Assignment 2.1

2) Winning Streak Problem Solution

Problem Setup

- After a win: $P(\text{next win}) = 0.8$, $P(\text{dinner}) = 0.7$
- After a loss: $P(\text{next win}) = 0.3$, $P(\text{dinner}) = 0.2$

(a) Long-run proportion of wins

Define states W (win), L (loss). Transition matrix:

$$P = \begin{bmatrix} P(W \to W) & P(W \to L) \\ P(L \to W) & P(L \to L) \end{bmatrix} = \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix}$$

Let π_W and π_L be the steady-state probabilities. Then:

$$\pi_W + \pi_L = 1, \quad 0.8\pi_W + 0.3\pi_L = \pi_W \Rightarrow \pi_L = \frac{2}{3}\pi_W \Rightarrow \pi_W = \frac{3}{5}, \quad \pi_L = \frac{2}{5}$$

Answer (a): $\boxed{60\%}$ of games are wins.

(b) Long-run proportion of games with dinner

$$P(\text{dinner}) = 0.7 \cdot \pi_W + 0.2 \cdot \pi_L = 0.7 \cdot 0.6 + 0.2 \cdot 0.4 = 0.5$$

Answer (b): $\boxed{50\%}$ of games result in dinner.

(c) Expected number of games until dinner

Let E_W, E_L be expected games until dinner starting from win/loss. Using first-step analysis:

$$E_W = 0.7(1) + 0.3[0.8(1 + E_W) + 0.2(1 + E_L)] = 1 + 0.24E_W + 0.06E_L \Rightarrow 0.76E_W - 0.06E_L = 1 \tag{1}$$

$$E_L = 0.2(1)+0.8[0.3(1+E_W)+0.7(1+E_L)] = 1+0.24E_W+0.56E_L \Rightarrow -0.24E_W+0.44E_L = 1 \quad (2)$$

Solving (1) and (2) gives:

$$E_W = \frac{25}{16}, \quad E_L = \frac{25}{8}$$

Expected games until dinner (averaging over steady state):

$$E = \pi_W \cdot E_W + \pi_L \cdot E_L = 0.6 \cdot \frac{25}{16} + 0.4 \cdot \frac{25}{8} = \frac{35}{16} \approx 2.19$$

Answer (c): $\boxed{\frac{35}{16} \approx 2.19}$ games.

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6) Substitution Shuffle - Markov Chain Solution

Part (a): Basic Swap Chain Analysis

Transition Probabilities:

- $P(g \to g) = 0$ (no self-transitions, always swap two different positions)
- $P(g \to h) = \frac{1}{\binom{26}{2}} = \frac{1}{325}$ if h differs from g by exactly one swap
- $P(g \to h) = 0$ otherwise

Stationary Distribution:

The chain is irreducible and finite. Since all permutations are symmetric under the swap operation, the stationary distribution is uniform:

$$\pi(g) = \frac{1}{26!} \quad \text{for all permutations } g$$

Part (b): Metropolis-Hastings Chain

Transition Probabilities $q(g, h)$ for $g \neq h$:

$$q(g,h) = \begin{cases} \frac{1}{325}, & \text{if } s(g) \leq s(h) \\ \frac{s(h)}{325 \cdot s(g)}, & \text{if } s(g) > s(h) \\ 0, & \text{otherwise} \end{cases}$$

Proving Reversibility:

We want to show that:

$$s(g) \cdot q(g,h) = s(h) \cdot q(h,g)$$

Case 1: $s(g) \leq s(h)$

$$s(g) \cdot q(g,h) = \frac{s(g)}{325}, \qquad s(h) \cdot q(h,g) = s(h) \cdot \frac{s(g)}{325 \cdot s(h)} = \frac{s(g)}{325}$$

Case 2: $s(g) > s(h)$

$$s(g) \cdot q(g,h) = s(g) \cdot \frac{s(h)}{325 \cdot s(g)} = \frac{s(h)}{325}, \qquad s(h) \cdot q(h,g) = \frac{s(h)}{325}$$

Stationary Distribution:
From the detailed balance condition:

$$\pi(g) \cdot q(g,h) = \pi(h) \cdot q(h,g)$$

We conclude:

$$\pi(g) \propto s(g) \quad \Rightarrow \quad \pi(g) = \frac{s(g)}{\sum_i s(i)} \quad \text{(sum over all permutations)}$$