Rotational dynamics continues…

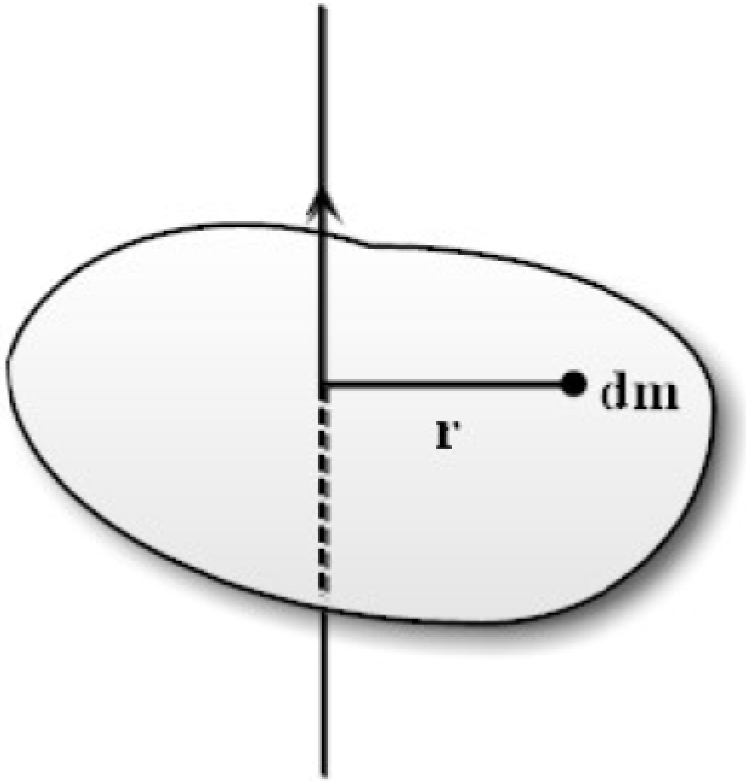
Moment of inertia of continuous mass distributions

If the body is continuous, its moment of inertia about a given axis can be obtained using the technique of integration. Consider a small element of the body of mass dm at perpendicular distance r from the axis of rotation. The moment of inertia of the element about the given axis is

$$dI = r^2 dm$$

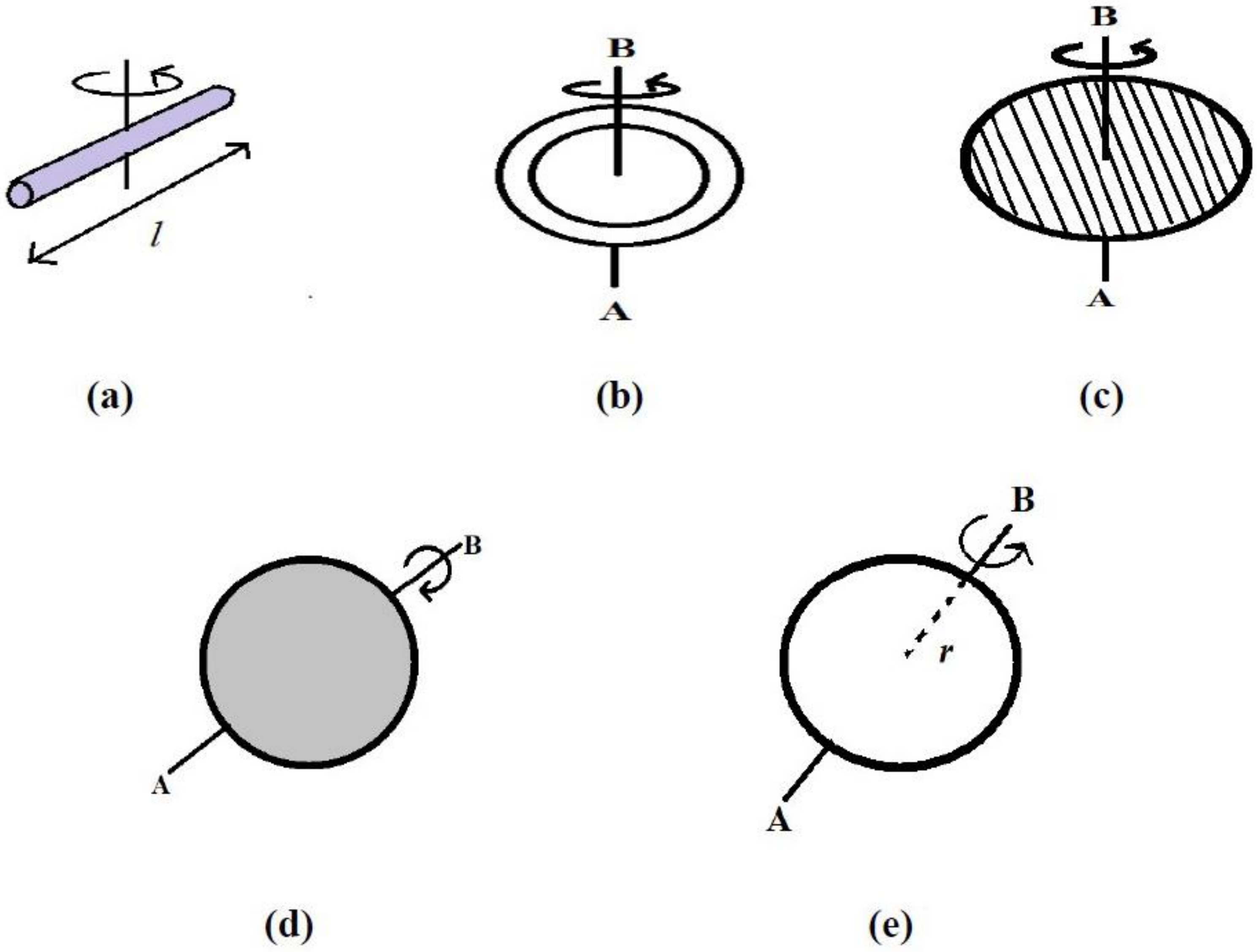
The moment of inertia of the rigid body about the given axis is obtained by integrating the above equation over appropriate limits to cover the whole body. Thus,

$$I = \int r^2 \, dm$$



Moment of inertia of a continuous body

Moment of inertia of (a) a thin road (b) a ring (c) a circular disc (d) solid sphere and (e) hollow sphere about the given axis



a) **Moment of inertia of a thin uniform rod, about an axis through its centre and perpendicular to its length.**

Consider a thin uniform rod of length l and mass M as shown in Fig. (a).Its moment of inertia about an axis through its centre and perpendicular to its length is given by

$$I = \frac{1}{12} Ml^2$$

b) **Moment of inertia of a ring, about an axis through the centre and perpendicular to its plane.**

Consider a circular ring of radius R and mass M as shown in Fig.(b). Its moment of inertia about an axis through the centre and perpendicular to its plane is given by

$$I = MR^2$$

c) MI of a uniform circular disc

c) Consider a uniform circular disc of radius R and mass M as shown in Fig. (c).Its moment of inertia about an axis through the centre and perpendicular to its plane is given by

$$I = \frac{1}{2} MR^2$$

d) Moment of inertia of a solid sphere, about any diameter.

Consider a solid sphere of radius R and mass M as shown in Fig. (d). Its moment of inertia about any diameter is given by

$$I = \frac{2}{5} MR^2$$

e) Moment of inertia of a hollow sphere, about any diameter.

Consider a hollow sphere of radius R and mass M. Its moment of inertia about any diameter is given by

$$I = \frac{2}{3} MR^2$$

Moment of Inertia(MI) of a circular Ring

A) about an axis passing through the center and perpendicular to its plane

Let M be the mass and R be the radius of the ring. Consider an infinitesimally small element of mass dm of the ring.

M.I of the element = R^2dm

Then MI of the ring about an axis perpendicular to the plane of the ring

and passing through the center of mass , I = $\int R^2 dm$

I = MR^2

B) About a diameter

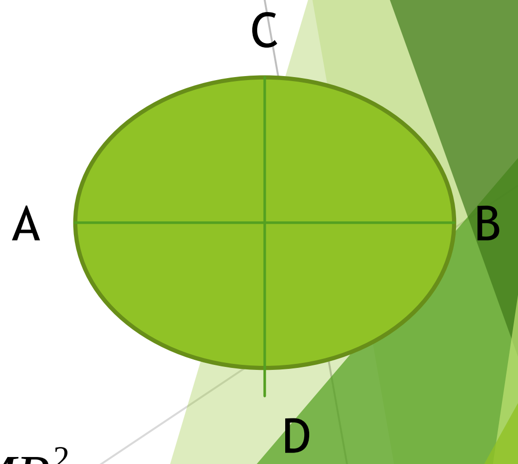
Let $$I_x = I_y = I_d$$

Using perpendicular axis theorem,

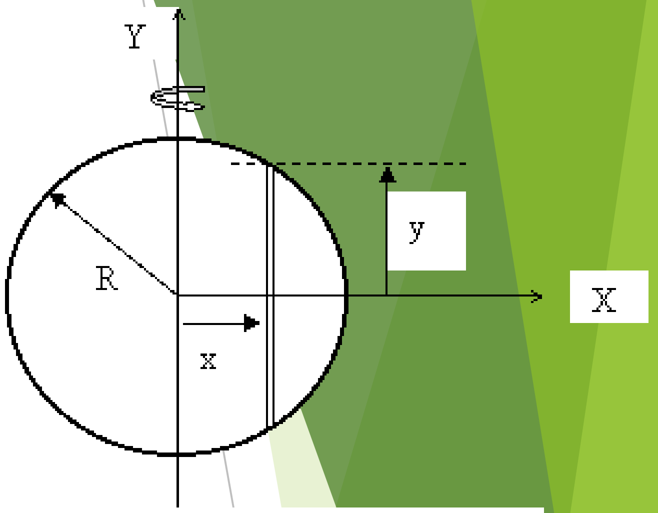
$$I_x + I_y = MR^2$$

Or,

$$2I_d = MR^2 \quad so \quad I_d = \frac{MR^2}{2}$$



MI of a uniform circular disc



A disc rotating about its diameter

Figure 5

- Consider a circular disc of mass M and radius R.
- The area of the disc is πR^2.
- Mass per unit area of
- the disc = $M/\pi R^2$
- Now, consider a small ring of breadth dx at a distance x from the centre.
- The area of the small ring= $2\pi x dx$
- The mass of the small ring, dm =

$$\frac{2\pi x dx M}{\pi R^2} = \frac{2M\,xdx}{R^2}$$

- Moment of inertia of this ring $=x^2dm$
- Assuming that the whole disc is made up of a number of small rings whose radii *x* varies from 0 to *R*, the MI of the whole disc can be obtained by integrating the expression within the limits *x*=0 to *x*=*R*.
- *i.e*

$$I=\int_0^R\left[\frac{2M}{R^2}\right]x^3dx$$

$$=\frac{2M}{R^2}\left[\frac{x^4}{4}\right]_0^R=\frac{2M}{R^2}\frac{R^4}{4}$$

- *Or*

$$I=\frac{MR^2}{2}$$

Rotational Dynamics

- A door opens slowly if we push too close to its hinges.

  The more massive the door, the more slowly it opens.

  - The more the force is applied from the pivot, the greater the angular acceleration;

    that angular acceleration is inversely proportional to mass.

These relationships are very similar to the familiar relationships among force, mass, and acceleration embodied in Newton’s second law of motion.

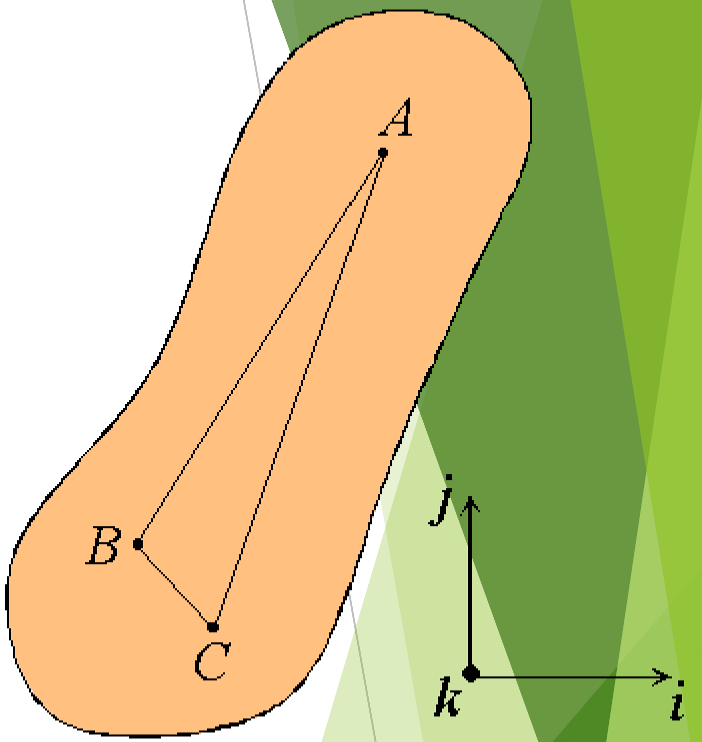
There are precise rotational analogs to both force and mass.

RIGIDBODY

- A solid **body** in which deformation is zero or so small it can be neglected is called a rigid body.

- The distance between any two given points on a **rigid body** remains constant in time regardless of external forces or moments exerted on it.

- Or, particles of the body should always remain in fixed positions relative to one another.



RIGID BODY MOTION

Linear Motion : change in motion only in linear direction
Rotation : the body rotates about an axis'

General Plane motion: consists of both linear and rotational motion- Eg: rolling wheel

Rigid body introduces the concept of Center of Mass

Centre of mass is the point at which all the mass of a body is considered to be concentrated.

Eg: A uniform sphere can be thought of as having all its mass concentrated at its center.

However an unsymmetric object such as a hammer has more mass towards one end and therefore the center of mass is towards the head of the hammer.

Center of Gravity is the average location of the weight distribution of a body. It is the point where all the weight of a body can be considered to be concentrated

Rotational dynamics

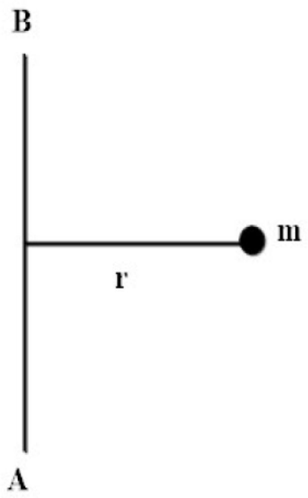
- Study of objects that are rotating or moving in a curved path
- It involves such quantities as torque, moment of inertia/**rotational** inertia, angular displacement (in radians or less often, degrees), angular velocity (radians per unit time), angular acceleration (radians per unit of time squared)
- **Characteristics of rigid body motion:**

All lines on a rigid body have the same angular velocity and the same angular acceleration .

Rigid motion can be decomposed into the translation of an arbitrary **point**, followed by a rotation about the **point**.

Moment of Inertia (Rotational Inertia) of a Particle

- Moment of inertia or angular mass or rotational inertia can be defined w.r.t. rotation axis.
- Consider a particle of mass m capable of rotation about an axis AB. Let r be the perpendicular distance of the particle from AB. The moment of inertia about the axis AB, $I = mr^2$



The SI unit of moment of inertia is kgm^2.

- The moment of inertia of a body depends on
  1. Mass of the body.
  2. The distribution of mass with respect to the axis of rotation
- The moment of inertia of a particle about a given axis is defined as the product of the mass of the particle and the square of the distance of the body from the axis.

Moment of Inertia (Rotational Inertia) of a Rigid Body

Consider a rigid body capable of rotation about an axis AB. Let us consider particles of masses m1, m2, m3,etc. of the body at distances r1, r2, r3,etc. respectively from the axis AB.

Moment of inertia of m_1 about AB = $m_1r_1^2$

Moment of inertia of m_2 about AB = $m_2r_2^2$

Moment of inertia of m_3 about AB = $m_3r_3^2$

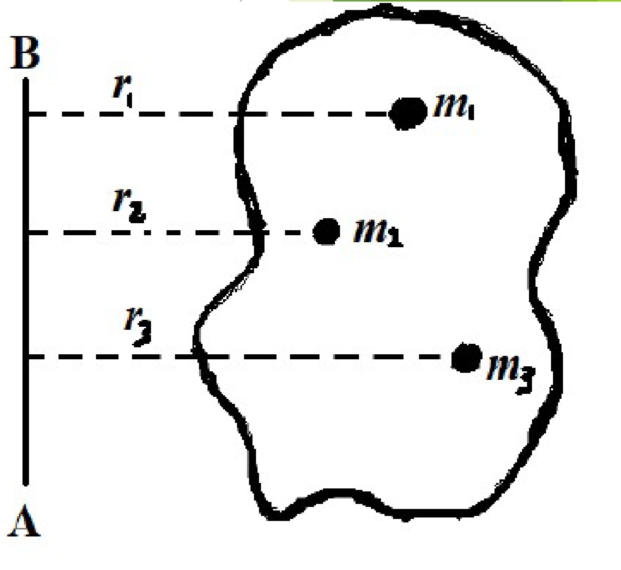
....

Moment of inertia of m_n about AB = $m_nr_n^2$

Therefore, the total moment of inertia of the body about the axis of rotation AB,

$$I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots\ldots m_nr_n^2$$

$$I = \sum_{i=1}^{n} m_1 r_1^2$$



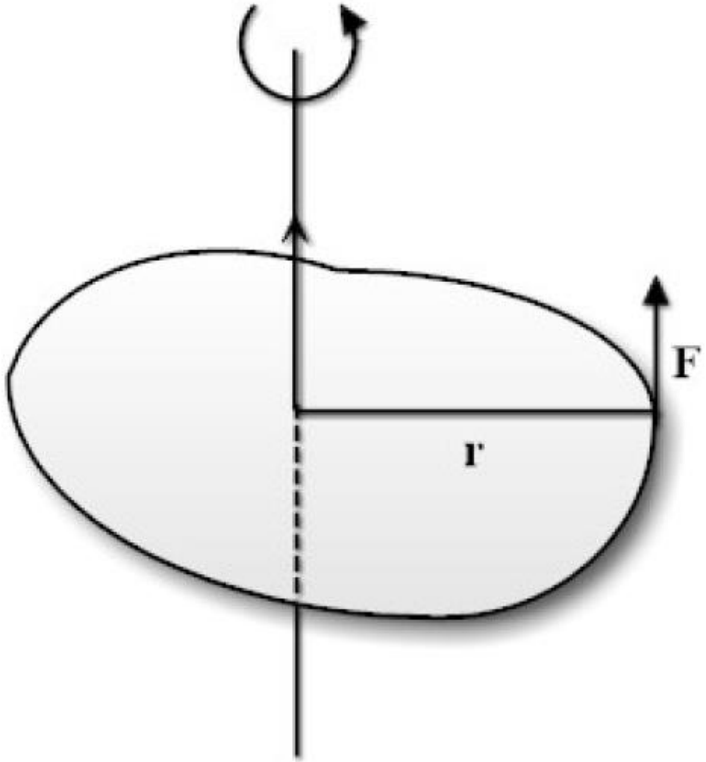
The moment of inertia is the sum of the Moment of inertia of the individual particles of the body about the axis of rotation.

Moment of a force or Torque

- In the case of linear motion, force is required to produce linear acceleration to a body. Similarly in rotational motion torque is required to produce angular acceleration.
- Force tends to change the motion of things. Torque tends to twist or change the state of rotation of things.
- Force should be applied to move a stationary object while torque is applied to rotate a stationary object. Hence, torque is the rotating effect of the force on a body.
- The rotating or turning effect produced by a force is called moment of a force. It is also called Torque or rotating force.

Torque is defined as the product of the force and the perpendicular distance between the line of action of the force and the axis of rotation.

- Consider a rigid body that is free to rotate about an axis. A force is applied to the rigid body at a perpendicular distance r from the axis of rotation as shown in the figure. The rotating effect of force on the rigid body about the axis of rotation depends on the magnitude of the force applied and the perpendicular distance of the point of application of the force from the axis of rotation.



$$\tau = Fr$$

The SI unit of torque is newton meter (Nm)

Torque also depends on the angle between the force and the line joining the point of application of the force and the axis of rotation. If the angle between r and F is θ, the perpendicular distance between the line of action of the force and axis of rotation becomes $r\sin\theta$.

Therefore, $\tau = Fr\sin\theta$

The torque is maximum when $\theta = 90^\circ$ or $sin\theta = 1$ and $\tau_{max} = Fr$.

Torque is minimum when $\theta = 0^\circ$ or $sin\theta = 0$ and $\tau_{min} = 0$.

Torque is the rotational equivalent of force. We can relate the torque on a rigid body to the angular acceleration.

$$\tau = Fr$$

From Newton's second law of motion,

$$F = ma$$

$$\therefore \quad \tau = mar$$

The relation between linear acceleration and angular acceleration is given by

$$a = r\alpha$$

$$\therefore \quad \tau = m\, r\alpha r$$

$$\tau = mr^2\alpha$$

Since the moment of inertia, $I = mr^2$

$$\tau = I\alpha$$

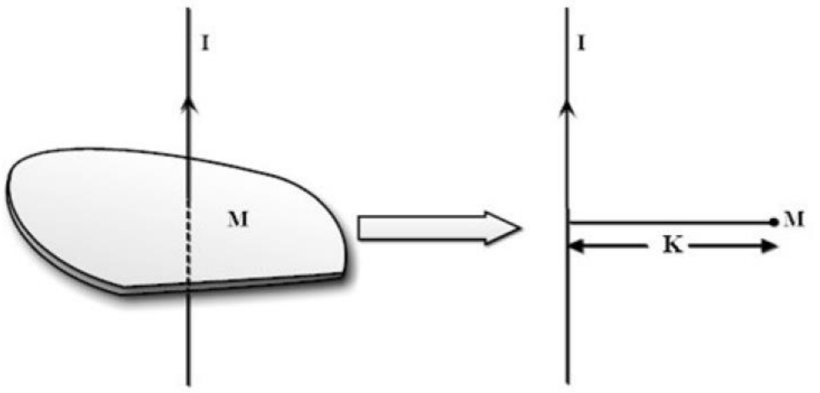
Comparison between linear motion and rotational motion

Linear motion		Rotational motion	
Linear displacement(s)		Angular displacement(θ)	
Linear velocity(v)		Angular velocity (ω)	
Linear acceleration (a)		Angular acceleration(α)	
Mass(m)		Moment of inertia(I)	$[I = mr^2]$
Linear momentum(p)	$[p = mv]$	Angular momentum(L)	$[L = I\omega]$
Force(F)	$[F = ma]$	Torque(τ)	$[\tau = I\alpha]$

Radius of Gyration

- Radius of gyration of a body about an axis of rotation is defined as radial distance of a point ,from the axis of rotation at which, if whole mass of any body is assumed to be concentrated, its moment of inertia about the given axis would be same as with its actual distribution of mass.
- Or , it is the effective distance of the particles from its axis of rotation and is denoted by ‘k’
- If the whole mass M of a body is supposed to be concentrated at a point of distance k from the axis such that Mk^2 has the same value as the MI about that axis, then k is called radius of gyration.
- Ie.,

$$Mk^2 = \sum mr^2$$



If M is the total mass of the body and K is the radius of gyration of the body about the axis of rotation, then the moment of inertia is given by

$$I = MK^2$$

$$K = \sqrt{\frac{I}{M}}$$

The SI unit of the radius of gyration is meter. The radius of gyration depends on

1. The distribution of mass from the axis of rotation.
2. The position and direction of the axis of rotation.

Theorems on Moment of Inertia

A) Parallel axes Theorem

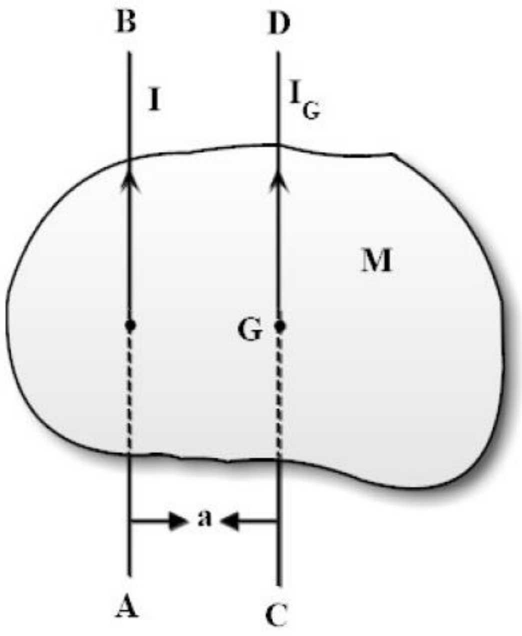
- *Parallel axes theorem states that the moment of inertia of any rigid body about a given axis is equal to the sum of its moment of inertia about a parallel axis passing through the centre of gravity and the product of the mass of the body and the square of the distance between the axes.*
- Let I be the moment of inertia of a body about an axis AB.

  Let I_G be the moment of inertia about another axis CD which is parallel to AB and passing through the centre of gravity G of the body.

  Let M be the mass and 'a' be the distance between the two axes.

  Then according to the parallel axes theorem,

$$I = I_G + Ma^2$$

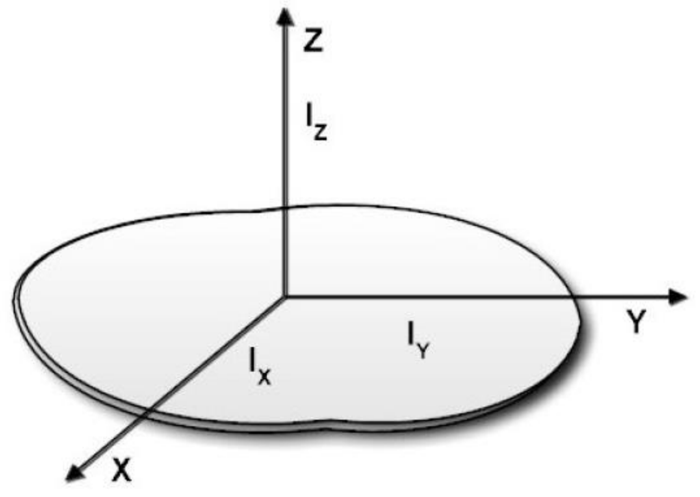


b)Perpendicular Axis theorem

- *Perpendicular axes theorem states that the sum of the moments of inertia of a plane lamina about two mutually perpendicular axes in its plane is equal to its moment of inertia about a perpendicular axis passing through the intersection of the first two axes.*

Let OX and OY be two mutually perpendicular axes in the plane of the lamina intersecting each other at point O. The axis OZ is perpendicular to both OX and OY. If IX, IY, and IZ are the moment of inertia about the axes OX, OY, and OZ respectively, then by perpendicular axes theorem

$$I_X + I_Y = I_Z$$



MODULE II
Rotational Motion

Dr.Jinchu.I

Basic Types of Motion

- Everything in the universe is moving and evolving. There are basically four different types of motion namely translational motion, rotational motion, oscillatory motion, and random motion.

a) Translational Motion

- Translational motion is a motion in which the location of the object changes with time.

Translational motion can again be classified into two types – rectilinear motion and curvilinear motion.

  - In rectilinear motion, the object moves along a straight line whereas, in curvilinear motion, the object moves along a curved path.
  - Circular motion is a special type of curvilinear motion in which an object moves along a circular path.
  - In translational motion, all objects are considered as point masses.

- Example: The motion of elevators in buildings (rectilinear motion)

  The motion of a basketball into the basket (curvilinear motion)

  The motion of satellites around the earth (circular motion).

b)Rotational motion

- Rotational motion is a motion in which the objects spin around an axis and the location of the object do not change with time. Rotational motion is always associated with rigid extended bodies.
- During rotational motion, each particle constituting the rigid body undergoes circular motion. Hence circular motion and rotational motion are closely related.

  Example: Motion of blades of the ceiling fan

c) Oscillatory motion

- Oscillatory motion is the to and fro motion of an object about a fixed point. It is a special

type of periodic motion – a motion that repeats itself in a regular interval of time.

Example: Oscillations of a pendulum

d) Random motion

- Random motion is a motion in which the particle moves in a zig-zag manner and the direction of motion changes continuously. This kind of motion is unpredictable in practice.

  Example: Motion of honey bee

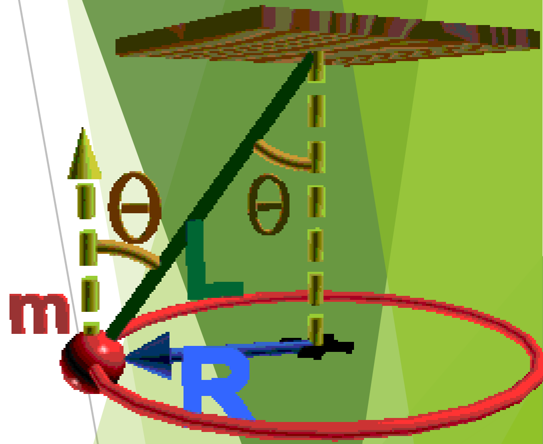
Circular Motion

circular motion is said to be uniform when a particle moves along a circular path with a constant speed.

The point or line that is the *center* of the circle is the *axis of rotation*.

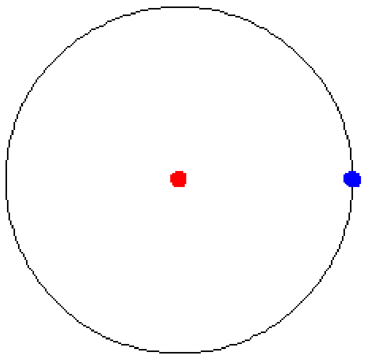
If the axis of rotation is *inside* the object, the object is *rotating (spinning)*.

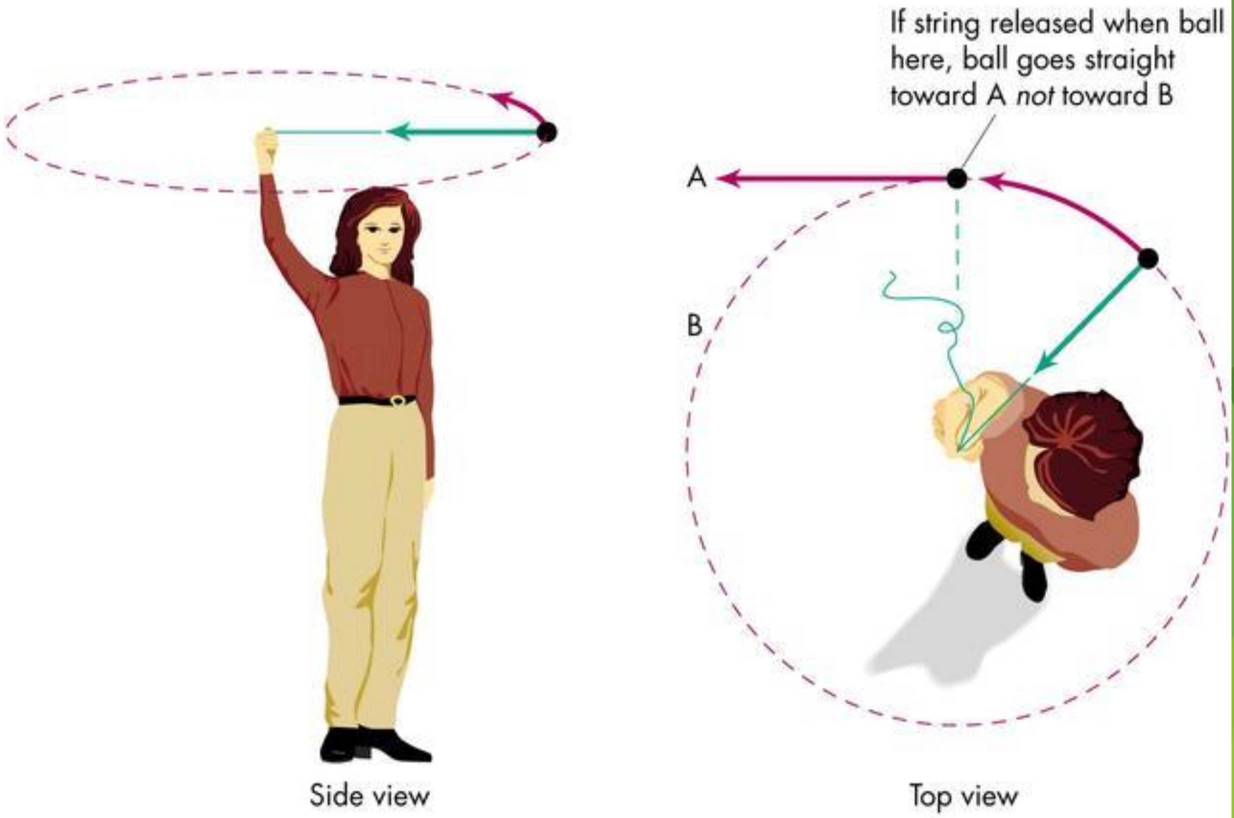
If the axis of rotation is *outside* the object, the object is *revolving*.



Uniform circular motion

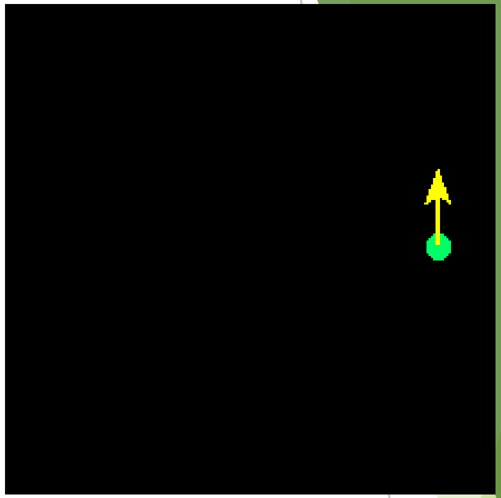
- An object which moves in a circle at constant speed is said to be executing uniform circular motion.
- Magnitude of the velocity remains constant, the direction of the velocity continuously changes.

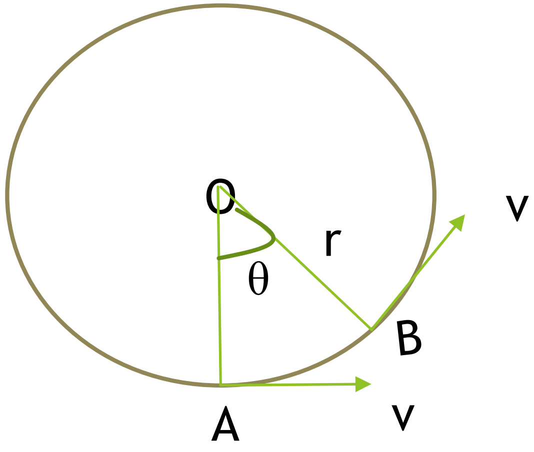


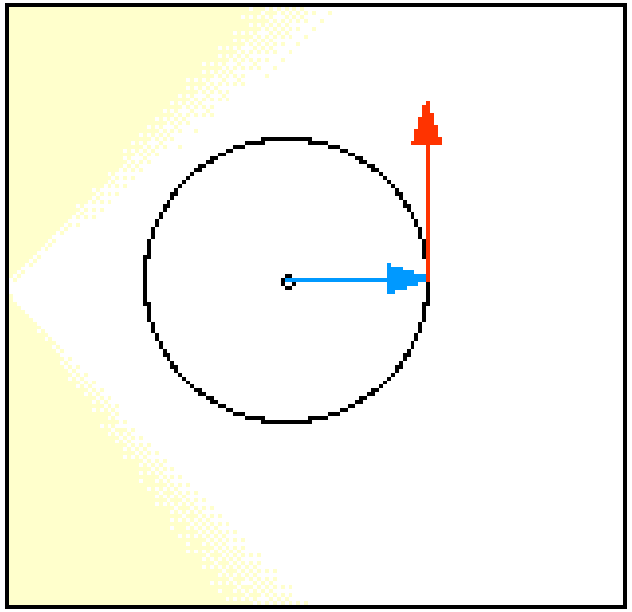


The direction of v changes continually!

- The instantaneous velocity is in a direction tangential to the circular path and hence called tangential velocity.
- Consider a body moving in a circle of radius r with a uniform speed v. The velocity at any instant is at right angles to the radius of the circular path.







The velocity is always tangent to the path

- During circular motion, the radius vector traces an angle at the centre. This angle is **angular displacement.**

  Let the body displaces from A to B in a time t.

  The angle traced is θ. Here angular displacement is θ

  Unit is rad.

  (1 radian is the angle subtended by an arc whose length is equal to the radius of the circle. 2л rad = 360

- Angular velocity (ω) : It is defined as the angular displacement in unit time.
- If θ is the angular displacement in a time t, $\omega = \theta/t$ rad/s

- Angular velocity of a rotating mechanism is expressed in ***revolution per minute (r.p.m)***.

  The angular velocity of a second hand in a clock is 1 r.p.m

  1 r.p.m = 1 revolution/1 minute = 2 л rad /60 second

  = л/30 rad/s

- ***Relation between linear velocity and angular velocity***
  - Let the body travels along a circle of radius r with uniform speed v. The body travels from A to B in a time t, the angular displacement being θ.

    $\omega = \frac{\theta}{t}$

  Since the body is travelling along AB in a time t, AB = v x t---------------(1)

    Applying the general formula

    Angle = $\frac{arc}{radius}$, for the sector AOB,

    $\frac{AB}{r}$ = θ or AB = rθ--------------------(2)

Equating the RHS of both equations, we get

$V\ t = r\theta$

Or $v = r(\frac{\theta}{t})$

ie, $v = r\omega$

Linear velocity = radius x angular velocity

Angular acceleration (α)

It is defined as the change in angular velocity in unit time.

If ω_1 and ω_2 are the initial and final angular velocities during time t,

Angular acceleration $(\alpha) = \frac{\omega_2 - \omega_1}{t}$

Relation between linear acceleration and angular acceleration

- Rate of change of linear velocity is linear acceleration.

  Rate of change of angular velocity is angular acceleration.

  Ie, a = $\frac{v_2 - v_1}{t}$ $\alpha = \frac{\omega_2 - \omega_1}{t}$

Consider a body moving along a circle of radius r with an angular velocity ω_1 , the corresponding linear velocity being v_1

Then $v_1 = r\omega_1$, $v_2 = r\omega_2$

We have $\alpha = (\omega_2 - \omega_1)/t = \{\frac{v_2}{r} - \frac{v_1}{r}\}/t$

$\frac{\frac{1}{r}(v_2 - v_1)}{t} = \frac{1}{r}a$ or $a = r\alpha$

- Period (T) : Period is the time taken for a single cycle.

  Unit is second

Frequency (f): The number of cycles completed in one second is called frequency.

Unit is Hertz

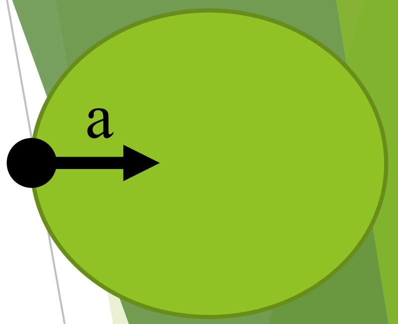
$f = 1/T$ and $T = 1/f$

Relation between T, ω and f

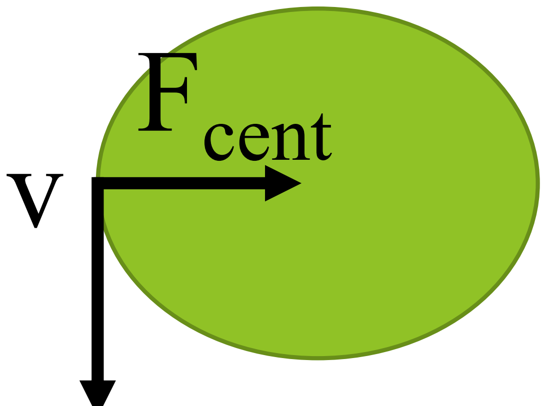
$$\omega = \frac{\theta}{t} \quad , \theta = 2\pi \text{ rad}$$

$$\omega = 2\pi/T \quad \text{or } \omega = 2\pi f$$

Centripetal Acceleration & Centripetal Force



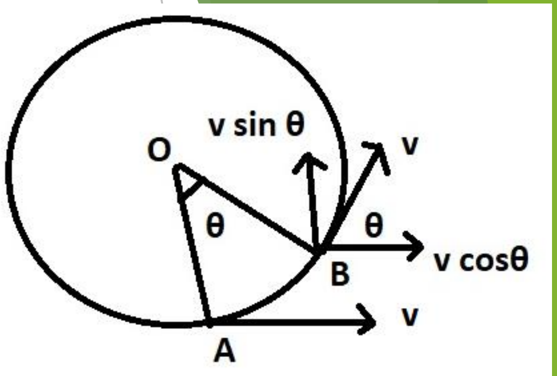
- During circular motion the magnitude of velocity remains constant, but its acceleration changes continuously. This change leads to an acceleration which is directed towards the centre. This acceleration is known as centripetal acceleration.
- A force is necessary to change the direction of a body. Therefore an external force must be acting inwards to keep the body in a circular path. This inward force acting on the body is called the centripetal force.



Magnitude of centripetal acceleration

- Imagine a body of mass m moving along a circle of radius r with a velocity v. Initially the body is at A, the velocity at this point of time does not have a component towards the centre, because v cos90 =0.
- After a small time interval t, let the body changes its position to B. θ is the angular displacement .

Angular velocity, $\omega = \theta/t$



At the position B, the tangential velocity can be resolved into two components. v cosθ along the horizontal direction and v sinθ along the vertical direction.

If the time interval is very small, the point B will not be far away from point A

- The angle θ also will be very small.

At point B the vertical component can be considered to be pointing towards the centre. This means that a change in velocity has occurred during transition from A to B

Change in velocity towards the centre = v sinθ-0 =v sinθ

For small values sinθ can be approximated to θ.

Therefore change in velocity = vθ

Centripetal acceleration = change in velocity /time

=vθ/t =vω

Since v =rω it can be also written as $r\omega^2$ or v^2/r

But F =ma Hence the expression for centripetal force becomes

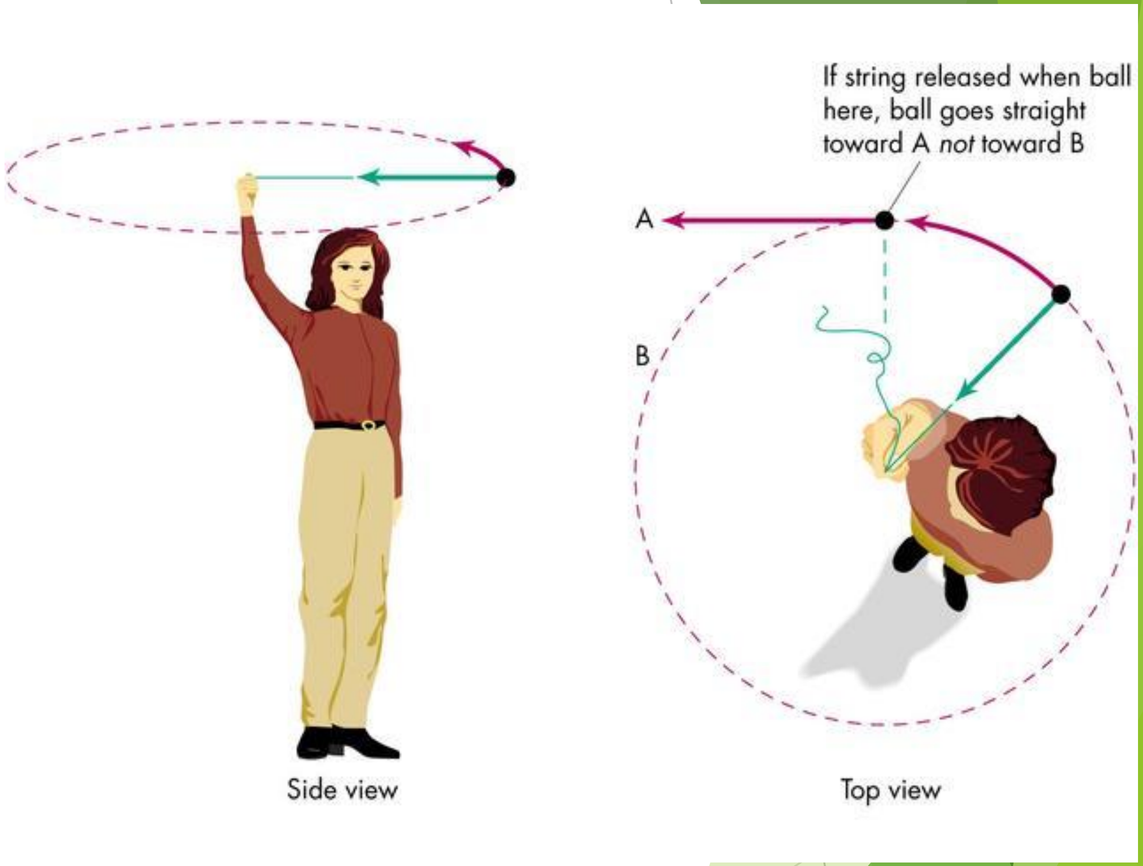
$F = mv\omega$ and $F = mv^2/r$ and $F = mr\omega^2$

The centripetal acceleration depends on:

The speed of the object.

The radius of the circle

- For a body execute uniform circular motion an agency should supply the necessary centripetal force.
- Example shown in figure
- In this case the centripetal force is supplied by the tension in the string.



Tie a stone to a string and whirl it around

Banking of Roads and Rails

- **Banking of roads** is defined as the phenomenon in which the edges are raised for the curved **roads** above the inner edge to provide the necessary centripetal force to the vehicles so that they take a safe turn. ... The angle at which the vehicle is inclined is defined as the **bank** angle.
- The angle of banking is the angle made by the elevated path with the horizontal. Let AB and AC represent the horizontal and banked paths respectively as shown in figure. Let θ be the angle of banking. Consider a vehicle of mass m takes a curved path of radius r with a speed v. The weight of the vehicle mg acts vertically downwards. The normal reaction N of the road on the vehicle will be perpendicular to the AC. The normal reaction can be resolved into vertical and horizontal components

$$\mathrm{Tan}\theta = \frac{v^2}{Rg} \quad where\ R\ is\ the\ radius\ of\ the\ curve$$

The vertical componentis equal to the weight of the body.

$$N cos\theta = mg$$

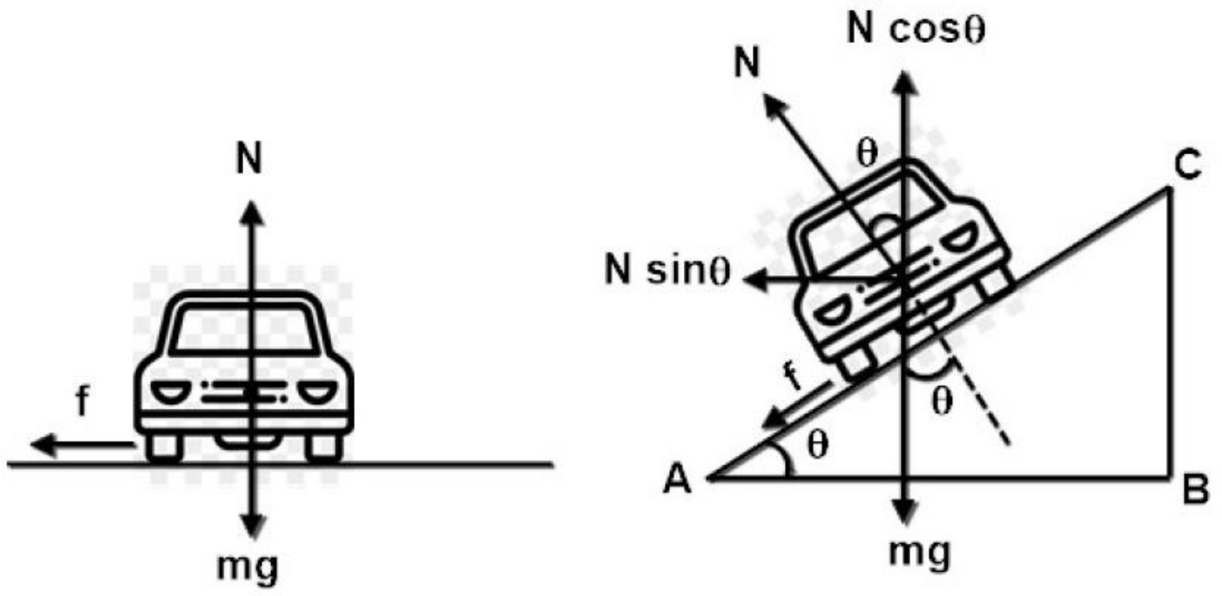
The horizontal component provides the centripetal force

$$N sin\theta = \frac{mv^2}{r}$$

Dividing the second equation by the first gives

$$tan\theta = \frac{v^2}{rg}$$

$$\theta = tan^{-1}\left\{\frac{v^2}{rg}\right\}$$



Horizontal curve

Banked curve

The angle of banking depends on the radius of the curve of the road and the speed of the vehicle.

- Roads are most often banked for the average speed of vehicles passing over them. Nevertheless, if the speed of a vehicle is lesser or more than this, the self-adjusting state friction will operate between tyre and road and vehicle will not skid.

Banking of Railway Tracks

- In the case of a train moving through a curved track, centripetal force is required towards the centre of the circular track.
- This force is provided by the thrust exerted by the side of the outer rail against the flange of the outer wheel.
- When a fast-moving train takes a curved path, it tends to move away tangentially off the track.
- To avoid this, the outer rail is raised above the level of the inner rail. This is known as the banking of railway tracks.
- The banking of railway tracks avoids skidding and reduces the wear and tear of the wheels. In the case of a curved railway track, the level of the outer rail is higher than that of the inner one.
- The height of the outer rail above the inner rail in the banked rail track is called super elevation (S). If d is the distance between the rails and θ be the angle of super elevation.

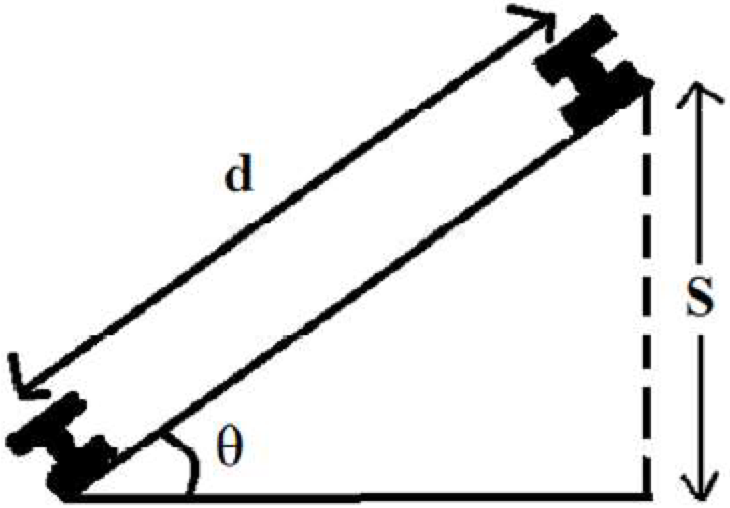
$$sin\theta = \frac{S}{d}$$

or

$$S = dsin\theta$$

Since θ is usually small for banked rail tracks, $sin\theta$ approximately equal to tanθ

$$tan\theta = \frac{S}{d}$$



But the equation for the angle of banking is given by

$$tan\theta = \frac{v^2}{rg}$$

$$\frac{S}{d} = \frac{v^2}{rg}$$

$$S = \frac{v^2 d}{rg}$$