In[ ]:= `ClearAll["Global`*"]; (* Fluids Exam #3 F22 *)`

In[ ]:= $h = H(1 + \epsilon \, \mathrm{Sin}[\omega t])$;
$V = \partial_t h$

Out[ ]= $H \epsilon \omega \, \mathrm{Cos}[t \omega]$

$$V = \frac{dh}{dt} = H\epsilon\omega \cos\omega t$$

In[ ]:= $\mathrm{Ovy} = H \epsilon \omega$; $\mathrm{Ovx} = H \epsilon \omega \frac{L}{H}$; $\mathrm{Odvxdx} = \frac{\mathrm{Ovx}}{L}$ (* O means order-magnitude *)
Ovxdvxdx = Ovx Odvxdx
$\mathrm{Odvxdy} = \frac{\mathrm{Ovx}}{H}$
Ovydvxdy = Ovy Odvxdy
Odvxdt = Ovx ω (* d/dt ~ order ω *)
$\frac{\mathrm{Ovxdvxdx}}{\mathrm{Odvxdt}}$ (* part a *)

$$V_y \sim H\epsilon\omega$$

$$V_x \sim \frac{L}{H} V_y \text{ (cont)} = L\epsilon\omega$$

$$\frac{\partial}{\partial t} \sim \omega$$

Out[ ]= $\epsilon \omega$

Out[ ]= $L \epsilon^2 \omega^2$

$$V_x \frac{\partial V_x}{\partial x} \sim L\epsilon\omega \frac{L\epsilon\omega}{L} = L\epsilon^2\omega^2$$

Out[ ]= $\frac{L \epsilon \omega}{H}$

Out[ ]= $L \epsilon^2 \omega^2$

$$\frac{\partial V_x}{\partial t} \sim L\epsilon\omega \cdot \omega = L\epsilon\omega^2$$

Out[ ]= $L \epsilon \omega^2$

Out[ ]= ϵ

$$a) \; \frac{V_x \frac{\partial V_x}{\partial x}}{\partial V_x / \partial t} \sim \frac{L\epsilon^2\omega^2}{L\epsilon\omega^2} \sim \epsilon$$

$\mathrm{Visc} = \frac{L \epsilon H \mu \omega}{H^3}$;
$\mathrm{Iner} = \frac{L \epsilon H \rho \omega^2}{H}$;
Iner / Visc (* part b *)

Out[ ]= $\frac{H^2 \rho \omega}{\mu}$

$$\mathrm{visc} \sim \mu \frac{\partial^2 V_x}{\partial y^2} \sim \mu \frac{L\epsilon\omega}{H^2}$$

$$b) \; \frac{\text{unsteady}}{\text{visc}} \sim \frac{\rho L \cancel{\epsilon} \omega^2}{\mu L \cancel{\epsilon} \omega H^2} \sim \frac{\rho \omega H^2}{\mu}$$

a Reyn number

In[ ]:= $\mathrm{vy} = V \left(a0 + a1 \frac{y}{h} + a2 \frac{y^2}{h^2} + a3 \frac{y^3}{h^3} \right)$;

In[ ]:= $vx = -\int \partial_y (vy)\, dx$

Out[ ]=

$$V_x = -\frac{3\,a3\,x\,y^2\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]}{H^2\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3} - \frac{2\,a2\,x\,y\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]}{H\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2} - \frac{a1\,x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]}{1+\epsilon\,\mathrm{Sin}[t\,\omega]}$$

(c)

$$vx = x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(-\frac{2\,a2\,y}{H\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2} - \frac{3\,a3\,y^2}{H^2\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3} - \frac{a1}{1+\epsilon\,\mathrm{Sin}[t\,\omega]}\right);$$ (* part c *)

y = 0 : vx = vy = 0; // y = h[t] : vy = dhdt = $H\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]$; (* part d *)

In[ ]:= a1 = a0 = 0;

(* solve for the a's *) $\longrightarrow$ $V_x(y=0) = 0 \Rightarrow a_1 = 0$ (d)

$V_y(y=0) = 0 \Rightarrow a_0 = 0$

In[ ]:= vy

vx

Out[ ]=

$$H\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(\frac{a3\,y^3}{H^3\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3} + \frac{a2\,y^2}{H^2\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)$$

Out[ ]=

$$x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(-\frac{3\,a3\,y^2}{H\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3} - \frac{2\,a2\,y}{H\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)$$

In[ ]:= vx /. y → h

Out[ ]=

$$x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(-\frac{3\,a3\,h^2}{H^2\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3} - \frac{2\,a2\,h}{H\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)$$

In[ ]:= Solve[(vx /. y → h) == 0, a3]

Out[ ]=

$$\left\{\left\{a3 \to -\frac{2\,a2}{3}\right\}\right\}$$

$V_x(y=h) = 0 \Rightarrow a_3 = -\frac{2a_2}{3}$

In[ ]:= $a3 = -\frac{2\,a2}{3}$;

vx

Out[ ]=

$$x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(\frac{2\,a2\,y^2}{H^2\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3} - \frac{2\,a2\,y}{H\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)$$

vy

Out[ ]=

$$H\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(-\frac{2\,a2\,y^3}{3\,H^3\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3} + \frac{a2\,y^2}{H^2\,(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)$$

In[ ]:= `Solve[(vy /. y → h) == V, a2]`

Out[ ]= `{{a2 → 3}}`

$V_y(y=h) = V$

$\Rightarrow a_2$

In[ ]:= `a2 = 3;`
`vx (* part e *)`

Out[ ]=
$$x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(\frac{6y^2}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}-\frac{6y}{H(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)$$

`vy (* part f *)`

Out[ ]=
$$H\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(-\frac{2y^3}{H^3(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}+\frac{3y^2}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)$$

In[ ]:=
$$vy = H\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(-\frac{2y^3}{H^3(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}+\frac{3y^2}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right);$$

In[ ]:=
$$vx = x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(\frac{6y^2}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}-\frac{6y}{H(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right);$$

In[ ]:=
$$\psi = -\int vy\,dx + f[y]$$

$V_y = -\frac{\partial\psi}{\partial x}$

Out[ ]=
$$f[y]+\frac{2xy^3\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}-\frac{3xy^2\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]}{H(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}$$

$\psi = -\int V_y\,dx + f(y)$

In[ ]:=
$$\psi = f[y] + x\,\epsilon\,\omega\,H\,\mathrm{Cos}[t\,\omega]\left(\frac{2y^3}{H^3(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}-\frac{3y^2}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right);$$

$V_x = \frac{\partial\psi}{\partial y}$

$vx == \partial_y\psi$

Out[ ]=
$$H\,x\,\epsilon\,\omega\,\mathrm{Cos}[t\,\omega]\left(\frac{6y^2}{H^3(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}-\frac{6y}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right)+f'[y]$$

$V_x(y=0) \Rightarrow f'(y)=0$

`(* compare to vx above, f=0 *)`

$f = \text{const}$

f arbitrary

set $f=0$

$$\psi = x\,\epsilon\,\omega\,H\,\mathrm{Cos}[t\,\omega]\left(\frac{2y^3}{H^3(1+\epsilon\,\mathrm{Sin}[t\,\omega])^3}-\frac{3y^2}{H^2(1+\epsilon\,\mathrm{Sin}[t\,\omega])^2}\right);\ \text{(* part g *)}$$

In[ ]:= `Expand[(vzB[x, y, z, t] + vzP[x, y, z, t]) ∂z((vxB[x, y, z, t] + vxP[x, y, z, t])^2)]`

`T1 = 2 vxB[x, y, z, t] × vzB[x, y, z, t] vxB^(0,0,1,0)[x, y, z, t]`
`T1av = 2 vxB[x, y, z, t]^2 vxB^(1,0,0,0)[x, y, z, t]`

$\Rightarrow \frac{\partial \bar{V}_x}{\partial z}$

`T2 == 2 vxP[x, y, z, t] × vzB[x, y, z, t] vxB^(0,0,1,0)[x, y, z, t]`
`T2av == 2 vzB[x, y, z, t] vxB^(0,0,1,0)[x, y, z, t] vxP[x, y, z, t] == 0`

$VXB \Rightarrow \bar{V}_x \qquad VXP = V_x'$ etc

$$\phi = (\bar{V}_z + V_z')\frac{\partial}{\partial z}\left(\bar{V}_x^{\,2} + 2\bar{V}_x V_x' + V_x'^{\,2}\right)$$

$$T_1 = 2\bar{V}_x \bar{V}_z \frac{\partial \bar{V}_x}{\partial z}$$

$$\not{d} = \bar{V}_z \left(2\bar{V}_x \frac{\partial \bar{V}_x}{\partial z} \right) + \ldots = 8 \text{ terms total}$$

```
T3 == 2 vxB[x, y, z, t] vxB(0,0,1,0) [x, y, z, t] vzP[x, y, z, t]
T3 == 2 vxB[x, y, z, t] vxB(0,0,1,0) [x, y, z, t] vzP[x, y, z, t]  = 0

T4 == 2 vxP[x, y, z, t] × vzP[x, y, z, t] vxB(0,0,1,0) [x, y, z, t]
T4av == 2 vxB(0,0,1,0) [x, y, z, t] Average[xP[x, y, z, t] × vzP[x, y, z, t]]

T5 == 2 vxB[x, y, z, t] × vzB[x, y, z, t] vxP(0,0,1,0) [x, y, z, t]
T5av == 2 vxB[x, y, z, t] × vzB[x, y, z, t] vxP(0,0,1,0) [x, y, z, t] = 0

T6 == 2 vxP[x, y, z, t] × vzB[x, y, z, t] vxP(0,0,1,0) [x, y, z, t]
T6 == 2 vzB[x, y, z, t] × Average[vxP[x, y, z, t] vxP(0,0,1,0) [x, y, z, t]]

T7 == 2 vxB[x, y, z, t] × vzP[x, y, z, t] vxP(0,0,1,0) [x, y, z, t]
T7 == 2 vxB[x, y, z, t] × Average[vzP[x, y, z, t] vxP(0,0,1,0) [x, y, z, t]]

T8 == 2 vxP[x, y, z, t] × vzP[x, y, z, t] vxP(0,0,1,0) [x, y, z, t]
T8 == Average[2 vxP[x, y, z, t] × vzP[x, y, z, t] vxP(0,0,1,0) [x, y, z, t]]

(* **************** )
```

av

$$\overline{\left(2\bar{V}_z \bar{V}_x \frac{\partial \bar{V}_x}{\partial z}\right)} = 2\bar{V}_z \bar{V}_x \frac{\partial \bar{V}_x}{\partial z} \quad \text{etc}$$

4th term

$$\overline{2 V_x' V_z' \frac{\partial \bar{V}_x}{\partial z}} = 2 \frac{\partial \bar{V}_x}{\partial z} \overline{V_x' V_z'}$$

etc.

In[ ]:= ClearAll["Global`*"];

$vS = \sqrt{\frac{\tau w}{\rho}}$; $yplus = \frac{vS\, y}{\nu}$; $dvdy = \frac{\tau w}{\mu}$;

$lmix = \kappa\, y \left(1 - Exp\left[-\frac{yplus}{A}\right]\right)$;

$\mu t = \rho\, lmix^2\, Abs[dvdy]$;

$\tau = (\mu + \mu t)\, dvdy$

Out[ ]=

$$\frac{\tau w \left(\mu + \left(1 - e^{-\frac{y\sqrt{\frac{\tau w}{\rho}}}{A\nu}}\right)^2 y^2 \kappa^2 \rho\, Abs\left[\frac{\tau w}{\mu}\right]\right)}{\mu}$$

$$yplus = y^+ = \frac{y V^*}{\nu}$$

$$V^* = \sqrt{\frac{\tau w}{\rho}}$$

sucessive substitutions

In[ ]:= $\nu = 1.5 \times 10^{-5}$; $\rho = 1.2$; $\mu = \rho\, \nu$;

A = 25.; U = 10.; κ = 0.41; τw = 0.2; yf = 0.012; $vS = \sqrt{\frac{\tau w}{\rho}}$;

In[ ]:= Plot[τ, {y, 0, yf}] (* part h *)

Out[ ]=