Class #6

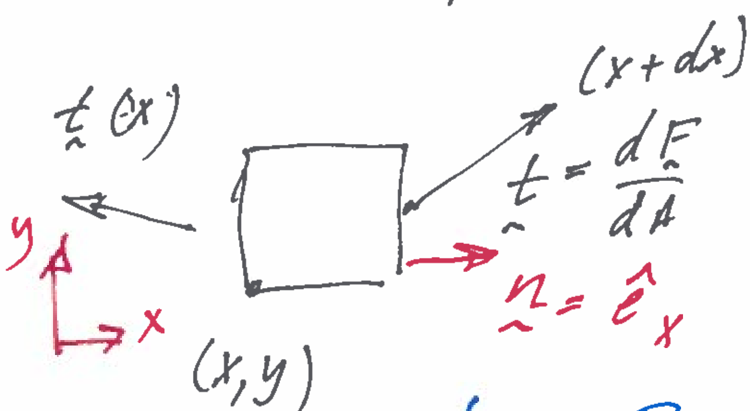
Viscous Stress Model

$$\frac{1}{V}\sum \underset{\sim}{F} = \underbrace{\frac{m}{V}}_{\rho}\ \underset{\sim}{a}_{\text{particle}}$$

$$\sum \underset{\sim}{F} = \underset{\sim}{F}_{body} + \underbrace{\underset{\sim}{F}_{surface}}_{\text{add forces/surface}}$$

$$\underbrace{\int(\ )\,dV}_{\text{gravity}}$$

$$\rho \underset{\sim}{g}$$



$$\underset{\sim}{t} = \underset{\sim}{n}\cdot\underset{\approx}{\sigma}$$

$$\underset{\approx}{\sigma} = -p\underbrace{\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}}_{\underset{\approx}{\delta}} + \underset{\approx}{\tau}$$

unit tensor
hydrostatic

extra stress
due to fluid
motion

$\underset{\approx}{\tau} = f(\underset{\sim}{V})$ simplest possible: $\underset{\approx}{\tau} = 0$

inviscid model... potential flow

next degree difficulty — linear

why not $\underset{\sim}{V}$ itself?

Want - every observer calculate same stress for same $\underset{\sim}{V}$

why not $\underset{\approx}{\tau} \overset{?}{=} \mu \underbrace{\underset{\sim}{\nabla} \underset{\sim}{V}}$ linear

proportionality const - viscocity

$\frac{N}{m^2} = $ Pa-s $\frac{1}{m} \frac{m}{s}$

$\frac{N - s}{m^2}$

$\underset{\sim}{\nabla}\underset{\sim}{V}$ - units $1/t \sim \frac{1}{s} = s^{-1} =$ reciprocal secs

$$\tau_{xx} = \mu \frac{\partial v_x}{\partial x}$$

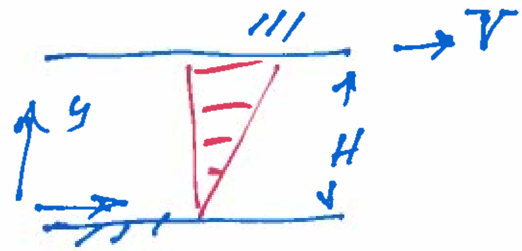
A model

$\underline{\underline{\tau}} = \mu \underline{\nabla} \underline{v}$

would say

$$\tau_{xy} = \mu \frac{\partial v_x}{\partial y}$$

$\vdots$



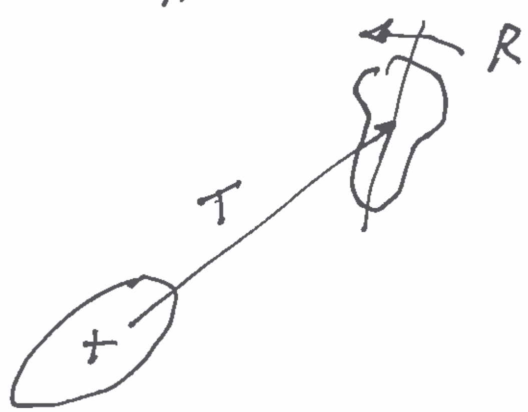
$$v_x = \frac{V}{H} y$$

OK special case

different (rotating obervers)
calculate diff stresses

model $fn\left(\frac{\partial v_x}{\partial y}\right)$ ... same for translating observers

translate + rotate + deform



want deformation to determine fluid model

Transpose

$$\underset{\approx}{Q} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \qquad \underset{\approx}{Q}^t = \begin{bmatrix} Q_{11} & Q_{21} \\ Q_{12} & Q_{22} \end{bmatrix}$$

Q_{ij}

Any tensor

$$\underset{\approx}{Q} + \underset{\approx}{Q}^T = \text{symmetric}$$

$$Q_{ij} = Q_{ji}$$

$$\underset{\approx}{Q} + \underset{\approx}{Q}^T = \begin{bmatrix} 2Q_{11} & Q_{12} + Q_{21} \\ Q_{21} + Q_{12} & 2Q_{22} \end{bmatrix}$$

$$\underset{\approx}{Q} - \underset{\approx}{Q}^T = \begin{bmatrix} 0 & Q_{12} - Q_{21} \\ Q_{21} - Q_{12} & 0 \end{bmatrix}$$ anti-symmetric

$$\underset{\sim}{\nabla}\,\underset{\sim}{V} = \begin{bmatrix} \dfrac{\partial V_x}{\partial x} & \dfrac{\partial V_x}{\partial y} \\[2ex] \dfrac{\partial V_y}{\partial x} & \dfrac{\partial V_y}{\partial y} \end{bmatrix}$$

some presentation

$$\underset{\sim}{\nabla}\,\underset{\sim}{V}^{T} = \begin{bmatrix} \dfrac{\partial V_x}{\partial x} & \dfrac{\partial V_y}{\partial x} \\[2ex] \dfrac{\partial V_x}{\partial y} & \dfrac{\partial V_y}{\partial y} \end{bmatrix}$$

sym.

$$\underset{\sim}{\nabla}\,\underset{\sim}{V} + \underset{\sim}{\nabla}\,\underset{\sim}{V}^{T} = \begin{bmatrix} 2\dfrac{\partial V_x}{\partial x} & \left(\dfrac{\partial V_x}{\partial y} + \dfrac{\partial V_y}{\partial x}\right) \\[2ex] \left(\dfrac{\partial V_x}{\partial y} + \dfrac{\partial V_y}{\partial x}\right) & 2\dfrac{\partial V_y}{\partial y} \end{bmatrix} = \underset{\sim}{\dot{\gamma}}$$

$$\underset{\sim}{\nabla}\,\underset{\sim}{V} - \underset{\sim}{\nabla}\,\underset{\sim}{V}^{T} = \begin{bmatrix} 0 & \dfrac{\partial V_x}{\partial y} - \dfrac{\partial V_y}{\partial x} \\[2ex] -\dfrac{\partial V_x}{\partial y} + \dfrac{\partial V_y}{\partial x} & 0 \end{bmatrix} = \underset{\sim}{W}$$

anti-sym

strain rate tensor

$\underset{\sim}{\dot{\gamma}}$ deformation rate tensor

stretch tensor

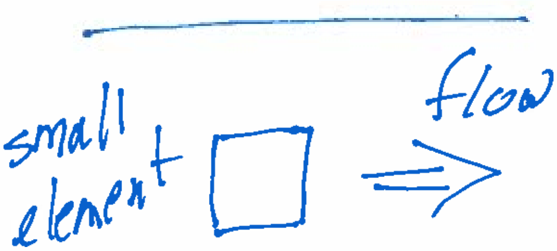
book ϵ

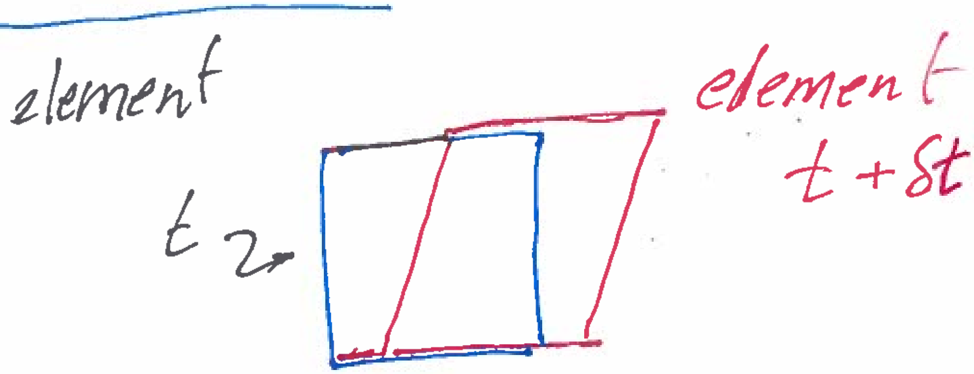
$\underset{\sim}{w}$ = spin tensor

vorticity tensor

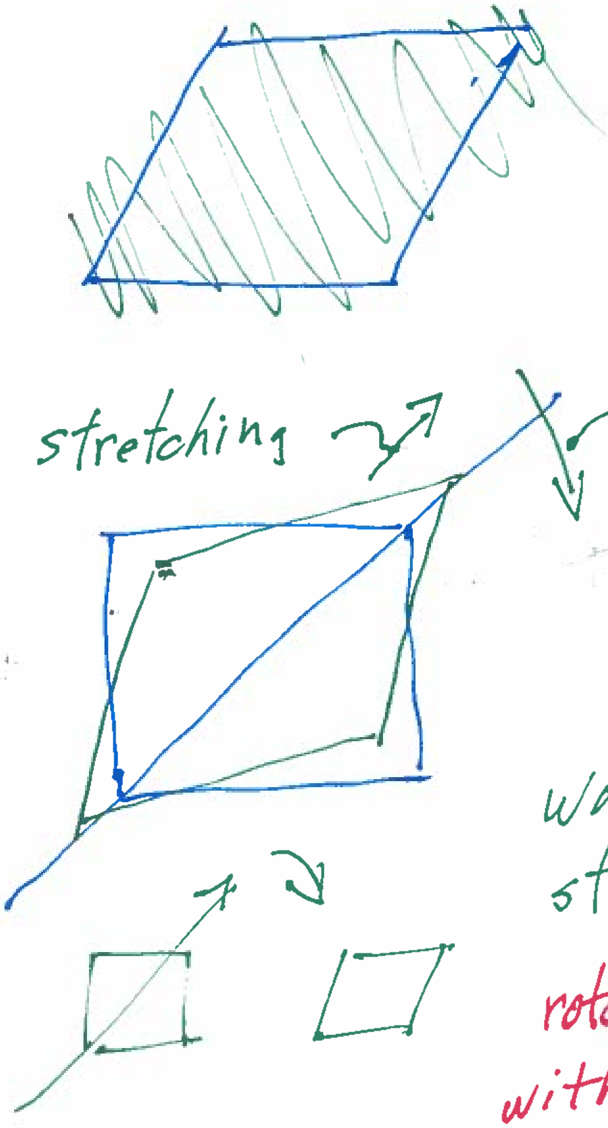
$\dot{\gamma}_{xy} = \dot{\gamma}_{yx}$ etc sym

$-w_{xy} = w_{yx}$ anti-sym





1-

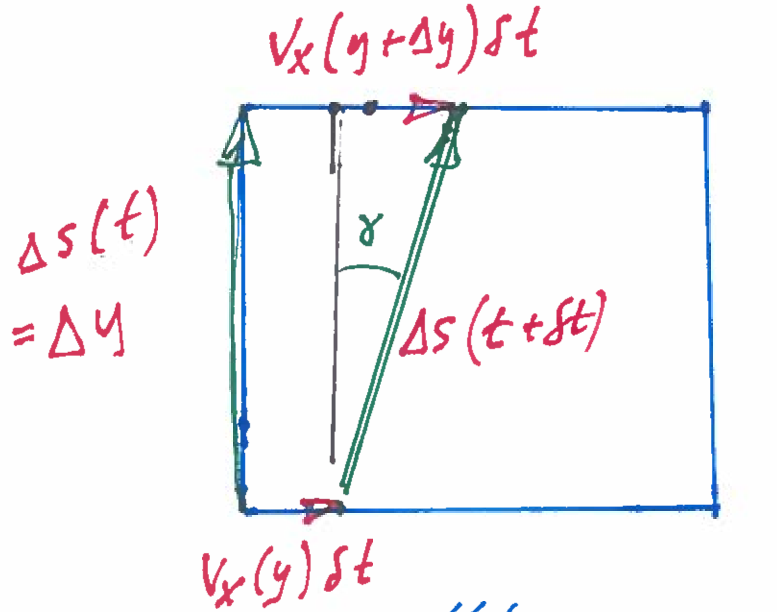


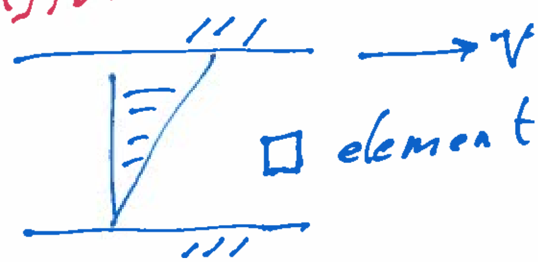
want only
streching part

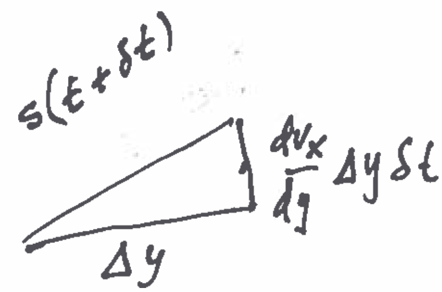
rotation varies
with rotating
observer

$-\delta$ $\qquad \dot\gamma = \dfrac{\partial v_x}{\partial y} \qquad \gamma \approx \dot\gamma\, \delta t$

want rate of change of length of green vector i.e. its streching







$$v_x(y+\Delta y) \approx v_x(y) + \frac{dv_x}{dy}\Delta y$$

$$\Delta s(t+\delta t) = \sqrt{\Delta y^2 + \left(\frac{dv_x}{dy}\delta t\,\Delta y\right)^2} = \Delta y\sqrt{1+\left(\frac{dv_x}{dy}\delta t\right)^2}$$

$$(1+\epsilon)^n \quad \epsilon \ll 1 \quad \approx 1 + n\epsilon$$

$$n = 1/2 \qquad (1+\epsilon)^{1/2} \approx 1 + \tfrac{1}{2}\epsilon$$

$$\Delta s(t+\delta t) \approx \Delta y\left(1 + \tfrac{1}{2}\left(\frac{dv_x}{dy}\delta t\right)^2\right)$$

$$\frac{ds}{dt} = \left.\frac{\Delta s(t+\delta t) - \Delta s(t)}{\delta t}\right|_{\delta t\to 0} = \tfrac{1}{2}\dot\gamma\,\gamma$$

stretching rate $\propto \dot\gamma$

anti sym part $\underset{\approx}{\omega}$

3-D

$$\underset{\approx}{\omega} = \begin{bmatrix} 0 & \omega_{xy} & \omega_{xz} \\ -\omega_{xy} & 0 & \omega_{yz} \\ -\omega_{xz} & -\omega_{yz} & 0 \end{bmatrix}$$

$\underset{\approx}{\nabla V} - \underset{\approx}{\nabla V}^T$

vorticity tensor

$$\omega_{xy} = -\omega_{yx} = \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)$$

$$\omega_{yz} = -\omega_{zy} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)$$

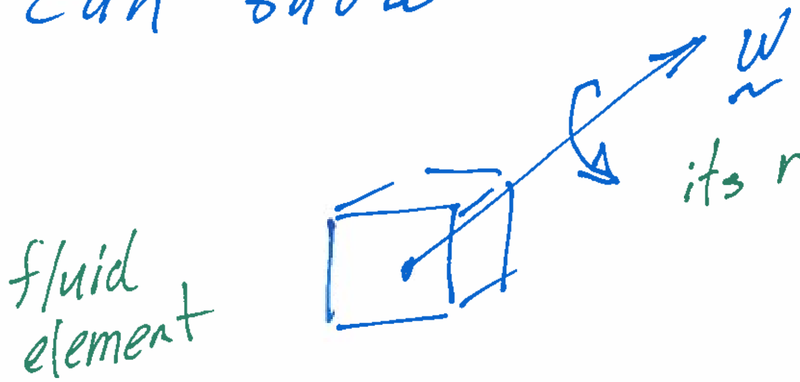
$$\omega_{zx} = -\omega_{xz} = \left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right)$$

$$\underset{\sim}{\omega} = \omega_x \hat{e}_x + \omega_y \hat{e}_y + \omega_z \hat{e}_z$$

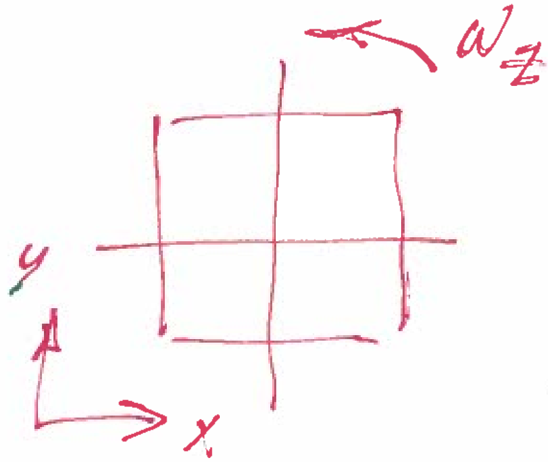
vorticity vector

$\omega_z = \omega_{xy}$ $\quad \omega_y = \omega_{zx}$ $\quad \omega_x = \omega_{yz}$

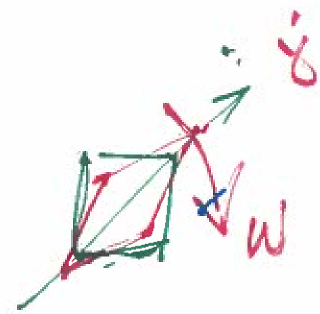
can show

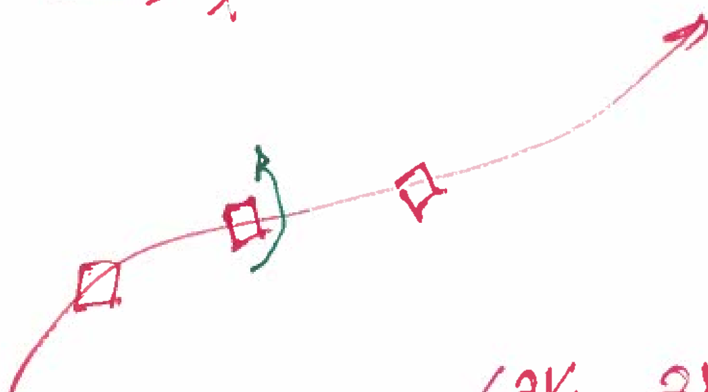


2-D



simple shear:





simple shear $\omega_z = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y} \right)$

$v_x = \frac{V}{H} y$ $\qquad \omega_z = -\frac{V}{H}$ CW

clockwise negative

$$w = \left(\frac{1}{2}\right)\left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)$$

Bird-Stewart Lightfoot

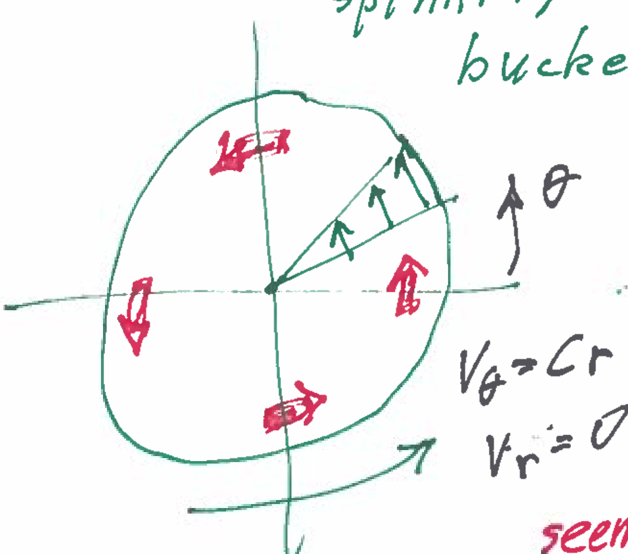
some authors ?

BST convention

2-D polar coords

$$w_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

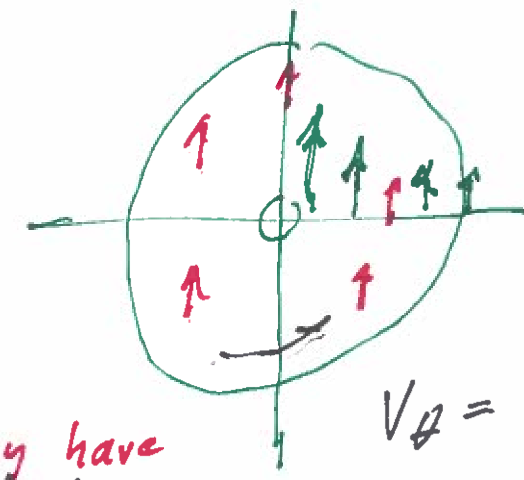
spinning bucket



$V_\theta = Cr$

$V_r = 0$

drain



$V_r \approx 0$

$V_\theta = \frac{c}{r}$

both seemingly have rotation

bucket

$$w_z = \quad = \frac{1}{r}\frac{\partial}{\partial r}(c\, r^2) = c\, 2r$$

particles rotate

drain

$$w_z = \frac{1}{r}\frac{\partial}{\partial r} \cancel{r}\frac{c}{\cancel{r}} = 0$$

particles don't rotate