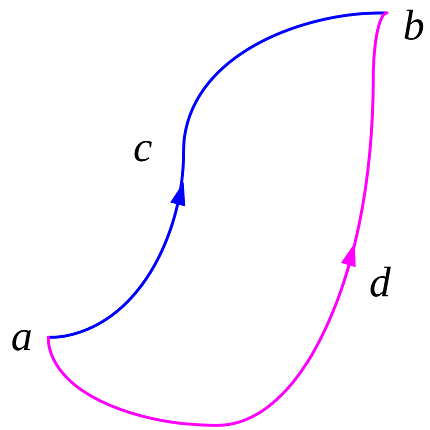
§ 3 Conservative Forces and Potential Energy

$$W = \int_{(L)} \vec{F} \cdot d\vec{r}$$



Work done by a force is a line integral or path integral. Generally, depends on the path followed by the particle. Different path corresponds to different work done by the same force.

A category of forces which have the special property, that the work done by such a force is independent of the path —— are conservative forces.

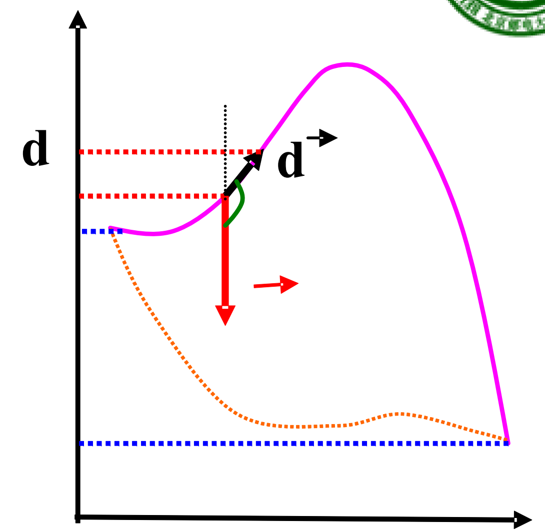
Work done by weight

$$\vec{G} = m\vec{g}$$

$$dW = \vec{G} \cdot d\vec{r} = G\cos\theta\, ds$$

$$= G\cos(180^\circ - \alpha) ds$$

$$= -mgds\cos\alpha = -mg\, dy$$



$$W = \int_a^b dW = \int_{y_a}^{y_b} -mg\, dy = -(mgy_b - mgy_a)$$

Only depends on the initial and final positions, and does not depend on the path taken by the particle.

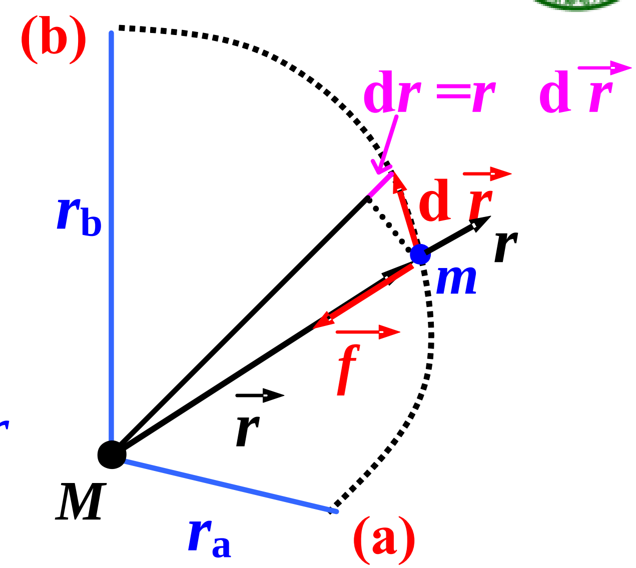
Work done by the universal gravitational force

$$\vec{f} = -G\frac{Mm}{r^2}\hat{r}$$

$$W = \int_a^b \vec{f}\cdot d\vec{r} = -\int_{r_a}^{r_b} G\frac{Mm}{r^2}\hat{r}\cdot d\vec{r}$$

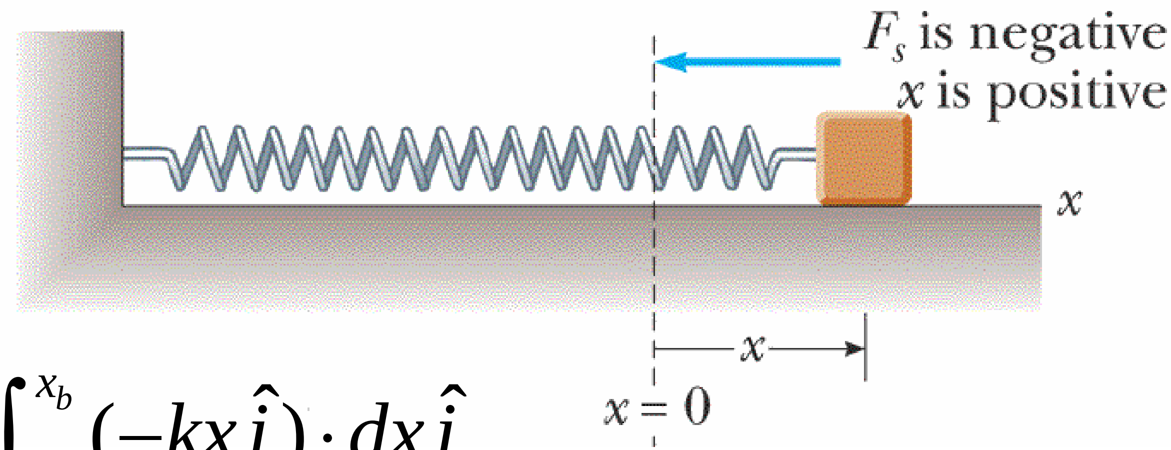
$$= -\int_{r_a}^{r_b} G\frac{Mm}{r^2}|d\vec{r}|\cos\theta = -\int_{r_a}^{r_b} G\frac{Mm}{r^2}dr$$

$$= -\left[\left(-G\frac{Mm}{r_b}\right) - \left(-G\frac{Mm}{r_a}\right)\right]$$



- **Only depends on the initial and final positions, and does not depend on the path taken by the particle.**

Work done by the spring force



$$\vec{F}_s = -kx\,\hat{i}$$

$$W = \int_{x_a}^{x_b} \vec{F}_s \cdot d\vec{r} = \int_{x_a}^{x_b} (-kx\,\hat{i}) \cdot dx\,\hat{i}$$

$$= -\int_{x_a}^{x_b} kx\,dx = -\left(\frac{1}{2}kx_b^2 - \frac{1}{2}kx_a^2\right)$$

- **Only depends on the initial and final positions, and does not depend on the path taken by the particle.**

- Conclusion: The conservative force has properties that

  - **The work done by a conservative force dose not depend on the path followed by the particle, and depends only on the initial and final positions.**

Equivalent statement:

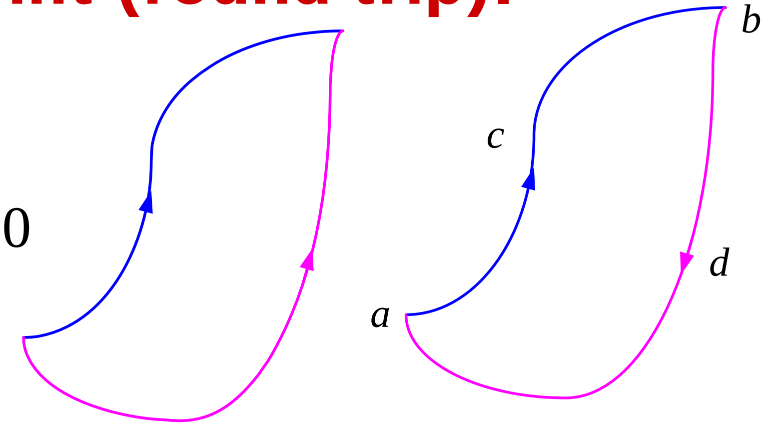
$$\boxed{W=\int_a^b \vec{F}\cdot d\vec{r}=-\left[U(\vec{r}_b)-U(\vec{r}_a)\right]}$$

  - **The total work done by a conservative force is zero, as the particle moves around a close path and returns to its starting point (round trip).**

$$\int_{acb} \vec{F}\cdot d\vec{r}=\int_{adb} \vec{F}\cdot d\vec{r}$$

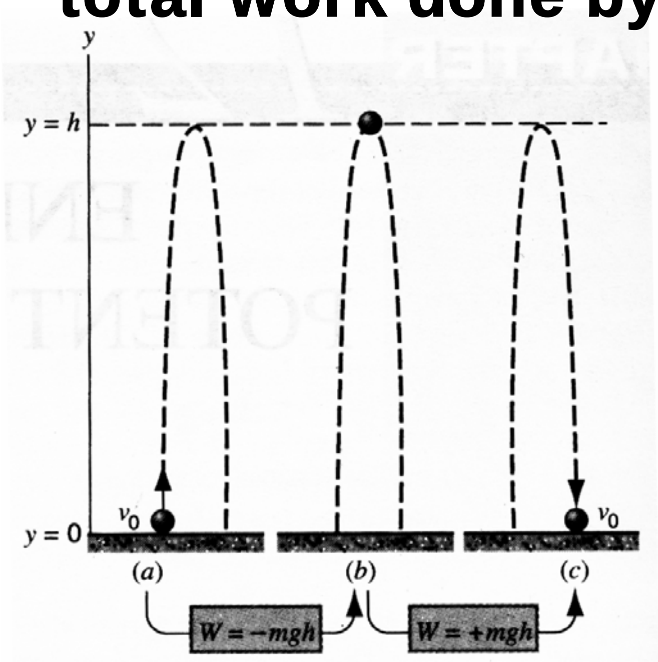
$$\int_{acb} \vec{F}\cdot d\vec{r}-\int_{adb} \vec{F}\cdot d\vec{r}=\int_{acb} \vec{F}\cdot d\vec{r}+\int_{bda} \vec{F}\cdot d\vec{r}=0$$

$$\oint \vec{F}\cdot d\vec{r}=0$$



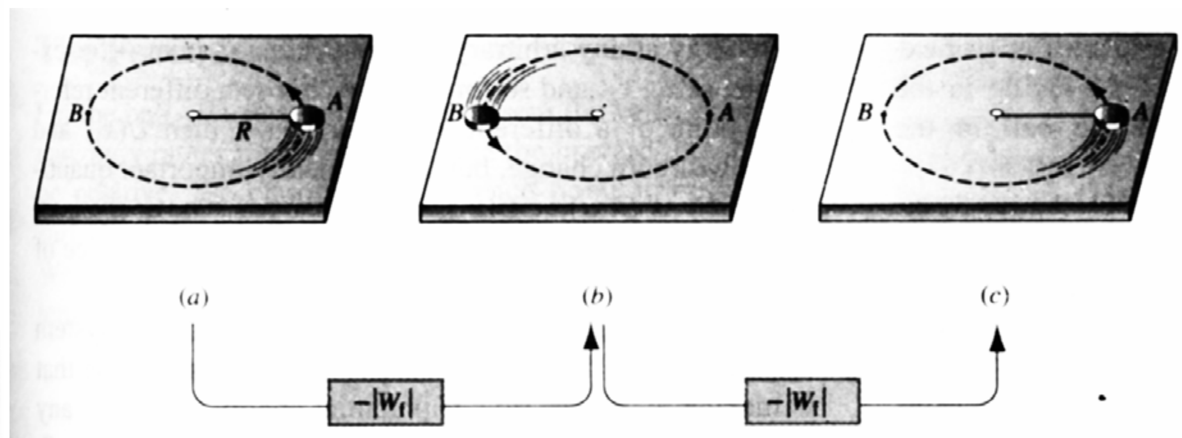
The conservative force and non-conservative force

- The total work done by a conservative force is zero as the particle moves along a round trip.
- But when the particle moves along a round trip, the total work done by a nonconservative force is not zero.



For the force of gravity:

$$W_{\text{round trip}} = 0$$



For the friction force:

$$W_{f\ \text{round trip}} = -2\pi R \mu_k mg$$

Why introduce potential energy?

$$\Delta U = U(\vec{r}_b) - U(\vec{r}_a) = -W = -\int_a^b \vec{F} \cdot d\vec{r}$$

- The work done by a conservative force can be represented in terms of the change in potential energy.
- Notice:
  - The potential energy ***U*** is the energy associated with the configuration of a system. Here "*configuration*" means how the parts of a system are located or arranged with respect to one another (the compression or stretching of the spring in the block-spring system, or height of the ball in the ball-Earth system.)
  - The potential energy belongs to the system. We should properly speak of "the elastic potential energy of the block-spring system" or "the gravitational potential energy of the ball-Earth system", not "the elastic potential energy of the spring" or "the gravitational energy of the ball".

How to get the absolute value of potential energy?

$$U(\vec{r}_b)-U(\vec{r}_a)=-\int_a^b \vec{F}\cdot d\vec{r}$$ the definition of potential energy only gives the change in potential energy, or the relative value of potential energy. We can choose a position $\vec{r}_0=\vec{r}_a$ as the reference point, define $U(\vec{r}_0)=0$ at the reference point. The choice of reference point is arbitrarily.

New definition of potential energy: $\boxed{U(\vec{r})=U(\vec{r})-0=-\int_{\vec{r}_0}^{\vec{r}} \vec{F}\cdot d\vec{r}=\int_{\vec{r}}^{\vec{r}_0} \vec{F}\cdot d\vec{r}}$

- For gravitational potential energy near the Earth's surface, it is accustomed to choose the reference point $y_0=0$ as surface of the Earth.

$$U(y)=mgy$$

- For gravitational potential energy associate with two particles, it is accustomed to take $U(r_0=\infty)=0$.

$$U(r)=-G\frac{Mm}{r}$$

- For elastic potential energy, it is accustomed to choose the reference position to be that in which the spring is in its relaxed state.

$$U(x)=\frac{1}{2}kx^2$$

The conservative force and potential energy

For an infinitesimal process,

$$-dU = -\left(\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz\right),$$

$$-dU = \vec{F}\cdot d\vec{r} = F_x dx + F_y dy + F_z dz,$$

$$\begin{cases} F_x = -\dfrac{\partial U}{\partial x} \\ F_y = -\dfrac{\partial U}{\partial y} \\ F_z = -\dfrac{\partial U}{\partial z} \end{cases}$$

$$\boxed{\vec{F} = -\left(\hat{i}\frac{\partial U}{\partial x} + \hat{j}\frac{\partial U}{\partial y} + \hat{k}\frac{\partial U}{\partial z}\right) = -\nabla U}$$

∇U means the gradient of the potential-energy function. The gradient of a scalar function is a vector function. ∇ is a gradient operator.

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

The conservative force and potential energy

- For force of gravity.

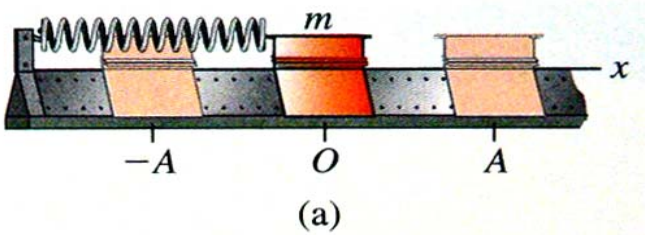
$$U(y)=mgy,\qquad F_y=-\frac{\partial U}{\partial y}=-mg$$

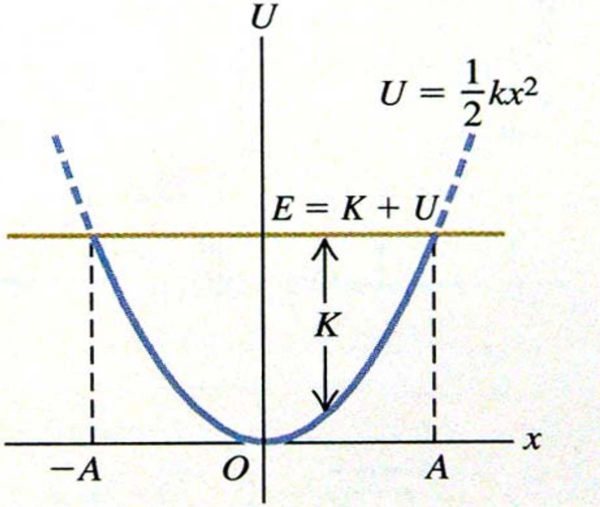
- For universal gravitational force.

$$U(r)=-\frac{GMm}{r},\qquad F_r=-\frac{\partial U}{\partial r}=-\frac{GMm}{r^2}$$

- For spring force.

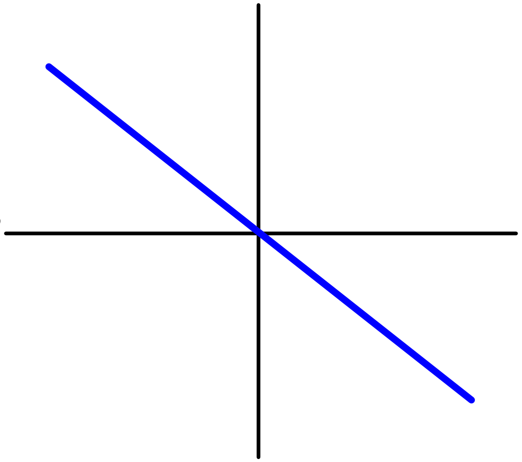
$$U(x)=\frac{1}{2}kx^2,\qquad F_x=-\frac{\partial U}{\partial x}=-kx$$



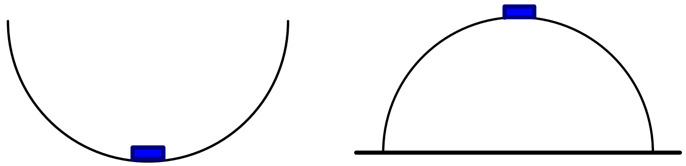


The force is equal to the negative of the slope of U(x)

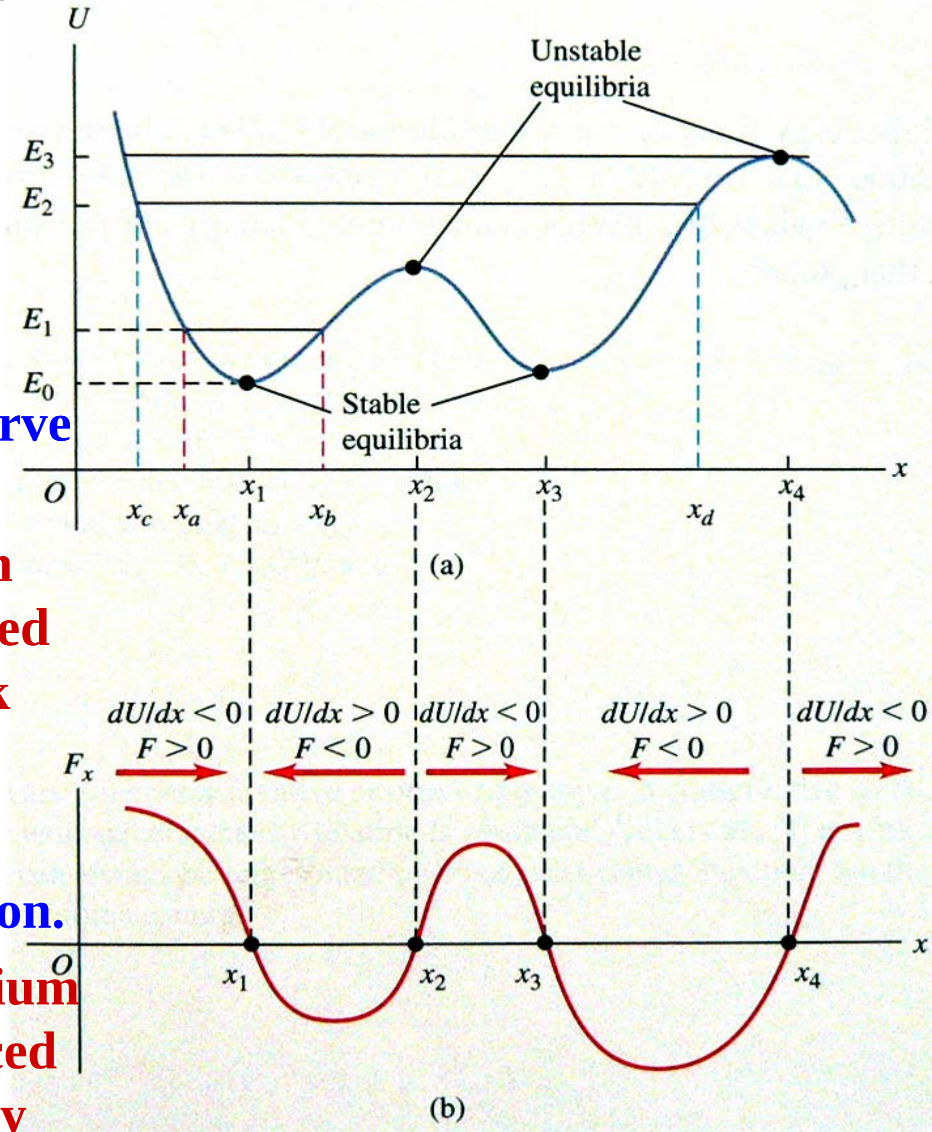
- **Because of conservation of mechanical energy, *E* as a function of *x* is a straight horizontal line** $E=K+U$
- **The glider can only move in the range between $x=\pm A$, since the kinetic energy in this range is positive.**
- **At $x=0$, the slope of $U(x)$ and the force are zero, so it is an equilibrium position.**



Stable and unstable equilibrium



The particle is in stable equilibrium (left) and in unstable equilibrium (right).



- **Any minimum in a potential-energy curve is a stable equilibrium position.**

  Points x_1 and x_3 are stable equilibrium points. When the particle is displaced to either side, the force pushes back toward the equilibrium point.

- **Any maximum in a potential-energy curve is an unstable equilibrium position.**

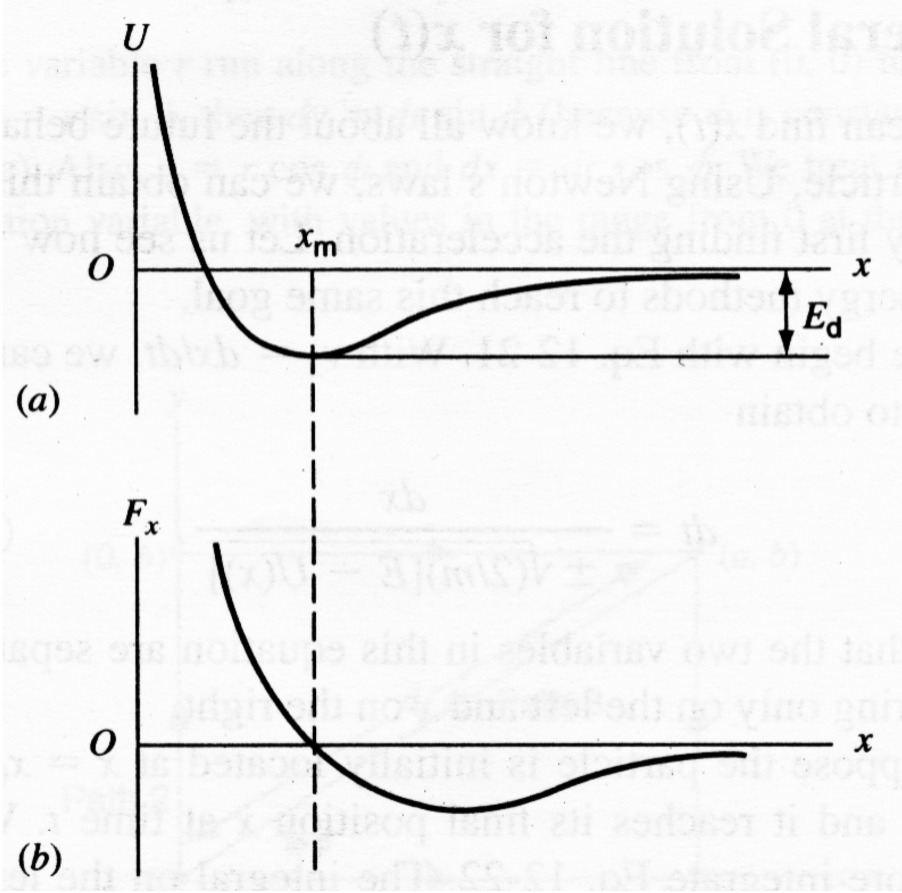
  Points x_2 and x_4 are unstable equilibrium points. When the particle is displaced to either side, the force pushes away from the equilibrium point.

Example

A commonly used potential function to describe the interaction between the two atoms in a diatomic molecule is the Lennard-Jones 6-12 potential

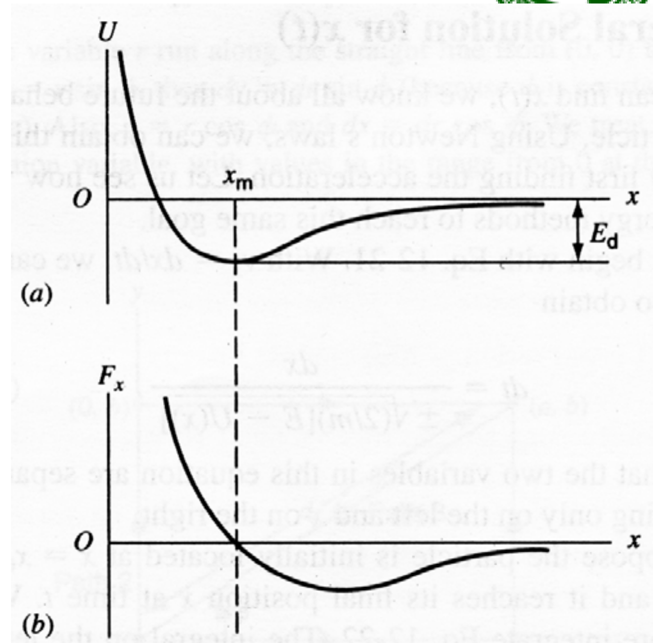
$$U(x) = \varepsilon\left[\left(\frac{x_0}{x}\right)^{12} - 2\left(\frac{x_0}{x}\right)^{6}\right]$$

Find (a) the equilibrium separation between the atoms, (b) the force between the atoms, (c) the minimum energy necessary to break the molecule apart.



Example

$$U(x)=\varepsilon\left[\left(\frac{x_0}{x}\right)^{12}-2\left(\frac{x_0}{x}\right)^{6}\right]$$



Solution: (a) Equilibrium occurs at the position where U(x) is minimum which is found from

$$\left(\frac{dU(x)}{dx}\right)_{x=x_m}=0 \qquad \varepsilon\left(-12\frac{x_0^{12}}{x_m^{13}}+12\frac{x_0^{6}}{x_m^{7}}\right)=0$$

$$x_m=x_0$$

(b) $$F(x)=-\frac{dU(x)}{dx}=12\varepsilon\left(\frac{x_0^{12}}{x^{13}}-\frac{x_0^{6}}{x^{7}}\right)$$

(c) The minimum energy needed to break up the molecule into separate atoms is called *dissociation energy*, E_d.

$$U(x_0)+E_d=0, \qquad E_d=-U(x_0)=\varepsilon$$

§ 5 Work-Energy Theorem and Conservation of Mechanical Energy

- Starting with work – kinetic energy theorem for the system of particles

$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal}} = K_f - K_i$$

  - The internal forces can be divided into conservative and non-conservative.

$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal-conserv}} + \sum W_{i-\text{internal-nonconserv}} = K_f - K_i$$

  - The work done by conservative forces can be described by the change in potential energy $\sum W_{i-\text{internal-conserv}} = -(U_f - U_i)$

$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal-nonconserv}} = (K_f + U_f) - (K_i + U_i)$$

  - Define $E_{\text{mech}} = K + U$ to be total mechanical energy of the system.

- Work – energy theorem:

$$\sum W_{i-\text{external}} + \sum W_{i-\text{internal-nonconserv}} = \Delta E_{\text{mech}}$$

  - The work done by all the external forces and internal forces other than internal conservative forces acting in a system of particles equals the change in total mechanical energy of the system.

Conservation of Mechanical Energy

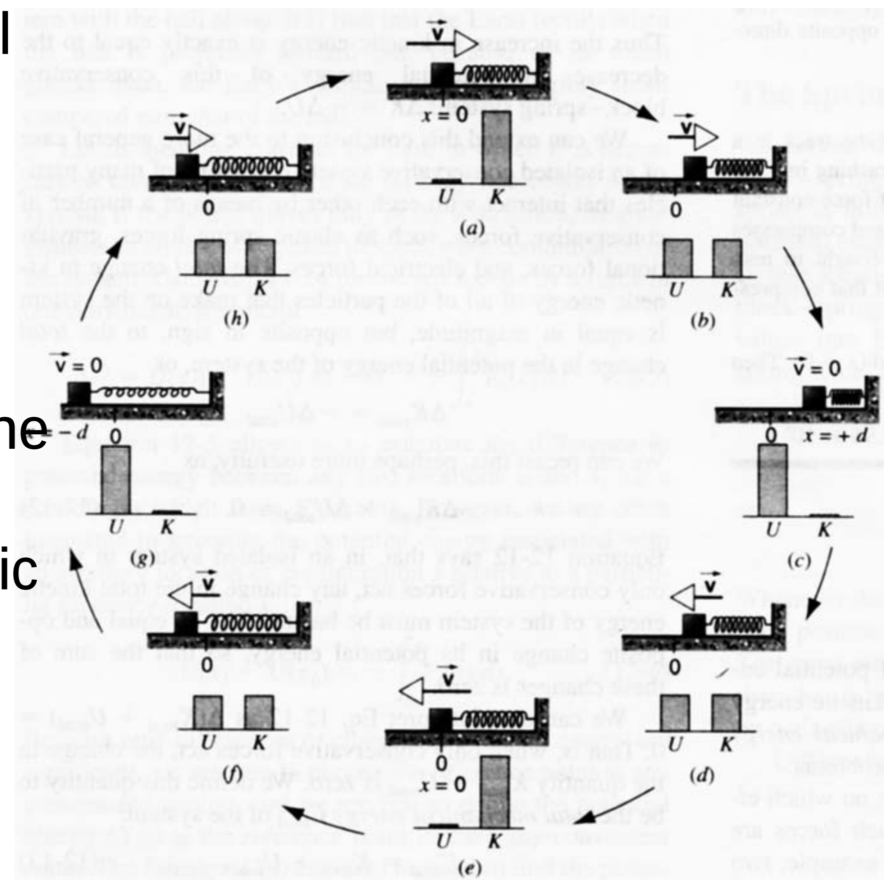
- Conservation of Mechanical Energy
  - For a system, if $\sum W_{i-\text{external}} + \sum W_{i-\text{internal-nonconserv}} = 0$

    then $\Delta E_{\text{mech}} = 0$ or $K_f + U_f = K_i + U_i = \text{constant}$

  - In a system in which only internal conservative forces act, the total mechanical energy remains constant.

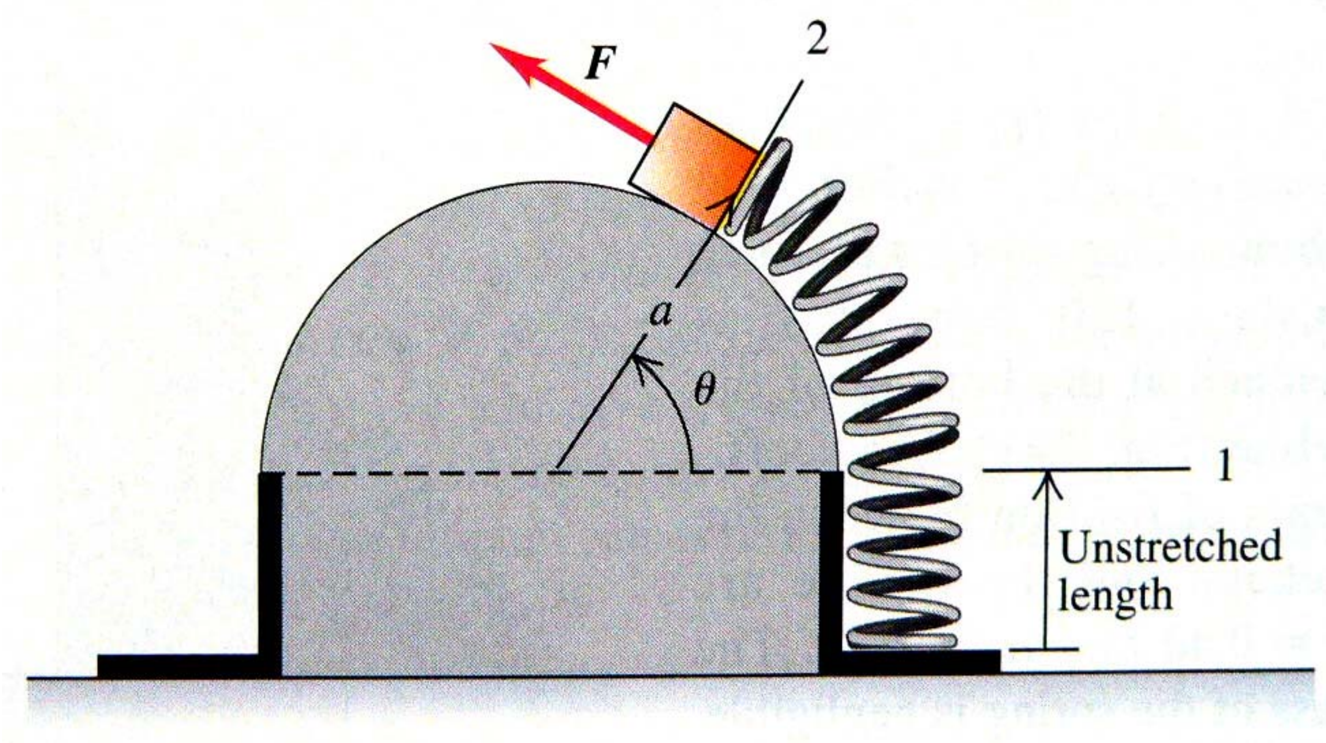
  - When $\Delta E_{\text{mech}}=0$, it is the internal conservative forces acting within the system that change kinetic into potential or potential into kinetic energy.

$$U \underset{W_{\text{conservative}}<0}{\overset{W_{\text{conservative}}>0}{\rightleftarrows}} K$$



Example

Variable force *F* is maintained tangent to a frictionless semicircular surface. By a slowly varying force *F*, a block with mass of *m* is moved, and spring to which it is attached is stretched from position 1 to position 2. The spring has negligible mass and force constant *k*. The end of the spring moves in an arc of radius *a*. Calculate the work done by the force *F* from position 1 to 2.(θ)



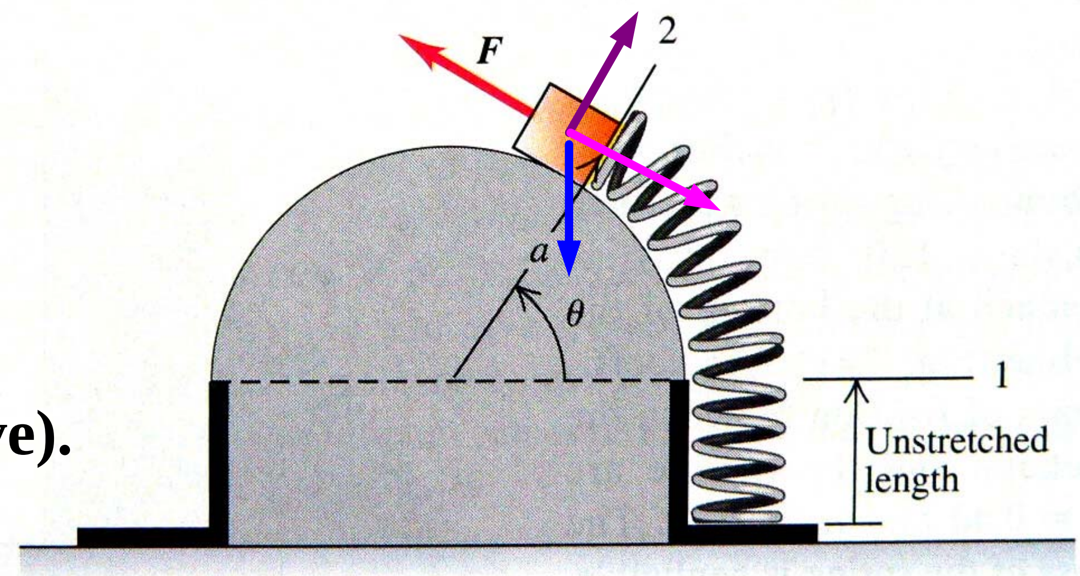
Solution II: by using work-energy theorem.

External force: *F*;

Internal forces:

N (non-conservative);

mg and F_s (conservative).



Choose the reference point at position 1 both for gravitational and elastic energy of block-spring-Earth system.

$$W_F = \Delta E = \Delta U = \frac{1}{2}ks^2 + mga\sin\theta = \frac{1}{2}ka^2\theta^2 + mga\sin\theta$$

Problem

- Ch7 (P164)
  - 38, 69, 70
- Ch8 (P194)
  - 35, 67, 79