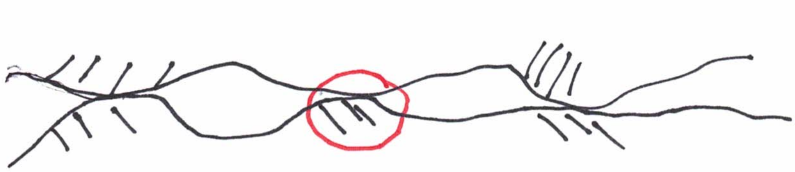
MANE 6520 - Fluid Mechanics

Class # 14 (Lubrication Theory)

3 types of Lubrication

Boundary Lube
chemical action (WD-40)

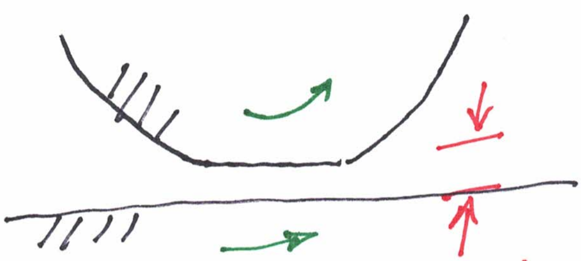
slow speeds
film ~ nm



asperity contact
no fluid action

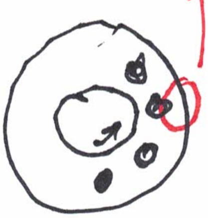
Elastohydrodynamic Lube

moderate speeds
~ 0.1 m/s



flattening
point or line contact

0.1 μm

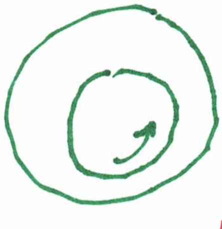


roller or ball bearing

fluid/solid problem
minimal surface contact

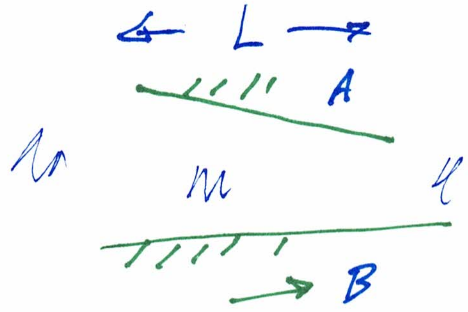
Hydrodynamic Lubrication

high speed
~ m/s

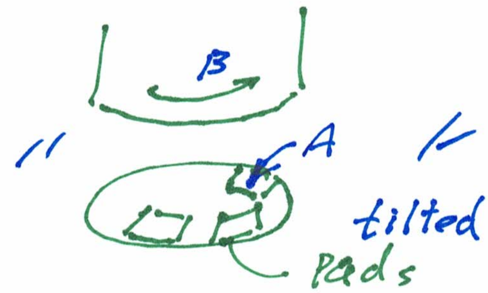


journal bearing
auto engines
nucl. power

only fluids problem
gap ~ μm - mm
no wear



slider (thrust) bearing



2-D, steady, incompressible
"thin," "slow" = low Reynolds number

$R_{shaft} \simeq 0.1\,m \qquad N = 1000\,rpm, \quad \omega \sim 100\,\frac{rad}{s}$

$V = \omega R \sim 10\,\frac{m}{s} \qquad \rho \sim 1000\,\frac{kg}{m^3} \qquad \mu \sim 0.01\,\frac{N\cdot s}{m^2}$

$= 1000\,\frac{N \cdot s^2}{m^4}$ (10x water)

$L \sim \frac{2\pi R}{4\,pads} \sim 0.1\,m$

$H \sim 0.1\,mm$

$Re = \frac{\rho V H}{\mu} = \frac{1000(10)(10^{-4})}{0.01} = 100$ (laminar)

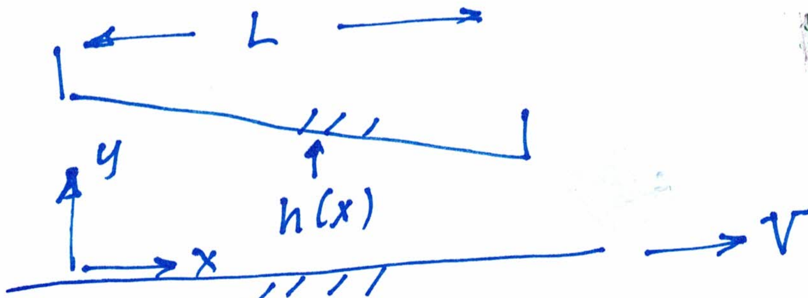
$Re^* = \frac{H}{L} Re = 0.1 \cdot$ effect of inertia

$\frac{H}{L} = 0.001$

-3-

$$\rho\cancel{\left(v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}\right)}=-\frac{\partial p}{\partial x}+\mu\left(\cancel{\frac{\partial^2 v_x}{\partial x^2}}+\frac{\partial^2 v_x}{\partial y^2}\right)$$

$$\rho\left(\cancel{v_x\frac{\partial v_y}{\partial x}}+\cancel{v_y\frac{\partial v_y}{\partial y}}\right)=-\frac{\partial p}{\partial y}+\mu\left(\cancel{\frac{\partial^2 v_y}{\partial x^2}}+\cancel{\frac{\partial^2 v_y}{\partial y^2}}\right)-\cancel{\rho g}$$



pressure doesn't vary across film

since $\frac{dp}{dx}=fn(x)$ can integrate

$$\frac{dp}{dx}=\mu\frac{\partial^2 v_x}{\partial y^2}$$

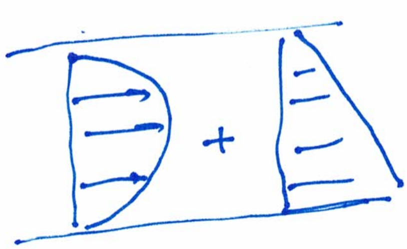
quadricll pressure drive

linear profile driven by sliding

$$v_x=\underbrace{\frac{h^2}{2\mu}\frac{dp}{dx}\left(\frac{y^2}{h^2}-\frac{y}{h}\right)}_{\text{Poiseuille}}+\underbrace{V\left(1-\frac{y}{h}\right)}_{\text{Couette}}\Leftarrow$$

$$\frac{\partial v_x}{\partial y}=\frac{1}{\mu}\frac{dp}{dx}y+c_1$$

$$v_x=\frac{1}{2\mu}\frac{dp}{dx}y^2+c_1y+c_2$$



BC's $y=0 \quad v_x=\cancel{0}\,V$

$y=h(x) \quad v_x=0$

$\frac{dp}{dx}<0\Rightarrow +v_x$

$c_2=V$

Inertia

$$\rho \cancel{V_x} \frac{\partial V_x}{\partial x}$$

$$\rho V \frac{V}{L}$$

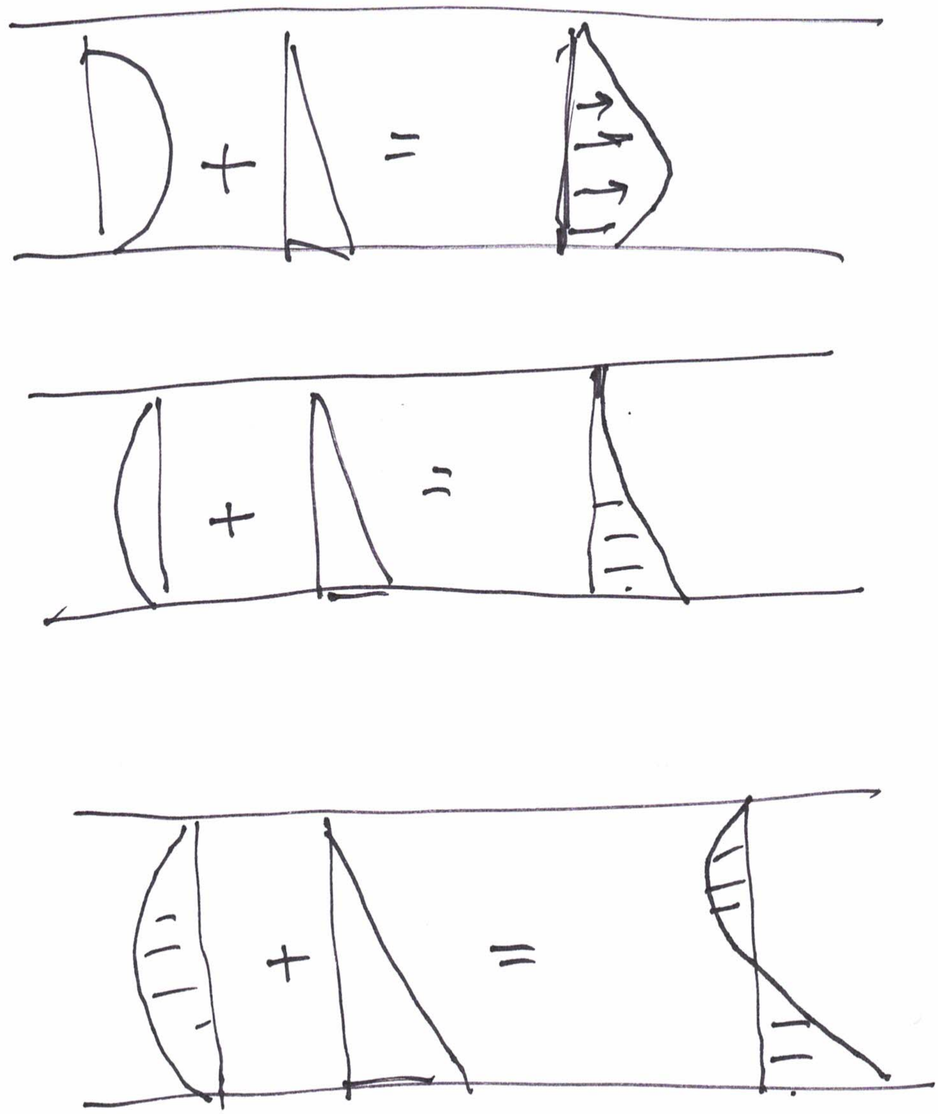
Viscous

$$\mu \frac{\partial^2 V_x}{\partial y^2}$$

$$\mu \frac{V}{H^2}$$

$$\frac{\text{Inertia}}{\text{Viscous}} = Re^* = \frac{\rho V H}{\mu} \frac{H}{L}$$

many possible flows



$\frac{dp}{dx} > 0$

no net flow

W into paper

mass flow rate $\dot{m} = \rho W \int_0^{h(x)} V_x \, dy = \rho W \left[\frac{-h^3}{12\mu} \frac{dp}{dx} + \frac{Vh}{2} \right] = \text{const}$

$$\frac{d\dot{m}}{dx} = 0 \qquad \frac{d}{dx}\left(\frac{h^3}{12\mu} \frac{dp}{dx} \right) = \frac{V}{2} \frac{dh}{dx} \qquad h = h(x)$$

Reynolds Equation

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx} \right) = 6\mu V \frac{dh}{dx}$$

$p = p_{atm}$ at $x = 0, L$

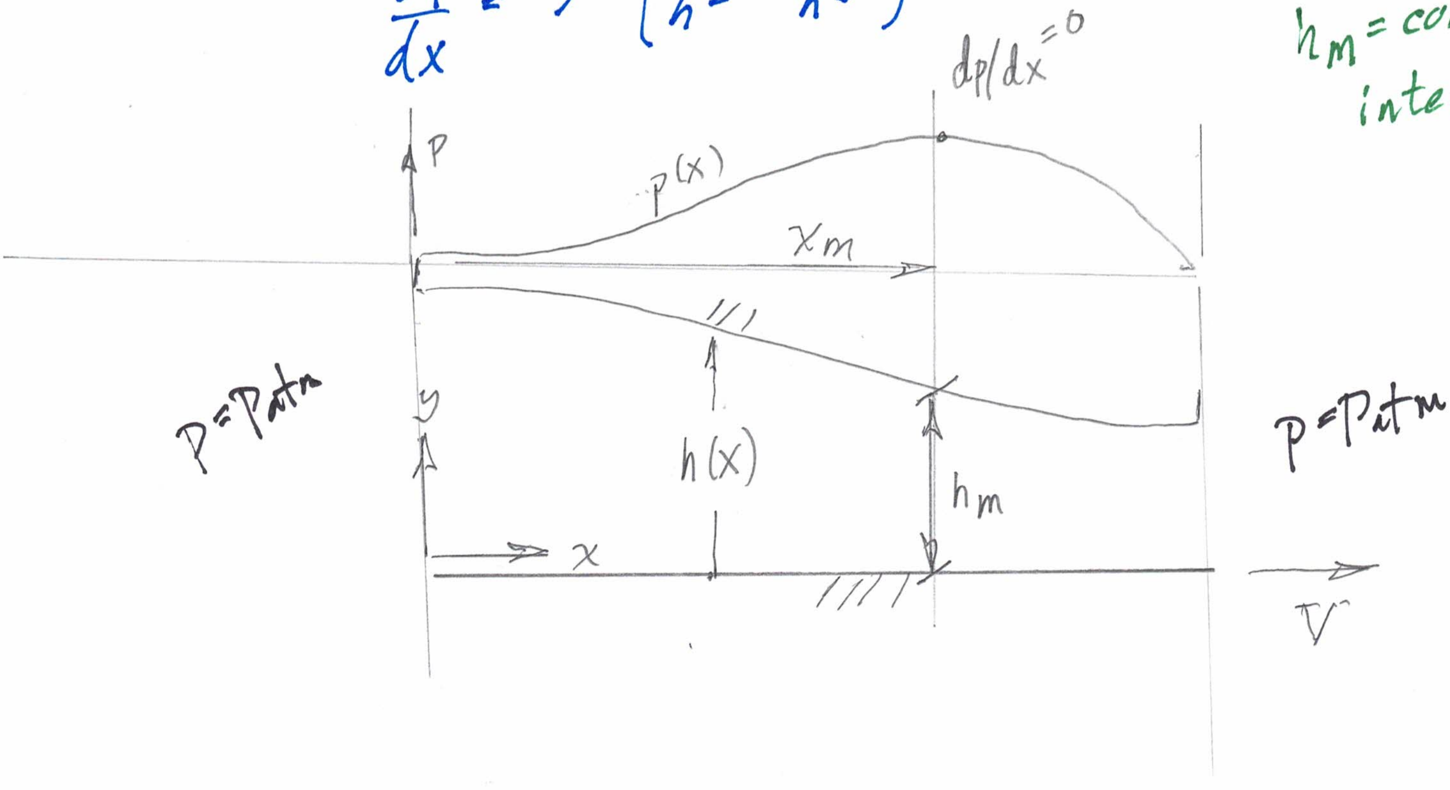
Solve

treat $= h(x)$ as known

$$h^3 \frac{dp}{dx} = 6\mu V h + C_1 \quad \longrightarrow \quad \frac{dp}{dx} = \frac{6\mu V}{h^2} + \frac{C_1}{h^3}$$

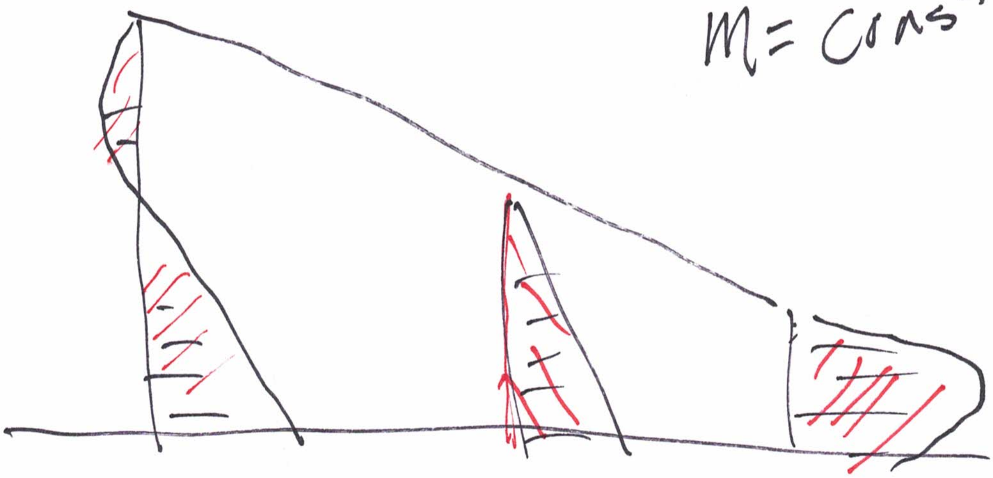
$$\frac{dp}{dx} = 6\mu V \left(\frac{1}{h^2} - \frac{h_m}{h^3} \right)$$

$h_m = \text{const}$ integr.



4a

$\dot{m} = \text{const}$



$\dot{m} \propto$ area under curve

5

In[99]:= `ClearAll["Global`*"]; (* solution Reynolds equation *)`

In[100]:= $$vx = dpdx \frac{h^2}{2\mu}\left(\frac{y^2}{h^2} - \frac{y}{h}\right) + V\left(1 - \frac{y}{h}\right);$$

geometry $L \gg H$

In[101]:= $\mu\, \partial_{y,y} vx$

Out[101]= dpdx

diff eq

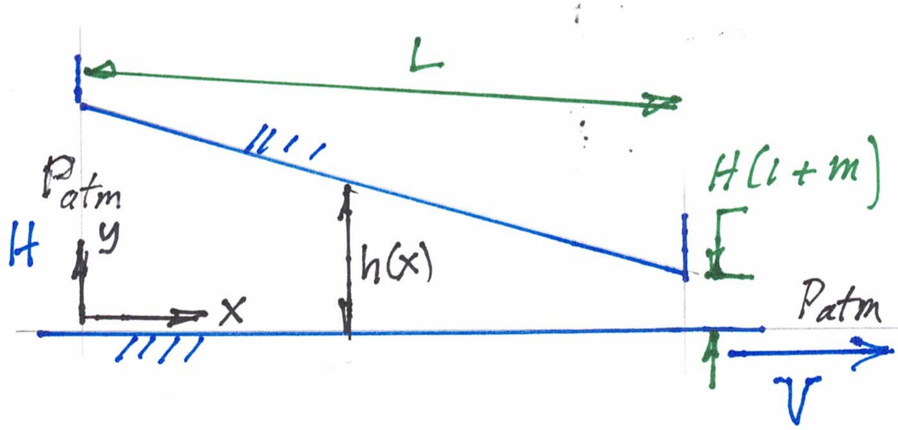
In[102]:= `vx /. y → 0`

`vx /. y → h`

Out[102]= V

Out[103]= 0

BCs



In[104]:= $$mdot = W\rho \int_0^h vx\, dy$$

Out[104]= $$W\left(\frac{h V}{2} - \frac{dpdx\, h^3}{12\mu}\right)\rho$$

$-1 < m < 0$

$m = -1$ touches

$m = 0$ flat

In[105]:= $$h = H\left(1 + m\frac{x}{L}\right);$$

$$dpdx = -\frac{6 L^2 m V (L - (2+m) x)\mu}{H^2 (2+m) (L + m x)^3};$$

$$p = pAtm - \frac{6 L m V (L - x) x \mu}{H^2 (2+m) (L + m x)^2};$$

In[108]:= $Simplify[(\partial_x p) - dpdx]$

Out[108]= 0

In[109]:= $$hm = H\frac{2(1+m)}{2+m};\ xm = \frac{1}{2+m};\ \ hL = H(1 + m L);\ dhdx = \frac{H}{L} m;$$

In[110]:= $Simplify[\partial_x (h^3\, dpdx) - 6 V \mu\, dhdx]$

Out[110]= 0

In[111]:=
```
vx0 = Simplify[vx /. x → 0];
vxm = Simplify[vx /. x → xm];
vxL = Simplify[vx /. x → L];
```

In[114]:=
```
V = 10.;
μ = 0.01; L = 0.1; pAtm = 1.×10^5;
H = 1.×10^-4;
m = -0.8;
W = 1.;
g = 9.81;
```

5a

$$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right) = 6\mu V\frac{dh}{dx}$$

$$h^3\frac{dp}{dx} = 6\mu V h + c_1$$

$$\frac{dp}{dx} = 6\mu V\frac{1}{h^2} + \frac{c_1}{h^3}$$

$$p = 6\mu V\int\frac{dx}{h^2} + \int\frac{c_1}{h^3} + C_2$$

Find c_1, c_2 from $p(x=0) = p_{atm}$

$p(x=L) = p_{atm}$

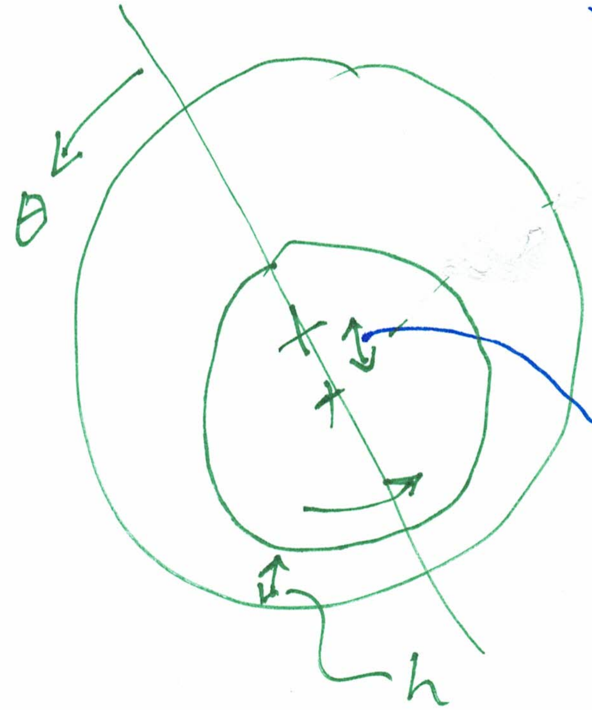
Involves integrals such as

$$\int\frac{dx}{h^2} \qquad \int\frac{dx}{h^3}$$

$$h = He^{mx/L} - HW$$

- 5b

Journal Bearing



$c = \text{gap} = R_o - R_i$

outside - inside

eccentricity e

$\epsilon = \dfrac{e}{c}$

$$h = c(1 + \epsilon \cos\theta)$$

Lots of complicated geometry

Need integrals like: $\displaystyle\int \frac{d\theta}{(1+\epsilon\cos\theta)^2}$ - Reynolds did these

we need

$$\int \frac{dx}{\left(1+m\frac{x}{L}\right)^2} \;\; \& \;\; \int \frac{dx}{\left(1+m\frac{x}{L}\right)^3}$$

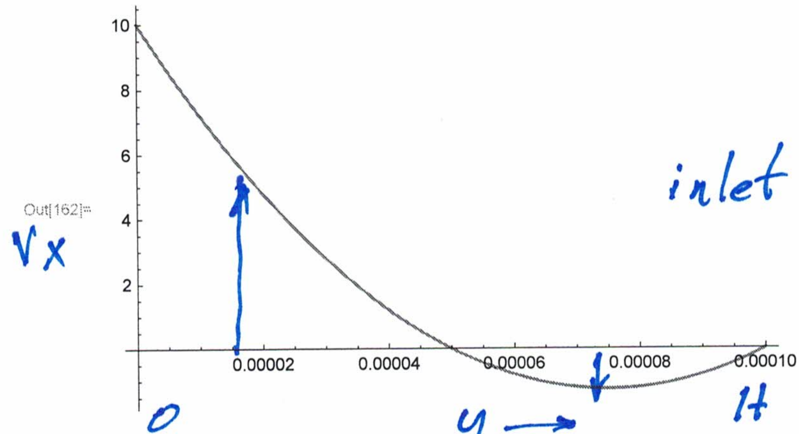
HW

you will need $\displaystyle\int \frac{dx}{(1+e^{mx})^3}$ etc $\displaystyle\int \frac{dx}{(1+e^{mx})^2}$

In[162]:= Plot[vx0 , {y, 0, H}]

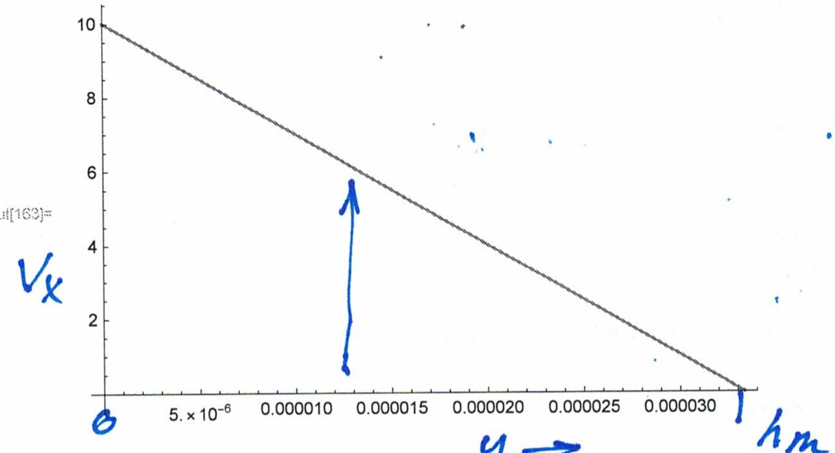
$m=-0.8$

Out[162]=



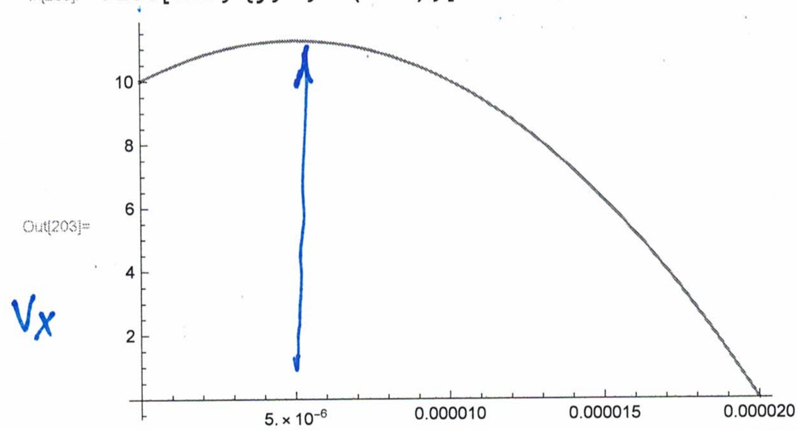
In[163]:= Plot[vxm , {y, 0, hm}]

Out[163]=



In[203]:= Plot[vxL , {y, 0, H (1 + m) }]

Out[203]=



$H(1+m)=0.2H$

In[204]:= hL / H

Out[204]= 0.2

2'

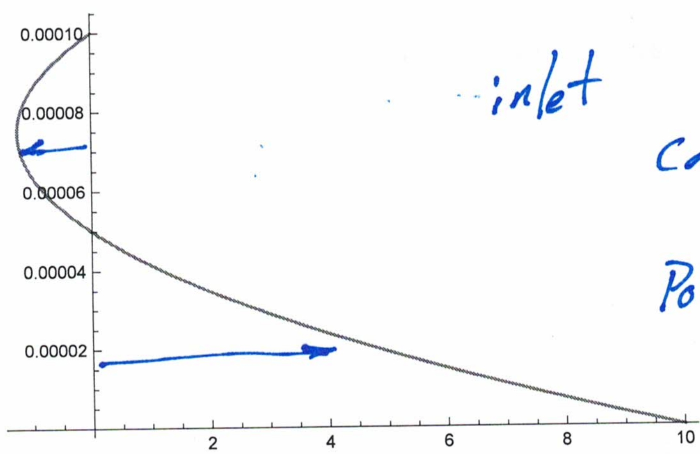

```
In[205]:= Ny = 101; Δy0 = h0/(Ny - 1); Δym = hm/(Ny - 1);  ΔyL = hL/(Ny - 1);
          y0i = Table[Δy0 (i - 1), {i, Ny}];
          ymi = Table[Δym (i - 1), {i, Ny}];
          yLi = Table[ΔyL (i - 1), {i, Ny}];

In[209]:= dpdx0 = dpdx /. {x → 0.0 };
          dpdxm = dpdx /. {x → xm};
          dpdxL = dpdx /. {x → L};

In[212]:= vx0i = Table[0, {i, Ny}];
          vxmi = Table[0, {i, Ny}];
          vxLi = Table[0, {i, Ny}];

In[215]:= Do[vx0i[[i]] = vx0 /. y → y0i[[i]], {i, Ny}];
          Do[vxmi[[i]] = vxm /. y → ymi[[i]], {i, Ny}];
          Do[vxLi[[i]] = vxL /. y → yLi[[i]], {i, Ny}];

In[218]:= plotv0 = Table[0, {i, Ny}, {j, 2}];
          Do[plotv0[[i, 2]] = y0i[[i]]; plotv0[[i, 1]] = vx0i[[i]], {i, Ny}];
          plotvm = Table[0, {i, Ny}, {j, 2}];
          Do[plotvm[[i, 2]] = ymi[[i]]; plotvm[[i, 1]] = vxmi[[i]], {i, Ny}];
          plotvL = Table[0, {i, Ny}, {j, 2}];
          Do[plotvL[[i, 2]] = yLi[[i]];
            plotvL[[i, 1]] = vxLi[[i]], {i, Ny}];

In[224]:= ListPlot[plotv0, Joined → True, PlotStyle → Thickness[0.005`], DisplayFunction → Identity]
```

to turn figures around

Out[224]=

H

y



v_x

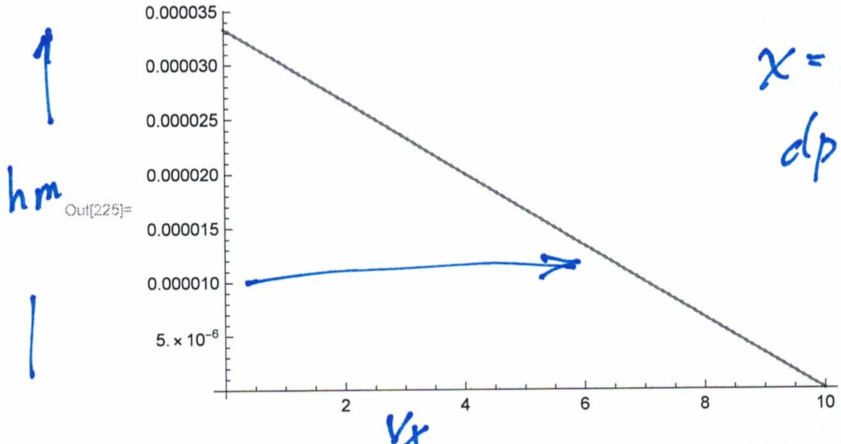
Couette (sliding)

Poiseuille (pressure)

$\frac{dp}{dx} > 0$

In[225]:= `ListPlot[plotvm, Joined → True, PlotStyle → Thickness[0.005`], DisplayFunction → Identity]`

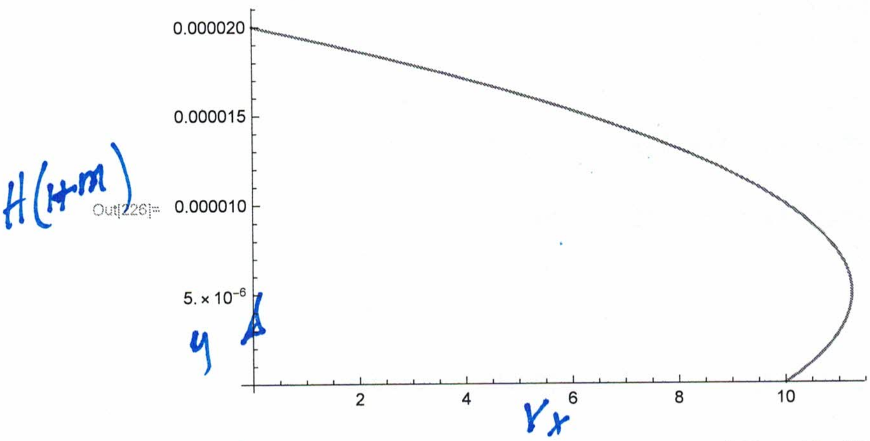
Out[225]=



$x = x_m$
$dp/dx = 0$
Couette

In[226]:= `ListPlot[plotvL, Joined → True, PlotStyle → Thickness[0.005`], DisplayFunction → Identity]`

Out[226]=

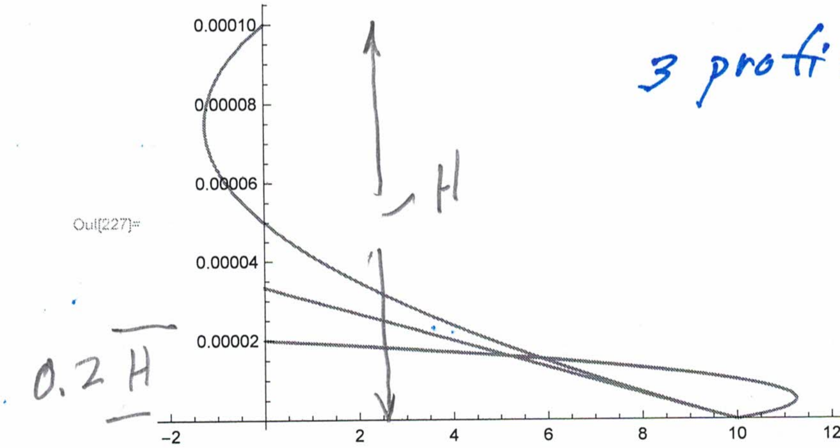


$x = L$ (exit)

$\frac{dp}{dx} < 0$

In[227]:= `Show[%, %%, %%%, PlotRange → {{-0.2 V, 1.2 V}, {0, H}}]`

Out[227]=



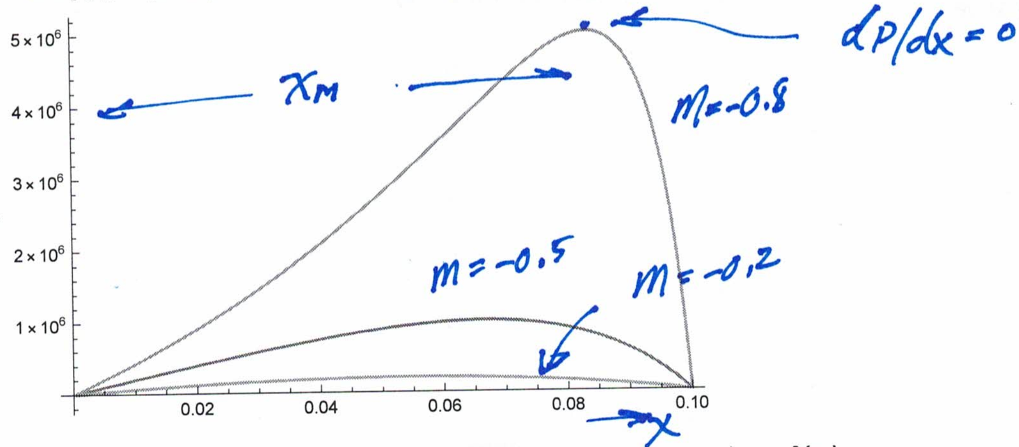
3 profiles together

`ClearAll[m]`

```
In[]:= Plot[{(p - pAtm) /. m → -0.5, (p - pAtm) /. m → -0.2, (p - pAtm) /. m → -0.8}, {x, 0, L}]
```

Out[]=

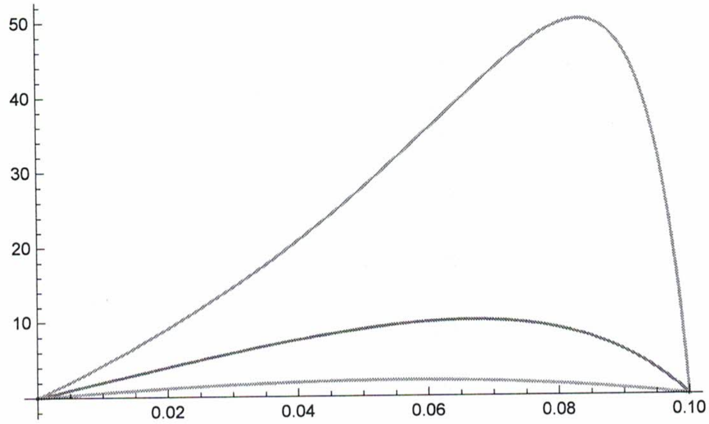
P (Pa)




```
In[]:= Plot[{(p - pAtm)/pAtm /. m → -0.5, (p - pAtm)/pAtm /. m → -0.2, (p - pAtm)/pAtm /. m → -0.8}, {x, 0, L}]
```

Out[]=

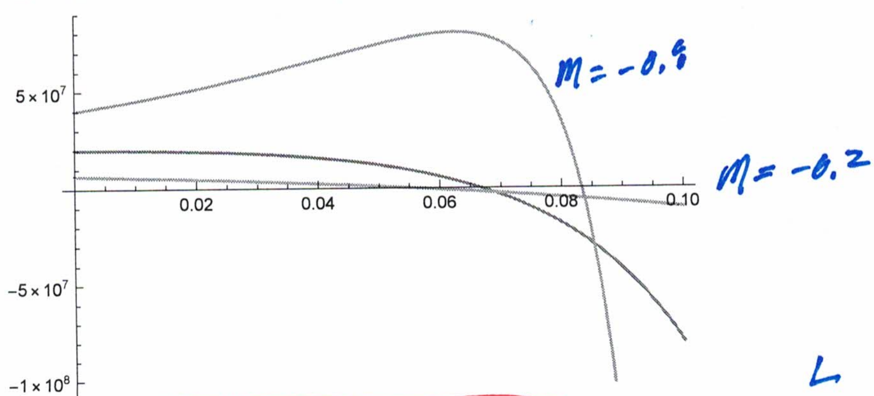
P (Bar)




```
In[]:= Plot[{dpdx /. m → -0.5, dpdx /. m → -0.2, dpdx /. m → -0.8}, {x, 0, L}]
```

Out[]=

$\frac{dp}{dx}$ $\left(\frac{N}{m^3}\right)$



In[]:=

$$\boxed{Fy = \frac{6 L^2 W V \mu}{H^2 m^2}\left(\frac{2 m}{2 + m} + \text{Log}\left[\frac{1}{1 + m}\right]\right);}$$

$$F_y = W\int_0^L (p - P_{atm})\,dx$$

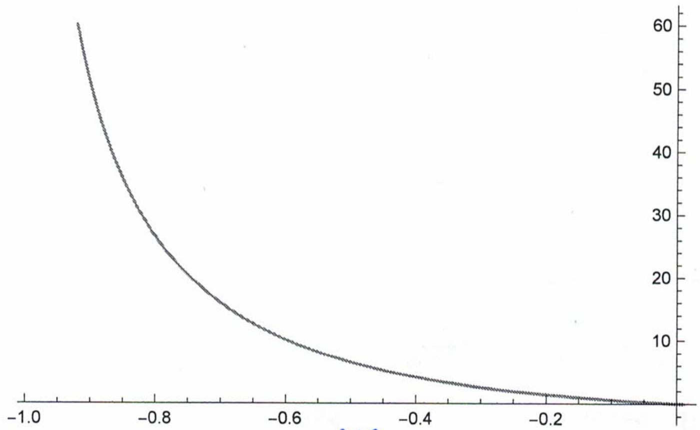
Note $\propto \left(\frac{L}{H}\right)^2$

9-

Load (Normal Force)

In[ ]:= Plot[$\frac{Fy}{g\,1000}$, {m, 0.01, -0.99}] (* in metric tons *)

Out[ ]=



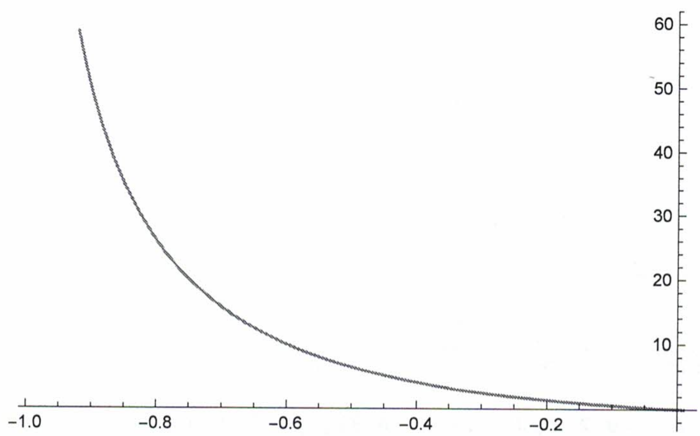
Fy (metric ton) = 1000kgf

~ English ton

m

In[ ]:= Plot[$\frac{Fy}{pAtm\,L\,W}$, {m, 0.01, -0.99}]

Out[ ]=



$P_{av} = \frac{F_y}{A} = \frac{F_y}{LW}$

(Bar)

$$F_y = W\int_0^L (P - P_{atm})\,dx$$