Understanding Options

I. Overview.

A. Definitions.

1. **Option** - contract that **entitles** holder to ***buy/sell*** a certain asset ***at or before*** a ***certain time*** at a ***specified price***.

Gives holder the right, but not the obligation, to do something.

Call - ... *buy* ...
Put - ... *sell* ...
European Option - ... *at* a certain time (not before) ...
American Option - ... *at or before* a certain time ...
Expiration / Maturity - the *certain time*, how long until maturity (T).
Exercise Price / Strike Price - the *specified price* (K).

A Call is in-the-money (itm) if $S > K$;
A Call is at-the-money (atm) if $S = K$;
A Call is out-of-the-money (otm) if $S < K$;

Premium - value or cost of option
Trade in round lots - 1 option is the right to buy 100 shares.

Overview of Options

B. Combinations.

1. Synthetic Call (Put-Call Parity).

2. Writing a Covered Call.

3. Straddle, Strangle.

4. Spreads (Bull, Bear, Butterfly).

C. Uses.

1. Options can be combined to create any payoff pattern desired; Possible variations are only limited by imagination.

2. Options have substantial "inherent leverage."

3. These characteristics make options powerful and useful tools for speculation or hedging.

Overview of Options

B. Buying gives you the right, not the obligation, to buy (call) or sell (put) an underlying instrument (for example, a share)

1. When you buy an option, you are NOT obligated to buy or sell the underlying instrument – you simply have the right to do so at the fixed (exercise or strike) price.
2. Your risk when you buy an option is simply the price you paid for it.

Calls

- *When you buy a call option, you are not obligated to buy the underlying instrument (such as shares of stock)*
- *Your risk, when you buy an option, is simply the price you paid for it.*
- *Your reward is potentially unlimited.*

For every call that you buy, there is someone else on the other side of the trade. The seller of an option is called an *option writer*.

Overview of Options

Calls

- *When you buy a call option, you are not obligated to buy the underlying instrument (such as shares of stock)*
- *Your risk, when you buy an option, is simply the price you paid for it.*
- *Your reward is potentially unlimited.*

For every call that you buy, there is someone else on the other side of the trade. The seller of an option is called an *option writer*.

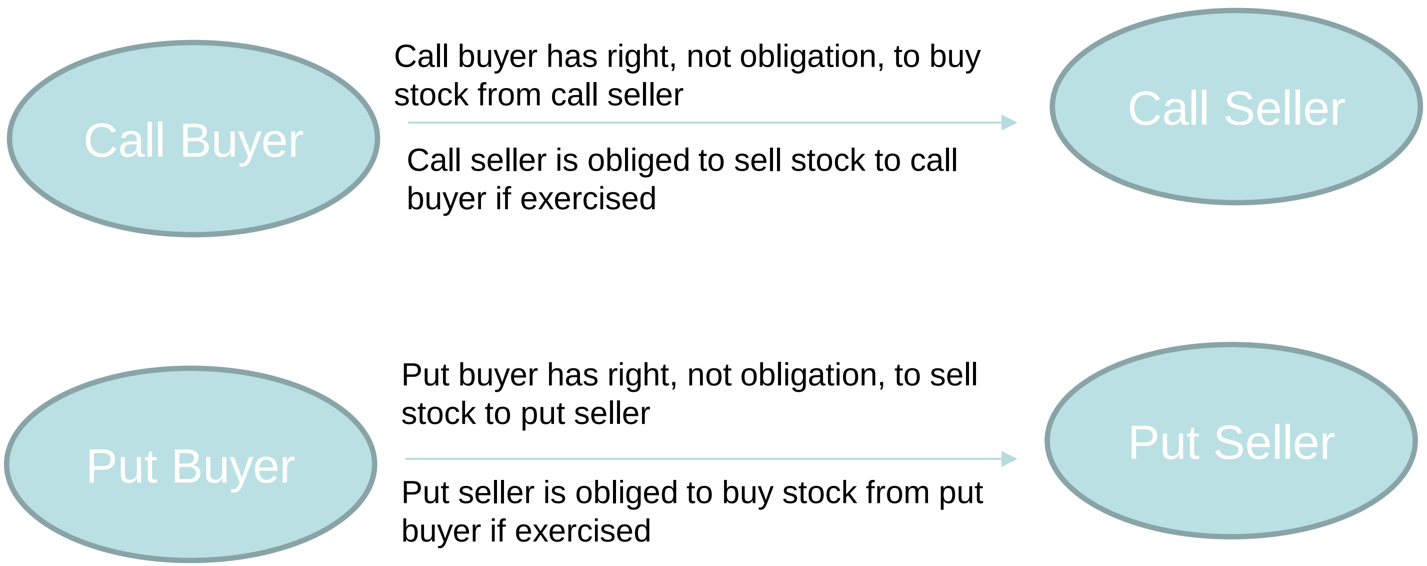
Puts

- *Buying a put option gives you the right, not the obligation, to sell an underlying instrument (such as shares of stock)*

Overview of Options

C. Selling (Naked) Imposes the obligation

- Selling an option (call or put) obliges you to buy (with sold puts) or deliver (with sold calls) to the option buyer if he or she exercises the option.
- Selling options naked (for example, when you have not bought a position in the underlying instrument or an option to hedge against it) gives you the unlimited risk profile.



Mechanics of Options

A. Strategy.

Option Buyer (Call)

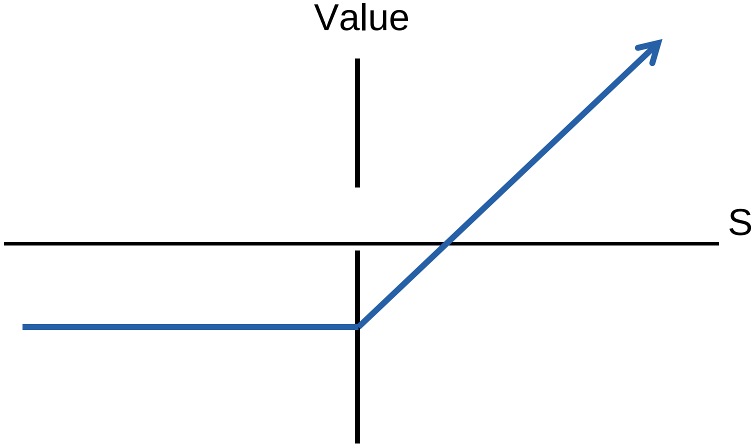
1. If you think S will ↑, **buy a call**;

a. If right (S↑), S > K, exercise.

(buy @ K, sell @ S, worth (S-K)).

b. If wrong (S↓), S < K,

lose price of call.



Option Writer (Call)

2. If you think S will ↓, **sell a call**;

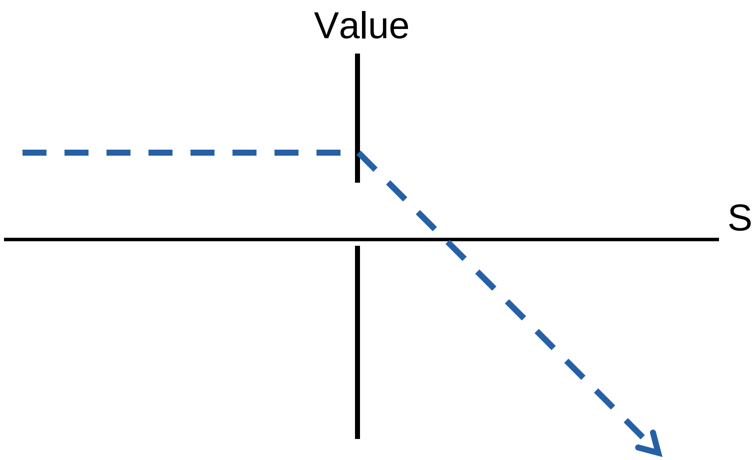
a. If right (S↓), S < K,

keep price of call.

b. If wrong (S↑), S > K,

will be exercised.

(must buy @ S, sell @ K, lose (S-K)).



Strategy - Mechanics of Options

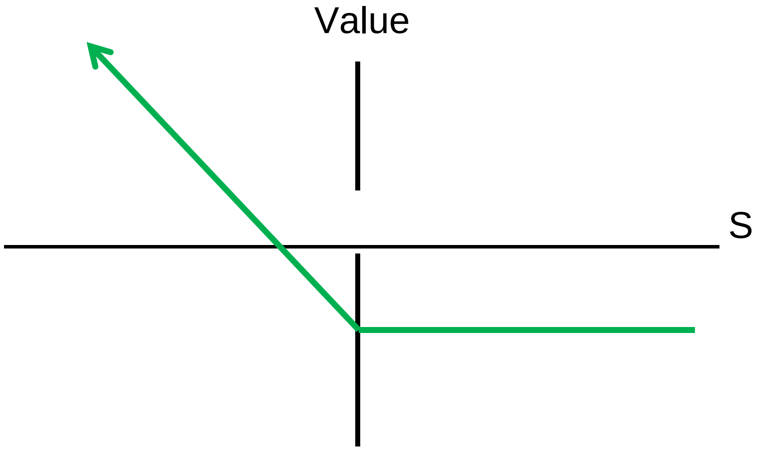
3. If you think S will $\downarrow$, **buy a put**;

a. If right ($S\downarrow$), $S < K$, exercise.

(buy @ S, sell @ K, worth (K-S)).

b. If wrong ($S\uparrow$), $S > K$,

lose price of put.



4. If you think S will $\uparrow$, **sell a put**;

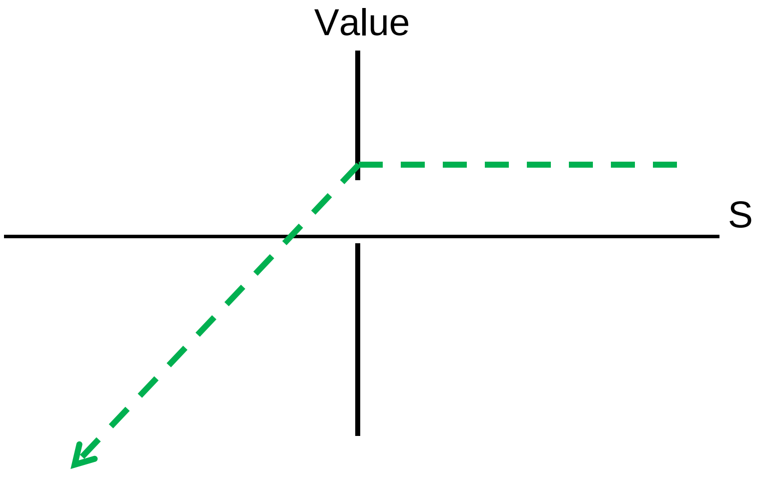
a. If right ($S\uparrow$), $S > K$,

keep price of put.

b. If wrong ($S\downarrow$), $S < K$,

will be exercised.

(must buy @ K, sell @ S, lose (K-S)).



Strategy - Mechanics of Options

5. Summary:

	Stocks	Calls	Puts	
Think S will ↑?	Buy	Buy	Sell	
Think S will ↓?	Sell	Sell	Buy	
Buy if you think:	S↑	S↑	S↓	Unlimited upside
Sell if you think:	S↓	S↓	S↑	Unlimited downside

Intrinsic and Extrinsic Values

1. Call option is:

	itm	atm	otm
if:	S > K	S = K	S < K
if:	(S-K) > 0	(S-K) = 0	(S-K) < 0

a. **Intrinsic value** of call = Max { S-K, 0 }.
(payoff if exercised now).

b. **Intrinsic value** of put = Max { K-S, 0 }.

c. An itm option will always be **exercised** @ expiration (if not early).

i. It may be optimal for holder to wait or sell, rather than exercise early.

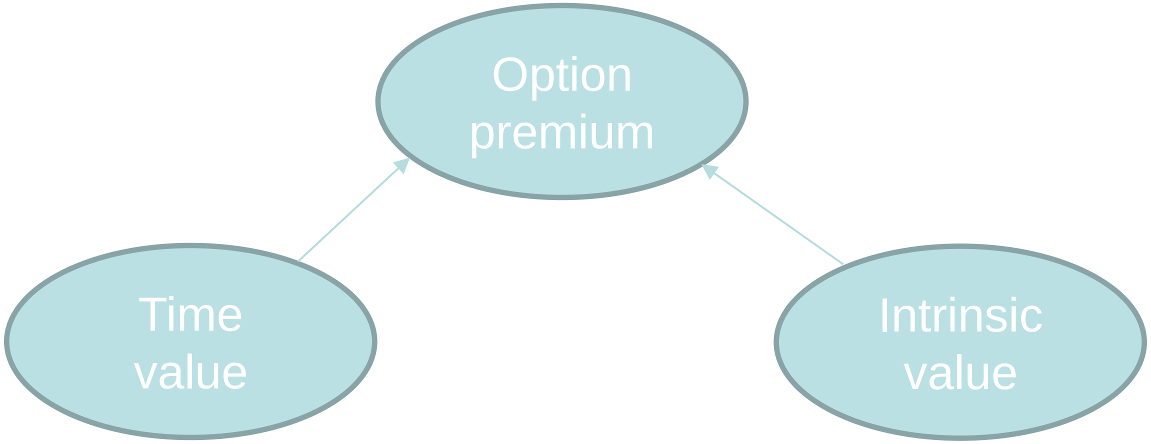
ii. In this case, option also has **Extrinsic value** (time value).

Total value of option = (**Intrinsic value**) + (**Extrinsic value**).

Overview of Options

D. Intrinsic value and time value

- ❑ Intrinsic value is that part of option's value that is in-the money (ITM)
- ❑ Time-value (hope value) is remainder of the option's value



Overview of Options

D. Intrinsic value and time value for calls

Call intrinsic value = stock price – strike price
Call time value = call premium – call intrinsic value
Minimum intrinsic value = 0

Call intrinsic value

Stock price	56.00
Call premium	7.33
Strike price	50
Time to expiration	2 months
Intrinsic value	= 56 – 50 = 6

Call time value

Stock price	56.00
Call premium	7.33
Strike price	50
Time to expiration	2 months
Intrinsic value	= 7.33 – 6.00 = 1.33

Margin Requirements

1. When **long term** call or put ***options*** *are* ***purchased***, (> 9 months mat), may borrow up to 25% of option price.

a. Longer term option acts more like the stock itself, especially if option is in-the-money.

b. Hence, margin purchases are allowed on longer term options.

2. When ***options*** *are* ***sold***, writer must maintain margin account.

a. Unlimited downside risk.

b. Broker & Exchange need assurance that writer will not default if option is exercised.

c. Size of margin depends on circumstances.

Naked options

1. A naked option (opened position option) cannot be combined with any offsetting position in the underlying stock.

2. Usually, the initial and maintenance margin required by the stock exchange for a written naked call option is greater than the following two calculations:

a). A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount, if any, by which the option is out of the money.

b). A total of 100% of the option proceeds plus 10% of the underlying share price.

Naked options

1. A naked option (opened position option) cannot be combined with any offsetting position in the underlying stock.

2. Similarly for a naked put option:

a). A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount, if any, by which the option is out of the money.

b). A total of 100% of the option proceeds plus 10% of the exercise price.

Margin

3. **Initial margin** for writing **naked options:**
the greater amount from two calculations:

a. [(100% of sale proceeds) + (20% of S) - (amount OTM)];

(If less OTM, will pay this.)

b. [(100% of sale proceeds) + (10% of S)]. (If more OTM, will pay this.)

4. **Example**: Investor writes 4 naked **calls**.

C = $5; K = $40; S = $38; (OTM: S - K = -$2).

Sale proceeds: $5 x 400 = $2,000 for calls.

Initial Margin is greater of two calculations:

a. 400 x [5 + .2 (38) - 2] = $4,240;

b. 400 x [5 + .1 (38)] = $3,520.

c. Thus, investor puts up $4,240 for this short position.

Margin

3. **Initial margin** for writing **naked options:**
the greater amount from two calculations:

a. [(100% of sale proceeds) + (20% of S) - (amount OTM)];
(If less OTM, will pay this.)

b. [(100% of sale proceeds) + (10% of S)]. (If more OTM, will pay this.)

Margin

3. A trader writes five naked put option contracts, with each contract being on 100 shares. The option price is \$10, the time to maturity is six months and the strike price is \$64.

a). What is the margin requirement if the stock price is \$58?

b). How would the answer to (a) change if the rules of index options applied.

(c). How would the answer to (a) change if the stock price were \$70?

(d). How would the answer to (a) change if the trader is buying instead of selling the options?

Margin

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(d). How would the answer to (a) change if the trader is buying instead of selling the options?

Answer:

(a). The margin requirement is the greater of 500* (10+0.2*58) = 10, 800 and 500* (10+0.1*64) = 8,200. It is $10,800.

(b). The margin requirement is greater than of 500* (10+0.15*58) = 9,350 and 500* (10+0.1*64) = 82,00. it is 9, 350.

(b). The margin requirement is greater than of 500* (10+0.2* 70 - 6) = 9,000 and 500* (10+0.1*64) = 82,00. it is 9, 000.

(d). No margin is required if the trader is buying.

Margin

4. **Initial Margin** for writing **Covered Calls**.

a. Sell call but own the shares; not so risky; worst result, must deliver your shares.

b. If covered call is **OTM** (less likely to be exercised), no initial margin is required.
Shares can be bought on 50% margin, and call price can be used to help pay.

c. If covered call is **ITM** (more likely to be exercised), again, no initial margin.
However, to calculate investor's equity position,
S is reduced by the extent to which the option is **ITM**.
This may limit amount investor can withdraw from margin account, if S↑ further.

d. **Example: C = $7; S = $63; K = $60; (ITM; S-K = $3).**
Want to buy 200 shares and write 2 calls.
Cost of shares = $63 x 200 = $12,600.
- Margin allowed on stock purchase = - $6,300.
- Price received for 2 calls = $7 x 200 = - $1,400. (Can use sale proceeds to pay.)

Therefore, minimum initial investment is: $12,600 - $6,300 - $1,400 = $4,900.

The Options Clearing Corporation (OCC)

1. Like the clearinghouse for futures markets.

Guarantees option writer will honor obligations.

Keeps record of all long & short positions.

2. OCC has members; all option trades must clear thru member.

a. If your brokerage house is not a member of OCC, they must arrange to clear your trades through a member.

b. Members are required to have minimum capital, and contribute to fund used to honor obligations if a member defaults.

3. When an option is purchased,

a. buyer must pay in full by morning of next business day;

b. funds are deposited with OCC;

c. writer maintains margin account with broker;

d. broker maintains margin account with OCC member;

e. OCC member maintains margin account with OCC.

The Options Clearing Corporation (OCC)

4. To exercise an option:

a. Investor (long position) notifies broker;

b. broker notifies OCC member who clears his/her trades;

c. OCC member places an exercise order with OCC;

d. OCC randomly selects member with outstanding short position;

e. This OCC member uses set procedure to select an investor who has written the option; this investor is said to be assigned.

f. If call, writer must sell @ K; If put, writer must buy @ K;

g. After an option is exercised, open interest declines by one.

i. At expiration, all ITM options should be exercised (automatic).

Basic Properties of Options

I. Notation and Assumptions:

A. Notation:

S:	current stock price;
K:	exercise price of option;
T:	time to expiration of option;
S_T:	stock price at time T;
σ_S:	volatility of stock returns;
r:	risk free nominal rate of interest for maturity at T;
C:	value of American call option to buy one share;
P:	value of American put option to sell one share;
c:	value of European call option to buy one share;
p:	value of European put option to sell one share.

B. Assumptions:

1. No transactions costs.
2. All net profits are subject to same tax rate.
3. Can borrow or lend at same risk-free rate (r).
4. Arbitrageurs take advantage of any opportunities.

Economic Factors that Affect Option Prices

A. Value of a Call = $\mathbf{C = f\{\overset{+}{S}, \overset{-}{K}, \overset{+}{T}, \overset{+}{r}, \overset{+}{\sigma_S}, \overset{-}{D}\}}$.

Call option is more valuable if:

1. underlying stock price (S) increases.
2. you have the right to buy at a lower strike price (K).
3. there is a longer time to maturity (T).
4. the risk-free rate (r) increases.
5. the underlying stock price (S) is more volatile (σ_S).
6. the dividend is smaller (D).

Summary: How Factors Affect Option Prices

Variable	European Call (c)	European Put (p)	American Call (C)	American Put (P)
S	+	-	+	-
K	-	+	-	+
T	+(?)	+(!)	+	+
r	+	-	+	-
σ_S	+	+	+	+
D	**-**	**+**	**-**	**+**

(?) – Long term European call *usually* costs more than short term call.
But suppose a ***dividend*** is paid after short term call expires.
Holder of short-term European call will get dividend.
Holder of long-term European call will not.
So short term call may be worth more than long term call.
Same argument does not apply to puts. More later.

Comparison of American call (C) and European call (c):
American call gives all rights in European call, plus *right to exercise early*.
Thus, American call must be at least as valuable as European call: $\mathbf{C \geq c}$.

Boundary Conditions for Option Prices

A. **Upper Bounds**:

1. Call gives holder right to buy one share @ K.
   Call cannot be more valuable than one share (S).
   $c \leq S$ and $C \leq S$.

2. Put gives holder right to sell one share @ K.
   Put cannot be more valuable than K.
   $p \leq K$ and $P \leq K$.

3. At *expiration*, we know $p \leq K$ (for European put).
   Thus, *today*, **$p \leq K e^{-rT}$** (and this is $< K$).

   If $p > K e^{-rT}$, sell European put today, invest @ r;
   at expiration, $p e^{rT} > K$; arbitrage.

   Not true for American put

Lower Bound for Euro Call on non-dividend-paying Stock

Consider two portfolios:

Portfolio A: Buy one call and K bonds that each pay \$1 at T.
Cash flows today: $-c$ $-Ke^{-rT}$ (will need \$K at exp.)

Portfolio B: Buy one share of stock.
Cash flows today: $-S$

Portfolio	Flows Today	Value at Expiration: If $S_T > K$	Value at Expiration: If $S_T < K$
Portfolio A:	**$-c$**	**$S_T - K$**	**0**
	$-Ke^{-rT}$	**K**	**K**
Total:	**$-c - Ke^{-rT}$**	**S_T**	**K**
Portfolio B:	**$-S$**	**S_T**	**S_T**

Obs. #1: If $S\uparrow$, both portfolios pay S_T; If $S\downarrow$, Portfolio A does better (hedged).
Thus, Portfolio A is worth at least as much as B:

$$c + Ke^{-rT} \geq S \quad \text{or} \quad c \geq S - Ke^{-rT}$$

European Call should not sell for less than $(S - Ke^{-rT})$.

Lower Bound for Euro Call on non-dividend-paying Stock

$$c \geq S - Ke^{-rT}$$

Obs. #2: If r higher, Ke^{-rT} lower; call is more valuable;
Then pay less today for bond that promises K at expiration;
If r increases, don't have to tie up as much $ today.

Thus, $c = f(r)$ --- if $r\uparrow$, $c\uparrow$.

Obs. #3: American call will not be exercised early (*if no dividend*).

$$C \geq S - Ke^{-rT} > S - K.$$

Want out? Can *exercise* American call early, & receive S - K;
→ Or can *sell* American call, & receive C [> S - K].

Will never exercise American call early (*if no dividend*) !
American call acts like European call (*if no dividend*) !
American and European call are worth same: **C = c.**

Lower Bound for Euro Call on non-dividend-paying Stock

Problem

What is the lower bound for the price of a six-month call option on a non-dividend-paying stock when the stock price is $80, the strike price is $75, and the risk-free interest rate is 10% per annum?

Answer: The lower bound is $80 – 75*e^{(-0.1*0.5)} = \$ 8.66$

Lower Bound for Euro Call on non-dividend-paying Stock

Problem

What is the lower bound for the price of a two-month put option on a non-dividend-paying stock when the stock price is $58, the strike price is $65, and the risk-free interest rate is 5% per annum?

Answer: The lower bound is $65 * e^{(-0.05*2/12)} - 58 = \$\ 6.46$

Lower Bound for Euro Put on non-dividend-paying Stock

Consider two portfolios.

Portfolio C: buy one put and one share of stock (S+P; protective put)

Cash flows today: -p -S

Portfolio D: buy K bonds that each pay \$1 at expiration.

Cash flows today: $-Ke^{-rT}$

Portfolio	Flows Today	Value at Expiration: If $S_T < K$	Value at Expiration: If $S_T > K$
Portfolio C:	**-p**	**$K - S_T$**	**0**
	-S	**S_T**	**S_T**
Total:	**-p - S**	**K**	**S_T**
Portfolio D:	**$-Ke^{-rT}$**	**K**	**K**

If S decreases, portfolios C and D both pay K.

If S increases, portfolio C does better than D (hedged).

Therefore, portfolio C should be worth more than D:

$$p + S > Ke^{-rT} \quad \text{or} \quad \mathbf{p > Ke^{-rT} - S}$$

Worst outcome, put finishes OTM; So $\mathbf{p > \max\{(Ke^{-rT} - S), 0\}}$

Arbitrage

Example:

A one-month European put option on a non-dividend-paying stock is currently selling for $2.50. The stock is $47, the strike price is $47, the strike price is $50 and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

Arbitrage

Example:

A one-month European put option on a non-dividend-paying stock is currently selling for $2.50. The stock is $47, the strike price is $47, the strike price is $50 and the risk-free interest rate is 6% per annum. What opportunities are there for an arbitrageur?

Answer:

The present value of the strike price is $50*e^{(-0.06*1/12)} = 49.75$.

Because $2.5 < 49.75 – 47$, the condition is violated.

An arbitrageur can borrow 49.50 at 6% for one month, buy the stock and buy the put option.

Arbitrage

Example:

- The present value of the strike price is $50 * e^{(-0.06*1/12)} = 49.75$.
- Because 2.5 < 49.75 – 47, the condition is violated.
- An arbitrageur can borrow 49.50 at 6% for one month, buy the stock and buy the put option.
- This generates a profit in all circumstances. If stock is above $50 in one month, the option expires worthless, but the stock can be sold for $50.
- A sum of $50 received in one month has a present value of 49.75 today.
- Profit = 50 – 49.75 = 0.25

Arbitrage

Example:

- If the stock is below $ 50 then put option is exercised and the stock owned is sold for $50 (or 49.75 in PV terms).
- Profit = 50 – 49.75 = 0.25

Arbitrage

Fixed relation between prices of European puts & calls with same maturity & asset.

If we know the price of a European put, can determine the price of a European call.

Recall: **S + p = B + c** or **S + p - B = c**. --(synthetic call)

Consider the combination, S + p - B;

Buy stock (+S), buy put (+p), & sell bond (-B) for Ke^{-rT}, maturing at expiration.

Portfolio	Flows Today	Value at Expiration: If $S_T > K$	Value at Expiration: If $S_T < K$
Portfolio A (buy synthetic call):			
buy stock	$-S$	S_T	S_T
buy put	$-p$	0	$K - S_T$
sell bond	$+Ke^{-rT}$	$-K$	$-K$
Total:	$-S - p + Ke^{-rT}$	$S_T - K$	0
Portfolio B (buy call):	$-c$	$S_T - K$	0

Outcomes are identical. Thus, initial cost of Portfolio A must be same as call:

$$-c = -S - p + Ke^{-rT} \quad \text{or} \quad c = S + p - Ke^{-rT}; \quad \textbf{Put - Call Parity}$$

Put-call parity

- It shows the relationship between the prices of European put and call options that have the same strike price and time to maturity.
- Portfolio A: one European call option plus a zero-coupon bond that provides a payoff of K at time T
- Portfolio C: one European put option plus one share of the stock.
- We also assume that the call and put options have the same strike price K and the same time to maturity T.

Put-call parity

- We assume that the Zero-coupon Bond (ZCB) will be worth K at time T.
- Strategy:
  - If the stock price S_T at time T proves to be above K, then the call option in portfolio A will be exercised.
  - It means that portfolio A is worth $(S_T – K) + K = S_T$ at time T in these circumstances.
  - If S_T proves to be less than K, then the call option in portfolio A will expire worthless and the portfolio will be worth K at time T.

Put-call parity

- We assume that the Zero-coupon Bond (ZCB) will be worth K at time T.
- Strategy:
  - In portfolio C, the share will be worth S_T at time T. If S_T proves to be below K, then the put option in portfolio C will be exercised.
  - This means that portfolio C is worth $(K - S_T) + S_T = K$ at time T in these circumstances.
  - If S_T proves to be greater than K, then the put option in portfolio C will expire worthless and the portfolio will be worth S_T at time T.

Put-call parity

- We assume that the Zero-coupon Bond (ZCB) will be worth K at time T.
- Strategy:

		$S_T > K$	$S_T < K$
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio C	Put option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

- Summarize: If $S_T > K$, both portfolios are worth S_T at time T; if $S_T < K$, both portfolios are worth K at time T.

Put-call parity

		$S_T > K$	$S_T < K$
Portfolio A	**Call option**	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio C	Put option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

Summarize: If $S_T > K$, both portfolios are worth S_T at time T; if $S_T < K$, both portfolios are worth K at time T.

Both are worth: Max (S_T , K)

when the options expire at time T.

Put-call parity

Summarize:

- Put-call parity is applicable only in case of European because the options cannot be exercised prior to time T.
- Since options are European, the portfolios have identical values at time T, they must have identical values today.

Arbitrage opportunity:

- If this were not the case, an arbitrageur could buy the less expensive portfolio and sell the more expensive one.
- Because the portfolios are guaranteed to cancel each other out at time T, this trading strategy would lock in an arbitrage profit equal to the difference in the values of the two portfolios.

Put-call parity

Summarize:

- If we take into account the lower bound of European call and put, the expression of Put-Call parity would be:

  - **Call (c) + Ke^{-rT} = Put (p) + S_0**

- This relationship is known as Put-Call Parity.
- It shows that the value of a European call with a certain exercise price and exercise date can be deduced from the value of a European put with the same exercise price and exercise date, and vice versa.

Put-call parity

Example:

- Suppose stock price (S_0) is at $31, the exercise price is $30, the risk-free interest rate is 10% per annum, the price of a three-month European call option is $3, and the price of a 3-month European put option is $2.25. What will be the arbitrage strategy and profit?

Solution:

- Put-Call parity condition:
- Call (c) + Ke^{-rT} = Put (p) + S_0

Call (c) + Ke^{-rT} = $3 + 30e^{-0.1*3/12}$ = 32.26

Put (p) + S_0 = 2.25 + 31 = 33.25

Put-call parity

Example:

Solution:

- Call (c) + Ke^{-rT} = $3 + 30e^{-0.1*3/12}$ = 32.26

  Put (p) + S_0 = 2.25 + 31 = 33.25

Since 33.25 > 32.26, arbitrage is possible because portfolio 2 is overpriced than the portfolio 1.

Put-call parity

Arbitrage strategy:

- The strategy involves buying the call and shorting both the put and the stock, generating a positive cash flow of 30.25 upfront.

  $$- 3 + 2.25 + 31 = 30.25$$

- When invested at the risk-free interest rate, this amount grows to $30.25e^{-0.1*0.25} = 31.02$ in 3 months

- If the stock price at expiration of the option is greater than \$30, the call will be exercised. If it is less than \$30, the put will be exercised.

- In either case, the arbitrageur ends up buying one share for \$30.

Put-call parity

Arbitrage strategy:

- If the stock price at expiration of the option is greater than \$30, the call will be exercised. If it is less than \$30, the put will be exercised.

- In either case, the arbitrageur ends up buying one share for \$30.

- This share can be used to close out the short position.
- The net profit is therefore 31.02 – 30.00 = 1.02
- 1.02 is the arbitrage profit.

Put-call parity

Example:

The price of a European call that expires in six months and has a strike price of $30 is $2. The underlying stock price is $29, and a dividend of $0.50 is expected in two months and again in five months. Risk-free interest rates for all maturities are 10%. What is the price of a European put option that expires in six months and has a strike price of $30?

Solution:

- Put-Call parity condition:
- Call (c) + Ke^{-rT} + D = Put (p) + S_0

Put-call parity

Solution:

- Put-Call parity condition:
- Call (c) + Ke^{-rT} + D = Put (p) + S_0
- If we solve for p then the expression becomes

  p = Call (c) + Ke^{-rT} + D - S_0

If we substitute the values, then we find that the

p = 2 + $30e^{-0.1*0.50} + (30e^{-0.1*2/12} + 30e^{-0.1*5/12}) - 29 = 2.51$

In other words, the put price is 2.51.

Put-call parity

Solution:

If we substitute the values, then we find that the

$p = 2 + 30e^{-0.1*0.50} + (30e^{-0.1*2/12} + 30e^{-0.1*5/12}) - 29 = 2.51$

In other words, the put price is 2.51.

Can we explain the arbitrage opportunity in this case when we assume that the European put price is $3.

Strategy:

- If the put price is $3.00, it is too high relative to the call price.
- An arbitrageur should buy the call, short the put and short the stock. It means he will have the cash flow of
  - 2 + 3+ 29 = $30 in cash which is invested at 10%.
- A profit of 3 – 2.51 is always possible

Put-call parity

Strategy:

- If the put price is $3.00, it is too high relative to the call price.
- **An arbitrageur should buy the call, short the put and short the stock.** It means he will have the cash flow of
  - 2 + 3+ 29 = $30 in cash which is invested at 10%.
- A profit of 3 – 2.51 is always possible
- If the stock price is above 30 in six months, the call option is exercised and put option expires worthless.
- The call option enables the stock to be bought for $30, or $30e^{-0.1*0.50}$ = 28.54 in PV terms. The call option helps buy the stock for $30.

Put-call parity

Strategy:

- The dividends on the short position cos:
- $(30e^{-0.1*2/12} + 30e^{-0.1*5/12}) = \$0.97$ in PV terms so that there is a profit with a present value of

  30 – 28.54 – 0.97 = $0.49

- Second, if the stock price is below $30 in six months, the put option is exercised and the call option expires worthless.
- The short put option leads to the stock being bought for $30 or $30e^{-0.10*6/12} = \$28.54$ in PV terms.
- The dividend costs $0.97 as above so that there is a profit of with a present value of 30 – 28.54 – 0.97 = $0.49.

Put-call parity

Strategy:

Three-month put price = $2.25	
Action now:	Action now:
• Buy call for $3 • Short put to realize $2.25 • Short the stock to realise $31 • Invest $30.25 for 3 months	• Borrow $29 for three months • Short call to realize $3 • Buy put for $1 • Buy the stock for $31
Action in 3 months if $S_T > 30$	Action in 3 months if $S_T > 30$
• Receive $31.02 from investment • Exercise call to buy stock for $30 • Net Profit = $1.02	• Call exercised: sell stock for $30 • Use 29.73 to repay loan • Net Profit = $0.27
Action in 3 months if $S_T < 30$	Action in 3 months if $S_T < 30$
• Receive $31.02 from investment • Put excercised buy stock for $30 • Net Profit = $1.02	• Exercise put to sell stock for $30 • Use $29.73 to repay loan • Net Profit = $0.27

American Options: Put-call parity

- Put–call parity holds only for European options. However, it is possible to derive some results for American option prices. When there are no dividends,

$$S_0 - K <= C - P <= S_0 - Ke^{-rT}$$

- The left side is the upper bound of an American option and right side is the lower bound.

American Options: Put-call parity

Example:

An American call option on a non-dividend-paying stock with strike price \$20.00 and maturity in 5 months is worth \$1.50. Suppose that the current stock price is \$19.00 and the risk-free interest rate is 10% per annum.

Solution:

$$S_0 - K <= C - P <= S_0 - Ke^{-rT}$$

$$19 - 20 <= C - P <= 19 - 20e^{-0.10*5/12}$$

After solving it, 1 >= P – C >= 0.18

It shows that the P – C lies between \$1.00 and \$0.18.

With C at \$1.50, P must lie between \$1.68 and \$2.50.

American Options: Put-call parity

Example:

An American call option on a non-dividend-paying stock with strike price \$20.00 and maturity in 5 months is worth \$1.50. Suppose that the current stock price is \$19.00, and the risk-free interest rate is 10% per annum.

Solution:

It shows that the P – C lies between \$1.00 and \$0.18.

With C at \$1.50, P must lie between \$1.68 and \$2.50.

- In other words, upper and lower bounds for the price of an American put with the same strike price and expiration date as the American call are \$2.50 and \$1.68.

American Options: Put-call parity

Example:

The price of an American call on a non-dividend-paying stock is $4. The stock price is $31, the strike price is $30, and the expiration date is in three months. The risk-free interest rate is 8%. Derive upper and lower bounds for the price of an American put on the same stock with the same strike price and expiration date.

Solution:

$S_0 - K <= C - P <= S_0 - Ke^{-rT}$

$31 - 30 <= 4 - P <= 31 - 30e^{-0.08*0.25}$

$1.00 <= 4.00 - P <= 1.59$

$2.41 <= P <= 3.00$

Upper and lower bounds for the price of an American put are therefore $2.41 and $3.00.

American Options: Calls on non-dividend paying stock

- Evidence suggests that it is never optimal to exercise an American call option on a non-dividend-paying stock before the expiration date.
- Consider an American call option on a non-dividend-paying stock with one month to expiration when the stock price is $70 and the strike price is $40.
- The option is deep in the money, and the investor who owns the option might well be tempted to exercise it immediately.
- However, if the investor plans to hold the stock obtained by exercising the option for more than one month, this is not the best strategy.

American Options: Calls on non-dividend paying stock

- Following European call option, we can write it as

  $c >= S_0 – Ke^{-rT}$

- Because the owner of an American call has all the exercise opportunities open to the owner of the corresponding European call, we must have

  $C >= c$

- $C >= S_0 – Ke^{-rT}$

  Given $r > 0$, it follows that $C > S_0 - K$ when $T > 0$. This means that C is always greater than the option's intrinsic value prior to maturity.

- If it were optimal to exercise at a particular time prior to maturity, C would equal the option's intrinsic value at that time. It follows that it can never be optimal to exercise early

American Options: Calls on non-dividend paying stock

- One relates to the insurance that it provides.

- A call option, when held instead of the stock itself, in effect insures the holder against the stock price falling below the strike price.

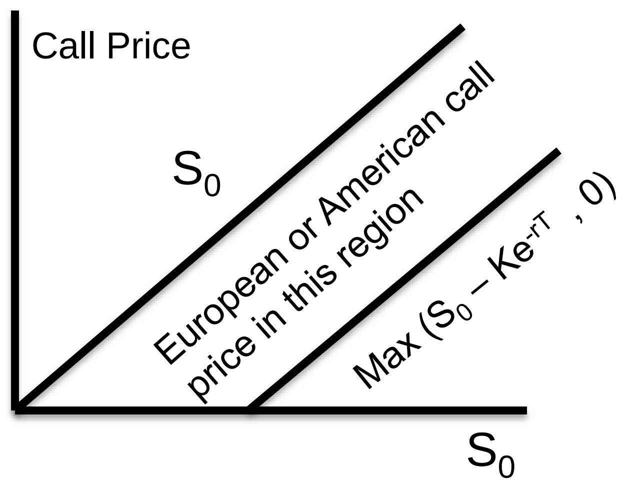
- Once the option has been exercised and the strike price has been exchanged for the stock price, this insurance vanishes.

American Options: Calls on non-dividend paying stock

- Bounds:
- Because American call options are never exercised early when there are no dividends, they are equivalent to European call options, so that C = c

  Max (S_0 – Ke^{-rT} , 0) and S_0

- Bounds for European and American call options when there are no dividends.



American Options: Puts on non-dividend paying stock

- It can be optimal to exercise an American put option on a non-dividend-paying stock early.

For instance,

- Suppose that the strike price is \$10 and the stock price is virtually zero.
- By exercising immediately, an investor makes an immediate gain of \$10.
- If the investor waits, the gain from exercise might be less than \$10, but it cannot be more than \$10, because negative stock prices are impossible.
- Furthermore, receiving \$10 now is preferable to receiving \$10 in the future.
- It follows that the option should be exercised immediately.
- A put option, when held in conjunction with the stock, insures the holder against the stock price falling below a certain level.

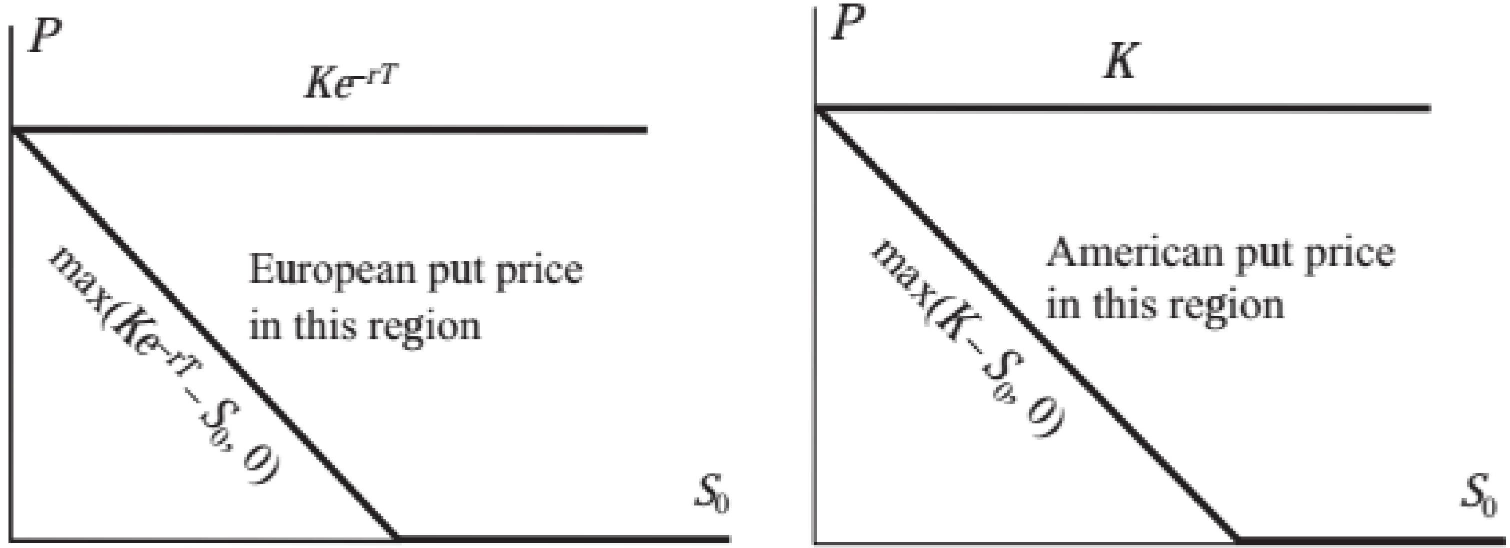
American Options: Puts on non-dividend paying stock

- The lower and upper bounds for a European put option when there are no dividends are given by

  $\text{Max}\,(Ke^{-rT} - S_0\,,\, 0) \;<=\; p \;<=\; Ke^{-rT}$

- For an American put option on a non-dividend-paying stock, the condition
  $P >= \text{Max}\,(K - S_0\,,\, 0)$ must apply because the option can be exercised at any time.

American Options: Puts on non-dividend paying stock



Bounds for European and American put options when there are no dividends.

Put - Call Parity

The price of European call and put options on a non-dividend paying stock with 12months to maturity, a strike price of $120 and an expiration date in 12 months are $20 and $5, respectively. The current stock price is 130. What is the implied risk free rate?

From put-call parity

$20 + 120e^{-r*1} = 5+130$

Solving this

$r = -\ln(115/120) = 0.0426 = 4.26\%$

Effect of Dividends

Let **D** = NPV(expected dividends during life of option).
Then all these relations hold, after adjusting for **D**.

A. Lower bound for European call on dividend paying stock:

$$\mathbf{c \geq S - (Ke^{-rT} + D)}$$

B. Lower bound for European put on dividend paying stock:

$$\mathbf{p \geq (Ke^{-rT} + D) - S}$$

C. Put-Call Parity; $c = S + p - Ke^{-rT}$:

$$\mathbf{c = S + p - (Ke^{-rT} + D)}$$

Early Exercise of American Puts

A. Recall, American call (C) will not be exercised early if no dividend is paid during its life. Thus, C = c.
(American call acts like a European call.)

B. American puts (P) are different.
It may be wise to exercise American put early (even if no div):

1. If S is low enough;
2. If put is deep ITM.
3. Thus, **P ≥ p**.

C. Example: Suppose K = \$10; and S ≈ \$0.

1. If you exercise, receive (K - S) ≈ \$10 today.
2. If you wait, cannot receive more than \$10 (S cannot ↓ below \$0).
3. Receiving \$10 now is better than later.
4. Such an option should be exercised early.

Early Exercise of American Puts

D. Intuition: Like a call, think of put as giving insurance.

1. Hold share (+S) plus put (+P); protective put.
   a. If S ↑, combination ↑. Stock more valuable.
   b. If S ↓, receive K, insures against downside.

2. However, unlike a call, it may be optimal:
   a. To forego this insurance & exercise put early;
   b. To receive $K immediately.

3. In general, early exercise of put is more attractive:
   a. If **S** ↓ (put deeper ITM; more intrinsic value);
   b. If **r** ↑ (more time value; (K-S) today is more attractive);
   c. If **σ** ↓ (less likely for S to ↓ further; less extrinsic value).

↑

(less reason to keep put alive)

Early Exercise of American Puts

E. Elaborate.

1. Recall lower bound for puts:
   a. For European put: $p \geq Ke^{-rT} - S$; For American put: $P \geq K - S$.

2. See graph. Shows how put values (p & P) vary with S:
   a. For European put, p may be $< K - S$.
   b. For American put, P may not be $< K - S$.

3. For American put,
   a. It is always optimal to exercise early, if S low enough (point A).

   b. Price curve merges into put's intrinsic value, (K - S), as S ↓ below A.

   c. For S values to right of point A, $P > K - S$ (there is extrinsic value!).
      i. Recall: Total Value = Intrinsic + Extrinsic.

   d. This extrinsic value increases if:
      i. r ↓; (lower interest, so K today less attractive; less time value)
      ii. σ ↑; (more likely for good things to happen (S↓); more extr value)
      iii. T ↑; (more likely for good things to happen (S↓); more extr value)

Early Exercise of American Puts

4. Compare American put (P) with European put (p).

   a. [$P \geq K - S$] **>** [$p \geq Ke^{-rT} - S$].

   b. At any time, S ***may*** decline enough so early exercise is good.

   c. American put is sometimes worth its intrinsic value ($P = K - S$).

   d. European put is always worth less than American put ($p < P$).

   e. Thus, European put is sometimes worth < Intrinsic Value.

      i.e., [p can be $< K - S$].

   f. See graph. Shows how European put value varies with S.

      i. At point B, $p = K - S$.

      ii. If $S <$ point B, $p < K - S$.

      iii. If $S >$ point B, $p > K - S$.

Trading Strategies with Options

I. Basic Combinations.

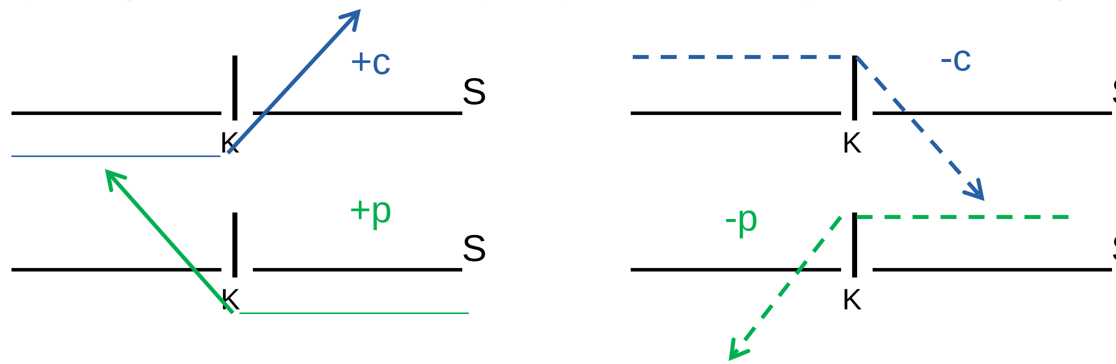
A. Calls & Puts can be combined with other **building blocks** (Stocks & Bonds) to give any payoff pattern desired.

1. Assume European options with same exp. (T), K, & underlying.

2. Already know payoff patterns for buying & selling calls & puts:

a. **Calls**.

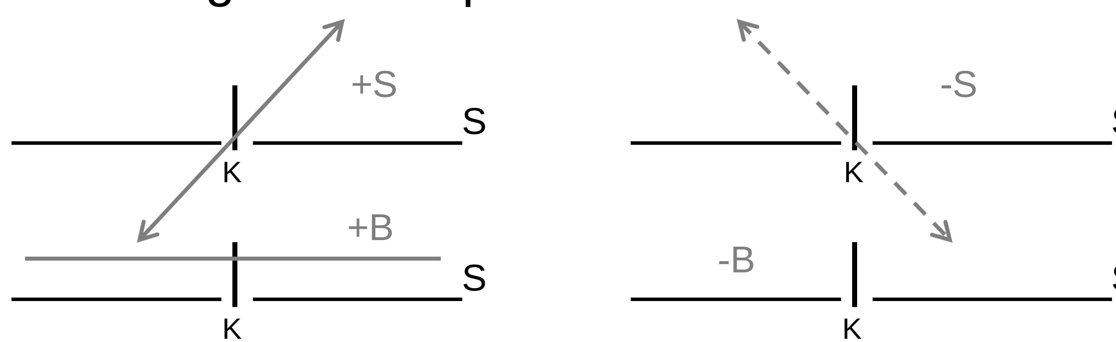
b. **Puts**.



3. Consider payoffs for long & short positions on:

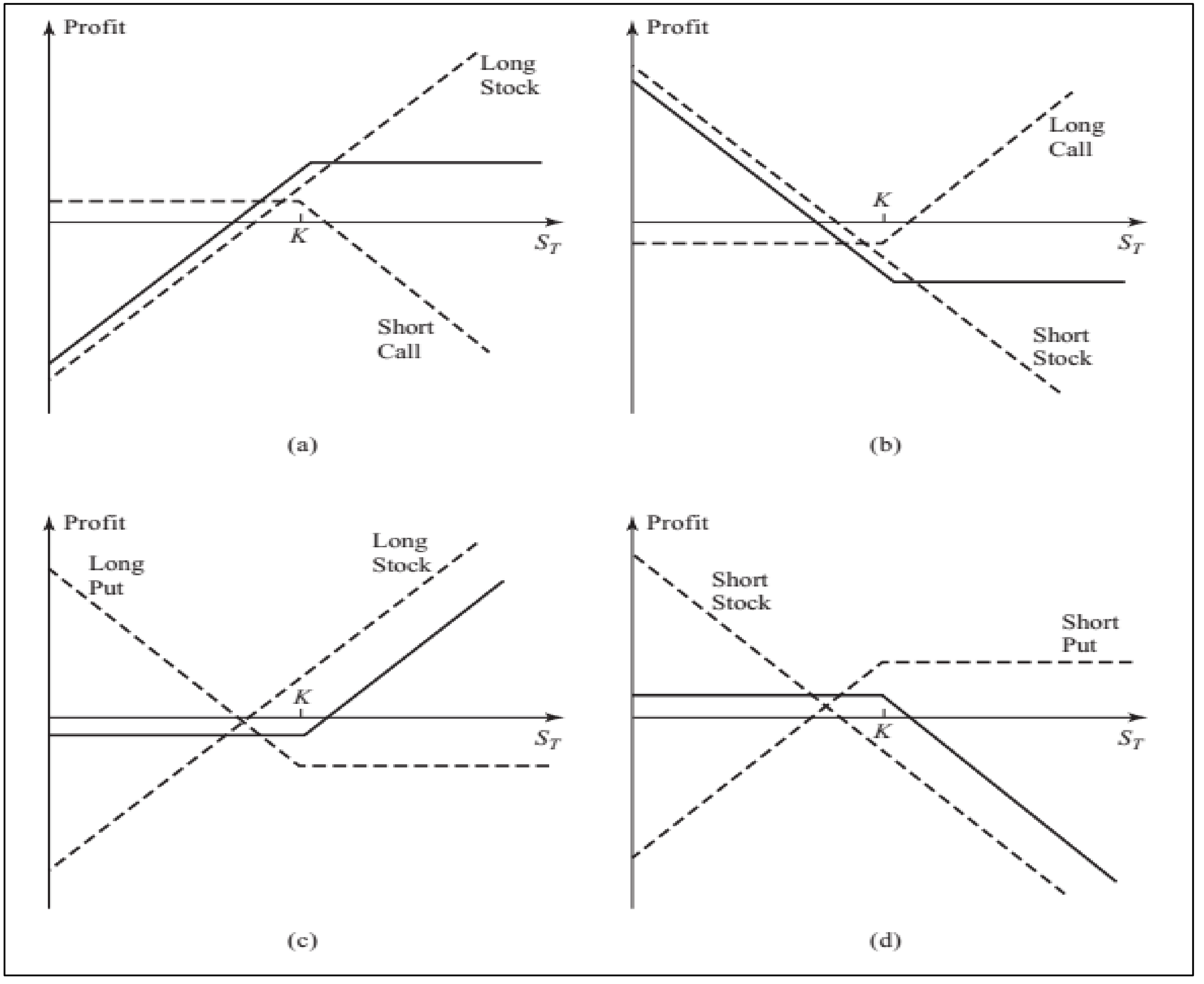
a. **Stocks**.

b. **Bonds**.



Trading Strategies with Options

- Trading an option and the underlying asset
- **Writing a covered call**: The portfolio consists of a long position in a stock plus a short position in a European call option.
- A short position in a stock is combined with a long position in a call option. This is the ***reverse of writing a covered call***.
- The investment strategy involves buying a European put option on a stock and the stock itself. This is referred to as a ***protective put strategy***



(a) long position in a stock combined with short position in a call;
(b) short position in a stock combined with long position in a call;
(c) long position in a put combined with long position in a stock;
(d) short position in a put combined with short position in a stock

Spreads

- A spread trading strategy involves taking a position in two or more options of the same type (i.e., two or more calls or two or more puts).

- **Bull Spreads**:
  - It is created by buying a European call option on a stock with a certain strike price and selling a European call option on the same stock with a higher strike price.

  - Both options have the same expiration date.

Spreads

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  - It is created by buying a European call option on a stock with a certain strike price and selling a European call option on the same stock with a higher strike price.

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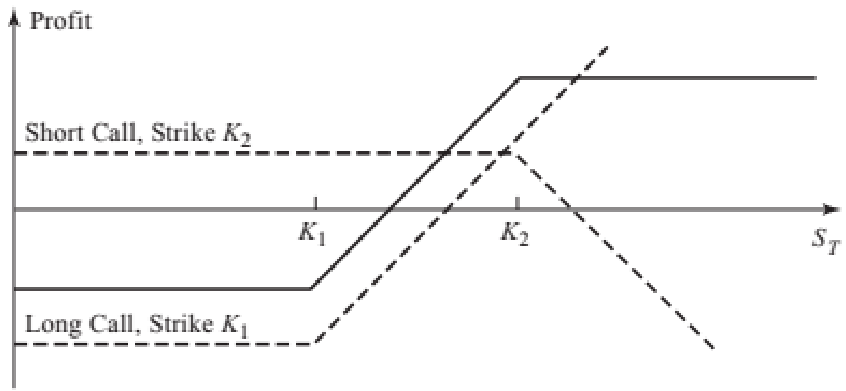


Spreads

▪ Bull Spreads:

- Suppose that K1 is the strike price of the call option bought, and K2 is the strike price of, the payoff from a bull strategy created using the calls:

Stock price range	Payoff from long call option	Payoff from short call option	Total payoff
$S_T \leq K_1$	0	0	0
$K_1 < S_T < K_2$	$S_T - K_1$	0	$S_T - K_1$
$S_T \geq K_2$	$S_T - K_1$	$-(S_T - K_2)$	$K_2 - K_1$

- A bull spread strategy limits the investor’s upside as well as downside risk

Spreads

- Example:
  - Suppose an investor buys 3-month European call at $9 with a strike price of $50 and sells for $5 a 3-month European call with a strike price of $60.
  - The payoff from this bull spread strategy is $10 if the stock price is above $60, and zero if it is below $50.
  - If the stock price is between $50 and $60, the payoff is the amount by which the stock price exceeds $60. The cost of the strategy is $9 - $5 = $4. So the profit is:

Answer:

Stock Price	Profit
$S_T \geq 60$	10
$50 \leq S_T < 60$	$S_T - 51$
$S_T \geq 60$	4

Spreads

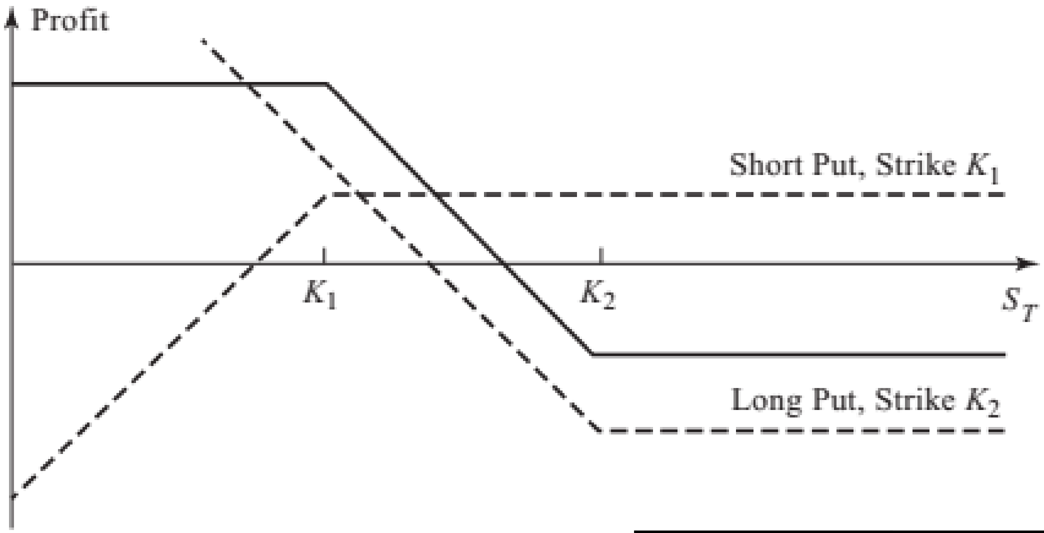
- **Bear Spreads**:
  - An investor who enters into a bear spread is hoping that the stock price will decline.
  - Bear spreads can be created by buying a European put with one strike price and selling a European put with another strike price.
  - The strike price of the option purchased is greater than the strike price of the option sold.

Spreads

- Bear Spreads:
  - A bear spread created from puts involves an initial cash outflow because the price of the put sold is less than the price of the put purchased.
  - In essence, the investor has bought a put with a certain strike price and chosen to give up some of the profit potential by selling a put with a lower strike price.
  - In return for the profit given up, the investor gets the price of the option sold.

Spreads

- Bear Spreads: Assume that the strike prices are K1 and K2, with K1 < K2.



The payoff from bear spread

Stock price range	Payoff from long put option	The payoff from short-put option	Total payoff
$S_T \leq K_1$	$K_2 - S_T$	$-(K_1 - S_T)$	$K_2 - K_1$
$K_1 < S_T < K_2$	$K_2 - S_T$	0	$K_2 - S_T$
$S_T \geq K_2$	0	0	0

Spreads

- Example:
  - Suppose an investor buys 3-month European put at $9 with a strike price of $50 and sells for $2 a 3-month European put with a strike price of $40.
  - The payoff from this bear strategy is 0 if the stock price is above $50, and $10 if it is below $40.
  - If the stock price is between $40 and $50, the payoff is 50 - S_T. The cost of the strategy is $9 - $2 = $7. So the profit is:

Answer:

Stock Price	Profit
$S_T \le 40$	$+9$
$40 < S_T < 50$	$43 - S_T$
$S_T \ge 50$	-7

Spreads

- **Example:**

Suppose that put options on a stock with strike prices $30 and $35 cost $4 and $7, respectively. How can the options be used to create (a) a bull spread and (b) bear spread? Construct a table that shows the profit and payoff for both spreads.

Answer:

- A bull spread is created by buying the $30 put and selling the $35 put. This strategy gives rise to to an initial cash inflow of $3. The outcome is as follows:

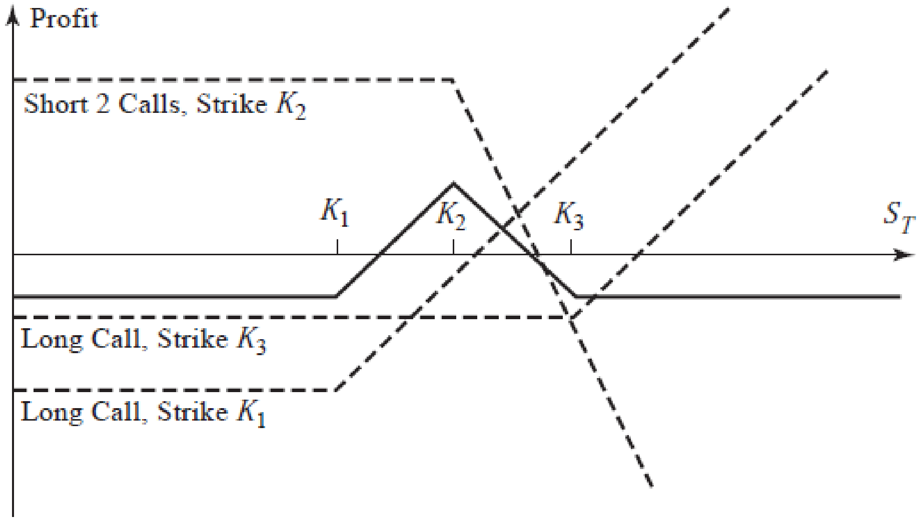
Stock Price	Payoff	Profit
$S_T \geq 35$	0	3
$30 \leq S_T < 35$	$S_T - 35$	$S_T - 32$
$S_T < 30$	-5	-2

- A bear spread is created by selling the $30 put and buying the $35 put. This strategy costs $3 initially. The outcome is as follows

Stock Price	Payoff	Profit
$S_T \geq 35$	0	-3
$30 \leq S_T < 35$	$35 - S_T$	$32 - S_T$
$S_T < 30$	5	2

Spreads

- **Butterfly Spreads**: A butterfly spread involves positions in options with three different strike prices. It can be created by buying a European call option with a relatively low strike price K_1.
- Profit from butterfly spread using call options



Spreads

- Butterfly Spreads: A butterfly spread involves positions in options with three different strike prices.
- It can be created by buying a European call option with a relatively low strike price K1, buying a European call option with a relatively high strike price K3, and selling two European call options with a strike price K2 that is halfway between K1 and K3 Payoff from a butterfly spread.
- K2 is close to the current stock price.
- A butterfly spread leads to a profit if the stock price stays close to K2, but gives rise to a small loss if there is a significant stock price move in either direction.

Stock price range	Payoff from first long call	Payoff from second long call	Payoff from short calls	Total payoff
$S_T \leq K_1$	0	0	0	0
$K_1 < S_T \leq K_2$	$S_T - K_1$	0	0	$S_T - K_1$
$K_2 < S_T < K_3$	$S_T - K_1$	0	$-2(S_T - K_2)$	$K_3 - S_T$
$S_T \geq K_3$	$S_T - K_1$	$S_T - K_3$	$-2(S_T - K_2)$	0

These payoffs are calculated using the relationship $K_2 = 0.5(K_1 + K_3)$

Spreads

- Butterfly Spreads: A butterfly spread involves positions in options with three different strike prices.
- **Example:** Suppose that a certain stock is currently worth $61. Consider an investor who feels that a significant price move in the next 6 months is unlikely. Suppose that the market prices of 6-month European calls are as follows:

Strike price ($)	Call price ($)
55	10
60	7
65	5

- The investor could create a butterfly spread by buying one call with a $55 strike price, buying one call with a $65 strike price, and selling two calls with a $60 strike price.
- It costs $10 + $5 – (2*$7) = $1 to create the spread.
- If the stock price in 6 months is greater than $65 or less than $55, the total payoff is zero, and the investor incurs a net loss of $1.

Spreads

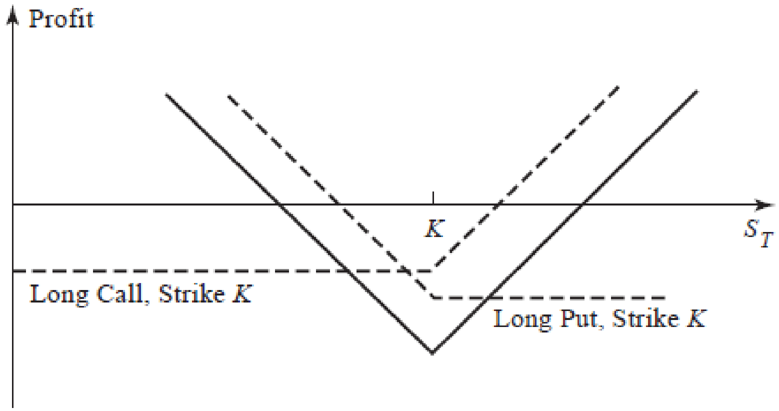
- Butterfly Spreads: A butterfly spread involves positions in options with three different strike prices.
- The investor could create a butterfly spread by buying one call with a $55 strike price, buying one call with a $65 strike price, and selling two calls with a $60 strike price.
- It costs $10 + $5 – (2*$7) = $1 to create the spread.
- If the stock price in 6 months is greater than $65 or less than $55, the total payoff is zero, and the investor incurs a net loss of $1.
- If the stock price is between $56 and $64, a profit is made.
- The maximum profit, $4, occurs when the stock price in 6 months is $60.

Combinations

- A combination is an option trading strategy that involves taking a position in both calls and puts on the same stock. We will consider
  - Straddles
  - Strips
  - Straps, and
  - Strangles.

Combinations

- **Straddles**: It involves buying a European call and put with the same strike price and expiration date.



- If the stock price is close to this strike price at expiration of the options, the straddle leads to a loss.
- If there is a sufficiently large move in either direction, a significant profit will result.

- Payoff from a straddle

Stock price range	Payoff from call	Payoff from put	Total payoff
$S_T \leq K$	0	$K - S_T$	$K - S_T$
$S_T > K$	$S_T - K$	0	$S_T - K$

Combinations

- **Example**: A call with a strike price of \$60 costs \$6. A put with the same strike price and expiration date costs \$4. Construct a table that shows the profit from a straddle. For what range of stock prices would the straddle lead to a loss?

- **Answer:**
- A straddle is created by buying both the call and the put. This strategy costs \$10. The profit/loss is shown in the following table:

Stock Price	Payoff	Profit
$S_T > 60$	$S_T - 60$	$S_T - 70$
$S_T \leq 60$	$60 - S_T$	$50 - S_T$

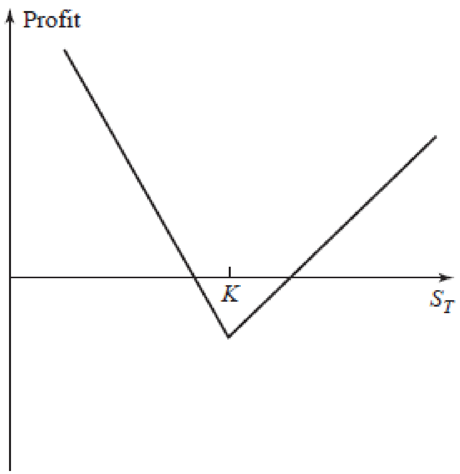
- This shows that the straddle will lead to a loss if the final stock price is between \$50 and \$70.

Combinations

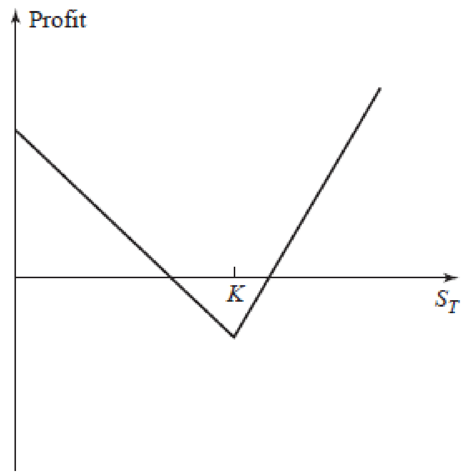
▪ Strips and Straps

- A *strip* consists of a long position in one European call and two European puts with the same strike price and expiration date.

- A *strap* consists of a long position in two European calls and one European put with the same strike price and expiration date.



Strip (one call + two puts)



Strap (two calls + one put)

- Profit from a strip and strap

Combinations

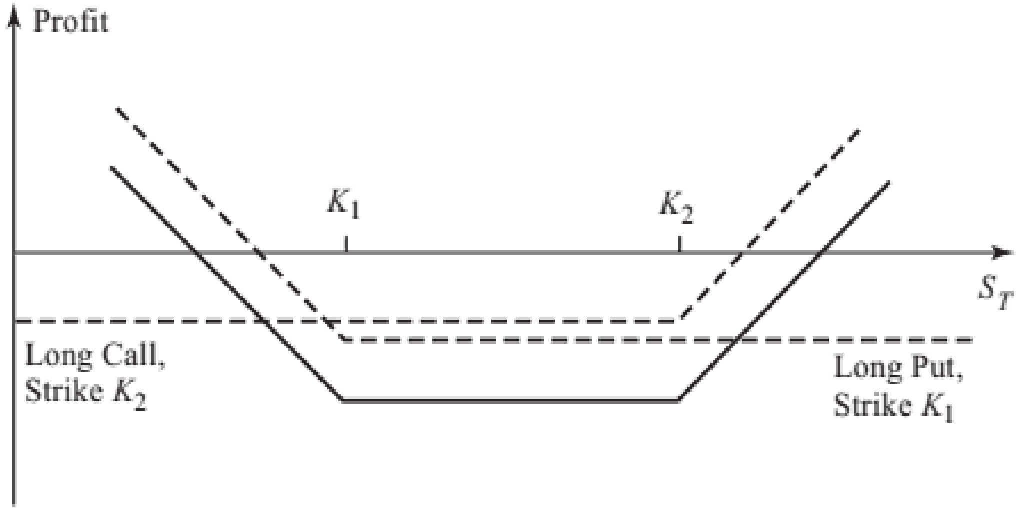
- Strips and Straps
- In a strip the investor is betting that there will be a big stock price move and considers a decrease in the stock price to be more likely than an increase.
- In a strap the investor is also betting that there will be a big stock price move.
- However, in this case, an increase in the stock price is considered to be more likely than a decrease.

Combinations

- Strangle: In a strangle, sometimes called a bottom vertical combination, an investor buys a European put and a European call with the same expiration date and different strike prices.
- A strangle is a similar strategy to a straddle.
- The investor is betting that there will be a large price move, but is uncertain whether it will be an increase or a decrease.

Combinations

- Strangle: In a strangle, sometimes called a bottom vertical combination, an investor buys a European put and a European call with the same expiration date and different strike prices.
- Profit from a strangle



Combinations

- Strangle: In a strangle, sometimes called a bottom vertical combination, an investor buys a European put and a European call with the same expiration date and different strike prices.
- Payoff from a strangle

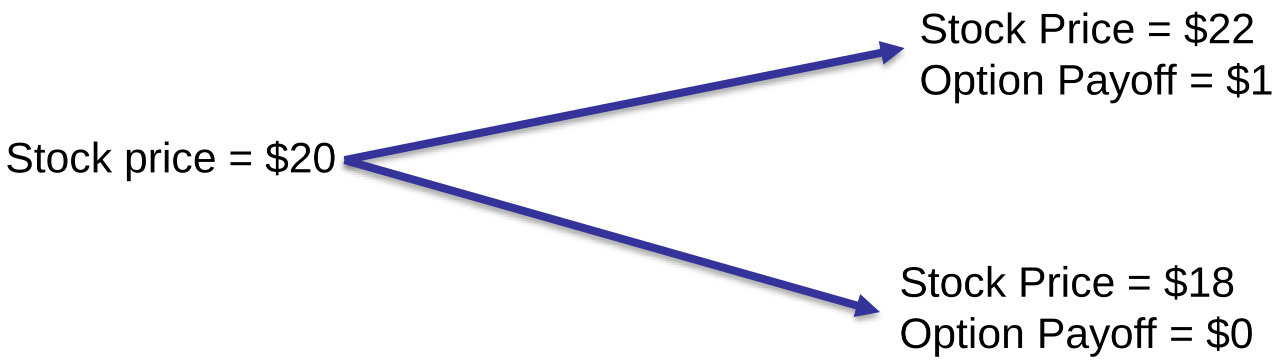
Stock price range	Payoff from call	Payoff from put	Total payoff
$S_T \leq K_1$	0	$K_1 - S_T$	$K_1 - S_T$
$K_1 < S_T < K_2$	0	0	0
$S_T \geq K_2$	$S_T - K_2$	0	$S_T - K_2$

Combinations

- Strangle: In a strangle, sometimes called a bottom vertical combination, an investor buys a European put and a European call with the same expiration date and different strike prices.
- **Example:** A trader sells a strangle by selling a call option with a strike price of \$50 for \$3 and selling a put option with a strike price of \$40 for \$4. For what range of prices of the underlying asset does the trader make a profit?
- **Solution:**
- The trader makes a profit if the total payoff is less than \$7.
- This happens when the price of the asset is between \$33 and \$57.

Binomial Trees

- One-step binomial model with a no-arbitrage argument
  - Suppose we have a European call option with the strike price \$20 expiring in 3 months.
  - The stock is currently trading at \$20.



Binomial Trees

- One-step binomial model with a no-arbitrage argument
  - The ups and downs in stock prices suggest that we can set-up a portfolio of stock and call options in such a way that there is no uncertainty about the value of the portfolio at the end of the 3 months.

  - Given these two scenarios, it is always possible to form a riskless portfolio.

Binomial Trees

- One-step binomial model with a no-arbitrage argument
  - Let’s assume that a portfolio consisting of a long position in Delta (Δ) shares of the stock and a short position in one call option.
  - We calculate the values of Δ in such a way that the portfolio becomes riskless.
  - If the stock price moves up from \$20 to \$22, the value of shares is 22Δ and the value of the option is 1, so the total value of the portfolio is $22\Delta - 1$.
  - If the stock price moves down from \$20 to \$18, the value of the share is 18Δ and the value of the option is 0.
  - $18\Delta - 0 = 18\Delta$

Binomial Trees

- One-step binomial model with a no-arbitrage argument
  - The portfolio is riskless if the value of Δ is chosen so that the final value the portfolio is the same as both the alternatives. It means

$22\Delta - 1 = 18\Delta$

$22\Delta - 18\Delta = 1$

$4\Delta = 1; \Delta = 0.25$

$\Delta = 0.25$

The 0.25 represents the proportion of the share. The riskless portfolio will now be

Long: 0.25 stocks

Short: 1 option

Binomial Trees

- One-step binomial model with a no-arbitrage argument
  - If the stock moves up to $22, the value of the portfolio is

    = 22*0.25 – 1 = 4.5
  - If the stock price moves down to $18, the value of the portfolio is

    = 18*0.25 = 4.5

Conclude: it appears that whether the stock price moves up or down, the value of the portfolio is always 4.5 at the end of the life of the option.

Binomial Trees

- One-step binomial model with a no-arbitrage argument

  Conclude: it appears that whether the stock price moves up or down, the value of the portfolio is always 4.5 at the end of the life of the option.

  - However, the argument could be what if we assume the riskless portfolio’s return equivalent to the risk-free rate especially when there is no arbitrage opportunity.

Binomial Trees

- One-step binomial model with a no-arbitrage argument
  - However, the argument could be what if we assume the riskless portfolio's return equivalent to the risk-free rate especially when there is no arbitrage opportunity.
  - Suppose that the risk-free rate is 12% per annum
  - It follows that the value of portfolio today must be the present value of 4.5 or $4.5e^{-0.12*3/12} = 4.367$.

  - In terms of present value, the value of the stock today is known to be \$20.
  - Suppose the option price is denoted by *f*. The value of the portfolio today is

$= 20*0.25 – f = 5 – f$

$= 5 - f = 4.367$

solve for $\boldsymbol{f = 0.633}$

Binomial Trees

- One-step binomial model with a no-arbitrage argument
  - Suppose the option price is denoted by f. The value of the portfolio today is

$= 20*0.25 – f = 5 – f$

$= 5 - f = 4.367$

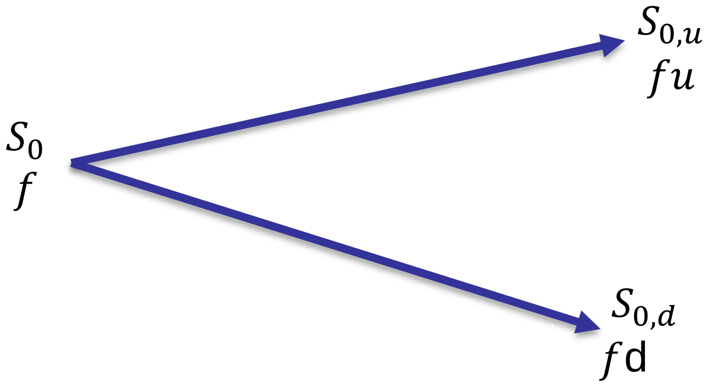
solve for $\mathbf{f = 0.633}$

  - This shows that in the absence of arbitrage opportunities, the current value of the option must be **0.633.**
  - If the value of the option were more than 0.633, the portfolio would cost less than 4.367 to set-up and would earn more than the risk-free rate.
  - If the value of the option were less than 0.633, shorting the portfolio would provide a way of borrowing the money at less than the risk free rate.

Binomial Trees: Generalization

- Suppose that S_0 is the stock price and option whose current price is f and expiration date is T.

- We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, S_0 u, where u > 1, or down from S_0 to a new level, S_0 d, where d < 1.



Binomial Trees: Generalization

- Suppose that S_0 is the stock price and option whose current price is f and expiration date is T.

- We suppose that the option lasts for time T and that during the life of the option the stock price can either move up from S_0 to a new level, $S_0 u$, where $u > 1$, or down from S_0 to a new level, $S_0 d$, where $d < 1$.
- The percentage increase in the stock price when there is an up movement is $u – 1$.
- The percentage decrease when there is a down movement is $1 - d$.
- If the stock price moves up to $S_0 u$, we suppose that the payoff from the option is f_u
- If the stock price moves down to $S_0 d$, we suppose the payoff from the option is f_d.

Binomial Trees: Generalization

- Now we create a portfolio which consists of a long position in Δ shares and a short position in one option.
- We calculate the value of Δ that makes the portfolio riskless.
- If there is an upward movement in the value of the portfolio at the end of life of the option is

$$S_0 u \Delta - f_u$$

- If there is down movement in the stock price, then the value becomes

$$S_0 d \Delta - f_d$$

- These two are equal when

$$S_0 u \Delta - f_u = S_0 d \Delta - f_d$$

If you solve for $\Delta = f_u – f_d / S_0 u - S_0 d$ (1)

Binomial Trees: Generalization

- These two are equal when

$$S_0 u \Delta - f_u = S_0 d \Delta - f_d$$

If you solve for $\Delta = f_u - f_d / S_0 u - S_0 d$

- In this case, portfolio is riskless and there to be no arbitrage opportunities, it must earn the risk free interest rate.

Δ = change in option price / change in the stock price (up and down)

- If we denote the risk-free rate by r, then the PV of the portfolio is

$$(S_0 u \Delta - f_u)e^{-rT}$$

- The cost of setting up the portfolio is:

$$(S_0 \Delta - f)$$

- It follows that

$$(S_0 \Delta - f) = (S_0 u \Delta - f_u) e^{-rT}$$

$$f = S_0 \Delta (1- u e^{-rT}) + f_u e^{-rT}$$

Binomial Trees: Generalization

- The cost of setting up the portfolio is:

  $(S_0 \Delta - f)$
- It follows that $(S_0 \Delta - f) = (S_0 u \Delta - f_u) e^{-rT}$

  $$f = S_0 \Delta (1- u e^{-rT}) + f_u e^{-rT}$$
- If we substitute the Delta (equation 1) value here than it becomes:

$$f = S_0 \left(\frac{f_u - f_d}{S_0 u - S_0 d}\right)(1 - ue^{-rT}) + f_u e^{-rT}$$

$$f = \frac{f_u(1 - de^{-rT}) + f_d(ue^{-rT} - 1)}{u - d} \qquad (2)$$

$$f = e^{-rT}[pf_u + (1-p)f_d]$$

$$where\ p = \frac{e^{rT} - d}{u - d} \qquad (3)$$

Expressions in (2) and (3) enable an option to be priced when stock price movements are given by a one-step binomial tree under the assumption there is no arbitrage opportunity.

Binomial Trees: Generalization

- Using the above example of stock price \$20 and strike price \$21.
- If we assume the u = 1.1, d = 0.9, r = 0.12, T = 0.25, fu = 1, fd = 0

We can first calculate the p

$$p = \frac{e^{-0.12*3/12} - 0.9}{1.1 - 0.9} = 0.6523$$

$$f = e^{-0.12*3/12}[0.6523 * 1 + (1 - 0.6533) * 0\,] = \mathbf{0.633}$$

Binomial Trees: Generalization

EXAMPLE

A stock price is currently $50. It is known that at the end of two months it will be either $53 or $48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a two-month European call option with a strike price of $49? Use no-arbitrage arguments.

Binomial Trees: Generalization

EXAMPLE

- At the end of two months the value of the option will be either \$4 (if the stock price is \$53) or \$0 (if the stock price is \$48).
- Consider a portfolio consisting of:

$$\begin{array}{lcl} +\Delta & : & \textit{shares} \\ -1 & : & \textit{option} \end{array}$$

- The value of the portfolio is 48Δ or $53\Delta - 4$ in two months. If

$$48\Delta = 53\Delta - 4$$

$$\Delta = 0.8$$

- The value of the portfolio is certain to be 38.4. For this value of Δ the portfolio is therefore riskless. The current value of the portfolio is:

$$0.8 \times 50 - f$$

- where f is the value of the option.
- Since the portfolio must earn the risk-free rate of interest

Binomial Trees: Generalization

EXAMPLE

- $(0.8 \times 50 - f)e^{0.10\times 2/12} = 38.4$
- $f = 2.23$
- The value of the option is therefore \$2.23

- This can also be calculated directly from equations (2) and (3). , so that

$$p = \frac{e^{0.10\times 2/12} - 0.96}{1.06 - 0.96} = 0.5681$$

- and

$$f = e^{-0.10\times 2/12} \times 0.5681 \times 4 = 2.23$$

Binomial Trees: Risk Neutral Valuation

- Two conditions of risk neutrality
  - Expected return on a stock (or any other investment) is the risk-free rate.
  - The discount rate used for the expected payoff on an option is the risk-free rate.
- The parameter *p* should be interpreted as the probability of an up movement in a risk-neutral world so that (1-p) is the probability of a down movement in this world.
- We assume $u > e^{rT}$ so that $0<p<1$, the expression becomes

$$f = [pf_u + (1-p)f_d]$$

is expected future payoff the option in a risk-neutral world.

- It states that the value of the option today is its expected future payoff in a risk-neutral world discounted at a risk-free rate.

Binomial Trees: Risk Neutral Valuation

- Suppose p is the probability of an upward movement, the expected stock price $E(S_T)$ at time T is given by:

$E(S_T) = [pS_0 u + (1 - p)S_0 d]$

- $E(S_T) = [pS_0(u - d)] + S_0 d$

If we substitute for $p = \frac{e^{-rT} - d}{u - d}$

- We obtain the expression of $E(S_T) = S_0 e^{-rT}$ *(4)*

- This shows that the stock price grows, on average, at the risk-free rate when p is the probability of an up movement.
- In other words, the stock price behaves exactly as we would expect it to behave in a risk-neutral world when p is the probability of an up movement.

Binomial Trees: Risk Neutral Valuation

Example:

Suppose that the stock price is currently \$20 and will move either up to \$22 or down to \$18 at the end of 3 months. The option considered is a European call option with a strike price of \$21 and an expiration date in 3 months. The risk-free interest rate is 4% per annum

Sol:

- We define p as the probability of an upward movement in the stock price in a risk neutral world.
- Alternatively, we can argue that the expected return on the stock in a risk-neutral world must be the risk-free rate of 4%.

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Sol:

- We define p as the probability of an upward movement in the stock price in a risk neutral world.
- Alternatively, we can argue that the expected return on the stock in a risk-neutral world must be the risk-free rate of 4%.
- $22p – 18 (1-p) = 20e^{0.04*3/12}$
- $4p = 20e^{0.04*3/12} - 18 = 0.5503$
- $P = 0.5503$

Binomial Trees: Risk Neutral Valuation

Example:

Suppose that the stock price is currently \$20 and will move either up to \$22 or down to \$18 at the end of 3 months. The option considered is a European call option with a strike price of \$21 and an expiration date in 3 months. The risk-free interest rate is 4% per annum

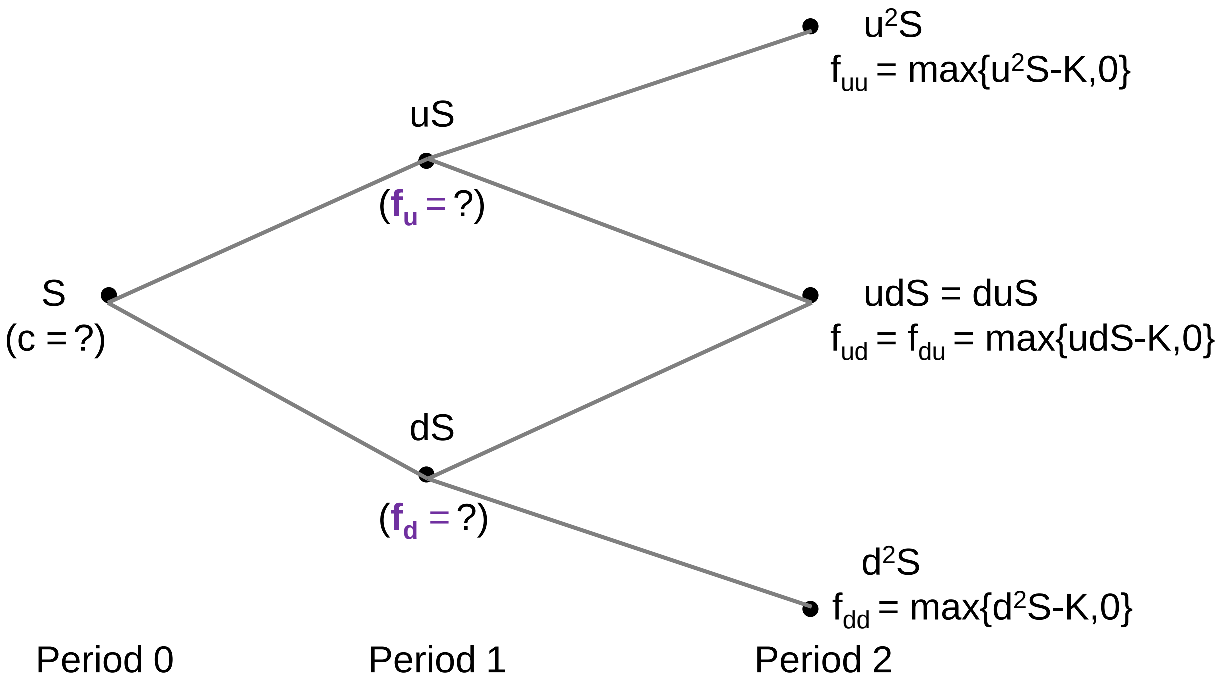
Sol:

- $4p = 20e^{0.04*3/12} - 18 = 0.5503$
- $P = 0.5503$
- At the end of the 3 months, the call option has a 0.5503 probability of being worth 1 and a 0.4497 probability of being worth zero.

- Its expected value is therefore 0.5503 * 1+0.4497=0.5503
- In a risk-neutral world this should be discounted at the risk-free rate. The value of the option today is therefore
- $0.5503ee^{0.04*3/12} = 0.545$

Two-Period Binomial Model

A. Suppose call expires in one year (as before).
Can extend framework to two periods by splitting year into two 6-month periods.



Each branch in Period 2 is like the One-Period Model: $f_u = [\, f_{uu}\, p + f_{ud}\,(1-p)\,]\, e^{-r\Delta T}$

$f_d = [\, f_{ud}\, p + f_{dd}\,(1-p)\,]\, e^{-r\Delta T}$

Likewise, for the first period:

$\mathbf{c = \{ f_u\, p + f_d\, (1-p) \}\, e^{-r\Delta T}}$ where ΔT = 6 months, and r = 6-mo risk-free rate.

Can work backward through the Tree.

Formula for Two-Period Binomial Model

1. Each branch in Period 2 can be examined using One-Period Binomial Model:

   $\mathbf{f_u} = [\, f_{uu}\, p + f_{ud}\,(1\text{-}p)\,]\, e^{-r\Delta T}$; $\mathbf{f_d} = [\, f_{ud}\, p + f_{dd}\,(1\text{-}p)\,]\, e^{-r\Delta T}$

   Likewise, in Period 1, $\mathbf{c} = \{\, \mathbf{f_u}\, \mathbf{p} + \mathbf{f_d}\, \mathbf{(1\text{-}p)}\, \}\, e^{-r\Delta T}$

   where ΔT = 6 months and r = 6-mo. riskfree rate.

2. Then we can solve for **c** by substituting for $\mathbf{f_u}$ and $\mathbf{f_d}$ above, as follows:

   $\mathbf{c} = \{\, \mathbf{[\, f_{uu}\, p + f_{ud}\,(1\text{-}p)\,]\, e^{-r\Delta T}\, p} + \mathbf{[\, f_{ud}\, p + f_{dd}\,(1\text{-}p)\,]\, e^{-r\,\Delta T}\, (1\text{-}p)}\, \}\, e^{-r\,\Delta T}$

Or: $\mathbf{c = \{\, f_{uu}\, p^2 + f_{ud}\,(1\text{-}p)p + f_{du}\, p(1\text{-}p) + f_{dd}\,(1\text{-}p)^2\, \}\, e^{-r\, 2\Delta T}}$

3. Simply the 1-Period Binomial Model applied twice.
   a. The same interpretation:

Call value is expected payoff over 2 periods discounted (twice) at (1-pd) risk-free rate.

*** 4. Note: Now **volatility of S** appears directly in formula! Now have all factors!
   a. $\boldsymbol{\sigma^2}$ for Binomial distrib. = N p (1-p): Thus, in 2-period model, $\boldsymbol{\sigma^2}$ = **2 p (1-p).**