A New way to calculate π

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Introduction

The mathematical constant π is one of the most well-known and fundamental constants in mathematics it is one of the most beautiful numbers and it was discovered more than 4,000 years ago. It is defined as the ratio of a circle's circumference to its diameter. In this paper, I propose a new method for calculating π.

The Geometry of a Circle

Let's take the equation of a circle of radius 1 centered at the origin:

$$x^2 + y^2 = 1.$$

The area of this circle is π (area = radius$^2\pi$). However, directly working with this equation is challenging because it does not represent a function. To address this, we can solve for y:

$$y = \pm\sqrt{1 - x^2}$$

This gives two functions, one representing the upper half of the circle and the other representing the lower half.

For simplicity, we will focus on the upper half of the circle. The equation for this is:

$$y = \sqrt{1 - x^2}, \quad -1 \leq x \leq 1.$$

Integral

By integrating the function $y = \sqrt{1 - x^2}$ from 0 to 1, we find the area of the semicircle. This value is $\frac{\pi}{4}$ because it is a quarter of a unit circle.

$$\frac{\pi}{4} = \int_0^1 \sqrt{1 - x^2}\, dx.$$

But taking the integral of this function gives a trigonometry answer where the operation uses pi in them like $\frac{1}{2}arcsin(x) + \frac{1}{2}x\sqrt{1-x^2} + C$, so we need a different way to look at the function.

binomial expansion

Another way to write the function of our semicircle is $(1-x^2)^{\frac{1}{2}}$. Now we can use the binomial expansion. The most well-known form of the binomial expansion is

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

But we can't use this form because our function is to the power of a fraction and a sum function takes only integers. So we will use the General Binomial Expansion for Fractional Powers.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

When we put the semicircle function in the Binomial Expansion we get

$$(1-x^2)^{\frac{1}{2}} = 1 + \frac{-x^2}{2} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(-x^2)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(-x^2)^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}(-x^2)^4 + ...$$

Now we will simplify

$$(1-x^2)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^4 - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^8 + ...$$

writing it in sum

After writing the series let's write it in a sum. So let's write again the series but we will change some parts so it will have repeating parts so it will be easier to write in sum. So 1 will be $\frac{1}{0!}$ and $\frac{1}{2}x^2$ will be $\frac{\frac{1}{2}}{1!}x^2$

$$(1-x^2)^{\frac{1}{2}} = \frac{1}{0!} - \frac{\frac{1}{2}}{1!}x^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}x^4 - \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}x^6 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}x^8 + ...$$

Now we can start writing the sum. We can see that the factorial is always going up and starting from one so we will start by writing.

$$\sum_{n=0}^{\infty} \frac{}{n!}$$

Also, something that is easy to see is that the series alternate between + and - starting from + so we can write it as $(-1)^n$, and x is just to the power of the even numbers so we can write it as x^2n so now it looks like

$$\sum_{n=0}^{\infty}(-1)^n\frac{}{n!}x^{2n}$$

Now we just need to find the product in the numerator but after some inspections, we can see that it is just $\prod_{k=0}^{n-1}(\frac{1}{2}-k)$

So:

$$\sqrt{1-x^2}=(1-x^2)^{\frac{1}{2}}=\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{k=0}^{n-1}(\frac{1}{2}-k)}{n!}x^{2n}$$

Taking the integral of the sum

now this is the last and easiest step and the one I got stuck on the most time. I thought it would be very hard to calculate the integral of a sum. But it has the same rules as always. So it will be:

$$\int\sqrt{1-x^2}\,dx=\sum_{n=0}^{\infty}((-1)^n\frac{\prod_{k=0}^{n-1}(\frac{1}{2}-k)}{n!(2n+1)}x^{2n+1})+C$$

last part

now we just insert 0 and 1 to x in the sum and it will be equal to $\frac{\pi}{4}$

$$\int_0^1\sqrt{1-x^2}\,dx=\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{k=0}^{n-1}(\frac{1}{2}-k)}{n!(2n+1)}=\frac{\pi}{4}$$

$$4\sum_{n=0}^{\infty}(-1)^n\frac{\prod_{k=0}^{n-1}(\frac{1}{2}-k)}{n!(2n+1)}=\pi$$

Conclusion

The value of π can be derived from many different perspectives. There are a lot of ways to calculate π each revealing a different aspect of its beauty and significance. This way demonstrated how calculus can π. There are a lot of better ways to calculate π for example After I discovered this way I also discovered that Newton found a similar equation $6\sum_{n=0}^{\infty}\frac{(2n)!}{2^{4n+1}(n!)^2(2n+1)}$ using a similar way but he was 23 when he did it and I'm just 15 so I beat him!