Problem Set 7: Computational Methods for Economists

Iddrisu Kambala Mohammed

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Dynamic Programming

The following dynamic program models individual consumer's behaviour when there is habit formation. That is, utility derived from consuming c_t today depends both on current consumption (c_t) and previous consumption (c_{t-1}). The general specification of the utility function is $u(c_t, c_{t-1})$.
The optimisation problem takes the form:

$$\max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log\left(c_t - \alpha c_{t-1}\right)$$

subject to:

$$c_t + A_{t+1} = Y_t + (1+r)A_t, t = 0, 1, 2, ...$$

$$c_t \geq 0$$

$$A_0 > 0 \, given$$

$$c_{-1} > 0 \, given$$

where c_t is consumption level in period t, A_t is consumer's assets/debts accumulated at time t, r is exogenous interest rate, and Y is also exogenous income (wages). We assume that the discount factor $\beta \epsilon (0, 1)$. The parameter $\alpha \epsilon (0, 1)$ measures the intensity of habit formation and also denotes the nonseparability of preferences over time.
When there is habit persistence, an increase in current consumption lowers the marginal utility of consumption in the current period but increases marginal utility in the next period. Intuitively, if the consumer eats a lot today, then she wakes up hungrier tomorrow. This is what the whole model is all about.
In this model, the choice variables are c_t and A_{t+1}. That is, the consumer

decides how much to consume today and how much to save (or borrow) for the future. When the consumer saves, he/she earns interest r. In other words, there is return on storage. The state variables are c_-1, and A_0. The consumer can be a borrower ($A_t < 0$) or lender ($A_t > 0$). However, $A_0 > 0$ is given. This means individuals do not start with debts at the initial period.

Solving for the Bellman Equation (BE):
We can write the value function as follows:

$$V(A_t, c_{t-1}) = \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t - \alpha c_{t-1})$$

subject to:

$$c_t + A_{t+1} = Y_t + (1+r)A_t, t = 0, 1, 2, ...$$

$$c_t \geq 0$$

$$A_{t+1} \geq 0,$$

And the Bellman Equation would be:

$$V(A, c_{-1}) = \max_{c, \widetilde{A}} \{\log(c - \alpha c_{-1}) + \beta V(\widetilde{A}, c)\}$$

$$\Rightarrow V(A, c_{-1}) = \max_{\widetilde{A}} \{\log(Y + (1+R)A - \widetilde{A} - \alpha c_{-1}) + \beta V(\widetilde{A}, c)\}$$

Note: $\widetilde{A} = A_{t+1} = Y_t + (1+r)A_t - c_t, t = 0, 1, 2, ...$ (derived from the budget constraint)

Assuming that the Inada conditions are satisfied and the constraints bind, **the First Order Conditions (FOCs) are:**

$$\mathcal{L} = \max_{\{c_t, A_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \log(c_t - \alpha c_{t-1}) + \lambda_t \left[Y_t + (1+r)A_t - A_{t+1} - c_t\right]$$

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{\beta^t}{(c_t - \alpha c_{t-1})} - \lambda_t - \frac{\alpha \beta^{t+1}}{(c_{t+1} - \alpha c_t)} = 0 \tag{1}$$

$$\Rightarrow \frac{\beta^t}{(c_t - \alpha c_{t-1})} - \frac{\alpha \beta^{t+1}}{(c_{t+1} - \alpha c_t)} = \lambda_t \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial A_{t+1}} = -\lambda_t + \lambda_{t+1}(1+r) = 0 \tag{3}$$

Combining (2) and (3), we have:

$$\frac{\alpha\beta^{t+1}}{(c_{t+1}-\alpha c_t)}-\frac{\beta^t}{(c_t-\alpha c_{t-1})}+\left[\frac{\beta^{t+1}}{(c_{t+1}-\alpha c_t)}-\frac{\alpha\beta^{t+2}}{(c_{t+2}-\alpha c_{t+1})}\right](1+r)=0 \quad (4)$$

$$\Rightarrow \frac{\alpha}{(c_{t+1}-\alpha c_t)}-\frac{1}{\beta\left(c_t-\alpha c_{t-1}\right)}=\left[\frac{\alpha\beta}{(c_{t+2}-\alpha c_{t+1})}-\frac{1}{(c_{t+1}-\alpha c_t)}\right](1+r) \quad (5)$$

(5) is the Euler equation.