Mathematical Methods for International Commerce

Week 14/1-14/2: Cramer's Rule (7.3)

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Agenda

1. Cramer’s Rule (7.3)
2. Group activity: Solving matrixes
3. Homework #2

1. Cramer's Rule (7.3)

Why It Matters in Economics, Business & Finance

- Many real-world economic systems are governed by **simultaneous equations**.
- Think of equilibrium conditions in macroeconomic models, or supply-demand analysis in trade.
- **Cramer's Rule** is a neat algebraic technique for solving linear systems using **determinants**, without explicitly computing the inverse matrix.

You'll learn to:

- See when and why Cramer's Rule is useful
- Use it to solve 2×2 and 3×3 systems
- Apply it to real examples in **macroeconomics** and **international trade**

Recap: Linear System as Matrix Equation

Given a system:

$$\begin{aligned} a_{11}x + a_{12}y &= b_1 \\ a_{21}x + a_{22}y &= b_2 \end{aligned}$$

It can be written as:

$$Ax = b, \quad \text{where} \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad x = \begin{bmatrix} x \\ y \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Usually, we solve this using the inverse:

$$x = A^{-1}b$$

> But when A^{-1} is hard to compute, **Cramer's Rule** offers a shortcut.

Cramer's Rule: The Concept

Let A be a square matrix with non-zero determinant. Then the solution x_i to the system $Ax = b$ is:

$$x_i = \frac{\det(A_i)}{\det(A)}$$

Where:

- A_i: the matrix formed by replacing the *i-th column* of A with the vector b
- $\det(A) \neq 0$: system has a unique solution

Example 1: Solve Using Cramer's Rule

Given:

$$
\begin{aligned}
4x + 3y &= 10 \\
2x + y &= 4
\end{aligned}
$$

Step 1: Coefficient matrix:

$$
A = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 10 \\ 4 \end{bmatrix}
$$

Step 2: Determinant of A:

$$
\det(A) = (4)(1) - (3)(2) = 4 - 6 = -2
$$

Example 1: Solve Using Cramer's Rule (cont)

Step 3: Determinants with substituted columns:

$$A_1 = \begin{bmatrix} 10 & 3 \\ 4 & 1 \end{bmatrix}, \quad \det(A_1) = 10 \cdot 1 - 3 \cdot 4 = 10 - 12 = -2$$

$$A_2 = \begin{bmatrix} 4 & 10 \\ 2 & 4 \end{bmatrix}, \quad \det(A_2) = 4 \cdot 4 - 10 \cdot 2 = 16 - 20 = -4$$

Final Solution:

$$x = \frac{-2}{-2} = 1, \quad y = \frac{-4}{-2} = 2$$

Example 2: Solve Using Cramer's Rule

Solve the system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ -4x_1 + x_2 + 6x_3 &= -9 \\ 2x_1 + 7x_2 + 5x_3 &= 13 \end{aligned}$$

Find x_1 using **Cramer's Rule**.

Step 1: Identify Matrices

Coefficient matrix A:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 1 & 6 \\ 2 & 7 & 5 \end{bmatrix}$$

Right-hand side vector b:

$$b = \begin{bmatrix} 9 \\ -9 \\ 13 \end{bmatrix}$$

Replace the **first column** of A with b to form A_1:

$$A_1 = \begin{bmatrix} 9 & 2 & 3 \\ -9 & 1 & 6 \\ 13 & 7 & 5 \end{bmatrix}$$

Step 2: Compute Determinants

Expand $\det(A_1)$ along the top row:

$$\det(A_1) = 9 \cdot \begin{vmatrix} 1 & 6 \\ 7 & 5 \end{vmatrix} - 2 \cdot \begin{vmatrix} -9 & 6 \\ 13 & 5 \end{vmatrix} + 3 \cdot \begin{vmatrix} -9 & 1 \\ 13 & 7 \end{vmatrix}$$

$$= 9(-37) - 2(-123) + 3(-76) = -333 + 246 - 228 = -315$$

Step 3: Compute $\det(A)$

$$\det(A) = 1 \cdot \begin{vmatrix} 1 & 6 \\ 7 & 5 \end{vmatrix} - 2 \cdot \begin{vmatrix} -4 & 6 \\ 2 & 5 \end{vmatrix} + 3 \cdot \begin{vmatrix} -4 & 1 \\ 2 & 7 \end{vmatrix}$$

$$= 1(-37) - 2(-32) + 3(-30) = -37 + 64 - 90 = -63$$

Step 4: Apply Cramer's Rule

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{-315}{-63} = 5$$

- Use same method to find x_2, x_3 by replacing 2nd or 3rd column of A.

Real-World Example: Static Macroeconomic Model

Consider a simple Keynesian model:

$$Y = C + I + G$$
$$C = 100 + 0.8Y$$

Solve for Y and C when $I = 50$ and $G = 20$

Step 1: Substitute:

$$Y = (100 + 0.8Y) + 50 + 20 \Rightarrow Y = 170 + 0.8Y$$

$$Y - 0.8Y = 170 \Rightarrow 0.2Y = 170 \Rightarrow Y = 850$$

$$C = 100 + 0.8 \cdot 850 = 780$$

Expressing this in matrix form:

$$A = \begin{bmatrix} 1 & -1 \\ -0.8 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} Y \\ C \end{bmatrix}, \quad b = \begin{bmatrix} 70 \\ 100 \end{bmatrix}$$

Cramer’s Rule Application

Apply Cramer’s Rule to confirm.

Example 2: Two-Country Trade Model

Country A and B produce goods X and Y.

- 2X + 3Y = 20 (A's resource constraint)
- 3X + Y = 18 (B's resource constraint)

Solve for quantities of X and Y that satisfy both.

Matrix Form:

$$A = \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 20 \\ 18 \end{bmatrix}$$

Compute determinants:

$$\det(A) = 2 \cdot 1 - 3 \cdot 3 = 2 - 9 = -7$$

$$A_1 = \begin{bmatrix} 20 & 3 \\ 18 & 1 \end{bmatrix}, \quad \det(A_1) = 20 \cdot 1 - 3 \cdot 18 = 20 - 54 = -34$$

Example 2: Two-Country Trade Model (cont)

$$A_2 = \begin{bmatrix} 2 & 20 \\ 3 & 18 \end{bmatrix}, \quad \det(A_2) = 2 \cdot 18 - 20 \cdot 3 = 36 - 60 = -24$$

Solution:

$$X = \frac{-34}{-7} \approx 4.86, \quad Y = \frac{-24}{-7} \approx 3.43$$

Practice Problems: Solve using Cramer's Rule

- Problem 1:

$$
\begin{aligned}
3x + 4y &= 10 \\
2x + y &= 5
\end{aligned}
$$

- Problem 2:

$$
\begin{aligned}
2x + 3y - z &= 5 \\
4x - y + 2z &= 6 \\
-3x + 2y + z &= -4
\end{aligned}
$$

- Problem 3: In a 3-sector macro model:

$$
\begin{aligned}
Y &= C + I + G \\
C &= a + bY
\end{aligned}
$$

  Set $a = 120, b = 0.75, I = 50, G = 30$. Solve for Y and C.

- Problem 4: Trade model:

$$
\begin{aligned}
4X + 5Y &= 40 \\
6X + 2Y &= 38
\end{aligned}
$$

  Use Cramer's Rule to find X, Y.

Summary

- Cramer's Rule offers a quick solution for small systems using determinants
- Works only if $\det(A) \neq 0$
- Particularly useful for economic models involving linear constraints
- Check numerical stability: not suitable for large or nearly singular matrices

Limitations: Only practical for small systems due to cost of computing determinants.

2. Group activity: Solving matrixes

3. Home work #2

Homework #2

- **Due Date:** June 13, 2025, before the start of class.
- **Submission Format:** Submit your solutions as a single PDF file via the Cyber Campus.

- **Instructions:**

    - Clearly show all steps and calculations.
    - Include explanations for your answers where applicable.
    - Ensure your submission is neat and well-organized.
    - Bring any questions to the office hours or email me.

- Work on your Home Assignment #2 (Jacques, 10th edition, Chapters 5-9):

    - Chapter 5.1: Exercise 5.1, Problem 5 (p. 411)
    - Chapter 5.2: Exercise 5.2, Problem 7 (p. 427)
    - Chapter 5.3: Exercise 5.3, Practice Problem inside the chapter (p. 432 only)
    - Chapter 5.4: Exercise 5.4, Problem 5 (p. 457)
    - Chapter 5.5: Exercise 5.5, Problems 6 and 7 (p. 467)
    - Chapter 5.6: Exercise 5.6, Problem 4 (p. 479)
    - Chapter 6.1: Exercise 6.1, Problem 3 (p. 506)
    - Chapter 6.2: Exercise 6.2, Problem 6 (p. 521)
    - Chapter 7.1: Exercise 7.1, Problem 5 (p. 553)
    - Chapter 7.2: Exercise 7.2, Problem 6 (p. 572)
    - Chapter 7.3: Exercise 7.3, Problem 5 (p. 584)
    - Chapter 8.1: Exercise 8.1, Problem 6 (p. 615)
    - Chapter 8.2: Exercise 8.2, Problem 3 (p. 626)
    - Chapter 9.1: Exercise 9.1, Problem 4 (p. 655)
    - Chapter 9.2: Exercise 9.2, Problem 3 (p. 670)

Good luck!

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (June 6: Recorded lecture) Linear Programming (8.1)