Comparisons of numerical methods applied to simple harmonic motion, and use of the most accurate of these to determine how a mass on a spring behaves under an applied force

James Clark

8929885

School of Physics and Astronomy

The University of Manchester

Second Year Computational Physics Report

April 2015

Abstract

In order to find out which numerical method of four would be best for finding how applying a force to a simple harmonic system of a mass attached to a spring, they were compared against the analytical solution in the absence of force. Out of the Euler, Improved Euler, Euler-Cromer, and Verlet numerical methods, Verlet was found to be the most accurate. This backwards algorithm was started by calculating the first displacement from the initial position with one thousand steps of the Euler-Cromer algorithm, at a thousandth of the step size. Constant forces were applied at different times in the oscillation of the mass, and the effect of adding a sinusoidal force was visualised as well.

Introduction

Numerical methods are commonly used when a problem can't be solved analytically. The problem in question is simple harmonic motion, and how a mass on a spring responds to different levels of damping and various externally applied forces. Four numerical methods (Euler, Improved Euler, Euler-Cromer, and Verlet) were compared with the analytical solution for the solved simple harmonic system with no applied force, and the one found to be most in accordance with the analytical solution after a given time was chosen to be used to explore the system when it could not be solved analytically.

Initially, the basic Euler method was used to see whether the displacement, velocity, and acceleration would be similar to that we expect from experiment. These three quantities all had the correct phase differences between each other, however this numerical method added energy to the system (increasing its maximum velocity and displacement as time progresses) even with small step sizes – so different methods were compared to find one accurate enough to use with an externally applied force.

The methods were compared at the end of twenty seconds, after about three oscillations. This was not a long enough time scale to be able to tell if the Euler-Cromer method or the Verlet method gave values closer to the analytical solution's position, however the other two methods had since been eliminated as being the most accurate. The Verlet method was the most accurate after two hundred seconds, with and without damping, so it was the method of choice for the analytically unsolvable system.

The effect of critical damping, under-damping, and over-damping was explored with the Verlet method, and then the effect of different forces (time varying or constant) provided at different times was explored.

This methodology, and maybe the code itself (after some modifications), could be applied to other areas of physics, such as electrical circuits, pendulums, and systems with non-linear restoring forces. Further comparisons of the algorithms could be done, to find how much computation time is required for a given accuracy – this information would be useful in simulations which require quick computation, such as video games or in driverless cars.

Theory

An in-depth derivation of the numerical methods used will not be provided, as they are well documented elsewhere. For example, Riley clearly demonstrates how the simple Euler method is formed (specifically for the differential equation $\frac{dy}{dx} = -y\,,\ y(0) = 1$) by considering the gradient being equivalent to the change in two adjacent y-coordinates, divided by a horizontal step size [1]. In the context of simple harmonic motion, the acceleration can be written in terms of velocity and step size h as

$$a = \frac{v(t+h) - v(t)}{h} \qquad , \qquad 1$$

and velocity can be similarly defined in terms of displacement. Rearrangement gives:

$$v_{n+1} = v_n + ha_n \qquad and \qquad 2$$

$$x_{n+1} = x_n + hv_n \qquad , \qquad 3$$

which can be used to calculate the displacement of a mass attached to a spring against time. Note that an acceleration has to be defined – this can be found by analysing forces. Inherent forces to the system are drag, b, and that due to the spring, k, and there are also externally applied forces, $F(t)$, - considering these gives

$$a_n = -\frac{b}{m}v_n - \frac{k}{m}x_n + \frac{F(t)}{m} \qquad . \qquad 4$$

By including another term from the Taylor expansion of position at some small time after a given time, the Euler method can be turned into the Improved Euler method by writing

$$x_{n+1} = x_n + hv_n + \frac{h^2}{2}a_n \qquad . \qquad 5$$

One problem with the Euler algorithm is that its energy increases with time – this can be seen by substituting equations 2 and 3 into the expression

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \qquad , \qquad 6$$

which will show that energy is added to the system on the order of the square of the step size. This can be overcome by modifying Euler's method, producing a symplectic (energy preserving) method known as the Euler-Cromer method, the changes being that now:

$$v_{n+1} = v_n - \frac{kh}{m}x_n \qquad , \quad and \qquad 7$$

$$x_{n+1} = x_n + hv_{n+1} \qquad . \qquad 8$$

All of these methods are “forward methods”. The “backwards” Verlet algorithm requires two previous positions to calculate the next one, and can be expressed firstly be considering centred derivatives, performing algebra to reach the equations

$$x_{n+1} = Ax_n + Bx_{n-1} \qquad , \qquad 9$$

where

$$A = \frac{2}{D}(2m - kh^2) \quad , \quad B = \frac{bh - 2m}{D} \quad , \quad D = 2m + bh$$

Table 1 summarises some key points about these methods.

Method name	Symplectic?	Order of error with step size	Forward or backward
Euler	No	h^2	Forward
Improved Euler	No	h^3	Forward
Euler-Cromer	Yes	h^2	Forward
Verlet	No	h^4	Backward

Table 1 – each method has its own advantages and disadvantages. Euler is fast but less accurate, and Euler-Cromer has energy which oscillates about the mean value, for examples.

To compare the effectiveness of these methods, they can be compared to the analytically derived solution to this unforced simple harmonic motion,

$$x = Aexp\left(i\omega - \frac{\gamma}{2}\right)t \qquad , \qquad 10$$

where ω is the angular frequency, γ is a mass-weighted damping term, t is time, and A is a constant. Associated quantities such as the natural angular frequency, the quality factor, and the value of critical damping, are closely related and are explained in Young & Freedman‘s *University Physics* textbook [2].

Checking phases of displacement, velocity, and acceleration with Euler's method

Despite not having a very small step size and the method's simplicity, it is able to reproduce aspects of the behaviour we know the simple harmonic systems possess. Figures 2 and 3 show how important a decrease in step size is for this method.

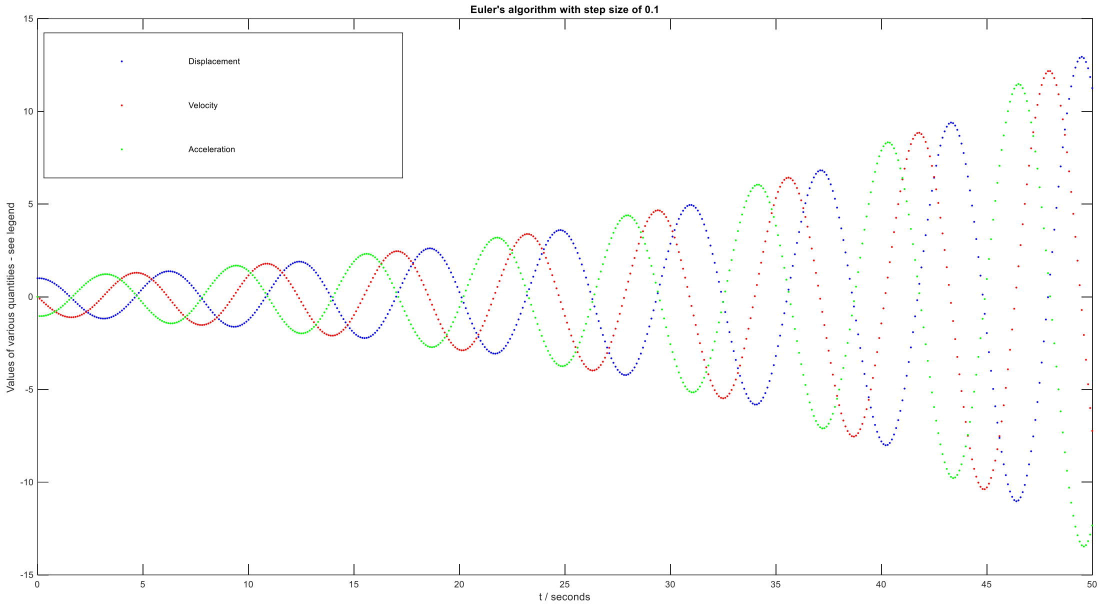

Figure 1: This figure shows how the displacement, velocity, and acceleration of the mass vary with time, calculated with the simple Euler method.

As shown in figure 1, when the mass has zero displacement it has a restoring force of zero, hence an acceleration of zero, and it possesses its maximum velocity. At its maximum displacement, the velocity is zero and the acceleration is a maximum – and for all these the directions are correct. Where the simulation fails is its preservation of energy. As can readily be seen, for a step size of 0.1 the amplitude of the displacement increases significantly – however this increase is able to be reduced easily, by reducing the step size to 0.01.

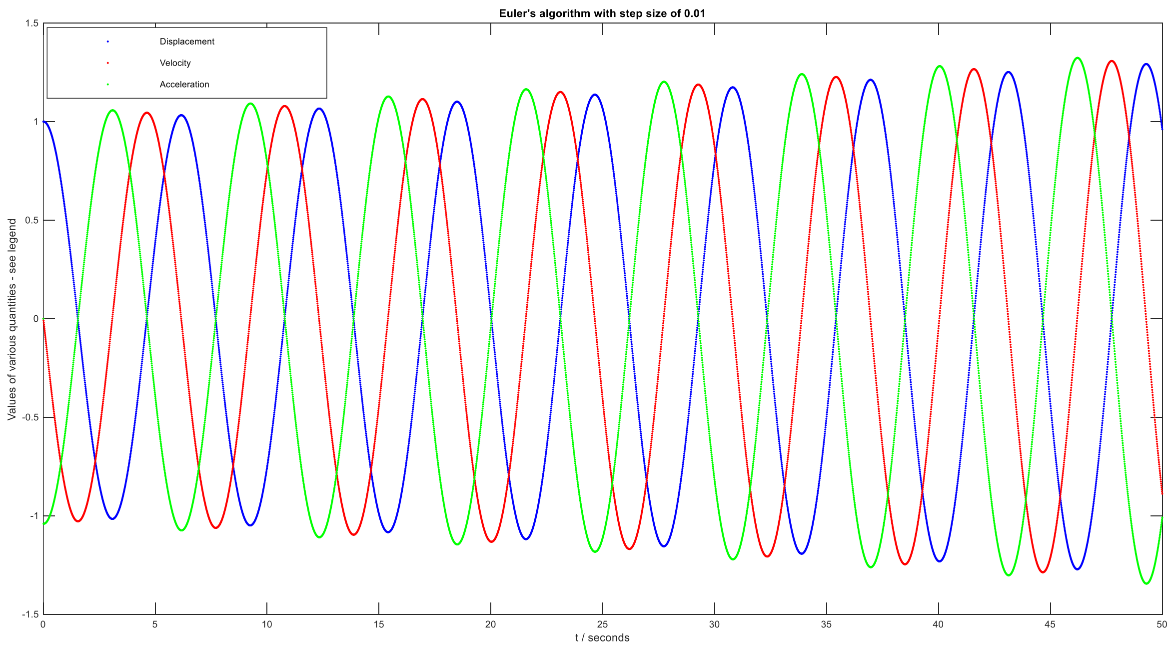

Figure 2: This figure demonstrates how a division of ten in the step size can produce much more accurate data. The maximum displacement should always be 1 metre, as there is no damping or external force.

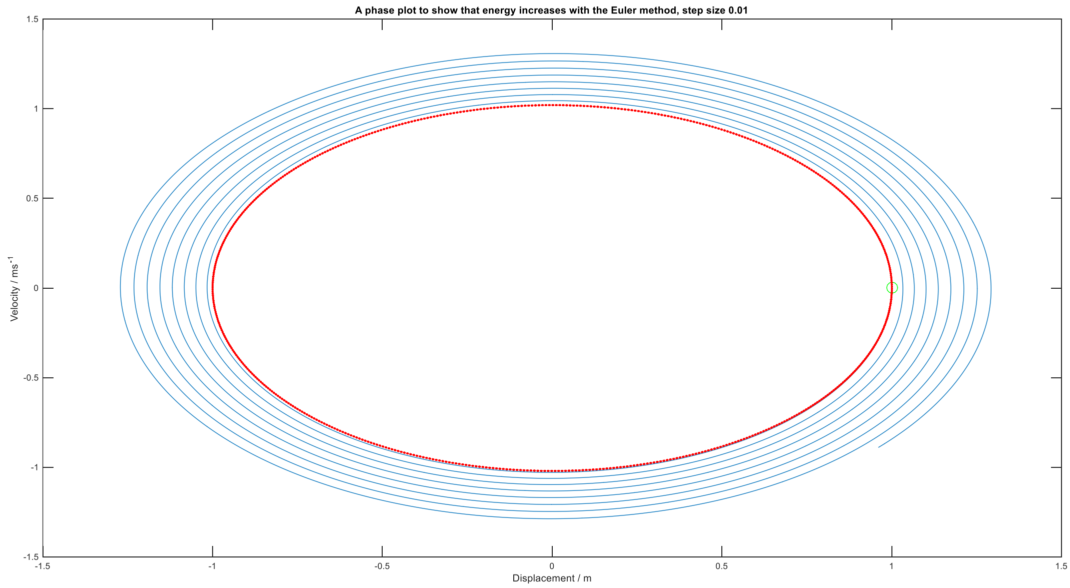

Figure 3 shows that the Euler algorithm adds energy artificially. The green circle represents the starting position of the mass, and the red ellipse represents the analytical solution's displacement and velocity. Every point enclosed by the red ellipse represents a possible pair of values of displacement and velocity that the mass can possess. The Euler data starts to spiral out from the bottom of the inner ellipse, as its energy increases.

The assigned values of m and k, 7.4 and 7.7 respectively, suggest that the amplitudes of displacement and velocity will be similar (using equation 6) – as is seen in figure 3.

Figures 1, 2, and 3 clearly demonstrate that the Euler algorithm, whilst giving a good solution in some aspects, lacks the accuracy required for finding how the system reacts to an external force. A more thorough comparison of numerical methods follows.

Comparing numerical methods to the analytical solution

All the numerical methods under comparison have been described in the theory section already, apart from how the Verlet function would be started – it needs two previous displacements to calculate the next one. The second position was found by using one thousand steps of the Euler-Cromer algorithm, at a thousandth of the step size to be used by the Verlet algorithm.

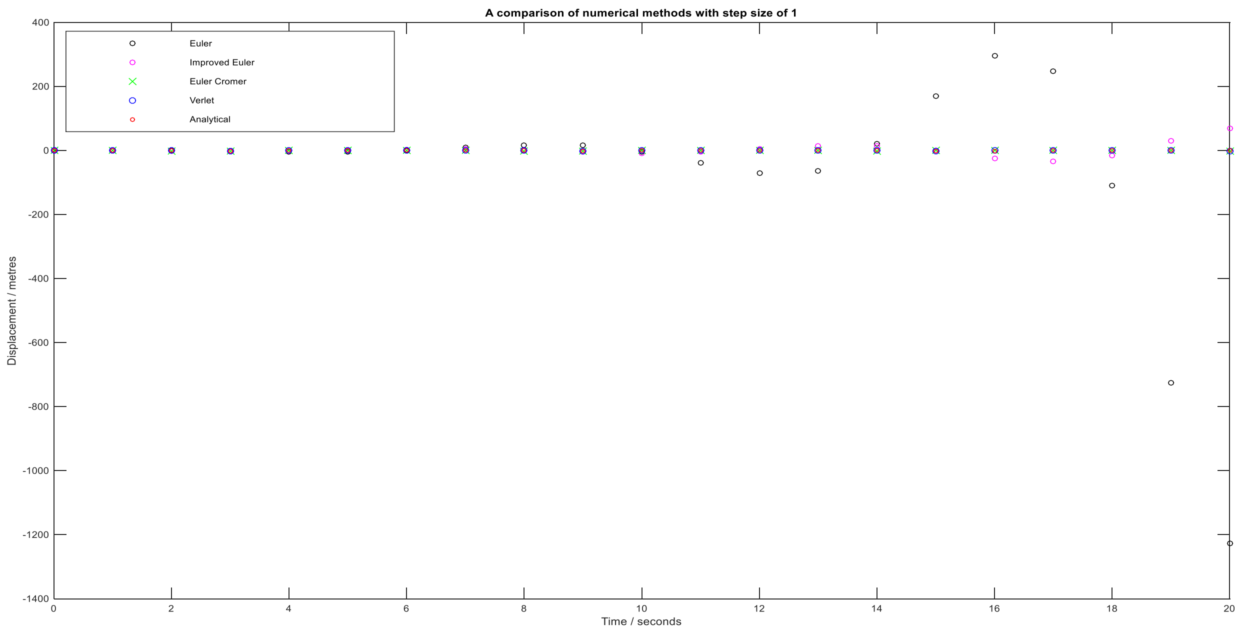

Figure 4: This plot uses a much higher step size than would be ideal – however, it shows that the Euler and Improved Euler methods have significant inaccuracy after only twenty seconds, and that the Verlet and Euler-Cromer methods are much more accurate. A smaller step size will differentiate between these two.

As was shown in the previous section, the basic Euler method does become much more accurate with a smaller step size – however, it is readily apparent that this and the Improved Euler methods are still much more inaccurate than the other two methods.

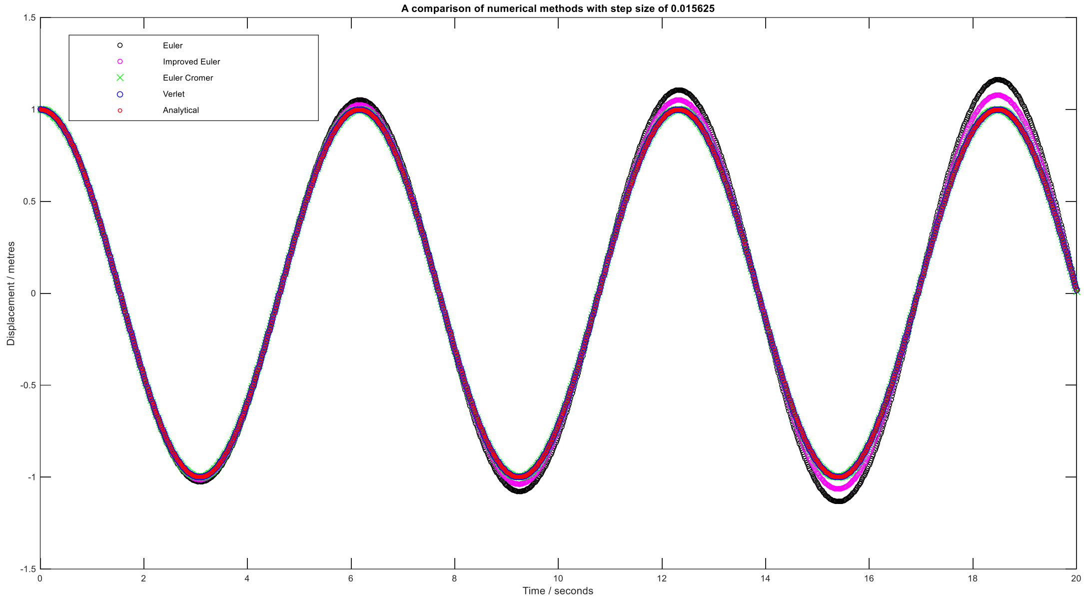

Figure 5: This plot does not make it clear whether the Verlet or Euler-Cromer method is more accurate. Even a close-up image of one of the peaks, for a smaller step size, doesn't make it clear which is more accurate.

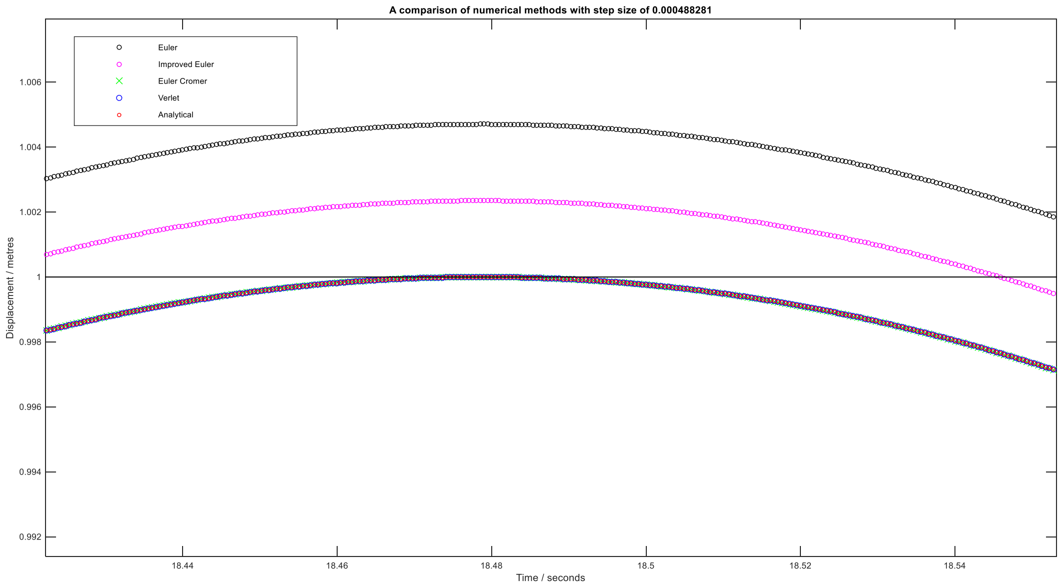

Figure 6: Even when zoomed in, at this step size it is too difficult to visually differentiate between the Euler-Cromer and Verlet methods – they both seem quite accurate for this short simulation time.

To make it clear which method is the most accurate, a 200 second simulation will be run – at the end of this, the difference from a method's final displacement value will be compared to the analytical solution's final value, and the method with the least difference will be chosen to see how an external force and various levels of damping affect the motion of the system.

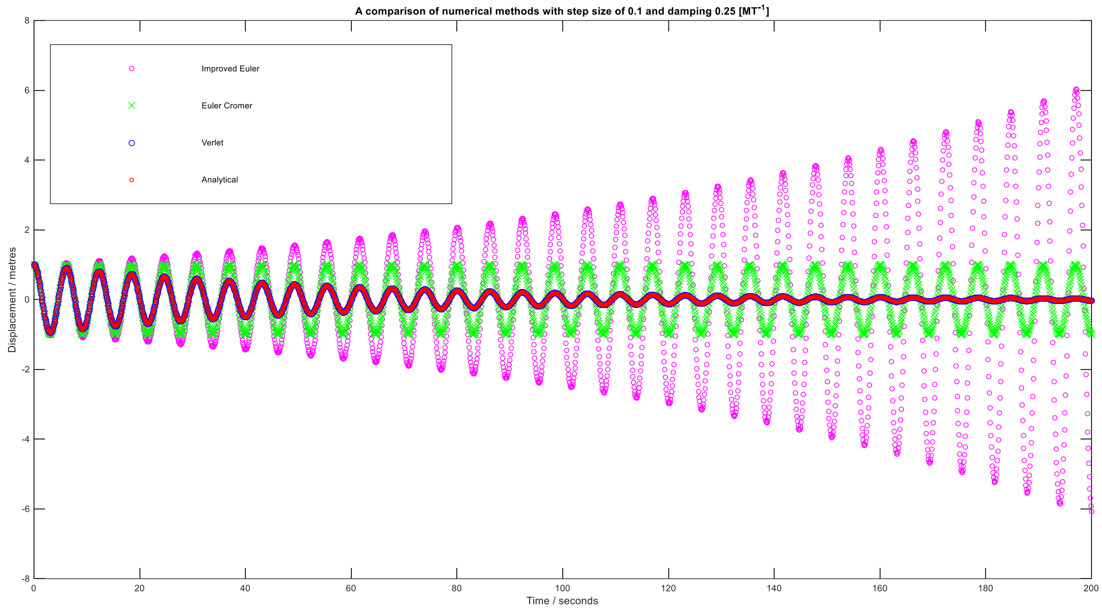

Figure 7: This plot shows that the Euler-Cromer method doesn't interact with the introduced damping term, whereas the Verlet method does – indicating that the Verlet method is the one to be used. A table of data is below, to compare accuracies of the methods without this damping term.

Step size	Euler	I. Euler	E-Cromer	Verlet
1	64667.58	3589.734	62.20717	43.59903
0.5	5371.082	268.0892	25.528	11.65051
0.25	280.8942	18.75911	9.705888	2.904809
0.125	23.5362	1.510018	**4.096717**	0.728499
0.0625	3.693222	0.086409	1.862919	0.183273
0.03125	0.887684	0.170346	0.885545	0.046293
0.015625	0.283612	0.109132	0.431346	0.011791
0.007813	0.10937	0.059712	0.212822	0.003054
0.003906	0.047409	0.031045	0.105699	0.000816
0.001953	0.021983	0.015808	0.052672	0.00023
0.000977	0.010573	0.007974	0.026291	7.07E-05
0.000488	0.005183	0.004004	0.013135	2.45E-05

Table 2 Shows values of accuracy for various step sizes, where this accuracy was calculated as the fractional change from the analytical position value, by using equation eq,

$$acc. = \frac{(x_{method} - x_{analytical})}{x_{analytical}} \quad . \qquad 11$$

As can be seen on the row of the bold value (E-Cromer, step size 0.125), apparently the Euler-Cromer method becomes less accurate than the Euler and Improved Euler methods for step sizes smaller then 0.25. The red value (Improved Euler, step size 0.0625) does not fit the trend suggested by the data, suggesting that maybe the total energy calculated by each method should have been used for these calculations, rather than the displacements. Alternatively, these comparisons could be made only at a peak in position, i.e. when there is no energy stored kinetically.

Critically damping, over-damping, and under-damping

The effect of three specifics levels of damping on the mass were calculated with the Verlet method, and are shown below.

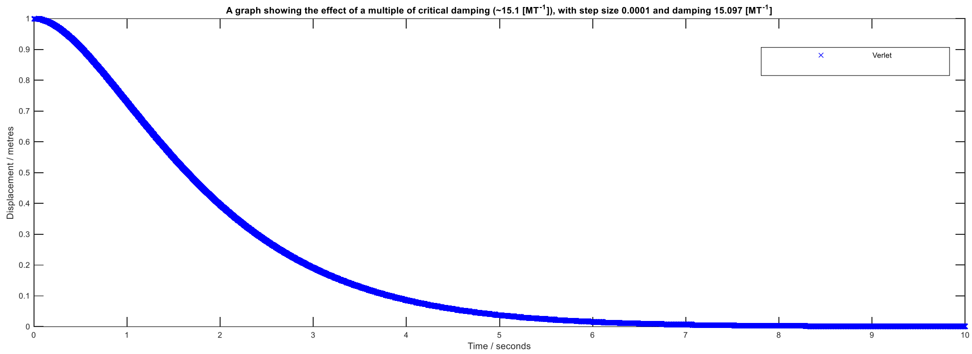

Figure 8: This curve shows how the simple harmonic system reacts to critical damping. Compared to figures 9 and 10, with under-damping and over-damping respectively, the critically damped system decays the fastest.

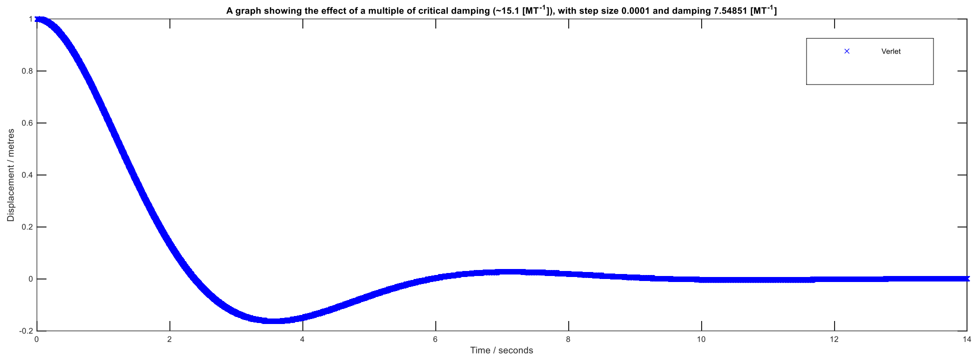

Figure 9: This curve shows an under-damped system. Noticeably, the mass manages to obtain negative displacement before coming to rest.

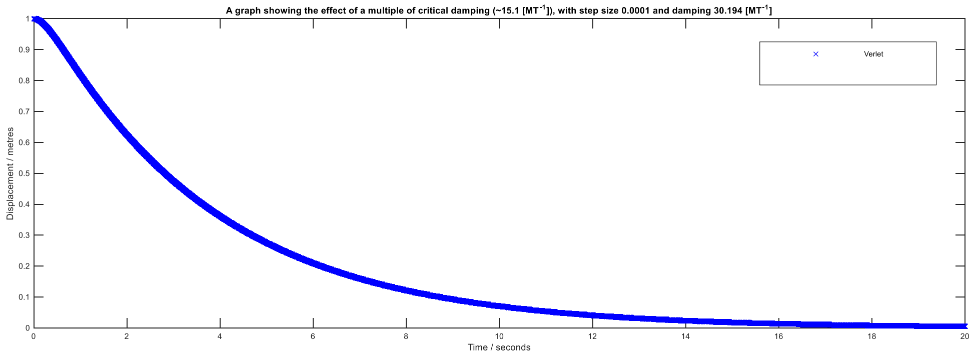

Figure 10 has an over-damped system – the displacement does not go negative, but the time it takes for the mass to come to rest is longer than with the critically damped system.

The observations from figures 8, 9, and 10 all agree with theory on vibrations and damping, and the graphs are of the same form as those in King's *Vibrations and Waves* [3].

Application of force to the system

The effect of applying a constant force at a certain part of an oscillation is shown with figures 11, 12, and 13. It is apparent that the time of application of a sudden constant force has a great effect on the resultant displacement of the mass.

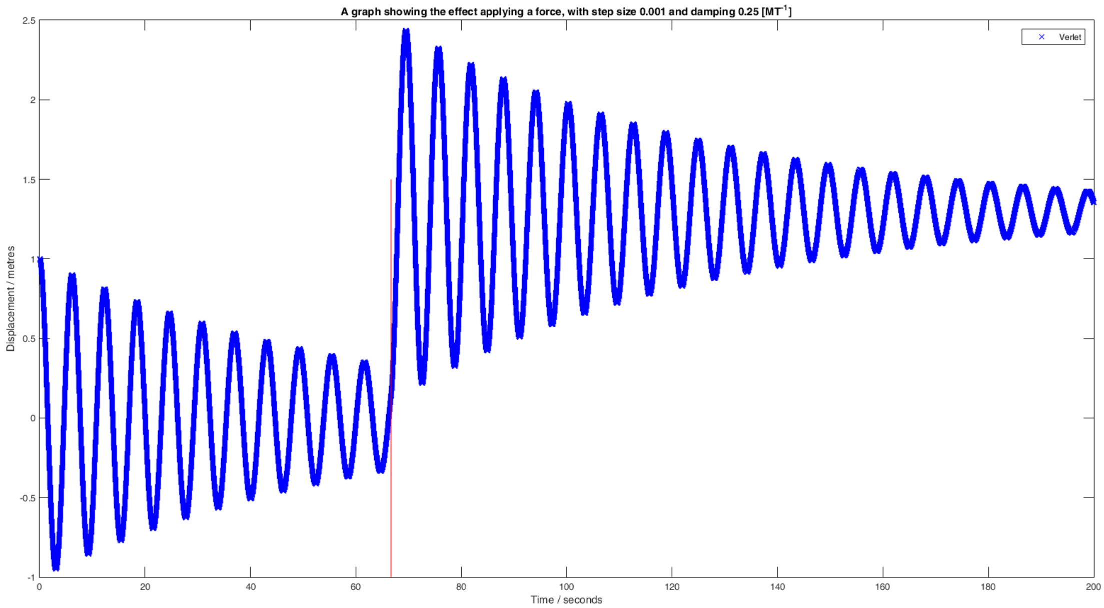

Figure 11 shows the force being applied when the mass is at a midpoint. The difference between the first maximum and minimum of the oscillation after applying the force is ~2.15

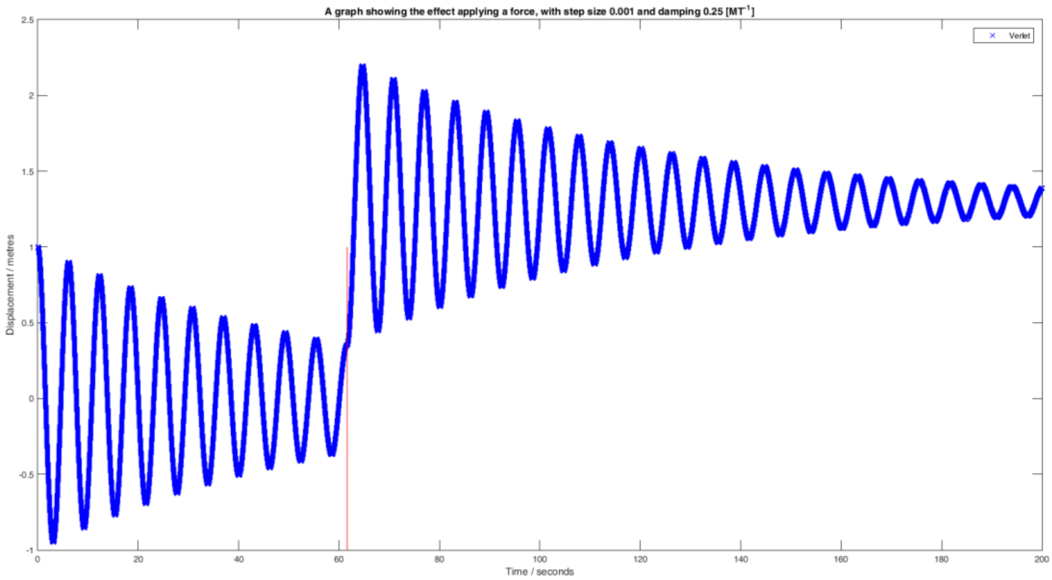

Figure 12 shows the force being applied when the mass is a maxima. The difference between the first maximum and minimum of the oscillation after applying the force is ~1.75

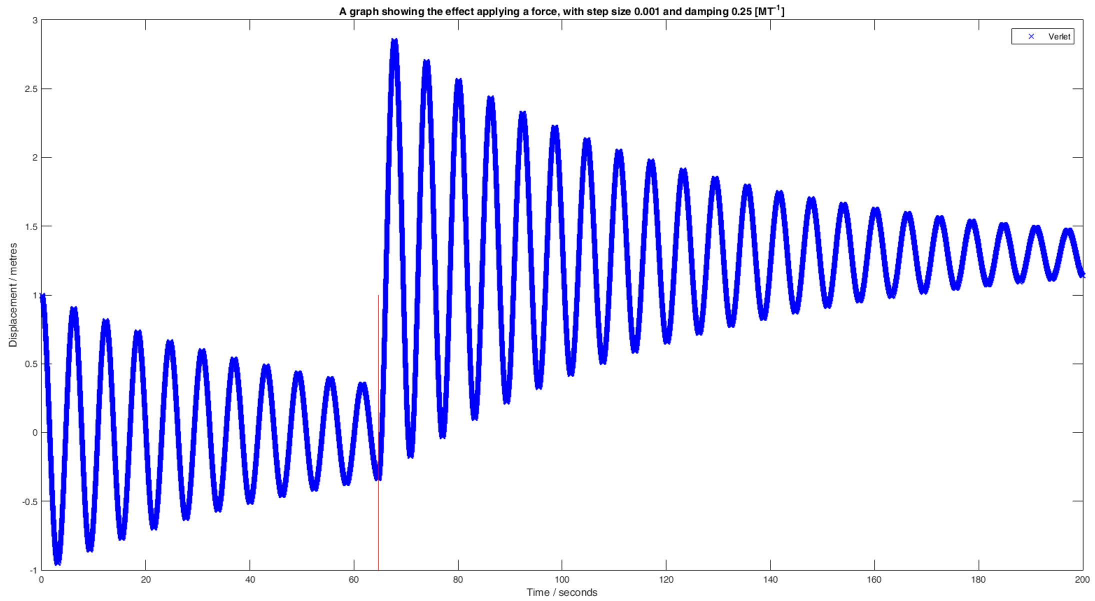

Figure 13 shows the force being applied when the mass is at a minima. The difference between the first maximum and minimum of the oscillation after applying the force is ~3.2

What is apparent is that applying a constant positive force whilst or just before the mass is about to move in the positive direction gives a larger variation in amplitude than when applying this force when the mass is about to move in the opposite direction. This seems intuitive, and the analogy of pushing a swing seems an apt comparison.

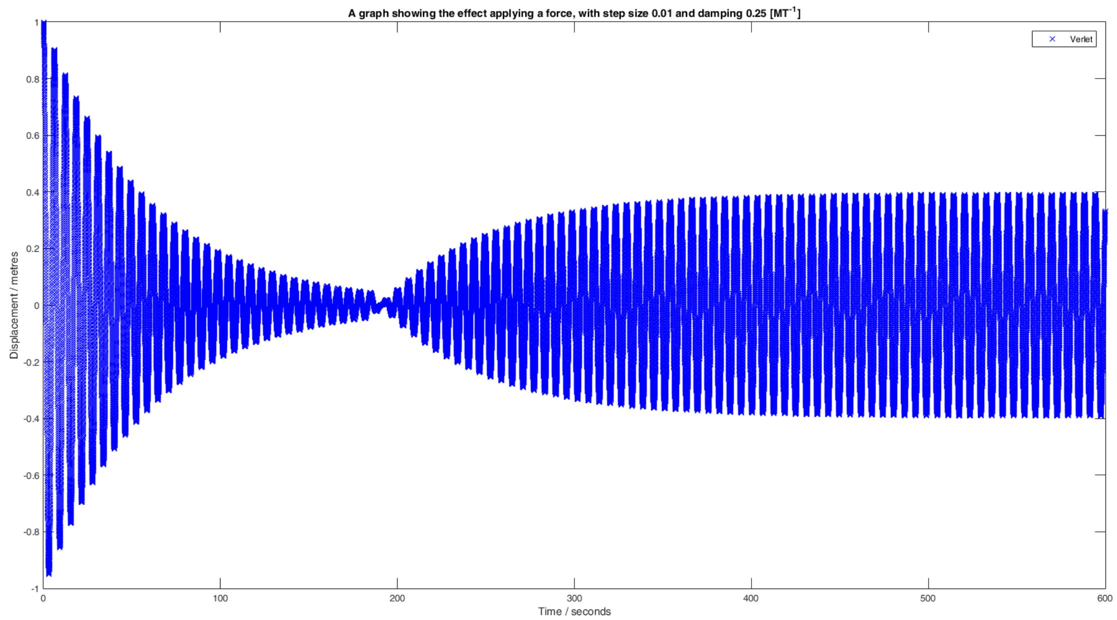

Figure 14 shows how the initial oscillation decays heavily over the first 184.8 seconds, then the amplitude of oscillation increases to a fixed amplitude after about twenty oscillations.

For figure 14, the force applied was sinusoidal, with the same frequency as the original wave. To explore resonance, the variation of the amplitude in the steady-state system (about 500 seconds onwards) with frequency of the driving force would be plotted.

Discussion

As has already been mentioned, to compare the numerical methods it may have been better to compare energy differences, rather than displacements.

A plot of the Euler-Cromer's energy against time would have been useful in demonstrating its symplectic nature – however this should not explain why the Euler-Cromer algorithm was unaffected by damping. This behaviour is likely due to an error in programming.

Table 2 indicates that it would have perhaps have been better to start off the Verlet algorithm with an Euler or Improved Euler series of small steps – however, the accuracy of the Verlet algorithm was high, so this is a minor thing to improve.

As has been mentioned, measuring the computing time against accuracy could be insightful.

Conclusion

By comparing numerical methods with an analytical solution to the simple harmonic system of a mass on a spring, the Verlet backwards method was chosen to be used to investigate how this system would behave when an external force is applied, and under certain damping conditions. It was found that applying a positive force when the displacement of the mass was the most negative would produce the largest vibrations, and applying it when the displacement was the most positive would produce the least vibrations - and both the pre- and post- force-application oscillations decay with under an exponential envelope. Applying a sinusoidal force caused an almost completely decayed oscillation to grow to a new amplitude, the amplitude depending on the magnitude of the force.

References

[1] Riley, K. *Mathematical Methods for Physics and Engineering*, *2nd Edition, Great Britain, Cambridge University Press, 1978*

[2] Freedman, R., Young, H. *University Physics*, *13th Edition*, United States of America, Pearson Education Limited, 2013

[3] King, G. *Vibrations and Waves*, 1st *Edition*, Great Britain, John Wiley & Sons Ltd, 2009