Mathematical Methods for International Commerce

Week 4/2: Geometric Series and Investment Appraisal

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Hello everyone!

I hope you all are doing well and ready for yet another **exciting session** of Mathematical Methods for International Commerce.

PART 1. Flipped Classroom – Pre-Class Video 2 Slides

Week 4/2: Geometric Series and Investment Appraisal

Why it matters (time value of money)

- Regular payments are geometric series
- Compounding grows money forward FV
- Discounting brings money back to today PV
- Finance decisions: loans, projects, bonds
    - Key idea: a dollar today is worth more than a dollar tomorrow

Geometric sequence and finite sum

- Sequence: $a, ar, ar^2, \ldots$
- Finite sum: $S_n = a\frac{1-r^n}{1-r}$ for $r \neq 1$
- Quick example: $5 + 10 + 20 + 40 = 75$

Here, `(a=5, r=2, n=4)`

Future value of regular savings (ordinary annuity)

- Assume deposit at end of each year:

$$FV = PMT \cdot \frac{(1+r)^n - 1}{r}$$

- Example: $PMT = 100,\ r = 0.05,\ n = 5$
- Factor: $(\frac{(1.05)^5 - 1}{0.05} \approx 5.526)$

$$FV \approx 552.56$$

PMT = payment each period, r = interest rate, n = number of periods

Common pitfall: “saved” vs “balance”

- Total contributions: $100 \times 5 = 500$
- Balance is larger because interest is earned
  - Vocabulary: “accumulated balance” not “saved amount”

It is common mistake to confuse total contributions with accumulated balance, just remember that interest earned increases the total balance over time.

Present value of an annuity (loan instalments)

- If you repay x each year for n years: $PV = x \cdot \frac{1-(1+r)^{-n}}{r}$
- Negative exponent means discounting: $(1 + r)^{-n} = 1/(1 + r)^n$

This formula calculates the present value of a series of future payments, taking into account the time value of money through discounting (r is a discount rate (opportunity cost of capital)).

Loan example

- Loan $PV = 2000, r = 10$
- Factor: $(\frac{1-(1.10)^{-4}}{0.10} \approx 3.1699)$

$(x = 2000/3.1699 \approx 631.3)$

So, the annual instalment to repay the loan is approximately $631.30.

Present value: discrete vs continuous

- Discrete: $PV = \frac{A}{(1+r)^n}$
- Continuous: $PV = Ae^{-rt}$
- Example: 1000 in 3 years at 6%
  - Discrete ≈ 839.62
  - Continuous ≈ 835.27

Continuous discounting assumes that interest is compounded continuously, leading to a slightly lower present value compared to discrete compounding.

NPV (project appraisal)

$$NPV = -C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+r)^t}$$

- Example: cost 2000, return 800 for 3 years, r=10%

NPV ≈ -10.52 (slightly negative)

NPV is Net Present Value. Since the NPV is negative, the project is not financially viable at a 10% discount rate.

Bond pricing intuition

- Bond value = PV(coupons) + PV(face value)
- If coupon rate = yield, price = face value
- Example: 50 for 10 years + 1000 at end, r=5%
- Price = 1000

Bond is a financial instrument that pays periodic interest (coupons) and returns the principal (face value) at maturity. When the coupon rate equals the market yield, the bond sells at its face value.

What you must be ready to do in class

- Compute FV of savings plan (and state timing assumption)
- Compute instalment x from PV annuity formula
- Compute PV/NPV with correct discounting
- Explain results in words (what changes PV/NPV and why)

Self check questions

1. Find the sum of: $3 + 6 + 12 + 24 + 48$

2. Find the final Savings Plan amount for $200/year, 4% interest, 10 years

3. A loan of $5000 is repaid in 5 annual instalments at 8%. Find the annual instalment.

Report via Cyber Campus by the class start.

PART 2. In-Class Presentation Slides

Indices and Logarithms, Exponential and Natural Log Functions

Why This Topic Matters

Understanding **geometric series** and **investment appraisal** is crucial in economics and finance:

- **Geometric series** model regular payments, compounding, and amortization schedules
- **Investment appraisal** tools help evaluate project viability, return potential, and loan value
- Core to decisions in **banking, government budgeting, business strategy, and personal finance**
- Equips you to analyze time-based value of money, compare alternatives, and manage risk

> "A dollar today is worth more than a dollar tomorrow — let's understand why."

Section 3.3: Geometric Series

What is a Geometric Progression?

A **geometric sequence** has the form:

$$a, ar, ar^2, ar^3, \ldots$$

Where:

- a = first term
- r = common ratio (can be fraction or negative)

Basically, each term is the previous term multiplied by the common ratio.

Example: $2, 4, 8, 16, 32$

$a = 2, r = 2$

Sum of a Finite Geometric Series

$$S_n = a \cdot \frac{1 - r^n}{1 - r}, \quad r \neq 1$$

- Example: $5 + 10 + 20 + 40$
  $a = 5, r = 2, n = 4$

$$S_4 = 5 \cdot \frac{1 - 2^4}{1 - 2} = 5 \cdot \frac{-15}{-1} = 75$$

Example: Regular Savings Plan

Annual deposit: $100, interest = 5%, duration = 5 years

It means that each year, you save $100 and earn 5% interest.

$$S_5 = 100 \cdot \frac{1.05^5 - 1}{0.05} \approx 100 \cdot 5.5256 = \boxed{\$552.56}$$

In 5 years, you will have saved $552.56.

tiny[Assume deposits are made at the end of each year. It's ordinary annuity.]

Your turn! Calculate the Sum

- Find the sum of the following geometric series:

$$2 + 4 + 8 + 16 + 32$$

$$3 + 6 + 12 + 24 + 48$$

- Find the final Savings Plan amount for $200/year, 4% interest, 10 years

Example: Loan Repayment by Instalments

Loan = $2000, repaid in 4 annual instalments at 10%

$$x \cdot \left(\frac{1 - (1.10)^{-4}}{0.10} \right) = 2000$$

Solve for x to find equal annual instalments.

$$x = 2000 \cdot \frac{0.4641}{0.10} \approx 928.20$$

This result is incorrect—here is the fix.

Example: Loan Repayment by Instalments (continued)

Correct Calculation

We use the **Present Value of an Ordinary Annuity** formula:

$$PV = x \cdot \frac{1-(1+r)^{-n}}{r}$$

PS: we have $-n$ power because we are calculating the present value of future payments.

$$(1+r)^{-n} = \frac{1}{(1+r)^n}$$

This tells us:

- **Raising to a negative power** means **discounting** — calculating the **present value** of a future amount.
- It reflects how **$1 received in the future is worth less today**.

Substitute known values:

- $PV = 2000$
- $r = 0.10$
- $n = 4$

Step-by-Step Solution

Step 1: Plug values into the formula:

$$2000 = x \cdot \frac{1-(1.10)^{-4}}{0.10}$$

$$(1.10)^{-4} = \frac{1}{1.4641} \approx 0.6830$$

$$\Rightarrow \frac{1-0.6830}{0.10} = \frac{0.3170}{0.10} = 3.170$$

Step 2: Solve for x

Now solve:

$$2000 = x \cdot 3.170 \Rightarrow x = \frac{2000}{3.170} \approx \boxed{631.30}$$

So your **annual repayment** is **$631.30**.

Your turn! Calculate the Instalments

- A loan of \$5000 is repaid in 5 annual instalments at 8%. Find the annual instalment.
- A loan of \$3000 is repaid in 3 annual instalments at 6%. Find the annual instalment.

Section 3.4: Investment Appraisal

Investment Appraisal

Stands for evaluating the financial viability of investments.

Present Value (Discrete)

- Present value PV is the current value of future cash flows.

$$PV = \frac{A}{(1+r)^n}$$

- 1000 in 3 years at 6%:

$$PV = \frac{1000}{1.06^3} \approx 839.62$$

Present Value (Continuous)

$$PV = A \cdot e^{-rt}$$

- 1000 in 3 years at 6%:

$$PV = 1000 \cdot e^{-0.18} \approx 835.27$$

What's the difference between discrete and continuous?

- Discrete: Values occur at specific intervals (e.g., yearly, monthly).
  Example: Annual interest compounding.
- Continuous: Values change smoothly over time, modeled using exponential functions.
  Example: Continuous compounding of interest.

Net Present Value (NPV)

- Net present value NPV is the sum of present values of cash flows.

$NPV = -C_0 + \sum_{t=1}^{n} \frac{C_t}{(1+r)^t}$

- Example: Cost = $2000, Returns = $800/year for 3 years, $r = 10\%$:

$$NPV \approx -2000 + \frac{800}{1.10} + \frac{800}{1.10^2} + \frac{800}{1.10^3} \approx 75.13$$

This result is incorrect—here is the fix.

Correct Calculation

$$= -2000 + 727.27 + 661.16 + 601.06 = \boxed{-10.51}$$

So, the project has a **negative NPV** of about **–$10.51**, meaning **it's not financially viable** at a 10% discount rate.

Present Value of an Annuity

- Annuity is a series of equal payments over time.

$$PV = PMT \cdot \frac{1-(1+r)^{-n}}{r}$$

- \$1000/year for 5 years at 6%:

$$PV = 1000 \cdot \frac{1-(1.06)^{-5}}{0.06} \approx 4212.36$$

Internal Rate of Return (IRR)

- IRR is the rate at which NPV = 0.
- We calculate IRR to compare investment options.

Find r such that $NPV = 0$:

$$0 = -C_0 + \sum \frac{C_t}{(1+r)^t}$$

Solve numerically or using software.

Present Value of Government Bonds

Bond pays \$50/year for 10 years, plus \$1000 at end.

$$PV = \sum_{t=1}^{10} \frac{50}{(1.05)^t} + \frac{1000}{(1.05)^{10}} \approx 925.81$$

This result is incorrect—here is the fix.

Correct Calculation

$$PV = \sum_{t=1}^{10} \frac{50}{(1.05)^t} + \frac{1000}{(1.05)^{10}} \approx \boxed{1000}$$

This makes sense: at a 5% rate, the bond is worth **its face value**.

Practice Problems

1. Sum of: $3 + 6 + 12 + 24$
2. \$200/quarter, 4% quarterly compounding for 3 years
3. PV of \$1500 in 2 years at 8% (continuous)
4. NPV: \$3000 cost, \$1200/year for 3 years, 7% rate
5. PV of \$500 annuity for 6 years at 5%
6. IRR: \$2000 cost, \$500/year for 5 years

Summary

- Geometric series = finance modeling
- Investment tools = PV, NPV, IRR
- Practical for decision-making in public/private sectors

Any QUESTIONS?

Thank you for your attention!

Next Class

- (March 31) The Derivative of Functions (4.1), Rules of Differentiation (4.2)

Next Friday (April 2) - Quiz #1

- Please review your Home Work #1, in-class problems and problems from the textbook.