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Multiple Angles.

$2\theta, 3\theta, 4\theta$ etc are multiple angles.

① $\sin 2\theta = \sin(\theta+\theta)$

$= \sin\theta\cos\theta + \cos\theta\sin\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$.

② $\cos 2\theta = \cos(\theta+\theta)$

$= \cos\theta\cos\theta - \sin\theta\sin\theta$.

$\cos 2\theta = \cos^2\theta - \sin^2\theta \rightarrow (1)$

③ $\tan 2\theta = \tan(\theta+\theta)$

$= \dfrac{\tan\theta + \tan\theta}{1-\tan\theta\tan\theta}$.

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

④ $\sin 2\theta$ and $\cos 2\theta$ in terms of $\tan\theta$.

$\sin 2\theta = 2\sin\theta\cos\theta$.

$= \dfrac{2\sin\theta\cos\theta}{1}$

$\sin 2\theta = \dfrac{2\sin\theta\cos\theta}{\sin^2\theta+\cos^2\theta}$.

Dividing numerator and denominator by $\cos^2\alpha$

(3) $\therefore \sin 2\alpha = \dfrac{\dfrac{2\sin\alpha\cos\alpha}{\cos^2\alpha}}{\dfrac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}}$.

$$= \frac{2\tan\alpha}{\frac{\sin^2\alpha}{\cos^2\alpha}+\frac{\cos^2\alpha}{\cos^2\alpha}} = \frac{2\tan\alpha}{\tan^2\alpha+1}$$

(4) $\therefore \sin 2\alpha = \dfrac{2\tan\alpha}{1+\tan^2\alpha}$.

$\therefore \cos 2\alpha = \cos^2\alpha - \sin^2\alpha$.

$$= \frac{\cos^2\alpha - \sin^2\alpha}{1}$$

$\therefore \cos 2\alpha \therefore = \dfrac{\cos^2\alpha - \sin^2\alpha}{\sin^2\alpha+\cos^2\alpha}$.

Dividing numerator and denominator by $\cos^2\alpha$.

$$\cos 2\alpha = \frac{\frac{\cos^2\alpha-\sin^2\alpha}{\cos^2\alpha}}{\frac{\sin^2\alpha+\cos^2\alpha}{\cos^2\alpha}}.$$

$$= \frac{\frac{\cos^2\alpha}{\cos^2\alpha} - \frac{\sin^2\alpha}{\cos^2\alpha}}{\frac{\sin^2\alpha}{\cos^2\alpha} + \frac{\cos^2\alpha}{\cos^2\alpha}}.$$

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(5) $\cos 2\alpha = \dfrac{1-\tan^2\alpha}{1+\tan^2\alpha} \rightarrow (2)$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = 1 - \sin^2\alpha - \sin^2\alpha$

(5A) $\cos 2\alpha = 1 - 2\sin^2\alpha \rightarrow (3)$

or

$1 - 2\sin^2\alpha = \cos 2\alpha$

$-2\sin^2\alpha = \cos 2\alpha - 1$

$2\sin^2\alpha = 1 - \cos 2\alpha$

(6) $\therefore \sin^2\alpha = \dfrac{1-\cos 2\alpha}{2}$

$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$

$= \cos^2\alpha - (1 - \cos^2\alpha)$

$= \cos^2\alpha - 1 + \cos^2\alpha$

(6A) $\cos 2\alpha = 2\cos^2\alpha - 1 \rightarrow (4)$

or $2\cos^2\alpha - 1 = \cos 2\alpha$

$2\cos^2\alpha = \cos 2\alpha + 1$

(7) $\therefore \cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$

$(6) \div (7)$

$\dfrac{\sin^2\alpha}{\cos^2\alpha} = \dfrac{(1-\cos 2\alpha)/2}{(1+\cos 2\alpha)/2}$

(8) $\tan^2\alpha = \dfrac{1-\cos 2\alpha}{1+\cos 2\alpha}$

1. If $\tan\alpha = 2$ find $\sin 2\alpha$, $\cos 2\alpha$ and $\tan 2\alpha$.

(1) $\sin 2\alpha = \frac{2\tan\alpha}{1+\tan^2\alpha}$, $\tan\alpha = 2$.

$= \frac{2\times 2}{1+4} = \frac{4}{5}$

(2) $\cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha}$.

$= \frac{1-4}{1+4} = -3/5$

(3) $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} = \frac{2\times 2}{1-4}$

$= \frac{4}{-3} = \cancel{8}\ -4/3$

or

$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{4/5}{-3/5}$

$= \frac{4}{5} \times \frac{-5}{3} = -4/3$

2. If $\tan A = 3/4$ (A is acute) and $\sin B = 5/13$ (B lies on 2^{nd} quadrant). Find $\sin 2A$ and $\sin 2B$.

① $\sin 2A = \dfrac{2\tan A}{1+\tan^2 A}$

Given $\tan A = 3/4$.

$\sin 2A = \dfrac{\cancel{2}\times 3/\cancel{4}_2}{1+9/16}$

$= \dfrac{3/2}{(16+9)/16}$.

$= \left(\frac{3}{2}\right)/(25/16)$

$= 3/\cancel{2} \times \dfrac{\cancel{16}^8}{25} = \dfrac{24}{25}$.

② $\sin 2B = 2\sin B\cos B$

Given $\sin B = 5/13$.

B lies on 2^{nd} quadrant

$\cos^2 B = 1-\sin^2 B$

$= 1-25/169$

$= \dfrac{169-25}{169} = \dfrac{144}{169}$.

$\cos B = \pm 12/13$

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B lies on 2^{nd} quadrant

$\cos B = -12/13$.

$\therefore \sin 2B = 2\sin B\cos B$

$= 2 \times \frac{5}{13} \times \frac{-12}{13}$.

$= \frac{-120}{169}$

3. Prove that $\frac{1+\cos 2A}{\sin 2A} = \cot A$ and deduce the value of $\cot 15^\circ$.

$L.H.S = \frac{1+\cos 2A}{\sin 2A}$.

$\cos^2 A = \frac{1+\cos 2A}{2}$

$\therefore 1+\cos 2A = 2\cos^2 A$.

$\sin 2A = 2\sin A\cos A$.

$L.H.S = \frac{2\cos^2 A}{2\sin A\cos A} = \cot A$

$= R.H.S$.

$\cot A = \frac{1+\cos 2A}{\sin 2A}$

$A = 15^\circ$.

$$\cot 15^\circ = \frac{1+\cos 2\times 15^\circ}{\sin 2\times 15^\circ}$$

$$= \frac{1+\cos 30^\circ}{\sin 30^\circ}$$

$$= \frac{1+\sqrt{3}/2}{1/2}$$

$$= \frac{(2+\sqrt{3})/2}{1/2} = \frac{2+\sqrt{3}}{2} \times \frac{2}{1}$$

$$= \underline{\underline{2+\sqrt{3}}}.$$

4. Prove that $\cos 4a = 1 - 8 \sin^2 a \cos^2 a$.

$\cos 2a = 1 - 2\sin^2 a$.

$a = 2a$.

$\cos 4a = 1 - 2\sin^2 2a$.

$= 1 - 2(\cancel{2}\sin 2a)^2$ $\qquad \sin^2 a$

$= 1 - 2(2\sin a \cos a)^2$. $\qquad = (\sin a)^2$.

$= 1 - 2(4\sin^2 a \cos^2 a)$

$= 1 - 8\sin^2 a \cos^2 a$

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Expressions of Sin3A, Cos3A

$\sin 3A = \sin(2A + A)$

$= \sin 2A \cos A + \cos 2A \sin A.$

$= 2\sin A \cos A \cdot \cos A + (1 - 2\sin^2 A)\sin A$

$= 2\sin A \cos^2 A + \sin A - 2\sin^3 A$

$= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A$

$= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A$

$\sin 3A := 3\sin A - 4\sin^3 A.$

$\cos 3A = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin A \cos A \cdot \sin A$

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2(\cos A - \cos^3 A)$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$\cos 3A = 4\cos^3 A - 3\cos A.$

5. Prove that $\dfrac{\cos 3A + \cos A}{\sin 3A - \sin A} = \cot A$

$L.H.S = \dfrac{\cos 3A + \cos A}{\sin 3A - \sin A}.$

$$= \frac{4\cos^3 A - 3\cos A + \cos A}{3\sin A - 4\sin^3 A - \sin A}$$

$$= \frac{4\cos^3 A - 2\cos A}{2\sin A - 4\sin^3 A}$$

$$= \frac{\cancel{2}\cos A(2\cos^2 A - 1)}{\cancel{2}\sin A(1 - 2\sin^2 A)}$$

$$= \frac{\cos A \cdot \cos 2A}{\sin A \cos 2A} = \frac{\cos A}{\sin A}$$

$$= \cot A = R.H.S$$

8. Prove that $\frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x} = 2.$

$$L.H.S = \frac{\sin 3x}{\sin x} - \frac{\cos 3x}{\cos x}.$$

$$= \frac{3\sin x - 4\sin^3 x}{\sin x} - \frac{4\cos^3 x - 3\cos x}{\cos x}$$

$$= \frac{\sin x(3 - 4\sin^2 x)}{\sin x} - \frac{\cos x(4\cos^2 x - 3)}{\cos x}$$

$$= 3 - 4\sin^2 x - (4\cos^2 x - 3)$$

$$= 3 - 4\sin^2 x - 4\cos^2 x + 3$$

$$= 6 - 4(\sin^2 x + \cos^2 x)$$

$$= 6 - 4 \times 1 = 6 - 4$$

$$= 2$$

Workout problems.

1. If $\cos\alpha = \frac{1}{2}$, α is acute, find $\sin 2\alpha$, $\cos 2\alpha$, $\sin 3\alpha$ and $\cos 3\alpha$.
2. If $\tan B = 1$, B lies in 3^{rd} quadrant, find $\sin 2B$, $\cos 2B$ and $\tan 2B$.
3. If $\tan\alpha = 2$, find $\cos 2\alpha$.
4. Prove that $\frac{\sin 2A}{1+\cos 2A} = \tan A$ and deduce the value of $\tan 15^\circ$.
5. If $\cos 2\alpha = -3/5$ find $\tan\alpha$.
6. Prove that $\frac{\sin 3A}{\sin A} + \frac{\cos 3A}{\cos A} = 4\cos 2A$.
7. Prove that $\cos^4 A - \sin^4 A = \cos 2A$