Assignment 1

Problem 1

1 → Solution →

Total no. of ways to assign N letters to N envelopes

$$= N!$$

$$\cancel{P(\text{at least one correct envelope}) = \frac{\text{Arrangements with atleast one match}}{N!}}$$

using inclusion - exclusion

Number of derangements $D(N)$ for N objects

$$D(N) = N! \times \sum_{k=0}^{N} \frac{(-1)^k}{k!}$$

Probability of at least one correct placed

$$= 1 - \frac{D(N)}{N!}$$

$\frac{(-1)^k}{k!}$ → This series converges very rapidly

(so $N = 50$ can be considered as large numbers)

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1} = \frac{1}{e}$$

$$\boxed{P = 1 - \frac{1}{e}}$$

2→ Solution:

$P(\text{Present 1 has } \$1000) = \frac{1}{3}$

$P(\text{Present 2 has } \$1000) = \frac{1}{3}$

$P(\text{Present 3 has } \$1000) = \frac{1}{3}$

Host opens Present 2 (empty)

$\Rightarrow$ Present 2 is eliminated

$P(\text{Present 1 has } \$1000) = \frac{1}{3}$

$P(\text{Present 3 has } \$1000) = \frac{2}{3} = \left(\frac{1}{3} + \frac{1}{3}\right)$

Expected winnings if switch to Present 3

$P(\text{Present 3 has } \$1000) = \frac{2}{3}$

$P(\text{Present 3 is empty}) = \frac{1}{3}$

$$\text{Expected winnings} = \frac{2}{3} \times (\$1000) + \frac{1}{3} \times 0$$
$$= \$666.67$$

3. (a) Solution →

$$P(A \cap B|C) = P(A|B \cap C)\, P(B|C)$$

$$\text{LHS} = P(A \cap B|C) = \frac{P((A \cap B) \cap C)}{P(C)} = \frac{P(A \cap B \cap C)}{P(C)}$$

$$\text{RHS} = P(A|B \cap C)\, P(B|C) =$$

$$= \left[\frac{P(A \cap B \cap C)}{P(B \cap C)}\right] \times \frac{P(B \cap C)}{P(C)}$$

$$= \frac{P(A \cap B \cap C)}{P(C)}$$

$$= \text{LHS}$$

True

(b) Solution

$$P(A \cap B) = P(A)\, P(B)$$

Example $P(A) = P(B) = \frac{1}{2}$, $P(C) = 3/4$.

$P(A \cap B) = \frac{1}{4}$, $P(A \cap C) = P(B \cap C) = \frac{1}{2}$

$P(A \cap B \cap C) = \frac{1}{4}$

LHS $\quad P(A \cap B|C) = \dfrac{P(A \cap B \cap C)}{P(C)} = \dfrac{(1/4)}{(3/4)} = 1/3$

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{(\frac{1}{2})}{(3/4)} = \frac{2}{3}$$

$$P(B|C) = \frac{P(B\cap C)}{P(C)} = \frac{\left(\frac{1}{2}\right)}{\left(3/4\right)} = \frac{2}{3}$$

$$P(A|C)\,P(B|C) = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$$

$$\frac{1}{3} \neq \frac{4}{9}$$

False

(C) Solution- $P(A|B) = P(A|B\cap D)\,P(D|B) + P(A|B\cap D^c)\,P(D^c|B)$

$P(A|B^c) = P(A|B^c\cap D)\,P(D|B^c) + P(A|B^c\cap D^c)\,P(D^c|B^c)$

Given $P(A|D\cap B^c) > P(A|D\cap B)$

$P(A|D^c\cap B^c) > P(A|D^c\cap B)$

$P(A|B) - P(A|B^c)$ can be negative using the above conditions

$\therefore$ $P(A|B) > P(A|B^c)$

is not always true

false

4) Solution

(a)

Discrete random variable with

$$P(x=k) = \frac{6}{\pi^2 k^2} \qquad k = \pm 1, \pm 2, \pm 3, \ldots$$

$E(x) = 0$ finite

$E(x^2) = \infty$ infinite

Constructed!

(b)

Continuous random variable with

$$f(x) = \frac{1}{|x|^3} \quad \text{for } |x| \geq 1$$

$$= 0 \quad \text{elsewhere}$$

$E(X) = 0$ finite

$E(x^2) =$ infinite

Constructed!

(c)

we need to find the random variable X such that :

$$E(X) = 1$$

$$E(e^{-X}) < \frac{1}{3}$$

The f^n e^{-x} is convex bcoz.

$$f'(x) = -e^{-x}$$

$$f''(x) = e^{-x} \text{ for all } x$$

Since $f(x)$ is convex by Jensen's inequality

$$E(f(X)) \geq f(E(X))$$

$$E(e^{-X}) \geq e^{-E(X)}$$

$$E(X) = 1, \Rightarrow E(e^{-X}) \geq e^{-1}$$

$$(= 0.368)$$

we need $E(e^{-X}) < \frac{1}{3}$

So it is not possible to construct.

5→

find $P(M=k)$

for the max. prize ~~to~~ be k

All n drawn tickets must have prizes $\leq k$

At least one ticket must have prize exactly k

$\therefore \quad P(M=k) = P(\text{all tickets} \leq k) - P(\text{all tickets} \leq k-1)$

Since drawing is with replacement)

$$P(M=k) = \left(\frac{k}{N}\right)^n - \left(\frac{(k-1)}{N}\right)^n$$

$$E(M) = \sum_{k=1}^{N} k \cdot P(M=k)$$

$$= \sum_{k=1}^{N} k\left[(k/N)^n - ((k-1)/N)^n\right]$$

$$= \sum_{k=1}^{N} k\,(k/N)^n - \sum_{k=1}^{N} k\left(\frac{(k-1)}{(N)}\right)^n$$

$$\left[\ \sum_{k=1}^{N} k\left[\frac{(k-1)}{N}\right]^n = \sum_{j=0}^{N-1} (j+1)\left(\frac{j}{N}\right)^n\right.$$

Since the $j=0$ term $=0$

$$\sum_{j=1}^{N-1} (j+1)\left(\frac{j}{N}\right)^n$$

$$= \sum_{j=1}^{N-1} j\left(\frac{j}{N}\right)^n + \sum_{j=1}^{N-1} \left(\frac{j}{N}\right)^n$$

$$E(M) = \sum_{k=1}^{N} k\left(\frac{k}{N}\right)^{n} - \left[\sum_{k=1}^{N-1} k\left(\frac{k}{N}\right)^{n} + \sum_{k=1}^{N-1}\left(\frac{k}{N}\right)^{n}\right]$$

$$= \sum_{k=1}^{N} k\left(\frac{k}{N}\right)^{n} - \sum_{k=1}^{N-1} k\left(\frac{k}{N}\right)^{n} - \sum_{k=1}^{N-1}\left(\frac{k}{N}\right)^{n}$$

$$= N - \sum_{k=1}^{N-1}\left(\frac{k}{N}\right)^{n}$$