Transform Newton To Taylor-Like

Let $p_N(t)$ be a polynomial of degree N in Newton representation

$$p_N(t) = \sum_{i=0}^{N} a_i r_i(t) \tag{1}$$

where $r_0(t) = 1$,

$$r_{i+1}(t) = (t - z_i)\, r_i(t), \qquad 0 \le i \le N-1, \tag{2}$$

and we want to write it in the following form

$$p_N(t) = \sum_{j=0}^{N} s_j \frac{t^j}{j!}. \tag{3}$$

$r_i(t)$ can be written as

$$r_i(t) = \sum_{j=0}^{i} q_{i,j} \frac{t^j}{j!}. \tag{4}$$

Hence

$$r_{i+1}(t) = \sum_{j=1}^{i+1} q_{i,j-1} \frac{t^j}{(j-1)!} - z_i \sum_{j=0}^{i} q_{i,j} \frac{t^j}{j!}. \tag{5}$$

Since

$$r_{i+1}(t) = \sum_{j=0}^{i+1} q_{i+1,j} \frac{t^j}{j!} \tag{6}$$

we get the following reccurence relation

$$q_{i+1,0} = -z_i q_{i,0}, \tag{7}$$

$$q_{i+1,j} = j q_{i,j-1} - z_i q_{i,j}, \qquad 1 \le j \le i, \tag{8}$$

$$q_{i+1,i+1} = (i+1)q_{i,i}. \tag{9}$$

Now, let us write

$$p_i(t) = \sum_{j=0}^{i} s_{i,j} \frac{t^j}{j!}. \tag{10}$$

Since these polynomials satisfy

$$p_{i+1}(t) = p_i(t) + a_{i+1} r_{i+1}(t) \tag{11}$$

we get

$$\sum_{j=0}^{i+1} s_{i+1,j} \frac{t^j}{j!} = \sum_{j=0}^{i} s_{i,j} \frac{t^j}{j!} + a_{i+1} \sum_{j=0}^{i+1} q_{i+1,j} \frac{t^j}{j!}. \tag{12}$$

Hence, upon defining $s_{i,i+1} = 0$, we get

$$s_{0,0} = a_0, \;\; q_{0,0} = 1 \tag{13}$$

and, for $1 \leq i \leq N-1$,

$$s_{i+1,j} = s_{i,j} + a_{i+1} q_{i+1,j}, \qquad 0 \leq j \leq i+1 \tag{14}$$

where $q_{i+1,j}$ are computed by (7)- (9)