Hypothesis Testing: t test

Student's 't' (*Definition*). If $x_1, x_2, \ldots, x_n$ is a random sample of size n from a normal population with mean μ and variance σ^2, then Student's t statistic is defined as :

$$\boxed{t = \frac{\bar{x} - \mu}{S/\sqrt{n}} = \frac{\bar{x} - \mu}{\sqrt{S^2/n}}}$$

where

$$\bar{x} = \frac{\sum x}{n}, \text{ is the sample mean}$$

and

$$\boxed{S^2 = \frac{1}{n-1} \sum (x - \bar{x})^2,}$$

is an unbiased estimate of the population variance σ^2. 't' defined in (19·1) follows Student's t-distribution with $\nu = (n-1)$ *d.f.* and with probability density function (*p.d.f.*)

Suppose we are interested to test :

(a) If the given normal population has a specified value of the population mean, say, μ_0.

(b) If the sample mean $\bar{x}$ differs significantly from specified value of population mean.

(c) If a given random sample $x_1, x_2, \ldots, x_n$ of size n has been drawn from a normal population with specified mean, μ_0.

Basically, all the three problems are same. We set up the corresponding null hypothesis as follows :

(a) $H_0 : \mu = \mu_0$, *i.e.*, the population mean is μ_0.

(b) H_0 : There is no significant difference between the sample mean and the population mean. In other words, the difference between $\bar{x}$ and μ is due to fluctuations of sampling.

(c) H_0 : The given random sample has been drawn from the normal population with mean μ_0.

Under H_0 the test-statistic is

$$t = \frac{\bar{x} - \mu_0}{S/\sqrt{n}} = \frac{\bar{x} - \mu_0}{\sqrt{S^2/n}} \sim t_{n-1} \qquad \ldots(19{\cdot}15)$$

where

$$\bar{x} = \frac{1}{n}\Sigma x \quad \text{and} \quad S^2 = \frac{1}{n-1}\Sigma(x - \bar{x})^2 \qquad \ldots(19{\cdot}15a)$$

and it follows Student's t-distribution with $(n - 1)$ degrees of freedom.

Ten cartons are taken at random from an automatic filling machine. The mean net weight of the 10 cartons is 11.8 oz. and standard deviation is 0.15 oz. Does the sample mean differ significantly from the intended weight of 12 oz? You are given that for $\nu = 9$ $t_{0.05} = 2.26$

$n=$	10		
$\nu =$	9	**$=n-1$**	degrees of freedom = sample size -1
$\mu =$	12		
$\bar{x} =$	11.8		
$s =$	0.15		

H0: $\mu = 12$ the sample mean does not differ significantly from the population mean

Ha: $\mu \neq 12$ TWO TAILED

The t statistics is given by

$\bar{x} - \mu =$ -0.2

$$t = \frac{\bar{x} - \mu_0}{\sqrt{S^2/n}} = \frac{\bar{x} - \mu_0}{\sqrt{s^2/(n-1)}} \sim t_{n-1}$$

$s^2 =$ 0.0225

$s^2/(n-1)$ 0.0225/9

sqrt($s^2/(n-1)$)= 0.05

t= -4

CONCLUSION

tabulated value of t for 9 d.f. at 5% level of significance for two tailed test is 2.26. Since calculated $|t|$ is much greater it is highly significant. Hence, null hypothesis is rejected at 5% LOS that the sample mean differs significantly from the mean $\mu = 12$ oz. at 5% level of significance

The mean weekly sales of the chocolate bar in candy stores was 146·3 bars per store. After an advertising campaign the mean weekly sales in 22 stores for a typical week increased to 153·7 and showed a standard deviation of 17·2. Was the advertising campaign successful ?

$n =$	22		
$\nu =$	21	**$=n-1$**	degrees of freedom = sample size - 1
$\mu =$	146.3		
$\bar{x} =$	153.7		
$s =$	17.2		

H0: $\mu = 146.3$ the sample mean does not differ significantly from the population mean (the advertisting campaign is not successful)

Ha: $\mu > 146.3$ the advertisting campaign is successful (RIGHT TAILED)

The t statistics is given by

$$t = \frac{\bar{x} - \mu_0}{\sqrt{S^2/n}} = \frac{\bar{x} - \mu_0}{\sqrt{s^2/(n-1)}} \sim t_{n-1}$$

$\bar{x} - \mu =$ 7.4

$s^2 =$ 295.84

$s^2/(n-1)$ 295.84/21

$$t = \frac{\bar{x} - \mu}{s / \sqrt{n-1}}$$

$\mathrm{sqrt}(s^2/(n-1)) =$ 3.75334771

$t =$ 1.9715733

$|t| =$ 1.9715733

CONCLUSION

tabulated value of t for 21 d.f. at 5% level of significance for right tailed test is 1.721 Since calculated |t| is greater than the critical value so its significant. Hence, null hypothesis is rejected at 5% LOS , So the advertisement campaign is successful 5% level of significance

Certain pesticide is packed into bags by a machine. A random sample of 10 bags is drawn and their contents are found to weigh (in kg.) as follows :

50, 49, 52, 44, 45, 48, 46, 45, 49, 45.

Test if the average packing can be taken to be 50 kg.

$n = 10$

$\nu = 9$ **=n-1** degrees of freedom = sample size -1

$\mu = 50$

H0: $\mu = 50$ the average packaging is 50 kg

Ha: $\mu \neq 50$ the average packaging is not 50 kg

X	d = X-A	d^2
50	2	4
49	1	1
52	4	16
44	-4	16
45	-3	9
48	0	0
46	-2	4
45	-3	9
49	1	1
45	-3	9
$\sum$	-7	69

A= 48

$\bar{x}$ =

S^2= 7.1222

$\bar{x}$ = 47.3

The t statistics is given by

$\bar{x}$ -μ = -2.7

S^2= 7.1222

S^2/n 7.1222/10 0.71222

sqrt(s^2/n)= 0.8439

t = -3.2

|t |= 3.2

$$\bar{x} = A + \frac{\sum d}{n}$$

$$S^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{(\sum d)^2}{n}\right]$$

$$t = \frac{\bar{x} - \mu_0}{\sqrt{S^2/n}}$$

CONCLUSION

tabulated value of t for 9 d.f. at 5% level of significance for Two tailed test is 2.262 Since calculated |t| is greater than the critical value so its significant. Hence, null hypothesis is rejected at 5% LOS , So the the average packaging is cannot be taken as 50 kg

A courier service advertises that its average delivery time is less than 6 hours for local deliveries. A random sample of 10 for the amount of time this courier takes to deliver packages to an addressee across the town produced the following times (rounded to the nearest hour) :

7, 3, 4, 6, 10, 5, 6, 4, 3, 8

Is this evidence sufficient to support the courier claim at 5% level of significance ?

n= 10

ν = 9 **=n-1** degrees of freedom = sample size -1

μ = 6

H0: μ >=6 the average delivery time is greater than or equal to 6 hours i.e. the courier services claim is **not** valid

Ha: μ < 6 the average delivery time is less than 6 hours, i.e. the courier services claim is valid

X	d = X-A	d^2
7	1	1
3	-3	9
4	-2	4
6	0	0
10	4	16
5	-1	1
6	0	0
4	-2	4
3	-3	9
8	2	4
∑	-4	48

$$\bar{x} = A + \frac{\sum d}{n}$$

$$S^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{(\sum d)^2}{n}\right]$$

A= 6

S^2= 5.1556

$\bar{x}$ = 5.6

The t statistics is given by

$$t = \frac{\bar{x} - \mu_0}{\sqrt{S^2/n}}$$

$\bar{x} - \mu =$	-0.4	
$S^2 =$	5.1556	
S^2/n	5.1556/10	0.51556
$\mathrm{sqrt}(s^2/n) =$		0.718
$t =$	-0.56	
$\lvert t \rvert =$	0.56	

An ambulance service company claims that on an average it takes 20 minutes between a call for an ambulance and the patient's arrival at the hospital. If in 6 calls the time taken (between a call and arrival at hospital) are $27, 18, 26, 15, 20, 32$. Can the claims be accepted?

Example 19·2. *A machine is designed to produce insulating washers for electrical devices of average thickness of 0·025 cm. A random sample of 10 washers was found to have an average thickness of 0·024 cm with a standard deviation of 0·002 cm. Test the significance of the deviation. Value of t for 9 degrees of freedom at 5% level is 2·262.*

[*C.A.* (*Intermediate*), *Nov. 1980*]

Solution. We are given :

$$n=10,\ \bar{x}=0{\cdot}024 \text{ cm},\ s=0{\cdot}002 \text{ cm}$$

Null Hypothesis. $H_0 : \mu=0{\cdot}025$ cm. *i.e.*, there is no significant deviation between sample mean $\bar{x}=0{\cdot}024$ and population mean $\mu=0{\cdot}025$.

Alternative Hypothesis. $H_1 : \mu\neq 0{\cdot}025$ cm

Under H_0, the test statistic is

$$t=\frac{\bar{x}-\mu}{S/\sqrt{n}}=\frac{\bar{x}-\mu}{s/\sqrt{n-1}}\sim t_{10-1}=t_9$$

Now
$$t=\frac{0{\cdot}024-0{\cdot}025}{0{\cdot}002/\sqrt{9}}=\frac{-0{\cdot}001\times 3}{0{\cdot}002}=-1{\cdot}5$$

Tabulated $t_{0\cdot05}$ for 9 *d.f.*$=2{\cdot}262$. Since $|t|<2{\cdot}262$, it is not significant at 5% level of significance. Hence the deviation $(\bar{x}-\mu)$ is not significant.

PAIRED t-TEST FOR DIFFERENCE OF MEANS

In the t-test for difference of means, the two samples were independent of each other. Let us now take a particular situation where

(*i*) The sample sizes are equal, *i.e.*, $n_1 = n_2 = n$, (say), and

(*ii*) The sample observations $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are not completely independent but they are dependent in pairs *i.e.*, the pairs of observations $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ correspond to the 1st, 2nd,..., nth unit respectively.

An IQ test was administered to 5 persons before and after they were trained. The results are given below :

Candidates →	I	II	III	IV	V
IQ before training	110	120	123	132	125
IQ after training	120	118	125	136	121

Test whether there is any change in IQ after the training programme.

use 1% LOS

n=	5		
ν =	4	**=n-1**	degrees of freedom = sample size -1

let x,y respectively denote the IQ scores before training and after training

Null Hypothesis: H0: $\mu1=\mu2$ the mean scores before and after training are same

*Alternative Hypothesis: Ha: **$\mu1 \neq \mu2$** the mean scores before and after training are **not** same*

x	y	d	d^2
110	120	-10	100
120	118	2	4
123	125	-2	4
132	136	-4	16
125	121	4	16
	$\sum$	-10	140

$\bar{d}$ $\sum d / n =$ -2

$S^2 =$ 30

$$S^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{(\sum d)^2}{n}\right]$$

$$t = \frac{-2}{2.4495}$$

$t =$ -0.816

$|t| =$ 0.8165

$$\boxed{t = \frac{\bar{d}}{S/\sqrt{n}} = \frac{\bar{d}}{\sqrt{S^2/n}} \sim t_{n-1}}$$

t tab at 4 df at 1% LOS for two tailed test is 4.6

as tcalc < ttab, we fail to reject the Null Hypothesis

We can conlcude that there is no change in IQ training programme

The sales data of an item in six shops before and after a special promotional campaign are as under :

Shops	*A*	*B*	*C*	*D*	*E*	*F*
Before campaign	53	28	31	48	50	42
After campaign	58	29	30	55	56	45

Can the campaign be judged to be a success ? Test at 5% level of significance.

$n =$ 6

$\nu =$ 5 **=n-1** degrees of freedom = sample size **-1**

let x,y respectively denote the sales data before and after special promotional campaign

Null Hypothesis: H0: μ1=μ2 the mean sales before and after special promotional campaign are same. The promotional campaign is **not** successful

Alternative Hypothesis: Ha: μ1< μ2 *the mean sales is increased after the special promotional campaign. i.e. the The promotional campaign is successful*

x	y	d	d^2
53	58	-5	25
28	29	-1	1
31	30	1	1
48	55	-7	49
50	56	-6	36
42	45	-3	9
	∑	-21	121

$\bar{d}$ $\sum d / n =$ -3.5

$S^2 =$ 9.5

$$S^2 = \frac{1}{n-1}\left[\sum d^2 - \frac{(\sum d)^2}{n}\right]$$

$t = \dfrac{-3.5}{1.2583}$

$$\boxed{t = \frac{\bar{d}}{S/\sqrt{n}} = \frac{\bar{d}}{\sqrt{S^2/n}} \sim t_{n-1}}$$

$t =$ -2.782

$||t| =$ 2.7815

t tab at 5 df at 5% LOS for one tailed test is 2.015

as tcalc > ttab, we reject the Null Hypothesis

We can conlcude that the mean sales is increased after the special promotional campaign. i.e. the The promotional campaign is successful

t-TEST FOR SIGNIFICANCE OF AN OBSERVED SAMPLE CORRELATION COEFFICIENT

Suppose that a random sample (x_i, y_i), $i = 1, 2, \ldots, n$ of size n has been drawn from a bivariate normal distribution and let r be the observed sample correlation coefficient. The problem is to find if this sample correlation coefficient r is significant of any correlation between the variables in the population or is it just due to fluctuations of sampling ? Prof. R.A. Fisher proved that under the null hypothesis $H_0 : \rho = 0$, *i.e.*, the variables are uncorrelated in the population, the statistic

$$t = \frac{r}{\sqrt{1 - r^2}} \times \sqrt{n - 2} \sim t_{n-2} \qquad \ldots(19{\cdot}23)$$

i.e., t follows Student's t-distribution with $(n - 2)$ *d.f.*, n being the sample size.

95% Confidence Limits for ρ are :

$$r \pm t_{n-2}(0{\cdot}025) \times S.E.\,(r) = r \pm t_{n-2}\,(0{\cdot}025) \times \frac{(1 - r^2)}{\sqrt{n}} \qquad \ldots(19{\cdot}24)$$

where $t_\nu\,(\alpha)$ is defined in (19·13).

99% Confidence Limits for ρ are :

$$r \pm t_{n-2}(0{\cdot}005) \times S.E.\,(r) = r \pm t_{n-2}\,(0{\cdot}005) \times \frac{(1 - r^2)}{\sqrt{n}} \qquad \ldots((19{\cdot}24a)$$

A random sample of **27** pairs of observations from a normal population gives a correlation coefficient of **0.42** **(a)** Is it likely that the variables in the population are uncorrelated? Use $\boldsymbol{\alpha} = \mathbf{0.05}$.

Method

Given a sample of n points (x_i, y_i) the correlation coefficient r is calculated from the formula

$$r = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\left[\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2\right]^{\frac{1}{2}}}.$$

To test the null hypothesis that the population value of r is zero, the test statistic

$$t = \frac{r}{\sqrt{1 - r^2}} \cdot \sqrt{n - 2}$$

is calculated and this follows Student's t-distribution with $n - 2$ degrees of freedom. The test will normally be two-tailed but in certain cases could be one-tailed.

Find the least value of r in samples of 18 pairs of observations from a bivariate normal population, which is significant at 5% level. (Value of t at 5% level for 16 d.f. is 2.12).

we are given

$n = 18$, The observed value ~~are~~ of sampl correlation coefficient r will be significant at 5% level of significance if

$$|t| = \left|\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}\right| > t_{n-2}(0.025) = t_{16}(0.025) \quad \text{----} (*)$$

we have $t_{16}(0.025) = 2.12$.

A coefficient of correlation of 0·2 is derived from a random sample of 625 pairs of observations :

(i) Is this value significant ?

(ii) What are the 95% and 99% confidence limits for the correlation coefficient in the population ?

Null Hypothesis: H0: $\rho=0$ the value of r is not significant

Alternative Hypothesis: Ha: $\rho \neq 0$ the value of r is significant

the test statistics is given by

$$t = \left| \frac{r\sqrt{n-2}}{\sqrt{1-r^2}} \right| \sim t_{n-2}$$

r=	0.2		
n=	625		
ν =n-2=	623	sqrt(n-2)=	24.96
$1-r^2$	0.96		
sqrt($1-r^2$)=	0.9798		
t=	5.0949		

As df= ν >60 the significant values of t are same as in case of normal distribution so

for two tailed test

t =	1.96	at 5 % LOS
t =	2.58	at 1 % LOS

Since the calculated value of t is much greater than these values it is significant, so we reject the Null Hypothesis at both the LOS

i.e. the value of r is significant

Next, to find the 95% confidence limits for ρ, the population correlation coefficient at 0.05 LOS

$$r \pm t_{n-2}(0.025) \times S.E.(r) = r \pm t_{n-2}(0.025) \times \frac{(1-r^2)}{\sqrt{n}}$$

$(1-r^2)/\sqrt{n}$ 0.0384

The 95% confidence limits at 0.05 LOS are 0.2 ± 1.96 x 0.0384

left confidence limit = 0.125 right confidence limit= 0.275

i.e. $\mathbf{0.125 \leq \rho \leq 0.275}$

Next, to find the 95% confidence limits for ρ, the population correlation coefficient at 0.01 LOS

$$r \pm t_{n-2}(0.025) \times S.E.(r) = r \pm t_{n-2}(0.025) \times \frac{(1-r^2)}{\sqrt{n}}$$

The 95% confidence limits at 0.01 LOS are 0.2 ± 2.58 x 0.0384

left confidence limit = 0.101 right confidence limit= 0.299

i.e. $\mathbf{0.101 \leq \rho \leq 0.299}$

A random sample of 27 pairs of observations from a normal

population gave a correlation coefficient of 0.6. Is this

significant of correlation in the population?