§ 1 Displacement Current and the Extended Ampére's Law

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} = \iint_S \vec{j}\cdot d\vec{A}$$

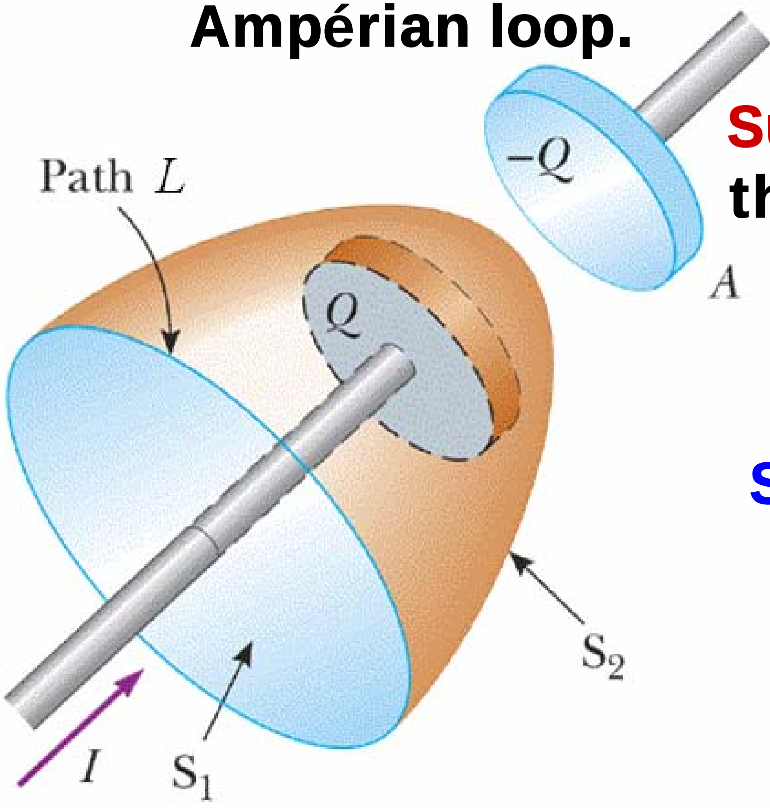
Electric current ⟶ Magnetic field

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Changing magnetic field ⟶ Electric field

Changing electric field $\xrightarrow{?}$ Magnetic field

- The contradiction in applying Ampére's law to a **charging capacitor**
  - Apply Ampére's law to a circular loop that surrounding the wire. Consider two surfaces bounded by the same Ampérian loop.



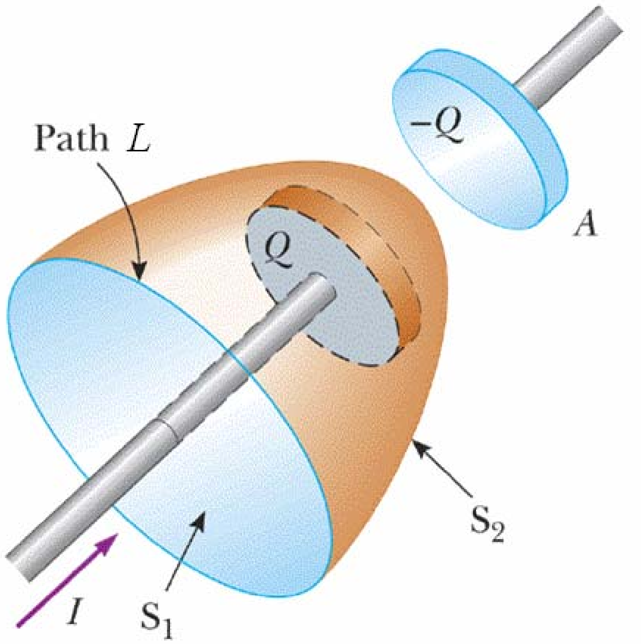
Surface S_1: the circular area in which the conduction current I penetrates.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{encl} = \mu_0 \iint_{S_1} \vec{j}\cdot d\vec{A} = \mu_0 I$$

Surface S_2: the paraboloid passing between the capacitor's plates.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{encl} = \mu_0 \iint_{S_2} \vec{j}\cdot d\vec{A} = 0$$

The Contradiction of Ampére's Law



Surface S_1:

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 \iint_{S_1} \vec{j}\cdot d\vec{A} = \mu_0 I$$

Surface S_2:

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 \iint_{S_2} \vec{j}\cdot d\vec{A} = 0$$

- **Question: Does Ampére's law need to be modified?**
  - **What is wrong with the Ampére's law?**
  - **How to treat the discontinuity of the current?**

  Ampére's law is valid only if the conduction current is continuous in space.

- **How to save Ampére's law from the contradiction?**

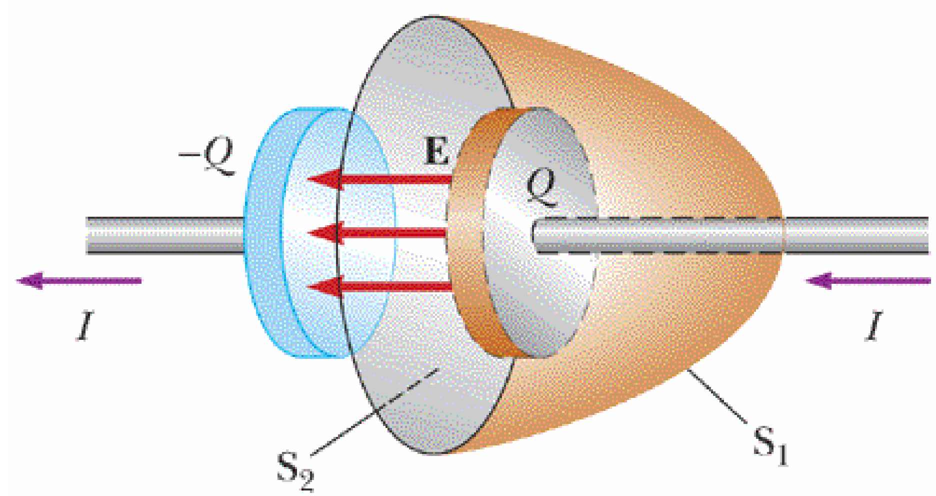
➡ **The contradiction comes from the discontinuity of the conduction current.**

The conduction current I is interrupted in the region between capacitor's two plates, there is also a changing electric field $\vec{E}$ or a changing electric flux Φ_E in this region.

$$I = \frac{dQ}{dt} = \frac{d(\sigma A)}{dt} = \frac{d\sigma}{dt} A$$

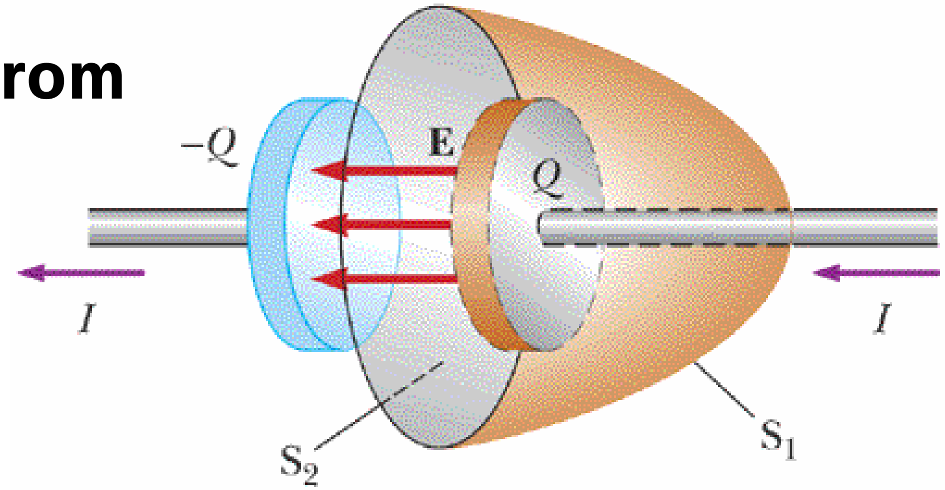
$$= \frac{d}{dt}(\varepsilon_0 E) A = \varepsilon_0 \frac{d}{dt}(EA)$$

$$= \varepsilon_0 \frac{d\Phi_E}{dt}$$



- **How to save Ampére's law from the contradiction?**

$$I = \frac{dQ}{dt} = \varepsilon_0 \frac{d\Phi_E}{dt}$$



- **To keep the continuity of the current, Maxwell made a postulation that there exists a fictitious current in the region between the plates, called the displacement current I_d.**

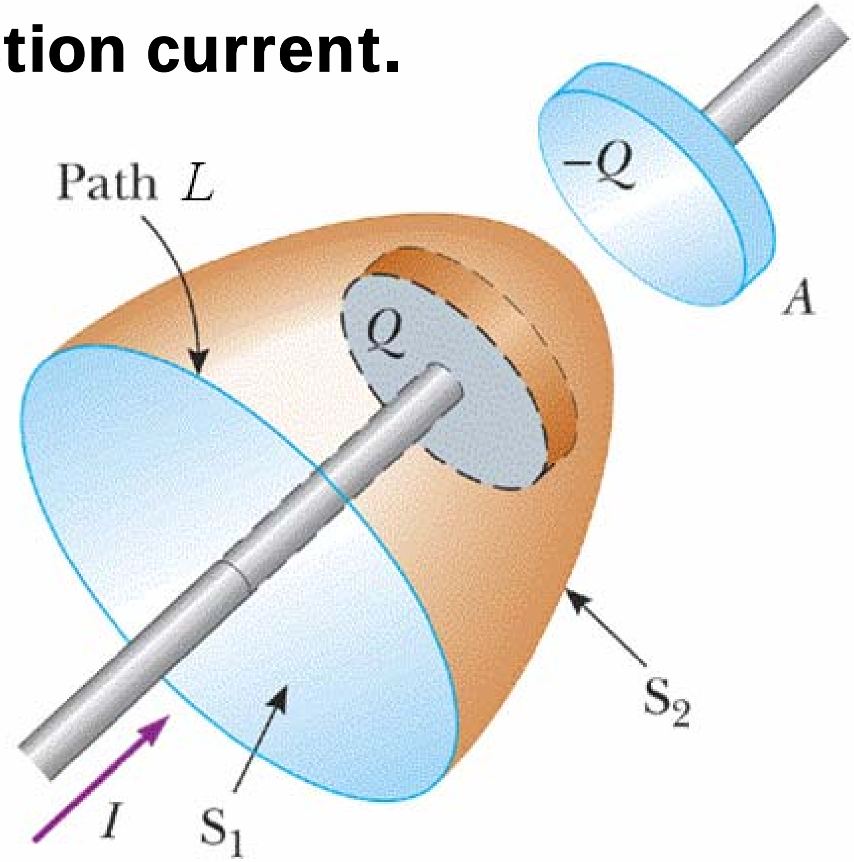
$$\boxed{I_d = \varepsilon_0 \frac{d\Phi_E}{dt}} = \varepsilon_0 \frac{d}{dt} \iint_S \vec{E} \cdot d\vec{A} = \iint_S \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \cdot d\vec{A}$$

- **Displacement current density:** $\vec{j}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t}, \quad I_d = \iint_S \vec{j}_d \cdot d\vec{A}$

- **Extended Ampére's law or Ampére-Maxwell law:**
  - **The postulation of displacement current solved the discontinuity of the conduction current.**

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 (I_c + I_d)_{\text{encl}}$$

$$= \mu_0 I_{\text{encl}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$



Extended Ampére's law

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}}$$

Electric current → Magnetic field

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Changing magnetic field → Electric field

Changing electric field → Magnetic field

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 (I_d)_{\text{encl}} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

Extended Ampére's law

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0(I_c + I_d)_{\text{encl}} = \mu_0 I_c + \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$

- **The displacement current is also a source of magnetic field**

Conduction current I_c

Displacement current I_d

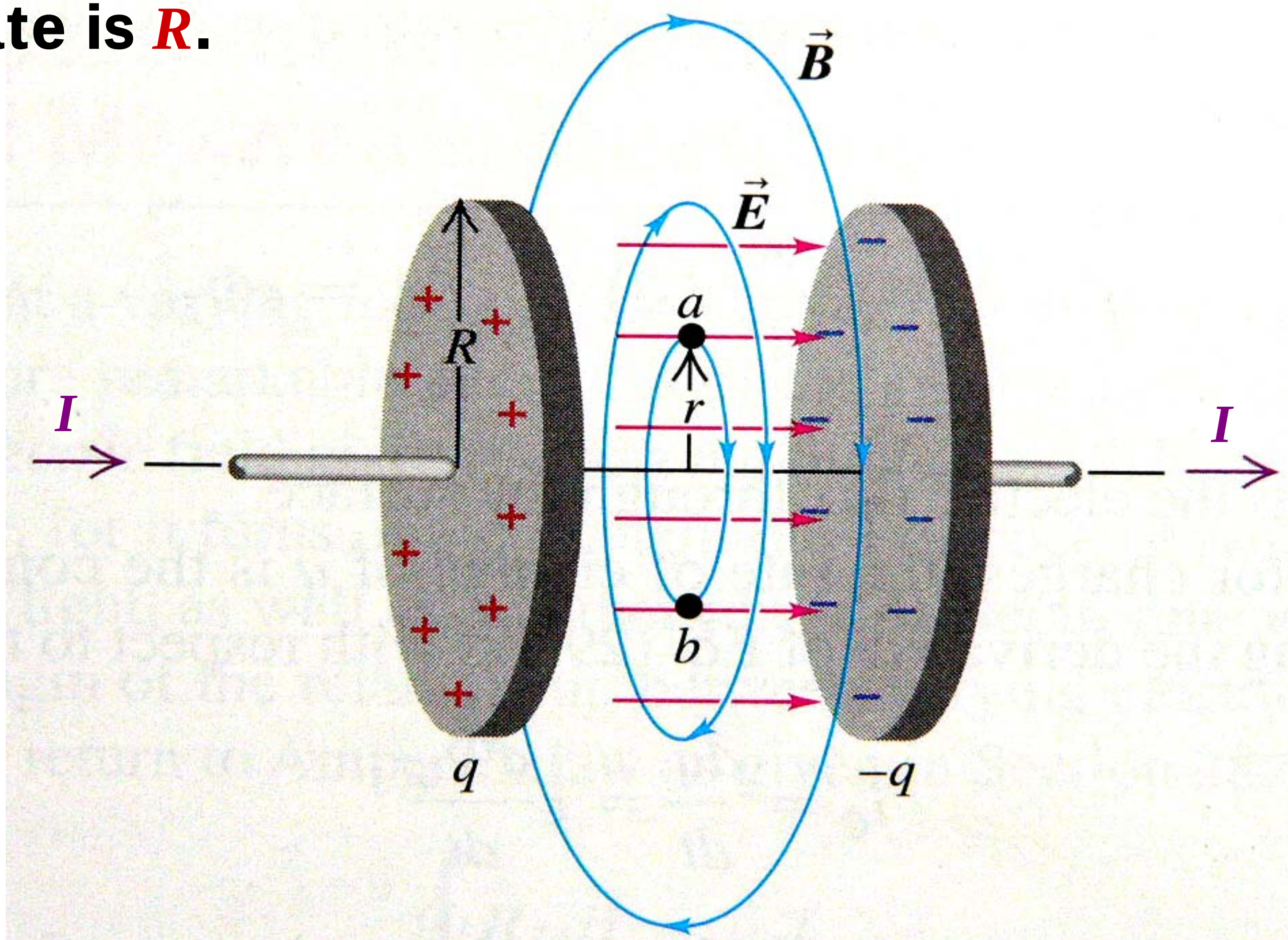
or changing electric field

$\rightarrow$ **Magnetic field**

- **Magnetic field are produced both by conduction current and by changing electric field.**

Example

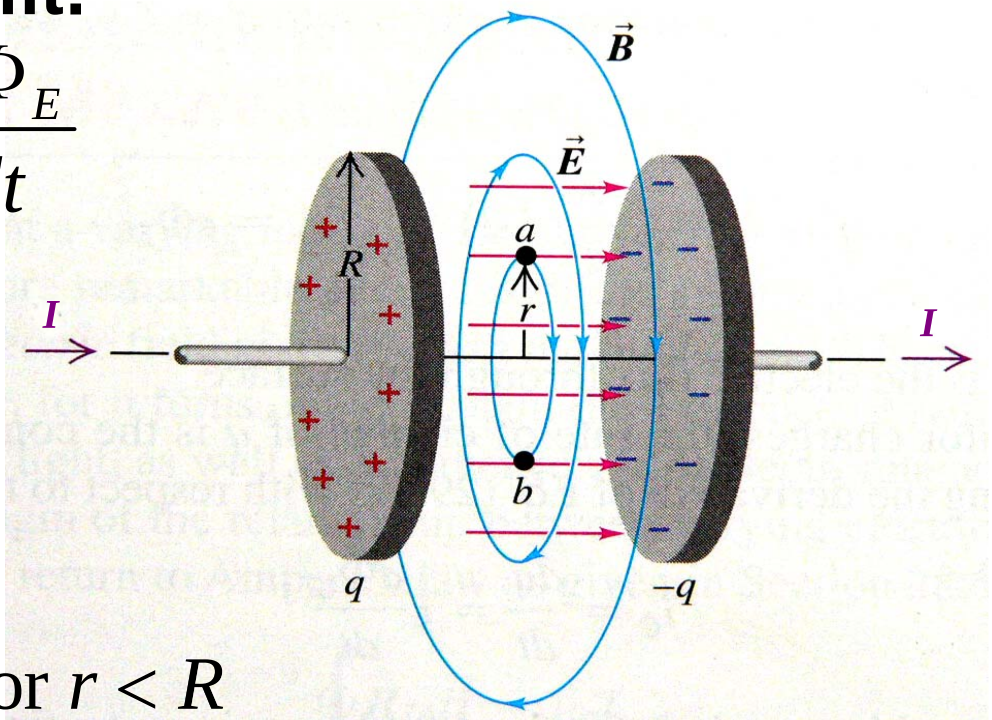
Calculate the magnetic field in the region between the two capacitor's plates while the capacitor is charging with a increasing current *I*. The radius of plate is *R*.



Example

Solution: For a point a distance *r* from the center, we apply Ampére's law to a circular path of radius *r* passing through the point.

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0\varepsilon_0\frac{d\Phi_E}{dt}$$



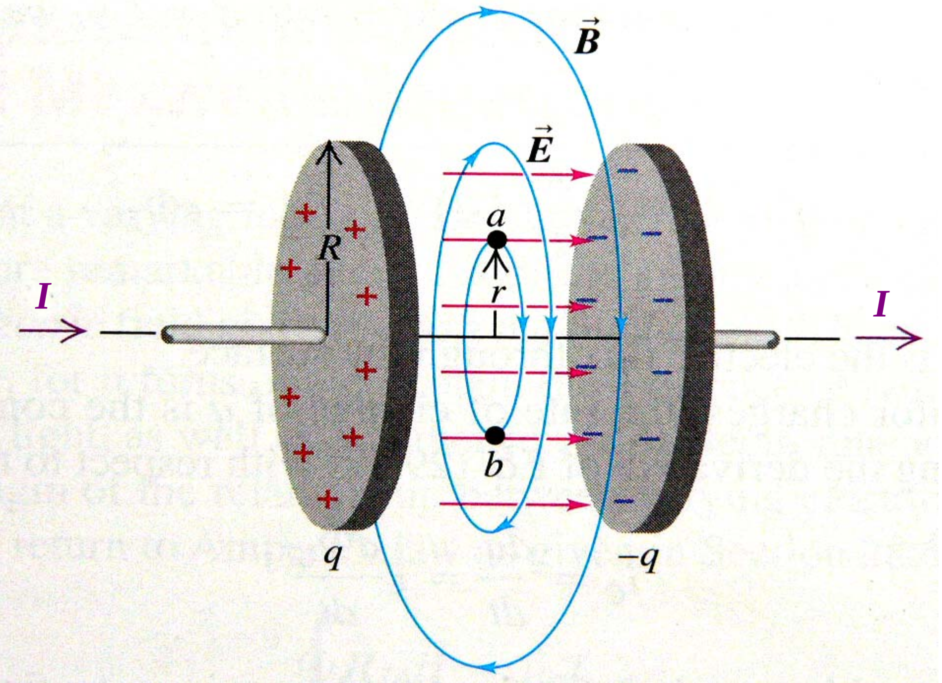
The electric field between the plates:

$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{\varepsilon_0 A}$$

$$\Phi_E = \begin{cases} E\pi r^2 = \dfrac{1}{\varepsilon_0}\dfrac{r^2}{R^2}Q, & \text{for } r < R \\ EA = \dfrac{Q}{\varepsilon_0}, & \text{for } r > R \end{cases}$$

Example

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0\varepsilon_0 \frac{d\Phi_E}{dt}$$



$$\Phi_E = \begin{cases} E\pi r^2 = \dfrac{1}{\varepsilon_0}\dfrac{r^2}{R^2}Q & \text{for } r < R \\ EA = \dfrac{Q}{\varepsilon_0} & \text{for } r > R \end{cases}$$

For $r < R$:

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 \frac{r^2}{R^2}\frac{dQ}{dt} = \mu_0 \frac{r^2}{R^2} I, \qquad B = \frac{\mu_0 I}{2\pi}\frac{r}{R^2}$$

For $r > R$:

$$\oint_L \vec{B}\cdot d\vec{s} = B(2\pi r) = \mu_0 \frac{dQ}{dt} = \mu_0 I, \qquad B = \frac{\mu_0 I}{2\pi r}$$

Example

Calculate the displacement current I_D between the square plates, 3.8 cm on a side, of a capacitor if the electric field is changing at a rate of 2.0×10^6 V/m·s.

(The permittivity of free space is $8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2}$.)

Solution:

The displacement current is

$$I_D = \epsilon_0 A\,(dE/dt) = (8.85\times10^{-12}\ \mathrm{C^2/N\cdot m^2})(0.038\ \mathrm{m})^2(2.0\times10^6\ \mathrm{V/m\cdot s})$$

$$= \boxed{2.6\times10^{-8}\ \mathrm{A.}}$$

§ 2 Maxwell's Equations

$$\oiint_S \vec{E}\cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$$

Gauss's law for electricity

$$\oiint_S \vec{B}\cdot d\vec{A} = 0$$

Gauss's law for magnetism

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

Faraday's law of induction

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0 \frac{d\Phi_E}{dt} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0 \iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

Ampére-Maxwell law

Maxwell's equations and Lorentz force give the fundamental relations of electromagnetism! They are fundamental in the sense that Newton's three laws are for mechanics.

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

Lorentz force

$$\oiint_S \vec{E}\cdot d\vec{A} = \frac{q_{\text{encl}}}{\varepsilon_0}$$

Charged particles create an electric field (electrostatic).

$$\oint_L \vec{E}\cdot d\vec{s} = -\frac{d\Phi_B}{dt} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

An electric field (non-electrostatic) can also be created by a changing magnetic field.

$$\oiint_S \vec{B}\cdot d\vec{A} = 0$$

There are no magnetic monopoles.

$$\oint_L \vec{B}\cdot d\vec{s} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0\frac{d\Phi_E}{dt} = \mu_0 I_{\text{encl}} + \varepsilon_0\mu_0\iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

A magnetic field can be created either by currents or by a changing electric field.

§ 3 Electromagnetic Waves

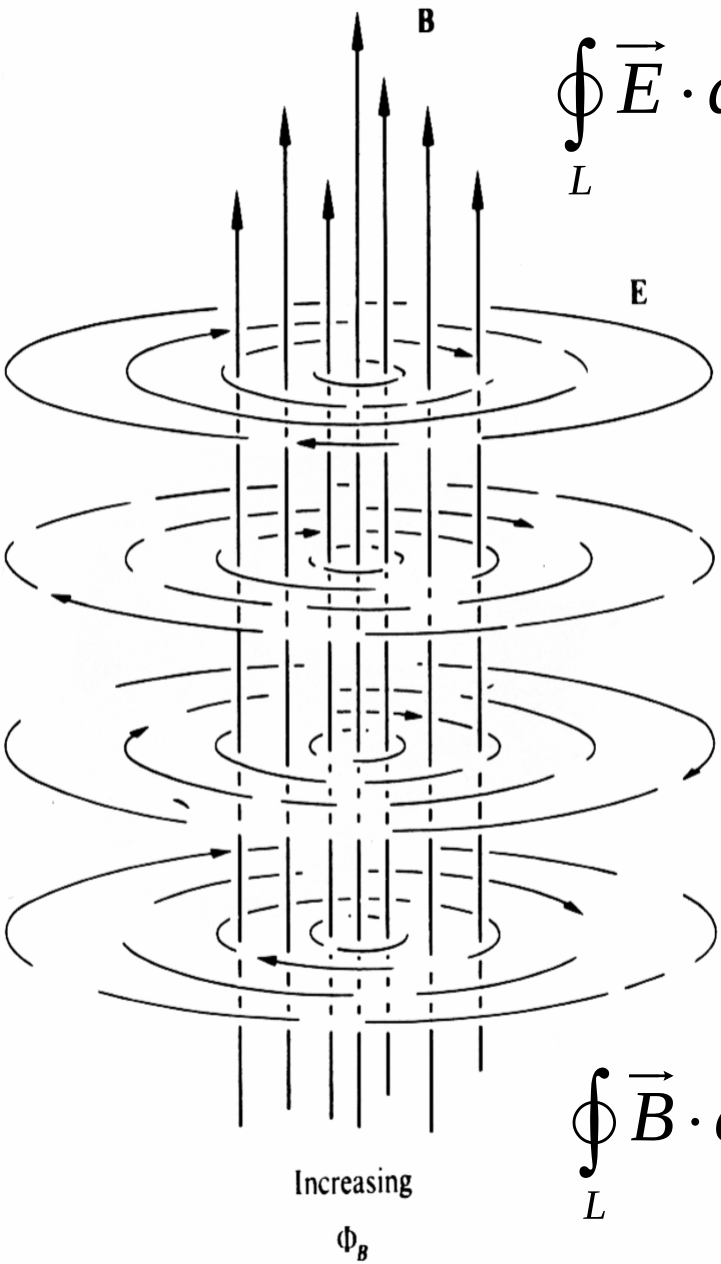
- The relationship between electric and magnetic field in **empty space**.

$$\oint_L \vec{E}\cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}, \qquad \oint_L \vec{B}\cdot d\vec{s} = \varepsilon_0\mu_0\iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

- A time varying magnetic field induces a electric field in neighboring regions;
- A time varying electric field induces a magnetic field in neighboring regions.

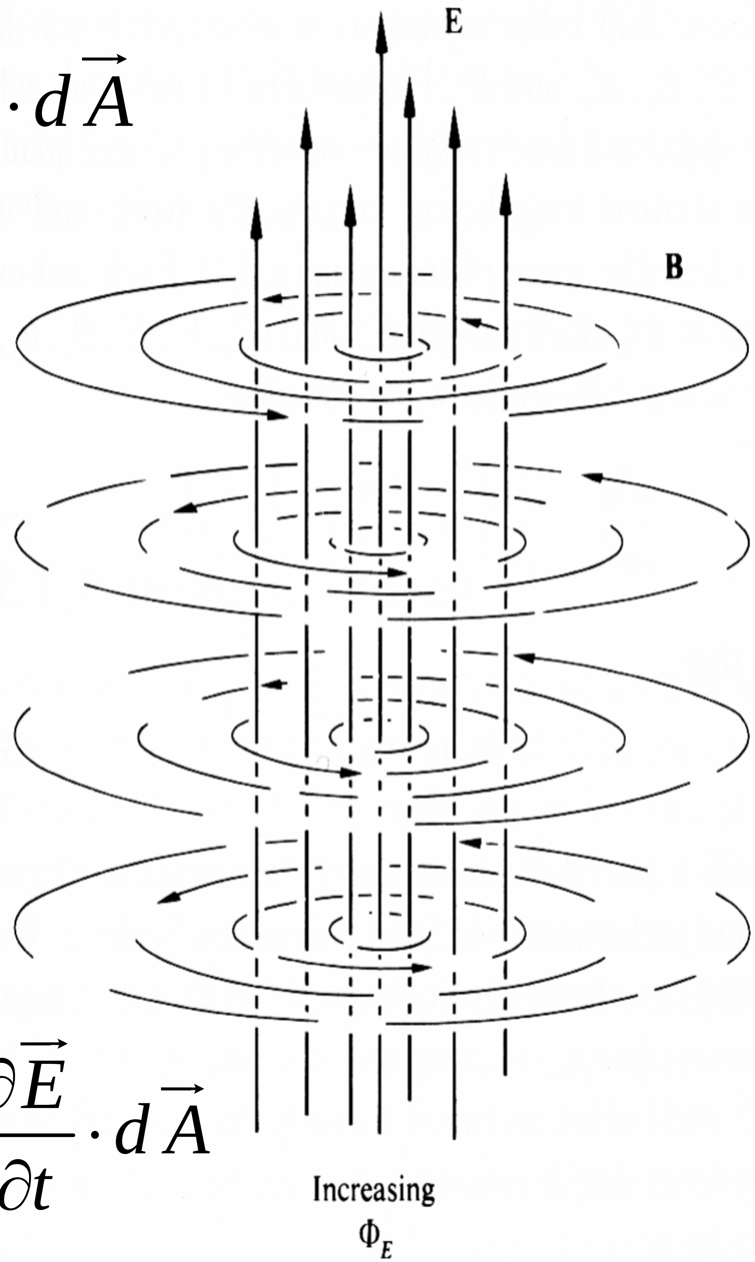
These relationships predicts the existence of **electromagnetic waves** consisting of time-varying electric and magnetic fields that travel from one region of space to another, even if no charge or current are present in space.

relationship between electric and magnetic field

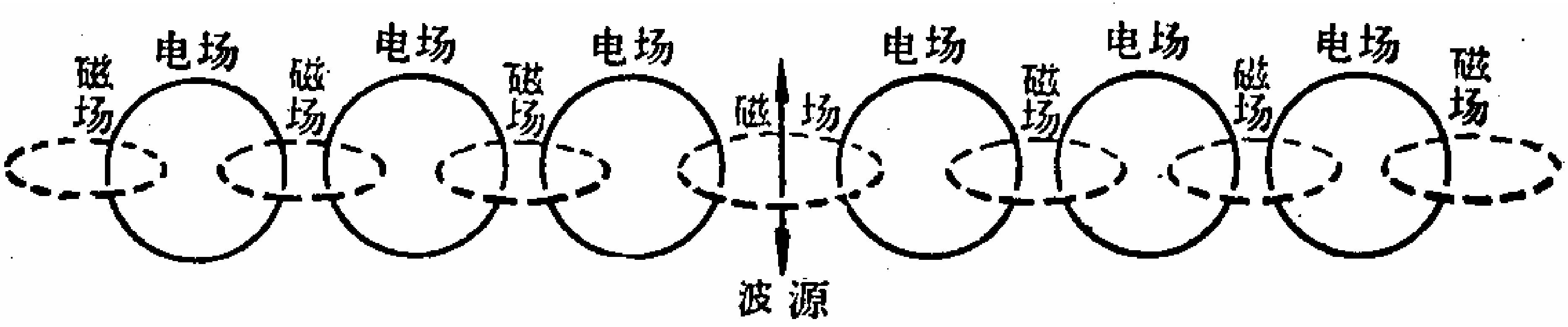


$$\oint_L \vec{E}\cdot d\vec{s} = -\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{A}$$

$$\oint_L \vec{B}\cdot d\vec{s} = \varepsilon_0\mu_0 \iint_S \frac{\partial \vec{E}}{\partial t}\cdot d\vec{A}$$

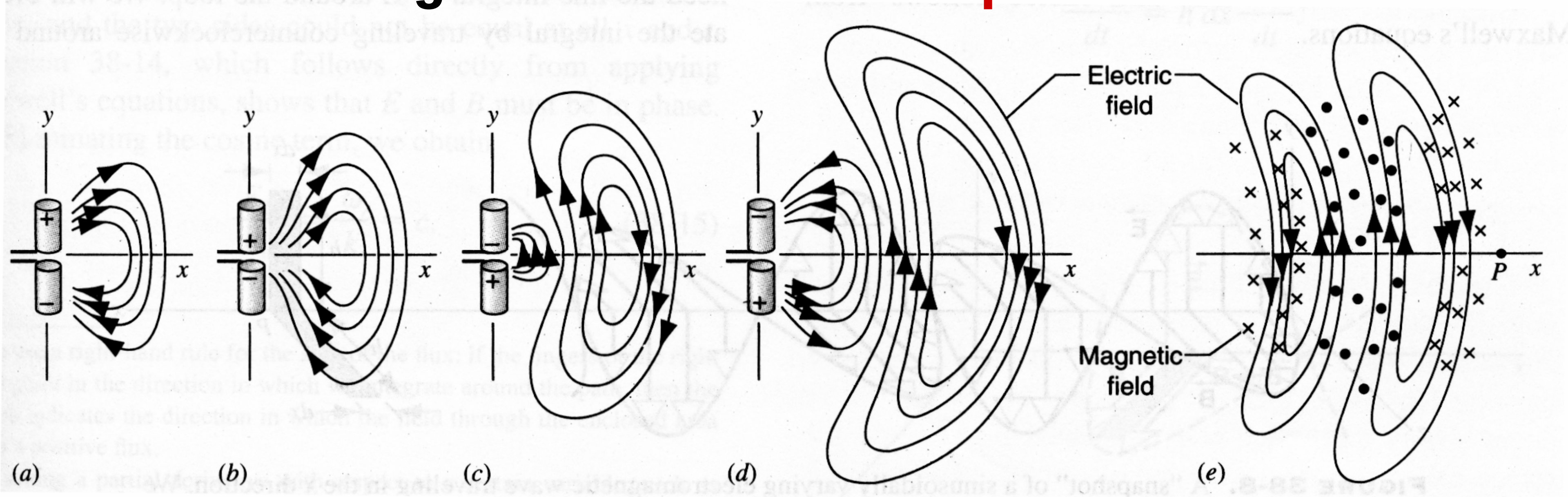


- **The mechanism for maintaining the propagation of the electromagnetic wave.**
  - **Unlike mechanical waves, which need a medium such as water or air to transit a wave, electromagnetic waves require no medium. The changing electric and magnetic fields create each other to maintain the propagation of the waves.**
  - **A exhibition map (not real) for propagation of electromagnetic waves**



The Propagation of the Electromagnetic Wave

- **The mechanism for maintaining the propagation of the electromagnetic wave.**
  - **The real stages in the emission of an electromagnetic wave from a dipole antenna**

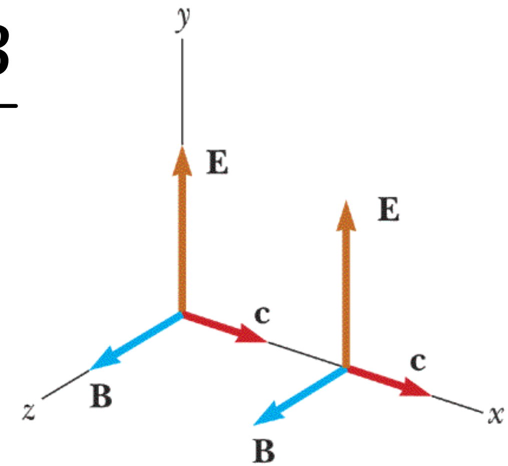


The important features of electromagnetic waves

- **The wave equation:**

  From Maxwell's equations, we can obtain the wave equation for a wave which propagates in x-direction

$$\frac{\partial^2 E}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2}, \qquad \frac{\partial^2 B}{\partial x^2} = \varepsilon_0 \mu_0 \frac{\partial^2 B}{\partial t^2}$$

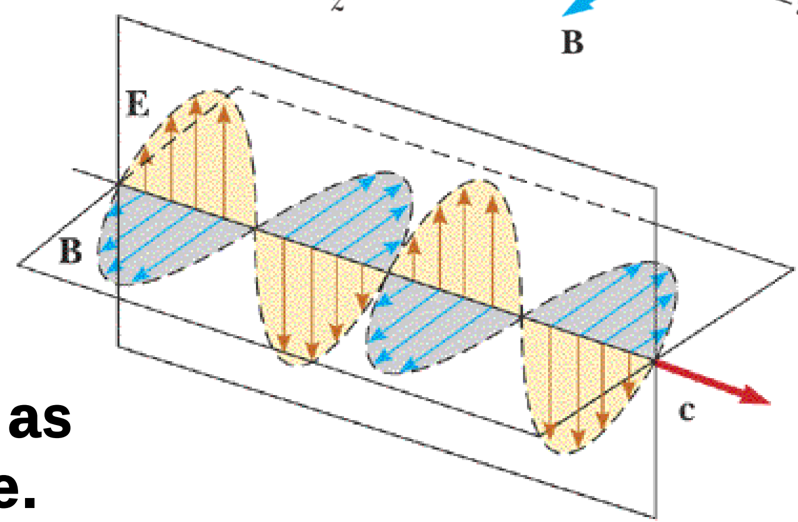


- **The wave speed:**

  Generally, the wave equation

$$\frac{\partial^2 \Psi(x,t)}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \Psi(x,t)}{\partial t^2}$$

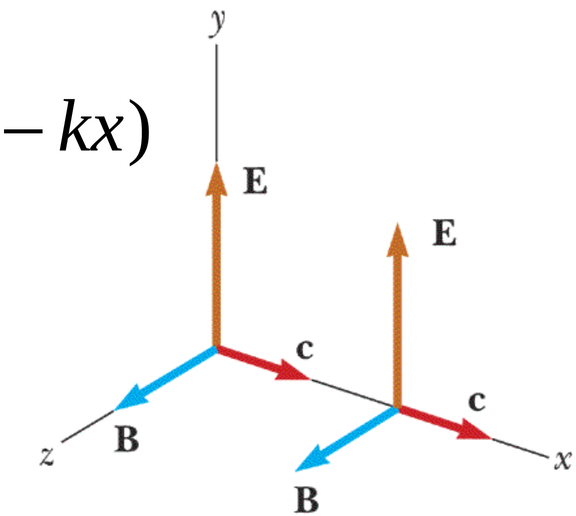
$$\boxed{v = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = 2.997 \times 10^8 \text{ m/s} = c}$$



This speed is precisely the same as the speed of light in empty space.

The important features of electromagnetic waves

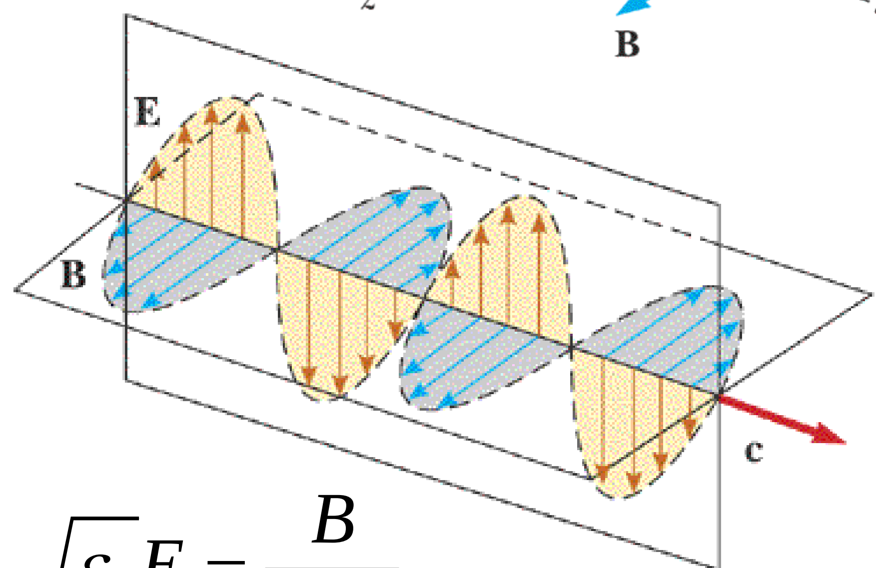
- **The sinusoidal plane wave is the simplest solution of the wave equations**



$$E = E_{\max}\cos(\omega t - kx), \qquad B = B_{\max}\cos(\omega t - kx)$$

- **The wave is transverse.**

Both $\vec{E}$ and $\vec{B}$ are perpendicular to each other, and to the direction of propagation. The direction of propagation is $\vec{E}\times\vec{B}$



- **$\vec{E}$ and $\vec{B}$ are in phase, and has a definite ratio**

$$\frac{E}{B} = \frac{E_{\max}}{B_{\max}} = c$$

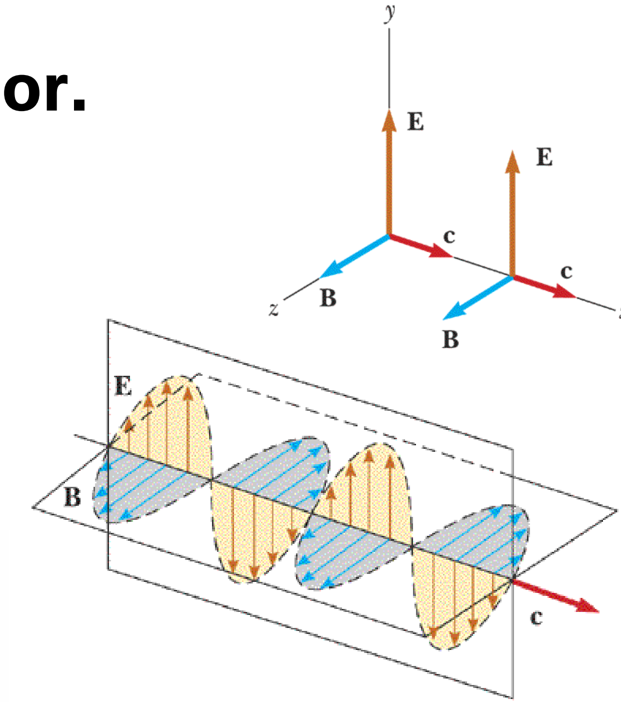
$$\boxed{E = cB}$$

$$\sqrt{\varepsilon_0}\,E = \frac{B}{\sqrt{\mu_0}}$$

- **Poynting vector**: **energy flow vector.**

The total energy density:

$$u = u_E + u_B = \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0} = \frac{EB}{\mu_0 c}$$



The energy current density:

$$S = uc = \frac{EB}{\mu_0}$$

$$\boxed{\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}}$$