Hull-White One Factor Model

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- **HJM Framework:**

$$\frac{dB(t,T)}{B(t,T)} = r_t\,dt + \Gamma(t,T).dW_t^{\mathbb{Q}} \qquad \left(B(t,T) := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T r_u du}\right]\right)$$

Where:

→ $(B(t,T))_{0\leq t\leq T}$: ZC Bond Price at $t\geq 0$ for maturity $T>t$

→ $(r_t)_{t\geq 0}$: Short Rate

→ $(\Gamma(t,T))_{0\leq t\leq T}$: d-dimension volatility vector

→ $(W_t^{\mathbb{Q}})_{t\geq 0}$: d-dimension Standard Brownian Motion vector under the Risk-Neutral measure $\mathbb{Q}$

- **Hull-White 1F (Simple Derivation of HJM):**

① HJM Adjustments: $d = 1$ (One Dimension) and $\Gamma(t,T) = \frac{\sigma(t)}{\lambda}\left(e^{-\lambda(T-t)} - 1\right)$

Where: λ is the mean reversion and $\sigma(t)$ is the short rate vol.

② Fwd Instantaneous Rate:

$$B(t,T) = e^{-\int_t^T f(t,s)ds} \iff f(t,T) = -\frac{\partial \ln(B(t,T))}{\partial T}$$

And hence:

$$f(t,T) = f(0,T) - \int_0^t \frac{\sigma(s)^2}{\lambda}.\left(e^{-2\lambda(T-s)} - e^{-\lambda(T-s)}\right)ds + \int_0^t \sigma(s).\,e^{-\lambda(T-s)}dW_s$$

Demo: ⚠ Itô's Lemma ⊕ T-Derivation

$(\Rightarrow f(t,T)$ is a Gaussian Process$)$

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③ Short Rate :

$$r_t = f(t,t) = f(0,t) - \int_0^t \frac{\sigma(s)^2}{\lambda}\left(e^{-2\lambda(t-s)} - e^{-\lambda(t-s)}\right) ds + \int_0^t \sigma(s)\, e^{-\lambda(t-s)} ds$$

($\Rightarrow r_t$ is a Gaussian Process)

④ HW-1F State Variable :

Let : $X_t := r_t - f(0,t)$ ($\Rightarrow X_0 = 0$ and X_t is a Gaussian Process)

Using ① and ③ :

$$dX_t = \left(\phi(t) - \lambda \cdot X_t\right) dt + \sigma(t)\, dW_t$$

(Vasicek Process)

where : $\phi(t) := \int_0^t \sigma(s)^2 \cdot e^{-2\lambda(t-s)} ds$ (Total Variance)

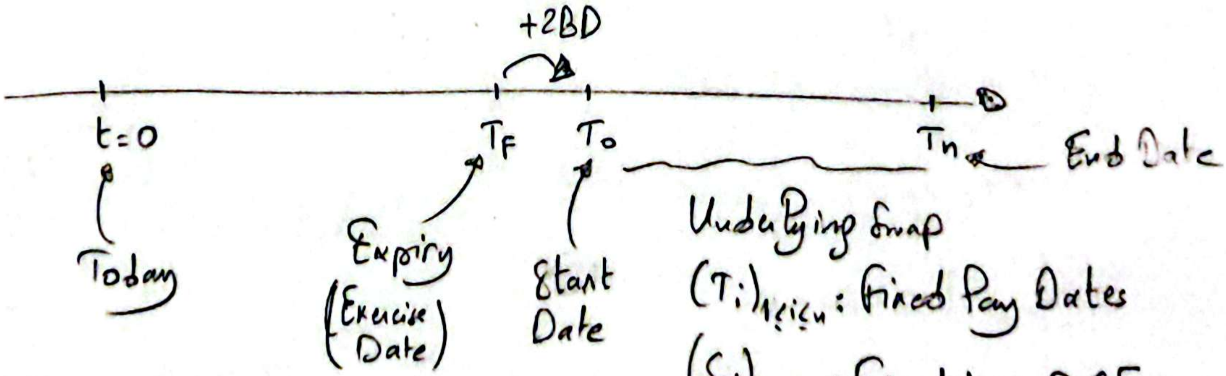
⑤ ZC Bond Price :

($\Rightarrow B(t,T)$ is LogNormal Process)

$$B(t,T) = \frac{B(0,T)}{B(0,t)} \times \exp\left[-\frac{1}{2}\beta(t,T)^2 \cdot \phi(t) - \beta(t,T) \cdot X_t\right]$$

where : $\beta(t,T) = \int_t^T e^{-\lambda(u-t)} du = \dfrac{1 - e^{-\lambda(T-t)}}{\lambda}$

- Hull-White Swaption Price (Payer Swaption) :



$(T_i)_{1\le i\le n}$: Fixed Pay Dates

$(\delta_i)_{1\le i\le n}$: Fixed Leg DCFs

K : Fixed Leg Rate

<u>Exercise Condition:</u>

IF : PV_{T_F}(Float Leg) > PV_{T_F}(Pay Leg) $\Rightarrow$ Enter the Underlying Swap

ELSE $\Rightarrow$ No Exercise.

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① HW-1F Pricer:

$$PV_t = B(t,T_F) \times \mathbb{E}_t^{Q^{T_F}}\left[\left(PV_{T_F}(\text{Float Leg}) - PV_{T_F}(\text{Fixed Leg})\right)^+\right]$$

where: Q^{T_F}: T_F - Forward Measure.

$$\Rightarrow PV_t = B(t,T_F) \times \mathbb{E}_t^{Q^{T_F}}\left[\left(B(T_F,T_0) - B(T_F,T_n) - K \times \sum_{i=1}^{n} \delta_i \times B(T_F,T_i)\right)^+\right]$$

$$= B(t,T_F) \times \mathbb{E}_t^{Q^{T_F}}\left[\left(B(T_F,T_0) - \sum_{i=1}^{n} c_i \, B(T_F,T_i)\right)^+\right]$$

⚠ Radon-Nikodym: Change of Numeraire $Q^{T_F} \rightarrow Q^{T_0}$

T_0-Fwd Measure

where: $\begin{cases} c_i = \delta_i \times K \; ; \; 1 \leq i \leq n-1 \\ c_n = 1 + \delta_n \times K \end{cases}$

$$\Rightarrow PV_t = B(t,T_0) \times \mathbb{E}_t^{Q^{T_0}}\left[\left(1 - \sum_{i=1}^{n} c_i \times \frac{B(T_F,T_i)}{B(T_F,T_0)}\right)^+\right]$$

(In HW-1F, the random variable of this payoff is $(X_t)_t$.)

Let: $$h(x) = \sum_{i=1}^{n} c_i \times \frac{B(T_F,T_i)(x)}{B(T_F,T_0)(x)}$$

$$= \sum_{i=1}^{n} c_i \times \frac{B(0,T_i)}{B(0,T_0)} \times \exp\left[-\frac{1}{2}\phi(T_F) \times \left(\beta(T_F,T_i)^2 - \beta(T_F,T_0)^2\right) - x \times \left(\beta(T_F,T_i) - \beta(T_F,T_0)\right)\right]$$

Since: $c_i > 0$ (Non-Negative Rates Universe)

And: $$\beta(T_F,T_i) - \beta(T_F,T_0) = \frac{e^{-\lambda(T_0-T_F)} - e^{-\lambda(T_i-T_F)}}{\lambda} > 0$$

Then: h is a continuous and a decreasing function $\Rightarrow$ **Bijective**

$\mathbb{R} \rightarrow]0;+\infty[$

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In this case:

$(\exists!\, x_0 \in \mathbb{R})$ such as:

$$\sum_{i=1}^{n} c_i \times \frac{B(T_F,T_i)(x_0)}{B(T_F,T_0)(x_0)} = 1 \quad \left(\Rightarrow \text{The swaption is exercised if } X_{T_F} > x_0.\right)$$

Let: $K_i := \dfrac{B(T_F,T_i)(x_0)}{B(T_F,T_0)(x_0)}$ $\left(\sum_{i=1}^{n} c_i \times K_i = 1\right)$

Then:

$$PV_t = B(t,T_0) \times \mathbb{E}_t^{Q^{T_0}}\left[\left(\underbrace{\sum_{i=1}^{n} c_i \times \left(K_i - \frac{B(T_F,T_i)}{B(T_F,T_0)}\right)}_{\text{n-weighted options}}\right)^+\right]$$

The n options $\left(K_i - \dfrac{B(T_F,T_i)}{B(T_F,T_0)}\right)^+$ have the same exercise frontier $\{X_{T_F} > x_0\}$

Therefore, the swaption PV_t can be written as following:

$$PV_t = B(t,T_0) \times \mathbb{E}_t^{Q^{T_0}}\left[\sum_{i=1}^{n} c_i \times \left(K_i - \frac{B(T_F,T_i)}{B(T_F,T_0)}\right)^+\right]$$

$$= \underbrace{B(t,T_0) \times \sum_{i=1}^{n} c_i \times \mathbb{E}_t^{Q^{T_0}}\left[\left(K_i - \frac{B(T_F,T_i)}{B(T_F,T_0)}\right)^+\right]}_{\text{Replication leg: ZC Forward Bond Put Option}}$$

Now, we have: ① $(B(t,T))_{0\le t \le T}$ is a LogNormal Process

② Risk-Neutral Measure: $dB(t,T) = r_t B(t,T)\,dt + \overbrace{\left(\frac{\sigma(t)}{\lambda}\cdot\left(e^{-\lambda(T-t)} - 1\right)\right)}^{\Gamma(t,T)} B(t,T)\,dW_t^{Q}$

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And since : $\left(\dfrac{B(t,T_i)}{B(t,T_0)}\right)_{0\le t\le T_F}$ is a Martingale Process under Q^{T_0}

Then, under Q^{T_0} :

$\left(\dfrac{B(t,T_i)}{B(t,T_0)}\right)_{0\le t\le T_F}$ $\boxed{\text{has no drift}}$ → We only need to find its volatility ⚠

Let's apply Ito's Lemma on it : $f(x,y) = \dfrac{x}{y}$

$f'_x = \dfrac{1}{y}$; $f''_x = 0$

$f'_y = -\dfrac{x}{y^2}$; $f''_{yy} = \dfrac{2x}{y^3}$

$f''_{xy} = -\dfrac{1}{y^2}$

$$d\left(\frac{B(t,T_i)}{B(t,T_0)}\right) = \frac{dB(t,T_i)}{B(t,T_0)} - \frac{B(t,T_i)}{B(t,T_0)^2}\cdot dB(t,T_0) + \frac{1}{2}\times\frac{2B(t,T_i)}{B(t,T_0)^3}\,d\langle B(\cdot,T_0)\rangle_t - \frac{d\langle B(t,T_0),B(t,T_i)\rangle}{B(t,T_0)^2}$$

We only care for volatility Part ⚠

Risk Neutral Measure

$$= (\dots)\times dt + \sigma(t,T_i)\times\frac{B(t,T_i)}{B(t,T_0)}\,dW_t^{Q} - \sigma(t,T_0)\times\frac{B(t,T_i)}{B(t,T_0)}\,dW_t^{Q}$$

$$= (\dots)\times dt + \big(\sigma(t,T_i) - \sigma(t,T_0)\big)\times\frac{B(t,T_i)}{B(t,T_0)}\,dW_t^{Q}$$

The instantaneous volatility of $\dfrac{B(t,T_i)}{B(t,T_0)}$ is : $\sigma(t,T_i) - \sigma(t,T_0)$

Chang of measure $Q \to Q^{T_0}$ (does not impact the vol part) :

$$d\left(\frac{B(t,T_i)}{B(t,T_0)}\right) = \big(\sigma(t,T_i) - \sigma(t,T_0)\big)\times\frac{B(t,T_i)}{B(t,T_0)}\,dW_t^{Q^{T_0}}$$

So : $\mathbb{E}_t^{Q^{T_0}}\left[\left(K_i - \dfrac{B(T_F,T_i)}{B(T_F,T_0)}\right)^+\right]$ is Black put forward price of vol :

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$$\sigma_{Black}^2 = \frac{1}{T_F} \times \int_0^{T_F} \left(\Gamma(t, T_i) - \Gamma(t, T_0) \right)^2 dt$$

$$= \frac{1}{T_F} \times \int_0^{T_F} \frac{\sigma(t)^2}{\lambda^2} \times \left(e^{-\lambda(T_i - t)} - e^{-\lambda(T_0 - t)} \right)^2 dt$$

$$= \frac{1}{\lambda^2 T_F} \times \int_0^{T_F} \sigma(t)^2 \times e^{-2\lambda(T_F - t)} \times \left(e^{-\lambda(T_i - T_F)} - e^{-\lambda(T_0 - T_F)} \right)^2 dt$$

$$= \frac{\left(e^{-\lambda(T_i - T_F)} - e^{-\lambda(T_0 - T_F)} \right)^2}{\lambda^2 \cdot T_F} \times \int_0^{T_F} \sigma(t)^2 e^{-2\lambda(T_F - t)} dt$$

$$= \frac{\left(e^{-\lambda(T_i - T_F)} - e^{-\lambda(T_0 - T_F)} \right)^2}{\lambda^2 T_F} \times \phi(T_F)$$