Mathematical Methods for International Commerce

Week 14/2-15/1: Linear Programming (8.1)

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Agenda

1. Linear Programming (8.1)
2. Group activity
3. Homework #2

1. Linear Programming (8.1)

Why It Matters in Economics & Business

Linear programming is a powerful tool to:

- Optimize **profits**, **costs**, **resource use**
- Solve **production**, **transportation**, **investment** problems
- Understand **feasibility** and **trade-offs** in economic models

Real-world applications include:

- Supply chain optimization
- Product mix decisions
- Agricultural planning
- Finance: portfolio risk constraints

> Linear programming helps make **data-driven** decisions in complex environments.

What is Linear Programming?

A **linear programming (LP)** problem:

- Optimize a linear objective function
- Subject to a set of linear inequalities (constraints)

General form:

$$\text{Maximize or Minimize } Z = c_1 x + c_2 y$$

Subject to:

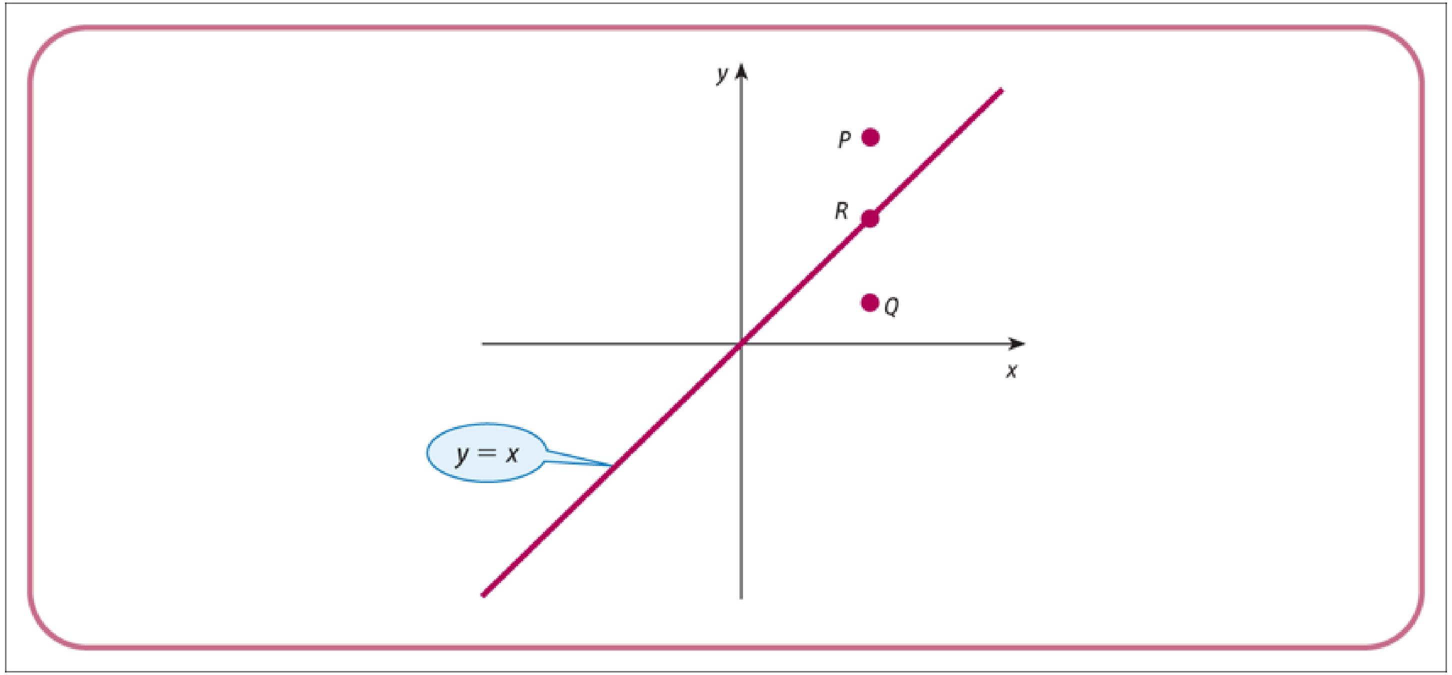
$$\begin{aligned} a_1 x + b_1 y &\leq d_1 \\ a_2 x + b_2 y &\leq d_2 \\ x, y &\geq 0 \end{aligned}$$

Interpreting Linear Inequalities Graphically

Linear inequality: $y \geq x$

- Points **above** the line $y = x$: satisfy $y > x$
- Points **on** the line: satisfy $y = x$
- Points **below** the line: satisfy $y < x$

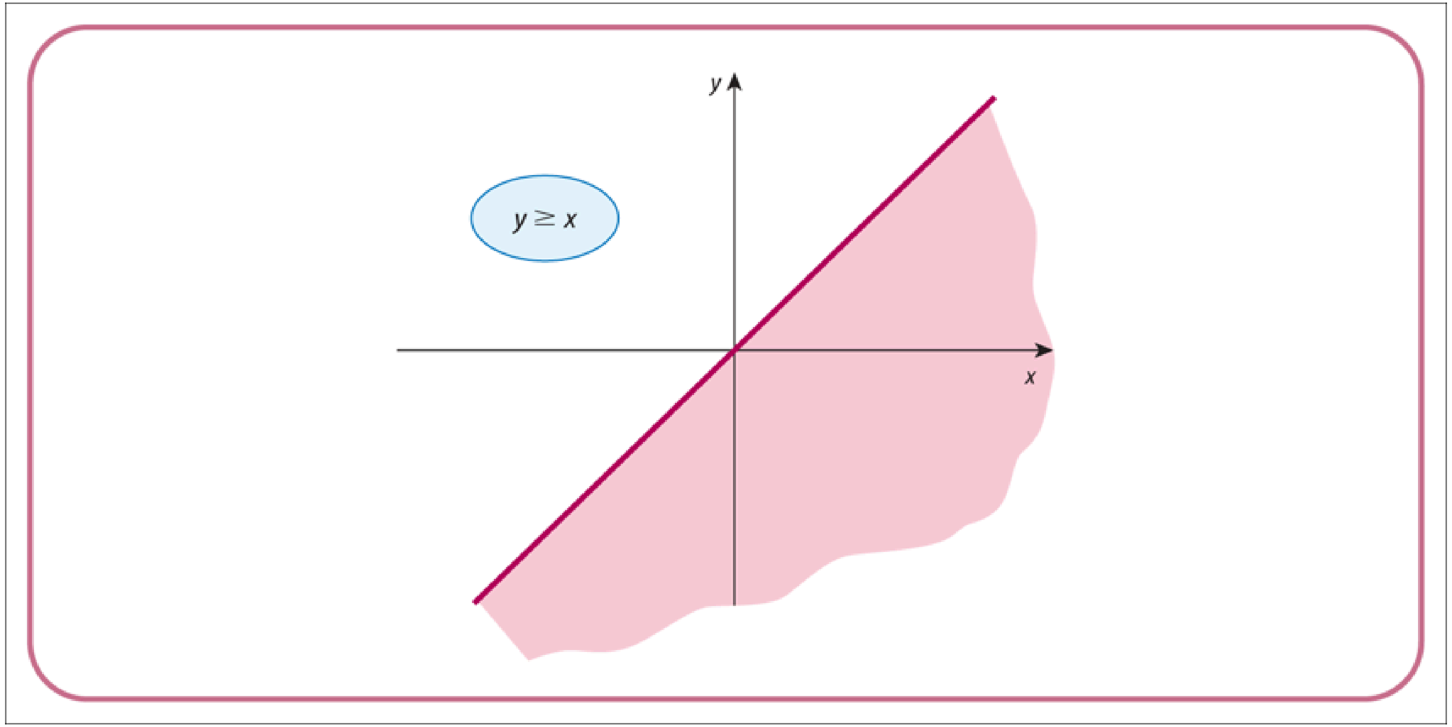
Graph of $y = x$ with Sample Points



- Point **P** lies above $y = x$
- Point **Q** lies below $y = x$
- Point **R** lies on $y = x$

Graph of $y \geq x$: Shading False Region

We **shade the region not satisfying** the inequality:



This helps highlight the feasible region clearly for optimization.

General Strategy

To sketch any linear inequality:

1. Draw the boundary line $dx + ey = f$

2. Use a test point to determine shading:

    - If inequality holds $\rightarrow$ region of interest
    - Otherwise $\rightarrow$ shade opposite side

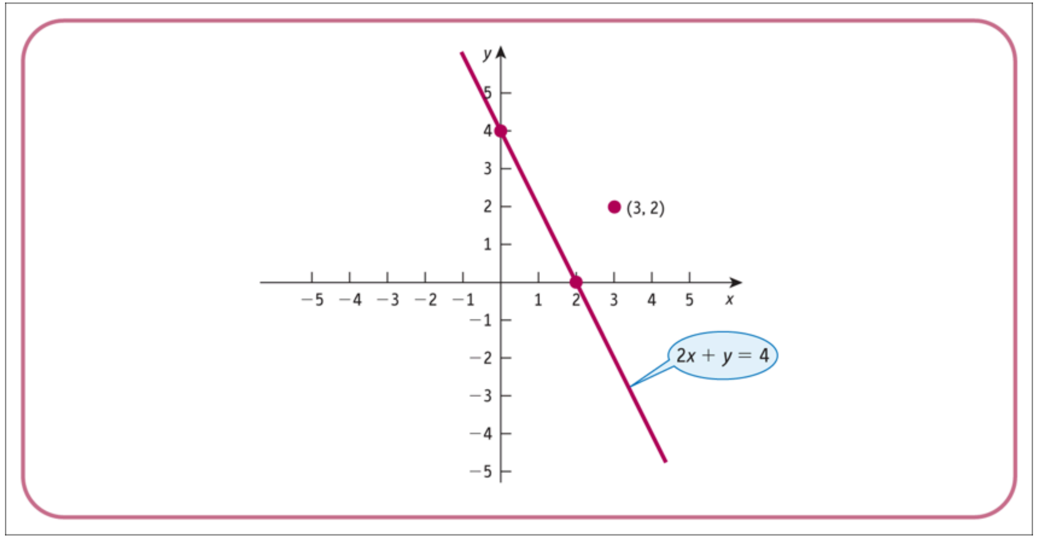
3. Use **solid line** for $\leq, \geq$, **dashed** for $<, >$

Example: $2x + y < 4$

Step 1: Plot $2x + y = 4$

- When $x = 0, y = 4$
- When $y = 0, x = 2$

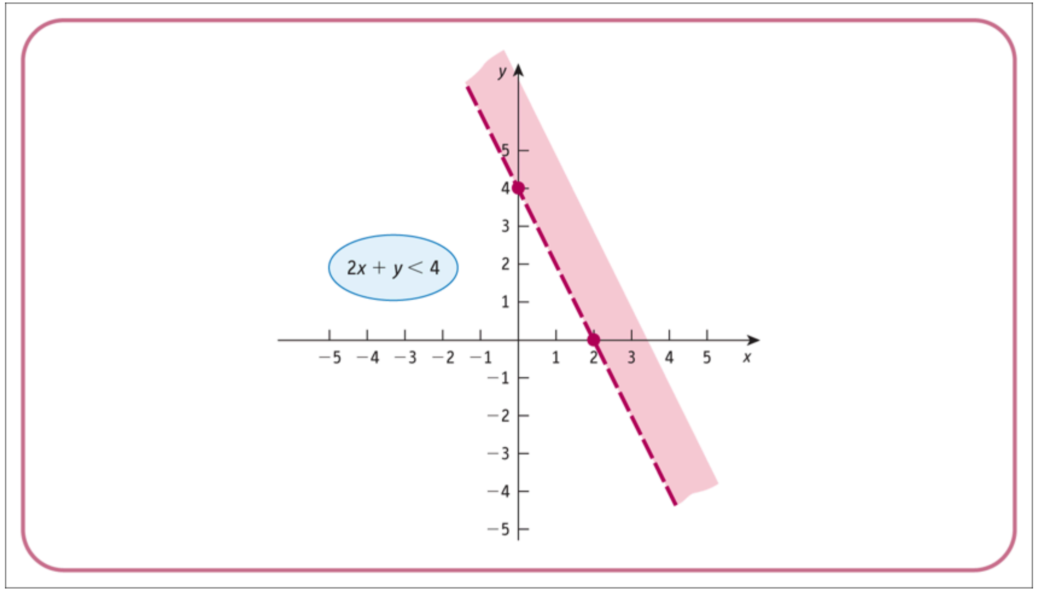
Plot line through $(0, 4)$ and $(2, 0)$



Step 2: Choose a Test Point

Test point: $(3, 2)$

- Evaluate: $2(3) + 2 = 8 \not< 4$
- So: region of interest lies **below** the line
- Use **broken line** for $<$

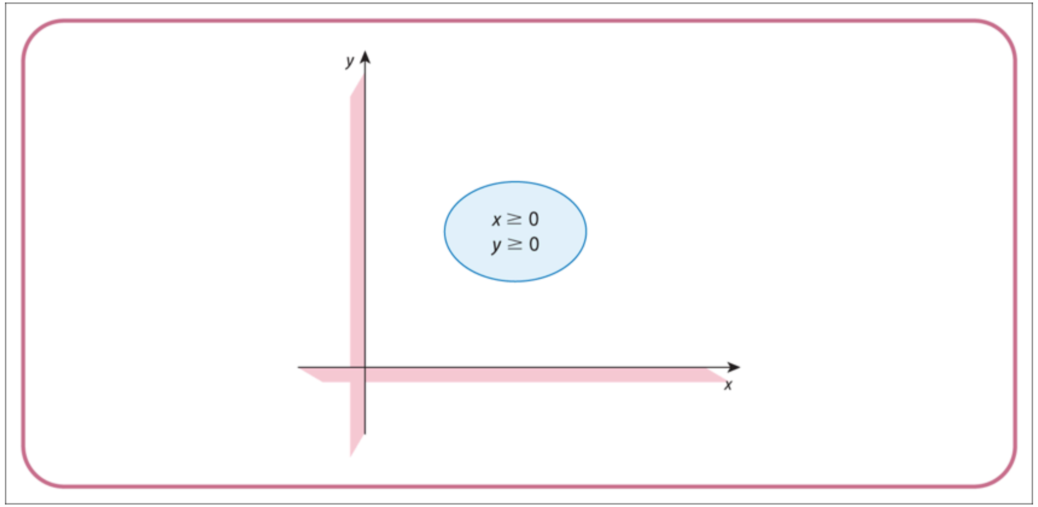


Feasible Region from Multiple Inequalities

To define a feasible region, plot and intersect:

- $x + 2y \leq 12$
- $-x + y \leq 3$
- $x \geq 0, y \geq 0$

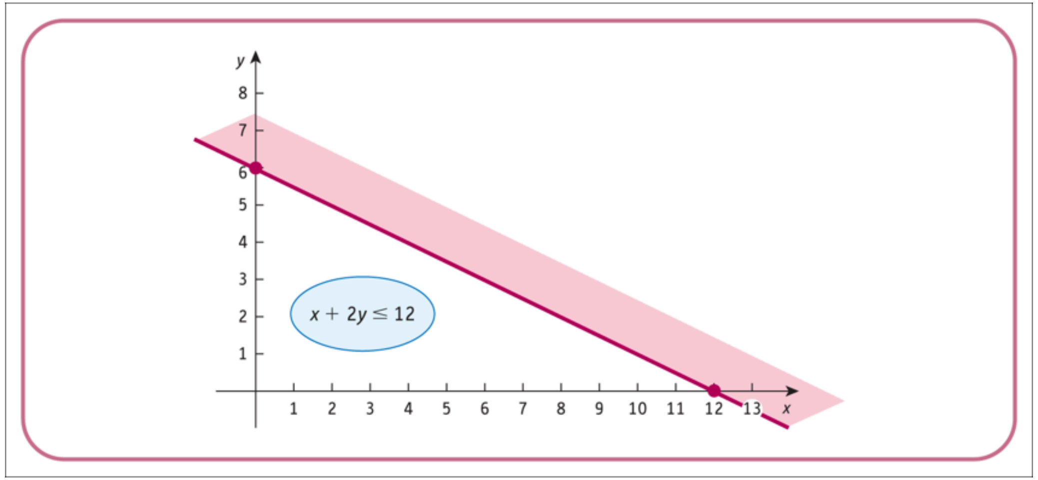
Non-Negativity Constraints



Top-right quadrant: restricts us to economically meaningful values of x and y.

Constraint $x + 2y \leq 12$

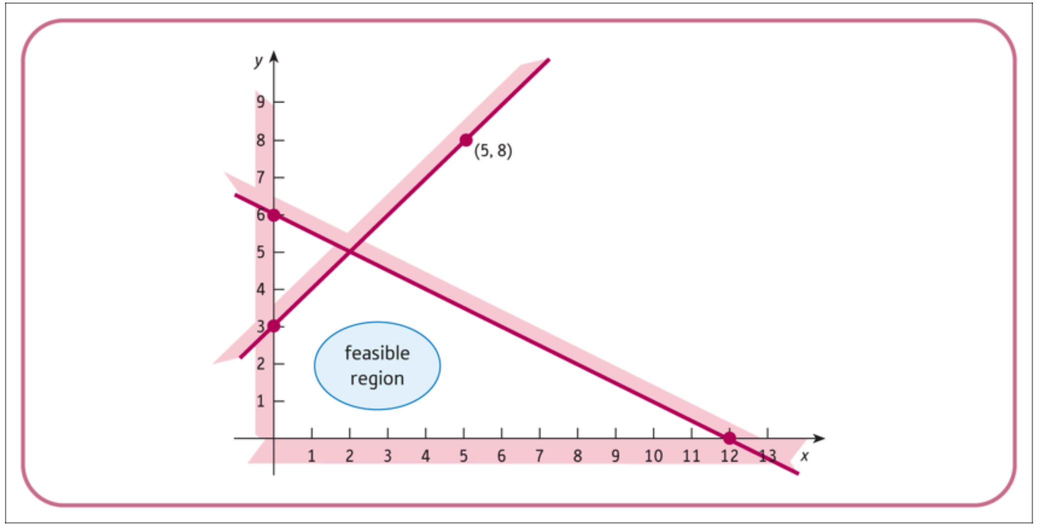
- Line passes through $(0, 6)$ and $(12, 0)$
- Test point $(0, 0)$: inequality holds → shade above



Final Constraint $-x + y \leq 3$

- Line passes through $(0, 3)$ and $(5, 8)$
- Test point $(0, 0)$: holds $\rightarrow$ shade above

The **feasible region** is the unshaded area satisfying all inequalities.



Summary: Graphical Approach (2 Variables)

Steps:

1. Express constraints as **equalities** to sketch lines
2. Identify **feasible region** (satisfying all inequalities)
3. Find **corner points** of feasible region
4. Evaluate **objective function** at each corner
5. Choose max or min value

> **Note:** Only works for 2-variable problems. For higher dimensions: use Simplex Method.

Sweep Method with Objective Lines

We now introduce a linear programming objective:

Minimize $-2x + y$

Subject to:

- $x + 2y \leq 12$
- $-x - y \leq 3$
- $x \geq 0$
- $y \geq 0$

Objective Function Interpretation

The objective function $-2x + y$ can be interpreted as a line in the xy-plane:

This corresponds to **lines of constant objective**: $-2x + y = c \Rightarrow y = 2x + c$

These lines are **parallel**, slope = 2, and shift with c

As c decreases, lines move across feasible region.

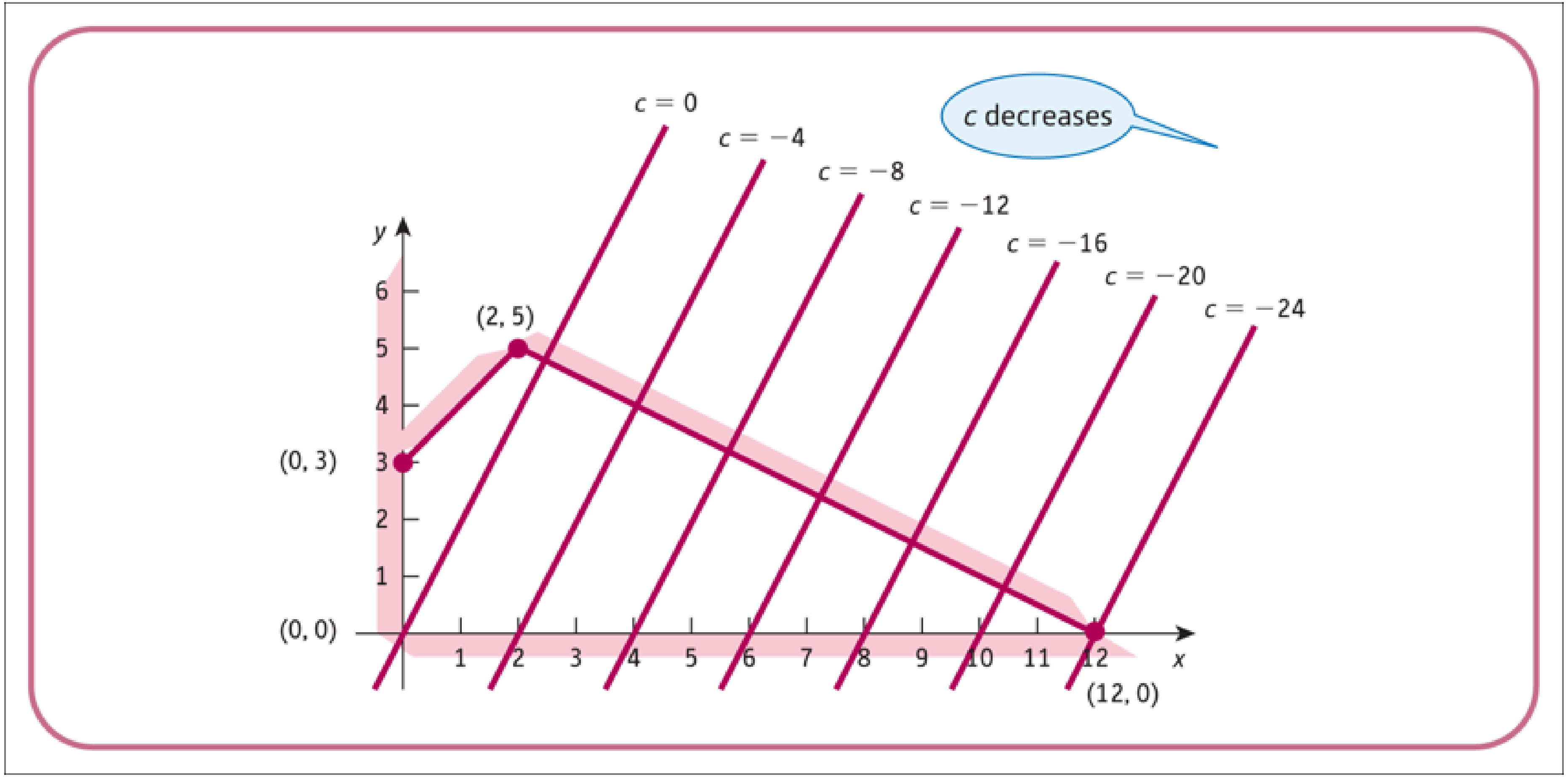
We are looking for the smallest c such that the line still touches the feasible region.

That line will be tangent at the **optimal solution**.

Minimum occurs at point $(12, 0)$

- Check: $-2(12) + 0 = -24$

Hence, the **minimum** value is **-24** at $(12, 0)$


c decreases
c = 0
c = −4
c = −8
c = −12
c = −16
c = −20
c = −24
y
x
(2, 5)
(0, 3)
(0, 0)
(12, 0)
1
2
3
4
5
6
7
8
9
10
11
12

Objective Values at the Corners

We evaluate the objective $-2x + y$ at corners of feasible region:

Corner	Objective function
(0, 0)	$-2(0) + 0 = 0$
(0, 3)	$-2(0) + 3 = 3$
(2, 5)	$-2(2) + 5 = 1$
(12, 0)	$-2(12) + 0 = -24$

Minimum = **−24** at $(12, 0)$, Maximum = **3** at $(0, 3)$

Example: Maximize Profit

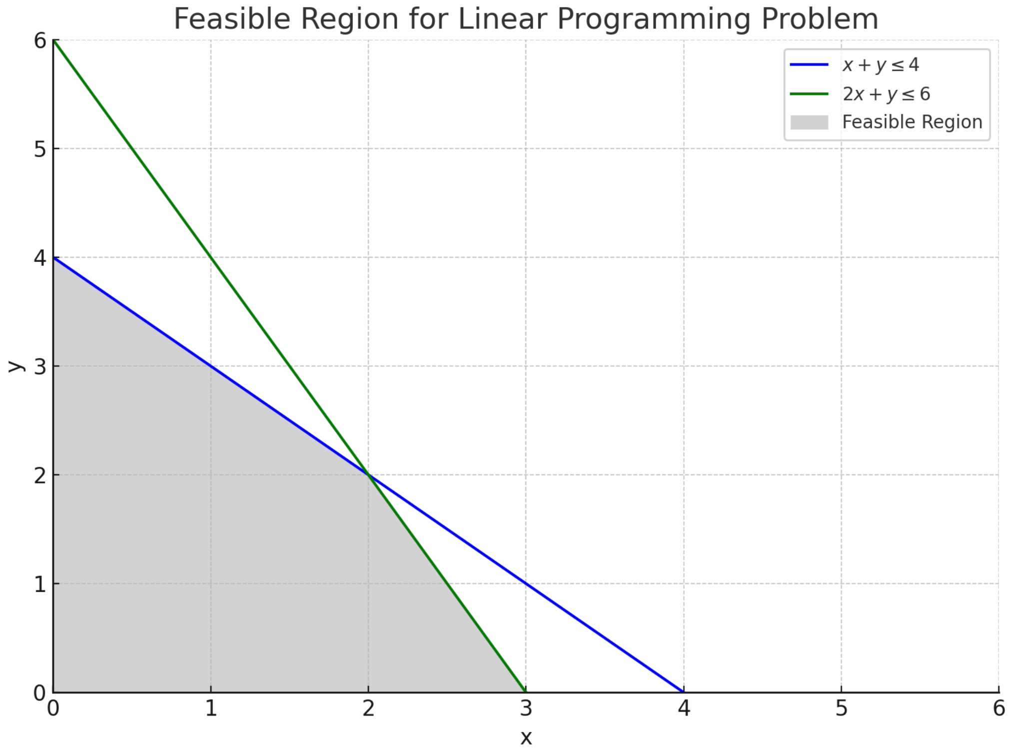
A firm produces two goods: x and y

Objective: Maximize $Z = 5x + 3y$

Subject to:

$$\begin{aligned} x + y &\leq 4 \\ 2x + y &\leq 6 \\ x, y &\geq 0 \end{aligned}$$

Step 1: Plot Constraints



Step 2-4: Feasible Region and Objective Evaluation

Feasible points: Intersections and axis cuts:

- (0, 0)
- (0, 4)
- (2, 2)
- (3, 0)

Evaluate $Z = 5x + 3y$ at each:

Point	Z
(0,0)	0
(0,4)	12
(2,2)	16 $\Leftarrow$ Max
(3,0)	15

Interpretation in Business Context

This tells the firm:

- To **maximize profits**, it should produce 2 units of x and 2 units of y
- Constrained by available **resources**

This helps with:

- **Resource allocation**
- **Product mix decisions**

Special Cases

1. No Solution (Infeasible)

- When constraints **conflict**

2. Infinite Solutions

- Objective function is **parallel** to a constraint edge

3. Unbounded Solution

- No upper limit; occurs when constraints don't bound the feasible region

Add visual plots to illustrate each.

Summary

- Linear programming optimizes linear functions under constraints
- Graphical method works for 2-variable problems
- Corner points of feasible region yield optimal solutions
- Special cases include infeasibility, infinite solutions, and unbounded solutions

2. Group activity

Practice Problem (Group)

Minimize $-x + y$

Subject to:

- $3x + 4y \leq 12$
- $x \geq 0$
- $y \geq 0$

(a) Sketch the feasible region.

(b) Sketch, on the same diagram, the five lines $y = x + c$ for $c = -4, -2, 0, 1, 3$. *Hint: Each line* $y = x + c$ *has slope 1 and passes through* $(0, c)$ *and* $(-c, 0)$*.*

(c) Use your answers to part (b) to solve the given linear programming problem.

Practice Problem (Group)

A bakery makes bread x and muffins y.

Profit: $Z = 3x + 4y$

Subject to:

$$\begin{aligned} x + 2y &\leq 8 \\ 5x + 3y &\leq 15 \\ x, y &\geq 0 \end{aligned}$$

Tasks:

- Graph the constraints
- Identify feasible region
- Compute profit at each corner point
- Find optimal solution

3. Home work #2

Homework #2

- **Due Date:** June 13, 2025, before the start of class.
- **Submission Format:** Submit your solutions as a single PDF file via the Cyber Campus.

- **Instructions:**

  - Clearly show all steps and calculations.
  - Include explanations for your answers where applicable.
  - Ensure your submission is neat and well-organized.
  - Bring any questions to the office hours or email me.

- Work on your Home Assignment #2 (Jacques, 10th edition, Chapters 5-9):

  - Chapter 5.1: Exercise 5.1, Problem 5 (p. 411)
  - Chapter 5.2: Exercise 5.2, Problem 7 (p. 427)
  - Chapter 5.3: Exercise 5.3, Practice Problem inside the chapter (p. 432 only)
  - Chapter 5.4: Exercise 5.4, Problem 5 (p. 457)
  - Chapter 5.5: Exercise 5.5, Problems 6 and 7 (p. 467)
  - Chapter 5.6: Exercise 5.6, Problem 4 (p. 479)
  - Chapter 6.1: Exercise 6.1, Problem 3 (p. 506)
  - Chapter 6.2: Exercise 6.2, Problem 6 (p. 521)
  - Chapter 7.1: Exercise 7.1, Problem 5 (p. 553)
  - Chapter 7.2: Exercise 7.2, Problem 6 (p. 572)
  - Chapter 7.3: Exercise 7.3, Problem 5 (p. 584)
  - Chapter 8.1: Exercise 8.1, Problem 6 (p. 615)
  - Chapter 8.2: Exercise 8.2, Problem 3 (p. 626)
  - Chapter 9.1: Exercise 9.1, Problem 4 (p. 655)
  - Chapter 9.2: Exercise 9.2, Problem 3 (p. 670)

Good luck!

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (June 14) Applications of Linear Programming (8.2)