Differential Equation

An equation which involves derivative or differential coefficients is called a differential equation.

example

$$\frac{dy}{dx} = \frac{y}{x} + \sin y/x, \quad \frac{dy}{dx} + 3y = 0.$$

order of a differential equation.

The order of a differential equation is the order of the highest order derivative appearing in the equation.

(1) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

The order of the highest derivative is 2.

(2). $\frac{d^3y}{dx^3} - 6\left(\frac{dy}{dx}\right)^2 - 4y = 0.$

the order of the highest derivative is 3

Degree of a differential equation

The degree of a differential equation is the power of the highest order derivative occurring in a differential equation.

① $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$

The highest derivative order is 2.
And its power is one. ∴ Its degree is one.

2. $4\frac{d^3y}{dx^3} - 3\left(\frac{dy}{dx}\right)^2 - 3y = 0$

The highest derivative is 2.
And its power is one. ∴ Its degree is one.

Methods of Solution.

I. Direct ~~dir~~ Integration.

Q. Solve $\frac{dy}{dx} + 5y = 0$.

$$\frac{dy}{dx} = -5y.$$

$$\frac{dy}{y} = -5\,dx.$$

Integrating on both sides.

$$\int \frac{dy}{y} = \int -5\,dx.$$

$$\log y = -5x + c.$$

$$y = e^{-5x+c}.$$

$$y = e^{-5x}.e^{c}$$

$$= e^{-5x}.k.$$

$$\underline{y = ke^{-5x}}.$$

$\log y = x.$
$y = e^{x}$

2. Solve: $\frac{dy}{dx} = ky.$

$$\frac{dy}{y} = k\,dx.$$

Integrating on both sides.

$$\int \frac{dy}{y} = \int k\,dx$$

$$\log y = kx + c.$$

$$y = e^{kx}.e^{c}$$

$$= e^{kx}A$$

$$y = \underline{Ae^{kx}}.$$

Q. Solve $\frac{dQ}{dt} = \frac{5}{2} Q$.

$$\frac{dQ}{Q} = \frac{5}{2} dt$$

Integrating on both sides-

$$\int \frac{dQ}{Q} = \int 5/2 \, dt$$

$$\log Q = 5/2 \, t + C$$

$$Q = e^{5/2 t + C}$$

$$Q = e^{5/2 t} \cdot e^{C}$$

$$Q = k e^{5/2 t}$$

Method 2

Solution of equations which are variable separable.

$$\frac{dy}{dx} = \frac{f(x)}{f(y)}$$

$$f(y) dy = f(x) dx$$

Q. Sole $dy = e^{3x+y} dx$.

$$dy = e^{3x} \cdot e^{y} dx$$

$$\frac{dy}{e^{y}} = e^{3x} dx.$$

$$\int e^{-y} dy = \int e^{3x} dx$$

$$-e^{-y} = \frac{e^{3x}}{3} + C$$

Q. Solve $\frac{dy}{dx} = \frac{xy^2+x}{yx^2+y}$.

$$\frac{dy}{dx} = \frac{x(y^2+1)}{y(x^2+1)}$$

$$\frac{y\,dy}{y^2+1} = \frac{x\,dx}{x^2+1}$$

Integrating on both sides.

$$\int \frac{y\,dy}{y^2+1} = \int \frac{x\,dx}{x^2+1}$$

$$\frac{1}{2}\log(y^2+1) + C_1 = \frac{1}{2}\log(x^2+1) + C_2$$

$$\frac{1}{2}\log(y^2+1) - \frac{1}{2}\log(x^2+1) = C$$

$$\frac{1}{2}\left[\log \cancel{y^2+1} - \log(x^2+1)\right] = C$$

$$\log(y^2+1) - \log(x^2+1) = 2C$$

$$\log\left(\frac{y^2+1}{x^2+1}\right) = 2C$$

$$\frac{y^2+1}{x^2+1} = e^{2C} = K$$

$$y^2+1 = K(x^2+1)$$

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Q. Solve $\frac{dy}{dx} + \frac{x\sqrt{1+y^2}}{y\sqrt{1+x^2}} = 0.$

$$\frac{dy}{dx} = -\frac{x\sqrt{1+y^2}}{y\sqrt{1+x^2}}$$

$$\frac{y\,dy}{\sqrt{1+y^2}} = \frac{-x\,dx}{\sqrt{1+x^2}}$$

Integrating on both sides

$$\int \frac{y\,dy}{\sqrt{1+y^2}} = -\int \frac{x\,dx}{\sqrt{1+x^2}}$$

$$\int \frac{\frac{1}{2}du}{\sqrt{u}} = -\int \frac{\frac{1}{2}dv}{\sqrt{v}}$$

$1+y^2 = u$

$2y\,dy = du$

$y\,dy = \frac{1}{2}du$

$$\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\int v^{-1/2}\,dv$$

$$\frac{1}{2}\frac{u^{1/2}}{1/2} + c_1 = -\frac{1}{2} \times \frac{v^{1/2}}{1/2} + c_2$$

$$\sqrt{1+y^2} = -\sqrt{1+x^2} + c$$

$$\sqrt{1+y^2} + \sqrt{1+x^2} = c$$

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Method 3.

Equations of the form $\frac{dy}{dx} + Py = Q$ —(1)
where P and Q are functions of x
(Linear differential equations).

A differential equation is said to be linear if the dependent variable and its differential coefficient occur only in first degree.

To find its solution, first to find out the integrating factor $e^{\int P dx}$
multiply the equation (1) by $e^{\int P dx}$.
Integrating both sides, we get the result $y \cdot e^{\int P dx} = \int Q \cdot e^{\int P dx} dx$

Q. Solve $\frac{dy}{dx} + y \cot x = \operatorname{cosec} x$.

Comparing the linear differential equation we get $P = \cot x$, $Q = \operatorname{cosec} x$.

Integrating factor $e^{\int P dx}$

$$e^{\int \cot x \, dx} = e^{\log \sin x} = \sin x.$$

The soln is $y e^{\int P dx} = \int Q e^{\int P dx} dx$
$(e^{\log x} = x)$

$$y \sin x = \int \operatorname{cosec} x \cdot \sin x \, dx$$
$$= \int \frac{1}{\sin x} \sin x \, dx = \int dx$$

$y \sin x = x + c$ is the Solution

Q Solve $x \frac{dy}{dx} + 3y = 5x^2$.

Divide the equation by x.

$\therefore \frac{1}{x}\left(x \frac{dy}{dx} + 3y = 5x^2\right)$.

$$\frac{dy}{dx} + \frac{3}{x} y = 5x.$$

Comparing the linear differential equation, we get

$P = 3/x$, $Q = 5x$.

Integrating factor $e^{\int P dx}$.

$$e^{\int 3/x \, dx} = e^{3\int 1/x \, dx}$$
$$= e^{3 \cdot \log x}$$
$$= e^{\log x^3}$$
$$= x^3$$

$n \log x = \log x^n$

$e^{\log x} = x$

Its Solution

$$y e^{\int P dx} = \int Q \cdot e^{\int P dx} dx$$

$$y \cdot x^3 = \int 5x \cdot x^3 dx$$
$$y x^3 = 5 \int x^4 dx$$
$$= \cancel{5} \cdot x^5 / \cancel{5} + c$$
$$y x^3 = x^5 + c$$ is the Solution

Q. Solve $\frac{dy}{dx} + y\tan x = \cos^2 x$

Comparing the linear differential equation, we get $P = \tan x$, $Q = \cos^2 x$.

Integrating factor, $e^{\int P\,dx}$.

$$e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x.$$

Its solution

$$y e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\,dx$$

$$y \cdot \sec x = \int \cos^2 x \cdot \sec x\,dx$$

$$= \int \cos^2 x \cdot \frac{1}{\cos x}\,dx$$

$$= \int \cos x\,dx = \sin x + c$$

Extra question

Q. $\frac{dy}{dx} = \sin x$.

$$dy = \sin x\,dx.$$

$$\int dy = \int \sin x\,dx$$

$$y = -\cos x + c$$

2. $\frac{dy}{dx} = \sec^2 x.$

$dy = \sec^2 x \, dx.$

$\int dy = \int \sec^2 x \, dx.$

$y = \tan x + c$

3. $\frac{dy}{dx} = x^2.$

$dy = x^2 \, dx.$

$\int dy = \int x^2 \, dx.$

$y = x^3/3 + c$

4. $\frac{dy}{dx} = 3x + 4.$

$dy = (3x + 4) dx.$

$\int dy = \int (3x+4) dx$

$y = 3 \cdot \frac{x^2}{2} + 4x + c.$

$y = 3/2 \, x^2 + 4x + c$

End of this semester