Black-Scholes Model Analytics

- <u>Model Dynamics:</u>
  The Black-Scholes model assumes the following SDE under the Risk-Neutral measure:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

  Where:
  - S_t is the underlying asset spot price
  - r is the zero-coupon rate
  - σ is the underlying asset volatility
  - W_t is a standard Brownian motion

  This SDE admits a unique solution:

$$S_t = S_0 \times e^{\left(r-\frac{\sigma^2}{2}\right)\times t+\sigma W_t};\ \forall t \geq 0$$

- <u>Black-Scholes Vanilla Price:</u>
  Let us assume a Vanilla Call paying the following cash-flow at $T > 0$: $MAX(S_T - K\ ; 0)$
  Its price C_0 under the Risk-Neutral measure is: $e^{-rT} \times E[MAX(S_T - K\ ; 0)]$, where $E[-]$ is the expectation operator
  The expected value is analytically computed under Black-Scholes, and hence, the price C_0:

$$C_0 = e^{-rT}\left(F_T N(d_1) - K\ N(d_2)\right)$$

$$F_t = E(S_t) = S_0 \times e^{rt}\ (\textit{The underlying asset T-forward price})$$

$$d_1 = \frac{\ln(F_T) + {\sigma^2 T}/{2}}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

  Using the Call-Put parity, the Vanilla Put price P_0 is:

$$P_0 = e^{-rT}\left(-F_T N(-d_1) + K\ N(-d_2)\right)$$

- <u>Black-Scholes Digital Price:</u>
  Let us assume a Digital Call paying the following cash-flow at $T > 0$: $1_{S_T > K}$
  Its price C_0 under the Risk-Neutral measure is: $e^{-rT} \times E\left[1_{S_T>K}\right] = e^{-rT} \times P(S_T > K)$
  The probability value is analytically computed under Black-Scholes, and hence, the price C_0:

$$C_0 = e^{-rT} N(d_2)$$

  Similarly, the Digital Put price P_0 is:

$$P_0 = e^{-rT} N(-d_2)$$

- <u>Black-Scholes Asian Price: *Moment matching method*</u>
  Let us assume an Arithmetic Asian Call paying the following cash-flow at $T > 0$:

$$MAX\left(\sum_{t_i \in Fixing\ Dates} \alpha_{t_i} \times S_{t_i} - K\ ;\ 0\right)$$

Where: $\left(\alpha_{t_i}\right)_i$ are the normalized weights

By approximating the arithmetic average with a log-normal distribution, Asian options can be priced via closed formulas in the Black-Scholes framework

Since:

$$m_1 = E\left(\sum_{t_i \in Fixing\ Dates} \alpha_{t_i} \times S_{t_i}\right)$$

$$= \sum\nolimits_{t_i \in Fixing\ Dates} \alpha_{t_i} E\left(S_{t_i}\right)$$

$$= \sum\nolimits_{t_i \in Fixing\ Dates} \alpha_{t_i} F_{t_i}$$

And:

$$m_2 = E\left(\left(\sum_{t_i \in Fixing\ Dates} \alpha_{t_i} \times S_{t_i}\right)^2\right)$$

$$= \sum\nolimits_{t_i \in Fixing\ Dates} \alpha_{t_i}{}^2\, E\left(\left(S_{t_i}\right)^2\right) + 2 \times \sum\nolimits_{t_i < t_j} \alpha_{t_i}\alpha_{t_j}\, E\left(S_{t_i}\, S_{t_j}\right)$$

$$= \sum\nolimits_{t_i \in Fixing\ Dates} \alpha_{t_i}{}^2\, F_{t_i}{}^2\, \mathrm{e}^{\sigma^2 t_i} + 2 \times \sum\nolimits_{t_i < t_j} \alpha_{t_i}\alpha_{t_j} F_{t_i} F_{t_j} \mathrm{e}^{\sigma^2 t_i}$$

$$= \sum\nolimits_{t_i \in Fixing\ Dates} \left[\alpha_{t_i} F_{t_i} \mathrm{e}^{\sigma^2 t_i} \times \left(\alpha_{t_i} F_{t_i} +\ 2 \times \sum\nolimits_{t_i < t_j} \alpha_{t_j} F_{t_j}\right)\right]$$

$$= \sum\nolimits_{t_i \in Fixing\ Dates} \left[\alpha_{t_i} F_{t_i} \mathrm{e}^{\sigma^2 t_i} \times \left(2 \times \sum\nolimits_{t_i \le t_j} \alpha_{t_j} F_{t_j} - \alpha_{t_i} F_{t_i}\right)\right]$$

Then, Asian Black-Scholes price formula is similar to the Vanilla's one with:

$$d_1 = \frac{ln\left(\frac{m_1}{K}\right) + \frac{ln\left(\frac{m_2}{m_1{}^2}\right)}{2}}{\sqrt{ln\left(\frac{m_2}{m_1{}^2}\right)}} = \frac{ln\left(\frac{\sqrt{m_2}}{K}\right)}{\sqrt{ln\left(\frac{m_2}{m_1{}^2}\right)}}$$

$$d_2 = d_1 - \sqrt{ln\left(\frac{m_2}{m_1{}^2}\right)}$$

- <u>Black-Scholes Basket Price: *Moment matching method*</u>
  Let us assume a Basket Call paying the following cash-flow at $T > 0$:

$$MAX\left(\sum_{i\in Bakset} \alpha_i \times S_T^i - K\ ;\ 0\right)$$

Where: $(\alpha_i)_i$ are the normalized weights and $\left(S^i\right)_i$ are $\rho -$ Correlated

Under the Black-Scholes model: $S_t^i = S_0^i \times e^{\left(r-\frac{\sigma_i^2}{2}\right)\times t+\sigma_i W_t^i}$, where: $d < W^i, W^j >_t = \rho_{i,j} dt$

By approximating the basket value with a log-normal distribution, Basket options can be priced via closed formulas in the Black-Scholes framework

Since:

$$m_1 = E\left(\sum_{i\in Bakset} \alpha_i \times S_T^i\right) = \sum_{i\in Bakset} \alpha_i \times E\left(S_T^i\right) = \sum_{i\in Bakset} \alpha_i \times F_T^i$$

And:

$$m_2 = E\left(\left(\sum_{i\in Bakset} \alpha_i \times S_T^i\right)^2\right)$$

$$= \sum_{i\in Bakset} \sum_{j\in Bakset} \alpha_i \times \alpha_j \times E\left(S_T^i S_T^j\right)$$

$$= \sum_{i\in Bakset} \sum_{j\in Bakset} \alpha_i\ \alpha_j\ F_T^i\ F_T^j\ e^{\sigma_i \sigma_j \rho_{i,j} T}$$

Then, Basket Black-Scholes price formula is similar to the Vanilla's one with:

$$d_1 = \frac{ln\left(\frac{m_1}{K}\right) + \frac{ln\left(\frac{m_2}{m_1{}^2}\right)}{2}}{\sqrt{ln\left(\frac{m_2}{m_1{}^2}\right)}} = \frac{ln\left(\frac{\sqrt{m_2}}{K}\right)}{\sqrt{ln\left(\frac{m_2}{m_1{}^2}\right)}}$$

$$d_2 = d_1 - \sqrt{ln\left(\frac{m_2}{m_1{}^2}\right)}$$

- <u>Black-Scholes Spread Price: *Kirk's approximation & Margrabe's formula*</u>
  Let us assume a Spread Call paying the following cash-flow at $T > 0$: $MAX(S_T^1 - S_T^2 - K\ ;\ 0)$
  This payoff can also be written as: $MAX\left(S_T^1 - S_T^{2^{Adj}}\ ;\ 0\right)$, where: $S_T^{2^{Adj}} = S_T^2 + K$
  In this case, under the absence of arbitrages assumption: $S_t^{2^{Adj}} = S_t^2 + Ke^{-r(T-t)}$

As a consequence:

$$dS_t^{2^{Adj}} = dS_t^2 + rKe^{-r(T-t)}dt$$

$$= r\left(S_t + Ke^{-r(T-t)}\right)dt + \sigma_2 S_t^2 dW_t^2$$

$$= rS_t^{2^{Adj}}dt + \sigma_2^{Adj}(t)S_t^{2^{Adj}}dW_t^2$$

Where: $\sigma_2^{Adj}(t) = \frac{S_t^2}{S_t^2+Ke^{-r(T-t)}}\sigma_2$

The Kirk's approximation stands on freezing the $\sigma_2^{Adj}(t)$ at $t = 0$

Therefore, $S_t^{2^{Adj}}$ will become a geometric Brownian motion:

$$dS_t^{2^{Adj}} = rS_t^{2^{Adj}}dt + \sigma_2^{Adj}S_t^{2^{Adj}}dW_t^2$$

Where:

$$\sigma_2^{Adj} = \frac{S_0^2}{S_0^2 + Ke^{-rT}}\sigma_2$$

Finally, the Margrabe's formula gives the following price C_0:

$$C_0 = S_0^1 N(d_1) - (S_0^2 + Ke^{-rT})\, N(d_2)$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^{Adj^2} - 2\sigma_1\sigma_2^{Adj}\rho_{1,2}}$$

$$d_1 = \frac{\ln\left(\frac{S_0^1}{S_0^2 + Ke^{-rT}}\right) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Similarly, the Spread Put price P_0 is: $C_0 = -S_0^1 N(-d_1) + (S_0^2 + Ke^{-rT})\, N(-d_2)$

- Monte-Carlo Simulation:

  - Single Asset:
    The Black-Scholes spot price solution is directly used for simulations:

$$S_{n+1} = S_n \times e^{\left(r-\frac{\sigma^2}{2}\right)\times(t_{n+1}-t_n)+\sigma(W_{n+1}-W_n)}$$

    Where: $(W_{n+1} - W_n)$ follows a normal distribution $N\left(\mu = 0, \sigma = \sqrt{t_{n+1} - t_n}\right)$

    Therefore:

$$S_{n+1} = S_n \times e^{\left(r-\frac{\sigma^2}{2}\right)\times(t_{n+1}-t_n)+\sigma\sqrt{t_{n+1}-t_n}G_{n+1}}$$

    Where: G_{n+1} follows a normal distribution $N(\mu = 0, \sigma = 1)$

  - Multi Asset:
    Let us assume $N\ \rho -$ Correlated assets. Under the Black-Scholes model:

$$S_t^i = S_0 \times e^{\left(r-\frac{\sigma_i^2}{2}\right)\times t+\sigma W_t^i};\ \forall t \geq 0\ \forall i \in [1, N]$$

Where: $d < W^i, W^j >_t = \rho_{i,j} dt$

The Cholesky decomposition of the correlation matrix $\rho = \left(\rho_{i,j}\right)_{1 \leq i,j \leq N}$ is used to simulate the N assets independently. To do so, the correlation matrix must be positive-definite, and hence, the values $\rho_{i,j}$ must be within the following interval: $\left] -\frac{1}{N-1}, 1\right[$

Therefore, a new vector of independent standard Brownian motions $(B_i)_{1 \leq i \leq N}$ is implied such as: $W_t = LB_t$, where: L is the Cholesky lower triangular matrix ($\rho = LL^T$)

Consequently: $S_t^i = S_0 \times e^{\left(r - \frac{\sigma_i^2}{2}\right) \times t + \sigma L_i B_t}$; $\forall t \geq 0 \; \forall i \in [1, N]$

Where: L_i is the i^{th} line of L

Finally: $S_{n+1}^i = S_n^i \times e^{\left(r - \frac{\sigma_i^2}{2}\right) \times (t_{n+1} - t_n) + \sigma_i \sqrt{t_{n+1} - t_n} L_i G_{n+1}}$

Where: G_{n+1} is an $N -$ Vector of independent normal distributions $N(\mu = 0, \sigma = 1)$