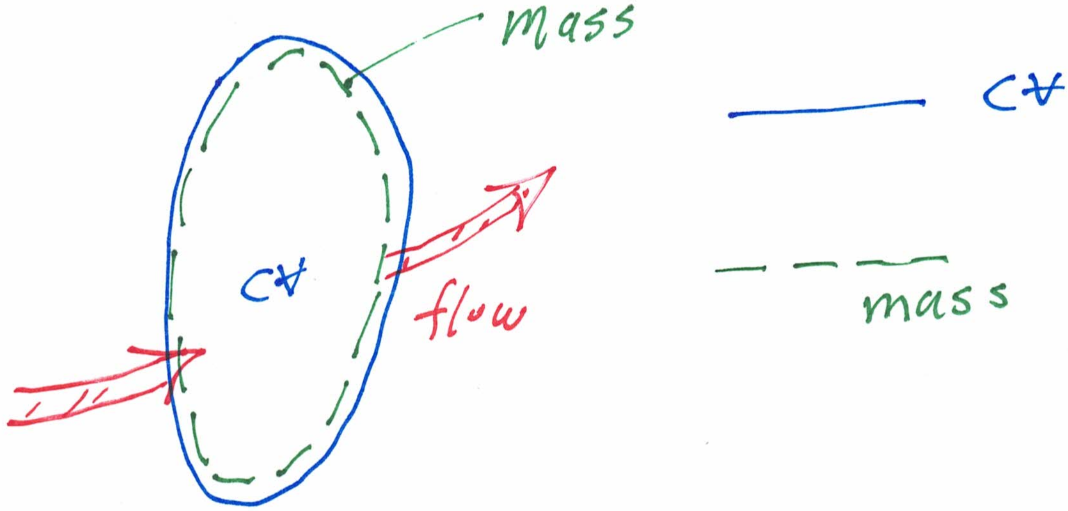
Class #9

Reynolds Transport Theorem

At instant mass fills $C\forall$



Φ any extensive property   mass, momentum

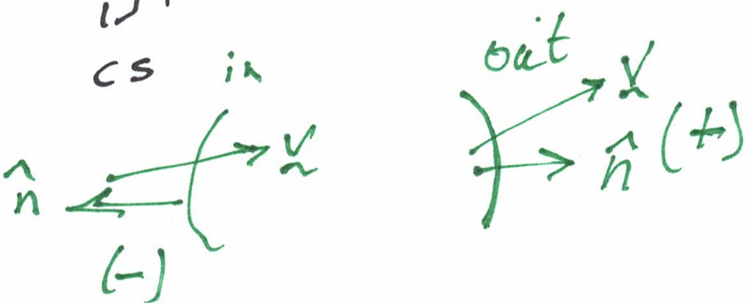
$\phi = \Phi / m$ intensive property

$$\left.\frac{d\Phi}{dt}\right|_{mass} = \left.\frac{d\Phi}{dt}\right|_{C\forall} + \dot{\Phi}$$

Net flow of Φ out

$$\frac{d}{dt}\underbrace{\int_{C\forall} \rho \phi \, d\forall}_{\Phi_{C\forall}} \qquad \int_{CS} \rho \phi (\underset{\sim}{V} \cdot \underset{\sim}{n}) \, dA$$

$d\underset{\sim}{A} = \underset{\sim}{n} \, dA$



1-aa

$$\Phi = m \qquad \phi = \frac{\Phi}{m} = 1$$

$$\Phi = m\underset{\sim}{V} \qquad \phi = \frac{\Phi}{m} = \underset{\sim}{V}$$

$$= \underset{\sim}{G}$$

momentum

$$\left.\frac{d\Phi}{dt}\right|_{mass} = 0 = \frac{d}{dt}\int_{CV} \rho \, d\forall$$

$\Phi = mass$

$$+ \int_{CS} \rho \, \underset{\sim}{V} \cdot \underset{\sim}{n} \, dA$$

$$\left.\frac{d\Phi}{dt}\right|_{mass} = \frac{d}{dt} m\underset{\sim}{V} = m\underset{\sim}{a} = \sum \underset{\sim}{F}$$

$$\sum \underset{\sim}{F} = \frac{d}{dt}\int_{CV} \rho \underset{\sim}{V} \, d\forall + \int_{CS} \rho \underset{\sim}{V} (\underset{\sim}{V} \cdot n) \, dA$$

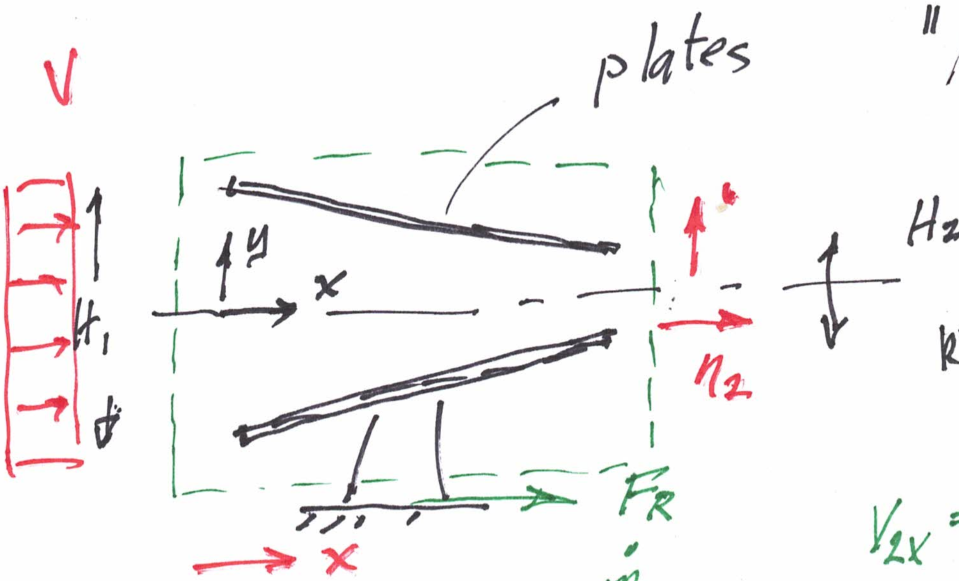
1-a Global Example

"Nozzle"

W into paper

on stand

know: Inlet H_1, W
density
mass flow rate $\dot{m}$



$$V_{2x} = V_{20}\left(1 - \frac{y^2}{(H_2/2)^2}\right)$$

steady
incomp.

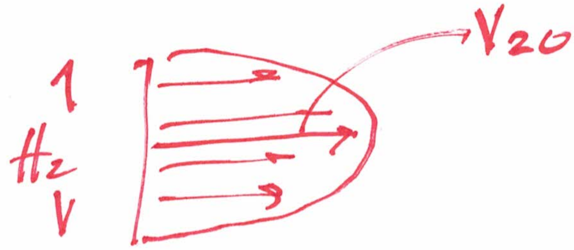
$$V_{x1} = \text{const} = \frac{\dot{m}}{\rho W H_1}$$

$$\dot{m}_2 = \int_{CS} \rho(\underset{\sim}{V} \cdot n)\, dA = \rho W \int_{-H_2/2}^{+H_2/2} V_{2x}\, dy$$

$$= \frac{2}{3} V_{20} H_2 W \rho$$

$$\sum F = F_R = + \int \rho V_x (\underset{\sim}{V} \cdot \underset{\sim}{n})\, dA$$

$$=$$



Find F_R

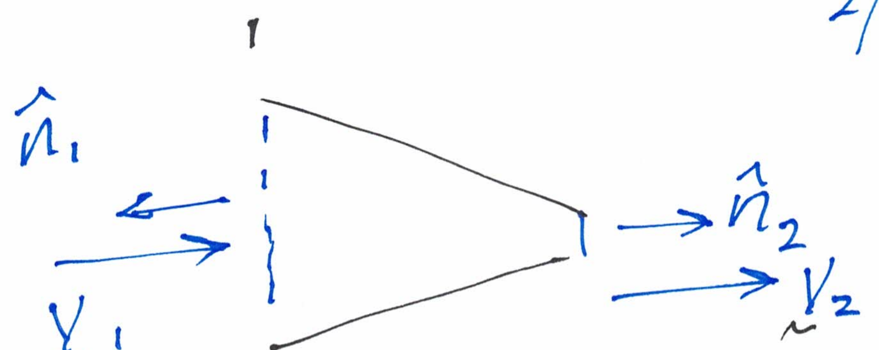
force on nozzle

Nozzle Example (cont)

mass

$$0 = \cancel{\frac{d}{dt} m_{cv}}^{\,0} + \int \rho \,(\underset{\sim}{V} \cdot n)\, dA$$

$$0 = \underbrace{-\rho V_1 W H_1}_{\dot{m}} + \underbrace{\rho W \int_{-H/2}^{H/2} V_{20}\left(1 - \frac{y^2}{(H/2)^2}\right) dy}_{2/3\, H_2 V_{20}}$$



$$V_{20} = \frac{3}{2} \frac{\dot{m}}{H_2 W \rho}$$

$$V_x = \frac{3}{2} \frac{\dot{m}}{\rho H_2 W} \left(1 - \frac{y^2}{(H/2)^2}\right)$$

1-c

x-momentum

$$\sum F = F_R = \cancel{\frac{d}{dt} m V \Big|_{CV}}^{0} + \int \rho V_x (\underset{\sim}{V} \cdot \underset{\sim}{n})\, dA$$

① $n_1 = -\hat{e}_x$

$\hat{n} + \hat{e}_x$

=

$$= \underbrace{-\rho \int_{-H/2}^{+H/2} V_{x_1}^2 W\, dy}_{\text{at } 1} + \underbrace{\rho \int_{-H/2}^{H/2} V_{x_2}^2 W\, dy}_{\text{at } 2}$$

⋮ algebra

$$F_R = \frac{(6 H_1 - 5 H_2)\, \dot{m}^2}{5 H_1 H_2 W \rho}$$

There is still a force if $H_1 = H_2$ due to different profiles

$$\int_{CS} \Rightarrow \int_{CV}$$

divergence theorem

just math changes volume $\int$ to surface $\int$

$$\int_{CS} \rho \phi (\underset{\sim}{V} \cdot \underset{\sim}{n}) \, dA = \int_{C\forall} \underset{\sim}{\nabla} \cdot (\rho \phi \underset{\sim}{V}) \, d\forall$$

turn global continuity to one volume $\int$

$$\int_{C\forall} \frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla} \cdot (\rho \underset{\sim}{V}) \,] \, d\forall = 0$$

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} \rho V_x + \frac{\partial}{\partial y} \rho V_y$$

Differential continuity

2-D

Incomp $\rho = \text{const}$

steady $\partial \rho / \partial t = 0$

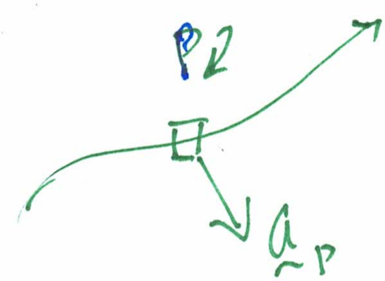
$$\boxed{\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0}$$

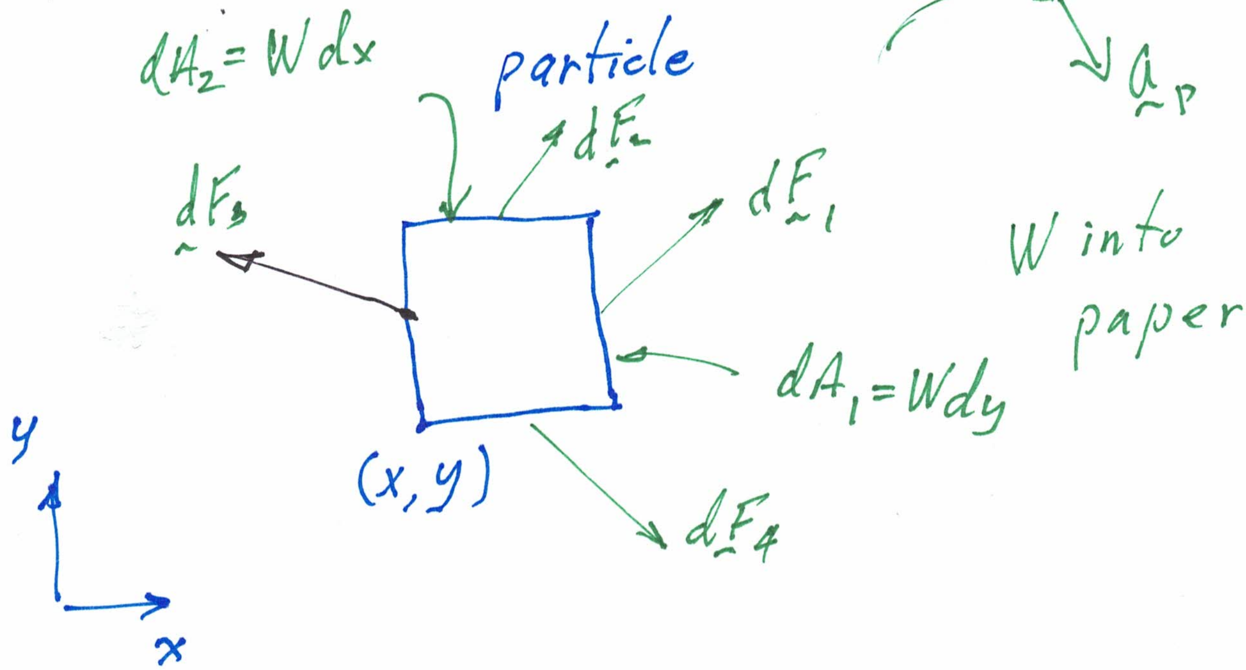
$$\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \operatorname{div} V = 0$$

Global Momentum Eq.

$$\sum \underset{\sim}{F} = \frac{d}{dt}\int_{CV} \rho \underset{\sim}{V}\, d\forall + \int_{CS} \rho \underset{\sim}{V} (\underset{\sim}{V}\cdot n)\, dA$$

Diffential Momentum Eq.





$$m \underset{\sim}{a}_p = \sum \underset{\sim}{F} \qquad \frac{1}{\forall}\left(m \underset{\sim}{a}_p = \sum \underset{\sim}{F}\right)$$

$$\frac{m}{\forall} = \rho \qquad \underset{\sim}{a}_p = \frac{D\underset{\sim}{V}}{Dt} = \underline{\frac{\partial \underset{\sim}{V}}{\partial t} + \underset{\sim}{V}\cdot \nabla \underset{\sim}{V}}$$

$$d\underset{\sim}{F}_1 = \underset{\sim}{t}_1\, dA_1 \qquad dA_1 = W\,dy$$

$$\underset{\sim}{n}_1 = \hat{e}_x$$

$$\underset{\sim}{t}_1 = \hat{e}_x \cdot \underset{\approx}{\sigma}$$

$$= \hat{e}_x \cdot \left(\sigma_{xx}\hat{e}_x\hat{e}_x + \sigma_{xy}\hat{e}_x\hat{e}_y + \sigma_{yx}e_y\hat{e}_x + \sigma_{yy}\hat{e}_y\hat{e}_y\right)$$

$$= \sigma_{xx}\hat{e}_x + \sigma_{xy}\hat{e}_y$$

$\frac{\text{force}}{\text{area}}$ on x-surfac in y-directi

$$\sigma_{xx}(x+dx) = \sigma_{xx}(x) + \frac{\partial \sigma_{xx}}{\partial x}dx$$

$$\sigma_{xx} = -p + \tau_{xx}$$

$$\Sigma\left(d\underset{\sim}{F}_1 + d\underset{\sim}{F}_2 + d\underset{\sim}{F}_3 + d\underset{\sim}{F}_4\right) =$$

$$\left(\frac{\partial \sigma_{xx}}{\partial x}\underbrace{dx\,\overbrace{W\,dy}^{dA_1}}_{d\forall} + \frac{\partial \sigma_{yx}}{\partial y}\underbrace{dy\,\underbrace{W\,dx}_{dA_2}}_{d\forall}\right)$$

$$m\underset{\sim}{a} = \Sigma\underset{\sim}{F}$$

$$\rho\underset{\sim}{a} = \Sigma\underset{\sim}{F}/\forall$$

3b

surface

Net force/volume

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y}\right)\hat{e}_x$$

$$+ \left(\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y}\right)\hat{e}_y$$

total stress

$$\underset{\approx}{\sigma} = -p\,\underset{\approx}{\delta} + \underset{\approx}{\tau}$$

hydrostatic pressure

$$\sigma_{xx} = -p + \tau_{xx}$$

$$\sigma_{xy} = \tau_{xy}$$

etc.

$\underset{\approx}{\tau}$ deviatoric (extra) stress

due to motion

-3c x-comp

$$\rho \frac{DV_x}{Dt} = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$$

$$\rho \frac{DV_y}{Dt} = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} - \rho g$$

$$F_{grav} = -mg \qquad \frac{F_{grav}}{Vol} = -\rho g$$

Any ~~...~~ flowing material

Newtonian fluid

$$\underset{\approx}{\tau} = \mu \underset{\approx}{\dot{\gamma}}$$ incompressible

μ = viscosity

$$\tau_{xx} = 2\mu \frac{\partial V_x}{\partial x}$$

$$\tau_{xy} = \tau_{yx} = \mu \left(\frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right)$$

$$\tau_{yy} = 2\mu \frac{\partial V_y}{\partial y} = \underbrace{-\tau_{xx}}_{\text{2-D incomp}}$$

4-

recall:

$$a_x^P = \frac{DV_x}{Dt} = \frac{\partial V_x}{\partial t} + V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y}$$

acceleration following particle

$$a_y^P = \frac{DV_y}{Dt} = \frac{\partial V_y}{\partial t} + V_x \frac{\partial V_y}{\partial x} + V_y \frac{\partial V_y}{\partial y}$$

$$\sum F_x / \forall = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y}$$

$$\sum F_x = m\, a_x^P$$

$$\sum F_x / \forall = \rho\, a_x^P$$

Navier-Stokes Equation

$$\rho\left(\frac{\partial V_x}{\partial t}+\underbrace{V_x\frac{\partial V_x}{\partial x}+V_y\frac{\partial V_x}{\partial y}}_{\text{non-linear}}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 V_x}{\partial x^2}+\frac{\partial^2 V_x}{\partial y^2}\right)$$

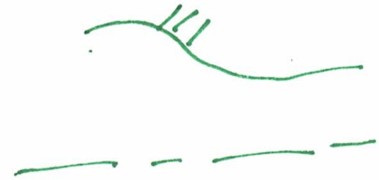
$$\rho\left(\frac{\partial V_y}{\partial t}+\underline{V_x\frac{\partial V_y}{\partial x}+V_y\frac{\partial V_y}{\partial y}}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 V_y}{\partial x^2}+\frac{\partial^2 V_y}{\partial y^2}\right)-\rho g$$

Cont $\qquad \dfrac{\partial V_x}{\partial x}+\dfrac{\partial V_y}{\partial y}=0$

$$\rho\frac{D\underset{\sim}{V}}{Dt}=-\underset{\sim}{\nabla}p+\underset{\sim}{\nabla}\cdot\underset{\approx}{\tau}-\rho\underset{\sim}{g}=-\underset{\sim}{\nabla}p+\underbrace{\underset{\sim}{\nabla}\cdot(\nabla\underset{\sim}{V})}_{\underset{\sim}{\nabla}^2V}-\rho\underset{\sim}{g}$$

Boundary & Initial Conditions

(to be continued)



1 IC

2 BC's ~~in~~ on X in y

$y=y_1(x)$

$V_x=--\,;\ V_y=$

$y=y_2(x)$

$V_x=--\,;\ V_y=$