(5).

matrix multiplication

Two matrices A and B are conformable for multiplication if the number of columns of A is equal to the number of rows of B.

If A is a matrix of order $m \times p$ and B is a matrix of order $p \times n$, AB is of order $m \times n$. $\boxed{(m \times p)(p \times n) = m \times n}$

Example.

1. $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 \end{bmatrix}$

$(1 \times 3) \times (3 \times 1) = 1 \times 1$

2. $\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix}$

$(1 \times 3) \times (3 \times 2)$

1×2

$= \begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_1y_1 + a_2y_2 + a_3y_3 \end{bmatrix} \rightarrow 1 \times 2$

3. $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 \\ b_1x_1 + b_2x_2 + b_3x_3 \end{bmatrix}$ 2×1

$(2 \times 3) \times (3 \times 1)$

4. $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_1y_1 + a_2y_2 + a_3y_3 \\ b_1x_1 + b_2x_2 + b_3x_3 & b_1y_1 + b_2y_2 + b_3y_3 \end{bmatrix}$

$(2 \times 3)(3 \times 2) = 2 \times 2$

(5). $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} a_1x_1 & a_1x_2 & a_1x_3 \\ a_2x_1 & a_2x_2 & a_2x_3 \\ a_3x_1 & a_3x_2 & a_3x_3 \end{bmatrix}$

$(3\times1)\times(1\times3) = \underline{\underline{3\times3}}$.

Q_1 If $A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ Evaluate AB.

$$AB = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1\times0-1\times-1 & 1\times1-1\times2 \\ 2\times0+1\times-1 & 2\times1+1\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 0+1 & 1-2 \\ 0-1 & 2+2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$$

Q_2. If $A = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 5 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 4 \end{bmatrix}$ find AB.

$(2\times3)\times(3\times2)$

$$AB = \begin{bmatrix} 3 & 2 & -1 \\ 4 & 5 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3\times1+2\times0-1\times-1 & 3\times2+2\times3-1\times4 \\ 4\times1+5\times0-3\times-1 & 4\times2+5\times3-3\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+1 & 6+6-4 \\ 4+0+3 & 8+15-12 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 7 & 11 \end{bmatrix}$$

Q3. If $A = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 4 & -1 \end{bmatrix}$ find AB and BA

$$AB = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & -1 \end{bmatrix}$$

$(3\times1) \times (1\times3)$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 2 & 8 & -2 \\ 3 & 12 & -3 \end{bmatrix} \Rightarrow 3\times3$$

$$BA = \begin{bmatrix} 1 & 4 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$$

$(1\times3)\times(3\times1)$

$$= [1\times0 + 4\times2 - 1\times3]$$

$$= [0 + 8 - 3] = 5 \Rightarrow (1\times1)$$

Q4. If $A = \begin{bmatrix} 5 & 6 \\ -1 & 2 \\ 3 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & -2 \end{bmatrix}$ find AB.

$$AB = \begin{bmatrix} 5 & 6 \\ -1 & 2 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 3 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 5\times 2+6\times 3 & 5\times -1+6\times 4 & 5\times 0+6\times -2 \\ -1\times 2+2\times 3 & -1\times -1+2\times 4 & -1\times 0+2\times -2 \\ 3\times 2+-4\times 3 & 3\times -1-4\times 4 & 3\times 0-4\times -2 \end{bmatrix}$$

$$= \begin{bmatrix} 10+18 & -5+24 & -12 \\ -2+6 & 1+8 & -4 \\ 6-12 & -3-16 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 28 & 19 & -12 \\ 4 & 9 & -4 \\ -6 & -19 & 8 \end{bmatrix}$$

Q5 If $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 1 \end{bmatrix}$ find AB and BA.

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times 1+2\times 3+3\times -1 & 1\times 2+2\times 4+3\times 1 \\ -4\times 1+5\times 3-1\times -1 & -4\times 2+5\times 4-1\times 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6-3 & 2+8+3 \\ -4+15+1 & -8+20-1 \end{bmatrix} = \begin{bmatrix} 4 & 13 \\ 12 & 11 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1\times1+2\times-4 & 1\times2+2\times5 & 1\times3+2\times-1 \\ 3\times1+4\times-4 & 3\times2+4\times5 & 3\times3+4\times-1 \\ -1\times1+1\times-4 & -1\times2+1\times5 & -1\times3+1\times-1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8 & 2+10 & 3-2 \\ 3-16 & 6+20 & 9-4 \\ -1-4 & -2+5 & -3-1 \end{bmatrix} = \begin{bmatrix} -7 & 12 & 1 \\ -13 & 26 & 5 \\ -5 & 3 & -4 \end{bmatrix}$$

~~Q6. If~~ $\cancel{A(\theta')} = \cancel{\begin{bmatrix} \cos\theta' & \sin\theta' \\ \sin\theta' & \cos\theta' \end{bmatrix}}$

Q6 If $A(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ Show that

$$A(\theta)\,A(\theta') = A(\theta+\theta')$$

$$A(\theta') = \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix}$$

$$A(\theta)\,A(\theta') = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta' & -\sin\theta' \\ \sin\theta' & \cos\theta' \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cos\theta' - \sin\theta\sin\theta' & -\cos\theta\sin\theta' - \sin\theta\cos\theta \\ \sin\theta\cos\theta' + \cos\theta\sin\theta' & -\sin\theta\sin\theta' + \cos\theta\cos\theta' \end{pmatrix}$$

$$= \begin{pmatrix} \cos(\theta+\theta') & -(\cos\theta\sin\theta' + \sin\theta\cos\theta') \\ \sin(\theta+\theta') & \cos\theta\cos\theta' - \sin\theta\sin\theta' \end{pmatrix}$$

$$\begin{pmatrix} \cos(\theta+\theta') & -\sin(\theta+\theta') \\ \sin(\theta+\theta') & \cos(\theta+\theta') \end{pmatrix}$$

$$= A(\theta+\theta').$$

$$\therefore A(\theta)\,A(\theta') = A(\theta+\theta')$$

Q. If $A = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$

Compute AB and hence show that $AB = 0$ even when $A \neq 0$, $B \neq 0$.

$$AB = \begin{bmatrix} 1 & 1 & -1 \\ -2 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1+6-5 & -2+12-10 & -1+6-5 \\ 2+18-20 & 4+36-40 & 2+18-20 \\ -3-12+15 & -6-24+30 & -3-12+15 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Q. Show that $[x \;\; y] \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = [ax^2 + 2hxy + by^2]$

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & h \\ h & b \end{bmatrix} = \begin{bmatrix} ax+hy & xh+yb \end{bmatrix}$$

$$\begin{bmatrix} ax+hy & xh+yb \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x(ax+hy)+y(xh+yb)$$
$$= ax^2+xhy+yxh+by^2$$
$$= ax^2+by^2+2xyh$$

Transpose of a matrix.

The matrix obtained by interchanging the rows and columns of a given matrix is called transpose of that matrix.

If a matrix of order $m \times n$, its transpose has order $n \times m$. It is denoted by A^T or A' (read as A transpose)

1. $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 4 \end{bmatrix} = A$, $A^T = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix}$

3×2 $\qquad$ 2×3

2. $B = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$, $B^T = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$

Properties of transpose

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(AB)^T = B^T A^T$
4. $(KA)^T = K \cdot A^T$ where K is a const.

1. If $A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$, $B = \begin{bmatrix}0 & -2\\ -3 & -3\end{bmatrix}$. find $(A+B)^T$

$$A+B = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} + \begin{bmatrix}0 & -2\\ -3 & -3\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Q. If $A = \begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 2\end{bmatrix}$, find AA^T

$$A^T = \begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 2\end{bmatrix}$$

$$AA^T = \begin{bmatrix}3 & 1 & -1\\ 0 & 1 & 2\end{bmatrix} \begin{bmatrix}3 & 0\\ 1 & 1\\ -1 & 2\end{bmatrix}$$

$$= \begin{bmatrix}9+1+1 & 0+1-2\\ 0+1-2 & 0+1+4\end{bmatrix} = \begin{bmatrix}11 & -1\\ -1 & 5\end{bmatrix}$$

Determinants.

Determinants of order two. (2×2).

The symbol $\begin{vmatrix}a & b\\ c & d\end{vmatrix}$ consisting of four elements, which are arranged in two rows and two columns is a determinant of order two. The value of $\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$.

1. $\begin{vmatrix}4 & 1\\ 2 & 3\end{vmatrix} = (4\times 3) - (2\times 1) = 12 - 2 = 10$ //

2. $\begin{vmatrix} -1 & -3 \\ 4 & -2 \end{vmatrix} = -1 \times -2 - -3 \times 4$
$= 2 + 12 = \underline{14}$.

3. $\begin{vmatrix} x & y \\ x^2 & y^2 \end{vmatrix} = \underline{xy^2 - yx^2}$.

4 $\begin{vmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{vmatrix} = \sin\theta \times \sin\theta - -\cos\theta \times \cos\theta$
$= (\sin\theta)^2 + (\cos\theta)^2$
$= \sin^2\theta + \cos^2\theta$
$= \underline{1}$

5 $\begin{vmatrix} \sec\theta & \tan\theta \\ \tan\theta & \sec\theta \end{vmatrix} = \sec\theta \times \sec\theta - \tan\theta \times \tan\theta$
$= (\sec\theta)^2 - (\tan\theta)^2$
$= \sec^2\theta - \tan^2\theta$
$= \underline{1}$

6. Solve for x if $\begin{vmatrix} x & 12 \\ 3 & x \end{vmatrix} = 0$.

$\begin{vmatrix} x & 12 \\ 3 & x \end{vmatrix} = x \times x - 12 \times 3 = 0$
$x^2 - 36 = 0$.
$x^2 = 36$
$x = \underline{\pm 6}$

7. If $\begin{vmatrix} 3x & 7 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix}$ find x.

$\begin{vmatrix} 3x & 7 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 4 & 2 \\ 2 & 2 \end{vmatrix}$

$3x \times 3 - 2 \times 7 = 4 \times 2 - 2 \times 2$
$9x - 14 = 8 - 4 = \underline{4}$
$9x = 4 + 14 = 18$

$x = 18/9 = \underline{\underline{2}}$

8. $\begin{vmatrix} x^2 & 3 \\ 4 & 1 \end{vmatrix} = \begin{vmatrix} 9 & 4 \\ 8 & 5 \end{vmatrix}$ find x.

$$\begin{vmatrix} x^2 & 3 \\ 4 & 1 \end{vmatrix} = \begin{vmatrix} 9 & 4 \\ 8 & 5 \end{vmatrix}$$

$$x^2 \times 1 - 3 \times 4 = 9 \times 5 - 4 \times 8.$$

$$x^2 - 12 = 45 - 32.$$

$$= 13$$

$$x^2 = 13 + 12 = 25$$

$$x = \sqrt{25} = \pm 5 //$$

Q. Solve for x. $\begin{vmatrix} 2x-1 & x+1 \\ x+2 & x-2 \end{vmatrix} = 0$.

$$\begin{vmatrix} 2x-1 & x+1 \\ x+2 & x-2 \end{vmatrix} = (2x-1)(x-2) - (x+1)(x+2)$$

$$= (2x \times x - 2x \times 2 - 1 \times x - 1 \times -2) -$$

$$(x \times x + 2 \times x + 1 \times x + 2) = 0$$

$$2x^2 - 4x - x + 2 - (x^2 + 2x + x + 2) = 0.$$

$$2x^2 - 5x + 2 - (x^2 + 3x + 2) = 0.$$

$$2x^2 - 5x + 2 - x^2 - 3x - 2 = 0.$$

$$x^2 - 8x = 0.$$

$$x(x-8) = 0$$

$$x = 0 \text{ or } x - 8 = 0$$

$$x = 0 \text{ or } x = 8$$