Solution of two Simultaneous linear equation.

Consider two Simultaneous linear equations $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$.
We can solve these equations using determinant method. In this method we have to evaluate three determinants Δ, Δ_1 and Δ_2.

Δ is the determinant of coefficients of x and y.

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

Δ_1 is the determinant, by replacing the first column of Δ by the columns of constants $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

$$\Delta_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} = c_1b_2 - c_2b_1.$$

Δ_2 is the determinants by replacing the second column of Δ by the column of constants:

$$\Delta_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} = a_1c_2 - a_2c_1$$

After finding Δ, Δ_1 and Δ_2

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}.$$

Q. Solve $x-2y+1=0$, $3x+2y=3$ using determinant method.

Rearrange the linear equations is

$$x-2y=-1$$
$$3x+2y=3.$$

$$\Delta = \begin{vmatrix} 1 & -2 \\ 3 & 2 \end{vmatrix} = (1\times2)-(-2\times3)$$
$$= 2-(-6) = 2+6 = 8$$

$$\Delta_1 = \begin{vmatrix} -1 & -2 \\ 3 & 2 \end{vmatrix} = (-1\times2)-(3\times-2).$$
$$= (-2)-(-6) = -2+6$$
$$= 4.$$

$$\Delta_2 = \begin{vmatrix} 1 & -1 \\ 3 & 3 \end{vmatrix} = (1\times3)-(3\times-1)$$
$$= 3-(-3) = 3+3 = 6$$

$$x = \frac{\Delta_1}{\Delta} = \frac{4}{8} = \frac{1}{2}$$

$$y = \frac{\Delta_2}{\Delta} = \frac{6}{8} = 3/4$$

Q. Solve $5x+2y=4$
$2x-y=7$

$$\Delta = \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = (5\times-1)-(2\times2)$$
$$= -5-4 = -9.$$

$$\Delta_1 = \begin{vmatrix} 4 & 2 \\ 7 & -1 \end{vmatrix} = (4\times-1)-(2\times7) = -4-14 = -18$$

$\Delta_2 = \begin{vmatrix} 5 & 4 \\ 2 & 7 \end{vmatrix} = (5\times7) - (4\times2)$
$= 35 - 8$
$= 27$ //

$x = \frac{\Delta_1}{\Delta} = \frac{-18}{-9} = 2$ //

$y = \frac{\Delta_2}{\Delta} = \frac{27}{-9} = -3$ //

Q. Solve $\frac{6}{x} + \frac{7}{y} = 5$, $\frac{2}{x} + \frac{5}{y} = 3$.

put $\frac{1}{x} = X$, $\frac{1}{y} = Y$.

$\therefore$ The equations becomes -

$6X + 7Y = 5$, $2X + 5Y = 3$ -

$\Delta = \begin{vmatrix} 6 & 7 \\ 2 & 5 \end{vmatrix} = (6\times5) - (2\times7)$
$= 30 - 14 = 16$

$\Delta_1 = \begin{vmatrix} 5 & 7 \\ 3 & 5 \end{vmatrix} = (5\times5) - (7\times3)$
$= 25 - 21 = 4$ //

$\Delta_2 = \begin{vmatrix} 6 & 5 \\ 2 & 3 \end{vmatrix} = (6\times3) - (2\times5)$
$= 18 - 10 = 8$ //

$X = \frac{\Delta_1}{\Delta} = \frac{4}{16} = \frac{1}{4}$, $x = \frac{1}{X} = \frac{1}{1/4} = 1\times4$ //
$= 4$

$Y = \frac{\Delta_2}{\Delta} = \frac{8}{16} = \frac{1}{2}$.

$y = \frac{1}{Y} = 1/(1/2) = 1 \times 2/1 = 2$.

Determinant of order three:

The symbol $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ consisting of 9 elements, which are arranged in three rows and three columns is a determinant of third order. Its value is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

(2) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

(1) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

(3) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Q. $\begin{vmatrix} 3 & 2 & -1 \\ 4 & 7 & 5 \\ -3 & 1 & 2 \end{vmatrix} = 3\begin{vmatrix} 7 & 5 \\ 1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 5 \\ -3 & 2 \end{vmatrix}$

$+ -1 \begin{vmatrix} 4 & 7 \\ -3 & 1 \end{vmatrix}$

$= 3((7\times2)-(1\times5)) - 2((4\times2)-(-3\times5))$

$-1((4\times1)-(-3\times7))$

$= 3[14-5] - 2[8--15] - 1[4--21]$

$= 3\times9 - 2[8+15] - 1[4+21]$

$= 27 - 2\times23 - 1\times25$

$= 27 - 46 - 25$

$= -44$

27
7
27

Q. $\begin{vmatrix} 3 & -2 & 2 \\ 1 & 4 & 5 \\ 6 & -1 & 2 \end{vmatrix} = 3\begin{vmatrix} 4 & 5 \\ -1 & 2 \end{vmatrix} - -2\begin{vmatrix} 1 & 5 \\ 6 & 2 \end{vmatrix} + 2\begin{vmatrix} 1 & 4 \\ 6 & -1 \end{vmatrix}$.

$= 3[(4\times2)-(-1\times5)] + 2[(1\times2)-(6\times5)]$

$+ 2[(1\times-1)-(6\times4)]$

$= 3[8--5] + 2[2-30] + 2[-1-24]$

$= 3[8+5] + 2[-28] + 2[-25]$

$= 3\times13 + 2\times-28 + 2\times-25$

$= 39 - 56 - 50$

$= 39 - 106$

$= -67$

Q Solve for x, if $\begin{vmatrix} 2 & 3 & 5 \\ 2 & x & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$.

$$\begin{vmatrix} 2 & 3 & 5 \\ 2 & x & 5 \\ 3 & -1 & 2 \end{vmatrix} = 0$$

$$2\begin{vmatrix} x & 5 \\ -1 & 2 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} + 5\begin{vmatrix} 2 & x \\ 3 & -1 \end{vmatrix} = 0$$

$$2[(2\times x) - (-1\times 5)] - 3[(2\times 2) - (3\times 5)] + 5[(2\times -1) - (3\times x)] = 0$$

$$2[2x - -5] - 3[4 - 15] + 5[-2 - 3x] = 0$$

$$2[2x + 5] - 3\times -11 + 5[-2 - 3x] = 0$$

$$4x + 10 + 33 + 5\times -2 - 5\times 3x = 0$$

$$4x + 10 + 33 - 10 - 15x = 0$$

$$-11x + 33 = 0$$

$$-11x = -33$$

$$x = \frac{-33}{-11} = 3$$

Q. If $\begin{vmatrix} 2 & 1 & x \\ 3 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix} = \begin{vmatrix} 4 & x \\ 3 & 2 \end{vmatrix}$ Find x.

$$\begin{vmatrix} 2 & 1 & x \\ 3 & -1 & 2 \\ 1 & 1 & 6 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ 1 & 6 \end{vmatrix} - 1\begin{vmatrix} 3 & 2 \\ 1 & 6 \end{vmatrix} + x\begin{vmatrix} 3 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2[(-1\times 6)-(1\times 2)] - 1[(3\times 6)-(1\times 2)] + x[(3\times 1)-(1\times -1)]$$

$$= \cancel{2[-6-2]}$$

$$= 2[-6-2] - 1[18-2] + x[3+1]$$

$$= 2\times -8 - 1\times 16 + 4x$$

$$= -16-16+4x = 4x-32$$

$$\begin{vmatrix} 4 & x \\ 3 & 2 \end{vmatrix} = 4\times 2 - 3\times x = 8-3x.$$

$$4x-32 = 8-3x$$

$$\therefore 4x+3x = 8+32 = 40$$

$$7x = 40$$

$$x = 40/7$$

Solution of a system of three linear equations in three unknowns.

Q Solve the following system of equation by cramer's rule $x+2y-z=-3$, $3x+y+z=4$ and $x-y+2z=6$

$$\Delta = \begin{vmatrix} 1 & 2 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 2 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 3 & 1 \\ 1 & -1 \end{vmatrix}$$

$$= 1(2 - -1) - 2(6-1) - 1(-3-1)$$

$$= 1(2+1) - 2(5) - 1(-4)$$

$$= 3 - 10 + 4 = -3$$

$$\Delta_1 = \begin{vmatrix} -3 & 2 & -1 \\ 4 & 1 & 1 \\ 6 & -1 & 2 \end{vmatrix} = -3\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 1 \\ 6 & 2 \end{vmatrix} - 1\begin{vmatrix} 4 & 1 \\ 6 & -1 \end{vmatrix}$$

$$= -3(2+1) - 2(8-6) - 1(-4-6)$$

$$= -3\times 3 - 2\times 2 - 1\times -10$$

$$= -9 - 4 + 10 = -3$$

$$\Delta_2 = \begin{vmatrix} 1 & -3 & -1 \\ 3 & 4 & 1 \\ 1 & 6 & 2 \end{vmatrix} = 1(8-6) + 3(6-1) - 1(18-$$

$$= 1\times 2 + 3\times 5 - 14$$

$$= 2 + 15 - 14 = 3$$

$$\Delta_3 = \begin{vmatrix} 1 & 2 & -3 \\ 3 & 1 & 4 \\ 1 & -1 & 6 \end{vmatrix} = 1(6+4) - 2(18-4) - 3(-3-1)$$

$$= 10 - 2\times 14 - 3\times -4$$

$$= 10 - 28 + 12 = -6$$

$$x = \frac{\Delta_1}{\Delta} = -3/-3 = 1$$

$$y = \frac{\Delta_2}{\Delta} = 3/-3 = -1$$

$$z = \frac{\Delta_3}{\Delta} = -6/-3 = 2$$

Q. Solve $2a-3b+c=-1$, $a+4b-2c=3$, $4a-b+3c=11$

$$\Delta = \begin{vmatrix} 2 & -3 & 1 \\ 1 & 4 & -2 \\ 4 & -1 & 3 \end{vmatrix} = 2(12-2)+3(3+8)+1(-1-16)$$

$$= 2\times10+3\times11+1\times-17$$
$$= 20+33-17$$
$$= 53-17 = 36$$

$$\Delta_1 = \begin{vmatrix} -1 & -3 & 1 \\ 3 & 4 & -2 \\ 11 & -1 & 3 \end{vmatrix} = -1(12-2)+3(9+22)+1(-3-44)$$

$$= -1\times10+3\times31+1\times-47$$

$$= -10+93-47 = -57+93$$
$$= 36$$

$$\Delta_2 = \begin{vmatrix} 2 & -1 & 1 \\ 1 & 3 & -2 \\ 4 & 11 & 3 \end{vmatrix} = 2(9+22)+1(3+8)+1(11-12)$$

$$= 2\times31+11-1$$
$$= 62+10 = 72$$

$$\Delta_3 = \begin{vmatrix} 2 & -3 & -1 \\ 1 & 4 & 3 \\ 4 & -1 & 11 \end{vmatrix} = 2(44+3) + 3(11-12) - 1(-1-16)$$

$$= 2 \times 47 + 3 \times -1 + -1 \times -17$$
$$= 94 - 3 + 17$$
$$= 111 - 3 = 108$$

$$\therefore a = \frac{\Delta_1}{\Delta} = \frac{36}{36} = 1$$

$$b = \frac{\Delta_2}{\Delta} = \frac{72}{36} = 2$$

$$c = \frac{\Delta_3}{\Delta} = \frac{108}{36} = 3$$

$\therefore a = 1, b = 2$ and $c = 3$

Minors and Cofactors -

Minor of an element in a matrix is the determinant obtained by deleting the row and column in which that element exists.

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

(1) Minor of $a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

(2) minor of $b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$.

$c_1 = \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$ and so on.

Cofactor of an element in a matrix is the minor with proper sign. Sign of an element in i^{th} row and j^{th} column is $(-1)^{i+j}$

$$\text{minor of } a_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

$$\text{Cofactor of } \underset{(1,1)}{a_1} = (-1)^{1+1} \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = (-1)^2 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$$

1^{st} row ← (1,1) → 1^{st} column

$$= \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} = b_2c_3 - b_3c_2.$$

$$\text{minor of } b_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$\text{Cofactor of } \underset{(1,2)}{b_1} = (-1)^{1+2} \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

1^{st} row ← (1,2) → 2^{nd} column

$$= (-1)^3 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$$

$$= -1\left(a_2c_3 - a_3c_2\right)$$

$$= \underline{\underline{-a_2c_3 + a_3c_2}}$$

and so on.

Q - Minor of 2 in $\begin{bmatrix} 2 & -1 \\ 8 & 5 \end{bmatrix}$ is 5

$$\text{Cofactor of } \underset{(1,1)}{2} = (-1)^{1+1}(5) = (-1)^2 \times 5$$

$$= 1 \times 5 = 5 //$$

Q2. Minor of 3 in $\begin{bmatrix} 1 & 3 & -2 \\ 0 & 0 & 5 \\ 1 & 1 & -3 \end{bmatrix}$ is $\begin{vmatrix} 0 & 5 \\ 1 & -3 \end{vmatrix}$

$$= (0 \times -3) - (1 \times 5)$$

$$= 0 - 5 = -5 //$$

Cofactor of $\underset{(1,2)}{3}$ $= (-1)^{1+2}(-5)$

$= (-1)^3(-5) = (-1)\times(-5)$

$= 5$

Q3 minor of 4 in $\begin{bmatrix} 0 & 1 & 8 \\ 1 & 4 & 3 \\ 1 & 2 & 2 \end{bmatrix}$ is $\begin{vmatrix} 0 & 8 \\ 1 & 2 \end{vmatrix}$

$= (0\times 2) - (1\times 8)$

$= 0 - 8 = -8$

Cofactor of $\underset{(2,2)}{4}$ $= (-1)^{2+2}(-8)$

$= (-1)^4(-8) = 1\times -8$

$= -8$

Q4 minor of 1 in $\begin{bmatrix} 2 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 4 & 0 \end{bmatrix}$ is $\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = 12 - 15$

$= -3$

Cofactor of $\underset{(3,1)}{1}$ $= (-1)^{3+1}(-3)$

$= (-1)^4(-3) = -3$

Cofactor matrix

The matrix obtained by taking the cofactors of each element in a matrix is known as cofactor matrix.

$A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$

Cofactor of $2 = C_{11}$

$= (-1)^2(4) = 4$

Cofactor of $3 = C_{12} = (-1)^3(-1) = (-1)\times(-1) = \underline{1}$

Cofactor of $(-1) = C_{21} = (-1)^3(3) = -3$

Cofactor of $4 = C_{22} = (-1)^4(2) = 1\times 2 = 2$