The Error Function

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1 Introduction

In mathematics, the error function (also called the Gauss error function) is a special function (non-elementary) of sigmoid shape that occurs in probability, statistics, and partial differential equations describing diffusion. It is defined as:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{1}$$

$$= \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{2}$$

In statistics, for nonnegative values of x, the error function has the following interpretation: for a random variable Y that is normally distributed with mean 0 and variance 0.5, erf(x) describes the probability of Y falling in the range [x, x]. There are several closely related functions, such as the complementary error function, the imaginary error function, and others.

2 Numerical solution

The error function can also be found numerically by solving the following differential equation:

$$u'(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \tag{3}$$

with the initial condition

$$u(0) = 0 \tag{4}$$

3 Plot visualization

Both the analytical and numerical solutions are shown in figure 1. The analytical solution (gsl erf(x)) is GSL's implementation of equation 2, while the numerical solution (myerf(x)) is computed by integrating equation 3 with 4.

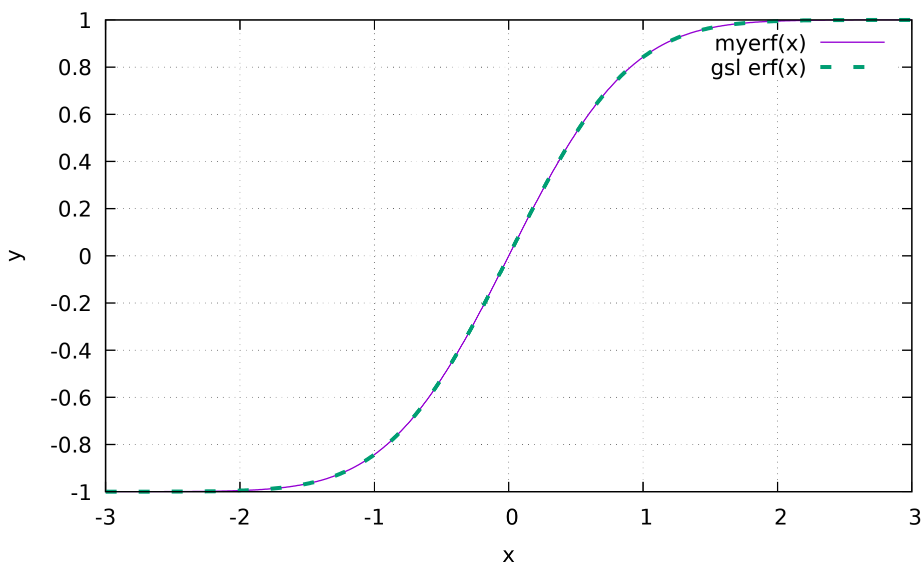

Figure 1: Numerical and analytical representations of the error function.