI^2$$



➡**Which one is the storehouse of the energy, the inductor or the magnetic field?**

The Energy Density in Magnetic Field



- **Energy stored in magnetic field.**
  - **Take a solenoid as an example.**

$$L = \mu_0 n^2 V, \quad B = \mu_0 n I,$$

$$U_B = \frac{1}{2} L I^2 = \frac{1}{2}\left(\mu_0 n^2 V\right)\left(\frac{B}{\mu_0 n}\right)^2 = \frac{B^2}{2\mu_0} V \propto \begin{cases} B^2 \\ V \end{cases}$$

  - **Energy is indeed stored in the space where the magnetic field exists.**

- **Energy density**

$$\boxed{u_B = \frac{U_B}{V} = \frac{B^2}{2\mu_0}}$$

  - **For a non-uniformed magnetic field**

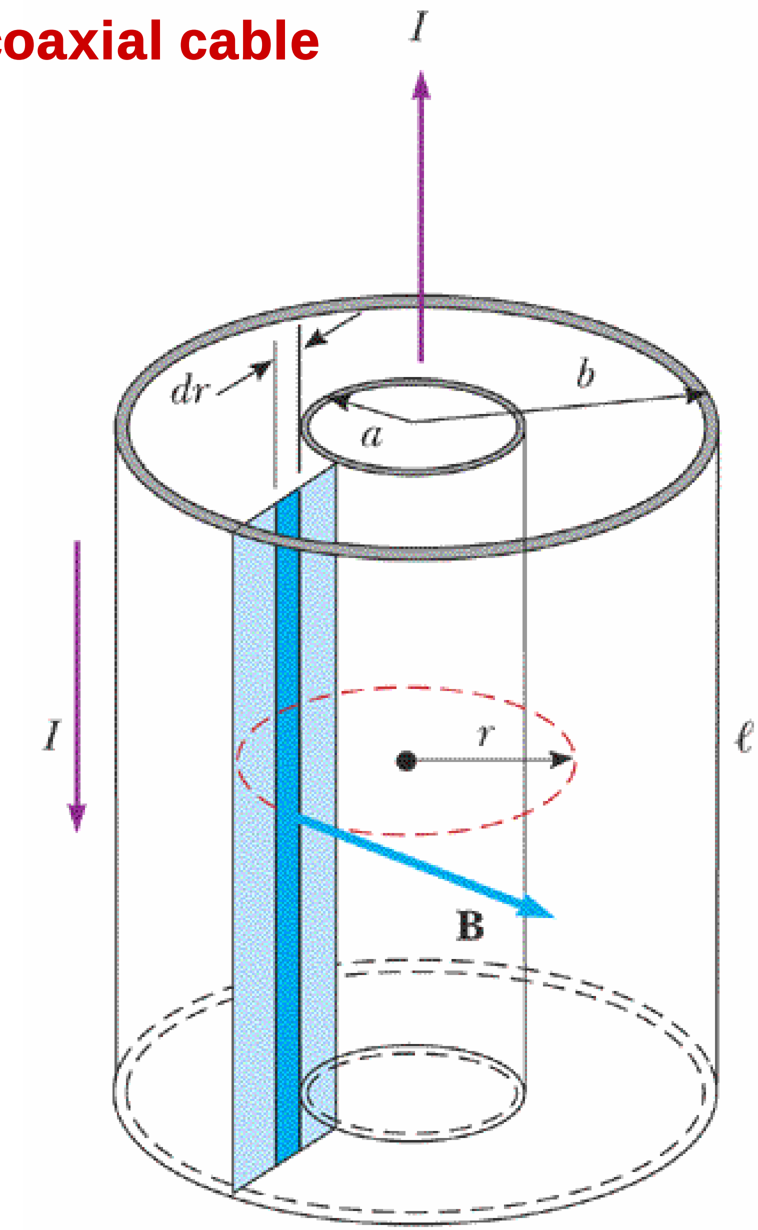
$$U_B = \iiint dU_B = \iiint_V \left(\frac{B^2}{2\mu_0}\right) dV$$

Energy in Electric and Magnetic Field

	Electric field	Magnetic field
Energy stored in the device	A capacitor stores energy $U_E = \frac{1}{2}C(\Delta V)^2$	An inductor stores energy $U_B = \frac{1}{2}LI^2$
Energy density in the field	$u_E = \frac{1}{2}\varepsilon_0 E^2$	$u_B = \frac{1}{2\mu_0}B^2$

The energy stored in a coaxial cable

A long coaxial cable consists of two concentric cylindrical conductors of radii *a* and *b* and length *l*. The conductors carry current *I* in opposite directions. Find the energy stored in this cable.



Solution:

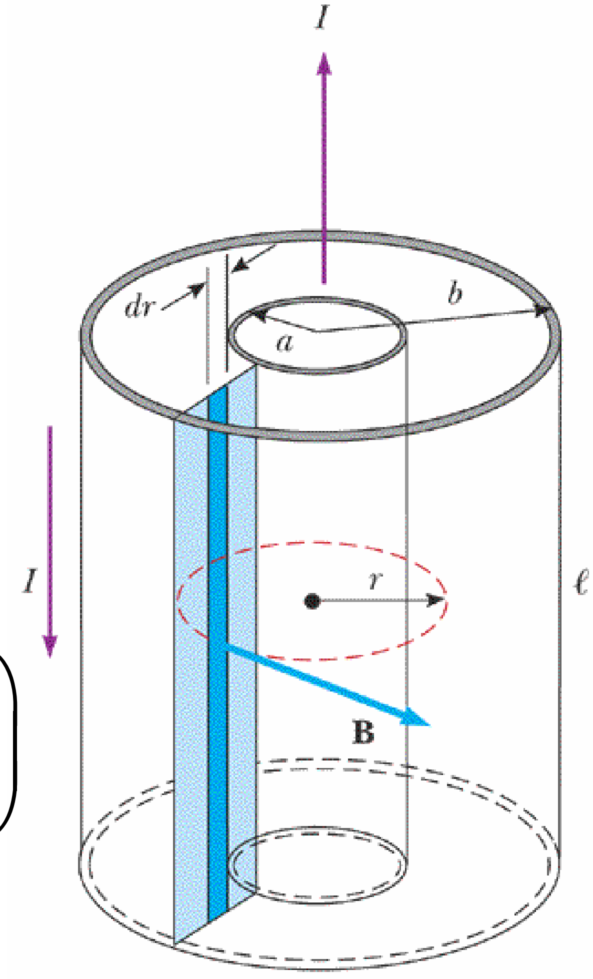
The magnetic field between the conductors is $B = \mu_0 I / 2\pi r$

The magnetic field is zero inside the inner conductor r<a, and outside the outer conductor r>b.

$$U_B = \iiint \left(\frac{B^2}{2\mu_0} \right) dV = \int_a^b \left[\frac{1}{2\mu_0} \left(\frac{\mu_0 I}{2\pi r} \right)^2 \right] (2\pi r l dr)$$

$$= \frac{\mu_0 I^2 l}{4\pi} \int_a^b \frac{dr}{r} = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right)$$

$$U_B = \frac{1}{2} L I^2 = \frac{\mu_0 I^2 l}{4\pi} \ln\left(\frac{b}{a}\right), \qquad L = \frac{\mu_0 l}{2\pi} \ln\left(\frac{b}{a}\right)$$



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