Stochastic Modelling of Financial Derivatives

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Total Marks: 100 **Deadline:** 30th May 2025

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1 Problems

1. **Derangement Problem (2 Marks)**
   N letters are to be put in N separate envelopes. Assuming an envelope can hold only a single letter. What is the probability that at least one letter is in the correct envelope? Find an approximation of this probability for $N = 50$.

2. **Showman (6 Marks)**
   You have 3 identical presents. The good gift has 1000 dollars, while others have nothing. The host of the party asks you to select a present. If you select the good gift, you keep it. You select Present 1. But the host opens the second present and it has nothing. Host knows where the money is, always reveals the blank prize. (Also assume that, when he chooses the prize gift, the host chooses one of the blank with equal probability.) You are given the option to switch your guess to the third present. What are your expected winnings if you switch?

3. **True or False (4 Marks each)**
   Let A, B, C and D be four events such that $\mathbb{P}(B \cap C) > 0$.

   (a) $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|B \cap C)\mathbb{P}(B|C)$

   (b) $\mathbb{P}(A \cap B|C) = \mathbb{P}(A|C)\mathbb{P}(B|C)$ for independent events A and B.

   (c) Given $\mathbb{P}(A|D \cap B^c) > \mathbb{P}(A|D \cap B)$ and $\mathbb{P}(A|D^c \cap B^c) > \mathbb{P}(A|D^c \cap B)$, $\mathbb{P}(A|B)$ must be greater than $\mathbb{P}(A|B^c)$.

4. **Construct the following or disprove its existence (4 + 4 + 7 Marks)**

   (a) A discrete random variable X for which $\mathbb{E}(X)$ is finite but $\mathbb{E}(X^2)$ is not finite.

   (b) A continuous random variable X for which $\mathbb{E}(X)$ is finite but $\mathbb{E}(X^2)$ is not finite.

   (c) A random variable X with $\mathbb{E}(X) = 1$, but $\mathbb{E}(e^{-X}) < 1/3$

5. **Expectation of Statistic (8 Marks)**
From N identical lotteries with prizes, $1, 2, ..., N$, $n \leq N$ tickets are drawn with replacement. You are allowed to keep only the maximal prize ticket. Let M = prize money obtained. Find $\mathbb{E}(M)$.

6. **Geometry of a line (5 Marks)**
Find the probability two points taken on a line segment of length d has distance between them less than $d/3$.

7. **Gossip Man (6 + 4 Marks)**
In a town of $(n+1)$ inhabitants, a person tells a rumor to a second person, who in turn tells it to a third one, and so on. At each step a random person is chosen to listen the rumor. Find the probability that the rumor will be told r times without

   (a) returning to the originator

   (b) being repeated to any persons

   Repeat when at each step the rumor is told to a gathering of N randomly chosen people.

8. **Bern Lee (Who?) Inequality (8 Marks)**
Let $A_1, A_2, \ldots, A_n$ be n independent events. Prove that
$$\mathbb{P}\left(\bigcap A_i^c\right) \leq e^{-\mathbb{P}(A_1)-\mathbb{P}(A_2)-\cdots-\mathbb{P}(A_n)}$$

9. **Convolution (6 Marks)**
Show that the convolution of two distribution functions is also a distribution function. (Read its definition online)

10. **Integral Identity of Expectation (8 Marks)**
Let X be a nonnegative random variable with cumulative distribution function $F(x) = \mathbb{P}\{X \leq x\}$. Show that
$$\mathbb{E}X = \int_0^\infty (1 - F(x))\, dx$$
by showing that
$$\int_\Omega \int_0^\infty \mathbb{I}_{[0,X(\omega))}(x)\, dx\, d\mathbb{P}(\omega)$$
is equal to both $\mathbb{E}X$ and $\int_0^\infty (1 - F(x))\, dx$.

11. **Moment Generating Function and Jensen's Inequality (8 Marks)**
Let u be a fixed number in $\mathbb{R}$, and define the convex function $\varphi(x) = e^{ux}$ for all $x \in \mathbb{R}$. Let X be a normal random variable with mean $\mu = \mathbb{E}X$ and standard deviation $\sigma = [\mathbb{E}(X-\mu)^2]^{1/2}$, i.e., with density
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

    (i) Verify that
    $$\mathbb{E}e^{uX} = e^{u\mu + \frac{1}{2}u^2\sigma^2}.$$

    (ii) Verify that Jensen's inequality holds (as it must):
    $$\mathbb{E}[\varphi(X)] \geq \varphi(\mathbb{E}X).$$

2 Practical Section

1. **Royal Revenge (20 Marks)**
   Edmond Dantes is on a maze consisting of the integer lattice points, $G = \{(x, y) : 0 \leq x, y \leq n\}$. He starts off at the origin $(0, 0)$ and he wants to reach his fiancée Mercedes up at (n, n). At every second he can take one step to the right moving from the point (u, v) to $(u + 1, v)$ or a step above to $(u, v + 1)$. However, there is a river running between the lines $y = x$ and $y = x+1$, so if he goes above the diagonal he will fall into the river. Let P_n denote the number of paths from $(0, 0) \to (n, n)$ that do *not* cross the diagonal i.e. the path never goes above the line $y = x$.

   Your mission should you choose to accept it, is to write a program that computes the value of $P_n \bmod (10^9 + 7)$ for $n = 1, 2, \ldots, 100$.

   Hint: It is easy to make a recurrence for the answer. However, if you try to write a recursive function to do this, then you run the risk of exploding your personal computational gizmo. This approach will compute the same answer repeatedly. *You might want to store all the answers that you compute in a matrix and look up the answer again whenever needed.*

   Bonus: What is the asymptotic behaviour for P_n? Can you come up with a closed-form expression for P_n?