Methods 2 Portfolio 1

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1)

a)

$$\vec{u} \cdot \vec{v}$$

$$\begin{aligned}(1, 1, 1) \cdot (2, 3, 1) && (1)\\ = \sum(2, 3, 1) && (2)\\ = 6 && (3)\end{aligned}$$

b)

$$\vec{u} \cdot \vec{w}$$

$$\begin{aligned}(1, 1, 1) \cdot (-1, -1, 2) && (4)\\ = \sum(-1, -1, 2) && (5)\\ = 0 && (6)\end{aligned}$$

c)

$$\vec{v} \cdot \vec{w}$$

$$\begin{aligned}(2, 3, 1) \cdot (-1, -1, 2) && (7)\\ = \sum(-2, -3, 2) && (8)\\ = -3 && (9)\end{aligned}$$

d)

$$\vec{u} \times \vec{v}$$

$$\begin{aligned}(1, 1, 1) \times (2, 3, 1) && (10)\\ = (\ (1 \cdot 1 - 1 \cdot 3),\ (1 \cdot 2 - 1 \cdot 1),\ (1 \cdot 3 - 1 \cdot 2)\ ) && (11)\\ = (-2, 1, 1) && (12)\end{aligned}$$

e)

$$\vec{u} \times \vec{w}$$

$$\begin{aligned} &(1,1,1) \times (-1,-1,2) && (13)\\ &= (\ (2 \cdot 1 - (-1 \cdot 1)),\ ((1 \cdot -1) - 1 \cdot 2),\ ((1 \cdot -1) - (1 \cdot -1)\ ) && (14)\\ &= (3,-3,0) && (15) \end{aligned}$$

f)

$$\vec{v} \times \vec{w}$$

$$\begin{aligned} &(2,3,1) \times (-1,-1,2) && (16)\\ &= (\ (3 \cdot 2 - (-1 \cdot 1)),\ ((1 \cdot -1) - 2 \cdot 2),\ ((2 \cdot -1) - (3 \cdot -1))\ ) && (17)\\ &= (7,-5,1) && (18) \end{aligned}$$

2)

“Vectors are *orthogonal* (or *perpendicular*) when their dot product is zero: $\vec{x} \perp \vec{y} \iff \vec{x} \cdot \vec{y} = 0$. A *unit vector* is a vector with norm 1: $\|\vec{x}\| = 1$.”

a)

“Find a unit vector that is perpendicular to both $\vec{u} = (1,0,1)$ and $\vec{v} = (1,2,0)$. To find the unit vector where $\vec{u}$ and $\vec{v}$ are orthogonal, we calculate the cross product between the two vectors according to the”right hand rule"

$$\begin{aligned} &(1,0,1) \times (1,2,0) && (19)\\ &= (\ (0 \cdot 0 - 1 \cdot 2),\ (1 \cdot 1 - 1 \cdot 0),\ (1 \cdot 2 - 0 \cdot 1)) && (20)\\ &= (-2,1,2) && (21) \end{aligned}$$

b)

“Find a vector that is orthogonal both to $\vec{u}_1 = (1,0,1)$ and $\vec{u}_2 = (1,3,0)$, and whose dot product with the vector $\vec{v} = (1,1,0)$ is equal to 8.”

$$\begin{aligned} \vec{u_3} &= \vec{u_1} \times \vec{u_2} && (22)\\ &= (1,0,1) \times (1,3,0) && (23)\\ &= (\ (0 \cdot 0 - 3 \cdot 1),\ (1 \cdot 1 - 1 \cdot 0),\ (1 \cdot 3 - 0 \cdot 1)) && (24)\\ &= (-3,1,3) && (25) \end{aligned}$$

The cross product $\vec{u_3}$ is orthogonal to $\vec{u_1}$ and $\vec{u_2}$. Since the vector does not need to be a unit vector we can extend the unit vector $\vec{u_3}$ so that it will meet the requirement of $\vec{u_3} \cdot \vec{v} = 8$ where $\vec{v} = (1,1,0)$.

The dot product of the cross vector $\vec{u_3}$ and the vector v is at this point -2 but we can extend the vector for it to be 8

$$(-3 \cdot 1) \cdot x + (1 \cdot 1) \cdot x + (3 \cdot 0) \cdot x = 8 \Leftrightarrow \tag{26}$$
$$-3 \cdot x + 1 \cdot x = 8 \Leftrightarrow \tag{27}$$
$$-2x = 8 \Leftrightarrow \tag{28}$$
$$x = \frac{8}{-2} \Leftrightarrow \tag{29}$$
$$x = -4 \tag{30}$$

Therefore the vector $\vec{u_3}$ must be multiplied by -4

$$\vec{u_3} \cdot -4 \tag{31}$$
$$= (-3, 1, 3) \cdot -4 \tag{32}$$
$$= (12, -4, -12) \tag{33}$$

The vector (12,-4,-12) is therefore orthogonal with $\vec{u_1}$ and $\vec{u_2}$ and also its dot product with $\vec{v}$ is 8

3)

"Prove the geometric formula for the dot product $\vec{x} \cdot \vec{y} = \|\vec{x}\|\|\vec{y}\| \cos \varphi$, where φ is the angle between the vectors $\vec{x}$ and $\vec{y}$."

Proof that:

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cdot Cos(\theta)$$

We apply the cosinus relation:

$$a^2 = b^2 + c^2 - 2bc \cdot cos(A)$$

In the cosinus relation a is equivalent to $||\vec{x} - \vec{y}||$

$$||\vec{x} - \vec{y}|| = ||\vec{x}||^2 + ||\vec{y}||^2 - 2 \cdot ||\vec{x}|| \cdot ||\vec{y}|| \cdot cos(\theta)$$

Using the properties of the dot product, the left side can be written as:

$$||\vec{x} - \vec{y}||^2 = (\vec{x} - \vec{y}) \cdot (\vec{x} - \vec{y}) \tag{34}$$
$$= \vec{x} \cdot \vec{x} - \vec{x} \cdot \vec{y} - \vec{y} \cdot \vec{x} + \vec{y} \cdot \vec{y} \tag{35}$$
$$= ||\vec{x}||^2 - 2 \cdot \vec{x} \cdot \vec{y} + ||\vec{y}||^2 \tag{36}$$

Our equation is then:

$$||\vec{x}||^2 - 2 \cdot \vec{x} \cdot \vec{y} + ||\vec{y}||^2 = ||\vec{x}||^2 + ||\vec{y}||^2 - 2 \cdot ||\vec{x}|| \cdot ||\vec{y}|| \cdot cos(\theta)$$

Which can be reduced to:

$$-2 \cdot \vec{x} \cdot \vec{y} = -2 \cdot ||\vec{x}|| \cdot ||\vec{y}|| \cdot cos(\theta)$$

divide by -2 to reduce further:

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cdot Cos(\theta)$$

Which is the equation sought to be solved.

4)

"For the matrix

$$X = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},$$

calculate X^n for $n = 2, 3, 4, 5$. Write a rule for calculating higher values of n."

Caclualating X^2

$$X^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Caclualating X^3

$$X^3 = X^2 \cdot X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$$

Caclualating X^4

$$X^4 = X^3 \cdot X = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

Caclualating X^5

$$X^5 = X^4 \cdot X = \begin{bmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 3 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 5 \\ 0 & 1 & 0 \\ 5 & 0 & 8 \end{bmatrix}$$

For calculating higher numbers of n in the matrix it follows the fibonacci numbers: Let $n \geq 1$ and f_n be the fibonacci numbers then

$$X^n = \begin{bmatrix} f_{n-1} & 0 & f_n \\ 0 & 1 & 0 \\ f_n & 0 & f_{n+1} \end{bmatrix}$$