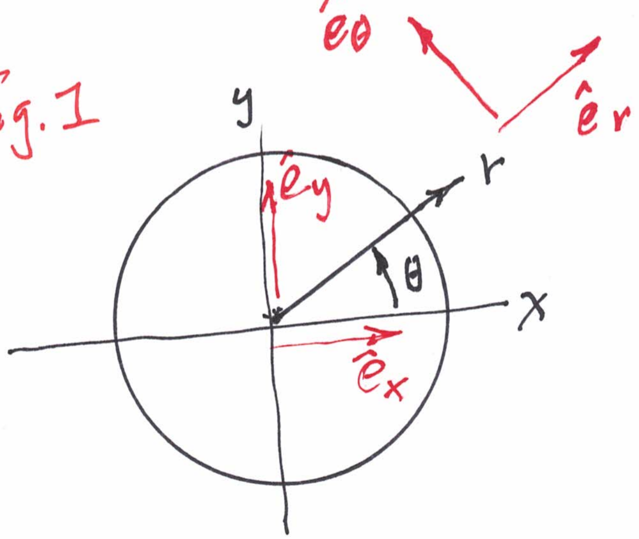
MANE 6520 Fluid Mechanics

Class #13 (cylindrical coords)

Fig. 1



(1)

$$x = r\cos\theta \qquad r = +\sqrt{x^2+y^2}$$

$$y = r\sin\theta \qquad \theta = \arctan(y/x)$$

$$z = z \qquad z = z$$

(2)

$$\frac{\partial}{\partial x} = \underbrace{\frac{\partial}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x}} \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

$$= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

chain rule: independent variables

These relation from geometry (in this simple case) fancy theory in general

$$\begin{cases} \hat{e}_r = \cos\theta\, \hat{e}_x + \sin\theta\, \hat{e}_y \\ \hat{e}_\theta = -\sin\theta\, \hat{e}_x + \cos\theta\, \hat{e}_y \end{cases} \qquad \hat{e}_z = \hat{e}_z$$

(3)
$$\begin{cases} \hat{e}_x = \cos\theta\, \hat{e}_r - \sin\theta\, \hat{e}_\theta \\ \hat{e}_y = \sin\theta\, \hat{e}_r + \cos\theta\, \hat{e}_\theta \end{cases}$$

(4) $$\underset{\sim}{V} = V_x \hat{e}_x + V_y \hat{e}_y = V_r \hat{e}_r + V_\theta \hat{e}_\theta$$

(5) $$\underset{\approx}{\sigma} = \sigma_{xx} \hat{e}_x \hat{e}_x + \sigma_{xy} \hat{e}_x \hat{e}_y + \sigma_{yx} \hat{e}_y \hat{e}_x + \dots + \sigma_{yy} \hat{e}_y \hat{e}_y$$

$$\underset{\approx}{\sigma} = \sigma_{rr} \hat{e}_r \hat{e}_r + \sigma_{r\theta} \hat{e}_r \hat{e}_\theta + \dots$$

Unit vectors not constant

(6) $$\frac{\partial}{\partial r} \hat{e}_r = 0 \qquad \frac{\partial}{\partial r} \hat{e}_\theta = 0 \qquad \frac{\partial}{\partial r} \hat{e}_z = 0$$

$$\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta \qquad \frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r \qquad \frac{\partial}{\partial \theta} \hat{e}_z = 0$$

$$\frac{\partial \hat{e}_r}{\partial z} = 0 \qquad \frac{\partial \hat{e}_\theta}{\partial z} = 0 \qquad \frac{\partial \hat{e}_z}{\partial z} = 0$$

Eq (2)

$$\underset{\sim}{\nabla} = \hat{e}_x \frac{\partial}{\partial x} + \hat{e}_y \frac{\partial}{\partial y} + \hat{e}_z \frac{\partial}{\partial z}$$

Eq (3)

subst. + rearrange

$$= \hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} + \hat{e}_z \frac{\partial}{\partial z}$$

incomp. continuity

$\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = 0$

Example (2-D) r-θ

$$\mathrm{div} V = \underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \left(\hat{e}_r \frac{\partial}{\partial r} + \hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}\right) \cdot \left(V_r \hat{e}_r + V_\theta \hat{e}_\theta\right)$$

$$= \underbrace{\cancel{\hat{e}_r \cdot \hat{e}_r}}_{\mathrm{I}}{}^{1} \frac{\partial V_r}{\partial r} + \underbrace{\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta} \cdot (V_r \hat{e}_r)}_{\mathrm{II}}$$

$$\underbrace{\hat{e}_r \frac{\partial}{\partial r} \cdot (V_\theta \hat{e}_\theta)}_{\mathrm{III}} + \underbrace{\hat{e}_\theta \frac{1}{r}\frac{\partial}{\partial \theta}(V_\theta \hat{e}_\theta)}_{\mathrm{IV}}$$

$$\mathrm{II} = \cancel{\hat{e}_\theta \cdot \frac{1}{r}\frac{\partial V_r}{\partial \theta} \hat{e}_r} + \hat{e}_\theta \cdot \frac{V_r}{r} \underbrace{\frac{\partial \hat{e}_r}{\partial \theta}}_{\hat{e}_\theta} = \frac{V_r}{r}$$

$$\mathrm{III} \quad \cancel{\hat{e}_r \cdot \hat{e}_\theta}^{\,0} \frac{\partial V_\theta}{\partial r}$$

$$\mathrm{IV} \quad \hat{e}_\theta \cdot \left[\left(\frac{1}{r}\frac{\partial V_\theta}{\partial \theta} \hat{e}_\theta\right) - \frac{V_\theta}{r} \underbrace{\frac{\partial \hat{e}_\theta}{\partial \theta}}_{\cancel{-\hat{e}_r}}\right] = \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$$

I + II + III + IV =

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{V} = \underbrace{\frac{\partial V_r}{\partial r} + \frac{V_r}{r}}_{\frac{1}{r}\frac{\partial}{\partial r}(r V_r)} + \frac{1}{r}\frac{\partial V_\theta}{\partial \theta}$$

compare Eq. (a), p. 4

-3a-

$$\hat{e}_r \frac{\partial}{\partial r} \cdot v_r \hat{e}_r$$

Term I

$$\hat{e}_r \cdot \frac{\partial}{\partial r}(v_r \hat{e}_r)$$

Eq (6)

$$\hat{e}_r \cdot \left[\frac{\partial v_r}{\partial r} \hat{e}_r + v_r \cancelto{0}{\frac{\partial \hat{e}_r}{\partial r}} \right] =$$

$$\hat{e}_r \cdot \hat{e}_r \frac{\partial v_r}{\partial r} = \frac{\partial v_r}{\partial r}$$

$$\frac{d}{dx} u v = u \frac{dv}{dx} + v \frac{du}{dx}$$

product rule

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Cylindrical Coordinates 3-D

the operator $(v \cdot \nabla) = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z}$

$$(\nabla \cdot v) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z}$$

$$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \frac{\partial^2 s}{\partial z^2}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r} \qquad \text{(D)} \qquad [\nabla \times v]_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}$$

$$[\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta} \qquad \text{(E)} \qquad [\nabla \times v]_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$

$$[\nabla s]_z = \frac{\partial s}{\partial z} \qquad \text{(F)} \qquad [\nabla \times v]_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

$$[\nabla \cdot \tau]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}$$

$$[\nabla \cdot \tau]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \frac{\partial}{\partial z}\tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r}$$

$$[\nabla \cdot \tau]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z} + \frac{\partial}{\partial z}\tau_{zz}$$

$$[\nabla^2 v]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}$$

$$[\nabla^2 v]_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}$$

$$[\nabla^2 v]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}$$

$$\{\nabla v\}_{rr} = \frac{\partial v_r}{\partial r}$$

$$\{\nabla v\}_{r\theta} = \frac{\partial v_\theta}{\partial r}$$

$$\{\nabla v\}_{rz} = \frac{\partial v_z}{\partial r}$$

$$\{\nabla v\}_{\theta r} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}$$

$$\{\nabla v\}_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

$$\{\nabla v\}_{\theta z} = \frac{1}{r}\frac{\partial v_z}{\partial \theta}$$

$$\{\nabla v\}_{zr} = \frac{\partial v_r}{\partial z}$$

$$\{\nabla v\}_{z\theta} = \frac{\partial v_\theta}{\partial z}$$

$$\{\nabla v\}_{zz} = \frac{\partial v_z}{\partial z}$$

N-S

$$\rho\left(\frac{D\underset{\sim}{V}}{Dt}\right) = -\underset{\sim}{\nabla} p + \mu \nabla^2 V$$

$$= \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}$$

$$= -\underset{\sim}{\nabla} p + \underset{\sim}{\nabla} \cdot \underset{\approx}{\tau}$$

$$\underset{\approx}{\tau} = \mu \underset{\approx}{\dot{\gamma}}$$

$$\underset{\approx}{\dot{\gamma}} = \underset{\sim}{\nabla}\underset{\sim}{V} + \underset{\sim}{\nabla}\underset{\sim}{V}^T$$

4 Cylindrical Coordinates

2-D

$r-\theta$

the operator $(\boldsymbol{v}\cdot\nabla) = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + \cancel{v_z \frac{\partial}{\partial z}}$

$V_z = 0$

$\frac{\partial}{\partial z} = 0$

$$(\nabla\cdot\boldsymbol{v}) = \frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \cancel{\frac{\partial v_z}{\partial z}} \quad \text{(a)}$$

$$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2} + \cancel{\frac{\partial^2 s}{\partial z^2}}$$

$$\{\nabla\boldsymbol{v}\}_{rr} = \frac{\partial v_r}{\partial r}$$

$$\{\nabla\boldsymbol{v}\}_{r\theta} = \frac{\partial v_\theta}{\partial r}$$

$$\{\nabla\boldsymbol{v}\}_{rz} = \cancel{\frac{\partial v_z}{\partial r}}$$

$$\{\nabla\boldsymbol{v}\}_{\theta r} = \frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}$$

$$\{\nabla\boldsymbol{v}\}_{\theta\theta} = \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r}$$

$$\{\nabla\boldsymbol{v}\}_{\theta z} = \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta}}$$

$$\{\nabla\boldsymbol{v}\}_{zr} = \cancel{\frac{\partial v_r}{\partial z}}$$

$$\{\nabla\boldsymbol{v}\}_{z\theta} = \cancel{\frac{\partial v_\theta}{\partial z}}$$

$$\{\nabla\boldsymbol{v}\}_{zz} = \cancel{\frac{\partial v_z}{\partial z}}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r} \quad \text{(D)}$$

$$[\nabla s]_\theta = \frac{1}{r}\frac{\partial s}{\partial \theta} \quad \text{(E)}$$

$$\cancel{[\nabla s]_z = \frac{\partial s}{\partial z}} \quad \text{(F)}$$

$$[\nabla\times\boldsymbol{v}]_r = \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta}} - \cancel{\frac{\partial v_\theta}{\partial z}}$$

$$[\nabla\times\boldsymbol{v}]_\theta = \cancel{\frac{\partial v_r}{\partial z}} - \cancel{\frac{\partial v_z}{\partial r}}$$

$$[\nabla\times\boldsymbol{v}]_z = \frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

$$[\nabla\cdot\boldsymbol{\tau}]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r} + \cancel{\frac{\partial}{\partial z}\tau_{zr}} - \frac{\tau_{\theta\theta}}{r}$$

$$[\nabla\cdot\boldsymbol{\tau}]_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta}) + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \cancel{\frac{\partial}{\partial z}\tau_{z\theta}} + \cancel{\frac{\tau_{\theta r} - \tau_{r\theta}}{r}}$$

$$[\nabla\cdot\boldsymbol{\tau}]_z = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz})} + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z}} + \cancel{\frac{\partial}{\partial z}\tau_{zz}}$$

$$[\nabla^2\boldsymbol{v}]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_r)\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \cancel{\frac{\partial^2 v_r}{\partial z^2}} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}$$

$$[\nabla^2\boldsymbol{v}]_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta)\right) + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}$$

$$[\nabla^2\boldsymbol{v}]_z = \cancel{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}$$

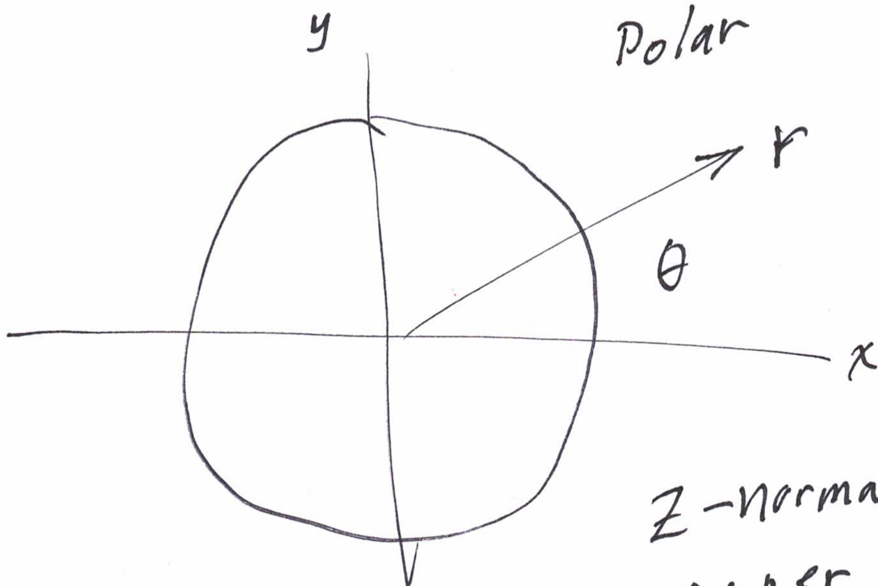
$\tau_{rr} = 2\mu\,(\nabla V)_{rr}$

$\tau_{\theta\theta} = 2\mu\,(\nabla V)_{\theta\theta}$

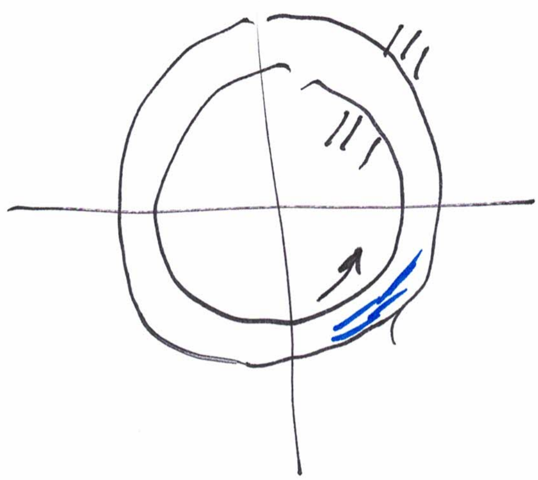
$\tau_{zz} = 0$

$\tau_{r\theta} = \tau_{\theta r} = \mu\left[(\nabla V)_{r\theta} + (\nabla V)_{\theta r}\right]$

2-D. : r-θ
Polar



Z-normal to paper



-5- Cylindrical Coordinates 2-D

$r-z$

$V_\theta = 0$

$\frac{\partial}{\partial \theta} = 0$

the operator $(v \cdot \nabla) = v_r \frac{\partial}{\partial r} + \cancel{\frac{v_\theta}{r} \frac{\partial}{\partial \theta}} + v_z \frac{\partial}{\partial z}$

(a) $$(\nabla \cdot v) = \frac{1}{r}\frac{\partial}{\partial r}(r v_r) + \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \frac{\partial v_z}{\partial z}$$

$$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial s}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 s}{\partial \theta^2}} + \frac{\partial^2 s}{\partial z^2}$$

$$\{\nabla v\}_{rr} = \frac{\partial v_r}{\partial r}$$

$$\{\nabla v\}_{r\theta} = \cancel{\frac{\partial v_\theta}{\partial r}}$$

$$\{\nabla v\}_{rz} = \frac{\partial v_z}{\partial r}$$

$$\{\nabla v\}_{\theta r} = \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta}{r}}$$

$$\{\nabla v\}_{\theta\theta} = \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}} + \frac{v_r}{r}$$

$$\{\nabla v\}_{\theta z} = \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta}}$$

$$\{\nabla v\}_{zr} = \frac{\partial v_r}{\partial z}$$

$$\{\nabla v\}_{z\theta} = \cancel{\frac{\partial v_\theta}{\partial z}}$$

$$\{\nabla v\}_{zz} = \frac{\partial v_z}{\partial z}$$

$$[\nabla s]_r = \frac{\partial s}{\partial r}$$

$$[\nabla s]_\theta = \cancel{\frac{1}{r}\frac{\partial s}{\partial \theta}}$$

$$[\nabla s]_z = \frac{\partial s}{\partial z}$$

(D) $$[\nabla \times v]_r = \cancel{\frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}}$$

(E) $$[\nabla \times v]_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}$$

(F) $$[\nabla \times v]_z = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}}$$

$$[\nabla \cdot \tau]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta r}} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}$$

$$[\nabla \cdot \tau]_\theta = \cancel{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \tau_{r\theta})} + \frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta\theta} + \cancel{\frac{\partial}{\partial z}\tau_{z\theta}} + \cancel{\frac{\tau_{\theta r} - \tau_{r\theta}}{r}}$$

$$[\nabla \cdot \tau]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \cancel{\frac{1}{r}\frac{\partial}{\partial \theta}\tau_{\theta z}} + \frac{\partial}{\partial z}\tau_{zz}$$

$$[\nabla^2 v]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}} + \frac{\partial^2 v_r}{\partial z^2} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}$$

$$[\nabla^2 v]_\theta = \cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)} + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}}$$

$$[\nabla^2 v]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}} + \frac{\partial^2 v_z}{\partial z^2}$$

$\tau_{rr} = 2\mu(\nabla V)_{rr}$

$\tau_{zz} = 2\mu(\nabla V)_{zz}$

$\tau_{\theta\theta} = 2\mu(\nabla V)_{\theta\theta}$

$\tau_{rz} = \tau_{zr} = \mu\left[(\nabla V)_{rz} + (\nabla V)_{zr}\right]$

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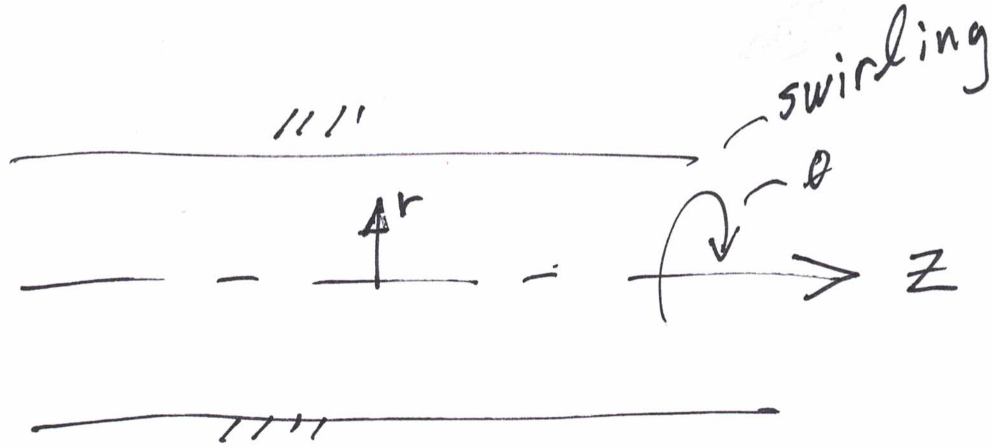

2-D, cylindrical coordinates

$r - z$ $\quad V_\theta = 0, \ \frac{\partial}{\partial \theta} = 0$

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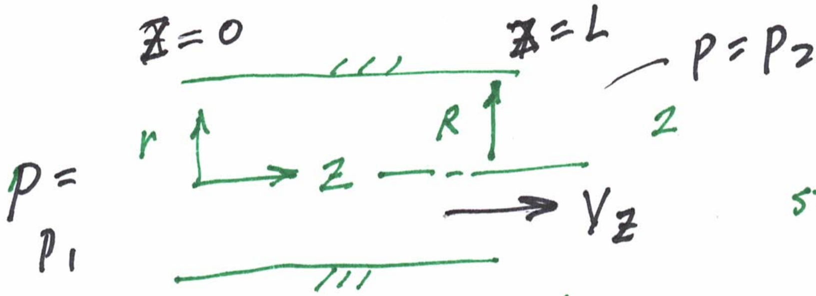

$V_z = V_z(r, \cancel{\theta}, \cancel{z}, \cancel{t})$

steady, incomp, no gravity

2-D r-z

$V_z = V_z(r)$

fully developed

Continuity $\quad \frac{\partial}{\partial z} = 0 \qquad \cancel{\frac{1}{r}} \frac{\partial}{\partial r}(r V_r) = 0$

p-5, Eq (a)

$r V_r = f(z) \qquad V_r = \frac{f(z)}{r} \Rightarrow f(z) = 0$

p. 8

$$\underset{=}{V_r = 0}$$

$$\rho \left[\cancel{\frac{\partial \underline{V}}{\partial t}} + \cancel{\underline{V} \cdot \nabla \underline{V}} \right] = -\nabla p + \mu \nabla^2 \underline{V}$$

$$0 = -\frac{\partial p}{\partial z} + \mu \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial V_z}{\partial r} \right)$$

$$0 = -\frac{\partial p}{\partial r} + \cancel{\mu (\nabla^2 V)_r}^{\,0} \quad \Rightarrow p = p(z)$$

$$\frac{\partial p}{\partial z} = \frac{dp}{dz} \qquad \frac{dp}{dz} = \frac{\mu}{r} \frac{d}{dr}\left(r \frac{dV_z}{dr} \right)$$

$$fn(z) = fn(r) \qquad = \text{const}$$

$$-\frac{dp}{dz} = \frac{P_1 - P_2}{L} \qquad \frac{P_2 - P_1}{\mu L} r = \frac{d}{dr}\left(r \frac{dV_z}{dr} \right)$$

$$\frac{P_2 - P_1}{\mu L} \frac{r^2}{2} + C_1 = r \frac{dV_z}{dr} \qquad C_1 = 0$$

$$\frac{r}{2} \frac{P_1 - P_2}{\mu L} = \frac{dV_z}{dr} \qquad V_z = \frac{P_1 - P_2}{4 \mu L} (R^2 - r^2)$$

integrate wrt r $\qquad$ BC $V_z(r = R) = 0$

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Cylindrical Coordinates 2-D r-z

the operator $(\boldsymbol{v}\cdot\nabla) = \cancel{v_r}\frac{\partial}{\partial r} + \cancel{\frac{v_\theta}{r}\frac{\partial}{\partial\theta}} + \cancel{v_z\frac{\partial}{\partial z}} = 0$

$V_\theta = 0$

$\frac{\partial}{\partial\theta} = 0$

+ tube flow $V_r = 0$

$\frac{\partial}{\partial z} = 0$

$$(\nabla\cdot\boldsymbol{v}) = \frac{1}{r}\frac{\partial}{\partial r}(r\cancel{v_r}) + \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}} + \cancel{\frac{\partial v_z}{\partial z}}$$

$$(\nabla^2 s) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial s}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 s}{\partial\theta^2}} + \cancel{\frac{\partial^2 s}{\partial z^2}}$$

$$\{\nabla\boldsymbol{v}\}_{rr} = \cancel{\frac{\partial v_r}{\partial r}}$$

$$\{\nabla\boldsymbol{v}\}_{r\theta} = \cancel{\frac{\partial v_\theta}{\partial r}}$$

$$\{\nabla\boldsymbol{v}\}_{rz} = \frac{\partial v_z}{\partial r}$$

$$\{\nabla\boldsymbol{v}\}_{\theta r} = \cancel{\frac{1}{r}\frac{\partial v_r}{\partial\theta}} - \cancel{\frac{v_\theta}{r}}$$

$$\{\nabla\boldsymbol{v}\}_{\theta\theta} = \cancel{\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}} + \cancel{\frac{v_r}{r}}$$

$$\{\nabla\boldsymbol{v}\}_{\theta z} = \cancel{\frac{1}{r}\frac{\partial v_z}{\partial\theta}}$$

$$\{\nabla\boldsymbol{v}\}_{zr} = \cancel{\frac{\partial v_r}{\partial z}}$$

$$\{\nabla\boldsymbol{v}\}_{z\theta} = \cancel{\frac{\partial v_\theta}{\partial z}}$$

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$$[\nabla s]_r = \frac{\partial s}{\partial r}$$

$$[\nabla s]_\theta = \cancel{\frac{1}{r}\frac{\partial s}{\partial\theta}}$$

$$[\nabla s]_z = \cancel{\frac{\partial s}{\partial z}}$$

(D) $$[\nabla\times\boldsymbol{v}]_r = \cancel{\frac{1}{r}\frac{\partial v_z}{\partial\theta}} - \cancel{\frac{\partial v_\theta}{\partial z}}$$

(E) $$[\nabla\times\boldsymbol{v}]_\theta = \cancel{\frac{\partial v_r}{\partial z}} - \frac{\partial v_z}{\partial r}$$

(F) $$[\nabla\times\boldsymbol{v}]_z = \cancel{\frac{1}{r}\frac{\partial}{\partial r}(rv_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial\theta}}$$

$$[\nabla\cdot\boldsymbol{\tau}]_r = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rr}) + \cancel{\frac{1}{r}\frac{\partial}{\partial\theta}\tau_{\theta r}} + \frac{\partial}{\partial z}\tau_{zr} - \frac{\tau_{\theta\theta}}{r}$$

$$[\nabla\cdot\boldsymbol{\tau}]_\theta = \cancel{\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\tau_{r\theta})} + \frac{1}{r}\frac{\partial}{\partial\theta}\tau_{\theta\theta} + \cancel{\frac{\partial}{\partial z}\tau_{z\theta}} + \cancel{\frac{\tau_{\theta r}-\tau_{r\theta}}{r}}$$

$$[\nabla\cdot\boldsymbol{\tau}]_z = \frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \cancel{\frac{1}{r}\frac{\partial}{\partial\theta}\tau_{\theta z}} + \cancel{\frac{\partial}{\partial z}\tau_{zz}}$$

$$[\nabla^2\boldsymbol{v}]_r = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cancel{v_r})\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_r}{\partial z^2}} - \cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}}$$

$$[\nabla^2\boldsymbol{v}]_\theta = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\cancel{v_\theta})\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_\theta}{\partial z^2}} + \cancel{\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}}$$

$$[\nabla^2\boldsymbol{v}]_z = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial\theta^2}} + \cancel{\frac{\partial^2 v_z}{\partial z^2}}$$

$\cancel{\tau_{rr}} = 2\mu(\nabla V)_{rr}$

$\cancel{\tau_{zz}} = 2\mu(\nabla V)_{zz}$

$\cancel{\tau_{\theta\theta}} = 2\mu(\nabla V)_{\theta\theta}$

$\tau_{rz} = \tau_{zr} = \mu\left[(\nabla V)_{rz} + \cancel{(\nabla V)_{zr}}\right]$

$= \mu\frac{\partial V_z}{\partial r}$

vol flow rate

$$Q = \int_0^R V_z(r)\, 2\pi r\, dr = \frac{P_1 - P_2}{8\mu L} \pi R^4$$

$$V_{AV} = \frac{Q}{\pi R^2} = \frac{(P_1 - P_2) R^2}{8 \mu L} = \frac{2}{3} V_{max}$$

A

$$Re_D = \rho \frac{V_{av} D}{\mu}$$

$$\tau = \mu \frac{\partial V_z}{\partial r} = -\frac{P_1 - P_2}{2L} r$$

$D = 2R$

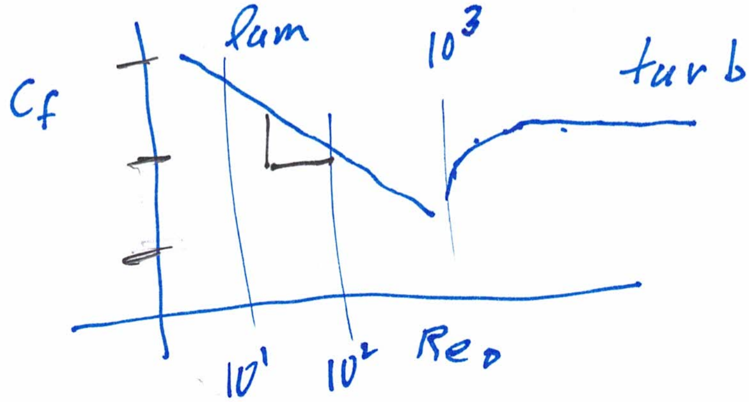
$$\tau_w = \tau(r = R) = -\frac{P_1 - P_2}{2L} R = -\frac{(P_1 - P_2)}{4L}$$

wall

algebra

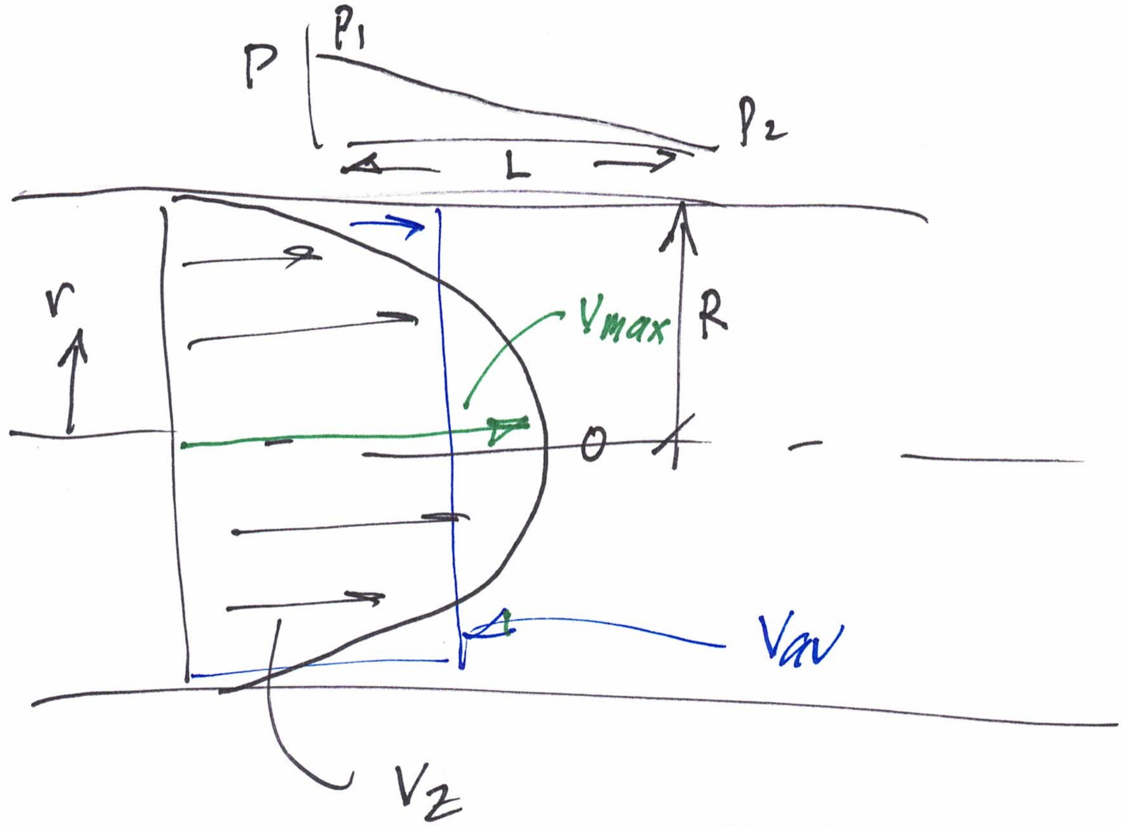
$$C_f \left(\text{Fanning Friction Factor}\right) = \frac{-\tau_w}{\frac{1}{2}\rho V_{AV}^2} = \frac{16}{Re_D}$$

$$\lambda = \text{Darcy Fric Factor} = \frac{-(P_1 - P_2)/4L}{\frac{1}{2}\rho V_{AV}^2} = \frac{64}{Re_D}$$



log slope = -1

-10-



$$V_{z\,max} = V_z(r=0) = \frac{P_1 - P_2}{4\mu L} R^2$$

$$\frac{dp}{dx} = \frac{P_2 - P_1}{L}$$