A presentation on:

The Beltrami framework with relevance to texture processing

Numerical Geometry of Images, 2014

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March 12, 2014

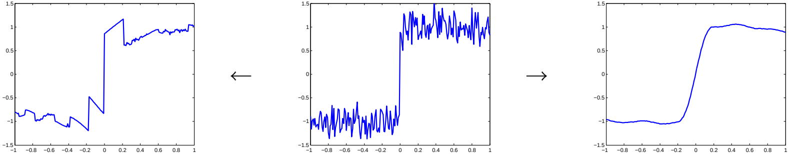
Background

Image filtering: Recover $\hat{I}$, best estimate for I_0 given I, where:

$$I(x,y) = I_0(x,y) + N(x,y), \tag{1}$$

and $N(x,y)$ is i.i.d. Gaussian noise.
Filtering in the case of Gray-level images ($I : \mathbb{R}^2 \to \mathbb{R}$):

- L_2–based filtering: Heat equation, Gaussian filtering.
- L_1–based filtering: Total–Variation.



What about multi-channel images?

Color images (three channels)

- Analysis of R,G,B separately.
  - Dependency between channels.
- Decomposing for Hue, Saturation, Brightness.
  - How to filter H, S channels?

Movies (multi "channel")

- How to analyze a sequence of N frames?

⋆ Better processing can be performed by incorporating dependencies between different channels.

⋆ Assumption: The gradients of the original image are aligned, and the gradients of the noise or artefacts are not.

Manifolds [Sochen et al., 1998, Sochen et al., Kimmel et al., 2000a]

- A 128×128 gray-level image has 2^{14} degrees of freedom.
- True dimension: Considerably less.
- Exploiting redundancy.

For instance, a surface:

$$U(x,y) = ax^2 + by^2, (x,y) \in (0,128)^2 \tag{2}$$

True dimension: 2, the parameter set (a, b).
Present color images as a manifold:

$$I : \underbrace{(x,y)}_{X} \mapsto \underbrace{(x,y,R,G,B)}_{Y},$$

a 2D surface embedded in 5D space.
A Riemannian manifold is a manifold coupled with a metric:

- The distance can be calculated on the coordinate space and on the feature space.
- $ds_X^2 = (x,y)\begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix}(x,y)^T = (x,y)g(x,y)^T$
- $ds_Y^2 = (x,y,R,G,B)h(x,y,R,G,B)^T$, $H \in \mathbb{R}^{5\times 5}$.
- $g, h \succeq 0$ and symmetric.

The energy functional

The energy on the manifold is defined as:

$$E(I) = \int dxdy \sqrt{g} \sum_{i,j} g^{i,j} \sum_{k,l} \frac{\partial y^k}{\partial x^i} \frac{\partial y^l}{\partial x^j} h_{k,l},$$

Example: $h_{i,j} = \delta_{i,j}$ and $h_{3,3} = \beta$, gray-level image, and demanding $ds_X^2 = ds_Y^2$ we obtain (via the pullback matrix):

$$g = \begin{pmatrix} 1+\beta^2 I_x^2 & \beta^2 I_x I_y \\ \beta^2 I_x I_y & 1+\beta^2 I_y^2 \end{pmatrix} \tag{3}$$

$$\sqrt{g} \triangleq \sqrt{\det(g)} = \sqrt{1 + \beta^2(I_x^2 + I_y^2)} = \sqrt{1 + \beta^2 |\nabla I|^2} \tag{4}$$

The energy integral yields:

$$E(I) = 2 \int \sqrt{1 + \beta^2 |\nabla I|^2} dxdy \tag{5}$$

- $\beta \ll 1 \Rightarrow$ minimizing area independently of image content.
- $\beta \gg 1 \Rightarrow$ Total-Variation minimization (in gray-level images).

Remark

Minimizing this energy yields an intermediate optimization between known schemes (such as the mean curvature flow or TV), generalized on manifolds.

The Beltrami flow

Optimization

Minimizing via Euler-Lagrange, we obtain a *Diffusion* flow:

$$R_t = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{i,j} \frac{\partial R}{\partial x^j} \right) \tag{6}$$

$$G_t = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{i,j} \frac{\partial G}{\partial x^j} \right) \tag{7}$$

$$B_t = \frac{1}{\sqrt{g}} \sum_{i,j} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{i,j} \frac{\partial B}{\partial x^j} \right) \tag{8}$$

Or, in short: $\boldsymbol{I}_t^k = \Delta_g \boldsymbol{I}^k$, where $\boldsymbol{I} = (R, G, B)$ and Δ_g is the laplacian w.r.t the manifold with metric $\boldsymbol{g}$.

$\star$ $\boldsymbol{g}$ represents the coupling between the different color channels.

Implementation

This tensor diffusion can be performed efficiently, similar to nonlinear diffusion.
Example: On smartphones [Wetzler and Kimmel, 2012]

Applications

Can be used directly for denoising:

$\beta = 0.001$ $\beta = 100$

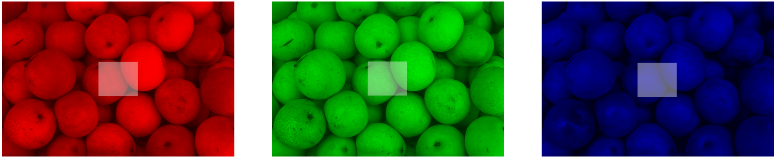
Can also be used as regularization for deblurring and other image processing problems.

Extensions (I) [Wetzler and Kimmel, 2012]

Patch-space:

$$(x, y) \mapsto (x, y, \{R_i\}, \{G_i\}, \{B_i\}), \tag{9}$$

where $\{X_i\}$ is a patch in the image.



In this case, $ds_Y^2 = dx^2 + dy^2 + \beta^2 \sum_{k \in (R,G,B)} \sum_{i,j} dI_{i,j}^{k\,2}$.

$\star$ Expansion: Space varying β with Gaussian kernel.

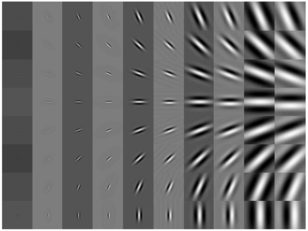
Extensions (I) [Wetzler and Kimmel, 2012]

Patch-space – example:

Left: Image with aliasing. Middle: Beltrami flow. Right: BM3D.

Extensions (II) [Kimmel et al., 2000b]

Texture processing via the Wavelet domain:



$$(x, y) \mapsto (x, y, \theta, R, J), \tag{10}$$

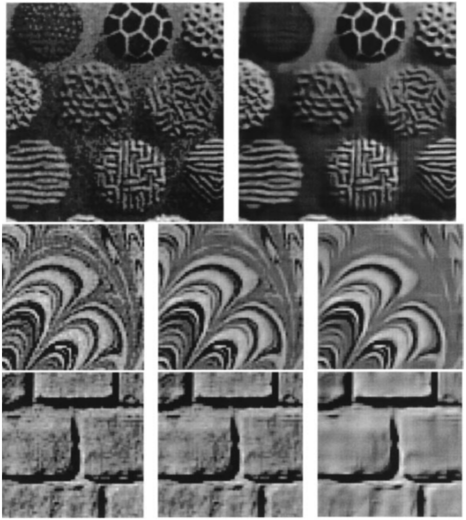
where θ is the angle, and $R + \sqrt{-1}J$ are the complex wavelet transform coefficients.

In this case, $ds_Y^2 = dx^2 + dy^2 + d\theta^2 + dR^2 + dJ^2$.

$\star$ The processing was performed independently in each wavelet scale and subsequently combined.

Extensions (II) [Kimmel et al., 2000b]

Texture processing via the Wavelet domain – example:



Extensions (III) [Sochen and Zeevi, 1998]

Representation of color images in a non-Euclidean space:

- Applying Karhunen-Loeve transform (KLT) on (R, G, B) to obtain uncorrelated color channels.
- $\underline{k} = M(R, G, B)^T$. k_1 is the luminance channel and k_2, k_3 are the chrominance channels.
- Additionally, a non-Euclidean metric is used:

$$ds^2 = dx^2 + dy^2 + \beta^2(k_1)dk_1^2 + c_1 dk_2^2 + c_2 dk_3^2, \tag{11}$$

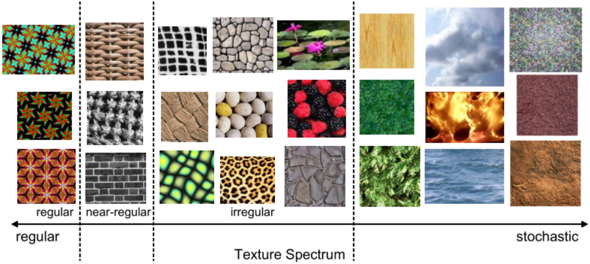
  where β^2 for the luminance channel (k_1) is an increasing function of k_1.
- Required the introduction of Levi-Civita connection coefficients, which yield the following diffusion flow:

$$k_{1,t} = \Delta_{g(\beta)} k_1 + \Gamma^1_{11}(g^{11}, {k_1}_x^2 + 2g^{12} {k_1}_x {k_1}_y + g^{22} {k_1}_y^2) \tag{12}$$
$$k_{2,t} = \Delta_{g(\beta)} k_2 \tag{13}$$
$$k_{3,t} = \Delta_{g(\beta)} k_3 \tag{14}$$

- High diffusivity in high luminence values.

Texture processing

Types of textures [Lin et al., 2004]



- Left-hand and middle: Edge-dominated or repetitive textures
  - L_1-based produces good results.
  - Sparsity-based with oscillatory dictionaries.
- Right-hand side: Stochastic
  - Gaussian statistics, high frequencies not dominated by edges.
  - Spectral resemblance to noise.

⋆ **Specific features for stochastic textures?**

Remaining tasks

- Develop a manifold for stochastic textures:
  - Dominant features
  - Optimization
- Implementation:
  - MATLAB
  - OpenCV on Android.
    - Partially done – framework for tensor diffusion.

Bibliography

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