Substitution Method

The substitution method

1. Guess a solution
2. Use induction to prove that the solution works

Substitution method

- Guess a solution
  - $T(n) = O(g(n))$
  - Induction goal: apply the definition of the asymptotic notation
    - $T(n) \leq d\, g(n)$, for some $d > 0$ and $n \geq n_0$

(strong induction)

  - Induction hypothesis: $T(k) \leq d\, g(k)$ for all $k < n$
- Prove the induction goal
  - Use the **induction hypothesis** to find some values of the constants d and n_0 for which the **induction goal** holds

Example 1

$$T(n) = c + T(n/2)$$

- Guess: $T(n) = O(\lg n)$
  - Induction goal: $T(n) \le d \lg n$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le d \lg(n/2)$
- Proof of induction goal:

$$T(n) = T(n/2) + c \le d \lg(n/2) + c$$

$$= d \lg n - d + c \le d \lg n$$

if: $-d + c \le 0,\ d \ge c$

Example 2

$$T(n) = T(n-1) + n$$

- Guess: $T(n) = O(n^2)$
  - Induction goal: $T(n) \le dn^2$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n-1) \le d(n-1)^2$ for all $k < n$
- Proof of induction goal:

$T(n) = T(n-1) + n \le d\,(n-1)^2 + n$

$= dn^2 - (2dn - d - n) \le dn^2$

if: $2dn - d - n \ge 0 \Leftrightarrow d \ge n/(2n-1) \Leftrightarrow d \ge 1/(2 - 1/n)$

  - For $n \ge 1 \Rightarrow 2 - 1/n \ge 1 \Rightarrow$ any $d \ge 1$ will work

Example 3

$$T(n) = 3T(n/4) + cn^2$$

- Guess: $T(n) = O(n^2)$
  - Induction goal: $T(n) \leq dn^2$, for some d and $n \geq n_0$
  - Induction hypothesis: $T(n/4) \leq d\,(n/4)^2$
- Proof of induction goal:

$$\begin{aligned} T(n) &= 3T(n/4) + cn^2 \\ &\leq 3d\,(n/4)^2 + cn^2 \\ &= (3/16)\,d\,n^2 + cn^2 \\ &\leq d\,n^2 \qquad \text{if: } d \geq (16/13)c \end{aligned}$$

- Therefore: $T(n) = O(n^2)$

Example 4

$$T(n) = 2T(n/2) + cn$$

- **Guess**: $T(n) = O(n \log n)$
- **Proof** by <u>Mathematical Induction</u>:

Prove that $T(n) \le d\, n \log n$ for $d>0$

$T(n) \le 2(d \cdot n/2 \cdot \log n/2) + cn$

(where $T(n/2) \le d \cdot n/2 \, (\log n/2)$ *by induction hypothesis*)

$\le dn \log n/2 + cn$

$= dn \log n - dn + cn$

$= dn \log n + (c-d)n$

$\le dn \log n$ if $d \ge c$

- Therefore, $\boldsymbol{T(n) = O(n \log n)}$

Example 5

$$T(n) = 2T(n/2) + n$$

- Guess: $T(n) = O(n\lg n)$
  - Induction goal: $T(n) \le dn \lg n$, for some d and $n \ge n_0$
  - Induction hypothesis: $T(n/2) \le dn/2 \lg(n/2)$
- Proof of induction goal:

$$T(n) = 2T(n/2) + n \le 2d\,(n/2)\lg(n/2) + n$$

$$= dn \lg n - dn + n \le dn \lg n$$

$$\text{if: } -dn + n \le 0 \Rightarrow d \ge 1$$