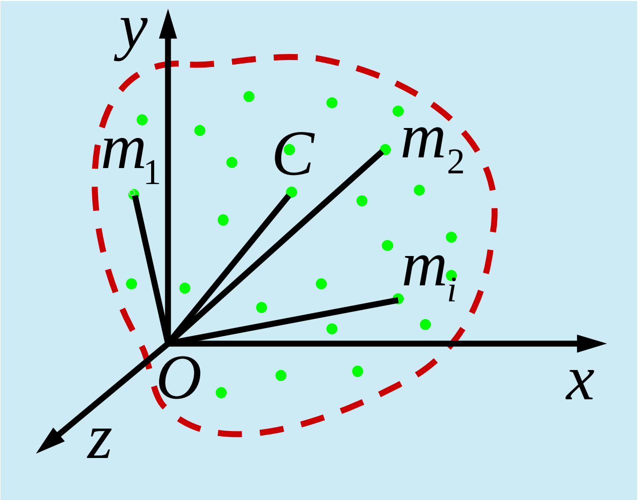
§ 3 Center of Mass

Describe the motion of a system of particles
- by every motion for individual particles
- by overall motion in terms of center of mass

- Center of mass
  - For the system of discretely distributed particles

$$\vec{r}_{\mathrm{CM}} = \frac{\sum_i m_i \vec{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \vec{r}_i}{M}$$



$$x_{\mathrm{CM}} = \frac{\sum_i m_i x_i}{M}, \quad y_{\mathrm{CM}} = \frac{\sum_i m_i y_i}{M}, \quad z_{\mathrm{CM}} = \frac{\sum_i m_i z_i}{M}$$

The Newton's Second Law for the motion of CM

(P219 § 9-9)

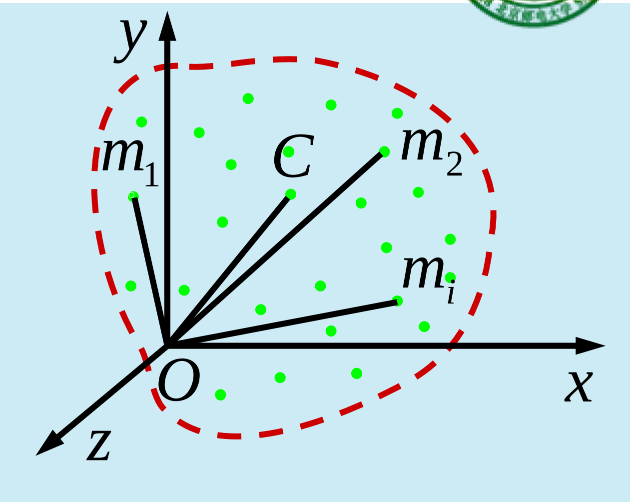
- **Motion of the center of mass**

Starting from

$$M\,\vec{r}_{\mathrm{CM}} = \sum_i m_i \vec{r}_i$$

by derivative

$$M\frac{d\vec{r}_{\mathrm{CM}}}{dt} = \sum_i m_i \frac{d\vec{r}_i}{dt}$$



$$M\,\vec{v}_{\mathrm{CM}} = \sum_i m_i \vec{v}_i = \sum_i \vec{p}_i = \vec{p}_{\mathrm{tot}}$$

The total momentum of the system of particles is equal to its total mass times the velocity of center of mass, just as though all the mass were concentrated at center of mass.

The Newton's Second Law for the motion of CM

$$M\vec{v}_{\mathrm{CM}} = \vec{p}_{\mathrm{tot}}, \qquad \sum_i \overrightarrow{F}_{i-\mathrm{ext}} = \frac{d\vec{p}_{\mathrm{tot}}}{dt} = M\frac{d(\vec{v}_{\mathrm{CM}})}{dt} = M\vec{a}_{\mathrm{CM}}$$

- **The Newton's Second Law for the motion of center of mass**

$$\boxed{\sum_i \overrightarrow{F}_{i-\mathrm{ext}} = M\vec{a}_{\mathrm{CM}}}$$

The overall translational motion of a system of particles can be analyzed using Newton's Law as if all the mass were concentrated at the center of mass and total external force were applied at that point.

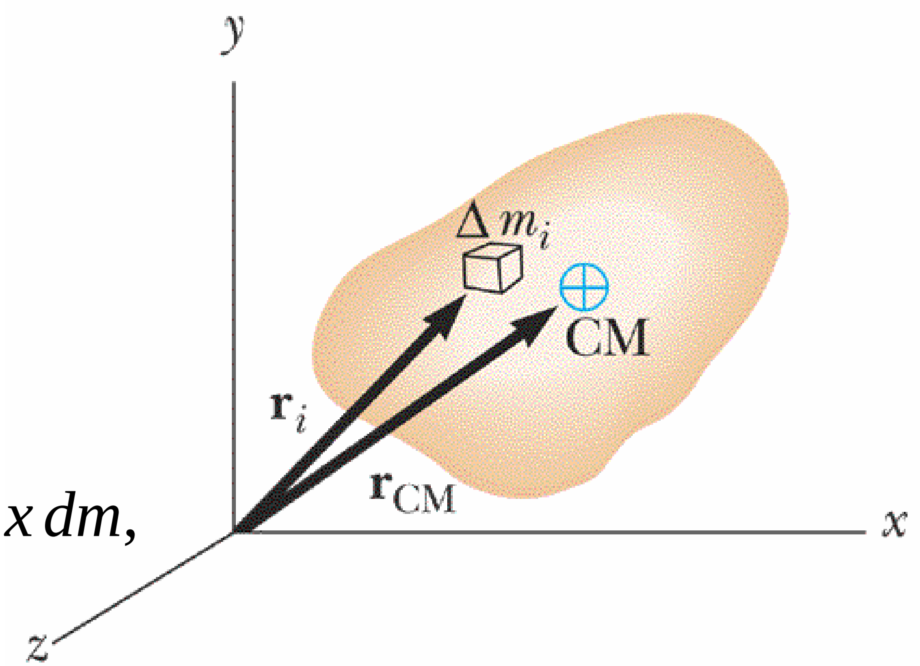
- **If net external force is zero, the center of mass moves with constant velocity**

$$\sum_i \overrightarrow{F}_{i-\mathrm{ext}} = 0 \quad \Rightarrow \quad \vec{p}_{\mathrm{tot}} = M\vec{v}_{\mathrm{CM}} = \text{constant}$$

- For the extended object with uniformly distribution of mass

$$\boxed{\vec{r}_{\mathrm{CM}} = \frac{1}{M}\int \vec{r}\, dm}$$

$$x_{\mathrm{CM}} = \frac{\lim\limits_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum\limits_{i=1}^{N} x_i \Delta m_i}{\lim\limits_{\substack{N\to\infty \\ \Delta m_i \to 0}} \sum\limits_{i=1}^{N} \Delta m_i} = \frac{\int x\, dm}{\int dm} = \frac{1}{M}\int x\, dm,$$

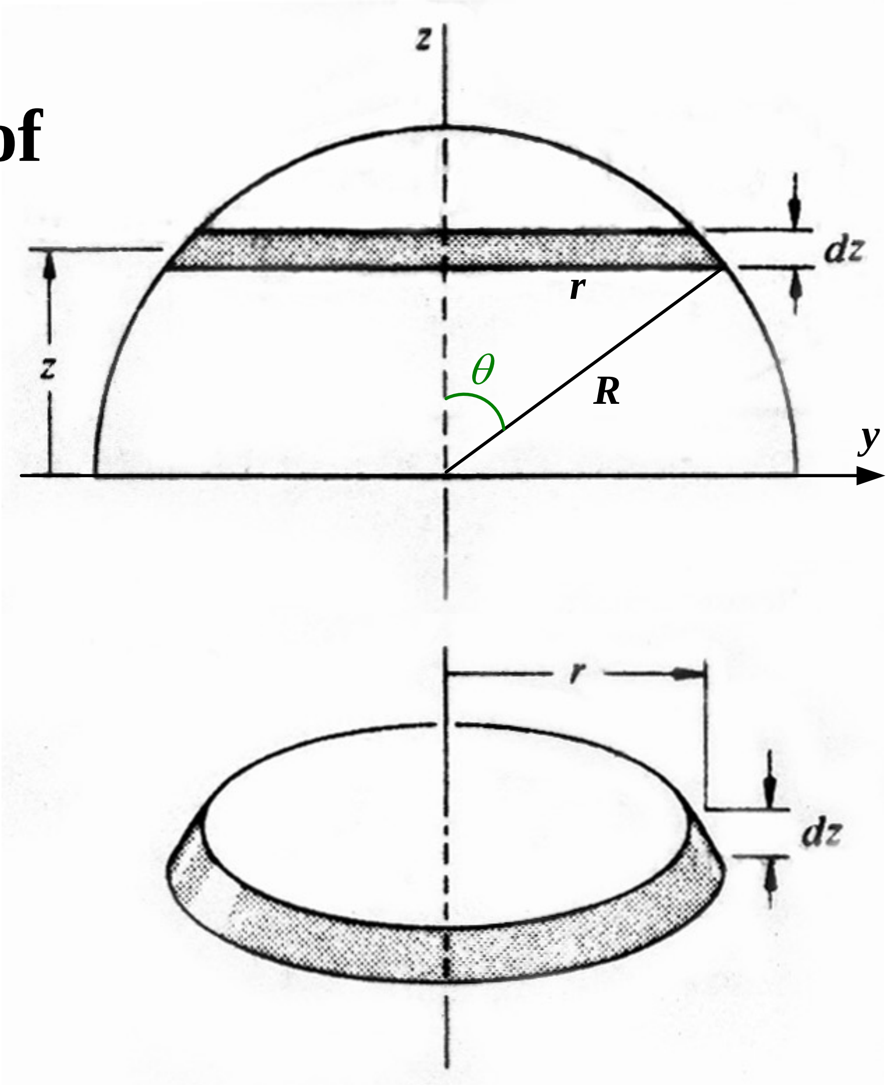


$$y_{\mathrm{CM}} = \frac{1}{M}\int y\, dm,$$

$$z_{CM} = \frac{1}{M}\int z\, dm$$

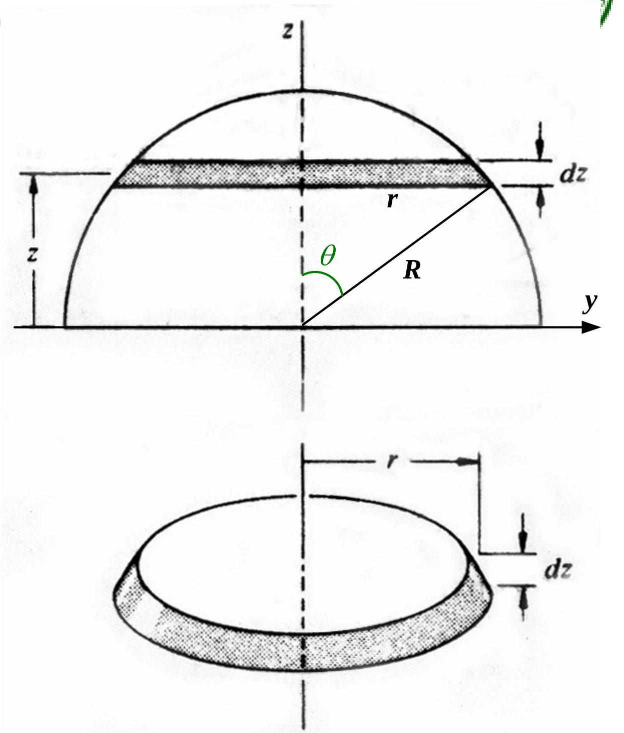
Example

Find the center of mass of a uniform solid hemisphere of radius ***R*** and mass ***M***.



Example

Solution: From symmetry it is apparent that the center of mass lies on the z axis. $\boldsymbol{x_{CM}=0}$, $\boldsymbol{y_{CM}=0}$.



$$z_{CM}=\frac{1}{M}\int zdm=\frac{1}{M}\int z\rho dV$$

The three-dimensional integral can be treated as an one-dimensional integral. Subdivide the hemisphere into a pile of thin disk.

$$\begin{cases} dV=\pi r^2 dz \\ \rho = M\Big/\left(\frac{2}{3}\pi R^3\right) \end{cases}$$

Find $\boldsymbol{r}$, $\boldsymbol{z}$ in terms of $\boldsymbol{\theta}$.

$$r=R\sin\theta,$$

$$z=R\cos\theta,$$

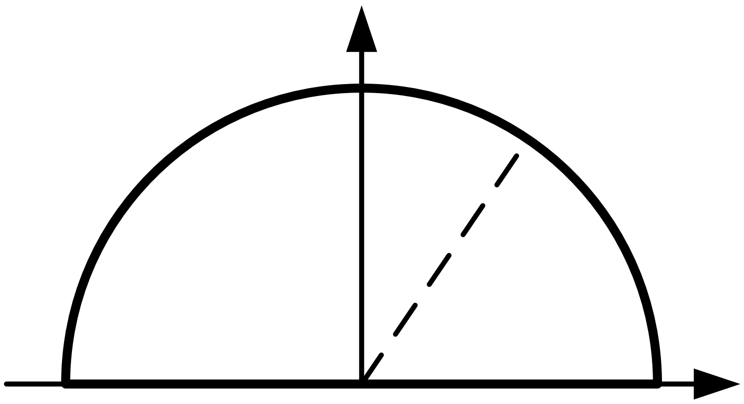
$$dz=-R\sin\theta d\theta$$

$$z_{CM}=\frac{3}{2R^3}\int_{\frac{\pi}{2}}^{0}R\cos\theta R^2\sin^2\theta(-R\sin\theta)d\theta$$

$$=\frac{3}{2}R\int_0^{\frac{\pi}{2}}\cos\theta\sin^3\theta d\theta=\frac{3}{2}R\int_0^{\frac{\pi}{2}}\sin^3\theta d(\sin\theta)$$

$$=\frac{3}{2}R\times\frac{1}{4}=\frac{3}{8}R$$

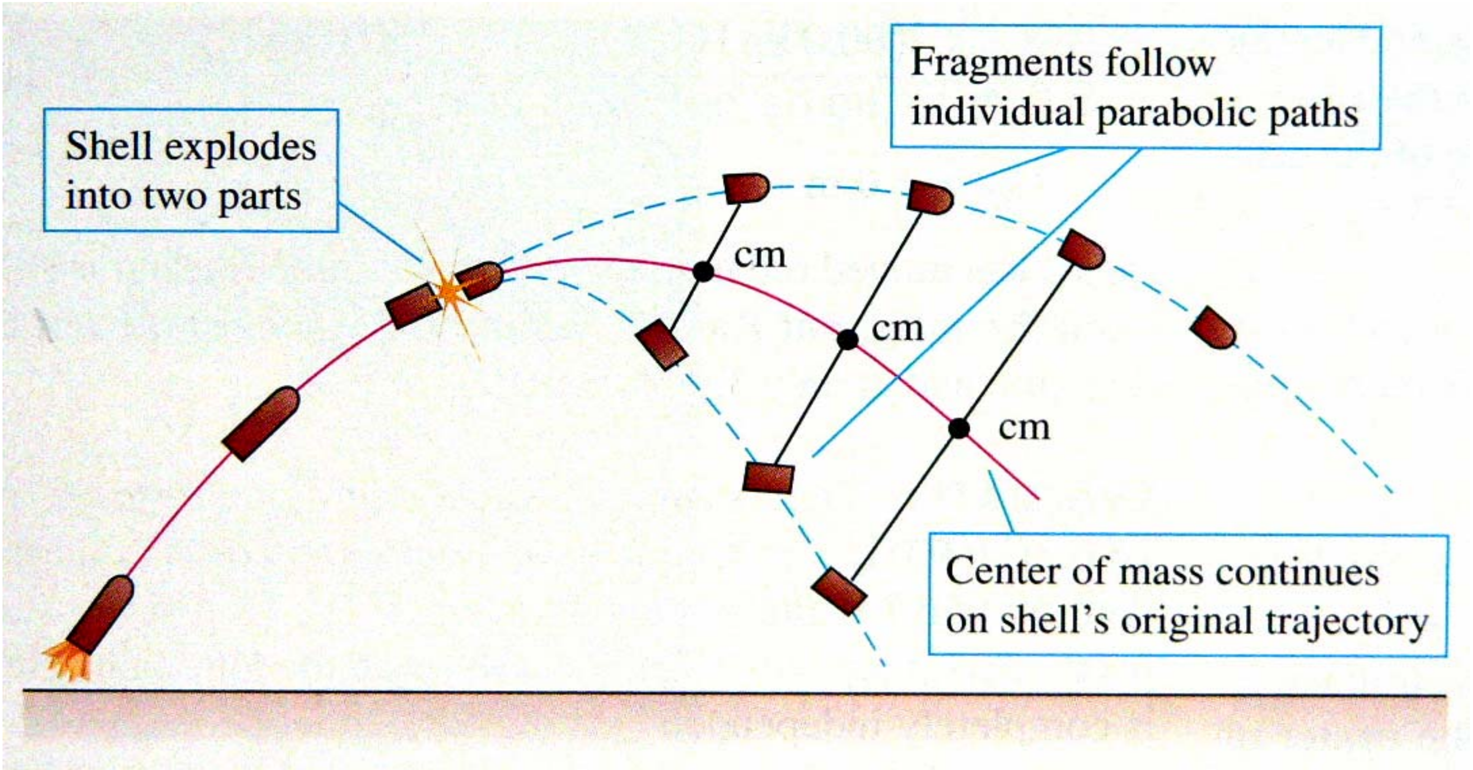
Supplement problem: Find the center of mass of a uniform semicircular plate of radius ***R*** and mass ***M***.



Applications of center of mass

- For a system of discrete particles

  A cannon shell in a parabolic trajectory explodes in flight, splitting into two fragments. The fragments follow new paths, but center of mass continues on the original parabolic trajectory.



Applications of center of mass

- For a **rigid** body

  We can describe a rigid body as a combination of translational motion of the center of mass and rotational motion about an axis through the center of mass.

Applications of center of mass

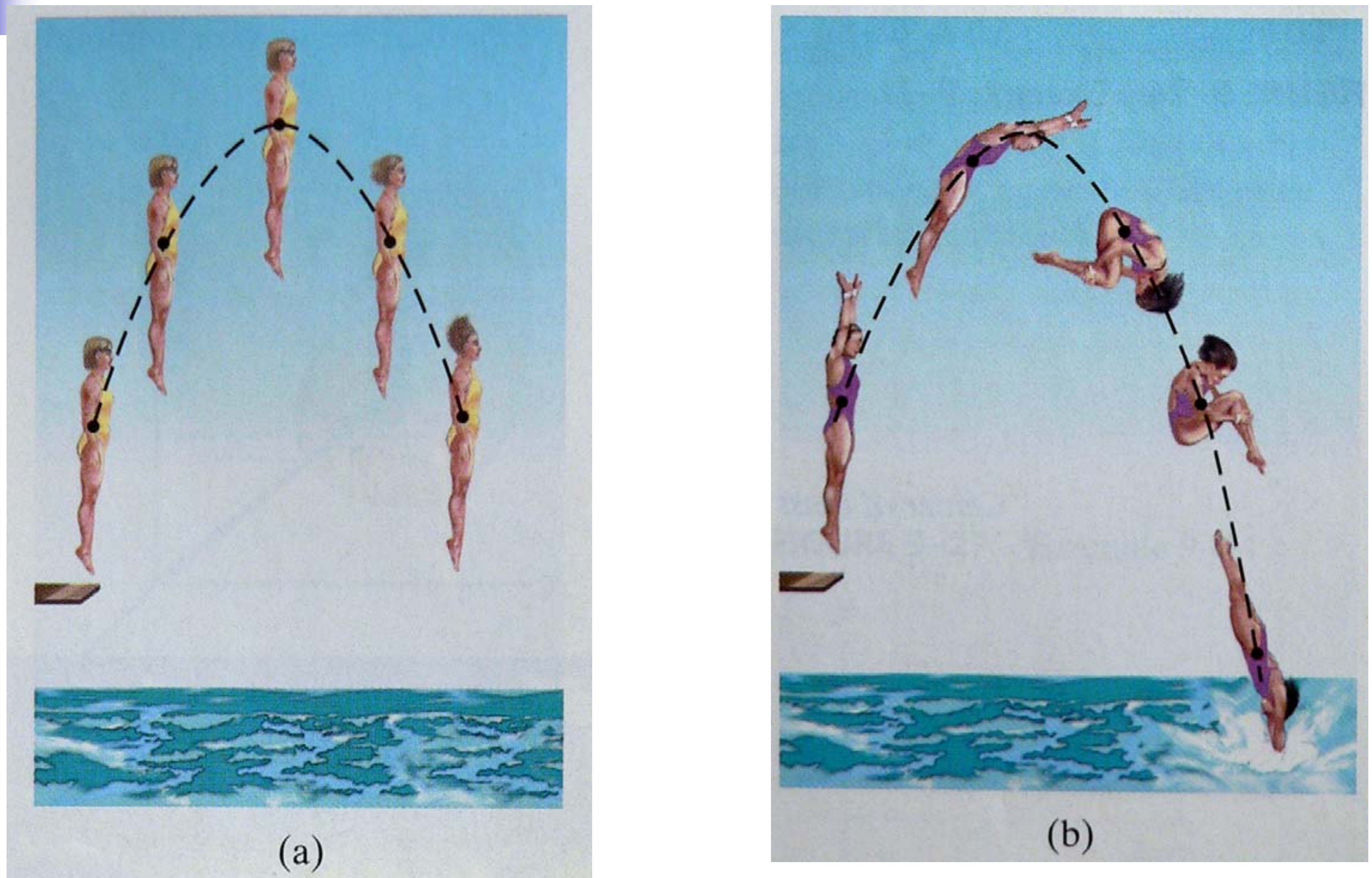
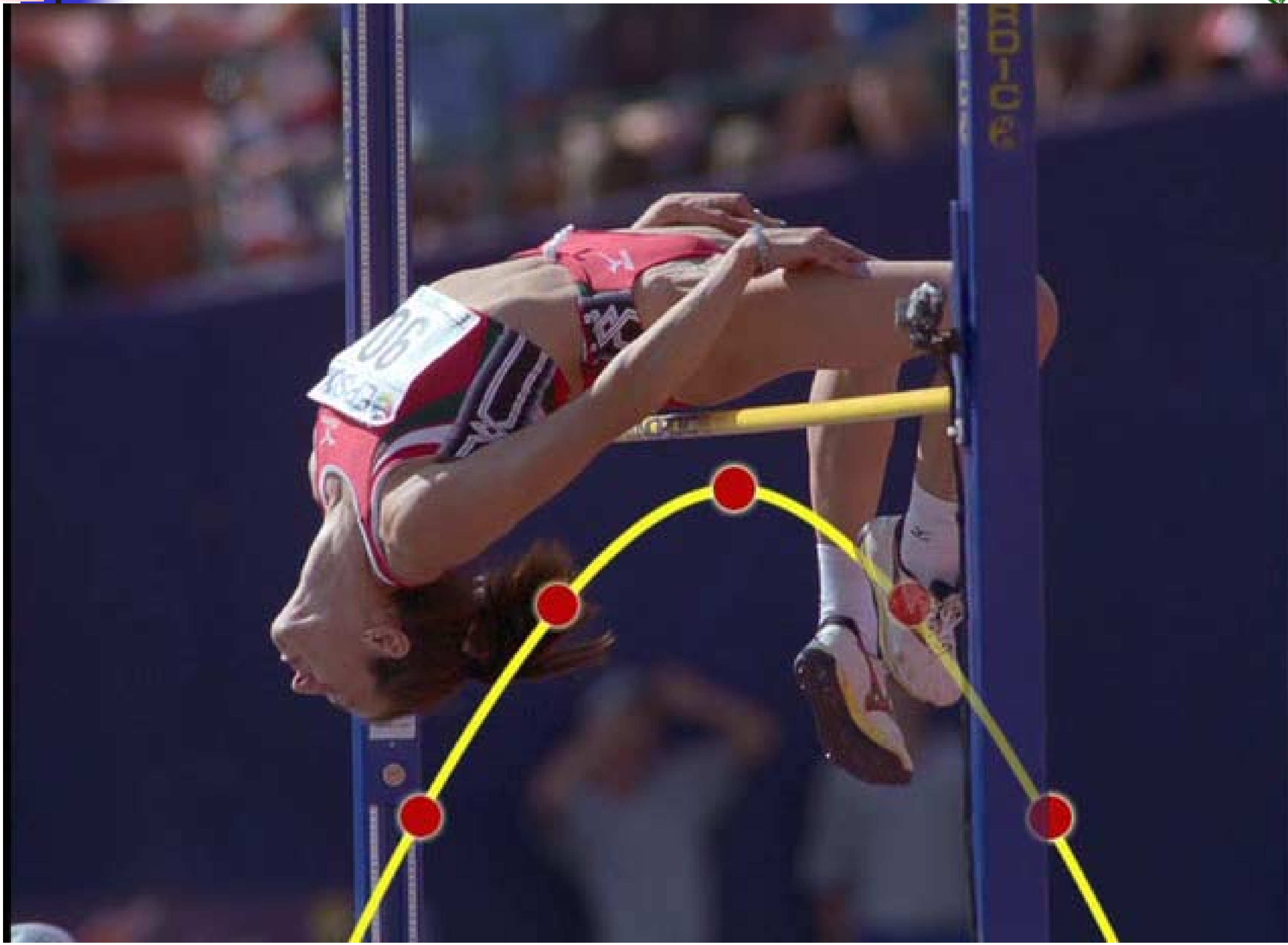


Fig. (a) The motion of the diver is pure translation.

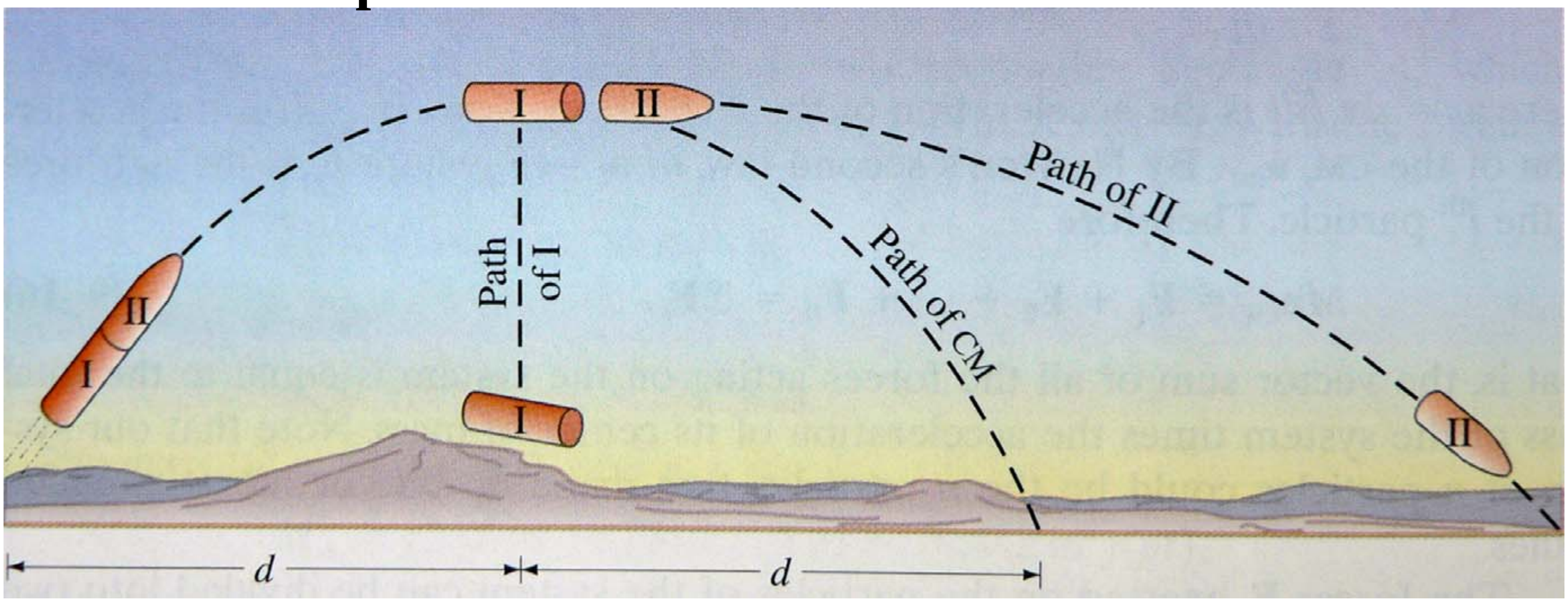
Fig. (b) The motion of the diver is translation plus rotation.

Applications of center of mass

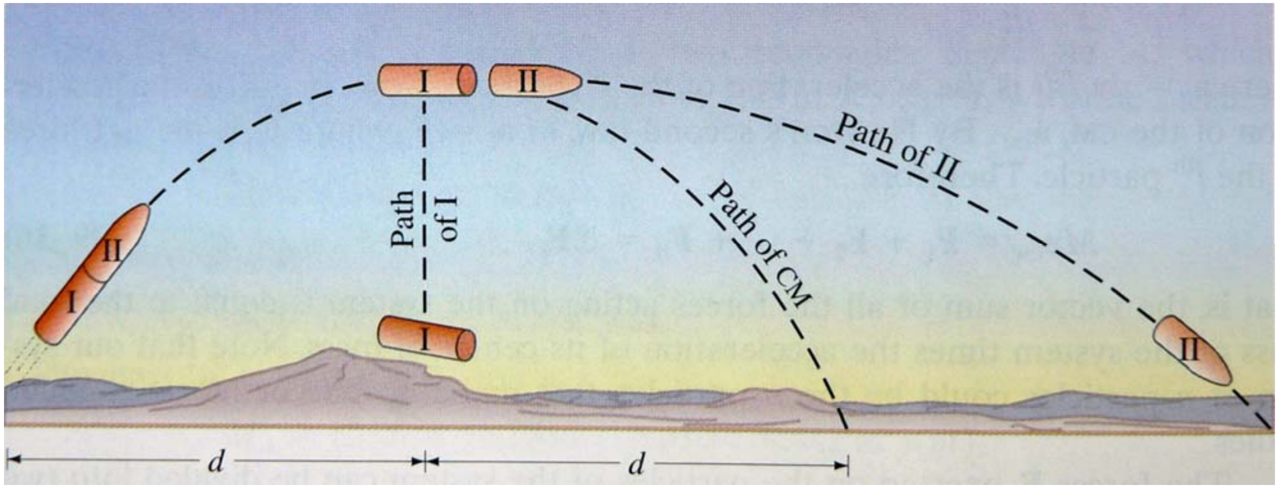


Example (P220 Ex. 9-16)

A rocket is fired into the air. At the moment it reaches its highest point, a horizontal distance *d* from its starting point, an explosion separates it into two parts of equal mass. Part I is stopped in midair by explosion and falls vertically to Earth. Where does part II land?



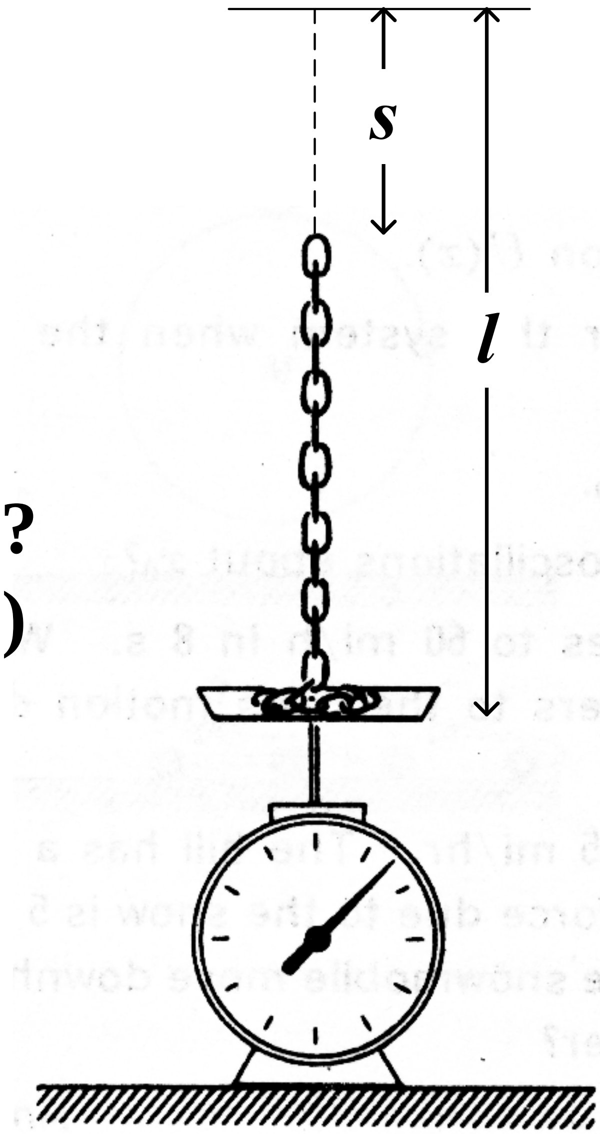
Example



Solution: After the rocket is fired, the path of the center of mass of the system continues to follow the parabolic trajectory of a projectile acted on only by a constant gravitational force. The center of mass will thus arrive at a point 2*d* from the starting point. Since the masses of I and II are equal, the center of mass must be midway between them. Therefore, II lands a distance 3*d* from the starting point.

Example

A chain of mass M length l is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. What is the reading of the scale when a length of chain, s, has fallen? (Neglect the size of individual links.)



Example

Solution (II): Using the Center of Mass:

$$N - Mg = M\frac{d^2 y_{CM}}{dt^2}$$

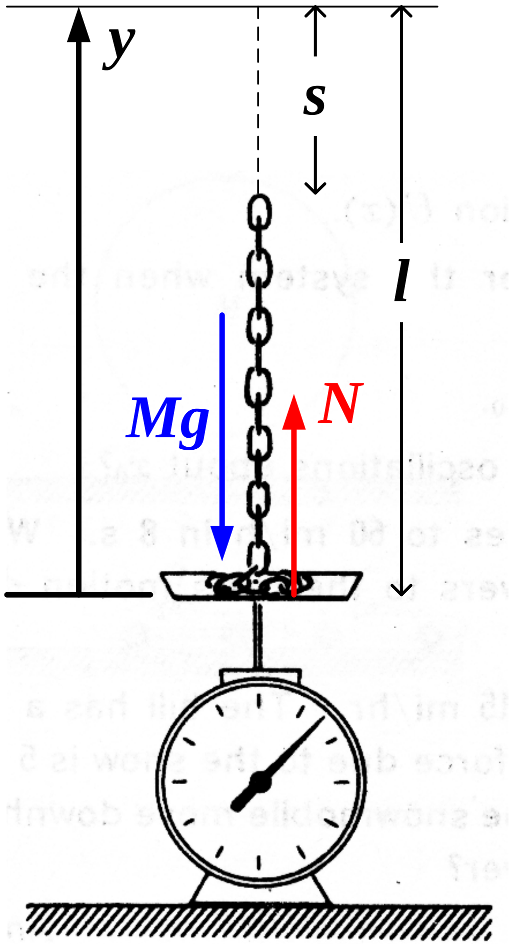
$$y_{CM} = \frac{M_1 y_1 + M_2 y_2}{M}$$

Two part: $M_1 = \lambda(l-s)$ $\qquad y_1 = (l-s)/2$

$M_2 = \lambda s \qquad y_2 = 0$

$\lambda = M/l$

$$y_{CM} = \frac{\lambda(l-s)(l-s)/2}{\lambda l} = \frac{(l-s)^2}{2l}$$



Example (continued)

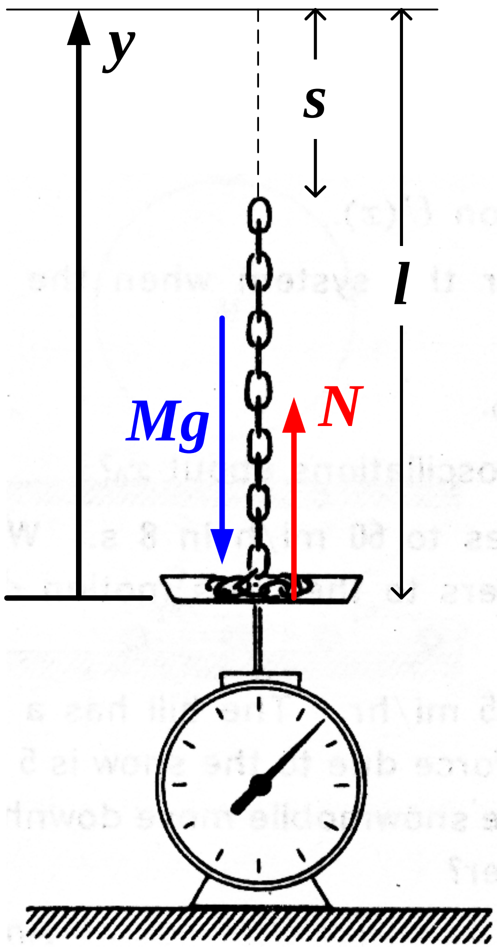
$$y_{\mathrm{CM}} = \frac{(l-s)^2}{2l}$$

For the part in the air: $s = \frac{1}{2}gt^2$

$$\frac{d\,y_{\mathrm{CM}}}{dt} = \frac{d}{dt}\left[\frac{(l-s)^2}{2l}\right] = -\frac{(l-s)}{l}\frac{ds}{dt}$$

$$= -\frac{gt}{l}\left(l - \frac{1}{2}gt^2\right)$$

$$\frac{d^2 y_{\mathrm{CM}}}{dt^2} = \frac{g(\frac{3}{2}gt^2 - l)}{l} = \frac{g(3s-l)}{l}$$



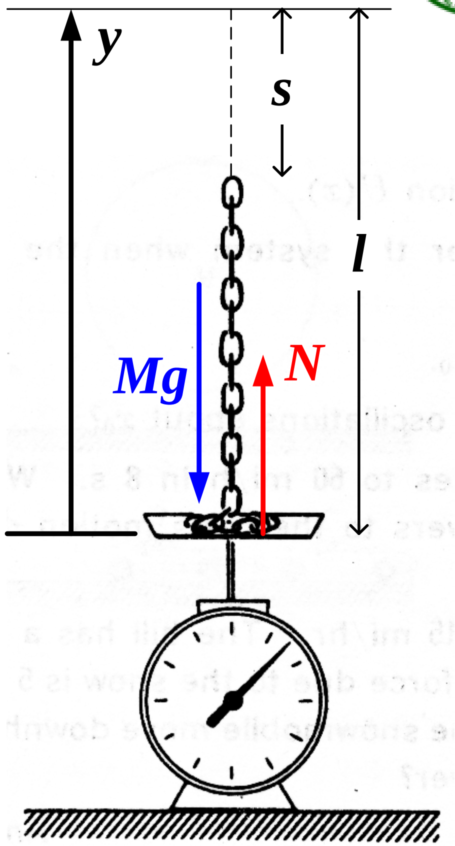
Example (continued)

$$\frac{d^2 y_{\mathrm{CM}}}{dt^2} = \frac{g(3s-l)}{l}$$

Newton's II law for CM:

$$N - Mg = M\frac{d^2 y_{\mathrm{CM}}}{dt^2} = Mg\left(\frac{3s}{l} - 1\right)$$

$$\boxed{N = 3Mg\frac{s}{l}}$$



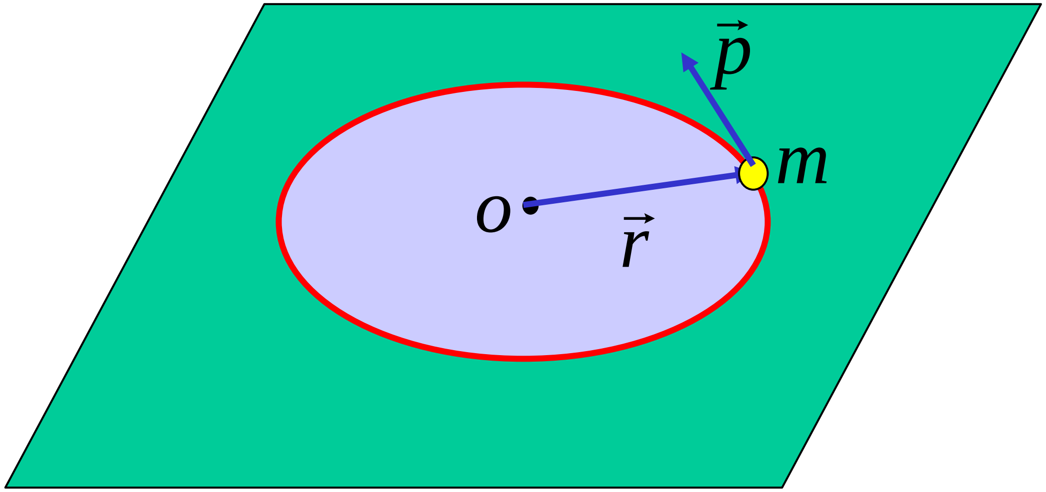
Problem

Ch9: 2, 5, 67, 70

§ 4 Angular Momentum

(P277 § 11-3)

Example

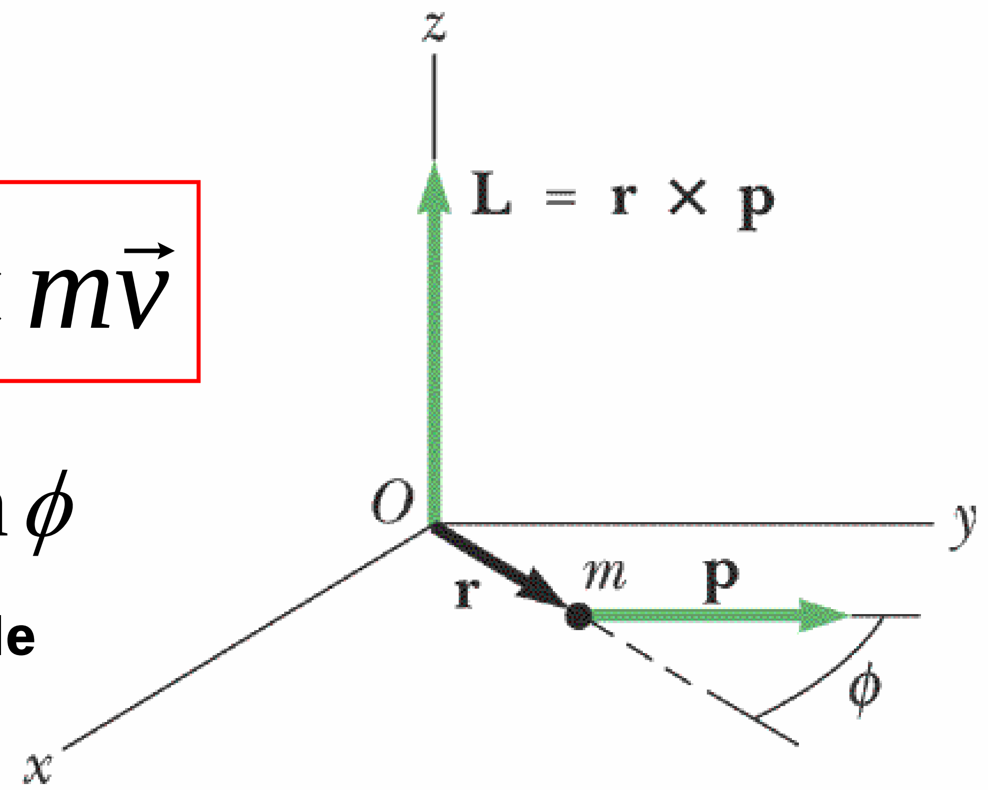


- **Definition**

$$\boxed{\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}}$$

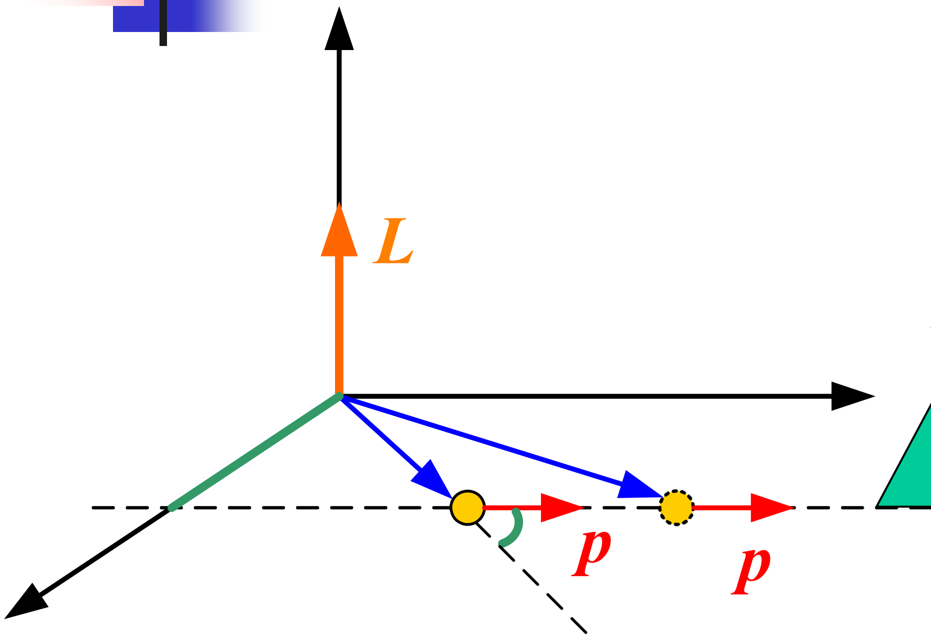
Magnitude: $L = mvr\sin\phi$

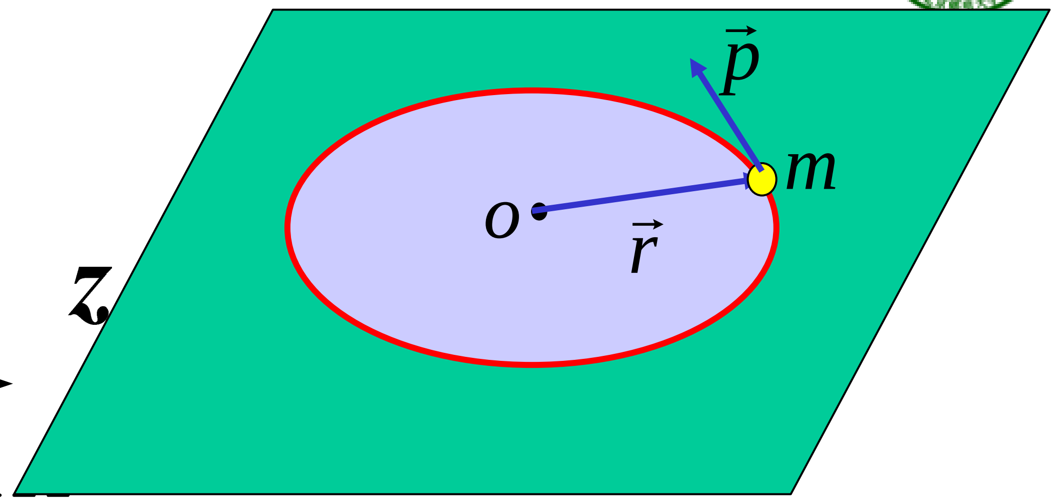
Direction: the right-hand rule



- **Depends on the choice of origin O.**
- **SI unit: kg•m²/s**

Example





The particle that moves in the circular orbit.

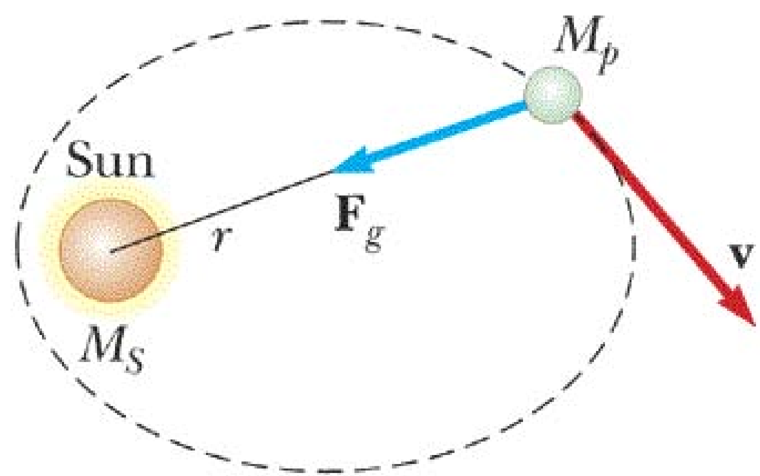
$$L = |\vec{r} \times \vec{p}| = mvr$$

The particle that moves in a straight line at constant velocity

$$L = |\vec{r} \times \vec{p}|$$

$$= mvr \sin\phi = mvd$$

$\vec{r}$



Torque

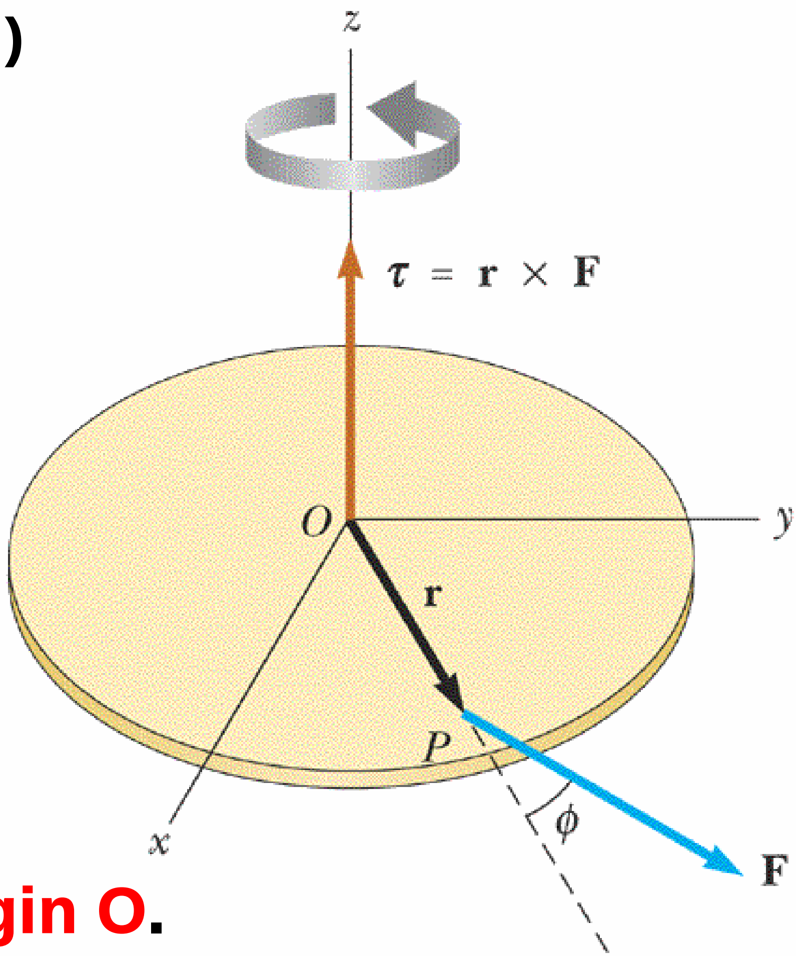
(P276 § 11-2)

- **Definition**

$$\boxed{\vec{\tau} = \vec{r} \times \vec{F}}$$

Magnitude: $\tau = rF \sin\phi$

Direction: the right-hand rule



- **Depends on the choice of origin O.**
- **SI unit: Newton • m.**

Torque-Angular Momentum Theorem

- **For one particle**

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times \frac{d\,\vec{p}}{dt}$$

$$\boxed{\vec{\tau} = \frac{d\,\vec{L}}{dt}}$$

$$\frac{d\,\vec{L}}{dt} = \frac{d}{dt}\left(\vec{r} \times \vec{p}\right) = \frac{d\,\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\,\vec{p}}{dt} = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\,\vec{p}}{dt} = \vec{r} \times \frac{d\,\vec{p}}{dt}$$

The torque acting on a particle is equal to the time rate of change of the particle's angular momentum.

- **Valid only if the origins of $\vec{L}$ and $\vec{\tau}$ are the same.**
- **Valid in inertial frame.**

$$\boxed{\vec{\tau} = 0 \quad \Rightarrow \quad \vec{L} = \text{const.}}$$

Kepler's Second Law (P284 Ex.11-6)

"The radius vector drawn from the Sun to any planet sweeps out equal areas in equal time intervals."

For a planet of mass M_p moving about the Sun

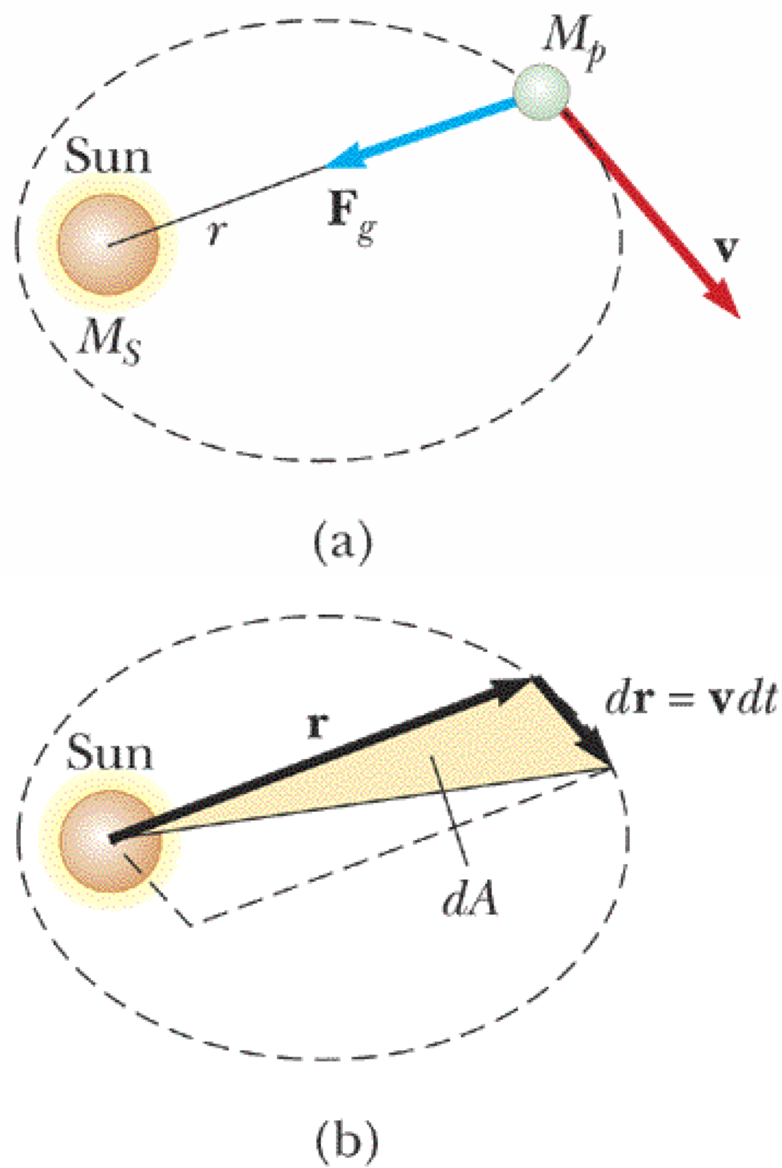
$$\vec{\tau} = \vec{r} \times \vec{F}_g = 0$$

$$\Downarrow$$

$$\vec{L} = \vec{r} \times \vec{p} = M_p \vec{r} \times \vec{v} = \text{constant}$$

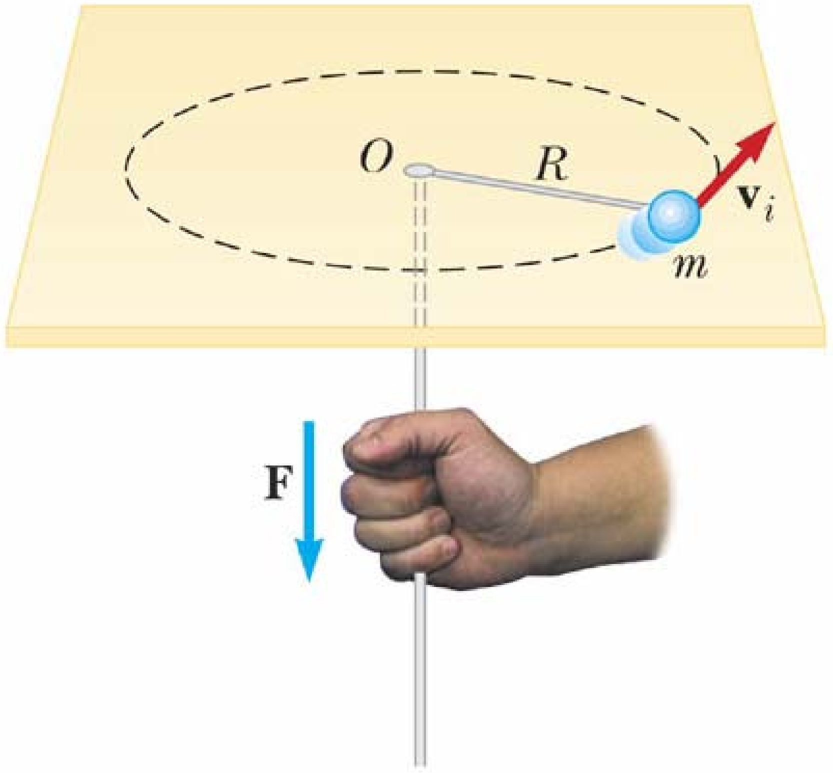
$$dA = \frac{1}{2}\left|\vec{r} \times d\vec{r}\right| = \frac{1}{2}\left|\vec{r} \times \vec{v}\,dt\right| = \frac{L}{2M_p}dt$$

$$\boxed{\frac{dA}{dt} = \frac{L}{2M_p} = \text{constant}}$$



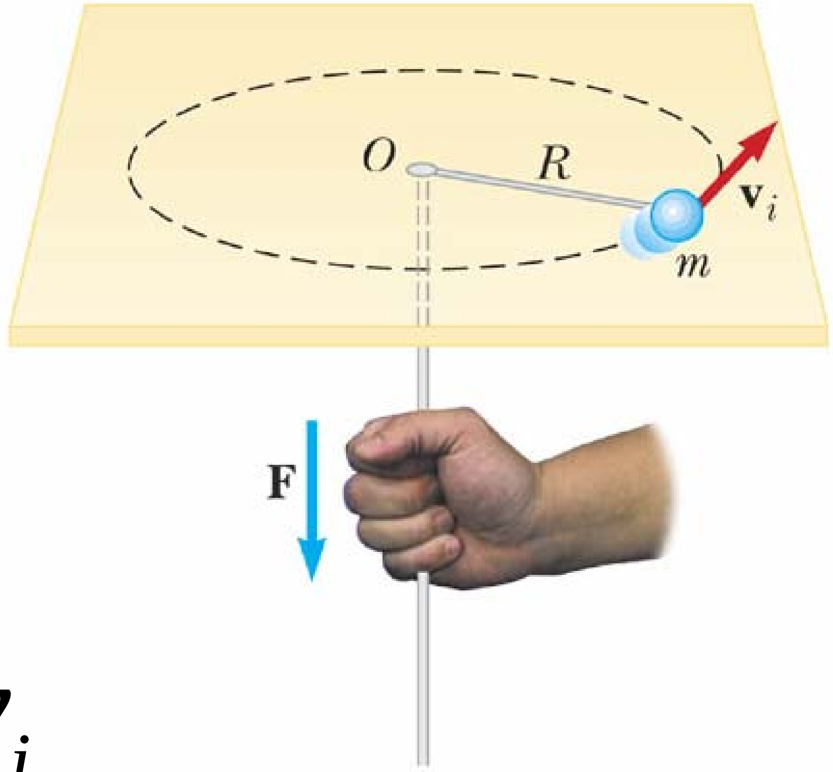
Example

A ball of mass ***m*** on a horizontal, frictionless table is connected to a string that passes through a small hole in the table. The ball is set into circular motion of radius ***R***, at which time its speed is v_i. If the string is pulled from the bottom so that the radius of the circular path is decreased to ***r***, what is final speed v_f of the ball?



Solution

The net torque for the ball system is zero. Therefore the angular momentum of ball remains constant.



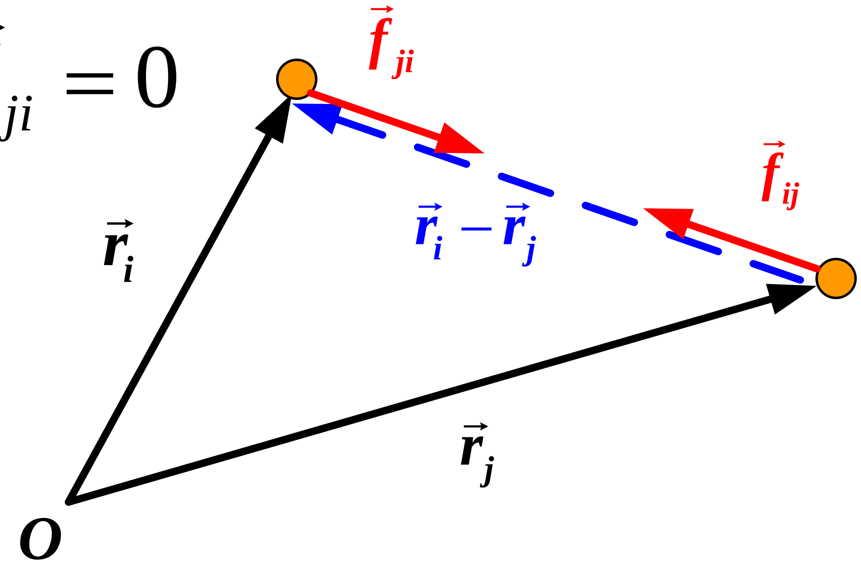
$$Rmv_i = rmv_f$$

$$v_f = \frac{R}{r} v_i > v_i$$

Torque-Angular Momentum Theorem for a system of particles

- The torques of each pair of internal forces are vanished.

$$\vec{r}_i \times \vec{f}_{ji} + \vec{r}_j \times \vec{f}_{ij} = (\vec{r}_i - \vec{r}_j) \times \vec{f}_{ji} = 0$$



$$\sum(\vec{\tau}_{\text{in}} + \vec{\tau}_{\text{ext}})$$

$$= \sum_i \frac{d\vec{L}_i}{dt} = \frac{d}{dt}\sum_i \vec{L}_i = \frac{d\vec{L}_{\text{tot}}}{dt}$$

$$\boxed{\sum \vec{\tau}_{\text{ext}} = \frac{d\vec{L}_{\text{tot}}}{dt}}$$

The net external torque acting on the system is equal to the time rate of change of the total angular momentum of the system.

Valid in inertial frame and the reference frame of the center of mass.
Valid only if all the origins of $\vec{L}$ and $\vec{\tau}$ in the system are the same.

Conservation of Angular Momentum

(P284 § 11-7)

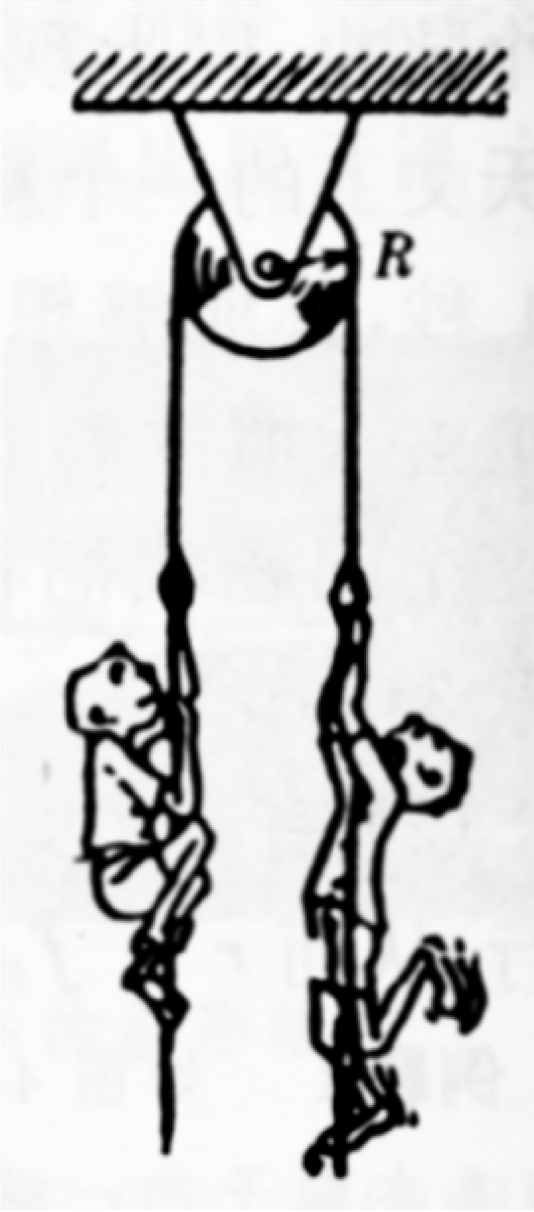
For a system of particles

$$\sum \vec{\tau}_{\text{ext}} = 0 \quad \Rightarrow \quad \frac{d\vec{L}_{\text{tot}}}{dt} = 0 \quad \textbf{or} \quad \vec{L}_{\text{tot}} = \text{constant}$$

The total angular momentum of a system remains constant if the net external torque acting on the system is zero.

Example

Two boys, with same mass of *m* , suspend to the two side of a pulley with a light rope. The boy on the left makes an effort to climb up, but the other boy keeps at rest without any action. Which boy is the first to approach pulley? Neglecting the mass of the pulley and the friction on the axis of the pulley.



Solution

For the two-boy system, the net external torque.

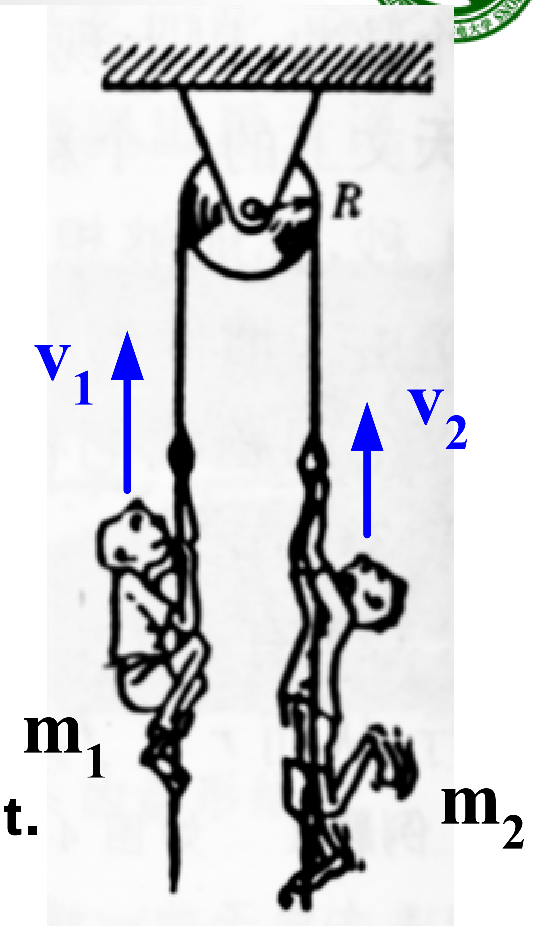
$$\sum \tau_{\text{ext}} = Rm_1 g - Rm_2 g = 0$$

(Take the direction of torque consistent with anti-clockwise.)

The angular momentum of two-boy system is conserved.

$$L_f = mR(v_2 - v_1) = L_i = 0$$

$v_2 - v_1 = 0$ **Two boy approach the pulley at same time, whoever makes an effort.**

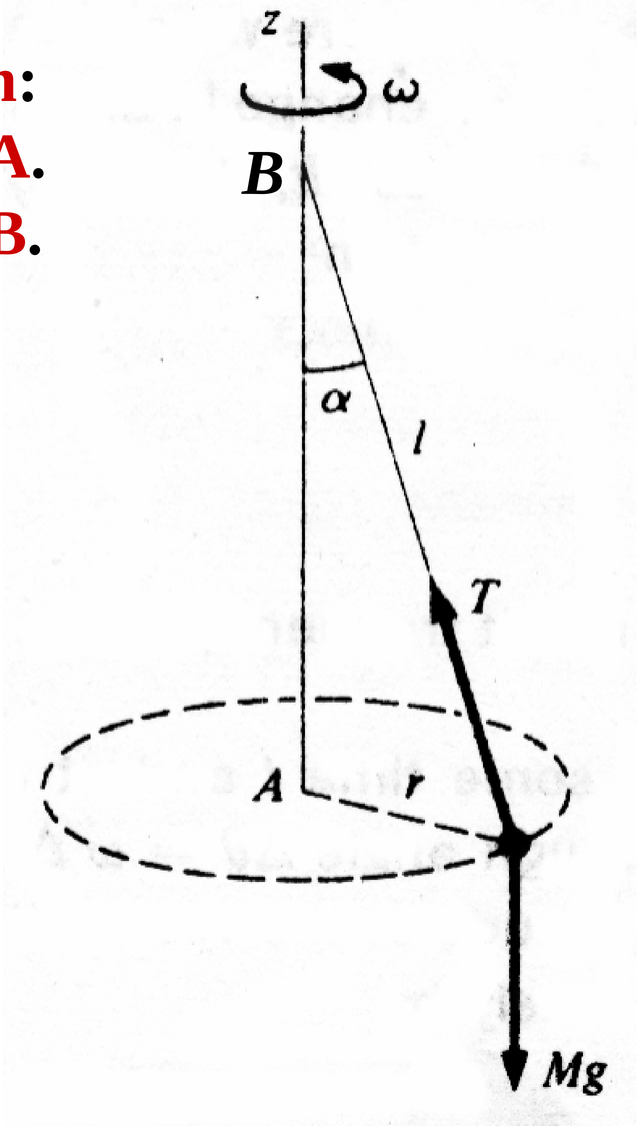


But if m_1>m_2, $\sum \tau_{\text{ext}} > 0$ $\quad \dfrac{dL}{dt} > 0,\ L_i = 0,\ L_f > 0,\ v_2 > v_1$

Angular momentum of the conical pendulum:

(1) The angular momentum about origin A.

(2) The angular momentum about origin B.



Solution

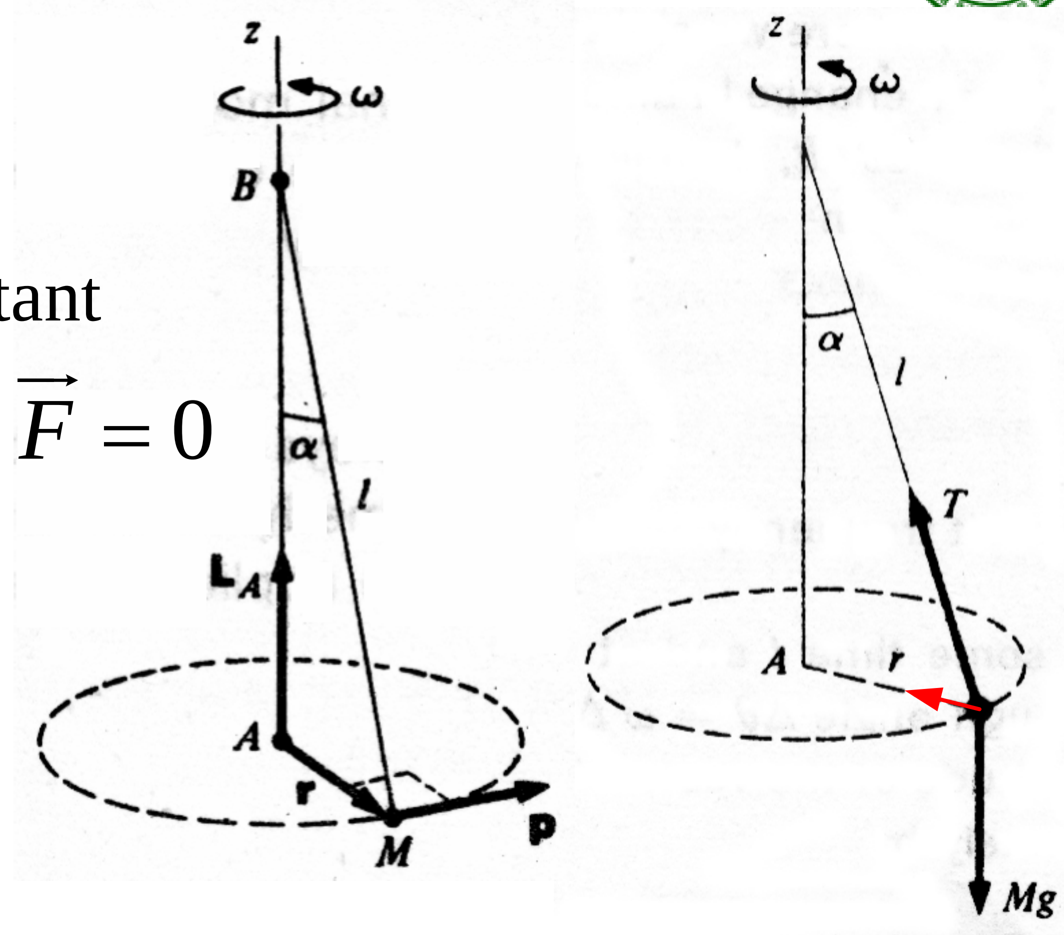
(1) For origin A

$$\vec{L}_A = \vec{r} \times \vec{p} = rMv\hat{k}$$

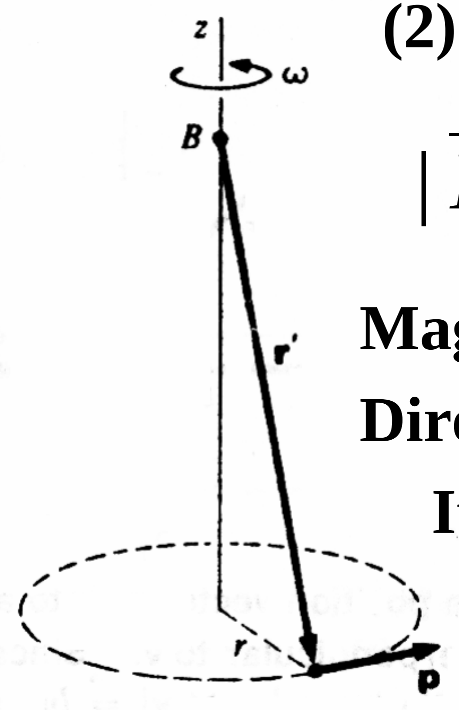
$$= r^2 M\omega\hat{k} = \text{constant}$$

$$\vec{\tau}_A = \vec{r} \times (\vec{T} + M\vec{g}) = \vec{r} \times \overrightarrow{F} = 0$$

$$\vec{\tau}_A = \frac{d\vec{L}_A}{dt} = 0$$



$\vec{L}_A$ **remains constant, both in magnitude and direction.**



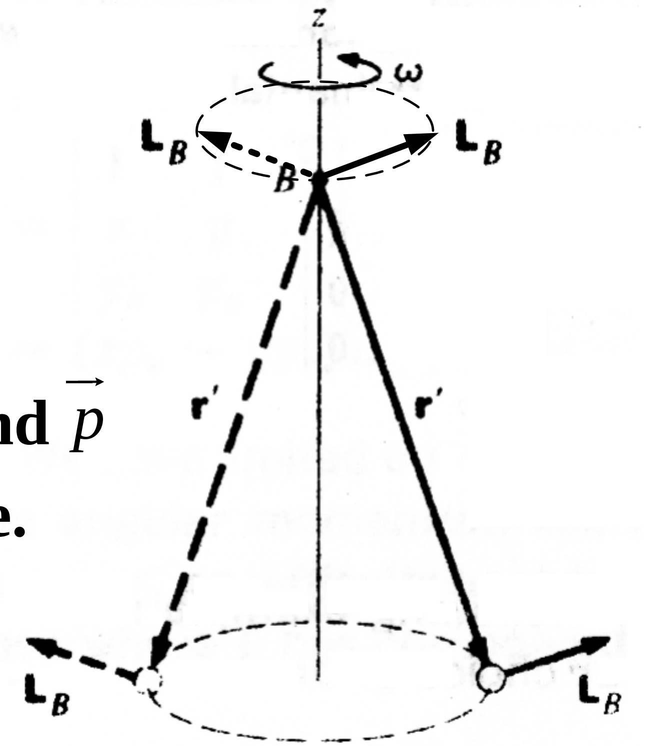
(2) For origin **B**

$$|\vec{L}_B| = |\vec{r}' \times \vec{p}| = l\,p = Mlr\omega$$

Magnitude: constant.

Direction: perpendicular to $\vec{r}'$ **and** $\vec{p}$

Its tip draws a horizontal circle.



$$\vec{\tau}_B = \vec{r}' \times (\vec{T} + M\vec{g}) = \vec{r}' \times \vec{F} \neq 0$$

$$\vec{\tau}_B = \frac{d\vec{L}_B}{dt} \neq 0$$

Problem

Ch11: 7, 10, 18