SMFD Assignment 1 (Q10-11)

Jatin Kawatra, Idhant Kadela, Aaditya Rathi

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Question 10: Integral Identity of Expectation (8 Marks)

Let X be a nonnegative random variable with cumulative distribution function $F(x) = \mathbb{P}(X \leq x)$. We aim to prove:

$$\mathbb{E}[X] = \int_0^\infty (1 - F(x))\, dx$$

We begin by recalling that:

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\, dx = \int_0^\infty (1 - F(x))\, dx$$

This identity follows from Fubini's Theorem. Consider:

$$\mathbb{E}[X] = \int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \int_\Omega \left(\int_0^{X(\omega)} 1\, dx \right) d\mathbb{P}(\omega)$$

Using Fubini's Theorem, we can change the order of integration:

$$\mathbb{E}[X] = \int_0^\infty \mathbb{P}(X > x)\, dx = \int_0^\infty (1 - F(x))\, dx$$

Thus, both forms are equal:

$$\boxed{\mathbb{E}[X] = \int_0^\infty (1 - F(x))\, dx}$$

Question 11: Moment Generating Function and Jensen's Inequality (8 Marks)

Let $u \in \mathbb{R}$ and define $\varphi(x) = e^{ux}$, a convex function. Let $X \sim \mathcal{N}(\mu, \sigma^2)$, i.e., a normal random variable with mean $\mu = \mathbb{E}[X]$ and standard deviation $\sigma = \sqrt{\mathbb{E}[(X - \mu)^2]}$. The density function is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x - \mu)^2}{2\sigma^2} \right)$$

(i) Verify that $\mathbb{E}[e^{uX}] = e^{u\mu+\frac{1}{2}u^2\sigma^2}$

This is the moment generating function of the normal distribution. For $X \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathbb{E}[e^{uX}] = M_X(u) = e^{u\mu+\frac{1}{2}u^2\sigma^2}$$

Thus,

$$\boxed{\mathbb{E}[e^{uX}] = e^{u\mu+\frac{1}{2}u^2\sigma^2}}$$

(ii) Verify that Jensen's Inequality holds: $\mathbb{E}[\varphi(X)] \geq \varphi(\mathbb{E}[X])$

Since $\varphi(x) = e^{ux}$ is convex and X is any random variable, Jensen's Inequality tells us:

$$\mathbb{E}[e^{uX}] \geq e^{u\mathbb{E}[X]}$$

Using the result from part (i):

$$\mathbb{E}[e^{uX}] = e^{u\mu+\frac{1}{2}u^2\sigma^2} > e^{u\mu} = \varphi(\mathbb{E}[X]) \quad \text{(for } \sigma > 0 \text{ and } u \neq 0\text{)}$$

Hence,

$$\boxed{\mathbb{E}[\varphi(X)] \geq \varphi(\mathbb{E}[X])}$$

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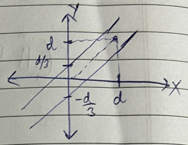
Q6. let $X, Y \rightarrow$ chosen independently in $[0, d]$
we want $P(|X-Y|) < d/3$

let define $P = \dfrac{\text{Area where } |X-Y| < d/3}{\text{Total Area}} = \dfrac{\text{Area of desired square}}{d^2}$

$|X-Y| < d/3$ lies in region between $Y = X + d/3$ and $Y = X - d/3$



Area where $|X-Y| < d/3$
= area b/w those two lines inside square

A = (area of square) − 2(area of triangles)
$= d^2 - \dfrac{2d^2}{3} = \dfrac{d^2}{3}$

$\Rightarrow P(E) = \dfrac{d^2/3}{d^2} = \dfrac{1}{3}$

Q7. People → $P_0\ P_1\ P_2 \ldots P_n$ = $n+1$ people
P_0 → start the rumour

(a) Probability that rumour is told r times without returning to originator

→ rumour is passed to n other people
Each step $= \dfrac{n-1}{n}$ → $P(E) = \left(\dfrac{n-1}{n}\right)^r$

(b) Prob that it is ~~not~~ repeated ⇒ only possible if rumour dies after first transmission.
$\Rightarrow P(E) = \dfrac{1}{n}$

~~Prove that the events repeated infinitely~~

Q8. $P\left(\bigcap_{i=1}^{n} A_i^c\right) \leq e^{-P(A_1) \dots -P(A_n)}$

Since events are independent

$$P\left(\bigcap_{i=1}^{n} A_i^c\right) = \prod_{i=1}^{n}\left(1 - P(A_i)\right) \leq \prod_{i=1}^{n} e^{-P(A_i)}$$

$$= e^{-\sum_{i=1}^{n} P(A_i)}$$

using $(1 - x \leq e^{-x})$

Q9. To prove - Convolution of two distribution function is also a distribution function

F, G - CDFs → Their convolution $H(x) = \int_{-\infty}^{\infty} F(x-y)\, G(y)$

Or, equivalently X, Y are independent random variables with CDFs F and G, then

$H(x) = P(X + Y \leq x)$

$H(x)$ = Non decreasing, Right continuous, $\lim_{x \to -\infty} H(x) = 0$, $\lim_{x \to \infty} H(x) = 1$

$\Rightarrow$ $H(x)$ is also a valid CDF.

$\Rightarrow$ Convolution of two CDFs also yields a CDF because it satisfies Monotonicity, limits and continuity