ECON 833 - Computational Methods for Economists

Fall 2021

Notes on Two-Sides Matching Models

Two-sided matching models

- These models consider a "relationship" between different agents
- Examples from economics:
  - Marriage markets
  - Medical residents and teaching hospital assignments
  - Venture capitalists and entrepreneurs
- Some models include transfers of utility (e.g., through the exchange of money), while others do not
- There are several types of matching problems:
  - One-to-one: when each agent matches with, at most, one other agent
    - e.g., Men and women in a marriage market
  - Many-to-one: when a number of agents match to the same agent
    - e.g., a bank making loans to numerous customers
  - Many-to-many: when agents (potentially) pair with more than one other agent
    - E.g., retailers to partner with different wholesalers, each of whom partners with multiple retailers

Solutions to two-sided matching models

The Gale-Shapely Algorithm

- This algorithm is appropriate for 2-sided matching models without utility transfers
- The GS algorithm finds "stable" matches
  - i.e., matches that agents would prefer to be in over alternative matches
- GS show that at least one set of stables matches exists in a wide class of 2-sided matching problems
- GS further show that these stable matches are weakly Pareto optimal
- A simple example: Consider a stylized marriage market. Suppose there are 4 men and 4 women, each of whom have rank ordered others of the opposite sex.
- Men's preferences are given by:

$$\text{Men's preferences} = \begin{bmatrix} 4 & 1 & 2 & 3 \\ 2 & 3 & 1 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 4 & 2 \end{bmatrix}$$

- Where each row is a different man and the columns have the number of the woman who that man prefers in rank-order (e.g., Man #1 prefers Woman #4 the most, followed by Woman #1 then Woman #2, and, lastly, Woman #3)

- Women's preferences are given by:

$$\text{Women's preferences} = \begin{bmatrix} 4 & 1 & 3 & 2 \\ 1 & 3 & 2 & 4 \\ 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{bmatrix}$$

- Given these preferences, the game proceeds as follows:
  - Each man makes a proposal to their highest valued partner
  - If the woman is single, she accepts the offer
  - The woman is engaged, she accepts if and only if she prefers the man making the proposal to her current partner
  - This continues until everyone is matched
- In our example, suppose Man #1 gets to make the first proposal, followed by Man #2 and so on. We'd have the following sequence of proposals:
  1. Man #1 proposes to Woman #4 (accepted)
  2. Man #2 proposes to Woman #2 (accepted)
  3. Man #3 proposes to Woman #2 (accepted, Man #2 is now single)
  4. Man #2 proposes to Woman #3 (accepted)
  5. Man #4 proposes to Woman #3 (rejected, she prefers Man #2)
  6. Man #4 proposes to Woman #1 (accepted)
- At this point, each man and woman is matched and the game ends with the following pairs: (M1, W4), (M2, W3), (M3, W2), (M4, W1)
- Now suppose that the order of the game is reversed, with Man #4 going first, followed by Man #3 and so on. We'd have the following sequence of proposals:
  1. Man #4 proposes to Woman #3 (accepted)
  2. Man #3 proposes to Woman #2 (accepted)
  3. Man #2 proposes to Woman #2 (rejected, she prefers Man #3)
  4. Man #2 proposes to Woman #3 (accepted, Man #4 is now single)
  5. Man #4 proposes to Woman #1 (accepted)
  6. Man #1 proposes to Woman #4 (accepted)
- At this point, each man and woman is matched and the game ends with the following pairs: (M1, W4), (M2, W3), (M3, W2), (M4, W1)
- Note that these are the same pairings as with the original order.
- In fact, we'll get the same stable pairings regardless of the order of the game! This is a nice property of the GS algorithm.
- Note that we would get different results if we changed the game such that women make the proposals.
- In addition, the amount of surplus captured by men and women depends on who gets to make the proposal, with the proposing party capturing more of the surplus.

The Becker-Shapely-Subik model

- The BSS model will be used to model 2-sided matching markets where there are transfers of utility (e.g., payments between agents)

- In this model, it will be useful to define the total value of a match to all parties involved. Call the value of the match between agents i and j V_{ij}.
  - Note that transfers between agents i and j will not affect the total value of the match, but only the division of this total value across agents
- Let's consider a very simple example to see how stable matches are determined in the BSS model.
- Again, consider a marriage market. Let's simplify so there are only two men and two women. We will use the following matrix to summarize the values to men and women of particular matches (the man's value comes first in the tuple):

		Woman	
		#1	#2
Man	#1	$(3,2)$	$(2,6)$
	#2	$(4,3)$	$(7,2)$

- Let's suppose (correctly) that (Man #1, Woman #2) and (Man #2, Woman #1) are the stable pairings.
- If this the case, then it must be that these pairing are preferred to the alternative pairings (i.e., (Man #1, Woman #1) and (Man #2, Woman #2))
- In other words, it must be the case that:

$$U_m(1,2) + U_w(2,1) \geq V_{11}$$
$$U_m(2,1) + U_w(1,2) \geq V_{22}$$

- The first condition says that Man #1 and Woman #1 (who are not a stable pair) cannot form a match that would improve upon the current pairings ((Man #1, Woman #2) and (Man #2, Woman #1))
  - If V_{11} were bigger than V_{12}, then there would be some bargaining solution that would allow both parties to be better off than they are in the current pairing.
- The second condition says the same about Man #2 and Woman #2 (they they cannot improve upon their current matches by pairing together).
- Since both of the above conditions must be satisfied, we can add them together to get:

$$\begin{aligned} & U_m(1,2) + U_w(2,1) + U_m(2,1) + U_w(1,2) \geq V_{11} + V_{22} \\ \Longrightarrow\ & \underbrace{U_m(1,2) + U_w(1,2)}_{V_{12}} + \underbrace{U_m(2,1) + U_w(2,1)}_{V_{21}} \geq V_{11} + V_{22} \\ \Longrightarrow\ & V_{12} + V_{21} \geq V_{11} + V_{22} \end{aligned}$$

- What this condition says is that the sum of the values from the stable pairs must be at least as large as the values of the non-stable pairings.
- So again, we find that the stable pairings are weakly Pareto optimal. They maximize total value over all possible matches.
- We can also see this by considering the counterfactual stable pairings, (Man #1, Woman #1) and (Man #2, Woman #2). Under these pairings the utility to Man #1 and Woman #2 is:

$$U_m(1,1) + U_w(2,2) = 3 + 2 = 5 \not\geq V_{12} = 8$$

- In other words, these can't be stable pairings because Man #1 and Woman #2 could be made better off by pairing together.

Example of Empirical Paper on Dating Markets

Hitch, Hortacsu, and Ariely, "Matching and Sorting in Online Dating," (*American Economic Review*, 2010)

- The authors seek to estimate the determinants to matches in a dating market to speak to the literature on assortative mating
- The authors use the Gale-Shapely algorithm to find stable matches
- They have data from a major online dating site and can observe interactions between users on the platform, as well as demographic information
  - Note that they do not have information on offline, or long term, outcomes such as marriage
- The model works as follows:
  - Suppose a user browses profiles of two potential mates, w and w'
  - If an email is sent to w but not w', we can infer that the user prefers w to w'
  - Let $U_M(m, w)$ be the expected utility user m gets from matching with user w
  - Let $\nu_M(m)$ be the value user m gets from not sending an email (the outside option, a reservation utility)
  - Then, if m visits the profile of w, he will send an email iff:

$$U_M(m, w) \geq \nu_M(m)$$

  - Incorporating women's decisions in the same manner, we can define the following indicator functions:

$$\mathcal{A}_W(m, w) = \mathbb{1}\{\text{woman } w \text{ accepts man } m\} = \mathbb{1}\{U_W(m, w) \geq \nu_W(w)\}$$
$$\mathcal{A}_M(m, w) = \mathbb{1}\{\text{man } m \text{ accepts woman } w\} = \mathbb{1}\{U_M(m, w) \geq \nu_M(m)\}$$

  - Which means the expected utility of a man m that meets a woman w is:

$$EU_M(m, w) = U_M(m, w)\mathcal{A}_W(m, w)\mathcal{A}_M(m, w) + \nu_M(m)(1 - \mathcal{A}_W(m, w))\mathcal{A}_M(m, w) + \nu_M(m)(1 - \mathcal{A}_M(m, w))$$

- To estimate this model, the authors propose a specific functional form for the utility function.
- The latent utility of man m from a match with woman w is parameterized as:

$$U_M(X_m, X_w; \theta_M) = x_w^{'}\beta_M + (|x_w - x_m|_+^{'})^{\alpha}\gamma_M^{+} + (|x_w - x_m|_-^{'})^{\alpha}\gamma_M^{-} + \sum_{k,l=1}^{N} \mathbb{1}\{d_{mk} = 1 \wedge d_{wl} = 1\}\vartheta_M^{kl} + \varepsilon_{mw}$$

  - The first term is a linear vector of the woman's characteristics
  - The second and third terms are differences on the man and woman's characteristics, allowing the coefficients to differ when those are positive of negative differences (e.g., maybe a large difference in age is not desirable unless the woman is younger than the man)
  - The fourth term relates preferences to categorial values of both the man and woman
- The authors then parameterize the reservation values, $\nu_M(m)$ and $\nu_W(w)$ as person-specific fixed effects, c_m and c_w, respectively.
- Assuming the error term, ε_{mw} is distributed iid from a Type 1 Extreme Value distribution, we can write the probability that man m contacts woman w as a logistic function:

$$Pr\{m \text{ contacts } w \mid m \text{ browses } w\} = \frac{exp(U_m(X_m, X_w; \theta_M) - c_m)}{1 + exp(U_m(X_m, X_w; \theta_M) - c_m)}$$

- This probability can be used to form a fixed effects logit estimator, to be estimated by MLE
- HHA further add some complications to this basic model (strategic behavior, costly communication)
- They use their estimates to see the degree to which there is assortative mating and how efficient the online dating market outcomes are (being able to perform welfare analysis like this is one advantage to using a structural model as is done here)