Mathematical Methods for International Commerce

Week 6/2: Further Rules of Differentiation, and Elasticity

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Hello everyone!

I hope you all are doing well and ready for yet another **exciting session** of Mathematical Methods for International Commerce.

PART 1. Flipped Classroom – Pre-Class Video 2 Slides

Week 6/2: Further Rules of Differentiation, and Elasticity

What you should already know before class

- Power rule (and constants)
- What a derivative means (slope / marginal change)
    - Today: chain, product, quotient + elasticity (arc vs point)

Chain rule (pattern recognition)

- If $y = f(u)$ and $u = g(x)$, then $\frac{dy}{dx} = f'(u) \cdot u'$
- Example: $y = (3x^2 + 2)^4 \Rightarrow y' = 24x(3x^2 + 2)^3$

Key habit: “outside derivative” times “inside derivative”

Product rule (two moving parts)

$\frac{d}{dx}[uv] = u'v + uv'$

- Example: $y = x^2 \ln x \Rightarrow y' = 2x \ln x + x$

Quotient rule (ratio)

$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v-uv'}{v^2}$

- Example: $y = \frac{\ln x}{x} \Rightarrow y' = \frac{1-\ln x}{x^2}$

Elasticity: what it measures

- Elasticity = “percent change in Q” relative to “percent change in P”
- Demand elasticity is typically negative (price up $\rightarrow$ quantity down)

Arc elasticity (two points, average)

$$E_d = \frac{(Q_2 - Q_1)/\left(\frac{Q_1+Q_2}{2}\right)}{(P_2 - P_1)/\left(\frac{P_1+P_2}{2}\right)}$$

- Quick example $10 \to 12, 50 \to 40$: $E_d \approx -1.22 = |-1.22| = 1.22$ (elastic)

Elasticity, especially price elasticity of demand, is mathematically negative due to the law of demand (price and quantity move inversely), but it's conventionally reported as a positive number using its absolute value to focus on the magnitude of responsiveness (elastic/inelastic) rather than the direction of change.

Point elasticity (a point, derivative)

$$E_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

- If $Q = 120 - 3P$, then $\frac{dQ}{dP} = -3$. At $P = 10$, $Q = 90$, so $E_d = |-0.33| = 0.33$ (inelastic)

Elasticity is below 1, indicating that quantity demanded is relatively unresponsive to price changes at this point on the demand curve.

Elasticity and total revenue (decision rule)

- If $|E_d| > 1$: lowering price increases TR
- If $|E_d| < 1$: lowering price decreases TR
- Unit elastic $|E_d| = 1$: TR maximized (for your linear example, at $P = 20$, $E_d = -1$)

Ready for class

In class you will:

- do mixed differentiation drills (choose the right rule)
- compute arc vs point elasticity
- interpret: “raise or lower price?” based on elasticity and revenue

Check your understanding

- What is the derivative of $y = (x^2 + 1)^3$?
- Compute the arc elasticity for a demand function where $P_1 = 10$, $P_2 = 12$, $Q_1 = 50$, and $Q_2 = 40$.
- If a product has a point elasticity of -0.5, what happens to total revenue if the price is lowered?

Report your answers via Cyber Campus by class time!

PART 2. In-Class Presentation Slides

Further Rules of Differentiation, and Elasticity

Agenda

1. Further Rules of Differentiations (4.4)
2. Elasticity (4.5)
3. Group Activity: Differentiation & Elasticity Challenge

Learning Objectives

Section 4.4 – Further Rules of Differentiation

- Use the **chain rule** to differentiate a function of a function
- Apply the **product rule** to differentiate the product of two functions
- Apply the **quotient rule** to differentiate the ratio of two functions
- Differentiate **complex functions** combining multiple rules

Section 4.5 – Elasticity

- Calculate **arc elasticity** (average)
- Calculate **point elasticity**
- Determine whether elasticity is **elastic**, **unitary**, or **inelastic**
- Understand **elasticity and total revenue**
- Analyze elasticity in **linear demand functions**

1. Further Rules of Differentiations (4.4)

Chain Rule

Used when you have a function **inside** another function:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

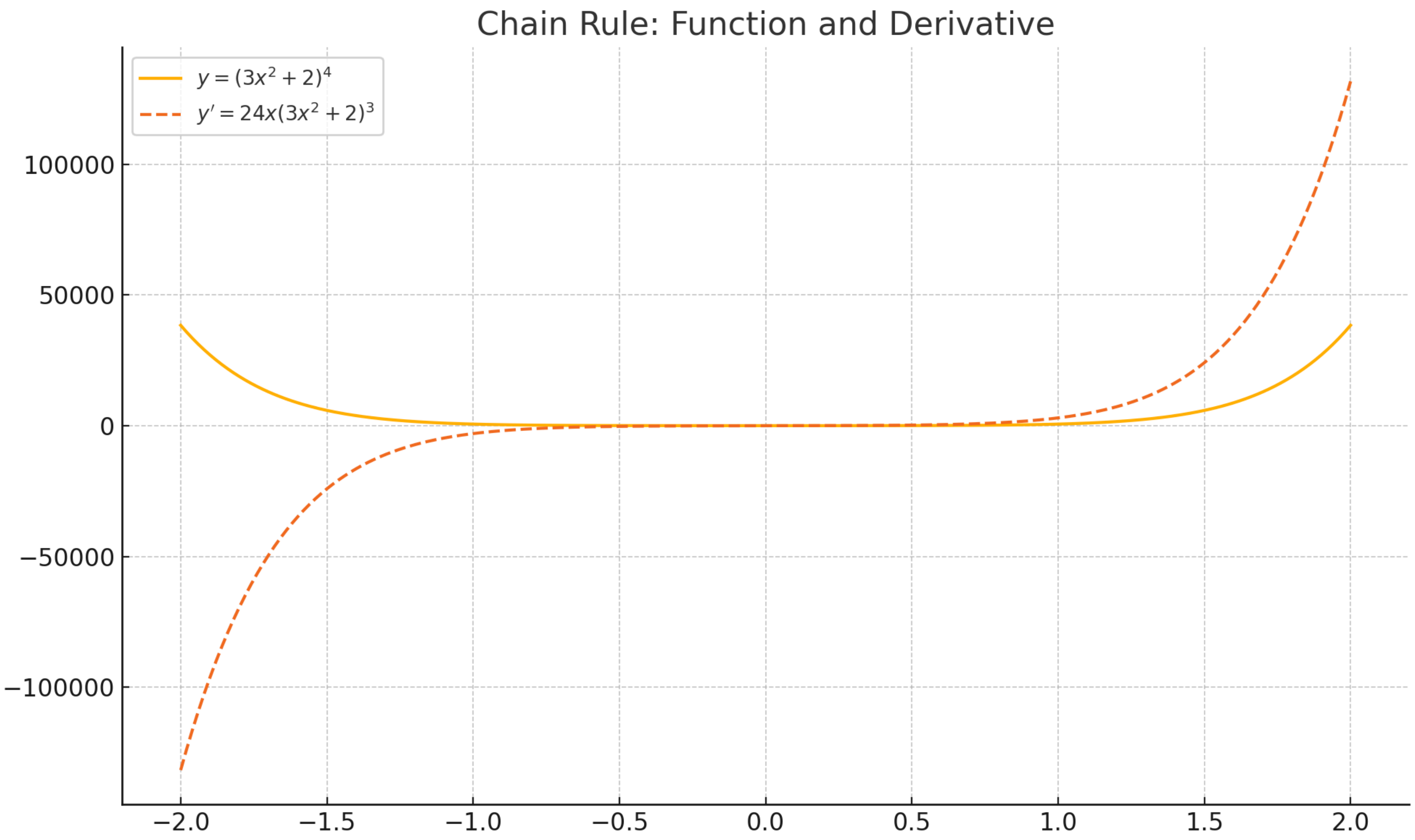
Example:

Let $y = (3x^2 + 2)^4$

- Set $u = 3x^2 + 2$
- $\frac{dy}{dx} = 4u^3 \cdot \frac{du}{dx} = 4(3x^2 + 2)^3 \cdot 6x = 24x(3x^2 + 2)^3$

Apply chain rule when exponents wrap a full expression

Illustration of Chain Rule



Product Rule

Used when differentiating a **product of two functions**:

$$\frac{d}{dx}[u(x) \cdot v(x)] = u'(x)v(x) + u(x)v'(x)$$

Example:

$y = x^2 \cdot \ln(x)$

- $u = x^2, \quad v = \ln(x)$
- $y' = 2x \cdot \ln(x) + x^2 \cdot \frac{1}{x} = 2x\ln(x) + x$

Both terms matter!

Quotient Rule

Used when differentiating a **quotient of two functions**:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{u'(x)v(x) - u(x)v'(x)}{[v(x)]^2}$$

Example:

$y = \frac{\ln(x)}{x}$

- $u = \ln(x), \quad v = x$
- $y' = \frac{\frac{1}{x} \cdot x - \ln(x) \cdot 1}{x^2} = \frac{1 - \ln(x)}{x^2}$

Useful for cost/revenue ratios!

Combination of Rules

Example:

$$y = \frac{(x^2+1)^3 \cdot \ln(x)}{x^2}$$

- Use **product rule** on numerator, **chain rule** on $(x^2+1)^3$
- Use **quotient rule** for the entire function

Many economic models require **layered differentiation**

2. Elasticity (4.5)

Elasticity of Demand & Supply

Arc Elasticity:

Average elasticity over an interval:

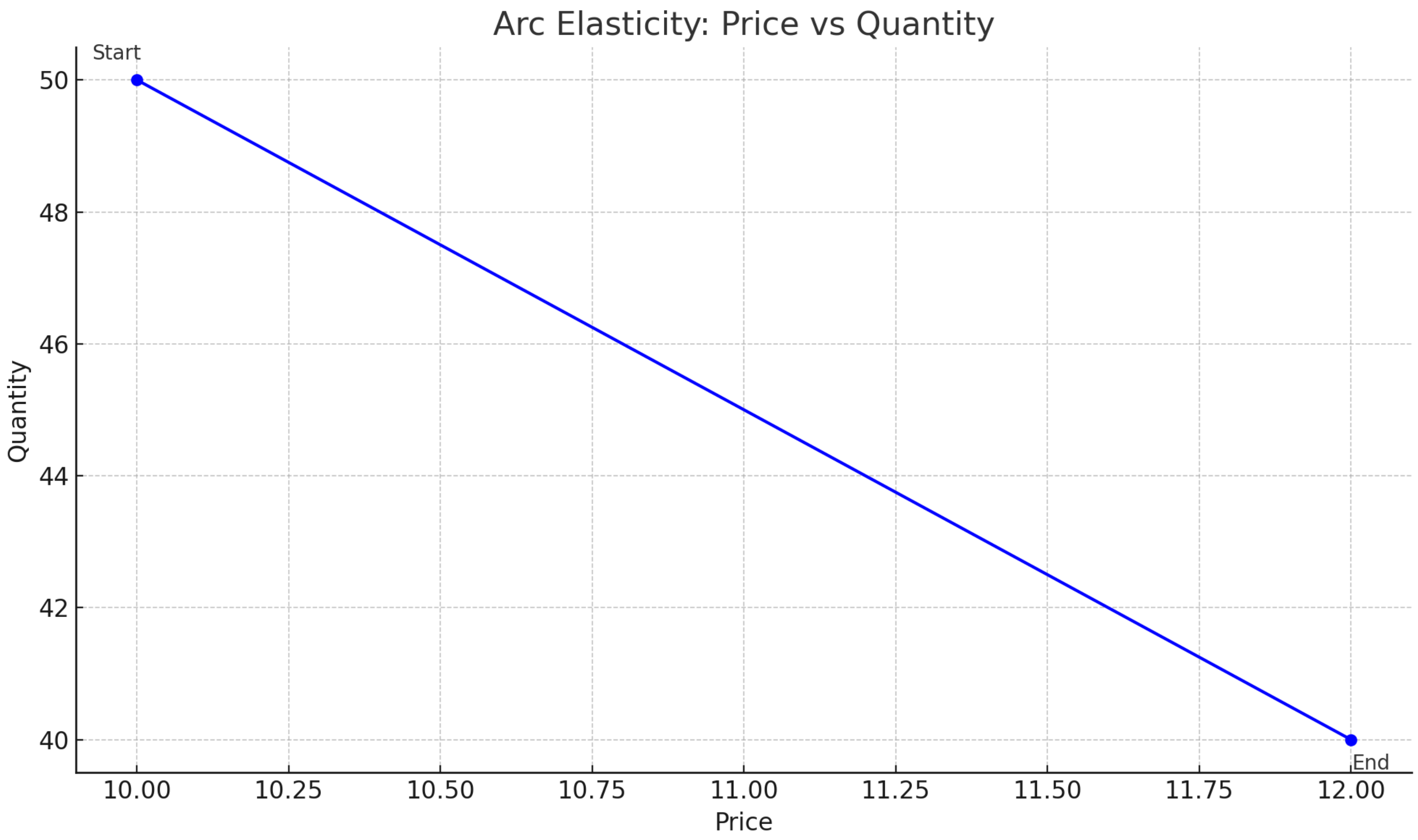
$$E_d = \frac{\Delta Q/\text{avg } Q}{\Delta P/\text{avg } P} = \frac{\frac{Q_2 - Q_1}{(Q_1+Q_2)/2}}{\frac{P_2 - P_1}{(P_1+P_2)/2}}$$

Example:

- Price increases from \$10 to \$12, quantity falls from 50 to 40
- $E_d = \frac{(40-50)/45}{(12-10)/11} = \frac{-10/45}{2/11} = -1.22$

Elastic demand (|E| > 1) meaning consumers are sensitive to price changes

Illustration of Arc Elasticity



Point Elasticity:

Use **derivative** and point values:

$$E_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

If $Q = 120 - 3P$, then $\frac{dQ}{dP} = -3$

- At $P = 10, Q = 90$

$E_d = -3 \cdot \frac{10}{90} = -0.33$

Inelastic demand meaning consumers are less sensitive to price changes

Elasticity & Revenue

Revenue rule of thumb:

- If $|E_d| > 1$: Lowering price (P) **increases** revenue (TR)
- If $|E_d| < 1$: Lowering price (P) **decreases** revenue (TR)
- At $|E_d| = 1$: Revenue (TR) is **maximized**

Graph it: Revenue = $P \times Q = P(120 - 3P) = 120P - 3P^2$

- Max revenue when $MR = 0 \Rightarrow$ elasticity = 1

Elasticity & Revenue

Total Revenue (TR)

$$TR(P) = P \cdot Q(P)$$

Key link:

$$MR = \frac{dTR}{dQ} = 0 \quad \Longleftrightarrow \quad |E_d| = 1 \text{ (unit elastic demand)}$$

Example (linear demand):

$$Q = 120 - 3P$$

$$TR(P) = P(120 - 3P) = 120P - 3P^2$$

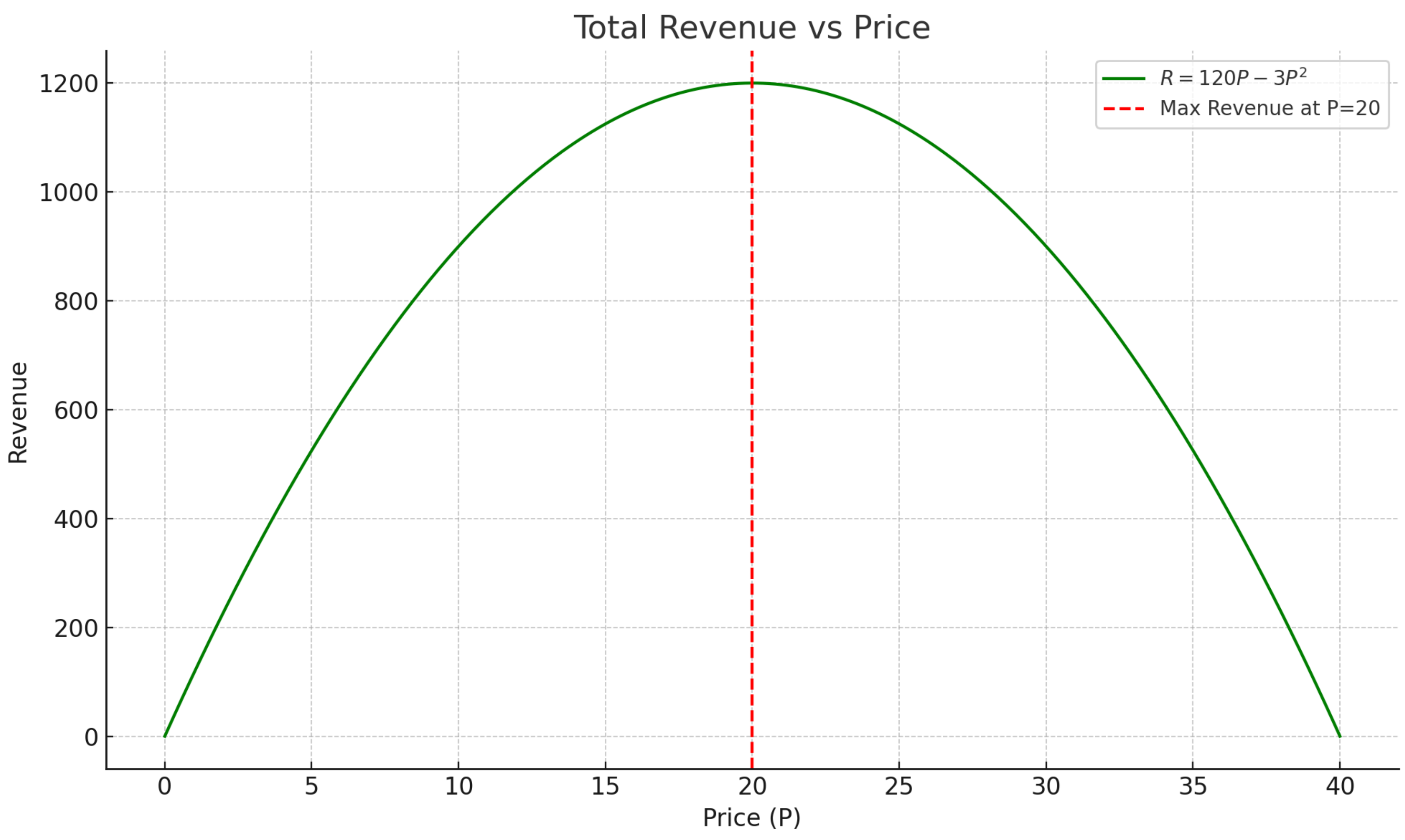
$$MR(P) = \frac{dTR}{dP} = 120 - 6P$$

Max revenue when (MR=0): $$ 120-6P=0 \Rightarrow P^*=20,\quad Q^*=120-3(20)=60 $$

Elasticity at the revenue-maximizing point:

$$E_d = \frac{dQ}{dP} \cdot \frac{P}{Q} = (-3) \cdot \frac{20}{60} = -1$$

Illustration of Revenue and Elasticity



Elasticity in Linear Demand

If $Q = a - bP$, then:

$$E_d = -b \cdot \frac{P}{Q}$$

- Easy to evaluate at any point
- At $P = 0$, elasticity = 0 (perfectly inelastic)
- At $Q = 0$, elasticity = ∞ (perfectly elastic)
- At midpoint, $E_d = -1$

Linear demand: Elasticity changes **along** the curve

Practice Problems

1. Differentiate using chain rule:
   - (a) $f(x) = (5x^2 + 1)^4$
2. Differentiate using product rule:
   - (a) $f(x) = x^2 e^x$
3. Differentiate using quotient rule:
   - (a) $f(x) = \frac{e^x}{x^2}$
4. Find arc elasticity:
   - Price rises from \$8 to \$10; Q falls from 60 to 48
5. For $Q = 100 - 4P$, find point elasticity at $P = 10$
6. Interpret elasticity:
   - When $E_d = -1$, what happens to total revenue if price increases?

Summary

- Use chain, product, and quotient rules to handle **complex expressions**
- Elasticity helps us understand **how responsive** quantity is to price
- Price elasticity relates directly to **revenue decisions**

> In economics, calculus lets us optimize decisions, and elasticity helps us understand consumer response.

3. Group Activity: Differentiation & Elasticity Challenge

Setup

- Class of **16 students** → **4 groups of 4**
- Each group receives **1 challenge card**
- Work collaboratively on whiteboards or paper

Time:

- 10 mins group work
- 2 mins presentation per group
- 5 mins wrap-up discussion

Group 1: Chain Rule in Production

Production Function:

$$Q = (5L^2 + 3)^3$$

1. Find the marginal product using the **chain rule**
2. Interpret: What does it say about productivity as labor increases?

Group 2: Product Rule in Revenue

Revenue Function:

$$R(x) = x \cdot \ln(x)$$

1. Find **marginal revenue** using the product rule
2. Evaluate MR at (x = 1)

Group 3: Elasticity Debate

Demand Function:

$$Q = 120 - 4P$$

1. Calculate **point elasticity** at (P = 10)
2. Should the firm **raise or lower the price** to increase revenue?

Group 4: Quotient Rule in Cost Analysis

Average Cost per unit:

$$C(x) = \frac{100 + 2x^2}{x}$$

1. Find the **the rate of change of average cost** using the quotient rule
2. What does it imply as output grows?

Any QUESTIONS?

Thank you for your attention!

Next Class

- (April 14) Optimization of Economic Functions (4.6, 4.7)