A Brief Review of The Hessian

The Hessian is a Matrix of a Function's Second Derivatives

Definition: Hessian of a function

If $f : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable at $\bar{x} \in \mathbb{R}^n$, then the Hessian of f at $\bar{x}$ is the matrix

$$\nabla^2 f(\bar{x}) \equiv H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_2 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n} & \frac{\partial^2 f}{\partial x_2 \partial x_n} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix},$$

where each of the partial derivatives is evaluated at $\bar{x}$.

We Will Rely on the Symmetry of the Hessian

Theorem: Symmetry of the Hessian

Consider $f : \mathbb{R}^n \to \mathbb{R}$. If the second-partial derivatives $\partial^2 f / \partial x_i \partial x_j$ are continuous at $\bar{x}$, for all i, j, then

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

and, therefore,

$$H^T = H.$$

We Will Also Need to Know About Positive Definite Matrices

Definition: Positive Definite

A matrix $H \in \mathbb{R}^{n \times n}$ is said to be positive definite if $p^T H p > 0$ for all $p \neq 0$.