Problem 17 - Arctangent function by numerical root finding

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1 Introduction

This report covers a numerical implementation of the arctangent function

$$a = arctan(x), \tag{1}$$

by finding roots of the equation

$$f(a) = tan(a) - x = 0. \tag{2}$$

That is, given x find a via eq. (2).

2 Implementation

The numerical solution is implemented via a root-finding algorithm in GSL, in this case the *gsl_multiroot_fdfsolver_hybridsj* solver which relies on analytical derivatives, here

$$\frac{d}{da}(tan(a) - x) = 1 + tan(a)^2. \tag{3}$$

The solver needs a starting point (henceforth called a_0), which if not chosen carefully can lead to undesirable results. All root-finding algorithms in the GSL multiroot library uses some variation of the newton iteration

$$a_1 = a_0 - J(a_0)^{-1} f(a_0), \tag{4}$$

with a_0 and $f(a_0)$ being vector quantities and $J(a_0)$ being the Jacobian matrix of the system. Looking at figure (1) we see why we must choose a_0 with care. For instance, setting $a_0 = 0$ is fine for small values of x, but the algorithm will soon run rampant as x exceeds $\pm\pi$ (for $a_0 = 0$), since the *arctan* function only is defined for the initial repetition of the tangent function

$$x = tan(a) \quad -\frac{\pi}{2} < a < \frac{\pi}{2}. \tag{5}$$

Too alleviate this problem, the following definition for a_0 is made

$$a_0 = \begin{cases} \dfrac{\pi}{2 + 1/x} & \text{if} \quad x > 0 \\ -\dfrac{\pi}{2 - 1/x} & \text{if} \quad x < 0 \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

This ensures that the algorithm will have a starting point that doesn't lead it beyond the limits in eq. (5) (at least up to $x = \pm 100000$).

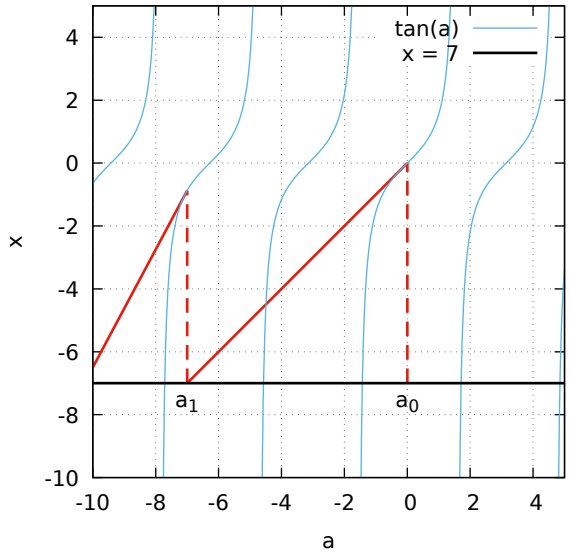

Figure 1: Progress of a Newton-Raphson algorithm starting with $a_0 = 0$ and $x = 7$

3 Results

The implemented numerical algorithm can be seen in figure (2), along with the standard atan definition from the c library math.h.

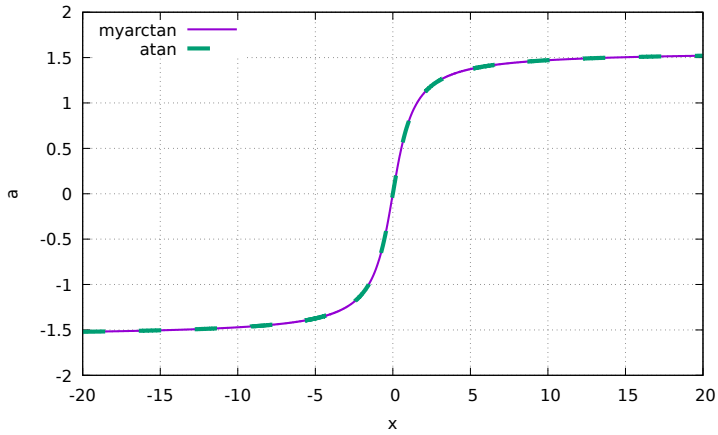

Figure 2: Visualization of the arctan function, showing both numerical implementation (myarctan) and the atan function from the math.h librabry