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# Integrated by Substitution method

1 Integrals of the form $\int x^{n-1} f(x^n)\,dx$.

put $x^n = u$

Q. $\int x \sin(x^2)\,dx$.

$u = x^2$ ✓

$\int \sin(u) \cdot du/2$. $\quad \frac{du}{dx} = 2x$.

$\frac{1}{2}\int \sin u\,du$ $\quad du = 2x\,dx$.

$du/2 = x\,dx$ ✓

$\frac{1}{2}(-\cos u) + C$

$\frac{1}{2} - \cos(x^2) + C \quad = -\frac{\cos(x^2)}{2} + C$

Q. $\int x^2 \cos(x^3)\,dx$.

put $u = x^3$ ✓

$\int \cos(u) \cdot du/3$. $\quad \frac{du}{dx} = 3x^2$.

$1/3 \int \cos u\,du$. $\quad du = 3x^2\,dx$

$du/3 = x^2\,dx$ ✓

$= 1/3 \sin u + C$

$= 1/3 \sin(x^3) + C$

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Q. $\int x^3 \sec^2(x^4)\, dx$.

$\int \sec^2(u) \cdot du/4$ $\quad u = x^4$

$= \frac{1}{4}\int \sec^2(u)\, du$. $\quad \frac{du}{dx} = 4x^3$

$du = 4x^3\, dx$.

$= \frac{1}{4} \tan u + C$ $\quad \frac{du}{4} = x^3\, dx$.

$= \frac{1}{4} \tan(x^4) + C$

Q. $\int x^2 e^{x^3}\, dx$. $\quad$ put $x^3 = u$.

$\frac{du}{dx} = 3x^2$.

$\int e^u \cdot du/3$

$= \frac{1}{3}\int e^u\, du$ $\quad du = 3x^2\, dx$.

$= \frac{1}{3} e^u + C$ $\quad \frac{du}{3} = x^2\, dx$.

$= \frac{1}{3} e^{x^3} + C$

Q. $\int \frac{2x^4}{1+x^{10}}\, dx$.

$= \int \frac{2x^4}{1+(x^5)^2}\, dx$. $\quad$ put $x^5 = u$

$\frac{du}{dx} = 5x^4$

$= \int \frac{2 \cdot du/5}{1+u^2}$ $\quad du = 5x^4\, dx$

$\frac{du}{5} = x^4\, dx$.

$= \frac{2}{5}\int \frac{du}{1+u^2}$.

$= 2/5 \tan^{-1} u + c$

$= 2/5 \tan^{-1}(x^5) + c.$

II Integrals of the form $\int (f(x))^n f'(x) dx$

put $f(x) = u$

Q. $\int \sin^3 x \cos x \, dx.$

$u = \sin x.$

$\int u^3 \, du.$

$du/dx = \cos x.$

$du = \cos x \, dx.$

$= \frac{u^{3+1}}{3+1} + c.$

$= \frac{(\sin x)^4}{4} + c. = \frac{\sin^4 x}{4} + c$

Q. $\int (3x^2 - 4x + 1)^3 (3x - 2) \, dx.$

$\int u^3 . du/2.$

$u = 3x^2 - 4x + 1$

$\frac{du}{dx} = 6x - 4$

$= 1/2 \int u^3 \, du.$

$\frac{du}{dx} = 2(3x - 2).$

$= \cancel{\frac{1}{2}[(3x^2-4x+1)^3}$

$du = 2(3x - 2) dx$

$= \frac{1}{2} . \frac{u^4}{4} + c.$

$du/2 = (3x - 2) dx.$

$= \frac{1}{2} \frac{(3x^2 - 4x + 1)^4}{4} + c = \frac{(3x^2 - 4x + 1)^4}{8} + c$

Q. $\int \frac{1+\cos x}{(x+\sin x)^3} dx$

$\int \frac{du}{u}$.

$x + \sin x = u$.

$\frac{du}{dx} = 1 + \cos x$

$du = (1+\cos x) dx$

$= \log u + c$

$= \log(x + \sin x) + c$

Q. $\int \frac{\cos x}{\sqrt{\sin x}} dx$.

$\sin x = u$

$\frac{du}{dx} = \cos x$.

$du = \cos x \, dx$.

$\int \frac{du}{\sqrt{u}}$

$= \int \frac{du}{u^{1/2}} = \int u^{-1/2} du$

$= \frac{u^{-1/2+1}}{-1/2+1} + c$

$= \frac{(\sin x)^{-1/2}}{-1/2} + c$.

$= -2(\sin x)^{-1/2} + c$

$= \frac{-2}{\sqrt{\sin x}} + c$

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1.4 Integrals of the form $\int \phi(f(x)) f'(x) dx$.
put $f(x) = u$

Q. $\int e^{\sin x} \cos x \, dx$.

put $\sin x = u$ ; $\frac{du}{dx} = \cos x$
$\cos x \, dx = du$.

$\int e^{u} du = e^{u} + c$
$= e^{\sin x} + c$

Q. $\int \frac{\sin(\log x)}{x} dx$.

put $u = \log x$, $\frac{du}{dx} = \frac{1}{x}$
$du = \cancel{\log x} \frac{1}{x} dx$.
$du = dx/x$.

$\int \sin(u) du$
$= -\cos u + c$
$= -\cos(\log x) + c$

Q. $\int (1 + e^{\tan x}) \sec^2 x \, dx$.

put $u = \tan x$, $\frac{du}{dx} = \sec^2 x$
$du = \sec^2 x \, dx$.

$\int (1 + e^{u}) du$

$= \int du + e^{u} du$
$= u + e^{u} + c$
$= \tan x + e^{\tan x} + c$

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Q. $\int e^x \sec^2(e^x)\,dx$.

$e^x = u,\ \frac{du}{dx} = e^x$
$e^x dx = du$.

$\int \sec^2 u \cdot du$
$= \tan u + c$
$= \tan(e^x) + c$

Q. $\int \frac{\sec^2 x\,dx}{\sqrt{1-\tan^2 x}}$.

$\tan x = u,\ \frac{du}{dx} = \sec^2 x$
$\sec^2 x\,dx = du$.

$\int \frac{du}{\sqrt{1-u^2}} = \sin^{-1} u + c$.
$= \sin^{-1}(\tan x) + c$

Integrals of the form $\int \frac{f'(x)}{f(x)}\,dx$.
put $f(x) = u$

Q. $\int \tan x\,dx$.

$\int \frac{\sin x}{\cos x}\,dx$

$\cos x = u,\ \frac{du}{dx} = -\sin x$
$-\sin x\,dx = du$.
$\sin x\,dx = -du$

$\int \frac{-du}{u} = -\int \frac{du}{u}$.

$= -\log(u) = -\log(\cos x) + c$.
$= \log(\cos x)^{-1} + c$.
$= \log\left(\frac{1}{\cos x}\right) + c$
$= \log \sec x + c$

Q. $\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx$.

$\int \frac{du}{u}$ put $u = \sin x$, $\frac{du}{dx} = \cos x$

$\frac{du}{dx} = \sin x$, $du = \cos x \, dx$.

$= \log u + c$

$= \log(\sin x) + c$

Q $\int \sec x \, dx$.

multiply Nr and Dr by $\sec x + \tan x$.

$\int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx$.

put $u = \sec x + \tan x$

$\frac{du}{dx} = \sec x \tan x + \sec^2 x$

$du = (\sec x \tan x + \sec^2 x) dx$

$du = \sec x(\tan x + \sec x) dx$

$\int \frac{du}{u}$

$= \log u + c$.

$= \log(\sec x + \tan x) + c$

Q. $\int \operatorname{cosec} x \, dx$.

multiply Nr and Dr by $\operatorname{cosec} x - \cot x$.

$\int \frac{\operatorname{cosec} x(\operatorname{cosec} x - \cot x)}{\operatorname{cosec} x - \cot x} dx$.

$$\int \frac{du}{u}$$

put $u = \operatorname{cosec} x - \cot x$.

$$= \log u + c.$$

$$\frac{du}{dx} = -\operatorname{cosec} x \cot x + \operatorname{cosec}^2 x$$

$$= \operatorname{cosec}^2 x - \operatorname{cosec} x \cot x$$

$$\frac{du}{dx} = \operatorname{cosec} x(\operatorname{cosec} x - \cot x)$$

$$= \log(\operatorname{cosec} x - \cot x) + c$$

$$du = \operatorname{cosec} x(\operatorname{cosec} x - \cot x)\, dx$$

Q. $\int \frac{2x}{x^2+1}\, dx.$

put $u = x^2+1$

$$\frac{du}{dx} = 2x$$

$$du = 2x\, dx.$$

$$\int \frac{du}{u} = \log u + c$$

$$= \log(x^2+1) + c$$

Problems

1. $\int x^2(x^3+1)\, dx.$

put $x^3+1 = u$.

$du/dx = 3x^2$

$du = 3x^2\, dx$

$du/3 = x^2\, dx.$

$$\int u \cdot du/3.$$

$$\frac{1}{3}\int u\, du$$

$$= \frac{1}{3}\left[\frac{u^2}{2}\right] + c$$

$$= \frac{1}{6}(x^3+1)^2 + c$$

or

$\int x^2(x^3+1)\,dx.$

$\int (x^2x^3 + x^2)\,dx$

$= \int x^2x^3\,dx + \int x^2\,dx.$

$u = x^3$

$= \int u\, du/3 + \int x^3/3 + c.$

$= 1/3 \int u\,du + x^3/3 + c.$

$= 1/3 \cdot \frac{u^2}{2} + x^3/3 + c$

$= \frac{1}{6} \cdot x^6 + x^3/3 + c$

Q $\int 3(x-1)^3\,dx.$

$x - 1 = u$

$du/dx = 1.$

$du = dx.$

$\int 3u^3\,du$

$= 3(u^4/4) + c$

$= 3/4\,(x-1)^4 + c$

Q. $\int \frac{2 + \sin x}{\cos^2 x}\,dx.$

$= \int \left[\frac{2}{\cos^2 x} + \frac{\sin x}{\cos x} \cdot \frac{1}{\cos x}\right] dx$

$= 2 \cdot \tan x + \sec x + c$

Q. $\int \sin^2(4x)\,dx$.

$\sin^2 x = \frac{1-\cos 2x}{2}$

$\sin^2(4x) = \frac{1-\cos 2(4x)}{2}$

$= \frac{1-\cos 8x}{2}$

$\int \frac{1-\cos 8x}{2}\,dx$.

$= \frac{1}{2}\left[x - \frac{\sin 8x}{8}\right] + C$ //

Q. $\int \sin x \sin 3x\,dx$.

$\sin A \sin B = \frac{-1}{2}[\cos(A+B) - \cos(A-B)]$

$\int -\frac{1}{2}[\cos 4x - \cos(-2x)]\,dx$

$= -\frac{1}{2}\int \cos 4x - \cos 2x\,dx$.

$= -\frac{1}{2}\left[\frac{\sin 4x}{4} - \frac{\sin 2x}{2}\right] + C$

Q. $\int \sin^3(2x)\,dx$.

$\sin^3 x = \frac{3\sin x - \sin 3x}{4}$

$\sin^3(2x) = \frac{3\sin 2x - \sin 3(2x)}{4}$

$= \frac{3\sin 2x - \sin 6x}{4}$

$\int \left(\frac{3\sin 2x - \sin 6x}{4}\right)dx$

$= \frac{1}{4}\left[3 \cdot \frac{-\cos 2x}{2} + \frac{\cos 6x}{6}\right] + C$

$= -\frac{3\cos 2x}{8} + \frac{\cos 6x}{24} + C$

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Q. $\int x \sec(x^2) \tan(x^2)\, dx$.

put $x^2 = u$

$2x = du/dx$

$du = 2x\, dx$

$du/2 = x\, dx$

$\int \sec(u) \tan(u) \cdot du/2$.

$1/2 \int \sec(u) \tan(u)\, du$

$= 1/2 \sec(u) + C$.

$= 1/2 \sec(x^2) + C$

Q. $\int \tan^5 x \sec^2 x\, dx$.

put $u = \tan x$.

$du/dx = \sec^2 x$.

$du = \sec^2 x\, dx$.

$\int u^5 du$.

$= u^6/6 + C$

$= 1/6 (\tan x)^6 + C$

$= 1/6 \tan^6 x + C$

Q. $\int (x+1)(x^2+2x+1)\, dx$.

put $u = x^2+2x+1$

$\frac{du}{dx} = 2x+2$.

$= 2(x+1)$

$\frac{du}{2} = (x+1)\, dx$

$\int u \cdot du/2$.

$1/2\ u^2/2 + C$

$= \frac{(x^2+2x+1)^2}{4} + C$

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Q. $\int \frac{x^2+1}{x^3+3x} dx$.

$\int \frac{1}{u} du/3$.

$u = x^3+3x$.

$\frac{du}{dx} = 3x^2+3$

$\frac{1}{3}\int \frac{du}{u}$.

$\frac{du}{dx} = 3(x^2+1)$

$\frac{du}{3} = (x^2+1)dx$

$= \frac{1}{3}\log u + c$

$= \frac{1}{3}\log(x^3+3x)+c$

Q. $\int \frac{(1-\sin x)dx}{(x+\cos x)^2}$.

put $u = x+\cos x$.

$\int \frac{du}{u}$.

$\frac{du}{dx} = 1-\sin x$.

$du = (1-\sin x)dx$.

$= \log u + c$

$= \log(x+\cos x)+c$

Q. $\int \frac{\cos x \, dx}{2+\sin x}$.

$2+\sin x = u$.

$\frac{du}{dx} = \cos x$.

$\int \frac{du}{u}$.

$du = \cos x \, dx$

$= \log u + c$

$= \log(2+\sin x)+c$

Q. $\int \frac{1}{x \log x} dx$.

$\int \frac{1}{\log x} \cdot \frac{1}{x} dx$

$u = \log x$.

$du/dx = 1/x$

$du = 1/x \, dx$.

$= \int \frac{1}{u} \cdot du$

$= \log u + c$

$= \log(\log x) + c$

Q. $\int x\sqrt{x^2+9} \, dx$.

$u = x^2 + 9$.

$\int \sqrt{u} \, du/2$.

$du/dx = 2x$

$du = 2x \, dx$

$\frac{1}{2}\int u^{1/2} du$.

$du/2 = x \, dx$

$= \frac{1}{2} \frac{u^{1/2+1}}{1/2+1} + c$

$= \frac{1}{2} \frac{u^{3/2}}{3/2} + c = \frac{\cancel{2} u^{3/2}}{\cancel{2} \times 3} + c$

$= \frac{(x^2+9)^{3/2}}{3} + c$

Q. $\int \frac{\sin(3 + 4\log x)}{x} dx$

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$\int \sin u \cdot du/4$ || put $u = 3 + 4 \log x$.
$du/dx = 4/x$.

$\frac{1}{4}\int \sin u \, du$ $\qquad \frac{du}{4} = \frac{1}{x} dx$.

$= -\frac{1}{4} \cos u + c$.

$= -\frac{1}{4} \cos(3 + 4 \log x) + c$

<u>Integration by parts</u>:

$\int uv \, dx = u \int v \, dx - \int \left[\frac{d}{dx}(u) \cdot \int v \, dx\right] dx$

$\int x \sin x \, dx$

$= x \int \sin x \, dx - \int \left[\frac{d}{dx}(x) \cdot \int \sin x \, dx\right] dx$.

$= x \cdot (-\cos x) - \int 1 \cdot (-\cos x) \, dx$

$= -x \cos x + \int \cos x \, dx$

$= -x \cos x + \sin x + c$

$\int x \cos x \, dx$.

$= x \cdot \int \cos x - \int \left[\frac{d}{dx}(x) \cdot \int \cos x \, dx\right] dx$

$= x(\sin x) - \int 1 \cdot \sin x \, dx$.

$= x \sin x + \cos x + c$

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Q. $\int x \sec^2 x \, dx$
$= x \cdot \int \sec^2 x \, dx - \int \left[\frac{d}{dx}(x) \cdot \int \sec^2 x \, dx\right] dx$

$= x \cdot \tan x - \int 1 \cdot \tan x \, dx$
$= x \tan x - \log(\sec x) + c$

Q. $\int x \log x \, dx$.
$= \int \log x \cdot x \, dx$
$= \log x \cdot \int x \, dx - \int \left[\frac{d}{dx}(\log x) \int x \, dx\right] dx$

$= \log x \cdot x^2/2 - \int \frac{1}{\cancel{x}} \cdot \frac{x^{\cancel{2}}}{2} \, dx$

$= \frac{x^2}{2} \log x - \int \frac{x}{2} \, dx$.

$= \frac{x^2}{2} \log x - \frac{1}{2} \int x \, dx$.

$= \frac{x^2}{2} \log x - \frac{x^2}{4} + c$

Q. $\int \log x \cdot dx$
$= \int \log x \cdot 1 \, dx$.
$= \log x \int 1 \, dx - \int \left[\frac{d}{dx}(\log x) \int dx\right] dx$

$= x \log x - \int \frac{1}{x} \cdot x \, dx$

$= x \log x - \int dx = x \log x - x + c$

Q. $\int x e^x dx$

$= x \cdot \int e^x dx - \int \left[\frac{d}{dx}(x) \cdot \int e^x dx\right] dx.$

$= x e^x - \int 1 \cdot e^x dx$

$= x e^x - e^x + c$

Q. $\int x \cdot \sin 2x \, dx$

$= x \cdot \int \sin 2x \, dx - \int \left[\frac{d}{dx}(x) \cdot \int \sin 2x \, dx\right] dx$

$= x \cdot \frac{-\cos 2x}{2} - \int 1 \cdot \frac{-\cos 2x}{2} dx$

$= -\frac{x \cos 2x}{2} + \frac{1}{2} \int \cos 2x \, dx.$

$= -\frac{x \cos 2x}{2} + \frac{1}{2} \frac{\sin 2x}{2} + c$

$= -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4} + c$