MATHEMATICS

Interesting implicit surfaces in $\mathbb{R}^3$

Asked 11 years, 6 months ago Modified 8 months ago Viewed 7k times

28

I have just written a small program in C++ and OpenGl to plot implicit surfaces in $\mathbb{R}^3$ for a Graphical Computing class and now I'm in need of more interesting surfaces to implement!

Some that I've implemented are:

- Basic surfaces like spheres and cylinders;
- Nordstrand's Weird Surface;
- Klein Quartic;
- Goursat's Surface;
- Heart Surface;

So, my question is, what are other interesting implicit surfaces in $\mathbb{R}^3$?

P.S.: I know this is kind of vague, but anything you find interesting will be of use. (:

P.P.S: Turn this into a community wiki, if need be.

algebraic-geometry multivariable-calculus visualization surfaces

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edited Jun 20, 2011 at 21:52

community wiki
8 revs, 5 users 64%
Leonardo Fontoura

2 There is a wealth of documentation on Wolfram's Mathworld: mathworld.wolfram.com/topics/Surfaces.html – barf Jun 19, 2011 at 1:38

Boy's surface.. – Cheerful Parsnip Jun 19, 2011 at 1:45

If you type "mathematical surfaces" into Google you get a near endless list, including lots of raytraced models, for example: math.rug.nl/models – Ryan Budney Jun 19, 2011 at 3:13

math.rug.nl/models/Clebsch.html is beautiful. Now I want one! Actually: can someone give me the equations of the lines in the Clebsh surface? – Mariano Suárez-Álvarez Jun 19, 2011 at 3:24

@Mariano: I can't but Oliver Lab can (in German) see in particular chapter 3, but the general form looks rather messy. He has some cool pictures on his page algebraicsurface.net – t.b. Jun 19, 2011 at 15:40

4 Answers

Sorted by: Highest score (default)

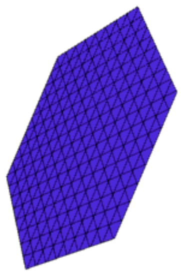
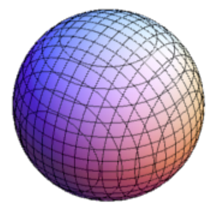
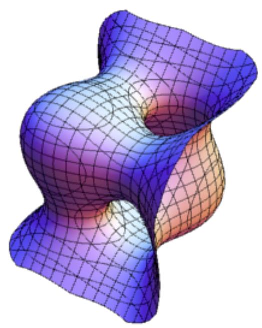
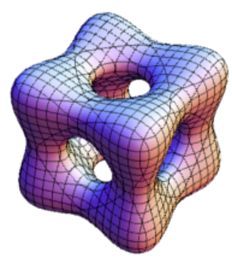
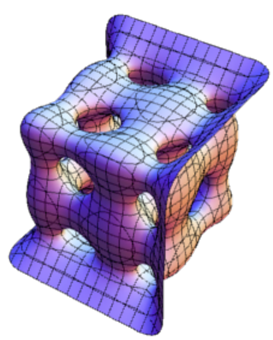
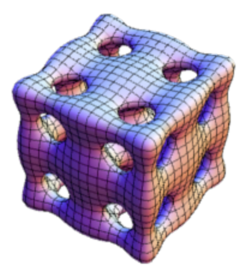
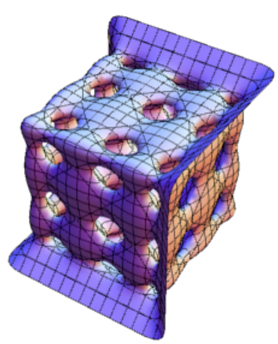
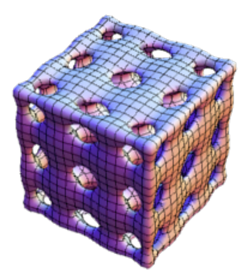
62

A really nice family of implicit surfaces in $\mathbb{R}^3$ are the Banchoff-Chmutov surfaces

$$BC_n := \{(x,y,z) \in \mathbb{R}^3\,;\;\; T_n(x) + T_n(y) + T_n(z) = 0\},$$

where T_n denotes the n-th Chebishev polynomial of the first kind, i.e.

$$T_n(x) := \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k}.$$

-  
  $\leftarrow BC_1 \quad BC_2 \rightarrow$
-  
  $\leftarrow BC_3 \quad BC_4 \rightarrow$
-  
  $\leftarrow BC_5 \quad BC_6 \rightarrow$
-  
  $\leftarrow BC_7 \quad BC_8 \rightarrow$

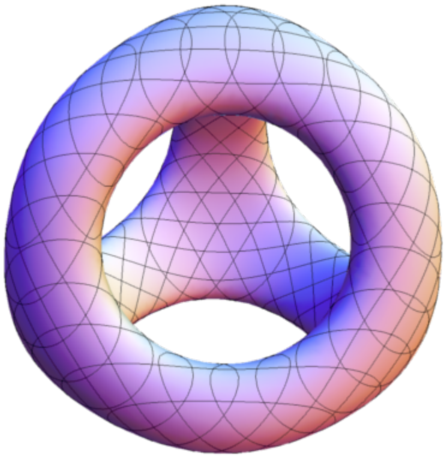
What is the genus of $\mathbf{BC_{2n+2}}$? Code for plotting surfaces BC_n in Mathematica 7:

```
BanchoffChmutov[n_] := ContourPlot3D[ChebyshevT[n,x]+ChebyshevT[n,y]+ChebyshevT[n,z], {x,-1.3,1.3}, {y,-1.3,1.3}, {z,-1.3,1.3},
Contours->0.02, AspectRatio->Automatic, Boxed->False, Axes->{False,False,False}, BoxRatios->Automatic, PlotRangePadding->None,
PlotPoints->30, ViewPoint->{-2,3,3}]
surfacesBCn = Table[BanchoffChmutov[i], {i, 8}]
Table[ChebyshevT[n,x]+ChebyshevT[n,y]+ChebyshevT[n,z], {n, 8}]
```

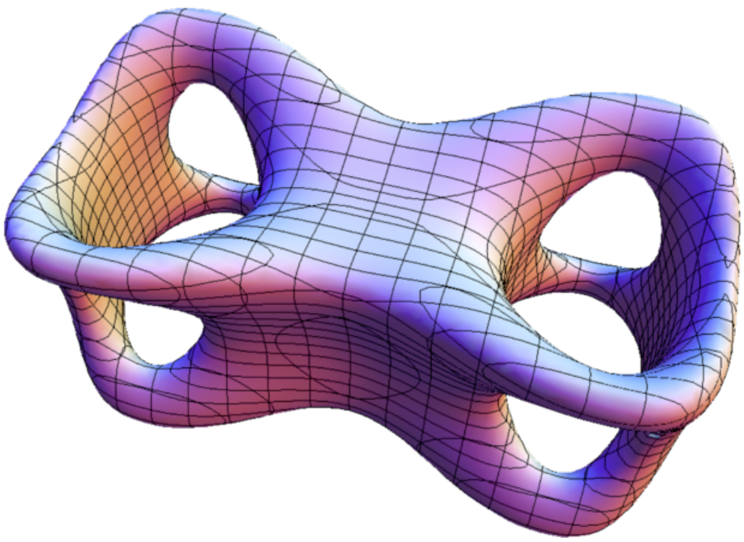
Other miscellaneous increasingly pretty examples, that I managed to construct myself, include:

$$\{(x,y,z) \in \mathbb{R}^3\,; \quad (x-2)^2(x+2)^2 + (y-2)^2(y+2)^2 + (z-2)^2(z+2)^2 +$$

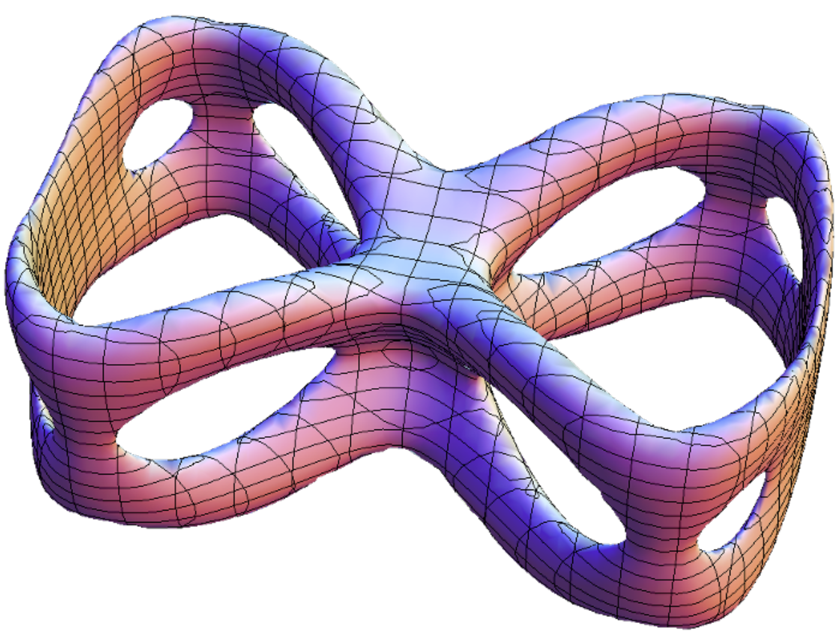
$$3(x^2y^2 + x^2z^2 + y^2z^2) + 6 * x * y * z - 10(x^2 + y^2 + z^2) + 22 = 0\}$$



$$\{(x, y, z) \in \mathbb{R}^3 ; \quad ((x-1)x^2(x+1) + y^2)^2 + ((y-1)y^2(y+1) + z^2)^2 + 0.1y^2 + 0.05(y-1)y^2(y+1) = 0\}$$

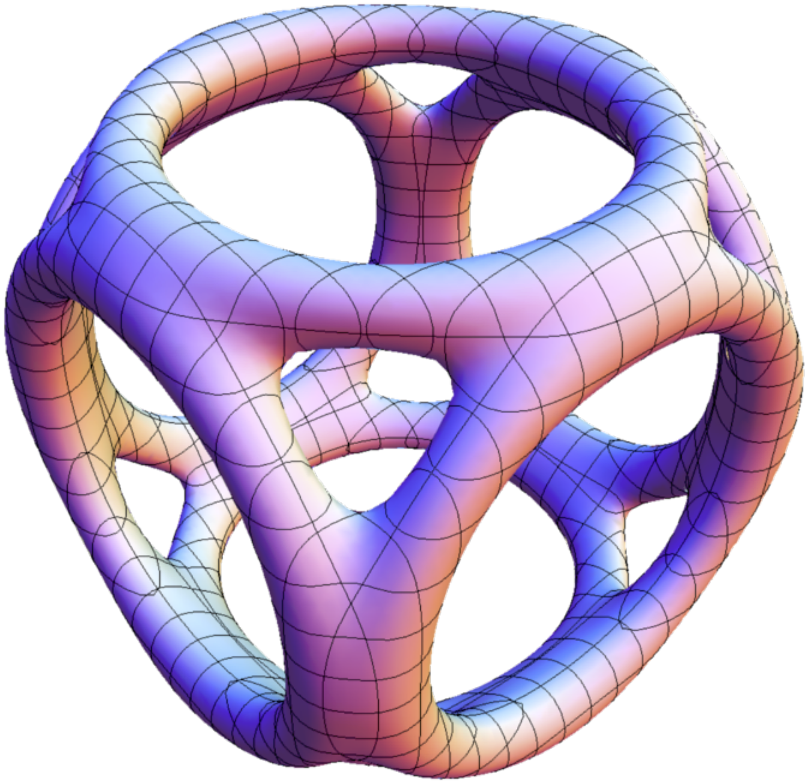


$$\{(x, y, z) \in \mathbb{R}^3 ; \quad 15((x-1.2)x^2(x+1.2) + y^2)^2 + 0.8(z-1)z^2(z+1) - 0.1z^2 + 20((y-0.8)y^2(y+0.8) + z^2)^2 + 0.8(x-1)z^2(x+1) - 0.1x^2 = 0\}$$

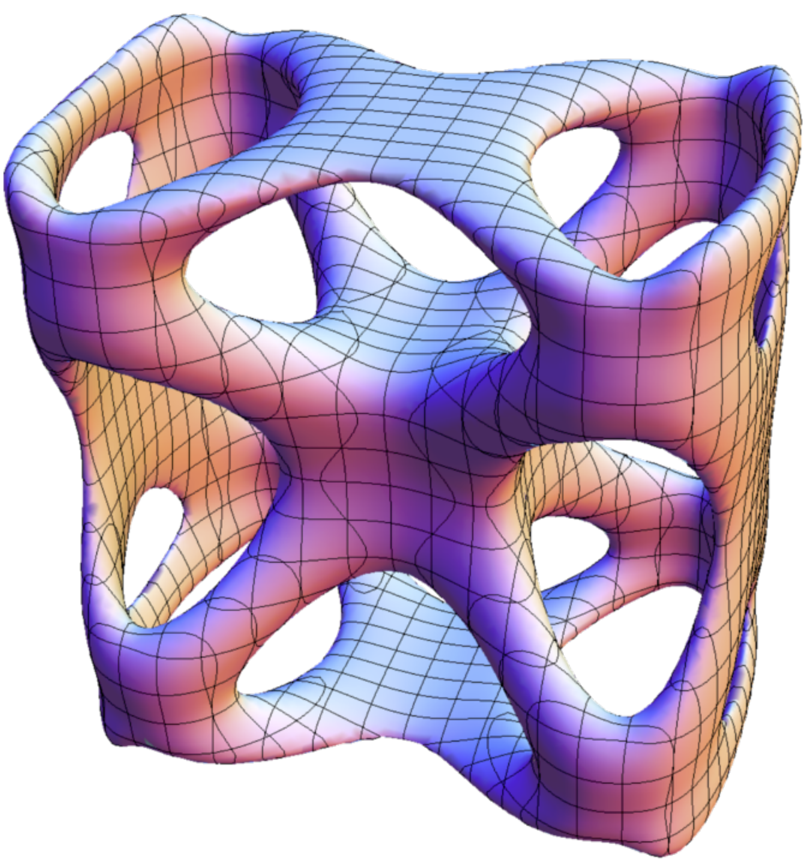


$$\{(x, y, z) \in \mathbb{R}^3 ; \quad ((x^2 + y^2 - 0.85^2)^2 + (z^2 - 1)^2) *$$

$$\begin{aligned}&((y^2+z^2-0.85^2)^2+(x^2-1)^2)*\\&((z^2+x^2-0.85^2)^2+(y^2-1)^2)-0.001=0\}\end{aligned}$$

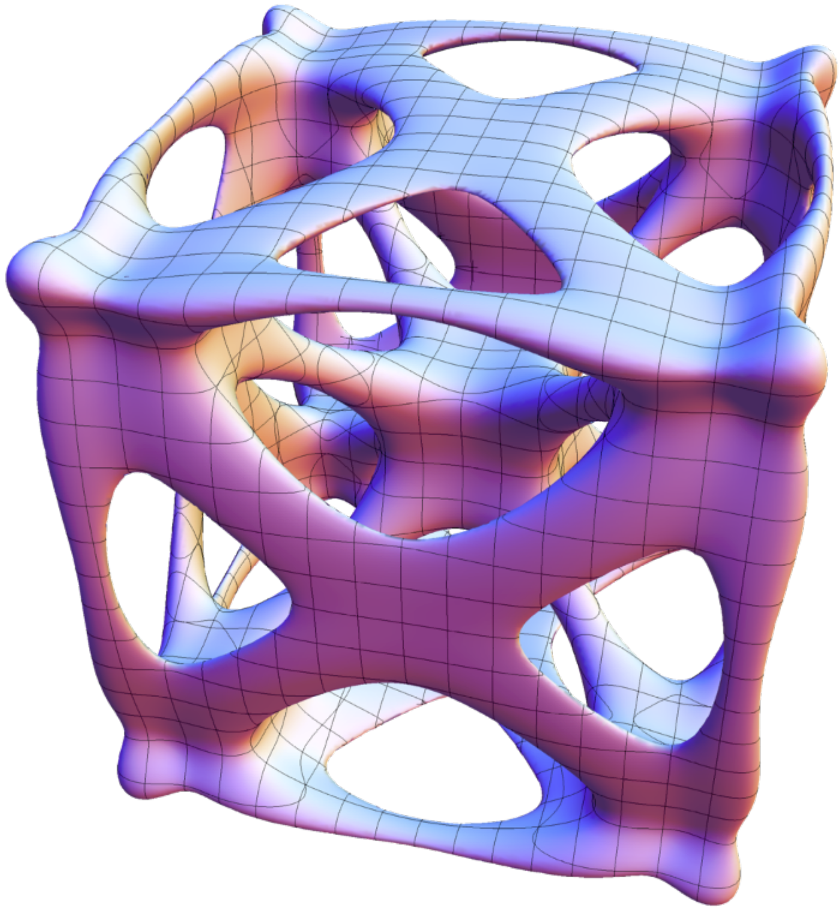


$$\{(x,y,z)\in\mathbb{R}^3\ ;\quad \begin{aligned}&(3(x-1)x^2(x+1)+2y^2)^2+(z^2-0.85)^2*\\&(3(y-1)y^2(y+1)+2z^2)^2+(x^2-0.85)^2*\\&(3(z-1)z^2(z+1)+2x^2)^2+(y^2-0.85)^2*-0.12=0\}\end{aligned}$$



$$\{(x,y,z)\in\mathbb{R}^3\ ;\quad \begin{aligned}&(2.92(x-1)x^2(x+1)+1.7y^2)^2*(y^2-0.88)^2+\\&(2.92(y-1)y^2(y+1)+1.7z^2)^2*(z^2-0.88)^2+\\&(2.92(z-1)z^2(z+1)+1.7x^2)^2*(x^2-0.88)^2-0.02=0\}\end{aligned}$$

$$(2.92(z-1)z^2(z+1)+1.7x^2)^2 * (x^2-0.88)^2 - 0.02 = 0\}$$



Hope you enjoy them...

Note also, that all of these examples have the form $\{(x,y,z)\in\mathbb{R}^3;\ P(x,y,z)=0\}=P^{-1}(0)$. They are 2-manifolds (surfaces). To see this not only with your eyes but also **in theory**, compute the matrix

$$\left[\frac{\partial P}{\partial x},\frac{\partial P}{\partial y},\frac{\partial P}{\partial z}\right]$$

at all the points of $P^{-1}(0)$ and find out (haven't done this myself) that the matrix is non-zero, which by implicit function theorem means that $P^{-1}(0)$ is a $3-1=2$ manifold.

Also, if you change the definition of $\{(x,y,z)\in\mathbb{R}^3;\ P(x,y,z)=0\}=P^{-1}(0)$ to

$$\{(x,y,z)\in\mathbb{R}^3;\ P(x,y,z)\leq 0\}$$

or

$$\{(x,y,z)\in\mathbb{R}^3;\ P(x,y,z)\geq 0\},$$

you get a 3-manifold (because 0 is a *regular value* of P) with the boundary $\{(x,y,z)\in\mathbb{R}^3;\ P(x,y,z)=0\}$, i.e. the surface $P^{-1}(0)$ with either it's interior or exterior "filled".

ADDITION (how to construct such surfaces): @Soarer, You never try to guess such a complicated polynomial. Here's the key to constructing such surfaces: step by step. You discover on the web, that the torus is

$$\{P(x,y,z):=(x^2+y^2-0.7^2)^2+z^2=0\}.$$

Then you learn that

$$\{P(x-a,y-b,z-c)=0\}$$

is torus, translated by $(a,b,c)\in\mathbb{R}^3$. Also, you notice, that

$$\{P(y,z,x)=0\}\text{ and }\{P(z,x,y)=0\}$$

is a torus with coordinate lines permuted (rotated by 90°) and

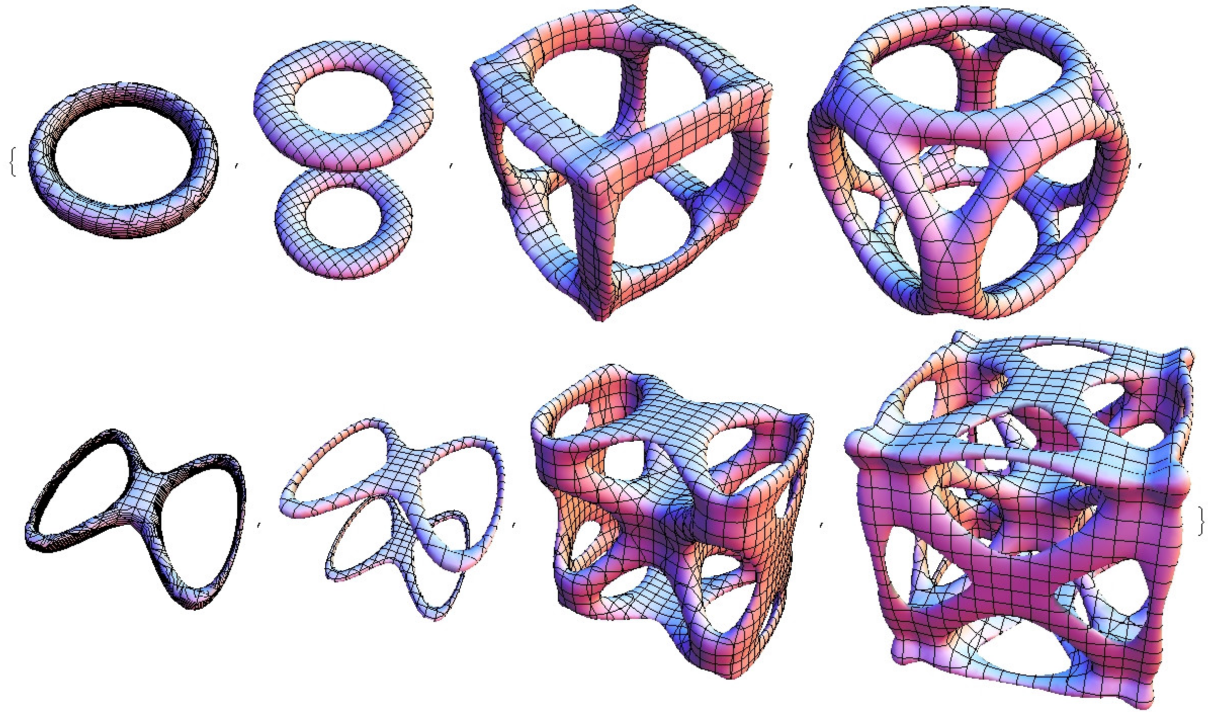
$$\{P(ax,by,cz)=0\}$$

is a torus, stretched in x-direction by factor $1/a$, etc. Next, you see that

$$\{P(x,y,z)\cdot Q(x,y,z)=0\}=\{P(x,y,z)=0\}\cup\{Q(x,y,z)=0\}.$$

Using all these techniques step by step, and being very patient, you manage to discover the following construction

```
implicitSurface[p_String, zoom_, quality_Integer, view_List] :=
  ContourPlot3D[ToExpression[p], {x, -zoom, zoom}, {y, -zoom, zoom}, {z, -zoom, zoom}, Contours → 0.02, Mesh → Automatic,
   AspectRatio → Automatic, Boxed → False, Axes → {False, False, False}, BoxRatios → Automatic, PlotRange → All,
   PlotRangePadding → None, PlotPoints → quality, ViewPoint → view];
surfaces = {
  implicitSurface["(x^2+y^2-0.7^2)^2+ z^2", 1.2, 10, {0.3, -0.8, 1}],
  implicitSurface["(x^2+y^2-0.7^2)^2+(z^2-1)^2-0.05", 1.2, 10, {1.3, -1.0, 2}],
  implicitSurface["(1.2y^2-1)^2*(x^2+y^2-1^2)^2+
                   (1.2z^2-1)^2*(y^2+z^2-1^2)^2+
                   (1.2x^2-1)^2*(z^2+x^2-1^2)^2-0.02", 1.5, 10, {1.3, -1.0, 1.5}],
  implicitSurface["((x^2+y^2-0.85^2)^2+(z^2-1)^2)*
                   ((y^2+z^2-0.85^2)^2+(x^2-1)^2)*
                   ((z^2+x^2-0.85^2)^2+(y^2-1)^2)-0.001", 1.2, 50, {1.2, -1.0, 1.3}],
  implicitSurface["(3(x-1)x^2(x+1)+y^2)^2+ 2z^2", 1.2, 10, {0.3, -0.8, 1}],
  implicitSurface["(3(x-1)x^2(x+1)+y^2)^2+(2z^2-1)^2-0.005", 1.2, 10, {0.3, -0.8, 1}],
  implicitSurface["(3(x-1)x^2(x+1)+2y^2)^2+(z^2-0.85)^2*
                   (3(y-1)y^2(y+1)+2z^2)^2+(x^2-0.85)^2*
                   (3(z-1)z^2(z+1)+2x^2)^2+(y^2-0.85)^2* -0.12", 2, 80, {1, -1.8, 1.7}],
  implicitSurface["(2.92(x-1)x^2(x+1)+1.7y^2)^2*(y^2-0.88)^2+
                   (2.92(y-1)y^2(y+1)+1.7z^2)^2*(z^2-0.88)^2+
                   (2.92(z-1)z^2(z+1)+1.7x^2)^2*(x^2-0.88)^2 -0.02", 2, 20, {0.4, -1.8, 1.3}]}
```



using the code (Mathematica 7):

```
implicitSurface[p_String, zoom_, quality_Integer, view_List] :=
  ContourPlot3D[ToExpression[p], {x,-zoom,zoom}, {y,-zoom,zoom}, {z,-zoom,
  zoom}, Contours->0.02, Mesh->Automatic, AspectRatio->Automatic, Boxed->
  False, Axes->{False,False,False}, BoxRatios->Automatic, PlotRange->All,
  PlotRangePadding->None, PlotPoints->quality, ViewPoint->view];
surfaces = {
  implicitSurface["(x^2+y^2-0.7^2)^2+ z^2", 1.2, 10, {0.3, -0.8, 1}],
  implicitSurface["(x^2+y^2-0.7^2)^2+(z^2-1)^2-0.05",
                  1.2, 10, {1.3, -1.0, 2}],
  implicitSurface["(1.2y^2-1)^2*(x^2+y^2-1^2)^2+
                  (1.2z^2-1)^2*(y^2+z^2-1^2)^2+
                  (1.2x^2-1)^2*(z^2+x^2-1^2)^2-0.02",
                  1.5, 10, {1.3, -1.0, 1.5}],
  implicitSurface["((x^2+y^2-0.85^2)^2+(z^2-1)^2)*
                  ((y^2+z^2-0.85^2)^2+(x^2-1)^2)*
                  ((z^2+x^2-0.85^2)^2+(y^2-1)^2)-0.001",
                  1.2, 50, {1.2, -1.0, 1.3}],
  implicitSurface["(3(x-1)x^2(x+1)+y^2)^2+ 2z^2",
                  1.2, 10, {0.3, -0.8, 1}],
  implicitSurface["(3(x-1)x^2(x+1)+y^2)^2+(2z^2-1)^2-0.005",
                  1.2, 10, {0.3, -0.8, 1}],
  implicitSurface["(3(x-1)x^2(x+1)+2y^2)^2+(z^2-0.85)^2*
                  (3(y-1)y^2(y+1)+2z^2)^2+(x^2-0.85)^2*
                  (3(z-1)z^2(z+1)+2x^2)^2+(y^2-0.85)^2* -0.12",
```

```
                  2, 80, {1, -1.8, 1.7}],
implicitSurface["(2.92(x-1)x^2(x+1)+1.7y^2)^2*(y^2-0.88)^2+
                  (2.92(y-1)y^2(y+1)+1.7z^2)^2*(z^2-0.88)^2+
                  (2.92(z-1)z^2(z+1)+1.7x^2)^2*(x^2-0.88)^2 -0.02",
                  2, 20, {0.4, -1.8, 1.3}]
}
```

Of course, some of the examples were obtained just with a lot of trying and intelligent guessing.

It is apparent that one can construct surfaces, as complicated as one wishes, via these steps, but each time a new component is added $(P^{-1}(0) \mapsto (P \cdot Q)^{-1}(0))$, the degree of the polynomial increases, which presents considerable numerical problems when plotting the surface.

P.S. I hope I haven't made any mistakes when I copied the code. If so, please inform me, to check with my original Mathematica file.

P.P.S The above surfaces were included, as examples, in my diploma.

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edited Jun 20, 2011 at 6:02

community wiki
11 revs
Leon Lampret

10 These are incredibly cool. – Leonardo Fontoura Jun 19, 2011 at 3:45

2 How did you write down the equations? Take your "first miscellaneous example" as an example, it's natural to imagine a surface like that, but I have no idea how to write down the equations.. – user325 Jun 19, 2011 at 4:52

1 @Soarer: I've added a fairly general "recipe" for construction. – Leo Jun 19, 2011 at 6:49

2 beautiful pictures! – Otis Aug 28, 2012 at 6:53

2 I can reproduce your last example but only by increasing the isolevel (e.g. 0.04 instead of 0.02). With 0.02, the result I get is not connex. – Stéphane Laurent Nov 23, 2018 at 8:01

6

Reposting my comment as requested:

There is a wealth of documentation on Wolfram's Mathworld: http://mathworld.wolfram.com/topics/Surfaces.html

I've also, not too long ago, seen a pdf file that had pictures and Mathematica code for a lot of these surfaces; if I can dig it out of my history I will post it as well.

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answered Jun 19, 2011 at 1:55

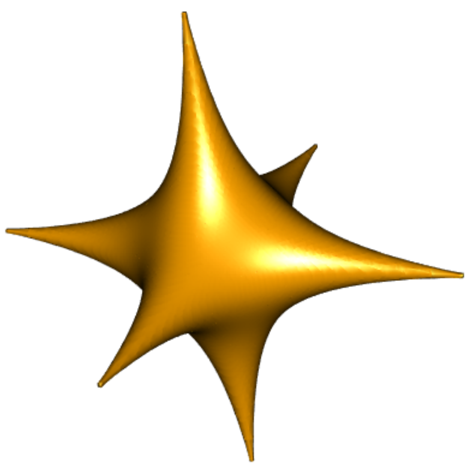
community wiki
barf

3

I like this "tetrahedron":

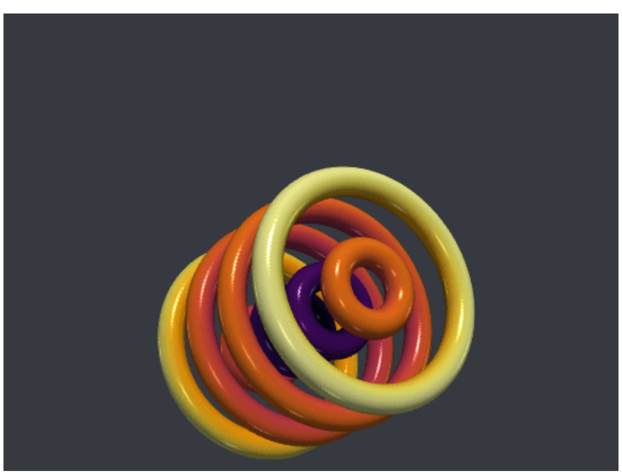
$$x^4+2x^2y^2+2x^2z^2+y^4+2y^2z^2+z^4+8xyz-10x^2-10y^2-10z^2+20=0.$$

The Entzensberger star:

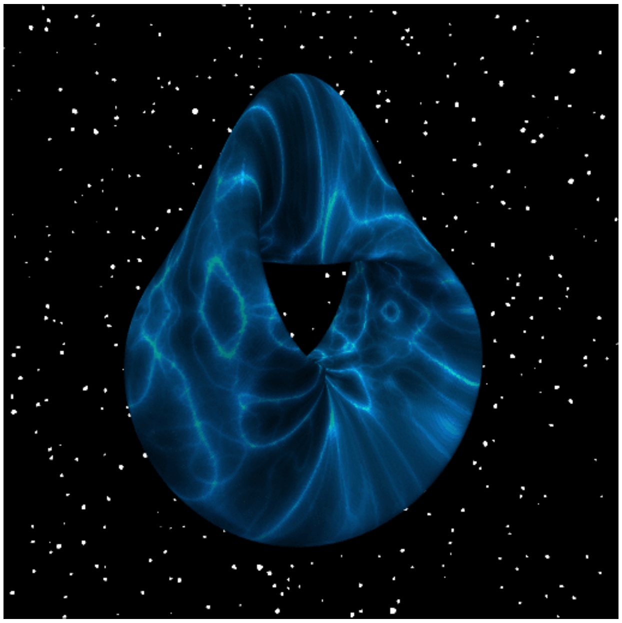


$$400(x^2y^2+y^2z^2+x^2z^2)=(1-x^2-y^2-z^2)^3.$$

On the higher space forum you can find amazing implicit surfaces, especially those of the member **ICN5D**. These are 3D slices of surfaces in higher dimension. I reproduced some of them with Python, like this one:



Solid Möbius strip, introduced in [this paper](#):



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edited Apr 30 at 18:14

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4 revs
Stéphane Laurent

2

Comlpete minimal surfaces, discovered circa 1985. For example, [Meeks' surfaces](#).

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answered Jun 19, 2011 at 7:13

community wiki
Andrew

Those are really nice, but what are the implicit equations? – Leo Jun 19, 2011 at 8:17

Maybe this link will give some more explicit information: [indiana.edu/~minimal/research/claynotes.pdf](#) – exchange Nov 1, 2017 at 22:51