HW #3 Solution

1

```
ClearAll["Global`*"]; (* HW #3  F23  *)
```

```
In[ ]:= (*  begin part 1  *)
```

In[ ]:= $vx = A[x]\frac{y^2}{H^2} + B[x]\frac{y}{H} + \cancel{C[x]};$

$v_x(y=0) = 0 \Rightarrow C = 0$

In[ ]:= $vx = A[x]\frac{y^2}{H^2} + B[x]\frac{y}{H};$

$v_x(y=H) = V_x = A\frac{H^2}{H^2} + B\frac{H}{H}$

$= A + B$

In[ ]:= `V == vx /. y → H`

Out[ ]= `V == A[x] + B[x]`

In[ ]:=
```
sol1 = First[Solve[(vx /. y → H) == V, B[x]]];
B[x] = Replace[B[x], sol1]
```

$B = V_x - A$

Out[ ]= `V - A[x]`

$v_x = \frac{y^2}{H^2}A + \frac{y}{H}(V_x - A)$

$= A(x)\left(\frac{y^2}{H^2} - \frac{y}{H}\right) + V_x\frac{y}{H}$

In[ ]:= `vx`

Out[ ]= $\frac{y\,(V - A[x])}{H} + \frac{y^2 A[x]}{H^2}$

In[ ]:=
```
vx /. y → 0
vx /. y → H
```

check

Out[ ]= `0`

Out[ ]= `V`

In[ ]:= $vx = V\frac{y}{H} + A[x]\left(\frac{y^2}{H^2} - \frac{y}{H}\right)$

Out[ ]= $\frac{V\,y}{H} + \left(-\frac{y}{H} + \frac{y^2}{H^2}\right)A[x]$

In[ ]:=
```
vx /. y → 0
vx /. y → H
```

Out[ ]= `0`

Out[ ]= `V`

$A' = \frac{dA}{dx}$

In[ ]:= $\partial_x vx$

Out[ ]= $\left(-\frac{y}{H} + \frac{y^2}{H^2}\right)A'[x]$

$\frac{\partial v_x}{\partial x} = A'\left(\frac{y^2}{H^2} - \frac{y}{H}\right)$

Cont : $\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$

In[ ]:= $vy = -\int \partial_x vx \, dy + C1$

Out[ ]=

$$C1 + \frac{\left(\frac{H y^2}{2} - \frac{y^3}{3}\right) A'[x]}{H^2}$$

$V_y = HA'\left(\frac{1}{2}\frac{y^2}{H^2} - \frac{1}{3}\frac{y^3}{H^3}\right) + C_1$

In[ ]:=
```
sol1 = First[Solve[(vy /. y → 0) == Vy, C1]];
C1 = Replace[C1, sol1]
```

$V_y(y=0) = V_y$

$C_1 = V_y$

Out[ ]=

Vy

In[ ]:= vy

Out[ ]=

$$Vy + \frac{\left(\frac{H y^2}{2} - \frac{y^3}{3}\right) A'[x]}{H^2}$$

In[ ]:= $vy = Vy + A'[x]\, H \left(\frac{y^2}{2 H^2} - \frac{y^3}{3 H^3}\right)$

Out[ ]=

$$Vy + H \left(\frac{y^2}{2 H^2} - \frac{y^3}{3 H^3}\right) A'[x]$$

$V_y = V_y + HA'\left(\frac{1}{2}\frac{y^2}{H^2} - \frac{y^3}{H^3}\right)$

In[ ]:= $\text{Simplify}[\partial_x vx + \partial_y vy]$

Out[ ]=

0

In[ ]:=
```
vy /. y → 0
vy /. y → H
```

check

Out[ ]=

Vy

Out[ ]=

$$Vy + \frac{1}{6} H A'[x]$$

In[ ]:=
```
Solve[0 == vy /. y → H, A'[x]] (*  key step to solve for A'[x] *)
```

Out[ ]=

$$\left\{\left\{A'[x] \to -\frac{6\, Vy}{H}\right\}\right\}$$

$y = H \quad V_y = 0$

In[ ]:= $A'[x] == -\frac{6\, Vy}{H};$

$0 = V_y + HA'\left(\frac{1}{2} - \frac{1}{3}\right)$

$A' = -6\frac{V_y}{H}$

In[ ]:=
$$vy = Vy + \frac{\left(\frac{H y^2}{2} - \frac{y^3}{3}\right)}{H^2}\left(-\frac{6\,Vy}{H}\right)$$

Out[ ]=
$$Vy - \frac{6\,Vy\left(\frac{H y^2}{2} - \frac{y^3}{3}\right)}{H^3}$$

subst. for A'

In[ ]:=
$$vy = Vy\left(1 - 3\frac{y^2}{H^2} + 2\frac{y^3}{H^3}\right);$$

$$V_y = V_y\left(1 - 3\frac{y^2}{H^2} + 2\frac{y^3}{H^3}\right)$$

Part 2

In[ ]:= vy /. y → 0 (* check vy BCs *)
vy /. y → H

Out[ ]=
Vy

Out[ ]=
0

check

$$A' = -6V_y/H$$

In[ ]:=
$$A = -\frac{6\,Vy}{H}x + C2;$$

$$A = -6\frac{V_y}{H}x + C_2$$

In[ ]:=
$$vx = V\frac{y}{H} + \left(-\frac{6\,Vy}{H}x + C2\right)\left(\frac{y^2}{H^2} - \frac{y}{H}\right);$$

$$V_x(x=0) = V_x\frac{y}{H}$$

In[ ]:= vx /. x → 0

Out[ ]=
$$\frac{V y}{H} + C2\left(-\frac{y}{H} + \frac{y^2}{H^2}\right)$$

$$V_x = \underbrace{A(0)}_{-6\frac{V_y x}{H} + \cancel{C_2}}\left(\frac{y^2}{H^2} - \frac{y}{H}\right) + V_x\frac{y}{H}$$

In[ ]:=
$$\text{Solve}\left[(vx\ /.\ x \to 0) == \frac{V y}{H}, C2\right]$$

Out[ ]=
{{C2 → 0}}

In[ ]:=
$$A = \left(-\frac{6\,Vy}{H}x\right);$$

$$V_x = -6\frac{V_y x}{H}\left(\frac{y^2}{H^2} - \frac{y}{H}\right) + V_x\frac{y}{H}$$

In[ ]:=
$$vx = V\frac{y}{H} - 6\,Vy\frac{x}{L}\frac{L}{H}\left(\frac{y^2}{H^2} - \frac{y}{H}\right);$$ (* part 1 *)

Part 1.

In[ ]:=
```
vx /. y → 0
vx /. y → H
vx /. x → 0 (*  check vx BCs*)
```

check BC's

Out[ ]=
0

Out[ ]=
V

Out[ ]=
$\frac{V y}{H}$

$V_x(x=0) = V_x \frac{y}{H}$

Part 3

In[ ]:=
```
Simplify[∂x vx + ∂y vy] (*  check continuity  *)
```

Out[ ]=
0

Both V_x and V_y contribute to O-M consideration

```
(*  begin part 3, orders of magnitude *)
vyOMinj = Vy; (*    *)
yOM = H;
xOM = L;
```

In[ ]:=
$\frac{vxOM}{xOM} + \frac{vyOM}{yOM} == 0$ (* continuity equation in terms of orders of magnitude *)

Out[ ]=
$\frac{vxOM}{L} + \frac{Vy}{H} == 0$

should check both, I allowed one or the other

In[ ]:=
Solve$\left[\frac{vxOM}{xOM} + \frac{vyOM}{yOM} == 0, vxOM\right]$ (* continuity equation $\frac{\partial}{\partial x}vx + \frac{\partial}{\partial y}vy = 0$ *)

Out[ ]=
$\left\{\left\{vxOM \to -\frac{L\,Vy}{H}\right\}\right\}$

Based on V_y

$V_x \sim V_y \frac{L}{H}$

In[ ]:=
vxOMinj = $\frac{L\,Vy}{H}$; (* OM of x-velocity due to injection *)

In[ ]:=
vxOMslide = $\frac{V}{H}$; (* OM of x-velocity due to sliding *)

(* really should check both, but in this case they are both same O-M *)

vxOM = $\frac{L\,Vy}{H}$

Based on V_x

In[ ]:=
InerOMx = ρ vxOM $\frac{vxOM}{xOM}$ (* first convective inertia term ρ vx$\frac{\partial}{\partial x}$vx *)

$V_x \sim V_x$

Out[ ]=
$\frac{L\,Vy^2\,\rho}{H^2}$

In[ ]:= InerOMy = ρ vyOM $\frac{vxOM}{yOM}$ (* 2nd convective inertia term ρ vy $\frac{\partial}{\partial y}$vx *)

Out[ ]= $\frac{L\,Vy^2\,\rho}{H^2}$

Inertia: $\rho v_x \frac{\partial v_x}{\partial x}$

based on V_y: $\rho V_y \frac{L}{H} V_y \frac{L}{H} \frac{1}{L}$

In[ ]:= (* both have same OM *)

In[ ]:= viscOMx = μ $\frac{vxOM}{xOM^2}$ (* viscous extension term $\mu\frac{\partial}{\partial x}\frac{\partial}{\partial x}$vx *)

Out[ ]= $\frac{Vy\,\mu}{H\,L}$

Based on V_x: $\rho V_x \frac{V_x}{L}$

In[ ]:= viscOMy = μ $\frac{vxOM}{yOM^2}$ (* viscous shear term $\mu\frac{\partial}{\partial y}\frac{\partial}{\partial y}$vx *)

Out[ ]= $\frac{L\,Vy\,\mu}{H^3}$

Viscous $\cancel{\frac{\partial^2 V_x}{\partial x^2}} + \frac{\partial^2 V_x}{\partial y^2}$

$\frac{H^2}{L^2} << 1$

In[ ]:= $\frac{viscOMy}{viscOMx}$ (* ratio of viscous terms, shear much larger than extension *)

Out[ ]= $\frac{L^2}{H^2}$

V_y: $\mu V_y \frac{L}{L} \cdot \frac{1}{H^2}$

In[ ]:= $\frac{InerOMx}{viscOMy}$ (* ratio of inertia (either term) to largest viscous, this is a Reynolds number, part 2 *)

$V_x \sim \mu \frac{V_x}{H^2}$

Out[ ]= $\frac{H\,Vy\,\rho}{\mu}$

$\frac{Inertia}{Visc} \sim \frac{\rho V_y H}{\mu}$ or $\frac{\rho V_x H}{\mu} \frac{H}{L}$

both are Reyn #

both must be << 1

In[ ]:= (* begin part ~~3~~ 4 *)

In[ ]:= dpdx = μ $\partial_{y,y}$vx (* discarding inertia, x-Navier Stakes left with *)

Out[ ]= $-\frac{12\,Vy\,x\,\mu}{H^3}$

X-NS

$0 = \frac{dp}{dx} + \mu \frac{\partial^2 V_x}{\partial y^2}$

In[ ]:= p = $-\frac{6\,Vy\,x^2\,\mu}{H^3}$ + C3

Out[ ]= C3 $-\frac{6\,Vy\,x^2\,\mu}{H^3}$

$\frac{\partial^2 V_x}{\partial y^2} = -\frac{12 V_y x \mu}{H^3}$

$p = -\frac{6 V_y x^2 \mu}{H^3} + C_3$

In[ ]:=
```
p = -6 (Vy μ/H) (L^2/H^2) (x^2/L^2) + p0 (* inlet pressure is p0, so C3=p0, part 3 *)
```

4

Out[ ]= $p0 - \frac{6\,Vy\,x^2\,\mu}{H^3}$

$p(x=0) = p_0 \quad C_3 = p_0$

$p = p_0 - \frac{6 V_y x^2}{H^3}\mu$

In[ ]:=
```
(* begin part 4 *)
```

5

In[ ]:= gradv = $\begin{pmatrix} \partial_x vx & \partial_x vy \\ \partial_y vx & \partial_y vy \end{pmatrix}$

$\underset{\sim}{\nabla} \underset{\sim}{V}$

Out[ ]= $\left\{\left\{-\frac{6\,Vy\left(-\frac{y}{H}+\frac{y^2}{H^2}\right)}{H}, 0\right\}, \left\{\frac{V}{H}-\frac{6\,Vy\,x\left(-\frac{1}{H}+\frac{2y}{H^2}\right)}{H}, Vy\left(-\frac{6y}{H^2}+\frac{6y^2}{H^3}\right)\right\}\right\}$

$\underset{\sim}{\nabla} \underset{\sim}{V}^T$

In[ ]:= gradvT = $\begin{pmatrix} \partial_x vx & \partial_y vx \\ \partial_x vy & \partial_y vy \end{pmatrix}$;

In[ ]:=
```
γdot = Simplify[gradv + gradvT] (* part 4 *)
```

5

$\underset{\approx}{\dot{\gamma}} =$

Out[ ]= $\left\{\left\{\frac{12\,Vy\,(H-y)\,y}{H^3}, \frac{H^2 V + 6\,H\,Vy\,x - 12\,Vy\,x\,y}{H^3}\right\}, \left\{\frac{H^2 V + 6\,H\,Vy\,x - 12\,Vy\,x\,y}{H^3}, -\frac{12\,Vy\,(H-y)\,y}{H^3}\right\}\right\}$

In[ ]:= ~~(* begin part 5 *)~~

In[ ]:= δ = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;

$\underset{\approx}{\sigma} = -p\,\underset{\approx}{\delta} + \mu\,\underset{\approx}{\dot{\gamma}}$

In[ ]:= σ = -p δ + μ γdot ~~(* part 5 *)~~

Out[ ]= $\left\{\left\{-p0+\frac{6\,Vy\,x^2\,\mu}{H^3}+\frac{12\,Vy\,(H-y)\,y\,\mu}{H^3}, \frac{(H^2 V + 6\,H\,Vy\,x - 12\,Vy\,x\,y)\,\mu}{H^3}\right\}, \left\{\frac{(H^2 V + 6\,H\,Vy\,x - 12\,Vy\,x\,y)\,\mu}{H^3}, -p0+\frac{6\,Vy\,x^2\,\mu}{H^3}-\frac{12\,Vy\,(H-y)\,y\,\mu}{H^3}\right\}\right\}$

upper surface

2

$\hat{n}_2 = \hat{e}_y$

In[ ]:= n2 = {0, 1}; ~~n1 = {0, -1}; (* part 6 *)~~

In[ ]:= ~~(* begin part 7 *)~~

$\underset{\approx}{\sigma}_2 = \underset{\approx}{\sigma}(y=H)$

In[ ]:=
```
σ2 = σ /. y → H (* stress tensor at upper surface *)
```

Out[ ]= $\left\{\left\{-p0+\frac{6\,Vy\,x^2\,\mu}{H^3}, \frac{(H^2 V - 6\,H\,Vy\,x)\,\mu}{H^3}\right\}, \left\{\frac{(H^2 V - 6\,H\,Vy\,x)\,\mu}{H^3}, -p0+\frac{6\,Vy\,x^2\,\mu}{H^3}\right\}\right\}$

$$\underset{\approx}{\sigma}_2 = \begin{bmatrix} -p_0 + \mu \frac{6 V_y x^2}{H^3} & \frac{\mu(H^2 V - 6 H V_y x)}{H^3} \\ \sigma_{xy} & \sigma_{yy} \end{bmatrix}$$

$\sigma_{xx} = \sigma_{yy} = -p$

$\sigma_{xy} = \sigma_{yx}$

$$\sigma_{xx} = -p_0 + 6\mu \frac{V_y x^2}{H^3} \qquad \sigma_{yx} = \frac{\mu}{H^3}\left(V_x H^2 - 6 x V_y H\right)$$

In[ ]:= `f2 = n2.σ2 (* traction vector on fluid at upper surface *)`

Out[ ]=

$$\left\{\frac{(H^2 V - 6 H Vy x)\mu}{H^3}, -p0 + \frac{6 Vy x^2 \mu}{H^3}\right\}$$

$$\underset{\sim}{f}_2 = \hat{n}_2 \cdot \underset{\approx}{\sigma}_2 = \hat{e}_y \cdot \left(\sigma_{xx}\hat{e}_x\hat{e}_y \text{ etc.}\right)$$

In[ ]:= `F2x = W ∫_0^L f2[[1]] dx (* x-component of force on fluid at upper surface, dA = W dx, x-component of traction f2[[1]] = f2x *)`

Out[ ]=

$$F_{2x} = W\left(\frac{L V \mu}{H} - \frac{3 L^2 Vy \mu}{H^2}\right)$$

$$\underset{\sim}{F}_2 = F_{2x}\hat{e}_x + F_{2y}\hat{e}_y \qquad f_{2x} = \sigma_{yx}$$

$$f_{2y} = \sigma_{xx}$$

$$F_{2x} = \int f_{2x}\, dA = W\int_0^L f_{2x}\, dx \quad \text{etc}$$

In[ ]:= `F2y = W ∫_0^L f2[[2]] dx`

Out[ ]=

$$F_{2y} = W\left(-L\, p0 + \frac{2 L^3 Vy \mu}{H^3}\right)$$

In[ ]:= `ex = {1, 0}; ey = {0, 1};`

In[ ]:= `F2 = F2x ex + F2y ey (* force vector on fluid oat upper surface *)`

Out[ ]=

$$\left\{W\left(\frac{L V \mu}{H} - \frac{3 L^2 Vy \mu}{H^2}\right), W\left(-L\, p0 + \frac{2 L^3 Vy \mu}{H^3}\right)\right\}$$

In[ ]:= `MatrixForm[F2]` ~~(* part 7 *)~~

Out[ ]//MatrixForm=

$$\begin{pmatrix} W\left(\frac{L V \mu}{H} - \frac{3 L^2 Vy \mu}{H^2}\right) \\ W\left(-L\, p0 + \frac{2 L^3 Vy \mu}{H^3}\right) \end{pmatrix} =$$

In[ ]:= `(* begin part` ~~8~~ 6 `*)`

$$\hat{n}_L = \hat{e}_x$$

In[ ]:= `v = {vx, vy}; (* velocity vector *)`

$$\underset{\sim}{V} = V_x\hat{e}_x + V_y\hat{e}_y$$

In[ ]:= `nL = {1, 0}; (* outer unit normal vector at exit *)`

In[ ]:= `vL = v /. x → L (* velocity vector vector at exit *)`

$$\underset{\sim}{V}_L = \underset{\sim}{V}(x=L)$$

Out[ ]=

$$\left\{\underbrace{\frac{V y}{H} - \frac{6 L Vy\left(-\frac{y}{H} + \frac{y^2}{H^2}\right)}{H}}_{V_{Lx}}, \underbrace{Vy\left(1 - \frac{3 y^2}{H^2} + \frac{2 y^3}{H^3}\right)}_{V_{Ly}}\right\}$$

$$\underset{\sim}{V}_L \cdot \hat{n}_L = V_x(x=L)$$

In[ ]:= `mdotL = ρ W ∫_0^H (vL.nL) dy (* part` ~~8~~ 6`, ṁ = ∫ρ (v·n̂) dA, dA = W dx *)`

$$dA = W\, dy$$

Out[ ]=

$$\left(\frac{H V}{2} + L\, Vy\right) W \rho$$

$$\dot{m}_L = \rho\int\left(\underset{\sim}{V}\cdot\hat{n}\right)_{x=L} dA = W\rho\left(\frac{H V_x}{2} + L V_y\right)$$

$$\hat{n}_H = \hat{e}_y$$

Part 7

$$\dot{m}_H = \int \underset{\sim}{V} \cdot \hat{n}_H \, dA$$

$$\doteq W\rho \int_0^L V_y(y=H)\,dy$$

In[ ]:= vH = v /. y → H (* velocity vector vector at upper surface *)

Out[ ]= {V, 0} $= \underset{\sim}{V}(y=H) = V_x \hat{i}$

In[ ]:= mdotH = ρ W $\int_0^L$ (vH.n2) dx (* part ~~8~~ 7, ṁ = ∫ρ (v·n̂)dA, dA = W dx *)

Out[ ]= 0

In[ ]:= p0 = 1. × 10^3; H = 0.005; L = 0.050; W = 0.1; ρ = 800.0; μ = 0.050; Vy = 0.001;
V = 0.01;

In[ ]:= $\frac{H\,Vy\,\rho}{\mu}$ (* check Reynolds number, should be small, it is *)

Out[ ]= 0.08

also $Re^* = \frac{\rho V_x H}{\mu} \frac{H}{L} = 0.08$ $\ll 1$

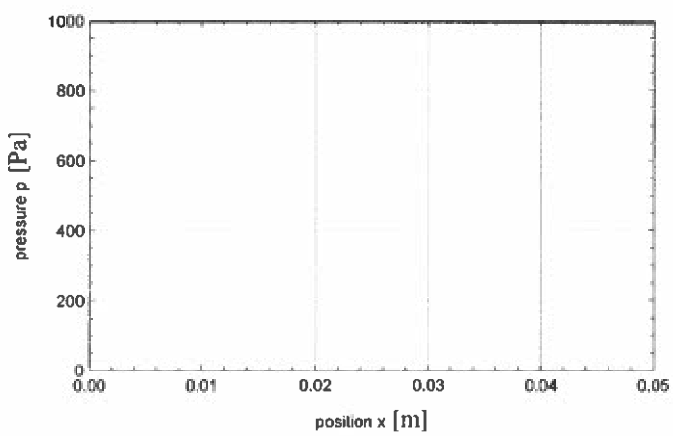
same in this case

In[ ]:= p

Out[ ]= 1000. − 2400. x^2

In[ ]:= Plot[{p}, {x, 0, L}, Frame → True, GridLines → Automatic,
FrameLabel → {"position x [m]", "pressure p [Pa]"}, PlotRange → {{0, 0.05}, {0, 1000}}]

Out[ ]=



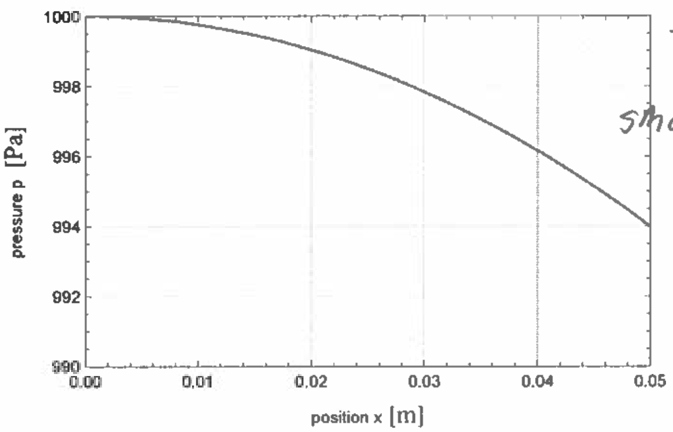
plot (almost flat)

Part 10

(Part 9, just below)

```
Plot[(p), {x, 0, L}, Frame → True, GridLines → Automatic,
 FrameLabel → {"position x [m]", "pressure p  [Pa]"}, PlotRange → {{0, 0.05}, {990, 1000}}]
 (* while correct, this graph is misleading   *)
```

Out[ ]=

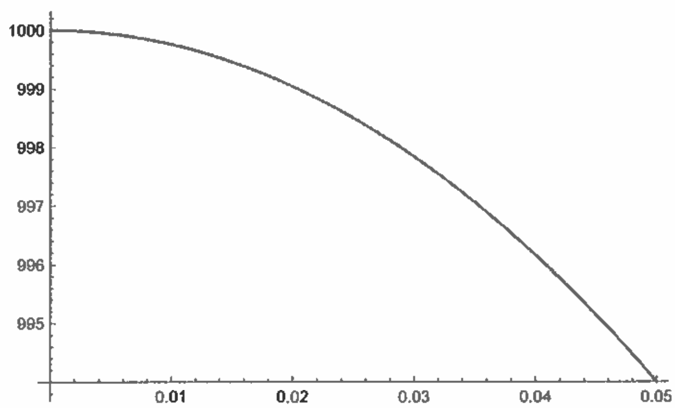


small

default y-axis misleading

In[ ]:= `Plot[p, {x, 0, L}]`

Out[ ]=



In[ ]:= `MatrixForm[F2] (* part ~~10~~ *)` 9

Out[ ]//MatrixForm=

$$\begin{pmatrix} -0.001 \\ -4.99 \end{pmatrix}$$

$\hat{F}_2$ (in N)

In[ ]:= `mdotL`

Out[ ]=

0.006

$\dot{m}_L$ (in $\frac{kg}{s}$)

In[ ]:= `p /. x → L (*  check pressure at exit, should be positive  *)`

Out[ ]=

994.

In[ ]:= $W \frac{4 L^3 Vy \mu}{H^3}$; `(*  check units of force term  *)`

$F_y = \frac{4 W L^3 V_y \mu}{h^3}$

$W \quad (L/h)^3 \quad V_y \quad \mu$

In[ ]:= $\text{Meter} \left(\frac{\text{Meter}}{\text{Meter}}\right)^3 \frac{\text{Meter}}{\text{Second}} \frac{\text{Newton Second}}{\text{Meter}^2}$

Out[ ]=

Newton

```
In[ ]:= W L p0; (*  check units of force term  *)
```

In[ ]:= $\text{Meter Meter} \frac{\text{Newton}}{\text{Meter}^2}$

Out[ ]=

Newton

```
In[ ]:= 2 L Vy W ρ; (*  check units of mass flow rate term  *)
```

In[ ]:= $\text{Meter} \frac{\text{Meter}}{\text{Second}} \text{Meter} \frac{\text{Kilogram}}{\text{Meter}^3}$

Out[ ]=

$\frac{\text{Kilogram}}{\text{Second}}$