SC1007
Data Structures and Algorithms

Week 9: Searching

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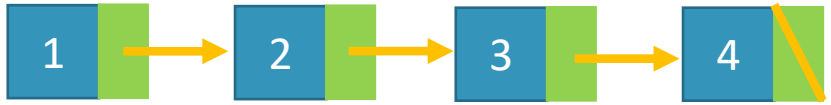
Overview

- Exhaustive Algorithm:
  - Sequential Search
- Decrease-and-conquer Algorithm:
  - Binary Search
  - Jump Search

Time Complexity of Sequential Search

```
def search(head, a):
    pt = head                                   ..... c1
    while pt is not None and pt.key != a:
        pt = pt.next                            ..... c2
    return pt
```



Assume that the search key ***a*** is in the list

1. Best-case analysis: c_1 when ***a*** is the first item in the list => $\Theta(1)$
2. Worst-case analysis:
3. Average-case analysis:

Time Complexity of Sequential Search

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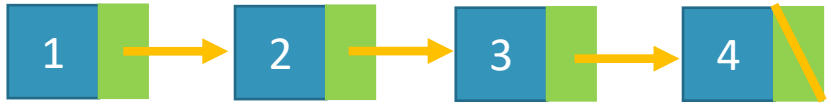
```

c_1

c_2 (n-1) iterations



Assume that the search key ***a*** is in the list

1. Best-case analysis: c_1 when ***a*** is the first item in the list => $\Theta(1)$
2. Worst-case analysis: $c_2 \cdot (n-1) + c_1$ => $\Theta(n)$ when ***a*** is the last item in the list
3. Average-case analysis $p_1 \times time\ to\ search\ for\ item\ 1 + p_2 \times time\ to\ search\ for\ item\ 2 + \cdots + p_n \times time\ to\ search\ for\ item\ n$

Time Complexity of Sequential Search

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def search(head, a):
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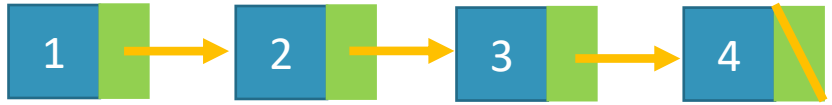
c_2 (n-1) iterations



Assume that the search key ***a*** is always in the list

1. Best-case analysis: c_1 when **a** is the first item in the list => Θ (1)
2. Worst-case analysis: $c_2 \cdot (n-1) + c_1$ => Θ (n) when **a** is the last item in the list
3. Average-case analysis: $\boldsymbol{p_1c_1 + p_2(c_1 + c_2) + p_3(c_1 + 2c_2) + \cdots + p_n(c_1 + (n-1)c_2)}$

Assume that every item in the list has an equal probability as a search key, i.e., $p_i = \frac{1}{n}$

$$\frac{1}{n}[c_1 + (c_1 + c_2) + (c_1 + 2c_2) + \cdots + (c_1 + (n-1)c_2)] = \frac{1}{n}\sum_{i=1}^{n}(c_1 + c_2(i-1))$$

$$= \frac{1}{n}[nc_1 + c_2 \sum_{i=1}^{n}(i-1)\,]$$

$$= c_1 + \frac{c_2}{n} \cdot \frac{n}{2}(0 + (n-1)) = c_1 + \frac{c_2(n-1)}{2} = \Theta(n)$$

Time Complexity of Sequential Search

```
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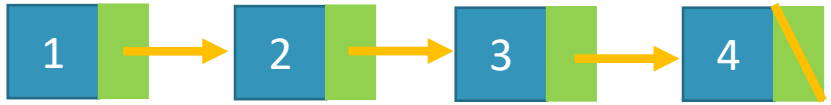
```



If the search key is in the list, on average: $c_1 + \frac{c_2(n-1)}{2} = \Theta(n)$

If the search key, a, is not in the list, then the time complexity is

$$c_1 + nc_2 = \Theta(n)$$

Since the probability of the search key is in the list is unknown, we only can have

$$f(n) = P(a \text{ in the list})(c_1 + \frac{c_2(n-1)}{2}) + (1 - P(a \text{ in the list}))(c_1 + nc_2)$$

It is still a linear function. **Θ (n)**

Time Complexity of Sequential Search

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```
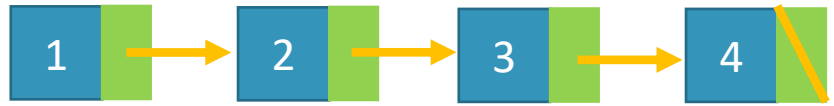
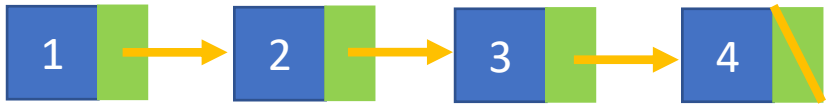



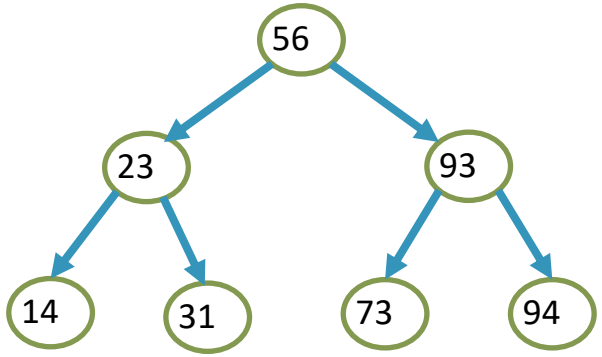
- The data is stored unordered
- To search a key, every element is required to read and compare
- This is a brute-force approach or a näive algorithm
- Its time complexity is **O(n)**
- How can we improve it?

Devide and Conquer: Binary Search

- Given a sorted list



- Whether a search key ***a*** is in the list?
  - Given a sorted list, e.g.,
    - 14, 23, 31, 56, 73, 93, 94
  - We can build a BST

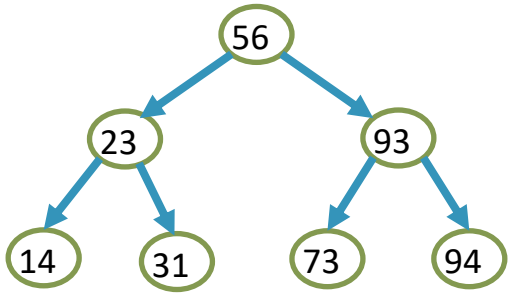


Time Complexity of Binary Search

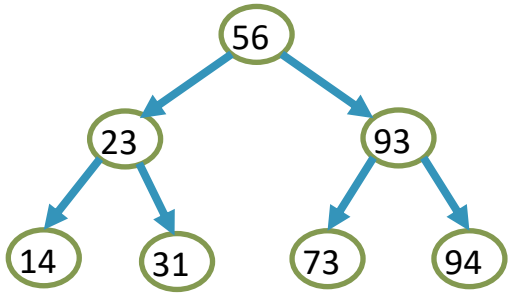
```
def binary_search_recursive(arr, left, right, target):
    if left > right:
        return -1
    mid = left + (right - left) // 2
    if arr[mid] == target:
        return mid
    elif arr[mid] < target:
        return binary_search_recursive(arr, mid + 1, right, target)
    else:
        return binary_search_recursive(arr, left, mid - 1, target)
```

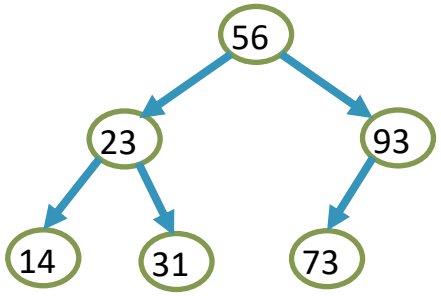
```
def binary_search(self, target, current_node):
    if current_node is None:
        return False
    elif target == current_node.data:
       return True
    elif target < current_node.data:
       return self.binary_search(target,current_node.left)
    else:
       return self.binary_search(target,current_node.right)
```

- Given a sorted list, e.g.,
  - 14, 23, 31, 56, 73, 93, 94
- We can build a BST



Terminology





- The Height of a tree: The number of **edges** on the longest path from the root to a leaf
- The Depth of a node: The number of edges from the node to the root of its tree.

For a complete binary tree with height *H*, we have:

$$2^H - 1 < n \leq 2^{H+1} - 1$$

where *n* is an integer and the size of the tree

$$2^H \leq n < 2^{H+1} \qquad \textbf{(e.g., } 7 < n \leq 15 \equiv 8 \leq n < 16\textbf{)}$$

$$H \leq \log_2 n < H+1$$

If H is an integer, H+1 must be the next integer.

Height = $\lfloor \log_2 n \rfloor$

Binary Search – Worst Case Time Complexity

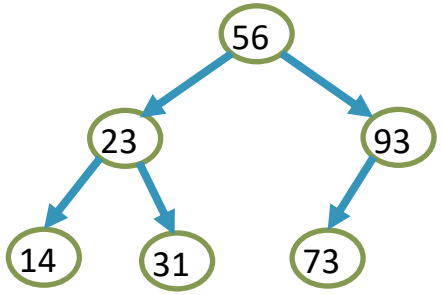
```
def binary_search(self, target, current_node):
    if current_node is None:
        return False
    elif target == current_node.data:
       return True
    elif target < current_node.data:
       return self.binary_search(target,current_node.left)
     else:
       return self.binary_search(target,current_node.right)
```

f(n)

Constant c

$f((n-1)/2)$

$f((n-1)/2)$



- Assume a complete binary tree

$$f(n) = f\left(\frac{n-1}{2}\right) + c \quad = f\left(\frac{\left(\frac{n-1}{2}\right)-1}{2}\right) + 2c = f\left(\frac{n-1-2}{2^2}\right) + 2c$$

$$= f\left(\frac{\frac{n-1-2}{2^2}-1}{2}\right) + 3c = f\left(\frac{n-1-2-2^2}{2^3}\right) + 3c$$

…

Binary Search – Worst Case Time Complexity

$$f(n) = f\left(\frac{n-1}{2}\right) + c$$

$$= f\left(\frac{n-(1+2+\cdots+2^{k-2}+2^{k-1})}{2^k}\right) + kc$$

$$= f\left(\frac{n-2^k+1}{2^k}\right) + kc$$

$$= f(1) + kc$$

$$= c + kc$$

$$= (\lfloor \log_2 n \rfloor + 1)c$$

$$= \Theta(\log_2 n)$$

From previous slide:

$2^k \leq n < 2^{k+1} \equiv$
$2^k - 1 < n \leq 2^{k+1} - 1$

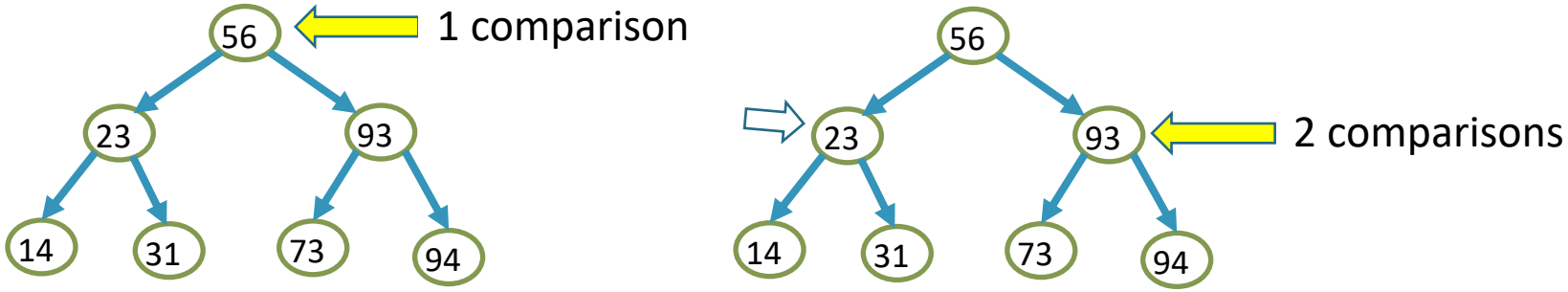
Therefore
$k = \lfloor \log_2 n \rfloor$

Binary Search – Average Case Time Complexity

- $A_s(n)$: # of comparisons for successful search
- $A_f(n)$: # of comparisons for unsuccessful search (worst case): $\Theta(\log_2 n)$

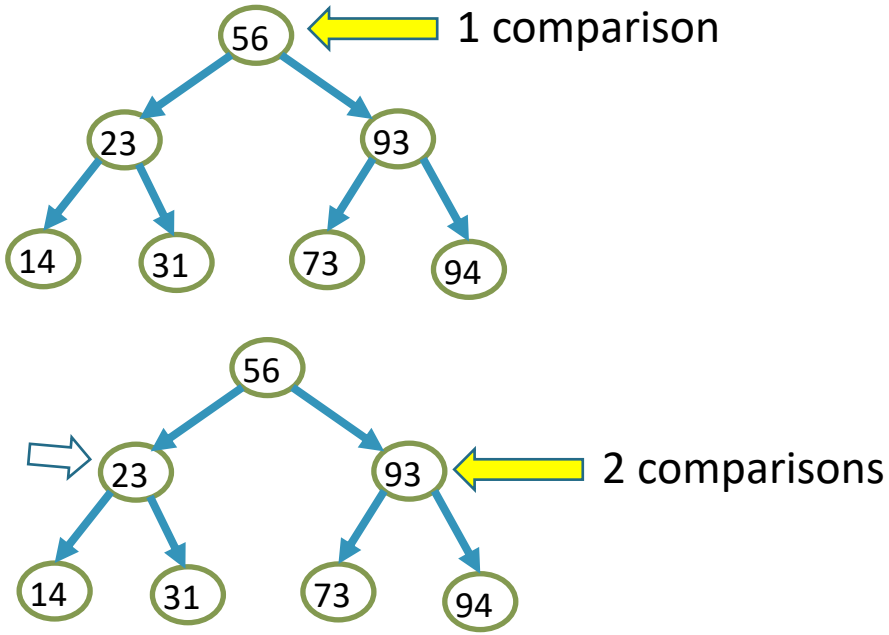
$$A(n) = qA_s(n) + (1-q)A_f(n)$$

For $A_s(n)$, we assume $n = 2^k - 1$ first



Binary Search – Average Case Time Complexity

$$A(n) = qA_s(n) + (1-q)A_f(n)$$

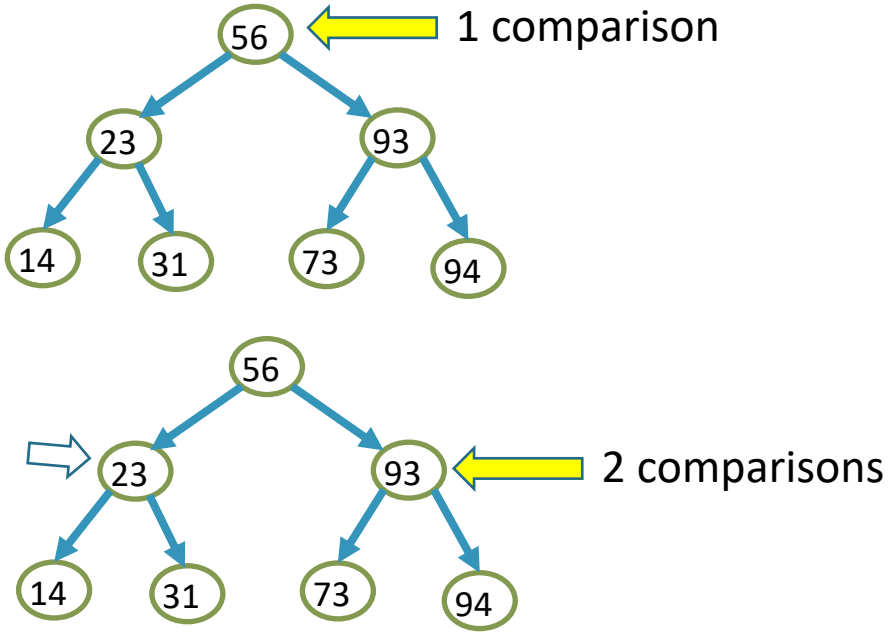


- For $A_s(n)$, we assume $n = 2^k - 1$ first
- We can observe that:
  - 1 position requires 1 comparison (level 1)
  - 2 positions requires 2 comparisons (level 2)
  - 4 positions requires 3 comparisons (level 3)
  - … …
  - 2^{t-1} positions requires t comparisons ((level t)

$$A_s(n) = \sum_{t=1}^{k} p_t \times (\#comparisons\ at\ level\ t)$$
$$= \sum_{t=1}^{k} \frac{1}{n} \times (\#positions\ at\ level\ t) \times (\#comparisons\ at\ level\ t)$$
$$= \sum_{t=1}^{k} \frac{1}{n} \times 2^{t-1} \times t$$

Binary Search – Average Case Time Complexity

$$A(n) = qA_s(n) + (1-q)A_f(n)$$



- Assuming $n=2^k-1$, we have

$$A_s(n) = \frac{1}{n}\sum_{t=1}^{k} t2^{t-1}$$

$$= \frac{(k-1)2^k + 1}{n} \quad \text{(geometric series)}$$

$$= \frac{[\log_2(n+1) - 1](n+1) + 1}{n}$$

$$= \log_2(n+1) - 1 + \frac{\log_2(n+1)}{n}$$

Binary Search – Average Case Time Complexity

- The time complexity is

$$A_q(n) = qA_s(n) + (1-q)A_f(n)$$
$$= q\left[\log_2(n+1) - 1 + \frac{\log_2(n+1)}{n}\right] + (1-q)(\log_2(n+1))$$
$$= \log_2(n+1) - \mathrm{q} + \mathrm{q}\frac{\log_2(n+1)}{n}$$
$$= \Theta(\log_2 n)$$

- q is probability which is always ≤ 1
- $\frac{\log_2(n+1)}{n}$ is very small especially when n >> 1
- Binary search does approximately $\log_2(n+1)$ comparisons on average for n elements.

Solution? Jump Search

Jump Search:

- Moves sequentially in blocks, reducing random accesses.
- Does fewer random accesses and switches to linear search only in a small block.
- Advantage: More cache-efficient and optimized for slow storage.

How Jump Search works?

- How Jump Search Works:
  - **Determine Block Size:** typically the square root of the array's length (√n).
  - **Jump Ahead:** starts at the beginning of the array and jumps forward by the determined block size
  - **Identify Block:** This jumping continues until an element is found that is greater than or equal to the target element. This indicates that the target element, if present, must lie within the previous block or the current block.
  - **Linear Search:** Once the potential block is identified, a linear search is performed within that smaller block to pinpoint the exact location of the target element.
- Example:
  - Consider a sorted array [0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610] and a target element 55. If the block size is 4 (=√16):
    - Jump to index 0, value 0. **0 < 55.**
    - Jump to index 4, value 3. **3 < 55.**
    - Jump to index 8, value 21. **21 < 55.**
    - Jump to index 12, value 144. **144 > 55**. This indicates that 55 must be in the block between index 8 and 12.
    - Perform a linear search from index 8 to 12 to find 55 at index 10.

Jump Search

```
def jump_search(arr, target):
    n = len(arr)
    step = int(math.sqrt(n))
    prev = 0

    while prev < n and arr[min(step, n) - 1] < target:
        prev = step
        step += int(math.sqrt(n))
        if prev >= n:
            return -1
    for i in range(prev, min(step, n)):
        if arr[i] == target:
            return i
    return -1
```

- When binary search is costly, e.g., searching for an element in a very large sorted dataset stored on a slow storage medium, like a database on disk or an external hard drive

Time Complexity of Jump Search

- Assume that the search key ***a*** is in the list
  1. Best-case: Θ (1)
  2. Worst-case: Θ $(\sqrt{n})$ + Θ$(\sqrt{n})$ = Θ $(\sqrt{n})$
  3. Average-case: $\sum_{i=1}^{\sqrt{n}} p_i$ Θ$(\sqrt{n}) = \sum_{i=1}^{\sqrt{n}} \frac{1}{\sqrt{n}}$ Θ$(\sqrt{n}) =$ Θ$(\sqrt{n})$

- Assume that the search key ***a*** is not in the list

Θ $(\sqrt{n})$ + Θ$(\sqrt{n})$ = Θ $(\sqrt{n})$

- On average, the time complexity of Jump Search is Θ$(\sqrt{n})$

Summary

- Exhaustive Algorithm: Sequential Search
  - Time complexity O(n)
- Decrease-and-conquer Algorithm:
  - Binary Search: Time complexity $O(\log_2 n)$
  - Jump Search: Time complexity $O(\sqrt{n})$

	Best Case	Average Case	Worst Case	Overall
Sequential	$\Theta(1)$	$\Theta(n)$	$\Theta(n)$	$O(n)$
Binary	$\Theta(1)$	$\Theta(\log_2 n)$	$\Theta(\log_2 n)$	$O(\log_2 n)$
Jump	$\Theta(1)$	$\Theta(\sqrt{n})$	$\Theta(\sqrt{n})$	$O(\sqrt{n})$