Dissipative Wave Equation

Let us consider the following equation

$$u_{tt} = u_{xx} - \mu u_t + s(x,t), \qquad u(x,t=0) = u^0, \qquad u_t(x,t=0) = u_t^0, \qquad \mu > 0. \tag{1}$$

After space discretization we get a system of second order differential equations

$$u_{tt} = Du - \mu u + s(t) \tag{2}$$

where, now, u and s are vectors and D is a matrix which carries out the space differentiation. Upon deining the vectors

$$u_1 = u, \qquad u_2 = u_t \tag{3}$$

the equation above can be written as

$$w_t = Gw + f \tag{4}$$

where

$$G = \begin{pmatrix} 0 & I \\ D & -\mu I \end{pmatrix},$$

$$w = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

$$f = \begin{pmatrix} 0 \\ s \end{pmatrix}.$$

It is easily verified that the eigenvalues of G are on the left side of the complex plane.

Numerical experiment

Let us consider the equation

$$u_{tt} = u_{xx} - \mu u_t + 10\mu \cos\left(10\left(x+t\right)\right), \qquad u(x,0) = \sin(10x), \qquad u_t(x,0) = 10\cos(10x). \tag{5}$$

The exact solution of this problem is

$$u(x,t) = \sin\left(10\left(x+t\right)\right). \tag{6}$$

We have solved this equation in the time interval [0 10]. It took RK4 12000 matrix-vector multiplications to compute the solution with accuracy 10^{-6}. On the other hand, we needed only 360 matrix-vector multiplications with the new algorithm.