Mathematical Methods for International Commerce

Week 10/2: Unconstrained Optimization (5.4)

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Why It Matters in Economics & Business

- Optimization helps firms maximize profit by adjusting inputs or pricing.
- It allows for strategic decision-making in pricing, production, and resource allocation.
- Applications in:
  - Cost minimization (minimizing production costs)
  - Profit maximization (maximizing revenue)
  - Price discrimination (optimizing prices in different markets)
  - Resource allocation (optimizing the use of limited resources)
  - Risk management (optimizing investment portfolios)

Learning Objectives

By the end of this class, you should be able to:

- Use first-order partial derivatives to find stationary points.
- Use second-order partial derivatives to classify stationary points.
- Maximize the profit of a firm producing two goods.
- Optimize profit for a firm using price discrimination in different markets.

Agenda

1. Unconstrained Optimization (5.4)

2. Class Activity

Finding Stationary Points

- Our first step is to find the stationary points of a function.
  - A stationary point is where the first derivative (or partial derivative) is zero.
- Why? - Because it indicates a potential maximum or minimum.
- We can find stationary points using the first-order conditions.

For a function of two variables $f(x, y)$, the first-order partial derivatives are:

$$f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}$$

Stationary points occur where both derivatives equal zero:

$$f_x = 0 \quad \text{and} \quad f_y = 0$$

Example:
Let $f(x, y) = 3x^2 - 2xy + y^2$

1. Calculate f_x and f_y.
2. Set $f_x = 0$ and $f_y = 0$ to find the stationary points.

Solution: Stationary Points

1. First-order derivatives:

$$f_x = 6x - 2y, \quad f_y = -2x + 2y$$

Setting them to zero:

$$6x - 2y = 0 \;\; -2x + 2y = 0$$

Solving simultaneously:

- $y = 3x$
- Substitute $y = 3x$ in the first equation:
  $6x - 6x = 0$

Stationary point: $(x, y) = (0, 0)$.

It means that at this point, the slope of the function is zero in both directions.

- This is a candidate for a local maximum, minimum, or saddle point.

Second-Order Partial Derivatives and Hessian Matrix

To classify the stationary points, we use the second-order partial derivatives:

$$f_{xx}, f_{yy}, f_{xy}$$

The determinant of the Hessian matrix D is given by:

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

- If $D > 0$ and $f_{xx} > 0$: Local Min
- If $D > 0$ and $f_{xx} < 0$: Local Max
- If $D < 0$: Saddle Point
- If $D = 0$: Inconclusive

Note 1: The second-order conditions are necessary but not sufficient for a maximum or minimum.

- They help us classify the nature of the stationary point.
- If $D = 0$, we cannot conclude anything about the nature of the stationary point.
- We may need to use higher-order derivatives or other methods to classify the point.

Note 2: The second-order conditions are based on the assumption that the function is twice differentiable.

- If the function is not twice differentiable, we may need to use other methods to classify the point.

Second-Order Partial Derivatives and Hessian Matrix (cont'd)

Hessian Matrix

What is the Hessian Matrix?

- The Hessian matrix, denoted by H_x, is a square matrix of second-order partial derivatives of a scalar-valued function $f(x_1, x_2)$.
- It provides information about the **curvature** of the function and helps in determining the nature of stationary points.
- The determinant of the Hessian matrix is used to classify stationary points.

Definition:

$$H_x = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

Explanation:

- **Diagonal Elements:** Measure the curvature with respect to each variable independently.
- **Off-Diagonal Elements:** Measure how the function curvature changes when both variables interact.

Second-Order Partial Derivatives and Hessian Matrix (cont'd)

Hessian Matrix

Why is it Important?

- The Hessian matrix helps to determine the nature of stationary points:
    - Positive determinant and positive diagonal elements: **Local minimum**.
    - Positive determinant and negative diagonal elements: **Local maximum**.
    - Negative determinant: **Saddle point**.
    - Zero determinant: **Inconclusive test**.

Example: Classification

Continuing from the previous example:

Second-order derivatives:

$f_{xx} = 6, \quad f_{yy} = 2, \quad f_{xy} = -2$

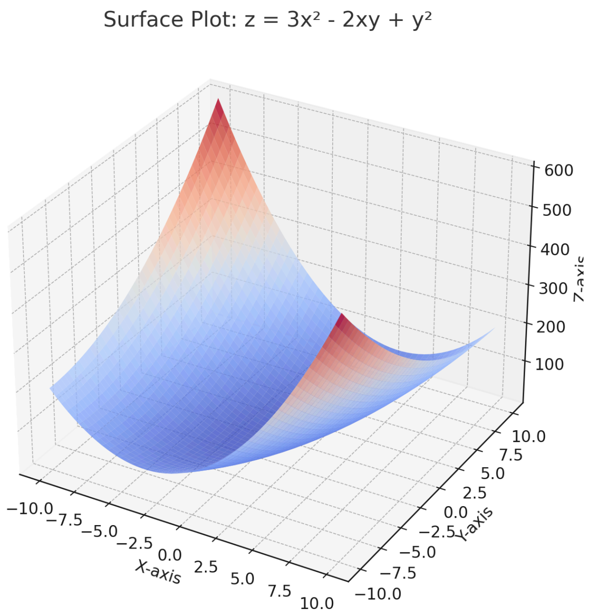
Determinant:

$$D = (6)(2) - (-2)^2 = 12 - 4 = 8$$

Since $D > 0$ and $f_{xx} > 0$, the point $(0, 0)$ is a **local minimum**. It means that the function has a minimum value at this point.

- The function is concave (curves inward like the interior of a circle or sphere) up in both directions at this point.
- The function is increasing in both directions away from this point.
- The function is "bowl-shaped" at this point.
- The function has a unique minimum value at this point.

Visualizing the Function



- The blue regions indicate lower values of z, while the red regions indicate higher values.
- The shape is not symmetric due to the interaction term -2xy, creating a slanted surface.
- The plot has a ridge or saddle point where the slope changes direction, indicating points of inflection or mixed concavity.

Economic Application (1): Profit Maximization

A firm produces two goods Q_1 and Q_2 with profit function:

$$\Pi(Q_1, Q_2) = 200Q_1 + 300Q_2 - 50Q_1^2 - 75Q_2^2 - 30Q_1Q_2$$

1. Determine the profit-maximizing quantities.

2. Classify the stationary points using the second-order derivatives.

Solution: Profit Maximization

1. First-order derivatives:

$$\Pi_{Q_1} = 200 - 100Q_1 - 30Q_2$$

$$\Pi_{Q_2} = 300 - 150Q_2 - 30Q_1$$

Setting to zero:

$$200 - 100Q_1 - 30Q_2 = 0\ 300 - 150Q_2 - 30Q_1 = 0$$

Solving simultaneously (corrected answer):

- $Q_1 \approx 1.49$
- $Q_2 \approx 1.70$

Solution: Profit Maximization (cont'd)

1. Second-order derivatives:

$$\Pi_{Q_1Q_1} = -100, \quad \Pi_{Q_2Q_2} = -150, \quad \Pi_{Q_1Q_2} = -30$$

Hessian Determinant:

$$D = (-100)(-150) - (-30)^2 = 15000 - 900 = 14100$$

Since $D > 0$ and $\Pi_{Q_1Q_1} < 0$, the point $(1.49, 1.70)$ is a **local maximum**.

It means that the firm maximizes its profit at this point.

2. Group Activity

Your turn: Profit Maximization (Group A)

A firm produces two goods Q_1 and Q_2 with the following profit function:

$$\Pi(Q_1, Q_2) = 150Q_1 + 250Q_2 - 40Q_1^2 - 60Q_2^2 - 20Q_1Q_2$$

Tasks:

1. **Determine the profit-maximizing quantities.**
2. **Classify the stationary points** using the second-order derivatives.
3. **Interpret the economic meaning** of the results.

Your turn: Cost Minimization (Group B)

A firm produces two goods using labor (L) and capital (K) with the following cost function:

$$C = 40L + 30K$$

The objective is to **minimize the cost** function without any production constraint.

Give economic interpretation of the results.

Your turn: Risk Management: Optimizing Investment Portfolios (Group C)

A firm is investing in two assets, A and B, with the following return function:

$$R = 0.08x + 0.12y - 0.5(0.06x^2 + 0.09y^2) - 0.04xy$$

where:

- x = Investment in Asset A
- y = Investment in Asset B

1. **Determine the optimal allocation** between assets A and B to maximize returns.
2. **Verify the optimal solution** using the second-order conditions.
3. **Interpret the economic meaning** of the results.

Your turn: Price Discrimination: Maximizing Profit in Multiple Markets (Group D)

A monopolist sells a product in two markets, A and B. The revenue functions for each market are:

$$R_A = 200Q_A - 5Q_A^2$$

$$R_B = 300Q_B - 10Q_B^2$$

The cost function for the firm is given by:

$$C = 50 + 20(Q_A + Q_B)$$

1. Determine the optimal quantities to sell in each market to maximize profit.
2. Verify the optimal solution using the second-order conditions.
3. Interpret the economic meaning of the results in the context of price discrimination.

Summary

- **Objective:** Maximize or minimize a function without constraints.
- **First-Order Derivative:** Used to find stationary points (set the derivative to zero).
- **Second-Order Derivative:** Determines the nature of the stationary points:
  - $f''(x) > 0$: Local minimum
  - $f''(x) < 0$: Local maximum
- **Applications in Economics:**
  - Cost minimization (input allocation)
  - Profit maximization (optimal production)
  - Risk management (investment allocation)
  - Price discrimination (market segmentation)
- **Key Takeaways:**
  - Identify objective functions and derive first and second derivatives.
  - Analyze the Hessian matrix for functions of two variables.
  - Apply economic interpretations to the mathematical solutions.

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (May 14) Constrained Optimization (5.5)