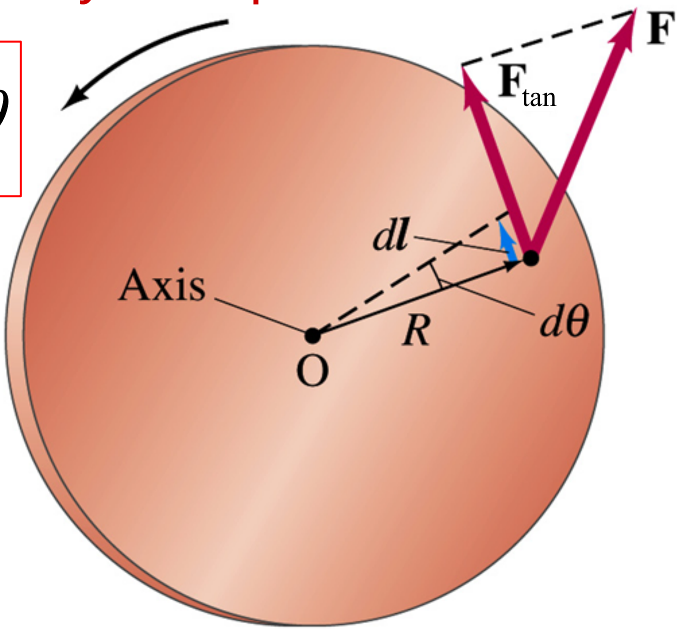
§ 5 Work-Energy Theorem for a Rigid Body

- Work done by a torque
  - For a fixed axis rotation of a rigid body, the work done by a force can appear in the form of torque —— work done by a torque.

$$W=\int_1^2 \vec{F}\cdot d\vec{l}=\int_1^2 F_{\tan}dl=\int_1^2 F_{\tan}Rd\theta=\int_{\theta_1}^{\theta_2}\tau d\theta$$



- The Power of a torque

$$P=\frac{dW}{dt}=\tau\frac{d\theta}{dt}=\tau\omega$$

- Rotational Kinetic Energy
  - For a fixed axis rotation of a rigid body, the kinetic energy can appear in another form:

$$K=\sum_i\left(\frac{1}{2}m_i v_i^2\right)=\sum_i\left(\frac{1}{2}m_i R_i^2\omega^2\right)=\frac{1}{2}\sum_i\left(m_i R_i^2\right)\omega^2=\frac{1}{2}I\omega^2$$

Work-Energy Theorem for a Rigid Body

- Work-kinetic energy theorem for a body rotating about a fixed axis
  - Starting from the rotational form of Newton's II law.

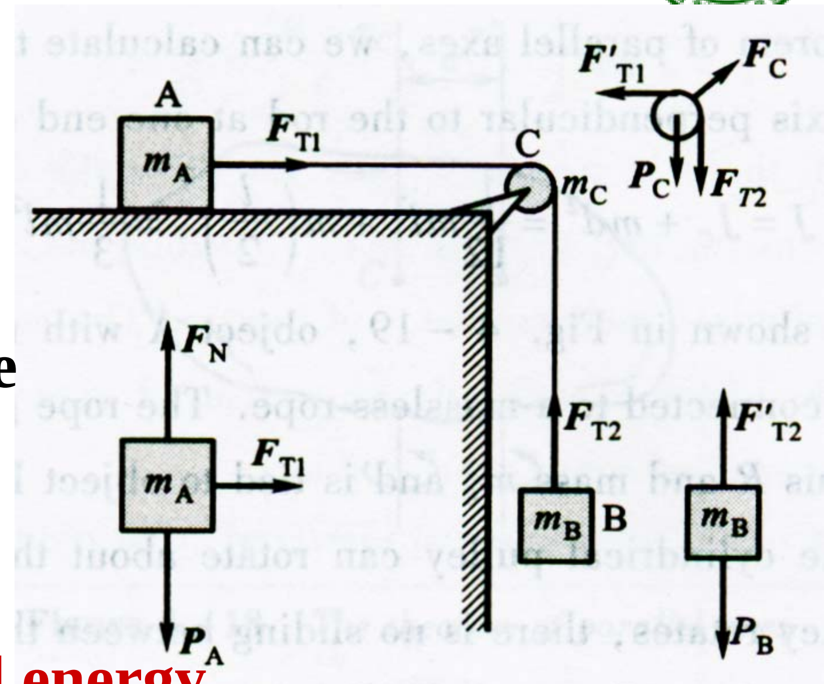
$$\tau_{\text{net}} = I\alpha = I\frac{d\omega}{dt} = I\frac{d\omega}{d\theta}\frac{d\theta}{dt} = I\omega\frac{d\omega}{d\theta}$$

$$\boxed{W_{\text{net}} = \int_{\theta_1}^{\theta_2} \tau_{\text{net}} d\theta = \int_{\omega_1}^{\omega_2} I\omega d\omega = \frac{1}{2}I\omega_2^2 - \frac{1}{2}I\omega_1^2}$$

➢**The work done in rotating a body through an angle $\theta_2 - \theta_1$ is equal to the change in rotational kinetic energy of the body.**

Example

Two blocks of masses m_A and m_B are connected by a light cord running over a pulley. The pulley are considered as a uniform cylindrical disk of mass m_C and radius R. There is no sliding between the pulley and the cord. Find the acceleration of two blocks.

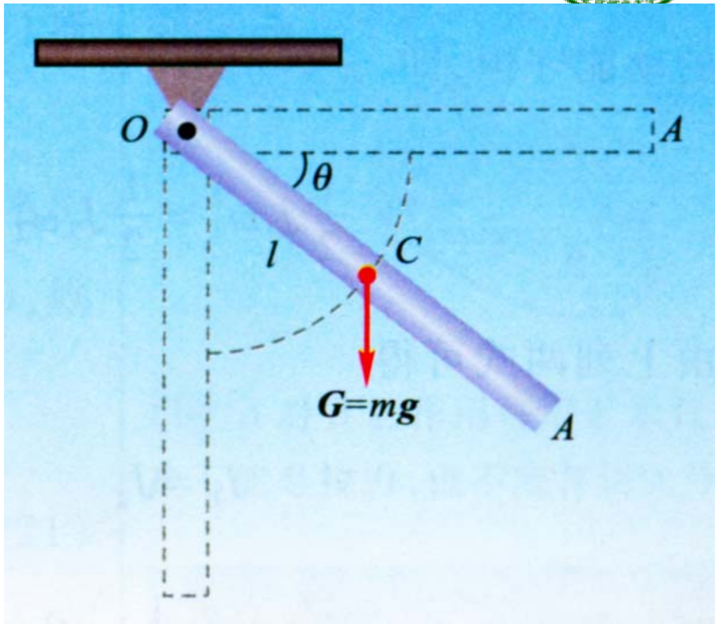


Solution (II): **conservation of mechanical energy**

$$0 = -m_B gh + \frac{1}{2} m_A \upsilon_A^2 + \frac{1}{2} m_B \upsilon_B^2 + \frac{1}{2} I_C \omega_C^2 \ , \quad I_C = \frac{1}{2} m_C R^2, \ \upsilon_A = \upsilon_B = \omega R$$

$$\omega = \frac{1}{R}\sqrt{\frac{2m_B gh}{m_A + m_B + \frac{1}{2} m_C}}, \qquad a = \frac{\mathrm{d}\upsilon}{\mathrm{d}t} = \frac{\mathrm{d}(\omega R)}{\mathrm{d}t} = \frac{m_B g}{m_A + m_B + \frac{1}{2} m_C}$$

A uniform rod of mass *m* and length *l* can pivot freely (no friction on the pivot) about a hinge to the ceiling. The rod is held horizontally and released. Determine the angular acceleration and angular velocity of the rod as the function of θ.



Solution:

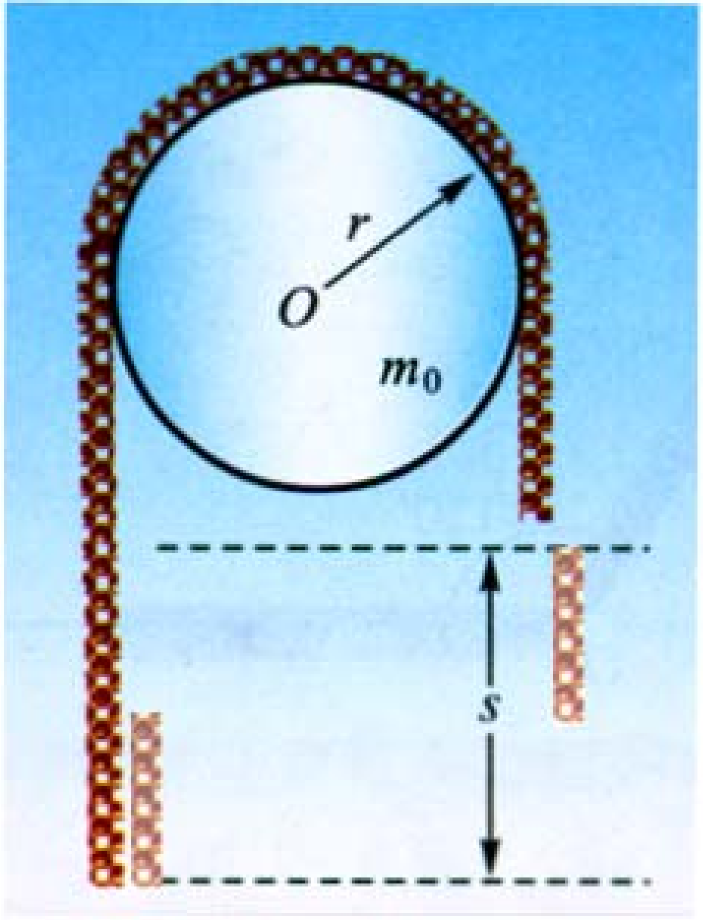
conservation of mechanical energy

$$0=\frac{1}{2}\left(\frac{1}{3}ml^2\right)\omega^2+\left(-mg\frac{l}{2}\sin\theta\right),\qquad \omega=\sqrt{\frac{3g}{l}\sin\theta}$$

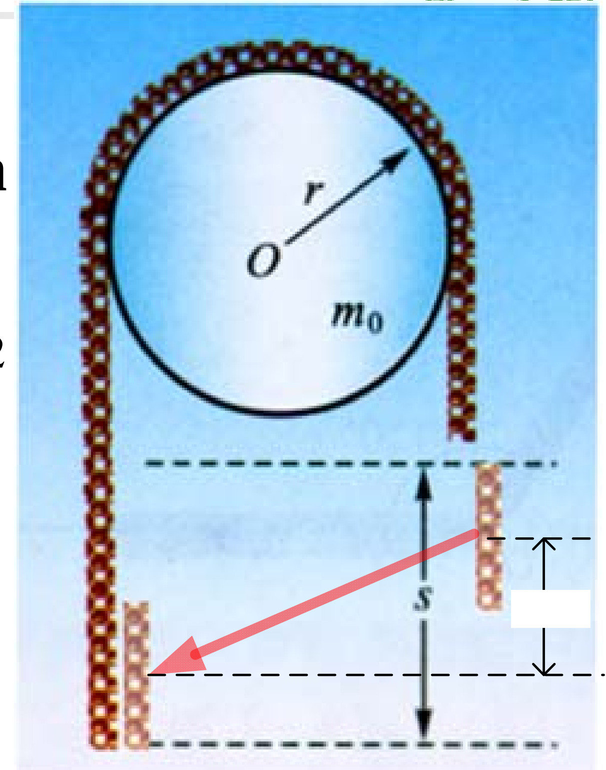
$$\alpha=\frac{d\omega}{dt}=\frac{d}{d\theta}\left(\sqrt{\frac{3g}{l}\sin\theta}\right)\frac{d\theta}{dt}=\sqrt{\frac{3g}{l}}\frac{\cos\theta}{2\sqrt{\sin\theta}}\sqrt{\frac{3g}{l}\sin\theta}=\frac{3g}{2l}\cos\theta$$

Example

A heavy steel chain of mass m and length l passes over a pulley of mass m_0 and radius r. The pulley is fixed with a frictionless pivot O. There is no slide between the chain and pulley. At beginning, the chain passes over the pulley with the lengths of both side equal. And then with a small perturbation, the chain slides to the left. Find the velocity and acceleration of the chain when the height difference of two end is s.



Solution



Take the chain, the pulley and the Earth as a system, the **mechanical energy** of the system is **conserved**.

$$0=-\left(\frac{m}{l}\frac{s}{2}\right)g\frac{s}{2}+\frac{1}{2}mv^2+\frac{1}{2}\left(\frac{1}{2}m_0r^2\right)\omega^2$$

$$v=\omega r,\qquad v=\sqrt{\frac{mgs^2}{2\left(m+\frac{1}{2}m_0\right)l}}$$

The acceleration: $a=\dfrac{dv}{dt}=\dfrac{dv}{ds}\dfrac{ds}{dt}=2v\dfrac{dv}{ds}$

$$=2\sqrt{\frac{mgs^2}{2\left(m+\frac{1}{2}m_0\right)l}}\cdot\sqrt{\frac{mg}{2\left(m+\frac{1}{2}m_0\right)l}}=\frac{mgs}{\left(m+\frac{1}{2}m_0\right)l}$$

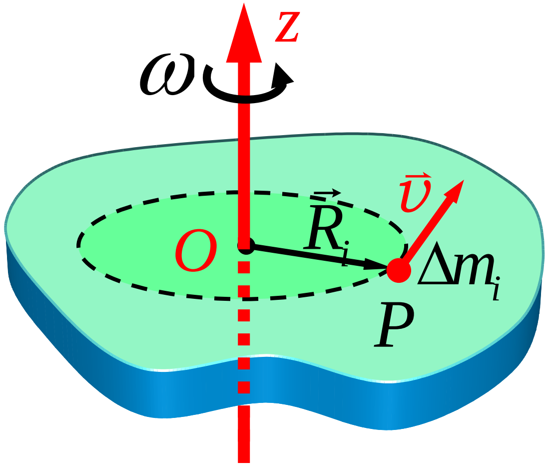
§ 6 Angular Momentum for a Rigid Body (P281)

- The total angular momentum L is the vector sum of l_i for each particle of the rigid body.

$$l_{i\omega} = \Delta m_i \upsilon_i R_i = \Delta m_i R_i^2 \omega$$

Sum over all the particles:

$$\boxed{L_\omega = \sum_i l_{i\omega} = \left(\sum_i \Delta m_i R_i^2 \right) \omega = I\omega}$$



(about a fixed axis)

Angular Momentum for a Rigid Body

- Rotational Form of Newton's II Law
  - Starting from the Torque-angular momentum theorem.

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \quad \Rightarrow \quad \boxed{\sum \tau_{\text{ext-axis}} = \frac{dL_\omega}{dt} = \frac{d}{dt}(I\omega) = I\alpha}$$

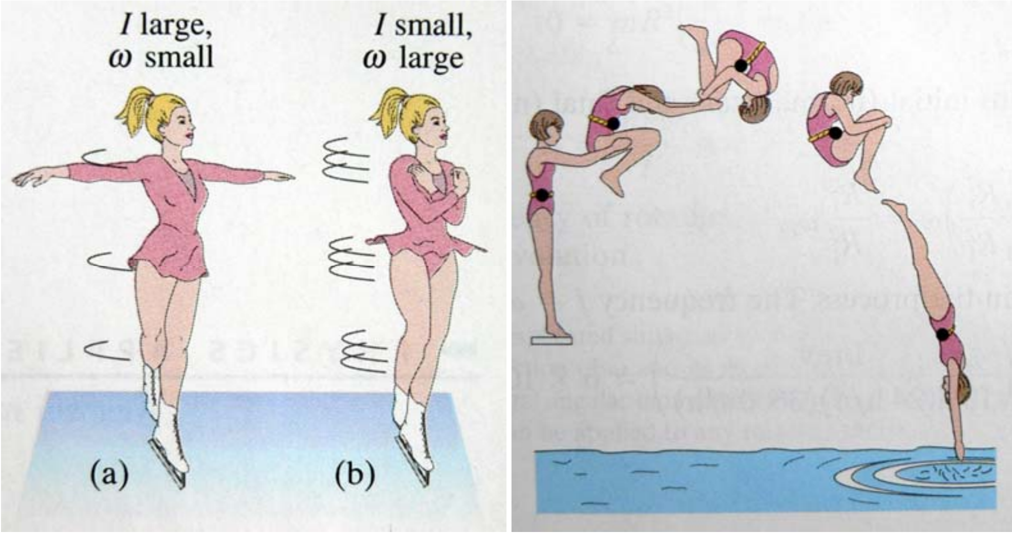
  - The Rotational Form of Newton's II Law can be considered as a special case of Torque-angular momentum theorem for a rigid body rotation about a fixed axis.

- The Conservation of Angular Momentum for Rigid Body
  - The total angular momentum of rotating body remains constant if the net external torque acting on it is zero.

If $\sum \tau_{\text{ext-axis}} = 0$

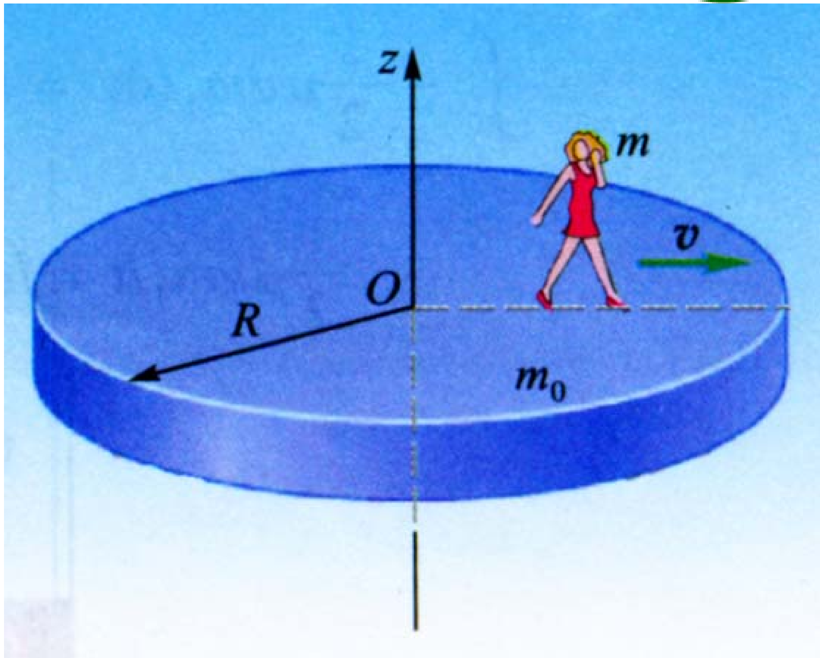
$$\Downarrow$$

$$\boxed{I\omega = I_0\omega_0}$$



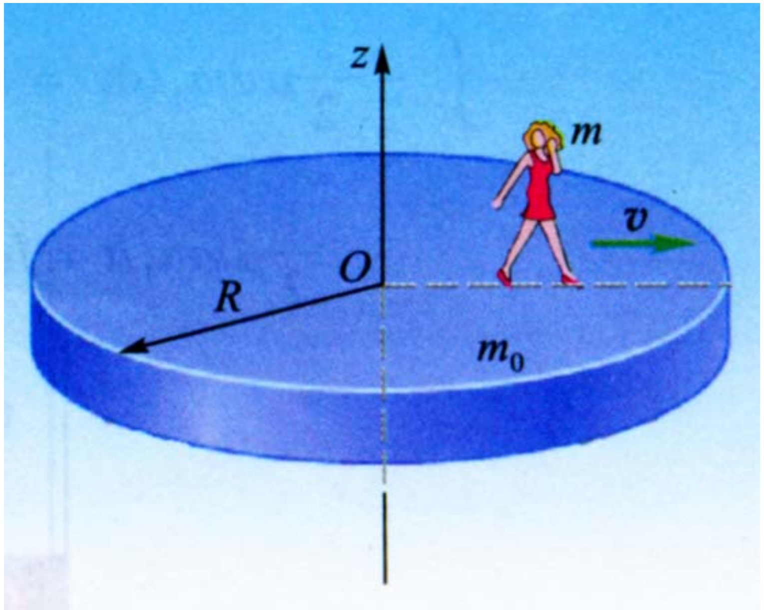
Example

A circular platform of mass m_0 and radius R rotates friction-free about an axis through its center. A woman standing on the platform a distance $R/2$ from the center. At beginning, the system of platform and woman rotates at the angular velocity ω_0 about the axis. The woman starts to walk to the edge of the platform. Determine the final angular velocity ω of the system when the woman arrives at the edge.



Solution

In the whole process that the woman walk to the edge of platform, the external torque is zero. Using the **conservation of angular momentum** of the system:



Initial state:

$$L_0 = \left(\frac{1}{2}m_0R^2\right)\omega_0 + m\left(\frac{R}{2}\right)^2\omega_0$$

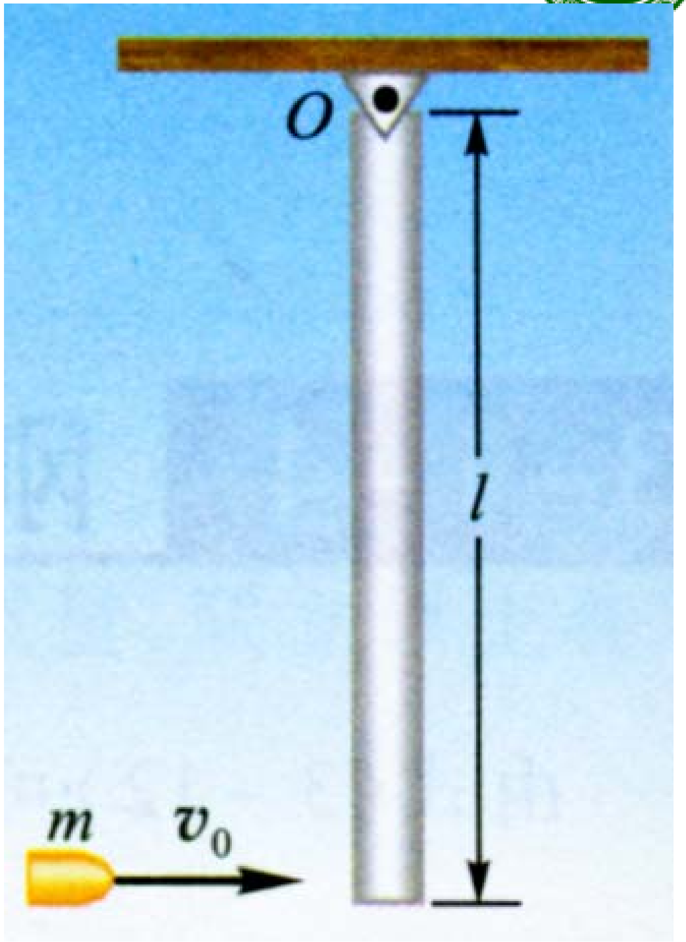
Final state:

$$L = \left(\frac{1}{2}m_0R^2\right)\omega + mR^2\omega$$

$$L_0 = L \Rightarrow \boxed{\omega = \frac{2m_0 + m}{2m_0 + 4m}\omega_0}$$

Example

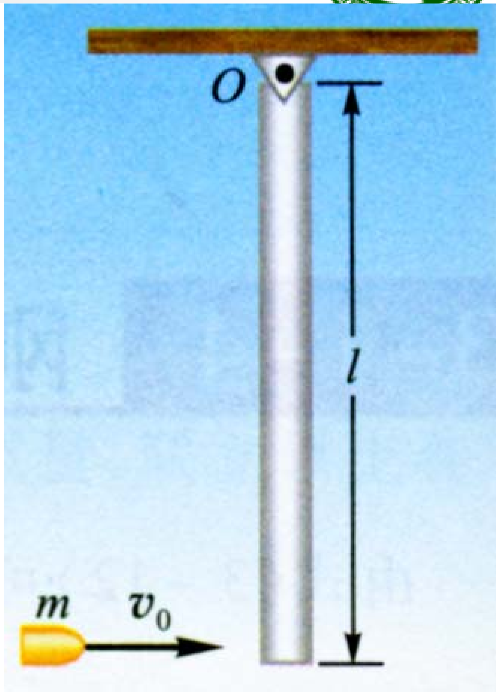
A rod of mass m' and length l can rotate about pivot O freely, a bullet of mass m and speed v_0 is shot into the lower end of the rod and embedded in the rod. What is the angle θ when the rod swings to its highest position?



Solution

(i) Take the bullet and the rod as a system.

The external forces: the constraint force exerted by the pivot; gravity. They go through the origin O. So the external torque about O is zero, and the angular momentum of the system should be conserved in the process of shouting.



$$lmv_0 = \left(\frac{1}{3}m'l^2 + ml^2\right)\omega \qquad \boxed{\omega = \frac{3mv_0}{(m'+3m)l}}$$

(ii) Take the bullet, the rod and the Earth as a system. In the process of the system swinging up, the mechanical energy is conserved.

$$\frac{1}{2}\left(\frac{1}{3}m'l^2 + ml^2\right)\omega^2 = mgl(1-\cos\theta) + m'g\frac{l}{2}(1-\cos\theta)$$

$$\boxed{\cos\theta = 1 - \frac{3m^2}{(m'+3m)(m'+2m)}\frac{v_0^2}{gl}}$$

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§ 6 Angular Momentum for a Rigid Body (P281)

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