Chapter 9, 10 and 11

Momentum, Collision and Rotation

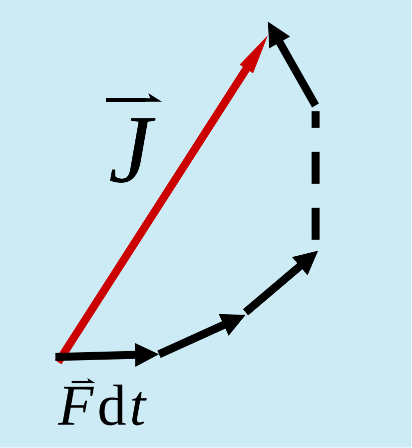
§ 1 Impulse and Momentum

- **Definition of impulse of a force $\vec{F}$**

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}\, dt$$

SI unit N•s

The impulse of a force is a vector. It depends on the strength of the force and on its duration.



Impulse and Momentum

- **Another form of Newton's second law in terms of momentum**

$$\vec{F} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$$

- **Definition of momentum or linear momentum of an object**

$$\vec{p} = m\vec{v}$$

SI unit kg•m/s

- **The form** $\vec{F} = m\vec{a}$ **is the special case for Newton's second law when the mass of the object remains constant.**

Impulse-momentum theorem (P206)

- **The impulse-momentum theorem for a particle**

$$\int_{t_i}^{t_f} \vec{F} dt = \int_{t_i}^{t_f} \frac{d\vec{p}}{dt} dt = \int_{\vec{p}_i}^{\vec{p}_f} d\vec{p} = \vec{p}_f - \vec{p}_i = \Delta \vec{p}$$

$$\boxed{\vec{J} = \vec{p}_f - \vec{p}_i = \Delta \vec{p}}$$

The impulse of the net force acting on a particle during a given time interval is equal to the change in momentum of the particle during that interval. (Valid only in inertial frame of reference)

Time - averaged impulsive force (P205 § 9-3)

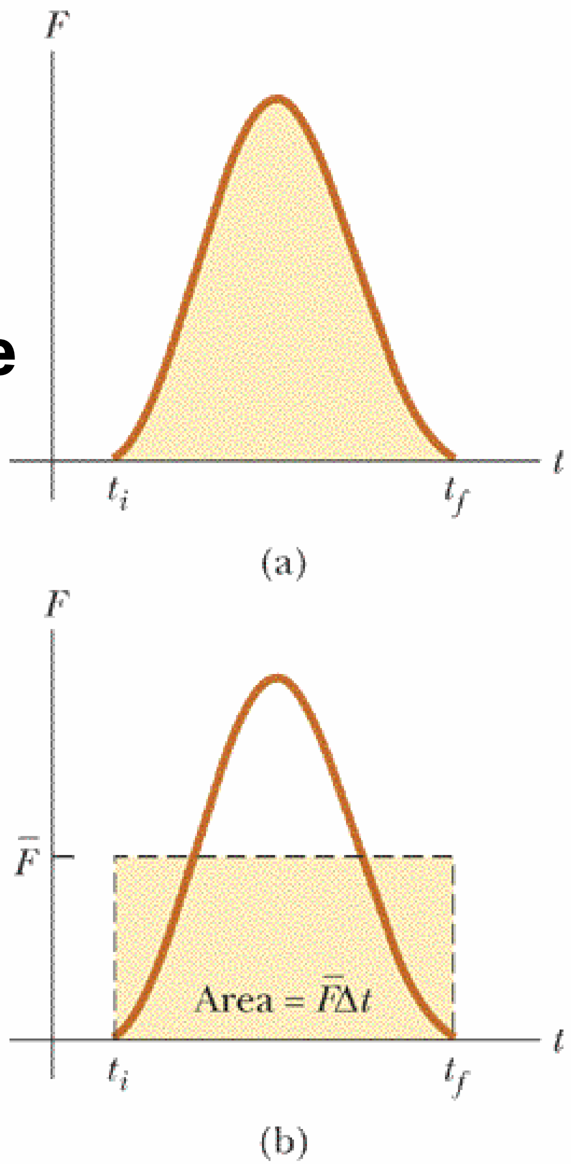
Impulsive force

- When a time-varying net force $\vec{F}(t)$ is difficult to measure, we can use a **time-averaged net force** as the substitute provided that it would give the same impulse to the particle in same time interval.

$$\overline{\vec{F}} = \frac{1}{\Delta t}\int_{t_i}^{t_f} \vec{F}\, dt = \frac{\Delta \vec{p}}{\Delta t}$$

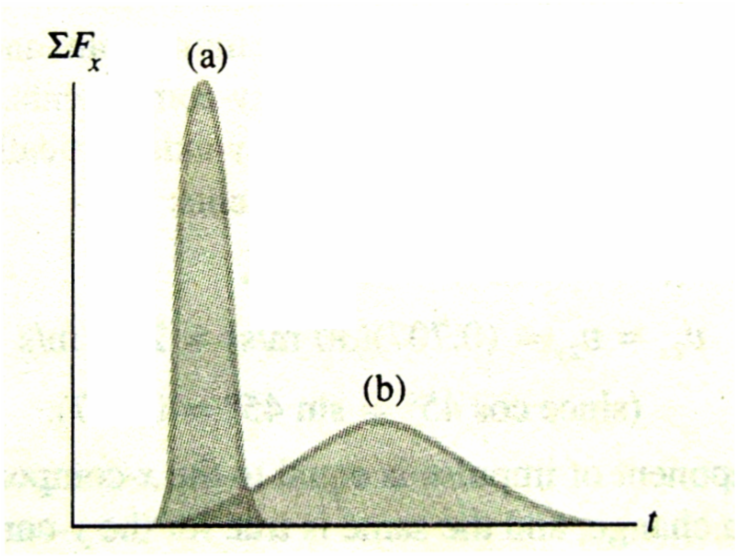
$$\vec{J} = \Delta\vec{p} = \overline{\vec{F}}\Delta t$$

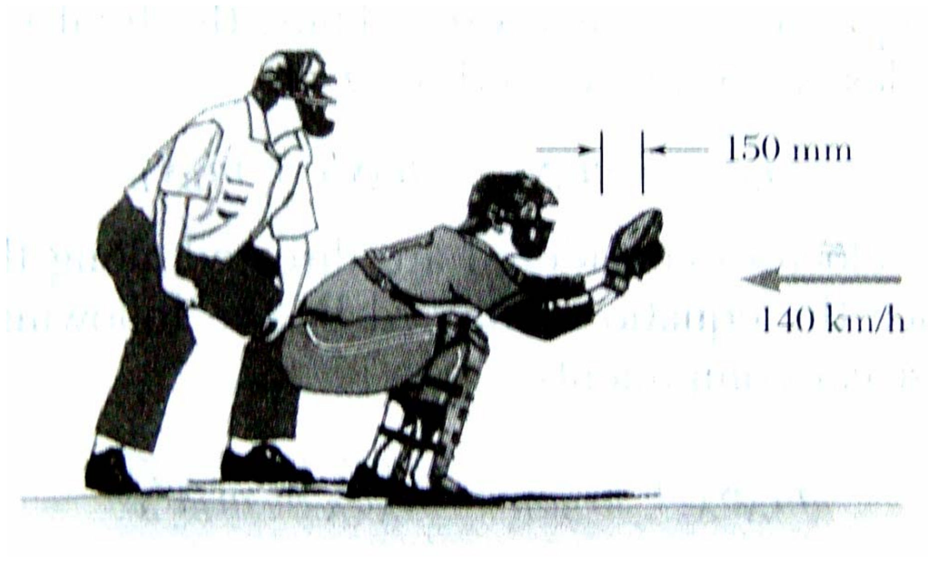
- When a particle experiences a impact in a very short time, the non-impulse forces such as gravitational force and friction force are **negligible** compared to impulsive force.



Time - averaged impulsive force

- For a given amount of momentum change, we can delay the time interval to decrease the impulsive force.
- A baseball player catching a ball can soften the impact by pulling his hand back.

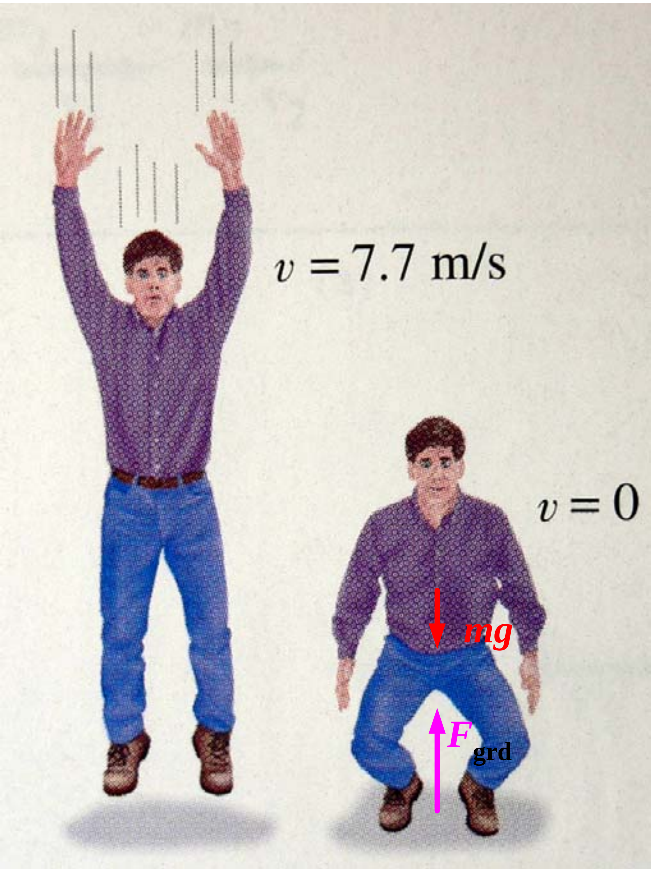




Example (P207 Ex. 9-6)

Bend your knees when landing.

(a) Calculate the impulse experienced when a 70kg person lands on firm ground after jumping from a height of 3.0m. Then estimate the average force exerted on the person's feet by the ground, if the landing is (b) stiff-legged (body moves 1.0cm during impact), and (c) with bent legs (about 50cm).



Solution

(a) $v = \sqrt{2gh} = 7.7\,\text{m/s}$

$$J = p_f - p_i = 0 - (70\text{kg})(7.7\text{m/s}) = -540\,\text{N}\cdot\text{s}$$

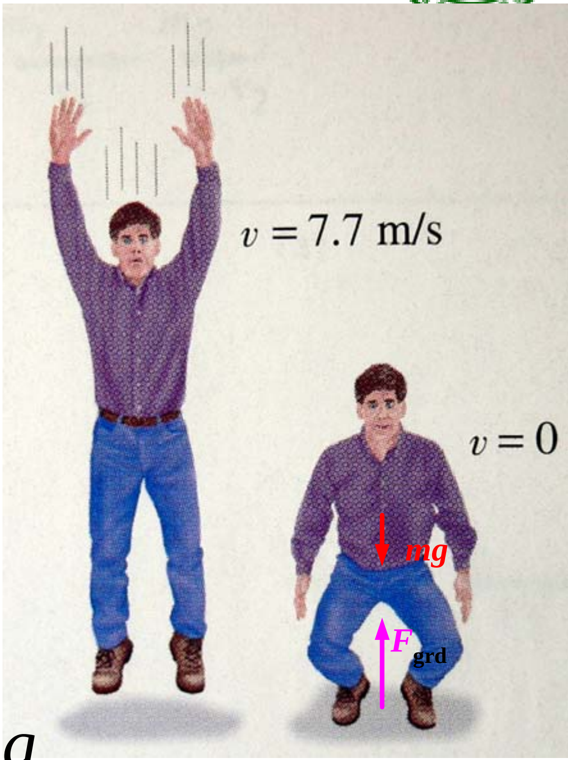
(b) $d = 1.0\text{cm} = 1.0\times10^{-2}\text{m}$

$$\bar{v} = (7.7+0)/2 = 3.8\text{m/s}, \quad \Delta t = d/\bar{v} = 2.6\times10^{-3}\text{s}$$

$$F_{\text{grd}} + mg = \frac{J}{\Delta t} = \frac{-540}{2.6\times10^{-3}} = -2.1\times10^{5}\,\text{N}$$

$mg = (70\text{kg})(9.8\text{m/s}^2) = 690\text{N}$

$$\left|F_{\text{grd}}\right| = 2.1\times10^{5}\,\text{N} + 690\,\text{N} \approx 2.1\times10^{5}\,\text{N} >> mg$$



The person's legs would likely break in such a stiff landing.

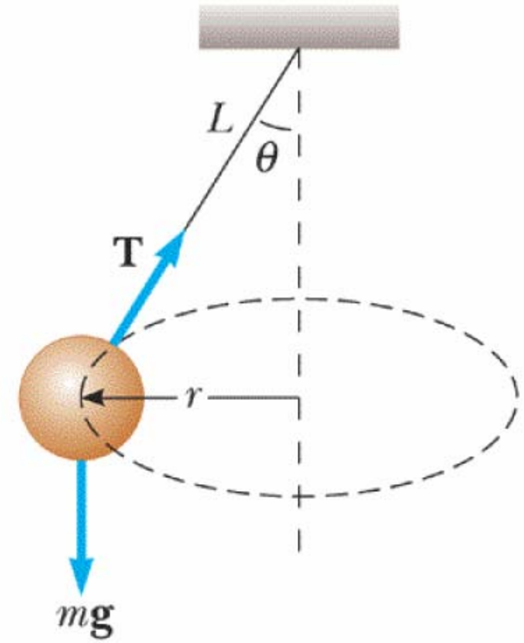
(c) d=0.50m, Δt=0.13s,

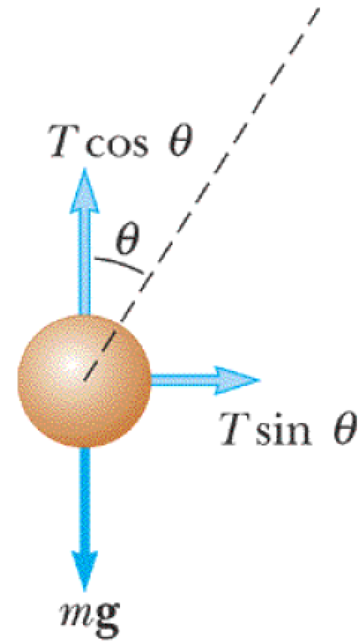
$$F_{\text{grd}} + mg = \frac{540}{0.13} = -4.2\times10^{3}\,\text{N}, \quad F_{\text{grd}} = -4.9\times10^{3}\,\text{N}$$

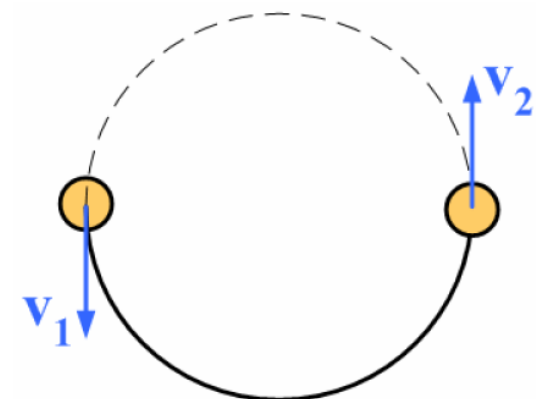
Example

Conical Pendulum.

A small object of mass m is suspended from a string. The object revolves in a horizontal circle of radium r with constant speed v. Determine the **impulse** exerted (1) by gravity, (2) by string tension on the object, during the time in which the object has passed half of the circle.







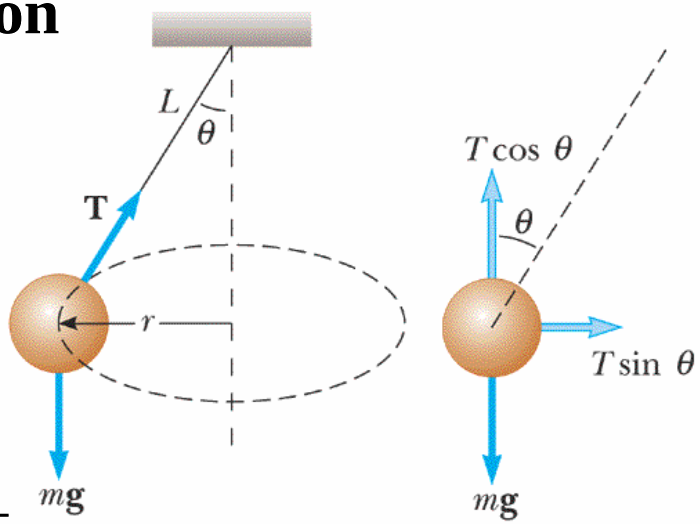
Example

Solution: (1) The impulse exerted by gravity on the object

$$\vec{J}_{mg} = \int_{t_1}^{t_2} m\vec{g}dt = m\vec{g}\left(\frac{1}{2}\frac{2\pi r}{v}\right) = \frac{\pi r}{v}m\vec{g}$$

(2) The impulse exerted by string intension on the object

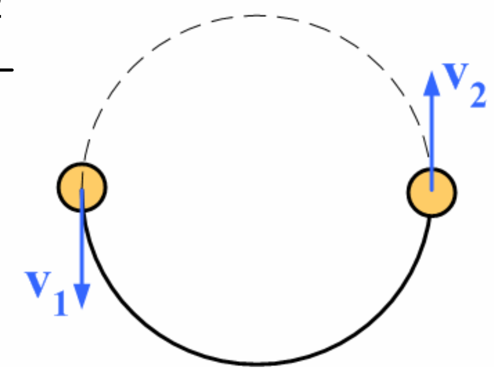
$$\vec{J}_T = \vec{J}_{net} - \vec{J}_{mg}$$



From impulse-momentum theorem

$$\vec{J}_{net} = \Delta\vec{p} = m\vec{v}_2 - m\vec{v}_1 = 2m\vec{v}$$

$$J_T = \sqrt{(2mv)^2 + \left(\frac{\pi rmg}{v}\right)^2} = m\sqrt{4v^2 + \frac{\pi^2 r^2 g^2}{v^2}}$$



Example

A particle of mass *m*=2kg moves in the velocity

$$\vec{v} = -3\sin(\frac{\pi}{2}t)\hat{i} + 3\cos(\frac{\pi}{2}t)\hat{j} \qquad \text{(SI)}$$

Find (1) the impulse of the net force acting on this particle during a time interval from *t*=0 to *t*=4s;
(2) the change in momentum of the particle during a time interval from *t*=0 to *t*=2s.

§ 2 Impulse-momentum theorem for a system of particles

Consider a system of N interacting particles

For i-th particle:

the net external force $\vec{F}_i$

the internal force exerted by j-th particle $\vec{f}_{ij}$

$$(\vec{F}_i + \sum_{j \neq i} \vec{f}_{ij})dt = d\vec{p}_i$$

For the system of particles: $\sum_i (\vec{F}_i + \sum_{j \neq i} \vec{f}_{ij})dt = \sum_i d\vec{p}_i$

According to Newton's third law, the internal forces cancel in pairs. $\sum_i \sum_{j \neq i} \vec{f}_{ij} = 0$

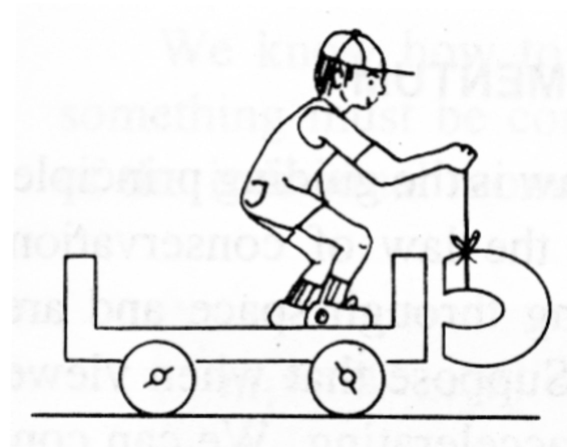
The total external force acting on the system: $\sum_i \vec{F}_i$

The total momentum of the system: $\vec{p}_{\text{tot}} = \sum_i \vec{p}_i$

Impulse-momentum theorem for a system of particles

$$\sum_i (\vec{F}_i + \underbrace{\sum_{j \neq i} \vec{f}_{ji}}_{\text{zero}})dt = \underbrace{\sum_i d\vec{p}_i}_{d\vec{p}_{\text{tot}}}$$

Can you get the wagon to move by hanging a huge magnet in front of you?



Conclusion:

The derivative form:

$$\boxed{\sum_i \vec{F}_{i-\text{ext}} = \frac{d\vec{p}_{\text{tot}}}{dt}}$$

The integral form:

$$\boxed{\int_{t_1}^{t_2} \sum_i \vec{F}_{i-\text{ext}}\, dt = \vec{p}_{\text{tot}2} - \vec{p}_{\text{tot}1}}$$

- **The total external force applied to a system of particles equals to the change in total momentum of the system.**
- **The internal forces can exchange the momenta between particles within system, but can not influence the total momentum of the system.**

Conservation of momentum

When $\sum_i \vec{F}_{i-\text{ext}} = 0$ $\qquad \dfrac{d\vec{p}_{\text{tot}}}{dt} = 0$ or $\vec{p}_{\text{tot}} = \sum_i \vec{p}_i = \text{constant}$

- **When the vector sum of external forces on a system is zero, the total momentum of the system is constant.**

- **Notice the difference between conservation of momentum and conservation of mechanical energy**

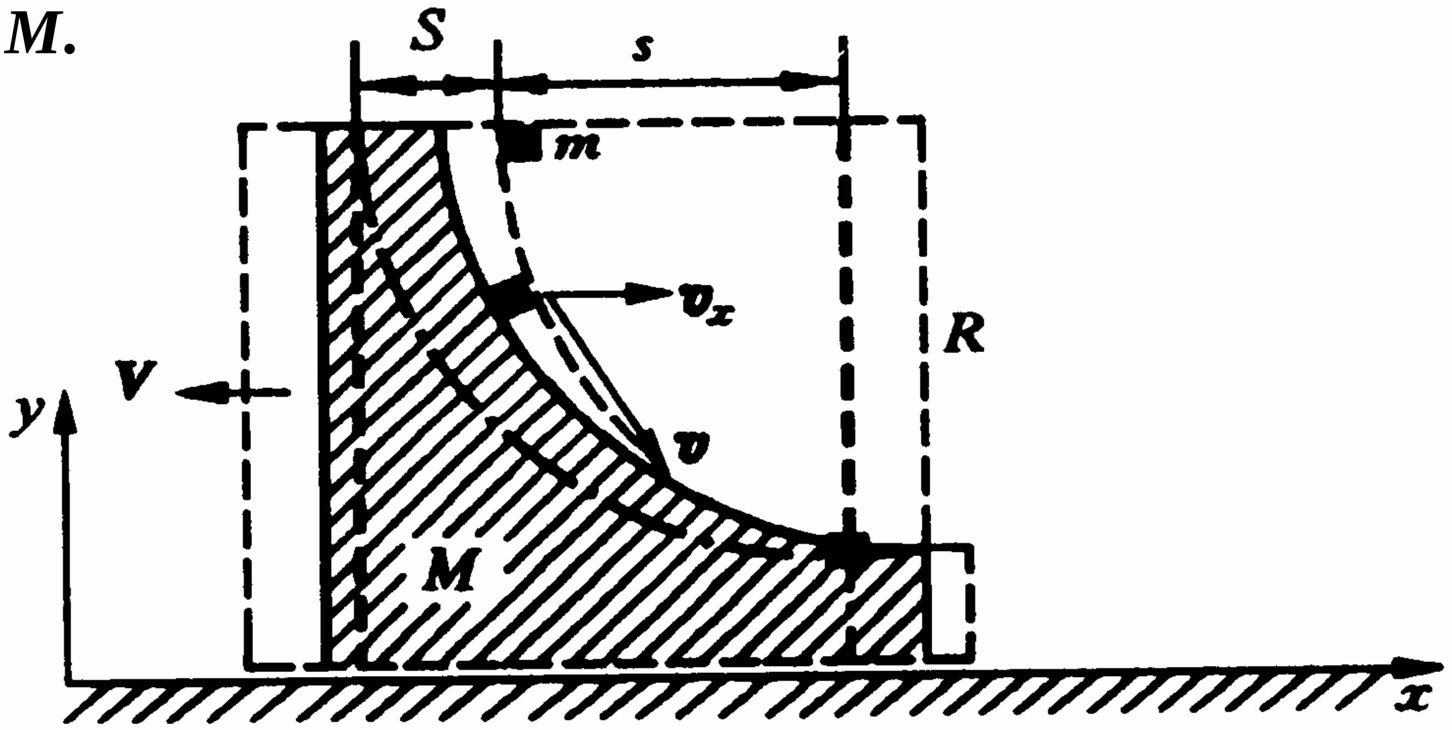
  For an isolated system, the mechanical energy is conserved only when the internal forces are conservative. But conservation of momentum is valid even when the internal forces are not conservative.

- **Conservation of momentum in component form**

When $\sum_i F_{i-\text{ext-}x} = 0$ $\qquad$ then $p_{\text{tot-}x} = \sum_i p_{i-x} = \text{constant}$

Example

A small cube of mass m slides down a circular path of radius R cut into a large block of mass M. M rests on a frictionless table. M and m are initially at rest. m starts from the top of the path. Find the distance traveled by M when the cube m leaves the block M.



Solution

No horizontal external force acts on the system consisting of the cube and the block. The total momentum of the system is conserved in horizontal direction.

$$0 = mv_x + M(-V) \Rightarrow mv_x = MV$$

Integrations on both side: $m\int_0^t v_x dt = M\int_0^t V dt, \quad ms = MS$ ①

In the reference frame of *M*: the horizontal displacement of *m* is

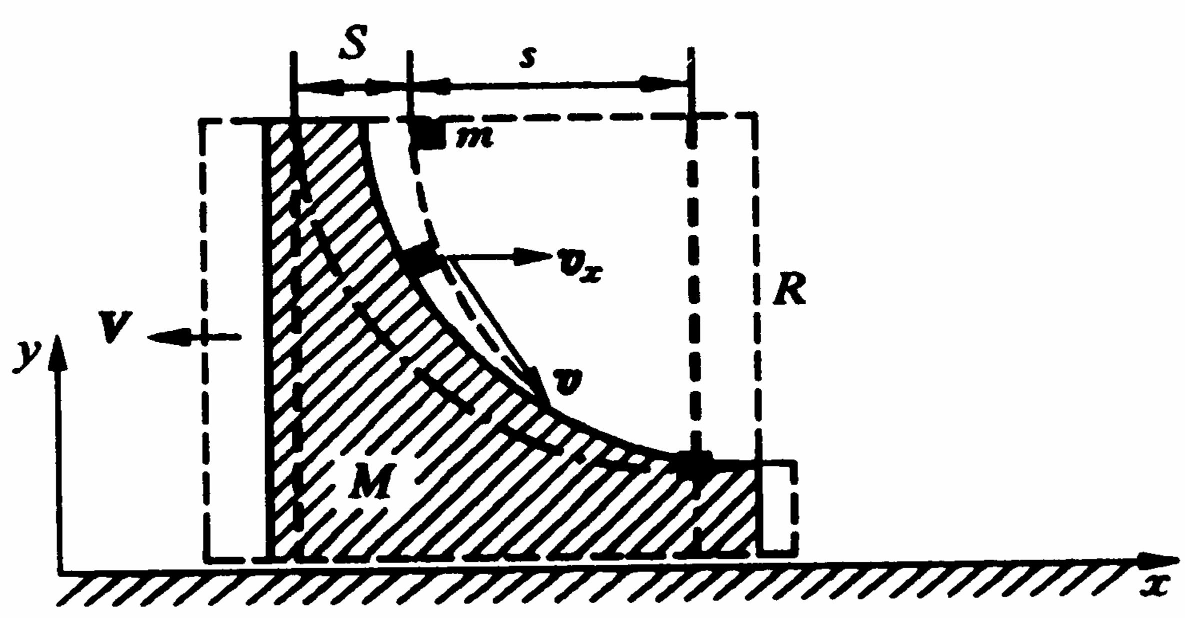
$$R = \int_0^t v'_x dt$$

$$= \int_0^t (v_x + V)dt$$

$$= s + S \qquad ②$$

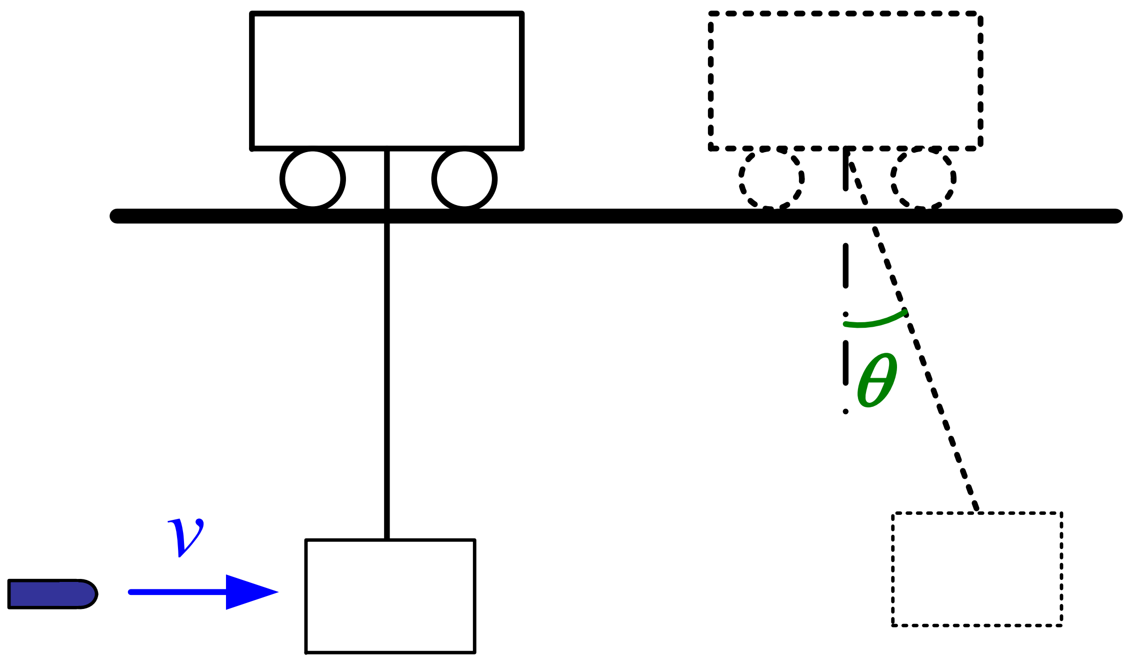
From ① and ②

$$\boxed{S = \frac{m}{m+M}R}$$



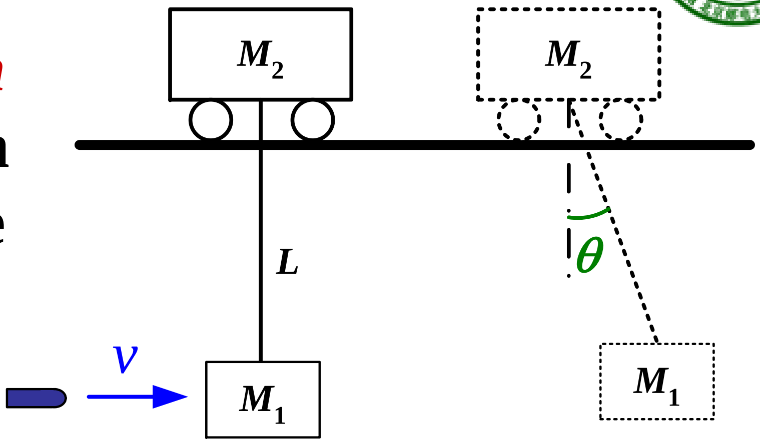
Example

A wooden block of mass M_1 is suspended from a cord of length L attached to a cart of mass M_2 which can roll freely on a frictionless horizontal track. A bullet of mass m is fired into the block from left. After the impact of the bullet, the block swings up with the maximum angle of θ. What is the initial speed v of the bullet?



M

Solution



Stage 1: **For the system consisting of** ***m*** **and** M_1**, the momentum is conserved in horizontal during a small interval time of impact.**

$$mv = (M_1 + m)v_1$$

Stage 2: **The block plus bullet swing up with initial speed** v_1**, and drive the cart sliding forward in the track. At the instant when the block-bullet swing at maximum angle,** (M_1+m)**,** M_2 **have the same horizontal speed of** v_2**, and the mechanical energy of the system of** (M_1+m) **and** M_2 **is conserved.**

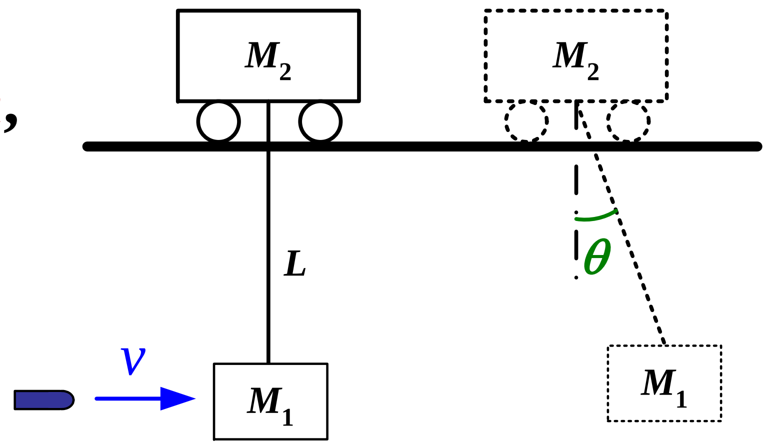
$$\frac{1}{2}(M_1 + m)v_1^2 = \frac{1}{2}(M_1 + M_2 + m)v_2^2 + (M_1 + m)gL(1 - \cos\theta)$$

Solution

$$mv = (M_1 + m)v_1 \qquad (1)$$

$$\frac{1}{2}(M_1 + m)v_1^2 = \frac{1}{2}(M_1 + M_2 + m)v_2^2 + (M_1 + m)gL(1 - \cos\theta) \qquad (2)$$

During the whole Stage1+Stage2: The momentum of system consisting of M_1, m, M_2 is conserved in horizontal.



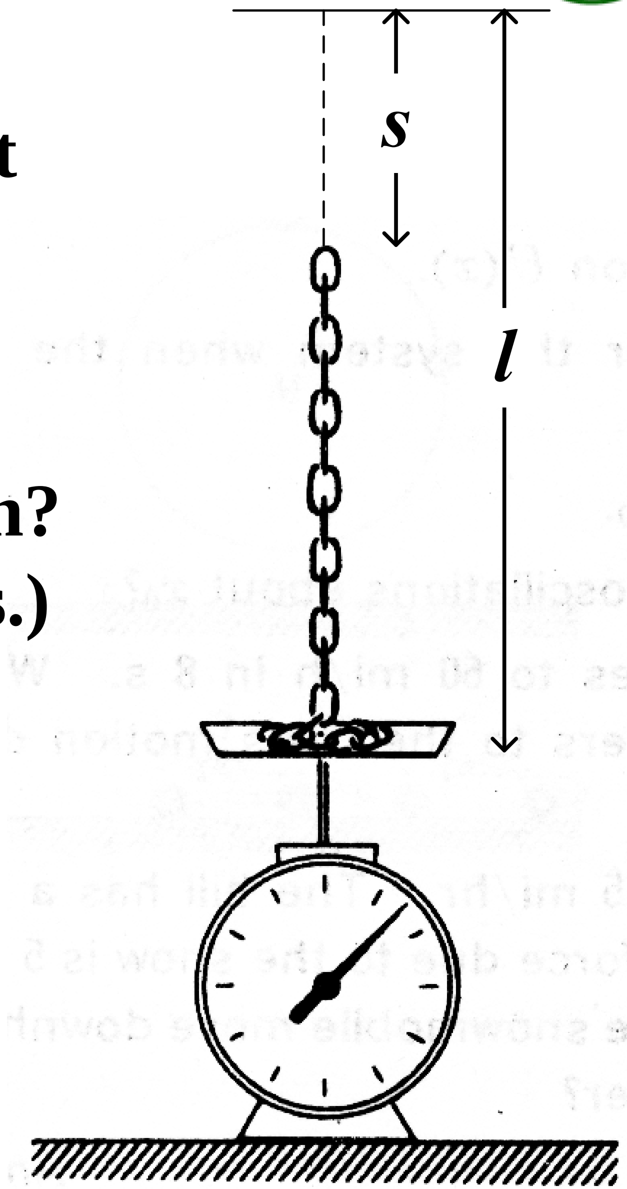
$$mv = (M_1 + M_2 + m)v_2 \qquad (3)$$

Final answer:

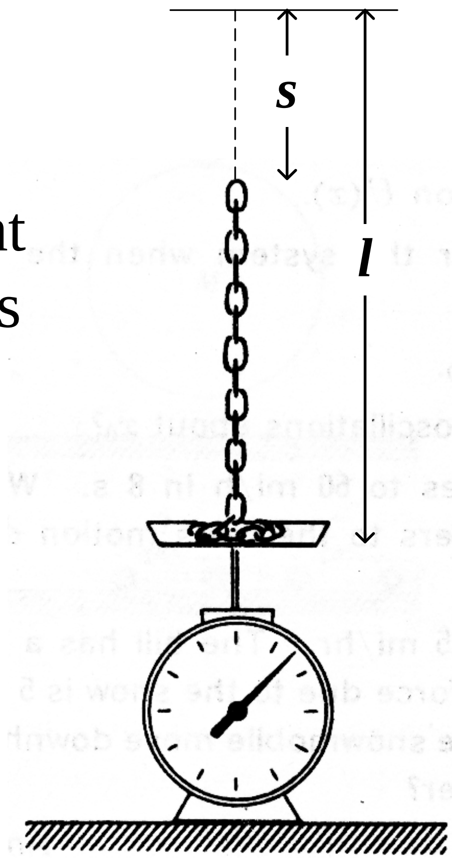
$$\boxed{v = \frac{M_1 + m}{m}\sqrt{\frac{M_1 + M_2 + m}{M_1} 2gL(1 - \cos\theta)}}$$

Example

A chain of mass ***M*** length ***l*** is suspended vertically with its lowest end touching a scale. The chain is released and falls onto the scale. What is the reading of the scale when a length of chain, ***s***, has fallen? (Neglect the size of individual links.)



Solution (I) : Using impulse-momentum theorem:



Assuming a length of chain ***s*** has been already in the scale. Take a infinitesimal process during ***dt***, a segment chain of length of ***ds*** impacts with the scale, and comes to a halt. The impulse that the surface of the scale acting on this segment is:

$$Fdt = 0 - vdm = -v\frac{M}{l}ds$$

$$F' = \frac{M}{l}v\frac{ds}{dt} = \frac{M}{l}v^2 = \frac{M}{l}(2gs) = 2Mg\frac{s}{l}$$

The reading of the scale

= the weight that has already in the scale + F'

$$= Mg\frac{s}{l} + 2Mg\frac{s}{l} = 3Mg\frac{s}{l}$$