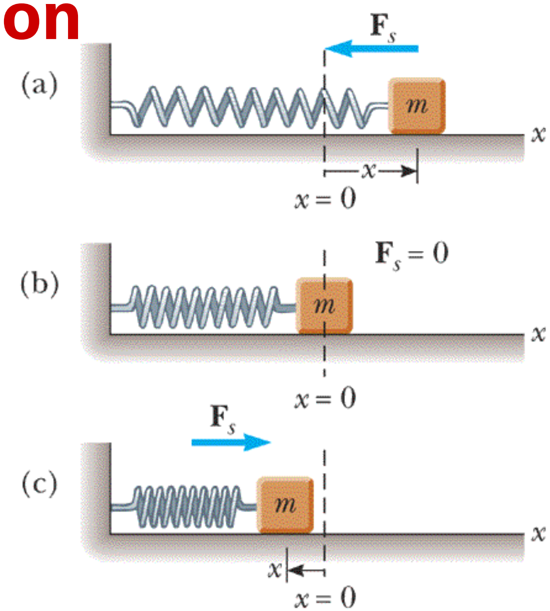
Chapter 12 Oscillations

§ 1 The Causes of Oscillation



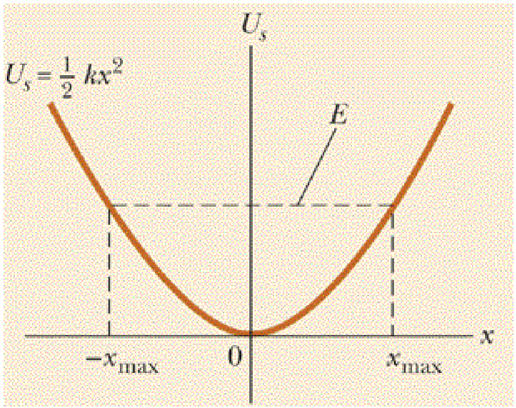
- The systems tends to return to equilibrium when slightly displaced
  - **Existence of a point of stable equilibrium**

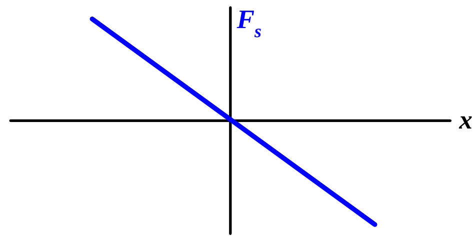
    For a block-spring system $$\boxed{U(x) = \frac{1}{2}kx^2}$$

  - **Existence of a restoring force**

    No matter what the direction of the displacement, the force always acts in a direction to restore the system to its equilibrium position.

    For a block-spring system $$\boxed{F = -\frac{dU}{dx} = -kx}$$

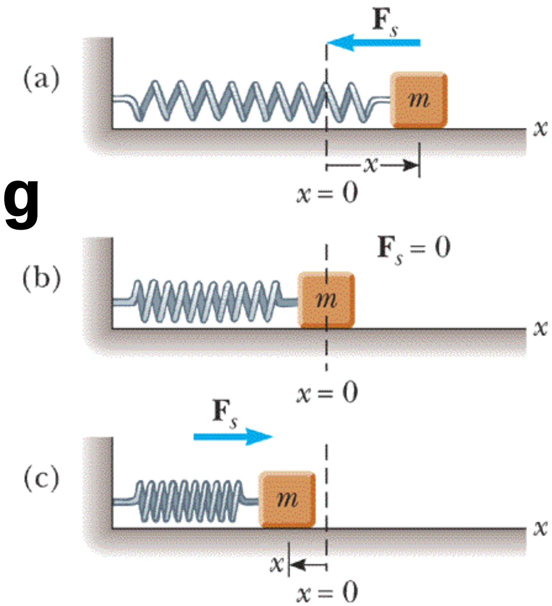




§ 2 Simple Harmonic Motion (SHM)

- **The block-spring system** (P299)
  - **Newton's second law for block-spring system**

$$-kx = m\frac{d^2x}{dt^2}$$



- **Dynamics' equation**

Denote the ratio *k*/*m* with symbol ω^2

$$\boxed{\frac{d^2x}{dt^2} + \omega^2 x = 0} \qquad \textbf{(1)}$$

Dynamics' equation for SHM

Take a tentative solution to Eq.(1)

$$\boxed{x = A\cos(\omega t + \phi)}$$

Kinematics' equation for SHM

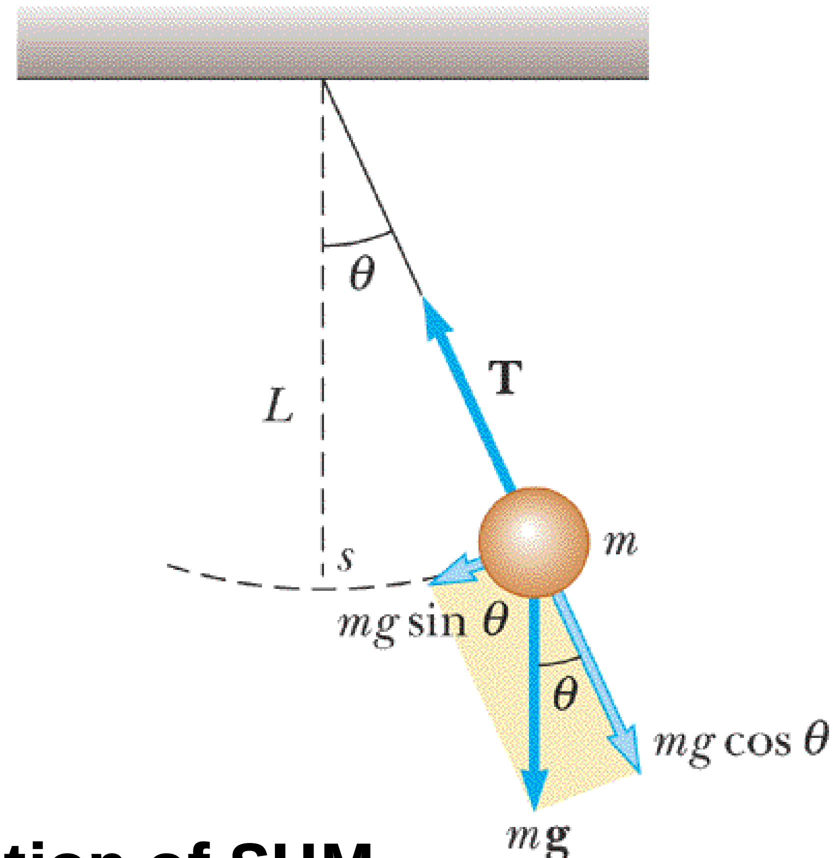
***A* and ϕ arise from the integral constants**

The simple pendulum (P307)

- **Newton's second law for the simple pendulum**

$$-mg\left(L\sin\theta\right)=\left(mL^2\right)\frac{d^2\theta}{dt^2}$$

$$\Rightarrow \quad \frac{d^2\theta}{dt^2}=-\frac{g}{L}\sin\theta$$



Let $\omega=\sqrt{\frac{g}{L}}$,

and for small angles $\sin\theta\approx\theta$

- **We get also a equation of motion of SHM**

$$\boxed{\frac{d^2\theta}{dt^2}+\omega^2\theta=0}$$

$$\boxed{\theta=\theta_m\cos(\omega t+\phi)}$$

The Physical Pendulum (复摆) (P308)

- **Newton's second law for rigid body:**

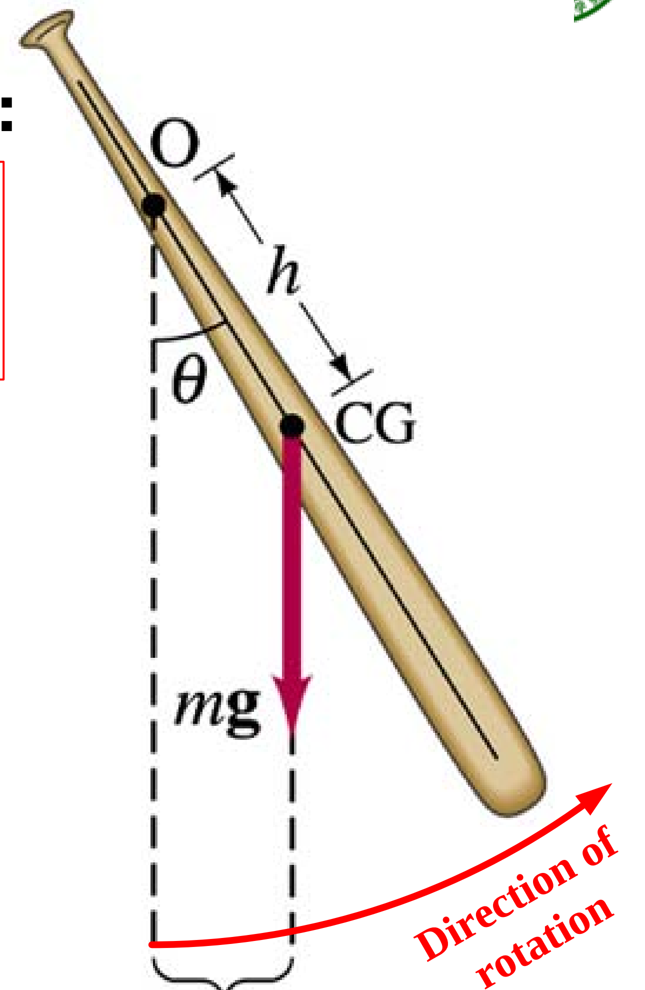
$$\tau_{\text{net-axis}} = I\alpha, \qquad \boxed{-mgh\sin\theta = I\frac{d^2\theta}{dt^2}}$$

It follows that:

$$\frac{d^2\theta}{dt^2} + \frac{mgh}{I}\sin\theta = 0, \qquad \sin\theta \approx \theta$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{mgh}{I}\right)\theta = 0 \quad \Rightarrow$$

$$\boxed{\theta = \theta_{\max}\cos(\omega t + \phi)} \quad \omega = \sqrt{\frac{mgh}{I}}, \quad d_{\perp}(= h\sin\theta)$$



The Torsion Pendulum (扭摆) (P309)

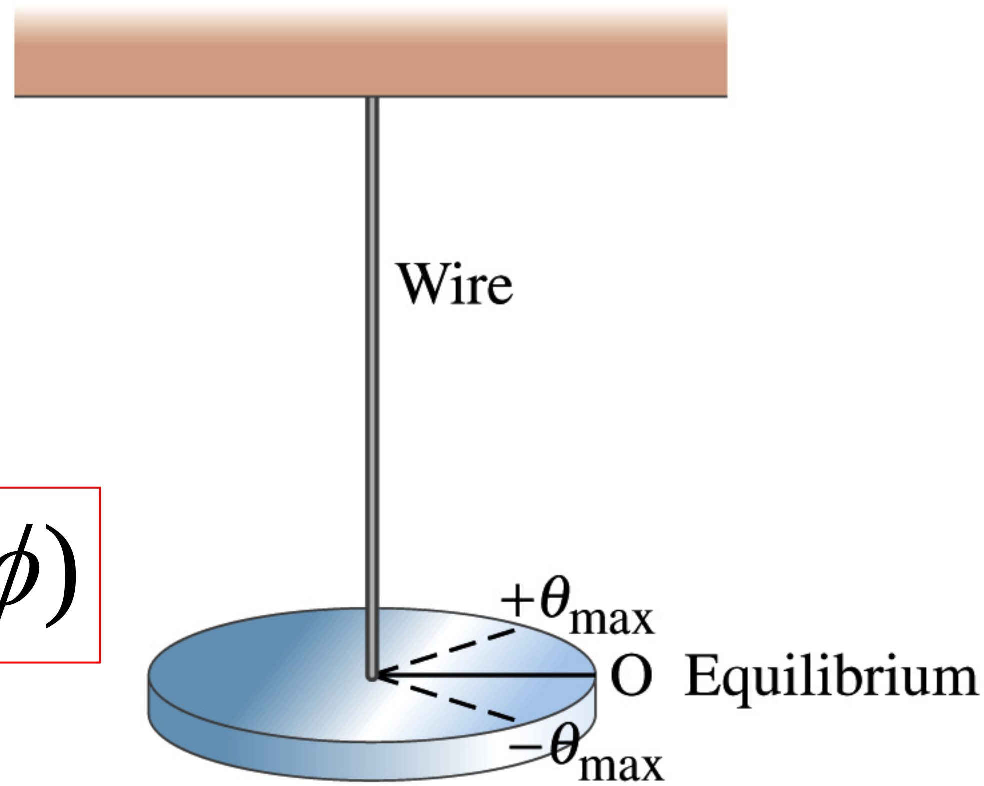
- **The restoring torque:** $\tau = -K\theta$

$$-K\theta = I\alpha = I\frac{d^2\theta}{dt^2}$$

$$\frac{d^2\theta}{dt^2} + \left(\frac{K}{I}\right)\theta = 0$$

$$\boxed{\theta = \theta_{\max}\cos(\omega t + \phi)}$$

$$\omega = \sqrt{\frac{K}{I}}$$



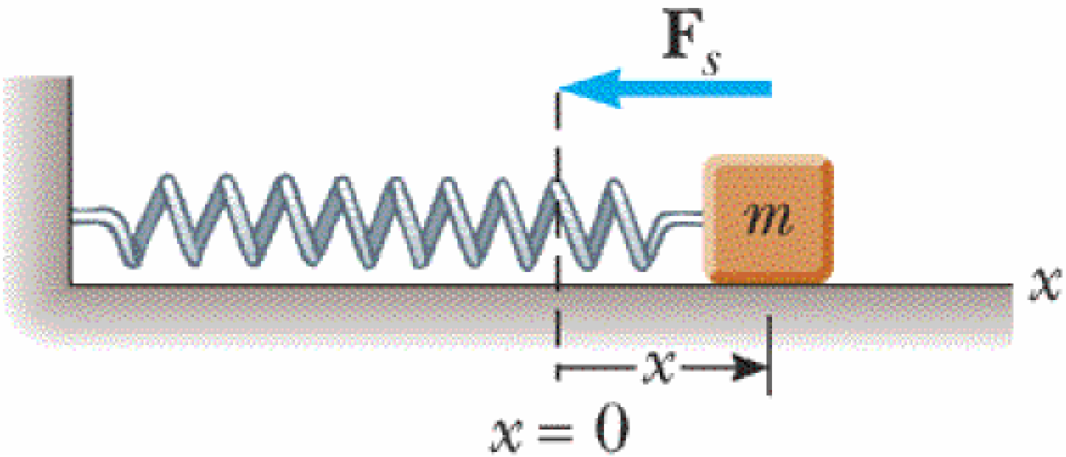
§ 3 The Characteristic Quantities for SHM

(P301)

$$\boxed{x = A\cos(\omega t + \phi)}$$

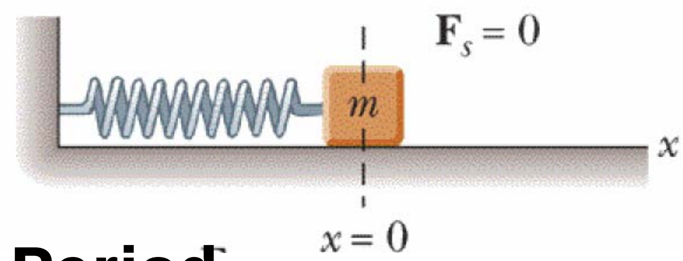
- **The amplitude *A***
  - **Maximum magnitude of displacement from equilibrium**

$$A = | x_{\max} |$$



The Characteristic Quantities for SHM

$$x = A\cos(\omega t + \phi)$$



- **Angular Frequency, Frequency, and Period**
  - **The period, T, is the time for oscillator to go though one cycle of motion**
  - **The frequency, f, is the number of cycles in a unit of time. (SI unit: Hz)**

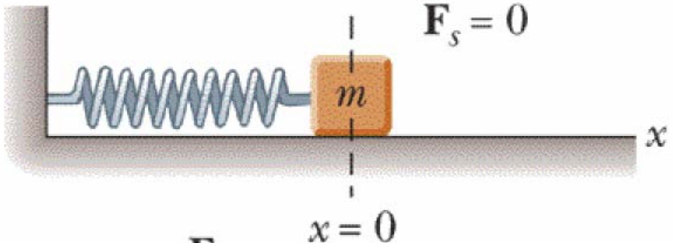
$$f = \frac{1}{T}$$

  - **The angular frequency, ω, is 2π times the frequency. (SI unit: rad/s)**

$$\omega = 2\pi f = \frac{2\pi}{T}$$

The Characteristic Quantities for SHM

$$x = A\cos(\omega t + \phi)$$



- ***T*, *f*, *ω*** **relate to the essential nature of an oscillator, which often called natural (intrinsic) period, natural frequency, and natural angular frequency.**

  - **For a block-spring oscillator:** $$\omega = \sqrt{\frac{k}{m}}$$

  - **For a pendulum:** $$\omega = \sqrt{\frac{g}{L}}$$ $$\omega = \sqrt{\frac{mgh}{I}}$$

All determined by the essential natures of two different oscillators.

- **The phase $(\omega t+\phi)$**
  - **The phase $(\omega t+\phi)$ can reflect entirely the motion state of an oscillator**

Phase —— $\omega t+\phi$ $\Longleftrightarrow$ $\begin{Bmatrix} x \\ v \end{Bmatrix}$ —— **State of motion**

$$x = A\cos(\omega t+\phi), \qquad v = \frac{dx}{dt} = -\omega A\sin(\omega t+\phi)$$

  - **When $t=0$, ϕ reflect the initial motion state of the oscillator**
  - **A and ϕ are determined by initial conditions (How the motion starts)**

When $t=0$, $x=x_0$, $v=v_0$

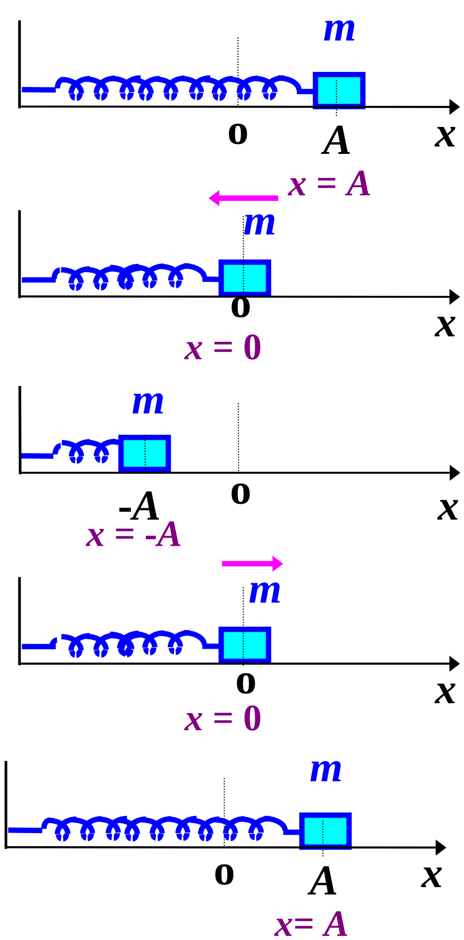
$$x_0 = A\cos\phi$$
$$v_0 = -\omega A\sin\phi$$

$$\Longrightarrow \boxed{A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}}, \quad \phi = \arctan\left(-\frac{v_0}{\omega x_0}\right)}$$

The relationship between motion state and phase

$$x(t) = A\cos(\omega t + \phi), \quad v = -\omega A\sin(\omega t + \phi)$$

Motion state ⟷ **Phase** $(\omega t + \phi)$



0

π/2

π

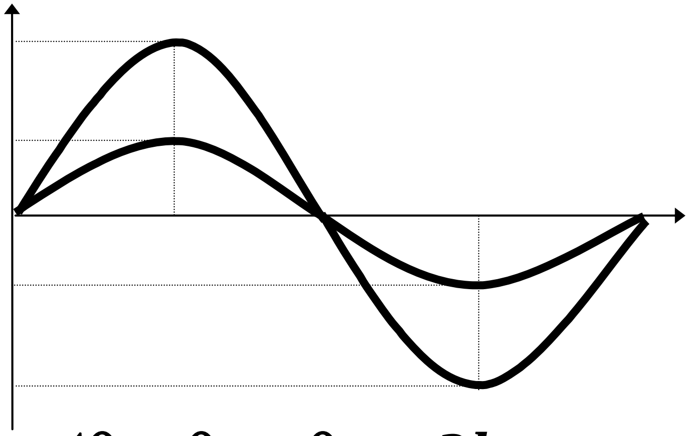
3π/2

2π

- **Phase difference play a an important role for oscillator**
  - **Two oscillators with phases:** $\theta_1 = \omega t + \phi_1$, $\theta_2 = \omega t + \phi_2$

$$\pi > \Delta\theta = \theta_2 - \theta_1 > 0, \qquad -\pi < \Delta\theta = \theta_2 - \theta_1 < 0$$

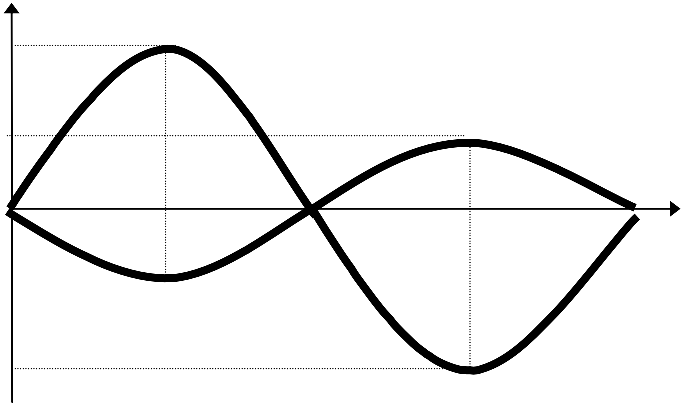
Ahead in phase **Lag in phase**



$$\Delta\theta = \theta_2 - \theta_1 = 2k\pi$$

$$k = 0, \pm 1, \pm 2 \cdots$$

In phase



$$\Delta\theta = \theta_2 - \theta_1 = (2k+1)\pi$$

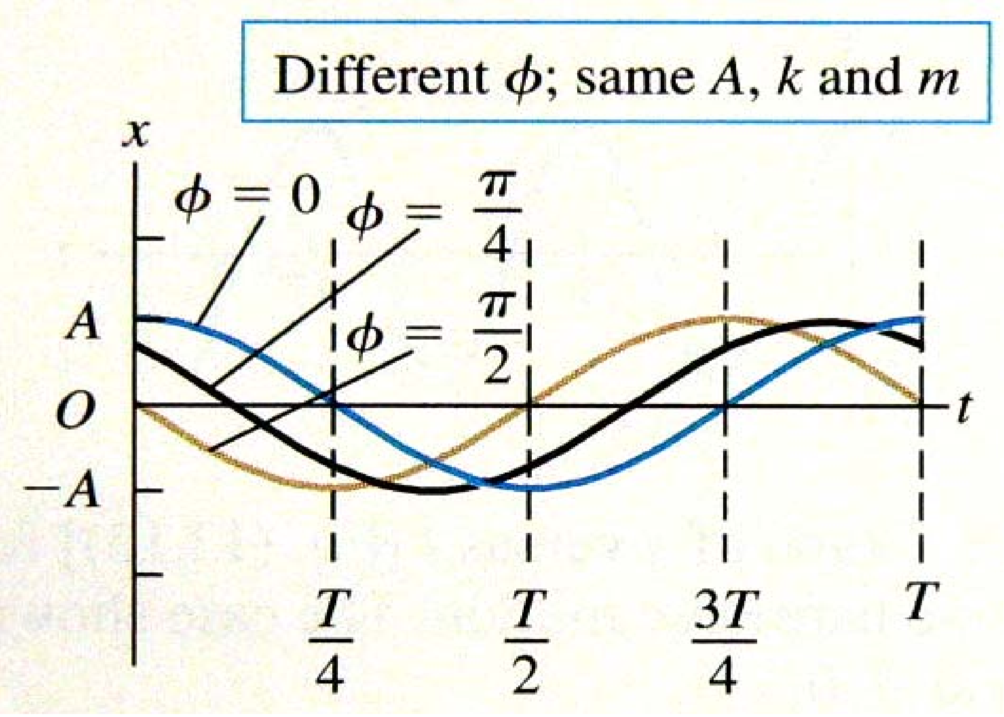
$$k = 0, \pm 1, \pm 2 \cdots$$

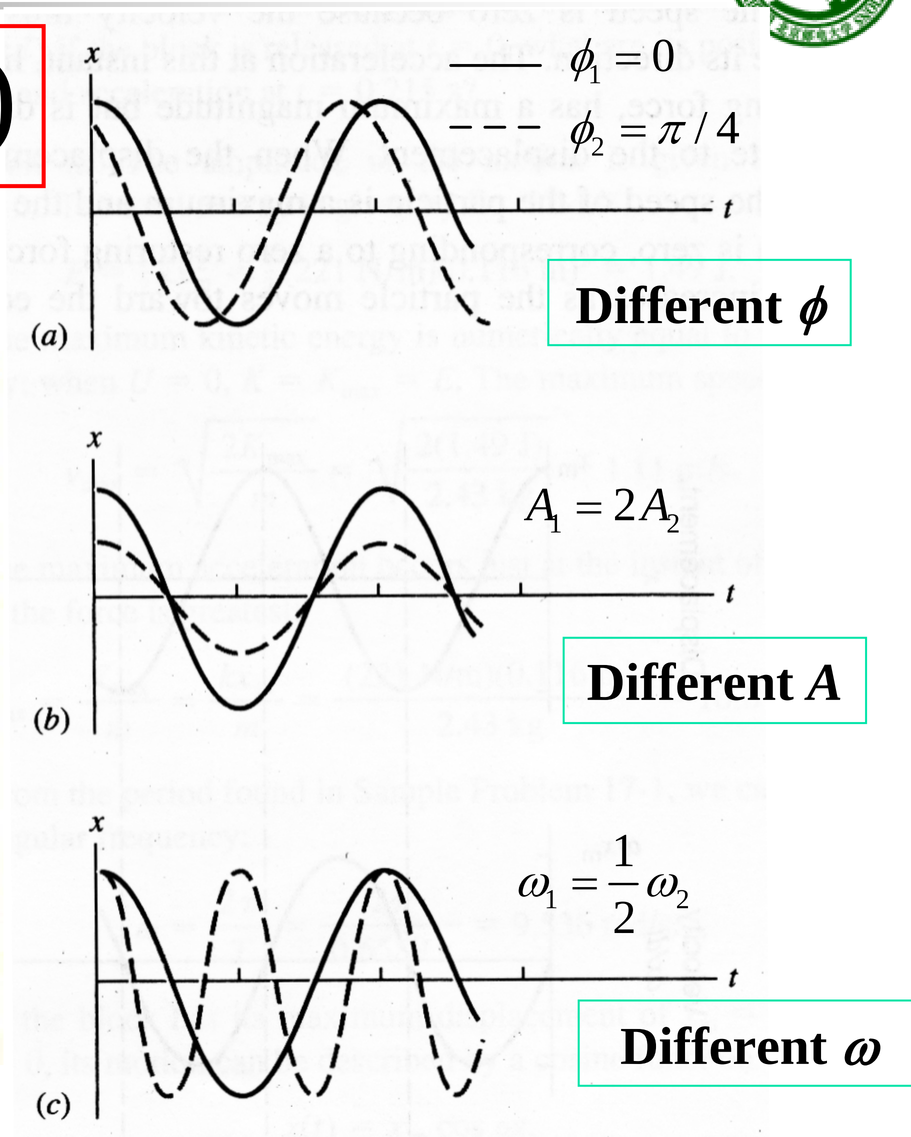
Out of phase

The Roles Characteristic Quantities

$$x = A\cos(\omega t + \phi)$$

Several SHM with different characteristic quantities





The relations among the position, velocity, and acceleration

$$x = A\cos(\omega t + \phi)$$

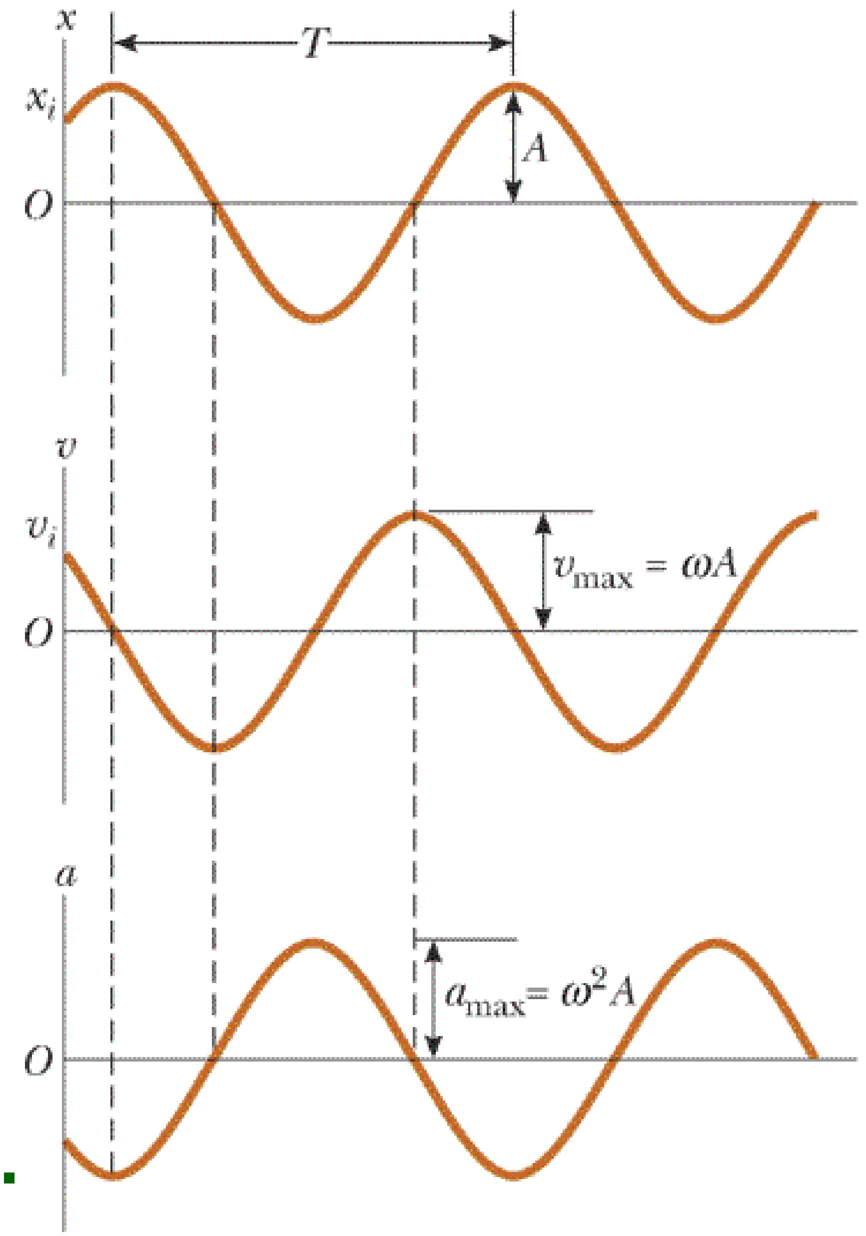
$$v = \frac{dx}{dt} = -\omega A\sin(\omega t + \phi)$$

$$= \omega A\cos(\omega t + \phi + \frac{\pi}{2})$$

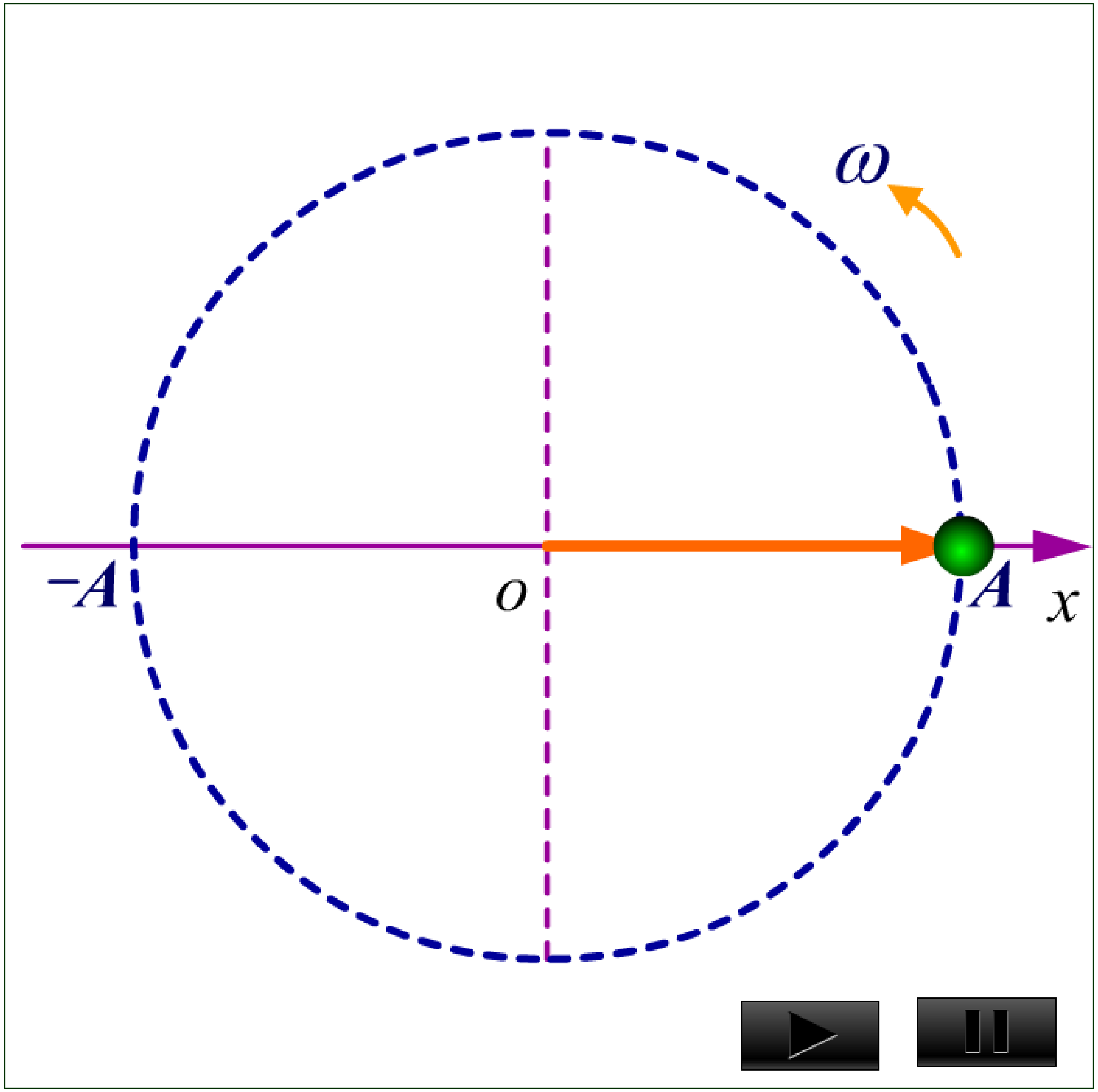
$$a = \frac{dv}{dt} = -\omega^2 A\cos(\omega t + \phi)$$

$$= \omega^2 A\cos(\omega t + \phi + \pi)$$

- **The velocity is $\pi/2$ ahead in phase of the position.**
- **The acceleration is π out of phase with the position.**

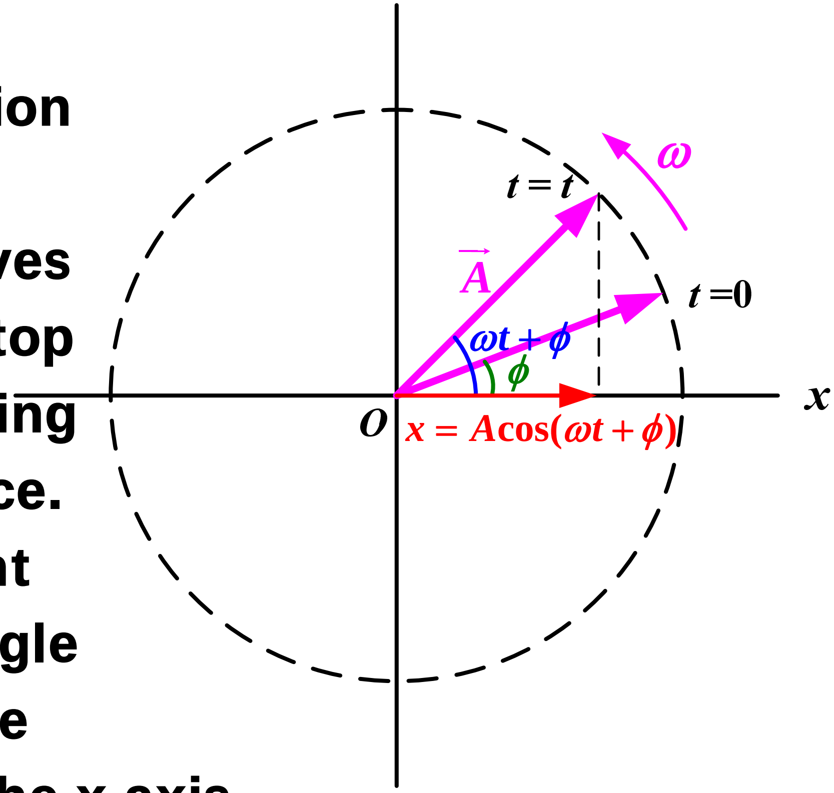


§ 4 The Circle of Reference (P306)



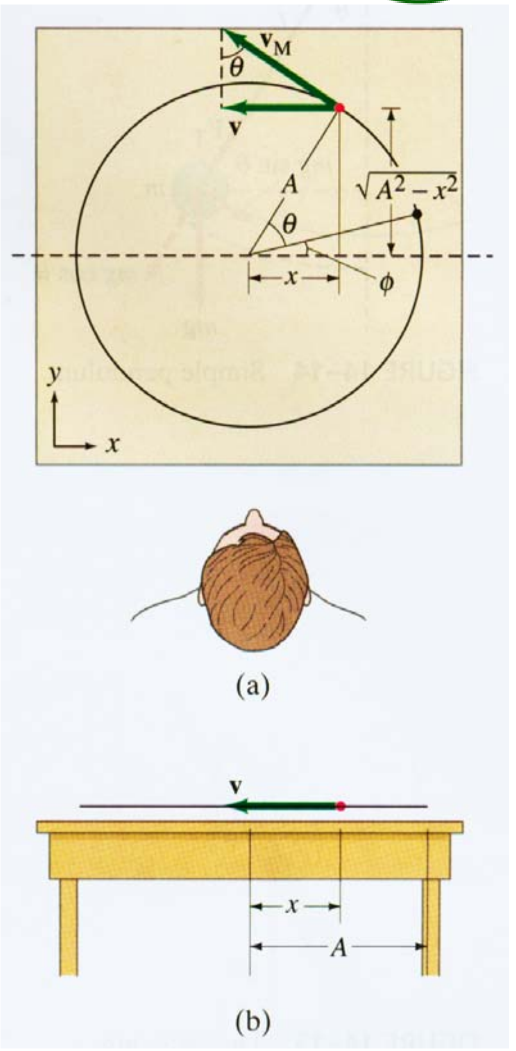
Circle of Reference or Phasor

- **The corresponding relation between SHM and uniform circular motion —— Circle of Reference (参考圆) or Phasor (旋转矢量)**
  - **Simple Harmonic Motion is the projection of uniform circular motion of phasor $\vec{A}$ onto x axis.**
  - **The circle in which the phasor moves so that the projection of phasor's top matches the motion of the oscillating body is called the circle of reference.**
  - **The phasor $\vec{A}$ rotates with constant angular speed ω, and makes an angle $\omega t+\phi$ with the x axis. When $t=0$, the phasor $\vec{A}$ makes an angle ϕ with the x axis.**



Corresponding Relation Between SHM and UCM

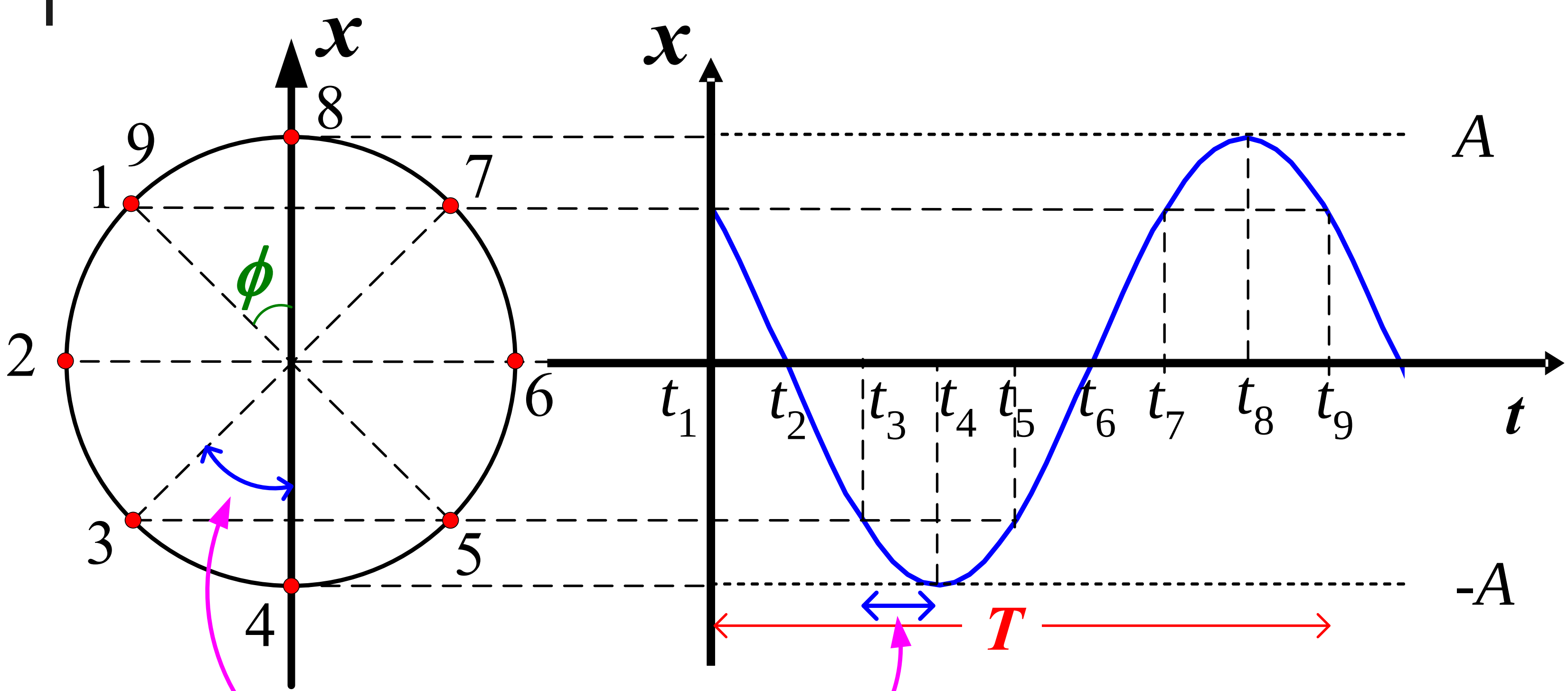
	For Simple Harmonic Motion	For Uniform Circular Motion
A	Amplitude	Radius
x	Displacement	Projection
ω	Angular Frequency	Angular Velocity
$\theta = \omega t + \phi$	Phase	Angle between Phasor and x axis



(a)
(b)

The simple harmonic motion is the side view of circular motion.

Draw x-t Graph Using Circle of Reference



$$\frac{|\Delta\phi|}{2\pi}T = \Delta t$$

Example

An object of mass 4 kg is attached to a spring of $\boldsymbol{k}$ =100N/m. The object is given an initial velocity of $\boldsymbol{v_0} = -5$m/s and an initial displacement of $\boldsymbol{x_0}$=1 m. Find the kinematics equation.

Solution: $$x = A\cos(\omega t + \phi)$$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{4}} = 5 \text{ rad/s}, \qquad A = \sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} = \sqrt{2} \text{ m}$$

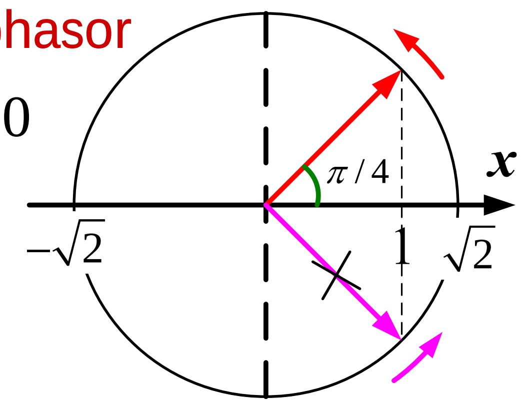
$$\because tg\phi = -\frac{v_0}{\omega x_0} = 1 > 0$$

ϕ locates in I or III quadrant $\quad \phi = \dfrac{\pi}{4}$ or $\dfrac{5\pi}{4}$

with $\quad v_0 = -\omega A \sin\phi < 0, \qquad \sin\phi > 0$

Using the phasor

$\boldsymbol{x_0}$=1, $\boldsymbol{v_0}$ <0



$$\therefore \phi = \frac{\pi}{4} \qquad \therefore x = \sqrt{2}\cos(5t + \frac{\pi}{4}) \quad \text{m}$$

Example

A particle undergoes SHM with A=4cm， f = 0.5Hz. The displacement x = –2cm when t = 1s, and is moving in the positive x-axis. Write the kinematics equation.

Solution: $A = 4\text{cm},\ f = 0.5\text{Hz},$

$$\omega = 2\pi f = \pi \text{ rad/s},$$

$$x = 0.04\cos(\pi t + \phi)\ \text{m}, \quad \phi = ?$$

When t=1s

$$-0.02 = 0.04\cos(\pi + \phi) = -0.04\cos\phi$$

$$\cos\phi = 1/2 \Rightarrow \phi = \pm\pi/3$$

locates in I or IV quadrant

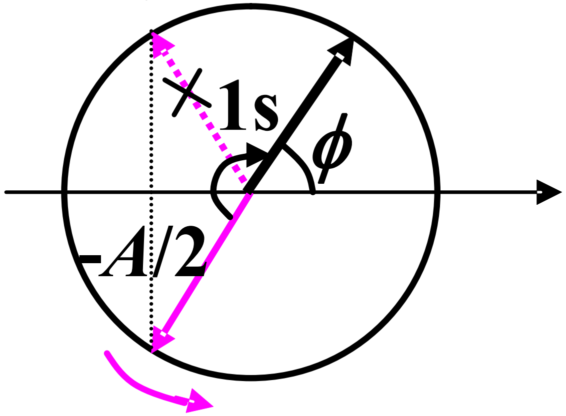
$$v = -0.04\pi\sin(\pi + \phi) = 0.04\pi\sin\phi > 0$$

ϕ locates in I quadrant. $\phi = \dfrac{\pi}{3}$

Too complicated !

Using the phasor:

Δt = 1 s corresponds to half a revolution.



$$v > 0$$

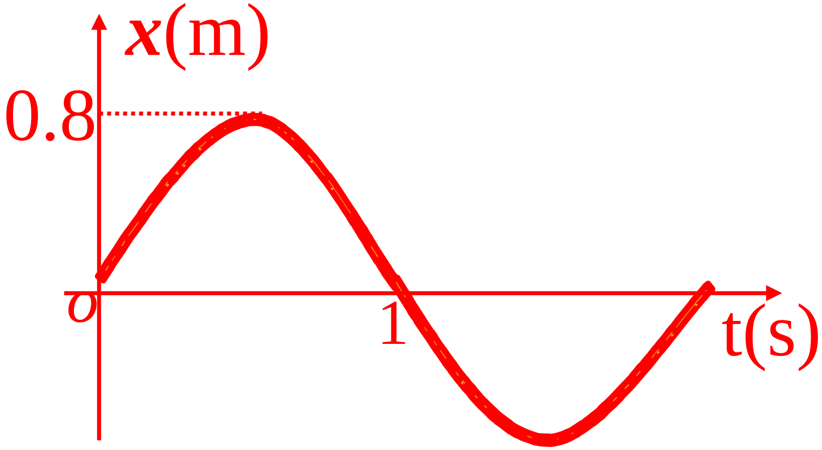
$$\Delta\phi + \phi = \omega\Delta t + \phi$$

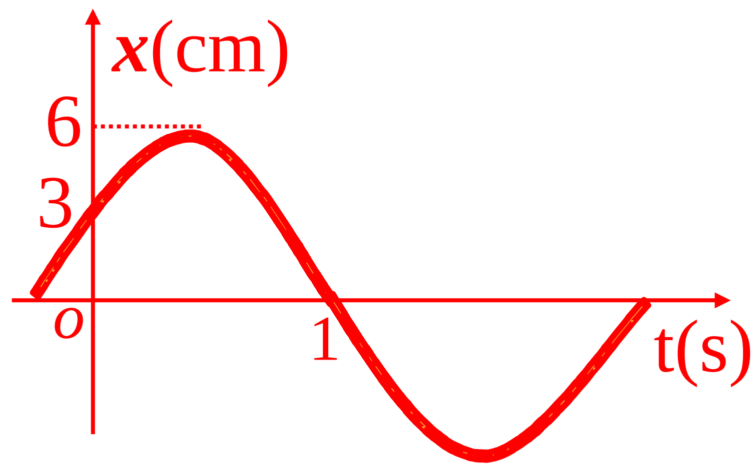
$$= \pi + \phi = 4\pi/3$$

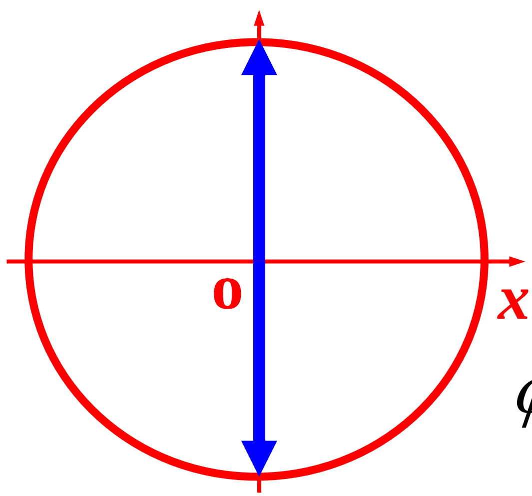
$$\Rightarrow \phi = \frac{\pi}{3}$$

Example

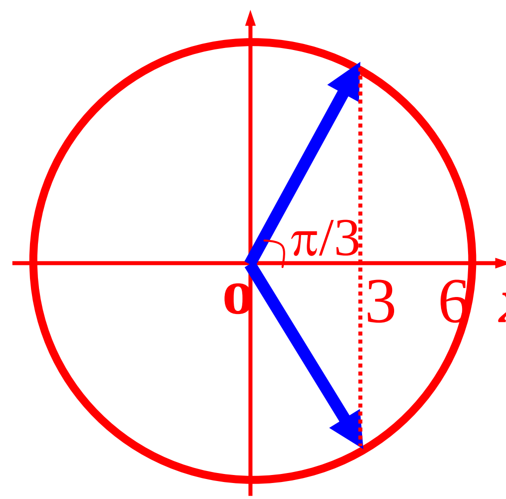
Find the initial phase of the two oscillations







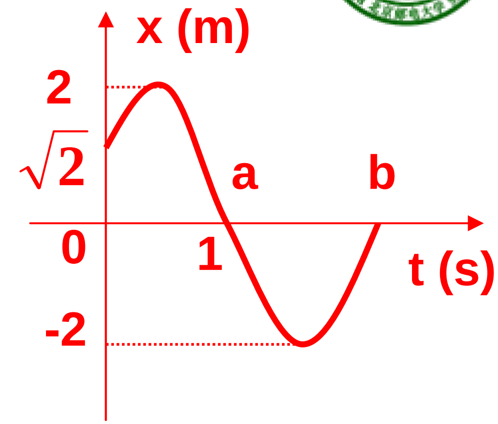
$$\phi = -\frac{\pi}{2}$$



$$\phi = -\frac{\pi}{3}$$

Example

SHM: From given *x-t* graph, find ϕ, θ_a, θ_b, and the angular frequency ω.

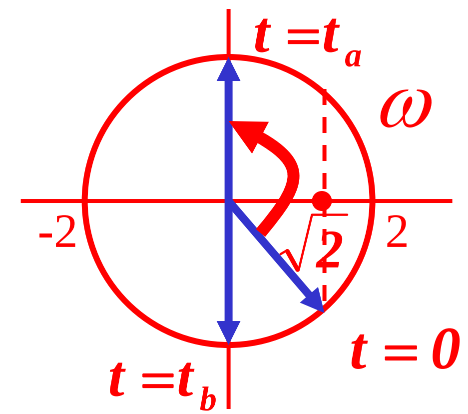


Solution:

From circle of reference

$$\therefore \phi = -\frac{\pi}{4}, \qquad \theta_a = \frac{\pi}{2}, \quad \theta_b = \frac{3\pi}{2}$$

$$\therefore \omega = \frac{\Delta\theta}{\Delta t} = \frac{\theta_a - \phi}{\Delta t} = \frac{\pi/2 - (-\pi/4)}{1} = \frac{3\pi}{4}\,\text{rad/s}$$

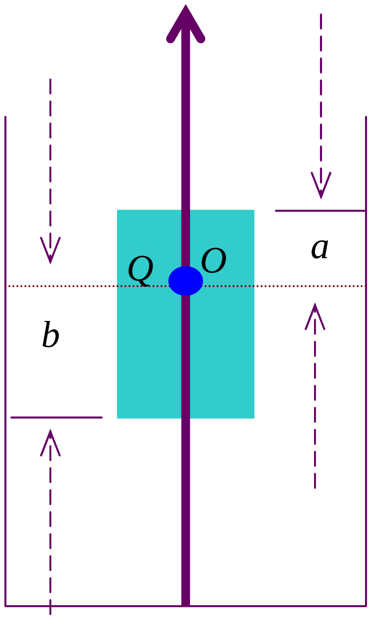


Example

A wooden block floats in water. We press it until its upper surface just under water, and release. Will the motion of the wooden block be SHM?

Example

Solution: **Take the point O at the surface of water to be the origin of *x*-axis. When the block is in equilibrium, the point Q of block coincides with origin point O.**



When block is in equilibrium

$$Sl\rho_{block}g = Sb\rho_{water}g$$

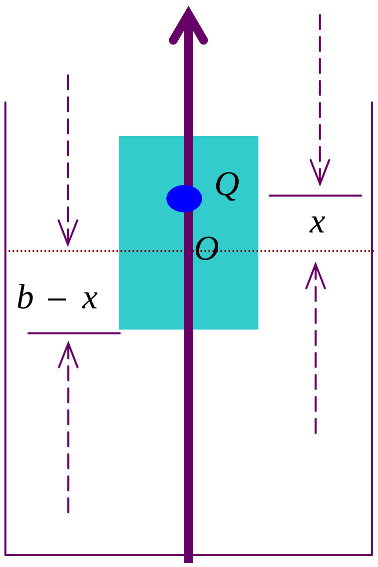
where *S* is the area of block's cross section, and *l=a+b*

The net force: $\sum F = S(b-x)\rho_{water}g - Sl\rho_{block}g$

$$= -Sx\rho_{water}g$$

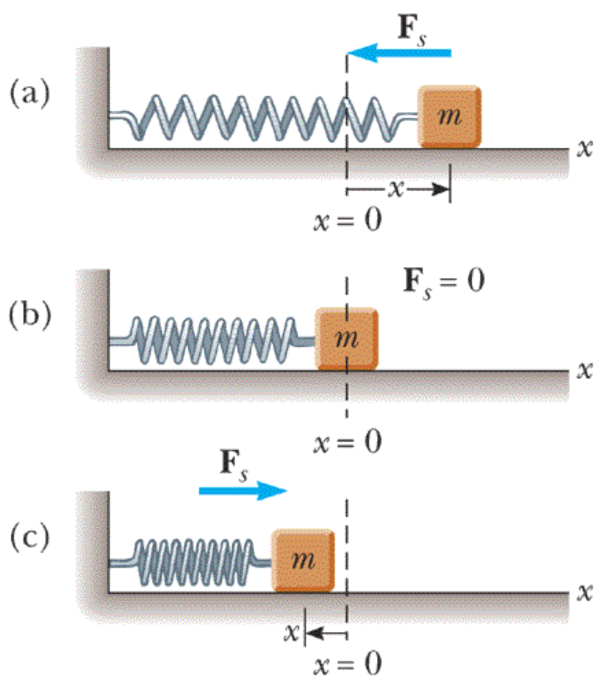
$$-S\rho_{water}gx = (Sl\rho_{block})\frac{d^2x}{dt^2} \quad \Rightarrow \quad \frac{d^2x}{dt^2} + \frac{g}{b}x = 0$$

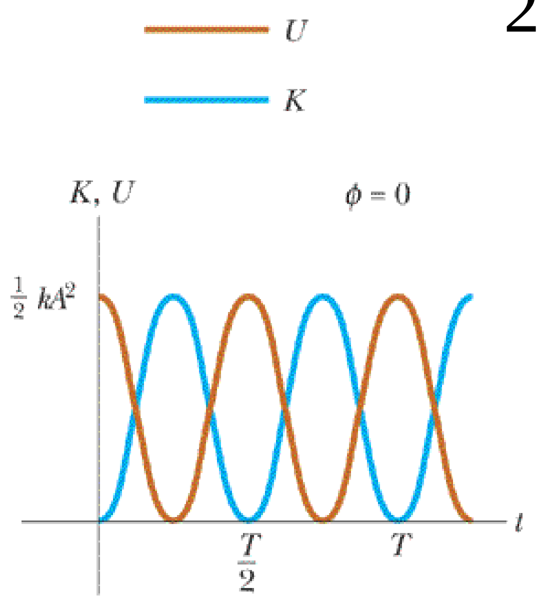
$$x = A\cos\left(\sqrt{\frac{g}{b}}t + \phi\right)$$



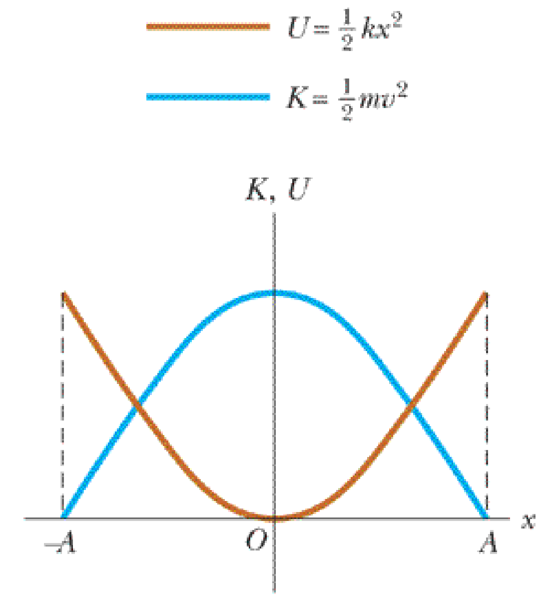
§ 5 Energy in Simple Harmonic Motion (P304)

- **The total mechanical energy for an isolated simple harmonic oscillator**
  - **Kinetic energy:** $$K = \frac{1}{2}mv^2 = \frac{1}{2}m\omega^2 A^2 \sin^2(\omega t + \phi)$$
  - **Potential energy:** $$U = \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \cos^2(\omega t + \phi)$$
  - **Total mechanical energy:** $$E = K + U = \frac{1}{2}kA^2 = \text{constant}$$





(a)

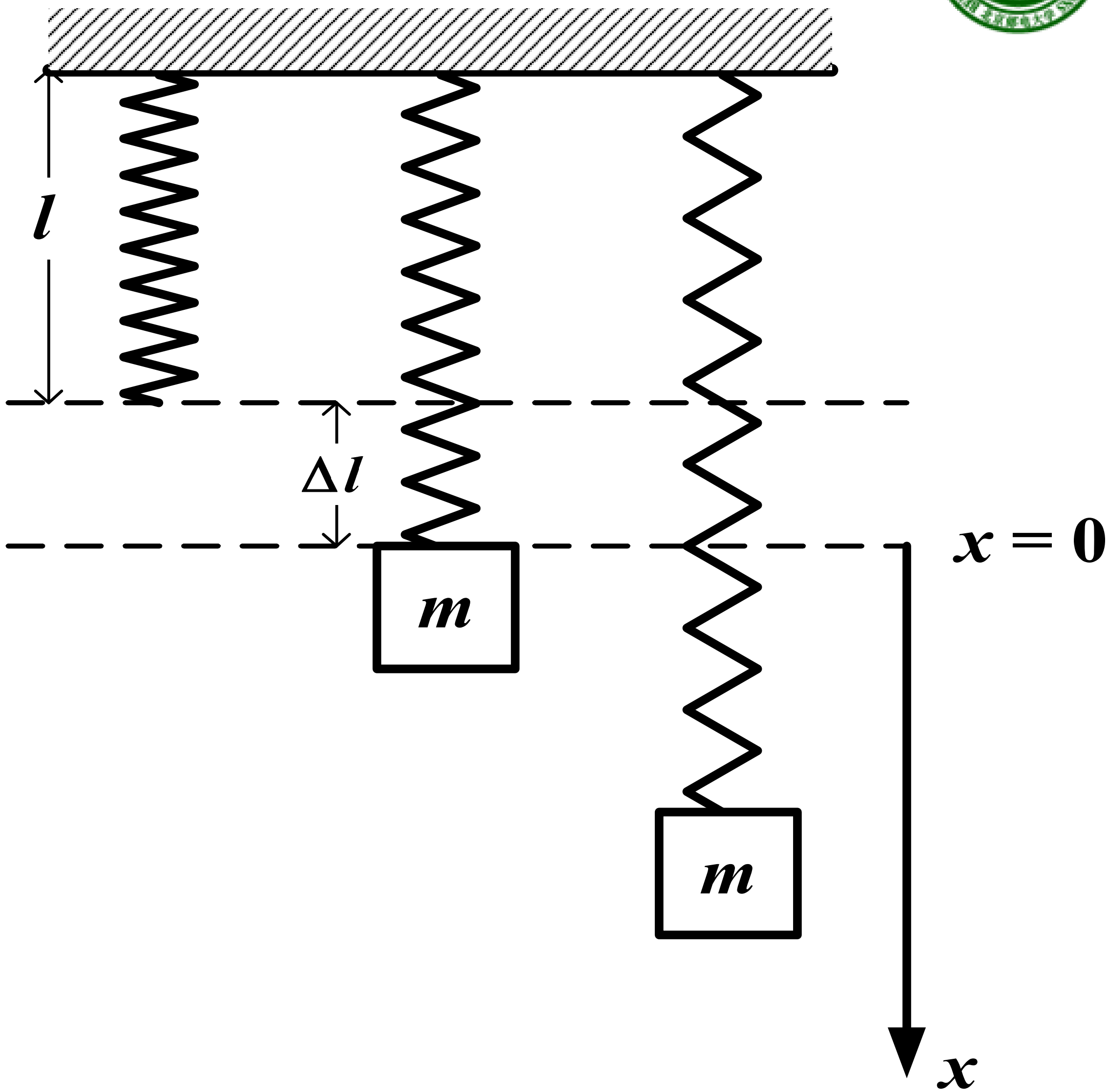


(b)

Example

Vertical SHM:

Suppose we hang a spring with force constant *k* and suspend from it a body with mass *m*. Oscillation will now be vertical. Will it still be SHM?

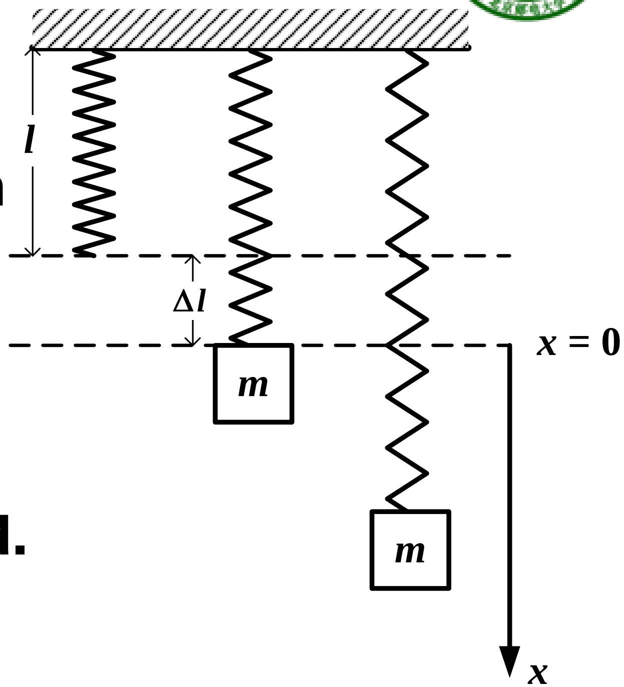


Example

Solution I: by Newton' second law

When the body hangs at rest, in equilibrium

$$k\Delta l = mg$$



Take x=0 to be the equilibrium position, and take the positive x-direction to be downward.

$$F_{net} = -k(x+\Delta l) + mg = -kx - k\Delta l + mg$$

$$= -kx = m\frac{d^2x}{dt^2}$$

The body's motion is still SHM with the angular frequency:

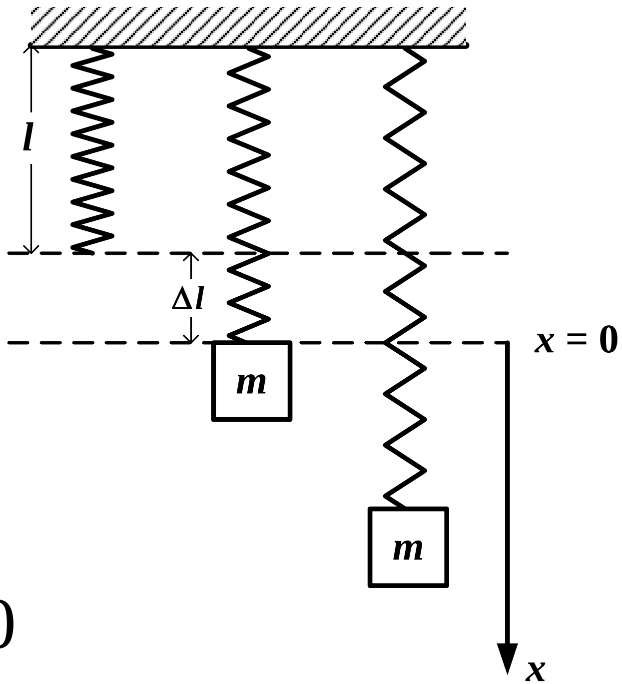
$$\frac{d^2x}{dt^2} + \frac{k}{m}x = \frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$\omega = \sqrt{\frac{k}{m}}$$

Example cont'd

Solution II: by energy analysis

When the body is at the position x, the total mechanical energy is



$$\frac{1}{2}mv^2+\frac{1}{2}k(x+\Delta l)^2-mgx=\text{costant}$$

by derivative on both sides

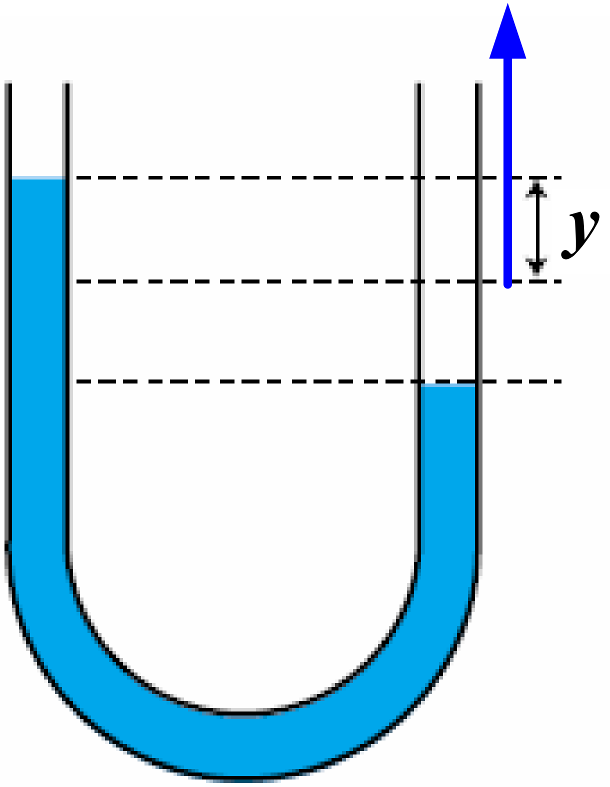
$$mv\frac{dv}{dt}+k(x+\Delta l)\frac{dx}{dt}-mg\frac{dx}{dt}=0$$

$$\frac{dv}{dt}=\frac{d^2x}{dt^2},\qquad \frac{dx}{dt}=v,\qquad m\frac{d^2x}{dt^2}+kx+(k\Delta l-mg)=0$$

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=\frac{d^2x}{dt^2}+\omega^2x=0$$

Example

Liquid in a U-tube: A liquid of density *ρ* is poured into a U-shaped tube with a cross-section of *S*. The total mass of the liquid is *m*. The liquid in the U-tube can undergoes vibration about equilibrium. Find the vibration period of the liquid.



$$T = 2\pi\sqrt{\frac{m}{2\rho S g}}$$

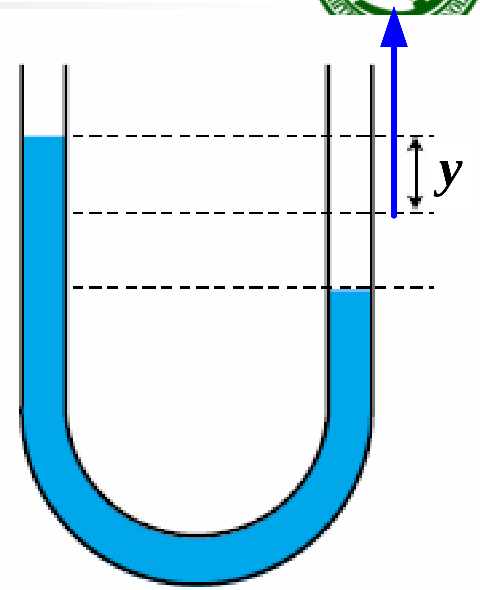
Example

Solution: **The potential energy**

$$U = (\rho g S y) y = \rho S g y^2$$

The kinetic energy:

$$K = \frac{1}{2} m \left(\frac{dy}{dt} \right)^2$$



$$K + U = \frac{1}{2} m \left(\frac{dy}{dt} \right)^2 + \rho S g y^2 = \text{const.}$$

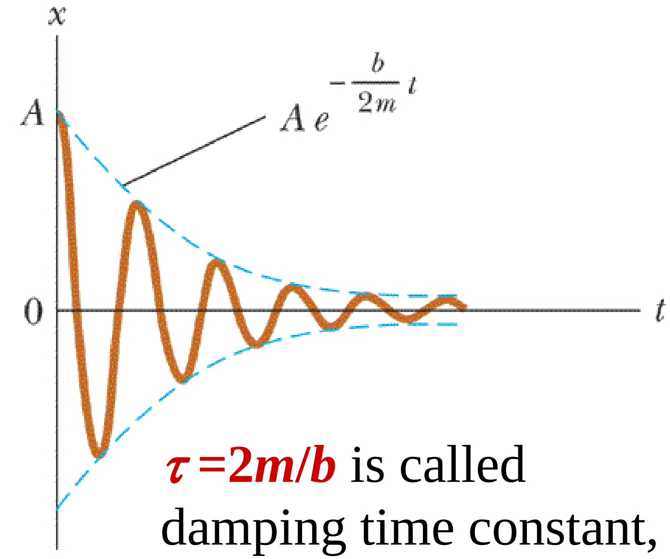
$$m \left(\frac{dy}{dt} \right) \left(\frac{d^2 y}{dt^2} \right) + 2 \rho S g y \left(\frac{dy}{dt} \right) = 0$$

$$\boxed{\frac{d^2 y}{dt^2} + \frac{2 \rho S g}{m} y = 0}$$

$$T = 2\pi \sqrt{\frac{m}{2 \rho S g}}$$

* § 6 Damped Oscillations (P310)

- **The dissipative force causes the decrease in amplitude —— damping, the corresponding motion is called damped oscillation.**



- **Restoring force:** $F_s = -kx$
- **Resistance force:** $R = -bv$
- **Newton's second law:** $\sum F = -kx - bv = ma$

$$\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = 0$$

$\tau = 2m/b$ is called damping time constant, or mean lift time.

- **The solution:**

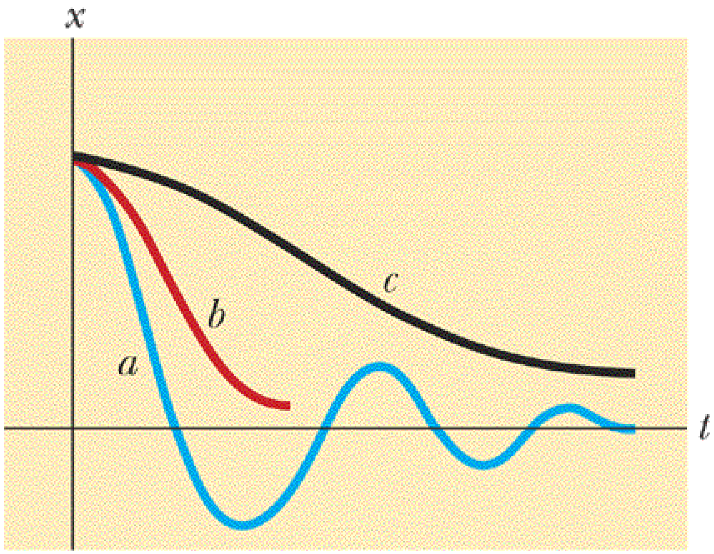
$$x = Ae^{-(b/2m)t}\cos(\omega t + \phi) = A'\cos(\omega t + \phi)$$

$$A' = Ae^{-(b/2m)t}, \qquad \omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2} = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

Damped Oscillations Cont'd

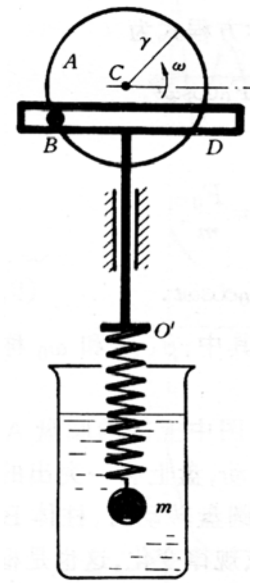
$$x = A'\cos(\omega t + \phi) \qquad \omega = \sqrt{\omega_0^2 - \left(\frac{b}{2m}\right)^2}$$

- When b=0, $\omega = \omega_0 = \sqrt{k/m}$ simple harmonic oscillator
- (a) If $\omega_0^2 > \left(\frac{b}{2m}\right)^2$ the system is underdamped (欠阻尼), oscillating with steadily decreasing amplitude.
- (b) If $\omega_0^2 = \left(\frac{b}{2m}\right)^2$ the system no longer oscillates, and is called critically damped (临界阻尼)
- (c) If $\omega_0^2 < \left(\frac{b}{2m}\right)^2$ the system is overdamped. (过阻尼)



* § 7 Forced Oscillations (P313)

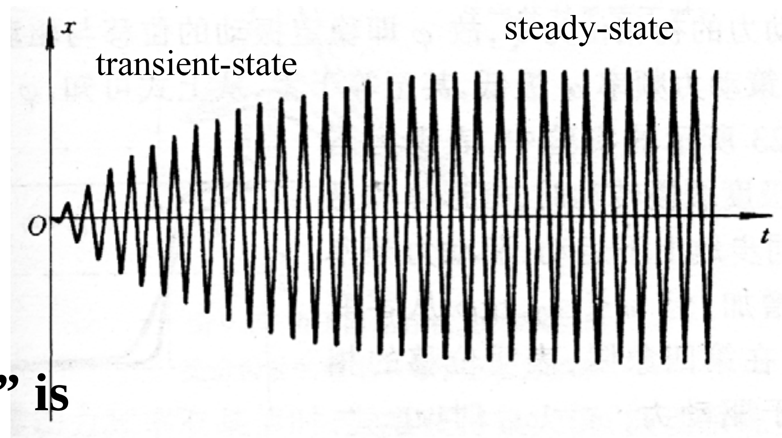
- A forced oscillator is damped oscillator driven by an external force that varies periodically.
  - A sinusoidally varying driving force: $F(t) = F_0 \sin \omega t$
  - Newton's Second Law: $F_0 \sin \omega t - b\dfrac{dx}{dt} - kx = m\dfrac{d^2x}{dt^2}$



$$\boxed{\frac{d^2x}{dt^2} + \frac{b}{m}\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin \omega t}$$

- The solution: $x = A\cos(\omega t + \phi)$

$$A = \frac{F_0/m}{\sqrt{\left(\omega^2 - \omega_0^2\right)^2 + \left(\dfrac{b\omega}{m}\right)^2}}$$



The forced oscillator in its "steady state" is oscillated with the frequency of driven force.

§ 8 Superposition of SHM

- **An object experiences two SHMs simultaneously.**
  - **Two SHMs**

$$x_1 = A_1 \cos(\omega t + \phi_1)$$

$$x_2 = A_2 \cos(\omega t + \phi_2)$$

- **Resultant motion which is superposed by the two SHMs is also a SHM**

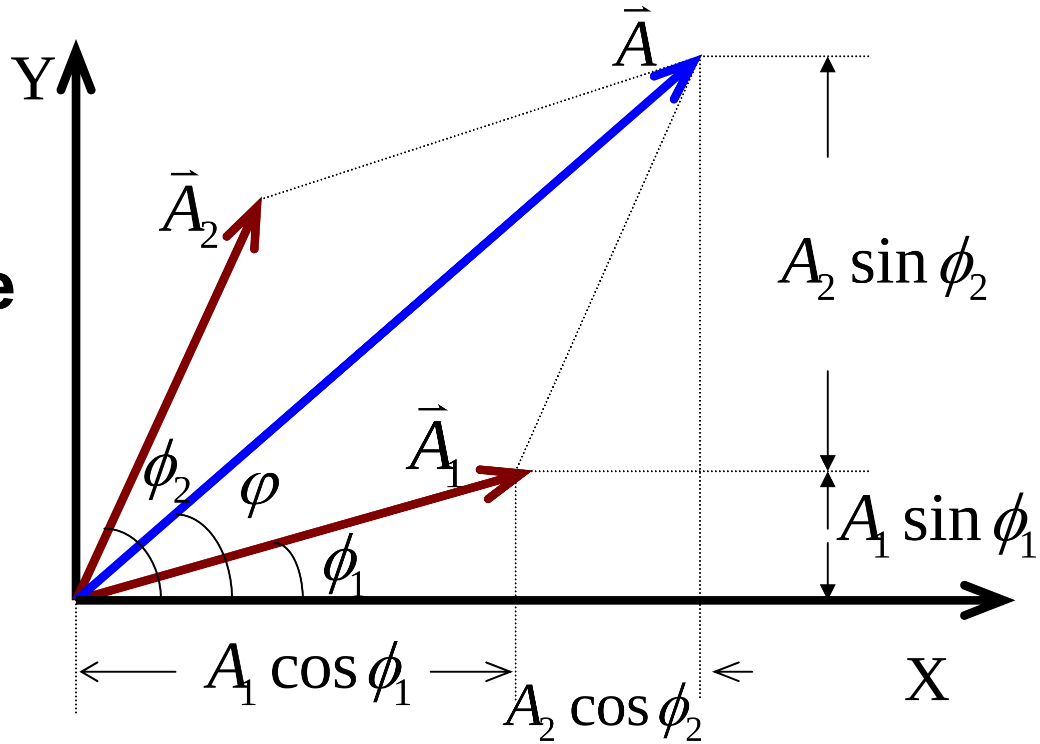
$$x = x_1 + x_2 = A\cos(\omega t + \phi)$$

Resultant Amplitude ?

Resultant Phase angle ?

Superposition of SHMs using phasor diagram

Using Circle of Reference



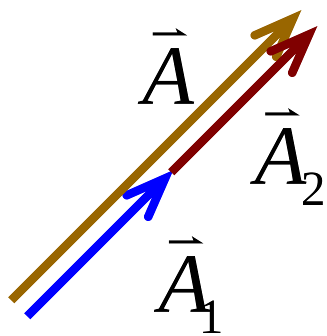
$$A = \sqrt{A_1^2 + A_2^2 + 2A_1A_2\cos(\phi_2 - \phi_1)}$$

$$\varphi = \arctan\frac{A_1\sin\phi_1 + A_2\sin\phi_2}{A_1\cos\phi_1 + A_2\cos\phi_2}$$

Superposition of SHMs under different phase differences

$$A=\sqrt{A_1^2+A_2^2+2A_1A_2\cos\Delta\phi}$$

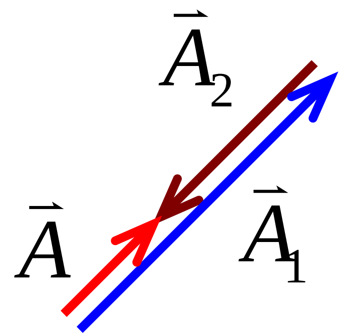
- **The phase difference $\Delta\phi=\phi_2-\phi_1$.**



  - **When $\Delta\phi=\phi_2-\phi_1=2k\pi$, k=0, $\pm$1, $\pm$2, …**

    The two SHMs are in phase, the resultant amplitude take its maximum.

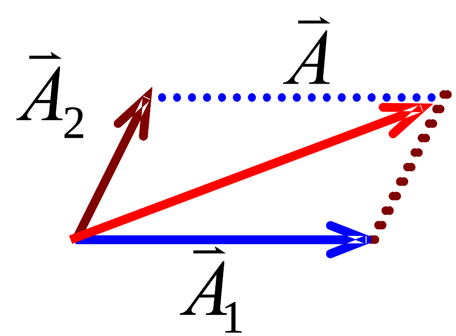
$$A=A_1+A_2$$



  - **When $\Delta\phi=\phi_2-\phi_1=(2k+1)\pi$, k=0, $\pm$1, $\pm$2, …**

    The two SHMs are out of phase, the resultant amplitude take its minimum.

$$A=|A_1-A_2|$$



  - **Generally, $\Delta\phi=\phi_2-\phi_1\neq k\pi$**

$$|A_1-A_2|<A<A_1+A_2$$

Example: $x_1=3\cos(2\pi t+\pi)$cm, $x_2=3\cos(2\pi t+\pi/2)$cm, find the superposition displacement of x_1 and x_2.

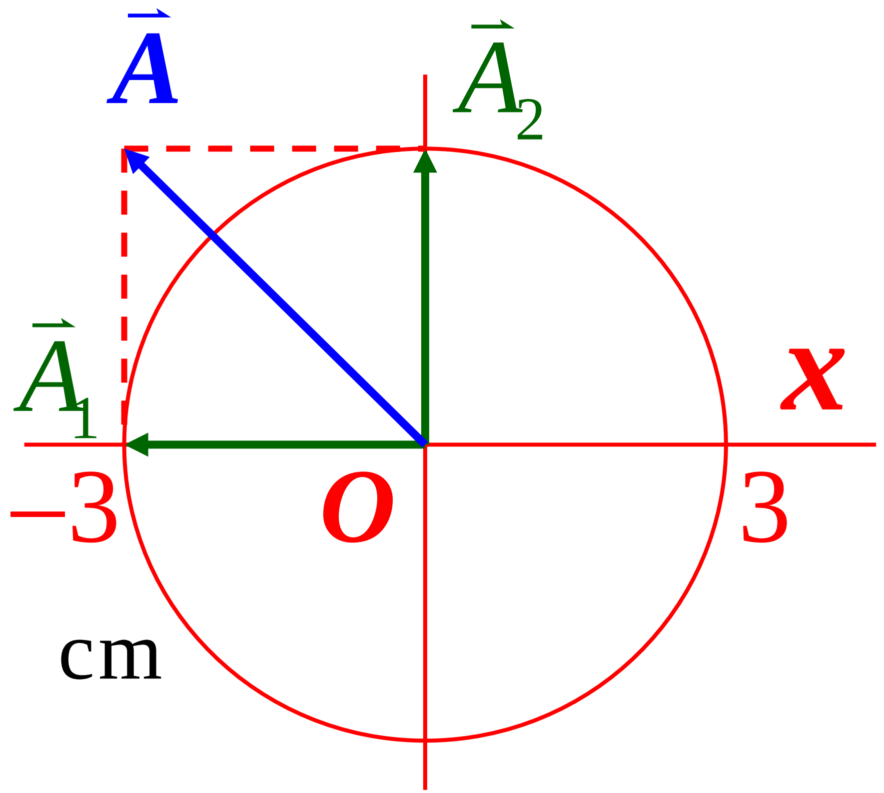
Solution:

Draw a circle of reference,

$$x = x_1 + x_2$$

$$= A\cos(\omega t + \phi)$$

$$= 3\sqrt{2}\cos(2\pi t + \frac{3\pi}{4}) \quad \text{cm}$$



Problem

Ch12: 4, 12, 14; 22,36