Stochastic Modelling of Financial Derivatives

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Total Marks: 100 (75 + Bonus) **Deadline:** 10^{th} June 2025

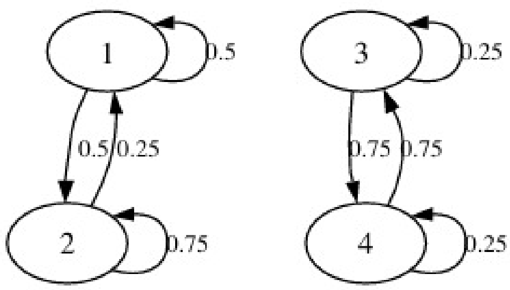
Week-2 Assignment: Markov Chains

1. **Two-State Loop (10 Marks)**
   Consider the Markov chain shown below, with state space $\{1, 2, 3, 4\}$, where the labels next to arrows indicate the probabilities of those transitions.

   (a) Write down the transition matrix Q for this chain.

   (b) Which states (if any) are recurrent? Which states (if any) are transient?

   (c) Find two different stationary distributions for the chain.



2. **Winning Streak (10 Marks)**
   Every time that the team wins a game, it wins its next game with probability 0.8; every time it loses a game, it wins its next game with probability 0.3. If the team wins a game, then it has dinner together with probability 0.7, whereas if the team loses then it has dinner together with probability 0.2.

   (a) In the long run, what proportion of the games do the team win?

(b) In the long run, what proportion of games result in a team dinner?

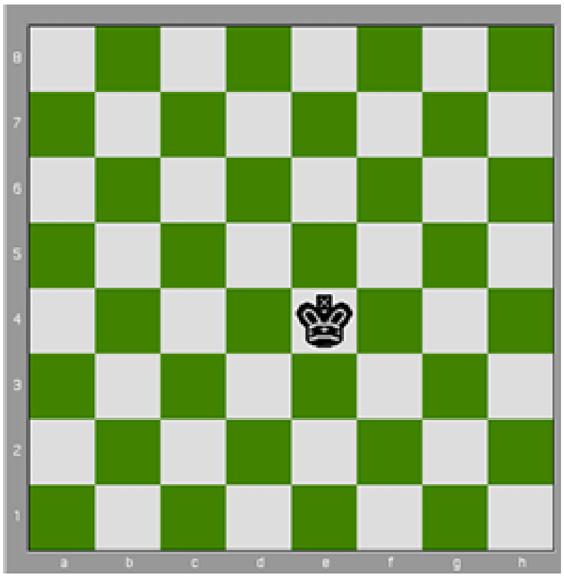
(c) What is the expected number of games the team needs to play for a dinner?

3. **Cat and Mouse Game (10 Marks)**
A cat and a mouse move independently back and forth between two rooms. - At each time step, the cat moves from its current room to the other with probability 0.8. - The mouse moves from Room 1 to Room 2 with probability 0.3, and from Room 2 to Room 1 with probability 0.6.

   (a) Find the stationary distributions of the cat chain and the mouse chain.

   (b) There are 4 possible (cat, mouse) states. Define Z_n to be the number of the current state at time n. Is $Z_0, Z_1, Z_2, \ldots$ a Markov chain?

4. **The Wandering King (20 Marks)**
In chess, a king can move one square in any direction (horizontal, vertical, diagonal). The king moves randomly on an 8×8 empty chessboard, and from each square all legal moves are equally likely.



Find the stationary distribution of this Markov chain. (Do not write the full 64-dimensional vector.) Classify the 64 squares into "types" and explain what the stationary probability is for each type.

5. **Stock Price Model (25 Marks)**
The tick size in a market refers to the smallest increment possible between price quotes. For NSE, this is Rs. 0.01, i.e., a stock whose value is Rs. 100 can take values such as 100.01 or 99.99. No quotes like 100.005, 100.003, 99.998, etc., are permissible. In other words, the stock price changes in increments of 0.01 only.

   Suppose the market is open from 10:00 am to 3:00 pm. Assume the tick size of stocks to be 0.01. A stock, ABC, starts trading at Rs. 120 at 10:00 am. Every 5 seconds, the probability of ABC moving up a tick is 0.1, staying at the same level is 0.85, and moving down a tick is 0.05.

(a) Is the stock price recurrent?

(b) Does the stationary distribution of the stock price exist?

(c) *American call option.* The American call option is a financial contract on some stock S with a payoff $\max(S_t - K, 0)$, where $t \leq T$. Here, S_t denotes the price of the stock at time t. The call option expires at time T. If the stock price S_t is greater than K, the strike price, the owner of the option can exercise it any time t before expiry to earn $S_t - K$.

As an example, suppose an American option on stock ABC with strike price $K = 110$ expires at the end of the day. If the owner of the option were to exercise it at 10:00 am, since the price is $S_t = 120$, she would earn $\max(120 - 110, 0) = 10$ at 10:00 am.

Suppose you own an American option on ABC with strike price $K = 125$. What is the probability that you will be able to earn a payoff of Rs. 5 before 1:00 pm? You will immediately exercise the option if the stock touches Rs. 130.

Hint: You may want to simulate this in code to get the answer here.

6. **Substitution Shuffle (25 Marks) (BONUS)**
Markov chains have recently been applied to codebreaking; this problem will consider one way in which this can be done. A *substitution cipher* is a permutation g of the letters from a to z, where a message is enciphered by replacing each letter each letter α by $g(\alpha)$. For example, if g is the permutation given by:

```
abcdefghijklmnopqrstuvwxyz
zyxwvutsrqponmlkjihgfedcba
```

where the second row lists the values $g(a), g(b), \ldots, g(z)$, we would then encipher the word `statistics` as `hgzgrhgrxh`. The state space consists of all $26! \approx 4 \cdot 10^{26}$ permutations of the letters a through z.

(a) Consider the chain that picks two different random coordinates between 1 and 26 and swaps those entries of the 2nd row, e.g., if we pick 7 and 20, then

```
abcdefghijklmnopqrstuvwxyz
zyxwvutsrqponmlkjihgfedcba
```

becomes

```
abcdefghijklmnopqrstuvwxyz
zyxwvugsrqponmlkjihtfedcba
```

Consider a Markov chain where two random positions in the second row are swapped. What is the probability of going from permutation g to h in one step (for all g , h)? What is the stationary distribution of this chain?

(b) Suppose we have a system that assigns a positive "score" $s(g)$ to each permutation g (*intuitively*, this could be a measure of how likely it would be to get the observed enciphered text, given that g was the cipher used). Consider the following Markov chain. Starting from any state g, generate a "proposal" h using the chain from (a). If $s(g) \leq s(h)$, then go to h (i.e., accept the proposal). Otherwise, flip a coin with probability $s(h)/s(g)$ of Heads. If Heads, go to h (i.e., accept the proposal); if Tails, stay at g. Show that this chain is reversible and has stationary distribution proportional to the list of all scores $s(g)$.

Hint : for $g \neq h$, let $q(g, h)$ be the probability of going from g to h in one step, and show that $s(g)q(g, h) = s(h)q(h, g)$. To compute $q(g, h)$, note that it is the probability of proposing h when at g, times the probability of accepting the proposal.