Module III Integration

Differentiation	Integration
1. $\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x\, dx = \sin x + c$
2. $\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x\, dx = -\cos x + c$
3. $\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x\, dx = \tan x + c$
4. $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$	$\int \operatorname{cosec}^2 x\, dx = -\cot x + c$
5. $\frac{d}{dx}(\sec x) \cancel{\tan x} = \sec x \tan x$	$\int \sec x \tan x\, dx = \sec x + c$
6. $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$	$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$
7. $\frac{d}{dx}(\log x) = 1/x$	$\int 1/x\, dx = \log x + c$
8. $\frac{d}{dx}(e^x) = e^x$	$\int e^x\, dx = e^x + c$
9. $\frac{d}{dx}(x^n) = n \cdot x^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + c$
10. $\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{\cancel{x^2}1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}\, dx = \sin^{-1} x + c$
11. $\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$	$\int \frac{-1}{\sqrt{1-x^2}}\, dx = \cos^{-1} x + c$
12. $\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + c$
13. $\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$	$\int \frac{-1}{1+x^2}\, dx = \cot^{-1} x + c$
14. $\frac{d}{dx}(\operatorname{cosec}^{-1} x) = \frac{-1}{x\sqrt{x^2-1}}$	$\int \frac{-1}{x\sqrt{x^2-1}}\, dx = \operatorname{cosec}^{-1} x$
15. $\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}\, dx = \sec^{-1} x$

16. $\int k\,dx$ $\frac{d}{dx}(k) = 0$. $\int k\,dx = kx + c$.

17. $\frac{d}{dx}(x) = 1$ $\int 1\,dx = x + c$.

Q1. $\int x^2\,dx = \frac{x^3}{3} + c$ $\int x^n = \frac{x^{n+1}}{n+1} + c$

2. $\int \sqrt{x}\,dx = \int x^{1/2}\,dx = \frac{x^{1/2+1}}{1/2+1}$

$= \frac{x^{3/2}}{3/2} = \frac{2x^{3/2}}{3} + c$.

3. $\int x\sqrt{x}\,dx = \int x \cdot x^{1/2}\,dx = \int x^{1+1/2}\,dx$

$= \int x^{3/2}\,dx$

$= \frac{x^{3/2+1}}{3/2+1} = \frac{x^{5/2}}{5/2}$

$= \frac{2x^{5/2}}{5} + c$.

4. $\int 2\sin x\,dx = 2\int \sin x\,dx$.

$= 2(-\cos x) + c$.

$= -2\cos x + c$.

5. $\int 5\sec^2 x\,dx = 5\int \sec^2 x\,dx$

$= 5\tan x + c$

6. $\int 5\operatorname{cosec} x \cot x\,dx = 5\int \operatorname{cosec} x \cot x\,dx$

$= -5\operatorname{cosec} x + c$

7. $\int 2e^x\,dx = 2e^x + c$

8. $\int (3x^2 + 4x + 6)\,dx = \int 3x^2\,dx + \int 4x\,dx + \int 6\,dx$

$= 3\int x^2\,dx + 4\int x\,dx + 6\int dx.$

$= 3.\frac{x^{2+1}}{2+1} + 4\frac{x^{1+1}}{1+1} + 6.x.$

$= \frac{3x^3}{3} + \frac{4x^2}{2} + 6x$

$= \underline{x^3 + 2x^2 + 6x}$

9. $\int 3\sec^2 x + 4\sin x + e^x.\,dx.$

$= \int 3\sec^2 x\,dx + \int 4\sin x\,dx + \int e^x dx$

$= 3\int \sec^2 x\,dx + 4\int \sin x\,dx + e^x$

$= \underline{3\tan x - 4\cos x + e^x + c}$

10 $\int 4e^x + \frac{2}{x\sqrt{x}}\,dx = 4\int e^x dx + 2\int \frac{1}{x\sqrt{x}}\,dx$

$= 4e^x + 2\int \frac{1}{x.x^{1/2}}\,dx$

$= 4e^x + 2\int \frac{1}{x^{3/2}}\,dx.$

$= 4e^x + 2\int x^{-3/2}\,dx$

$= 4e^x + 2.\frac{x^{-3/2+1}}{-3/2+1} + c$

$= 4e^x + \frac{2x^{-1/2}}{-1/2} + c.$

$= 4e^x - 2.2x^{-1/2} + c$

$= 4e^x - 4x^{-1/2} + c$

$= 4e^x - \frac{4}{x^{1/2}} + c = \underline{4e^x - \frac{4}{\sqrt{x}} + c}$

11. $\int (3x+4)(2x-1)\,dx$

$= \int (6x^2 - 3x + 8x - 4)\,dx$

$= \int (6x^2 + 5x - 4)\,dx$

$= \int 6x^2\,dx + \int 5x\,dx - \int 4\,dx$

$= 6\int x^2\,dx + 5\int x\,dx - 4\int dx$

$= 6 \cdot x^3/3 + 5 \cdot x^2/2 - 4x + c.$

$= 2x^3 + 5/2\, x^2 - 4x + c$

12. $\int \sqrt{x}(1-x)\,dx = \int (\sqrt{x} - x\sqrt{x})\,dx$

$= \int \sqrt{x}\,dx - \int x\sqrt{x}\,dx$

$= \int x^{1/2}\,dx - \int x \cdot x^{1/2}\,dx$

$= \dfrac{x^{1/2+1}}{1/2+1} - \int x^{1+1/2}\,dx$

$= \dfrac{x^{3/2}}{3/2} - \int x^{3/2}\,dx.$

$= \dfrac{2x^{3/2}}{3} - \dfrac{x^{3/2+1}}{3/2+1} + c$

$= 2/3\, x^{3/2} - \dfrac{x^{5/2}}{5/2} + c$

$= 2/3\, x^{3/2} - 2/5\, x^{5/2} + c$

13 $\int \frac{3x-1}{x^4}\,dx = \int\left(\frac{3x}{x^4} - \frac{1}{x^4}\right)dx$

$= \int\left(\frac{3}{x^3} - \frac{1}{x^4}\right)dx$

$= 3\int \frac{1}{x^3}\,dx - \int \frac{1}{x^4}\,dx$

$= 3\int x^{-3}\,dx - \int x^{-4}\,dx$

$= 3\,\frac{x^{-3+1}}{-3+1} - \frac{x^{-4+1}}{-4+1}$

$= \frac{3x^{-2}}{-2} - \frac{x^{-3}}{-3} + c.$

$= \frac{-3}{2x^2} + \frac{1}{3x^3} + c$

14 $\int \frac{x^3+5x^2-4}{x}\,dx = \int \frac{x^3}{x} + \frac{5x^2}{x} - \frac{4}{x}\,dx$

$= \int x^2 + 5x - 4/x\,dx.$

$= \int x^2\,dx + \int 5x\,dx - \int 4/x\,dx$

$= \frac{x^{2+1}}{2+1} + 5\cdot\frac{x^{1+1}}{1+1} - 4\int 1/x\,dx$

$= x^3/3 + 5x^2/2 - 4.\log x + c$

15 $\int \frac{3\cos x + 4}{\sin^2 x}\,dx.$

$= \int\left(\frac{3\cos x}{\sin^2 x} + \frac{4}{\sin^2 x}\right)dx.$

$= 3\int \frac{\cos x\,dx}{\sin x\cdot\sin x} + 4\int \frac{1}{\sin^2 x}\,dx.$

$= 3\int \cot x\cdot\operatorname{cosec} x\,dx + 4\int \operatorname{cosec}^2 x\,dx$

$= 3(-\text{cosec}\,x) + 4(-\cot x) + c$

$= -3\,\text{cosec}\,x - 4\cot x + c$

16. $\int \frac{2+3\sin x}{\cos^2 x}\,dx$

$= \int \frac{2}{\cos^2 x} + \frac{3\sin x}{\cos^2 x}\,dx$

$= 2\int \frac{1}{\cos^2 x}\,dx + 3\int \frac{\sin x}{\cos x \cdot \cos x}\,dx$

$= 2\int \sec^2 x\,dx + 3\int \tan x \cdot \sec x\,dx$

$= 2\tan x + 3\sec x + c$

17 $\int \frac{\cos 2x}{\cos^2 x \sin^2 x}\,dx$ $\quad \cos 2x = \cos^2 x - \sin^2 x$

$\int \frac{\cos^2 x - \sin^2 x}{\cos^2 x \sin^2 x}\,dx$

$= \int \frac{\cos^2 x}{\cos^2 x \sin^2 x}\,dx - \int \frac{\sin^2 x}{\cos^2 x \sin^2 x}\,dx$

$= \int \frac{1}{\sin^2 x}\,dx - \int \frac{1}{\cos^2 x}\,dx$

$= \int \text{cosec}^2 x\,dx - \int \sec^2 x\,dx$

$= -\cot x - \tan x + c$

18 $\int \frac{\sin^3 x + \cos^3 x}{\sin^2 x \cos^2 x}\,dx$

$$= \int \frac{\sin^3 x}{\sin^2 x \cos^2 x}\,dx + \int \frac{\cos^3 x}{\sin^2 x \cos^2 x}\,dx$$

$$= \int \frac{\sin x}{\cos^2 x}\,dx + \int \frac{\cos x}{\sin^2 x}\,dx$$

$$= \int \frac{\sin x}{\cos x \cdot \cos x}\,dx + \int \frac{\cos x}{\sin x \cdot \sin x}\,dx$$

$$= \int \tan x \cdot \sec x\,dx + \int \cot x \cdot \operatorname{cosec} x\,dx$$

$$= \sec x - \operatorname{cosec} x \cancel{dx} + c$$

$\tan^2 x = \sec^2 x$
$\tan^2 x = \sec^2 x - 1$

Q. $\int \cot^2 x\,dx$

$\cot^2 x = \operatorname{cosec}^2 x - 1$

$$\int (\operatorname{cosec}^2 x - 1)\,dx = \int \operatorname{cosec}^2 x\,dx - \int dx$$
$$= -\cot x - x + c$$

Q. $\int \tan^2 x\,dx$

$\tan^2 x = \sec^2 x - 1$

$$\int (\sec^2 x - 1)\,dx = \int \sec^2 x\,dx - \int dx$$
$$= \tan x - x + c$$

Q. $\int (\tan x + \cot x)^2\,dx$

$$= \int (\tan^2 x + 2\tan x \cot x + \cot^2 x)\,dx$$
$$= \int (\sec^2 x - 1 + 2 + \operatorname{cosec}^2 x - 1)\,dx$$

$$= \int \sec^2 x \cancel{-2+2} + \operatorname{cosec}^2 x\,dx$$

$= \int \sec^2 x \, dx + \int \operatorname{cosec}^2 x \, dx$

$= \tan x - \cot x + c$

Methods of integration

Integrals of the form $\int f(ax+b)\,dx$.

Q. $\int (ax+b)^n \, dx$.

put $u = ax+b$.

$\frac{d}{dx}(u) = \frac{d}{dx}(ax+b)$

$\frac{du}{dx} = \frac{d}{dx}(ax) + \frac{d}{dx}(b)$

$= a.\frac{d}{dx}(x) + 0$.

$\frac{du}{dx} = a$.

$du = a\,dx$.

$dx = \frac{1}{a} du$

$\int u^n . \frac{1}{a} du$

$\frac{1}{a}\int u^n du$.

$\frac{1}{a} \frac{u^{n+1}}{n+1} + c$

$\frac{1}{a} . \frac{(ax+b)^{n+1}}{n+1} + c$

Q. $\int \sin(ax+b)\,dx$

put $u = ax+b$

$du = a\,dx$

$du/a = dx$

$\int \sin u \times du/a$.

$\frac{1}{a}\int \sin u \, du$

$= \frac{1}{a}(-\cos u) + c$

$= -\frac{\cos(ax+b)}{a} + c$

Q. $\int \sec^2(ax+b)\,dx$

$= \frac{1}{a}\tan(ax+b) + c$

Q. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$

Q. $\int \cos(ax+b)\, dx = \frac{1}{a} \sin(ax+b) + c$

Q. $\int (5x-1)^3 dx$.

$u = 5x-1$

$\int u^3 \cdot du/5$ $\quad du/dx = \frac{d}{dx}(5x-1)$

$= \frac{1}{5} \frac{u^4}{4} + c$ $\quad \frac{du}{dx} = 5 \cdot \frac{d}{dx}(x) - \frac{d}{dx}(1)$

$= \frac{1}{5} \frac{(5x-1)^4}{4} + c$ $\quad \frac{du}{dx} = 5 - 0 = 5$

$du/5 = dx$

$= \frac{(5x-1)^4}{20} + c$

Q. $\int \sin(2x+3)\, dx = \frac{1}{2} -\cos(2x+3) \cdot \cancel{2} + c$

$= -\cos(2x+3) + c$

Q. $\int \sec^2(2x)\, dx = \frac{1}{2} \tan 2x + c$

Q. $\int e^{5x+4} dx = \frac{1}{5} e^{5x+4} + c$.

Q. $\int \frac{1}{2x-3} dx$

$u = 2x-3$

$du = (2x-3)\, dx$

$\int \frac{du/2}{u}$

$du = 2\, dx$.

$\frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \log u$ $\quad du/2 = dx$

$$= \frac{1}{2}\log(2x-3)+c$$

Q. $\int \sqrt{4x+3}\, dx$

$4x+3 = u$
$4\,dx = du$
$dx = du/4$

$\int \sqrt{u}\, du/4$

$$= \int u^{1/2}\frac{du}{4} = \frac{1}{4}\frac{u^{3/2}}{3/2}+c$$

$$= \frac{1}{4}\times \frac{2}{3}(4x+3)^{3/2}+c$$

$$= \frac{1}{6}(4x+3)^{3/2}$$

Integration using trigonometric transformation:

Q- $\int \sin^2 x\, dx$.

$\sin^2 x = \frac{1-\cos 2x}{2}$.

$$\int \frac{1-\cos 2x}{2}\, dx = \frac{1}{2}\int 1-\cos 2x\, dx$$

$$= \frac{1}{2}\left[\int dx - \int \cos 2x\, dx\right]$$

$$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]+c.$$

Q- $\int \cos^2 x\, dx$

$\cos^2 x = \frac{1+\cos 2x}{2}$.

$$\int \frac{1+\cos 2x}{2}\, dx = \frac{1}{2}\int (1+\cos 2x)\, dx$$

$$= \frac{1}{2}\left[x+\frac{\sin 2x}{2}\right]+c$$

$$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$$

$$\cos^3 x = \frac{\cos 3x - 3\cos x}{4}$$

Q. $\int \sin^3 x \, dx = \int \frac{3\sin x - \sin 3x}{4} \, dx$

$$= \frac{1}{4}\left[\int 3\sin x \, dx - \int \sin 3x \, dx\right]$$

$$= \frac{1}{4}\left[3 \cdot -\cos x - -\frac{\cos 3x}{3}\right] + c$$

$$= \frac{1}{4}\left[-3\cos x + \frac{\cos 3x}{3}\right] + c$$

Q. $\int \cos^3 x \, dx = \int \frac{\cos 3x - 3\cos x}{4} \, dx$

$$= \frac{1}{4}\left[\int \cos 3x \, dx - \int 3\cos x \, dx\right]$$

$$= \frac{1}{4}\left[+\frac{\sin 3x}{3} - 3\sin x\right] + c$$

Note $\sin^2 x = \frac{1 - \cos 2x}{2}$

$$\sin^2 2x = \frac{1 - \cos 2(2x)}{2} = \frac{1 - \cos 4x}{2}$$

$$\cos^2 2x = \frac{1 + \cos 4x}{2}$$

Q. $\int \sin 3x \cos x \, dx$

$= \int \frac{1}{2}[\sin 4x + \sin 2x]\, dx$

$= \frac{1}{2}\left[\int \sin 4x \, dx + \int \sin 2x \, dx\right.$

$= \frac{1}{2}\left[-\frac{\cos 4x}{4} + -\frac{\cos 2x}{2}\right] + c$

Q. $\int \cos 3x \cos 2x = \int \frac{1}{2}[\cos(5x) + \cos(x)]$

$= \frac{1}{2}\left[\int \cos 5x \, dx + \int \cos x \, dx\right]$

$= \frac{1}{2}\left[+\frac{\sin 5x}{5} + \sin x\right] + c$