Second-Order Sufficient Conditions

Recall the First-order Necessary Conditions

Theorem: First-order Necessary Conditions [NW06]

If x^* is a local minimizer and f is continuously differentiable (in an open neighborhood of x^*), then

$$\nabla f(x^*) = 0.$$

What Would Be Sufficient Conditions?

Assume x^* is a strict local minimizer in some $\mathcal{N}_\epsilon(x^*)$:

- So $f(x^* + \epsilon p) > f(x^*)$, for all $(x^* + \epsilon p) \in \mathcal{N}_\epsilon(x^*)$.
- Also, we know $\nabla f(x^*) = 0$.

What Would Be Sufficient Conditions? (cont.)

A Point That Satisfies the Second-order Sufficient Conditions Is a Minimizer

Theorem: Second-order Sufficient Conditions [NW06]

Suppose that $\nabla^2 f$ is continuous in an open neighborhood of x^* and that $\nabla f(x^*) = 0$ and $p^T \nabla^2 f(x^*) p > 0$ for all $p \neq 0$. Then x^* is a strict minimizer of f.

Example

Use the second-order sufficient conditions to classify the stationary points of the following 1D functions:

1. $f(x) = \frac{\pi}{6}x^2$
2. $f(x) = x^4$

References

J. Nocedal and S. J. Wright, *Numerical Optimization*, second ed., Springer–Verlag, Berlin, Germany, 2006.