Mathematical Methods for International Commerce

Week 7/1: Optimization of Economic Functions (4.6, 4.7)

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Agenda

1. Optimization of Economic Functions (4.6, 4.7)

2. Group Activity: Optimize the Business!

Learning Objectives

Section 4.6 – Optimization of Economic Functions

- Use **first-order derivatives** to find stationary points
- Use **second-order derivatives** to classify max/min
- Apply calculus to **maximize or minimize economic functions**
- Sketch graphs using critical points

Section 4.7 – Further Optimization Concepts

- At max profit: **MR = MC**
- At max profit: **slope of MR < slope of MC**
- Optimize profits **with/without price discrimination**
- Show **APL = MPL** at max APL
- Derive the **EOQ** formula in inventory management

1. Optimization of Economic Functions (4.6, 4.7)

Optimization: Economic Motivation

In economics, we often want to:

- Maximize **profit**, **revenue**, or **utility**
- Minimize **costs**, **waste**, or **risk**

These require:

- Finding **stationary points** of a function
- Classifying them using **second-order derivatives**

Note: Stationary points are where the first derivative is zero or undefined.

- They indicate where a function **changes direction** (max/min)

Step-by-Step: Find Stationary Points

1. First Derivative Test

Let:

$$f(x) = -2x^2 + 40x - 100$$

Find:

$$f'(x) = -4x + 40$$

Set derivative = 0:

$$-4x + 40 = 0 \Rightarrow x = 10$$

2. Second Derivative Test

$$f''(x) = -4 < 0$$

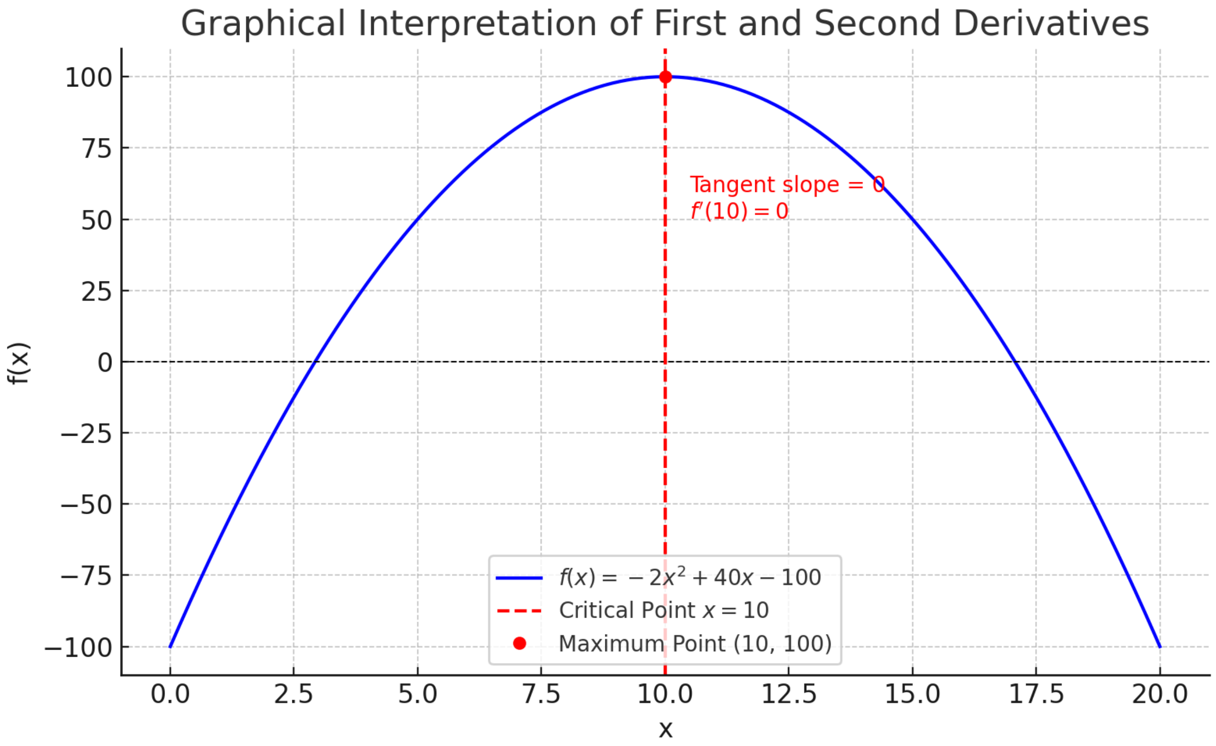
→ Maximum point (because it's negative)

- If $f''(x) > 0$, it's a minimum point

Maximum profit at $x = 10$, $f(10) = -2(100) + 400 - 100 = 200$

Graphical Interpretation

- First derivative = slope of tangent
- Second derivative = curvature
- Stationary points: where slope = 0
- Max if concave down, min if concave up



MR = MC: Condition for Max Profit

If:

$$\Pi(x) = R(x) - C(x)$$

Then:

$$\Pi'(x) = R'(x) - C'(x) = 0$$

$\rightarrow$ So:

$$MR = MC$$

Also:

$$\text{If } MR' < MC' \Rightarrow \text{Local Max}$$

Example: Profit Maximization

Let:

$$R(x) = 60x - x^2, \quad C(x) = 10x + 20$$

Profit:

$$\Pi(x) = R(x) - C(x) = 60x - x^2 - 10x - 20 = -x^2 + 50x - 20$$

1. First derivative:

$$\Pi'(x) = -2x + 50$$

→ Set to 0 → $x = 25$

2. Second derivative:

$$\Pi''(x) = -2 < 0$$

→ Max point

Max profit at $x = 25$, $\Pi(25) = -625 + 1250 - 20 = 605$

APL = MPL: Max Avg Productivity

Let:

$$Q = 20L - L^2$$

- APL = $\frac{Q}{L} = 20 - L$
- MPL = $\frac{dQ}{dL} = 20 - 2L$

Set APL = MPL:

$$20 - L = 20 - 2L \Rightarrow L = 0$$

$\rightarrow$ Verify using derivative of APL

Note: APL stands for **Average Product of Labor** and MPL for **Marginal Product of Labor**

- APL = MPL at $L = 0 \rightarrow$ Max APL

EOQ Formula (Economic Order Quantity)

- Economic Order Quantity (EOQ) is a key concept in inventory management.
- It helps determine the optimal order quantity that minimizes total inventory costs, which include ordering and holding costs.

Let:

- D = annual demand
- C_o = ordering cost per order
- C_h = holding cost per unit/year
- Q = order quantity

Total Cost:

$$TC(Q) = \frac{D}{Q}C_o + \frac{Q}{2}C_h$$

Minimize $TC(Q)$ by:

$$\frac{dTC}{dQ} = -\frac{DC_o}{Q^2} + \frac{C_h}{2} = 0$$

EOQ Formula (Economic Order Quantity) (cont'd)

Solve:

$$Q^* = \sqrt{\frac{2DC_o}{C_h}}$$

This gives optimal order size

EOQ Formula (Economic Order Quantity) (cont'd)

Example:

A company sells 1,000 units of a product per year.
Each order costs \$50 to place.
The holding cost per unit is \$2 per year.

$$EOQ = \sqrt{\frac{2 \cdot 1000 \cdot 50}{2}} = \sqrt{50000} \approx 223.6$$

Optimal order size = 224 units (rounded)

Interpretation

- The company should order **224 units each time** to minimize total inventory costs.
- This balances the **ordering cost** and the **holding cost** efficiently.

Practice Problems

1. Maximize $f(x) = -x^2 + 12x - 15$ using first and second derivatives.
2. For $R(x) = 30x - 0.5x^2$, $C(x) = 5x + 10$, find profit-maximizing output.
3. If $Q = 100L - 2L^2$, find when APL = MPL.
4. Derive EOQ for $D = 1000$, $C_o = 50$, $C_h = 2$

Summary

- Use first and second derivatives to **optimize** economic functions
- **MR = MC** is critical for profit max
- Derivatives help identify where functions **turn** and **peak**
- APL = MPL and EOQ have clear, testable formulas

> In economics, **optimization** helps us allocate resources most efficiently

2. Group Activity: Optimize the Business!

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Setup:

- Form **4 teams** of **4 students** each.
- Each team represents a **small business** deciding **how many units** of a product to produce.

Your Business Setup:

You sell eco-friendly water bottles. Your revenue and cost functions are:

- $R(x) = 80x - x^2$ (Revenue)
- $C(x) = 20x + 100$ (Cost)

Your Task:

1. Derive the **profit function**: $\Pi(x) = R(x) - C(x)$
2. Use **first-order** and **second-order derivatives** to find:
   - The **profit-maximizing output**
   - The **maximum profit**
3. Sketch a **graph** of $R(x), C(x), \Pi(x)$
4. Present your findings in **3–4 minutes**

Rules:

- You can use your notes and calculators.
- One person from each group presents.
- Bonus points for the most creative graph sketch!

Any QUESTIONS?

Thank you for your attention!

Next Class

- (April 18) Easter Holiday (Recorded lecture): The Derivative of the Exponential and Natural Log Functions (4.8)
- (April 23) No Class (Midterm Exam Week)
- (April 25) Mid term exam (in class):
  - Review all material from the beginning of the semester
  - Pay attention to the examples in the slides, HW #1, Quiz #1 and the exercises in the textbook