(3320)Digital Signal Processing|Matlab Exercise

Itay Chairman (322977372) Ido Sar Shalom (212410146)

June 10, 2022

Part 1- Implementing FFT

In this part we'll implement the FFT algorithm and it's inverse.
To do so we'll use the recursive method, we know that the algorithm takes a signal, split it into odd and even parts, find their DFT and connect them back. So what we'll do is split the signal, for each split we'll call the FFT function recursively and then connect the resulted splits.
The stopping condition would be a signal with length 1, and we know that the transform of a signal with length 1 equals to the signal itself.
If we get a signal with length which is not a power of 2 we'll pad it with zeros to fill to the closest power of 2. Afterwards we'll cut the signal to be with the original length.
In order to implement the inverse DFT we'll use the relation

$$x[n] = \frac{1}{N} DFT\left\{X^*[k]\right\}$$

so what we'll do is call the FFT function with the conjugated signal divided by it's length.

Part 2- Image Processing

In this part we'll process an image using tools we learned in class.
An image is a two dimensional signal, so the theory we learned is still valid but we need to consider the second dimension.
The two dimensional DTFT is defined as

$$X\left(e^{j\omega_1}, e^{j\omega_2}\right) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[n,m]\, e^{-j(\omega_1 n + \omega_2 m)}$$

and the two dimensional DFT is defined as

$$X[k_1,k_2]=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}x[n,m]\,e^{-j\left(\frac{2\pi n}{N}k_1+\frac{2\pi m}{M}k_2\right)}$$

where n is the row index and m is the column index.

(a) **Prove mathematically that we can find the two dimensional DTFT by applying a one dimensional DTFT on each column and afterwards another DTFT on each row:**

We know that

$$X\left(e^{j\omega_1},e^{j\omega_2}\right)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x[n,m]\,e^{-j(\omega_1 n+\omega_2 m)}=\sum_{n=-\infty}^{\infty}e^{-j\omega_1 n}\sum_{m=-\infty}^{\infty}x[n,m]\,e^{-j\omega_2 m}$$

Let's assume we're at a row n, now we'll define the signal of the columns at this row to be $x_n[m]=x[n,m]$. This signal has a single dimension, that's because n is constant in our case.

The DTFT of this signal is

$$X_n\left(e^{j\omega_2}\right)=\sum_{m=-\infty}^{\infty}x_n[m]\,e^{-j\omega_2 m}$$

We can notice that in the result of $X\left(e^{j\omega_1},e^{j\omega_2}\right)$ we got an inner sum and outer sum. Inside the inner sum the n parameter is constant, hence $x[n,m]=x_n[m]$.

It means that

$$X\left(e^{j\omega_1},e^{j\omega_2}\right)=\sum_{n=-\infty}^{\infty}e^{-j\omega_1 n}\sum_{m=-\infty}^{\infty}x_n[m]\,e^{-j\omega_2 m}=\sum_{n=-\infty}^{\infty}\underbrace{DTFT_m\{x[n,m]\}}_{X_n\left(e^{j\omega_2}\right)}e^{-j\omega_1 n}$$

Now we can do a similar thing and notice that inside the sum ω_2 is constant but n is not, so we can consider $X_n\left(e^{j\omega_2}\right)$ as a single dimensional signal of n.

We can easily see that this sum we got will give us the DTFT of $X_n\left(e^{j\omega_2}\right)$ according to n so we can conclude that

$$\boxed{X\left(e^{j\omega_1},e^{j\omega_2}\right)=DTFT_n\{DTFT_m\{x[n,m]\}\}}$$

as asked.

(b) **Prove mathematically that the two dimensional DFT is a sample of the two dimensional DTFT:**

Let's assume that we're talking about a finite image (which is a reasonable assumption, image shouldn't be infinite).

We'll get that the DTFT is

$$X\left(e^{j\omega_1},e^{j\omega_2}\right)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x\left[n,m\right]e^{-j(\omega_1 n+\omega_2 m)}=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}x\left[n,m\right]e^{-j(\omega_1 n+\omega_2 m)}$$

and we can notice that

$$X\left(e^{j\omega_1},e^{j\omega_2}\right)\Big|_{\omega_1=\frac{2\pi k_1}{N},\omega_2=\frac{2\pi k_2}{M}}=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}x\left[n,m\right]e^{-j\left(\frac{2\pi n}{N}k_1+\frac{2\pi m}{M}k_2\right)}$$

and we easily see that

$$\boxed{X\left(e^{j\omega_1},e^{j\omega_2}\right)\Big|_{\omega_1=\frac{2\pi k_1}{N},\omega_2=\frac{2\pi k_2}{M}}=X\left[k_1,k_2\right]}$$

Which means that the DFT is a sample of the DTFT at $\omega_1=\frac{2\pi k_1}{N}$ and $\omega_2=\frac{2\pi k_2}{M}$ as asked.

(c) **Prove that for $x\left[n,m\right]=y\left[n\right]z\left[m\right]$ we'll get $X\left(e^{j\omega_1},e^{j\omega_2}\right)=Y\left(e^{j\omega_1}\right)Z\left(e^{j\omega_2}\right)$:**

From definition we know that

$$X\left(e^{j\omega_1},e^{j\omega_2}\right)=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}x\left[n,m\right]e^{-j(\omega_1 n+\omega_2 m)}$$

$$=\sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty}y\left[n\right]z\left[m\right]e^{-j(\omega_1 n+\omega_2 m)}=\sum_{n=-\infty}^{\infty}y\left[n\right]e^{-j\omega_1 n}\sum_{m=-\infty}^{\infty}z\left[m\right]e^{-j\omega_2 m}$$

which means

$$\boxed{X\left(e^{j\omega_1},e^{j\omega_2}\right)=Y\left(e^{j\omega_1}\right)Z\left(e^{j\omega_2}\right)}$$

as asked.

(d) Assume the image

$$x\left[n,m\right]=\begin{cases}1 & 0\leq n<B_1,0\leq m<B_2\\ 0 & B_1\leq n<N,B_2\leq m<M\end{cases}$$

Find using analytical and numeric methods the DFT of the given image:

Let's start by defining

$$y\left[n\right]=\begin{cases}1 & 0\leq n<B_1\\ 0 & B_1\leq n<N\end{cases}$$

and

$$z\left[m\right]=\begin{cases}1 & 0\leq m<B_2\\ 0 & B_2\leq m<M\end{cases}$$

We can easily see that

$$x[n,m] = y[n]\,z[m]$$

which means that

$$X\left(e^{j\omega_1}, e^{j\omega_2}\right) = Y\left(e^{j\omega_1}\right) Z\left(e^{j\omega_2}\right)$$

and because we know that DFT is a sample of the DTFT we can say that

$$X[k_1,k_2] = Y[k_1]\,Z[k_2]$$

So all we need to do is find the DFT of $y[n]$ and $z[m]$.
We can see that both $y[n]$ and $z[m]$ are very similar, so we can find a general DFT and infer about each of them.
Let's look at a general window signal

$$x[n] = \begin{cases} 1 & 0 \le n < B \\ 0 & B \le n < N \end{cases}$$

and we'll get

$$X[k] = \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k}{N}n} = \sum_{n=0}^{B-1} e^{-j\frac{2\pi k}{N}n} = \frac{e^{-j\frac{2\pi k}{N}B} - 1}{e^{-j\frac{2\pi k}{N}} - 1}$$

$$= \frac{e^{-j\frac{\pi k}{N}B} - e^{j\frac{\pi k}{N}B}}{e^{-j\frac{\pi k}{N}} - e^{j\frac{\pi k}{N}}} \cdot \frac{e^{-j\frac{\pi k}{N}B}}{e^{-j\frac{\pi k}{N}}} = \frac{\sin\left(\frac{\pi k}{N}B\right)}{\sin\left(\frac{\pi k}{N}\right)} e^{-j\frac{\pi k}{N}(B-1)} = D\left(\frac{2\pi k}{N}, B\right) e^{-j\frac{\pi k}{N}(B-1)}$$

just like we'd expect of a window function.
Now we can find the two dimensional DFT

$$\boxed{X[k_1,k_2] = D\left(\frac{2\pi k_1}{N}, B_1\right) e^{-j\frac{\pi k_1}{N}(B_1-1)} D\left(\frac{2\pi k_2}{M}, B_2\right) e^{-j\frac{\pi k_2}{M}(B_2-1)}}$$

Let's run it in MATLAB and observe the results.
First we'll execute a numeric calculation, we'll create the vectors $y[n]$ and $z[m]$ which were defined earlier, and find their DFT using MATLAB's command fft.
After getting the DFT we can derive the image by multiplying the transformed vectors.
Next we'll plot the numeric solution for comparison, we'll calculate it using the result we found (we'll put the formula in MATLAB and let it find the values).
We decided to visualize the images as a heat map, this will help us see them better.
We'll get the following results

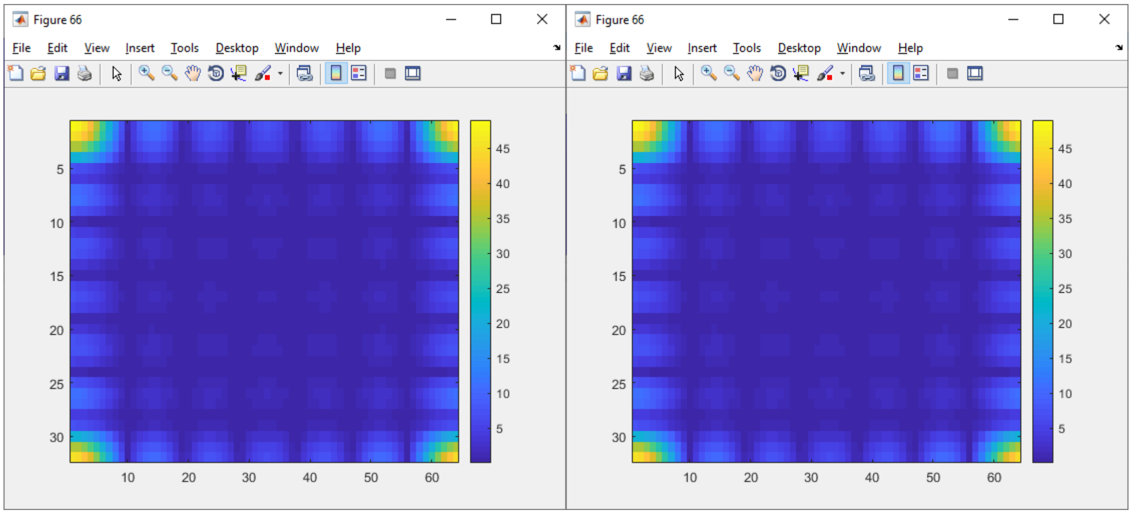

Figure 1: Numerical result (left) and analytical result (right)

We can easily see that the pictures look alike, which is a good sign.
Let's look at the difference between the pictures

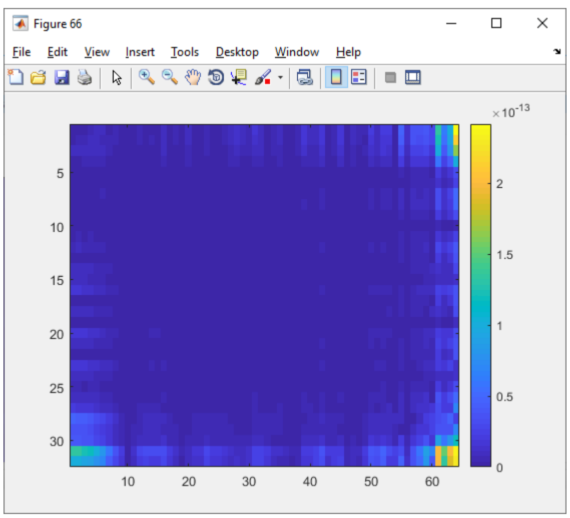

Figure 2: The difference between the numerical and analytical results

We can easily see that the picture has extremely low values (order of 10^{-13}) so we can conclude that the difference between the results accures only because of numerical errors of the calculation which are inevitable.
(e) Now we'll load the given images and work with them.
We'll get four images, the first two are images we got through a channel ($y_1[n,m]$ and $y_2[n,m]$), the next is a noised image ($y_3[n,m]$) and the last is the impulse respond of the channel ($h[n,m]$).
Let's look at the first column of the impulse respond $h_0[n] = h[n,0]$.
Calculate numerically $H_0\left(e^{j\omega}\right)$ at $\omega = 0, \frac{2\pi}{6}, 2\frac{2\pi}{6}, 4\frac{2\pi}{6}$:

Let's start by looking at the signal and we'll see that it has a length of 3.
We can also see that $h_0[n] = 0.0909 \cdot \delta[n-2]$ but we need to give a numerical solution which independent on the choice of $h_0[n]$ so we won't use it now, only at the end to check that our result is as expected.
We know that the DFT is a sample of the DTFT, so we can use the DFT to find the wanted value.
It's important to notice that we're looking at samples with the wrong frequencies, the DFT of the signal would give us a sample of the DTFT at $\omega = 0, \frac{2\pi}{3}, 2\frac{2\pi}{3}$ and not the wanted frequencies.
To handle this we'll pad the signal with zeros to reach the length of 6 and we'll get

$$\tilde{h}_0[n] = \begin{cases} h_0[n] & 0 \le n < N \\ 0 & N \le n < 2N \end{cases}$$

$$\tilde{H}_0[k] = \sum_{n=0}^{2N-1} \tilde{h}_0[n] e^{-j\frac{2\pi n}{2N}k} = \sum_{n=0}^{N-1} h_0[n] e^{-j\frac{2\pi n}{2N}k} = H_0\left(e^{j\cdot\frac{2\pi k}{2N}}\right)$$

where in our case $N = 3$.
So what we'll need to do is pad h_0 with 3 zeros, find the DFT and print the 0,1,2,4 results.
The result is a set of four complex numbers which is found in the code.
We expect the results to be

$$H_0\left(e^{j\frac{2\pi}{6}k}\right) = \sum_{n=-\infty}^{\infty} h_0[n] e^{-j\frac{2\pi n}{6}k} = \sum_{n=0}^{2} 0.0909\delta[n-2] e^{-j\frac{2\pi n}{6}k} = 0.0909 e^{-j\frac{2\pi}{3}k}$$

and we can find these theoretical values and easily see that they equal to the numerical values.
(f) **Find the circular convolution with period 32 between $\boldsymbol{h_0[n]}$ and $\boldsymbol{w[n] = \delta[n] + \delta[n-29]}$:**
We can notice that we're calculating a convolution between $h_0[n]$ and deltas.
We have to remember that when we're doing the convolution we need to pad the signals with zeros to reach the wanted length (in our case pad $h_0[n]$ with zeros to reach length 32 instead of 3).
In order to do the convolution we'll use the convolution theorem, which means that we can do a multiplication in the domain and it would be a circular convolution in time.
We'll find the DFT of $h_0[n]$ and $w[n]$, multiply them and go back to the time and we'll print the results.
We'll get that

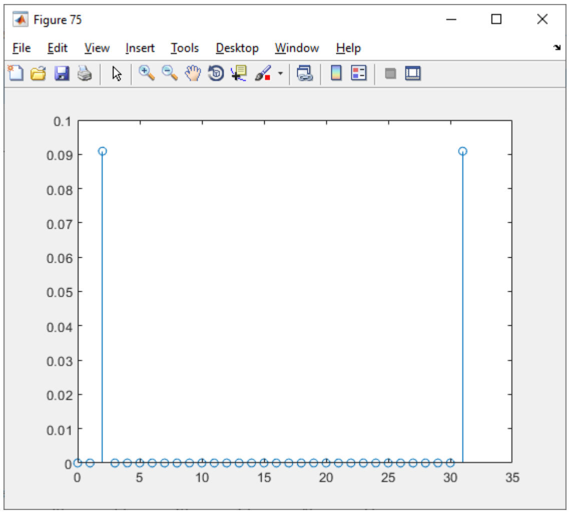

Figure 3: Result of convolution between $h_0[n]$ and $w[n]$

Let's compare this to the theoretical value to make sure we get what we expect.
From the properties of delta function we know that $h_0[n] \circledast w[n] = h_0[n] + h_0[\langle n-29\rangle_{32}]$ and we know that $h_0[n] = 0.0909\delta[n-2]$ so we'll expect to get

$$h_0[n] \circledast w[n] = 0.0909(\delta[n-2] + \delta[n-29-2]) = 0.0909(\delta[n-2] + \delta[n-31])$$

just like we got.
Another way to validate the results (more relevant if the theoretical calculation isn't that simple) is to check the output of cconv and make sure the results look the same.
(g) Now we'll try to restore the images.
The images we got are images which passed through the channel so what we see is a convolution between the image and the impulse respond of the channel, and a cut of the image to it's original value. Mathematically we can write

$$y[n,m] = \begin{cases} x[n,m] * h[n,m] & 0 \le n \le N-1, 0 \le m \le M-1 \\ 0 & otherwise \end{cases}$$

We'll assume the strong assumption that the image we got is a circular convolution between the impulse respond of the channel.

Find the original image under this assumption:

We know that just like in single dimension signals, circular convolution in time is multiplication in DFT domain.
What we'll do is find the DFT of the image, divide the image by the DFT of the impulse response and go back to the image space.
If our assumption is correct we're supposed to get the image before it's passed through the channel.

Let's observe the results for the first image we'll get

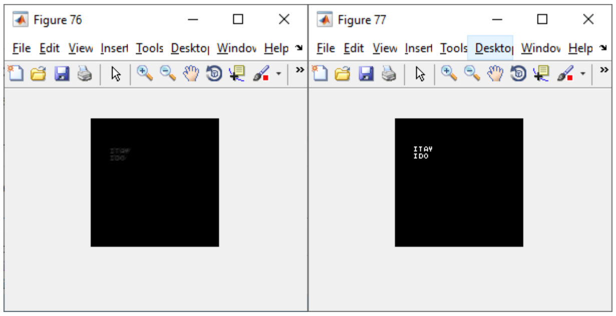

Figure 4: The original first image (left) and the restored first image (right)

and for the second image we'll get

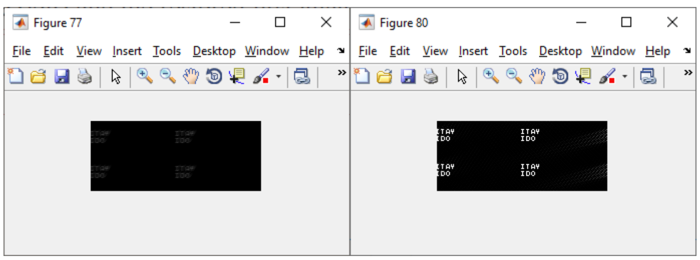

Figure 5: The original second image (left) and the restored second image (right)

(h) We're given that

- The picture $x_1[n,m]$ is created from a basic image which is padded by at least 20 zeros at each direction.
- The picture $x_2[n,m]$ is created from a basic image which is duplicated four times (2 times vertically and 2 times horizontally).

Explain the results:

Let's recap to our assumption, we assumed that the distorted images were generated from a cyclic convolution of the original images and the impulse response of the channel.
Of course, in real life this is not the case and what really happens is a linear convolution of the original images and the impulse response.
The two images we got are present two ways to handle this problem of the linear convolution.
In the first method we pad the image with many zeros, as we've seen in class padding a signal with zeros can turn a linear convolution of finite signal to a circular convolution (if we have a signal with length N and another with length M we can pad them to a length of $N+M-1$ and the cyclic convolution would be the same as the linear, the generalization to images is simple). So the goal of this method is to pad the signal enough such that the cyclic and linear convolution would be the same.

In the second image we duplicated the image and by that made the signal periodic (but finite). What it means is that if the original image is $x[n,m]$ and the duplicated image is $\tilde{x}[n,m]$, the relation between both of the images would be $\tilde{x}[n,m] = x\left[\langle n\rangle_N, \langle m\rangle_M\right]$ (of course for n, m in the defined regions of $\tilde{x}$). The reason this is so useful to us is that now the linear convolution of $\tilde{x}[n,m]$ and $h[n,m]$ would act as a circular convolution of $x[n,m]$ and $h[n,m]$.
So we can conclude that the images were build such that the linear convolution would be equal to the cyclic convolution, so we expect to reach good results in the restoration.
After looking at the results we can detect is the basic image the images are based of, we can see it a black image with our names in white.
The restored first image contains the basic image at the left corner and the rest of the image is black, which represent the padded area.
The restored second image contains four replicas of the basic image.
The results look the way we expected them to be, which insures that these images do satisfy our basic assumption and we can restore them using a cyclic convolution instead of linear.
(i) **Think of a different way to restore $\boldsymbol{x_2[n,m]}$:**
We know that $x_2[n,m]$ is a duplicate of four identical images.
It means that we can represent it as

$$x_2[n,m] = \begin{cases} x[n,m] & 0 \le n < N, 0 \le m < M \\ x[n-N,m] & N \le n < 2N, 0 \le m < M \\ x[n-N,m-M] & N \le n < 2N, M \le m < 2M \\ x[n,m-M] & 0 \le n < N, M \le m < 2M \end{cases}$$

Let's find it's DFT

$$X_2[k_1,k_2] = \sum_{n=0}^{2N-1}\sum_{m=0}^{2M-1} x_2[n,m]\, e^{-j\left(\frac{2\pi k_1}{2N}n + \frac{2\pi k_2}{2M}m\right)}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} x[n,m]\, e^{-j\left(\frac{2\pi k_1}{2N}n + \frac{2\pi k_2}{2M}m\right)} + \sum_{n=N}^{2N-1}\sum_{m=0}^{M-1} x[n-N,m]\, e^{-j\left(\frac{2\pi k_1}{2N}n + \frac{2\pi k_2}{2M}m\right)}$$

$$+ \sum_{n=N}^{2N-1}\sum_{m=M}^{2M-1} x[n-N,m-M]\, e^{-j\left(\frac{2\pi k_1}{2N}n + \frac{2\pi k_2}{2M}m\right)} + \sum_{n=0}^{N-1}\sum_{m=M}^{2M-1} x[n,m-M]\, e^{-j\left(\frac{2\pi k_1}{2N}n + \frac{2\pi k_2}{2M}m\right)}$$

$$= \sum_{n=0}^{N-1}\sum_{m=0}^{M-1} x[n,m]\, e^{-j\left(\frac{2\pi k_1}{2N}n + \frac{2\pi k_2}{2M}m\right)} \left(1 + (-1)^{k_1} + (-1)^{k_1+k_2} + (-1)^{k_2}\right)$$

Let's divide into cases.

If one of the indices is odd (without lose of generality let's assume $k_1 \to 2k_1+1$) we'll get

$$X_2\left[2k_1+1,k_2\right]=\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}x\left[n,m\right]e^{-j\left(\frac{2\pi k_1}{N}n+\frac{2\pi k_2}{2M}m\right)}\left(1-1-(-1)^{k_2}+(-1)^{k_2}\right)=0$$

If both of the indices are even we'll get

$$X_2\left[2k_1,2k_2\right]=4\sum_{n=0}^{N-1}\sum_{m=0}^{M-1}x\left[n,m\right]e^{-j\left(\frac{2\pi k_1}{N}n+\frac{2\pi k_2}{M}m\right)}=4X\left[k_1,k_2\right]$$

This tells us that

$$Y_2\left[k_1,k_2\right]=\begin{cases}0 & k_1 \text{ or } k_2 \text{ odd}\\ 4X\left[\frac{k_1}{2},\frac{k_2}{2}\right]H\left[k_1,k_2\right] & k_1 \text{ and } k_2 \text{ even}\end{cases}$$

So we can conclude that all of the information is found in the even places, so we can look at

$$Y_2\left[2k_1,2k_2\right]=4X\left[k_1,k_2\right]H\left[2k_1,2k_2\right]$$

and this would just be applying the filter on the basic image and multiplying by 4 (the filter $H\left[2k_1,2k_2\right]$ is the same filter then doing DFT half of the length).
Let's conclude what we just proved, the information of the duplicated signal is located in the places where k_1,k_2 are even.
This information is exactly the same information (by factor 4) contained in the basic image $X\left[k_1,k_2\right]$.
The information in the distorted image is the same information as the information of the distorted basic image.
What we can infer from all of this is that we can apply the same process we did before, but this time only on quarter of the image (which is the basic distorted image) and duplicate the results four times and we'll get the same results.
We can do this idea and we'll get the following result

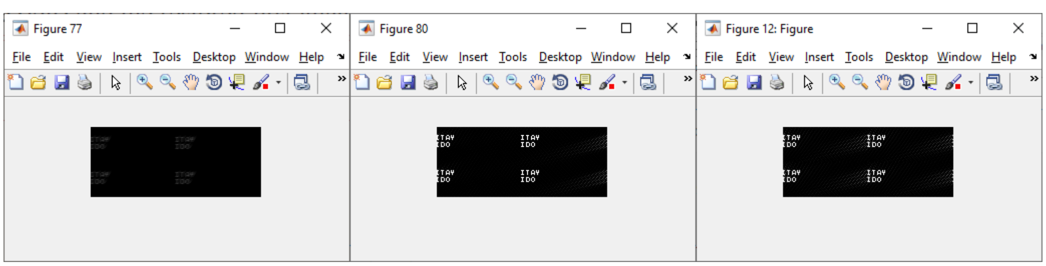

Figure 6: The original image (left) the image after restoring in the first way(middle) the image after restoring in the second way (right)

We can see that the restoration is almost identical, we can see that the second method is more noisy, that's because we took the noise we got from the restoration of the basic image and duplicated it four times.

Part 3- Generating and Processing of a Speech Signal

In this part we'll examine the effect of noise added to an audio signal.
We'll work with an audio signal which sampled at 16 kHz.
First, we'll take a speech signal, and cut it to be at length of 2^{16} values.
(a) **Calculate the average power of the signal:**
The average power is

$$P = \frac{1}{N}\sum_{n=0}^{N-1}(x[n])^2$$

we can easily do it using the build-in bandpower() function of MATLAB.
(b) Lets assume the noise disturbance signal

$$z[n] = 50\sqrt{P_x}\left[\cos(\omega_1 n) + \cos(\omega_2 n) + \cos(\omega_2 n)\right]$$

where $\omega_1 = 1.6 + 0.1d_1$, $\omega_2 = 1.6 + 0.1d$ and $\omega_3 = 3$.
We'll define the input signal $y[n] = x[n] + z[n]$
Listen to $\boldsymbol{y}[\boldsymbol{n}]$ and assess the quality of the speech signal:
hearing the input signal y[n] sounds just like noise.
we can't distinguish the original speech signal $x[n]$ from it i.e. it is poor quality speech signal.
(c) **Calculate and draw the input signal $\boldsymbol{y}[\boldsymbol{n}]$:**
We know that by definition:

$$y[n] = x[n] + z[n] = x[n] + 50\sqrt{P_x}\left[\cos(\omega_1 n) + \cos(\omega_2 n) + \cos(\omega_3 n)\right]$$

and by knowing $x[n]$ we can calculate in MATLAB the value of $y[n]$ for every n (just like was done in the previous part).
We'll get the following result

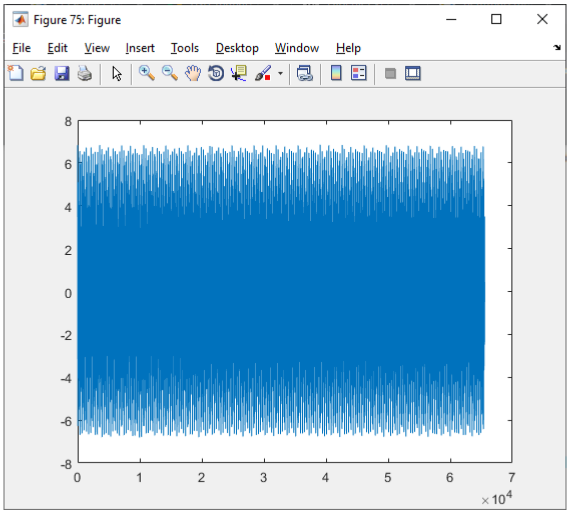

Figure 7: The noised signal as a function of the time

(d) **Calculate and draw the discrete time Fourier transform $Y\left(e^{j\omega}\right)$of $y\left[n\right]$:**
We'll calculate it using the Fourier transform's linearity property:

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) + Z\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) + \mathscr{F}\left\{50\sqrt{P_x}\left[\cos\left(\omega_1 n\right) + \cos\left(\omega_2 n\right) + \cos\left(\omega_3 n\right)\right]\right\}$$

Recall that DTFT of a cosine function is:

$$\mathscr{F}\left\{cos\left(\omega_0 n\right)\right\} = \pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

Using this formula and linearity of DTFT we can state:

$$Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) + 50\pi\sqrt{P_x}\left[\delta\left(\omega-\omega_1\right)+\delta\left(\omega+\omega_1\right)+\delta\left(\omega-\omega_2\right)+\delta\left(\omega+\omega_2\right)+\delta\left(\omega-\omega_3\right)+\delta\left(\omega+\omega_3\right)\right]$$

Therefore we expect to get six Kronecker delta functions in the corresponding frequencies $\pm\omega_i \forall i \in \{1,2,3\}$added to the transform of the original signal.
We can do it in MATLAB and we'll get

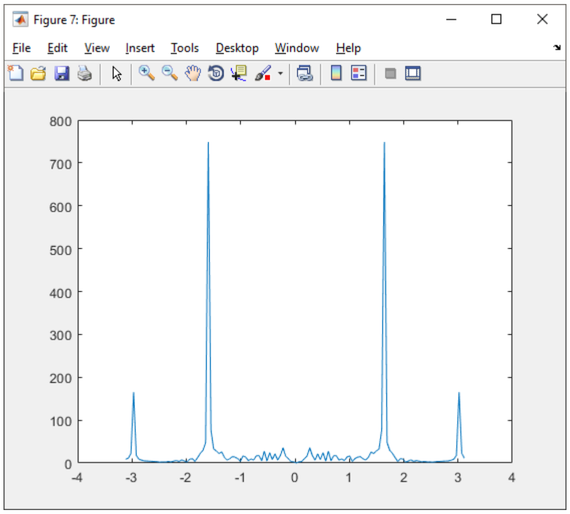

Figure 8: The noised signal as a function of the time

In the figure we can see that we have only four deltas, that's because two of the deltas are close, and we sample the signal in the domain (plotting only $\omega = \frac{2\pi}{128}k$ where $k \in \mathbb{Z}$) so we're missing two deltas in the sample. It's important to note that this sample occurs only when plotting, and the signal is not affected by it.

Anther thing this sampling creates is a height difference between the deltas, we can see that the deltas don't have the size size, even though they should have. That's because we didn't sample exactly at the right value, and the numerical errors couldn't give us a function with exactly delta, that's why we got a smaller delta than expected.

(e) The signals $y_2[n]$ and $z_2[n]$ are obtained by decimation with ratio of 2 of $y[n]$ and $z[n]$.

Write mathematical expression of $z_2[n]$ and its DTFT $Z_2(e^{j\omega})$:

decimation by factor of 2 means keeping only every second sample of the signal.

i.e.

$$z_2[n] = z[2n] = 50\sqrt{P_x}\left[\cos(2\omega_1 n) + \cos(2\omega_2 n) + \cos(2\omega_3 n)\right]$$

So we'd expect to get deltas at $\pm 2\omega_i$ for $i = 1, 2, 3$.

We can easily see that all of these frequencies are greater than pi, so we'll get the deltas which originally were between π and 3π.

We'll get that the deltas will be at $\pm(2\omega_i - 2\pi) = \pm[3.04, 3.07, 0.28]$

We can also see that from the formula, recall that decimation by factor of M is:

$$y[n] = x[nM] \iff Y\left(e^{j\omega}\right) = \frac{1}{M}\sum_{m=0}^{M-1} X\left(e^{j\frac{\omega - 2\pi m}{M}}\right)$$

hence, in the domain the $Z_2(e^{j\omega})$ is equal to:

$$Z_2\left(e^{j\omega}\right)=\frac{1}{2}\sum_{m=0}^{1}Z\left(e^{j\frac{\omega-2\pi m}{2}}\right)=\frac{Z\left(e^{j\frac{\omega}{2}}\right)+Z\left(e^{j\frac{\omega-2\pi}{2}}\right)}{2}$$

In our case, between $-\pi$ and π $Z\left(e^{j\frac{\omega}{2}}\right)=0$ which means that

$$\boxed{Z_2\left(e^{j\omega}\right)=\frac{1}{2}Z\left(e^{j\frac{\omega-2\pi}{2}}\right)}$$

Let's look at what we got

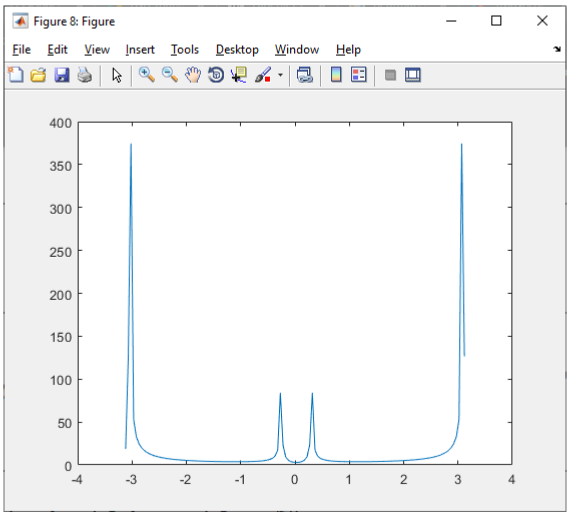

Figure 9: The signal $Z_2\left(e^{j\omega}\right)$

We can say a similar thing about the sampling as said before, which is why we got only two deltas with different heights.

(f) **Draw $y_2[n]$ and its DTFT:**

We can plot the signals and we'll get

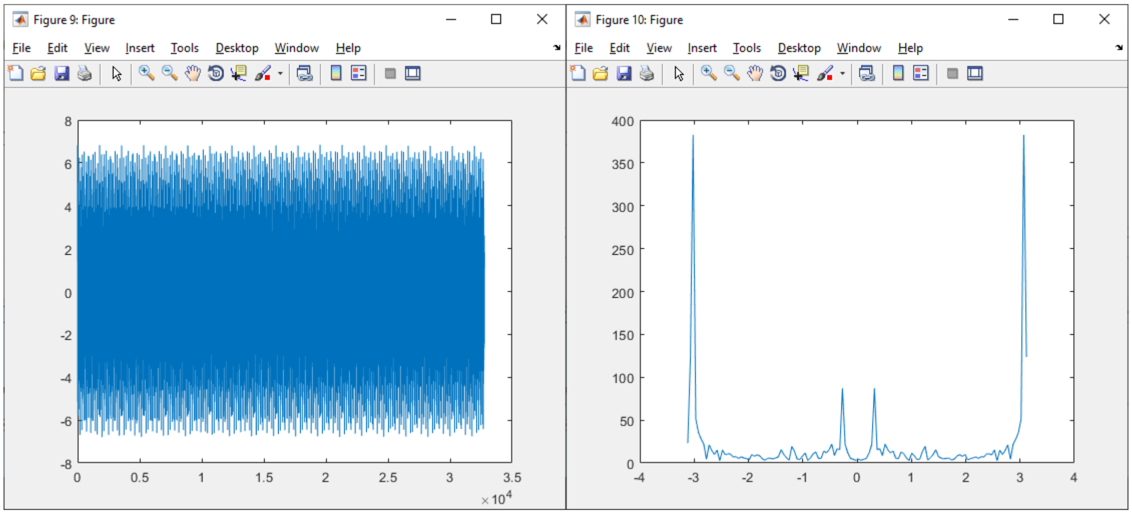

Figure 10: The noised signal in time (left) and it's DTFT (right)

The results do make sense, in time we can't really understand the signal but in the domain we can see the deltas of $Z_2\left(e^{j\omega}\right)$ added to the signal $X\left(e^{j\omega}\right)$.

Listen to $\boldsymbol{y}\left[\boldsymbol{n}\right]$, assess the quality of the speech signal:

After listening to the $y_2\left[n\right]$ signal we can sense an improvement compared to hearing $y\left[n\right]$ signal.
Nevertheless, the quality of the speech signal is poor, it still sounds like a noise.