Mathematical Methods for International Commerce

Week 13/2: Basic Matrix Operation (7.1)
Matrix Inversion (7.2)

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Agenda

1. Basic Matrix Operation (7.1)

2. Matrix Inversion (7.2)

3. Homework #2

Why Matrices Matter in Economics

- Matrix algebra allows economists to:
  - Solve systems of equations
  - Model economic input-output relationships
  - Optimize production & costs
  - Forecast using linear models
- Applications include:
  - Input-output models
  - Markov chains
  - Linear regression models
  - Economic equilibrium analysis

1. Basic Matrix Operation (7.1)

Example: Sales Table

Suppose that a firm produces three types of goods (G1, G2, G3) and sells them to two customers (C1 and C2). The matrix:

$$A = \begin{array}{cc} & \begin{array}{ccc} \mathrm{G1} & \mathrm{G2} & \mathrm{G3} \end{array} \\ \begin{array}{c} \mathrm{C1} \\ \mathrm{C2} \end{array} & \begin{bmatrix} 7 & 3 & 4 \\ 1 & 5 & 6 \end{bmatrix} \end{array}$$

represents monthly sales:

- Row 1: customer C1
- Row 2: customer C2
- Columns: goods G1, G2, G3

This format allows for compact storage and easy operations like summing totals or multiplying by price vectors.

Basic Matrix Terminology

- A matrix is a rectangular array of numbers:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

- **Order**: number of rows $\times$ number of columns (dimensions)
- **Row vector**: 1 row, n columns
- **Column vector**: n rows, 1 column
- **Element**: a_{ij} is the element in row i, column j

- **Square matrix**: same number of rows and columns (e.g., 2×2)

- **Zero matrix**: all elements are zero

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

General Matrix Notation

A general matrix D of order 3×2:

$$D = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \\ d_{31} & d_{32} \end{bmatrix}$$

A general matrix E of order 3×3:

$$E = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$$

Basic Matrix Operations

Transpose of a Matrix

- **Transpose:** A^T flips rows and columns

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

- Rows become columns
- Used frequently in optimization and econometrics

Basic Matrix Operations (continued)

Matrix Addition & Subtraction

Two matrices of the same order:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}, \quad A - B = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}$$

Basic Matrix Operations (continued)

Scalar Multiplication

Multiply each element:

$$2A = 2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Basic Matrix Operations (continued)

Matrix Multiplication

- Only defined if inner dimensions match: $A_{m\times n} \cdot B_{n\times p}$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 \cdot 2 + 2 \cdot 1 & 1 \cdot 0 + 2 \cdot 5 \\ 3 \cdot 2 + 4 \cdot 1 & 3 \cdot 0 + 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} 4 & 10 \\ 10 & 20 \end{bmatrix}$$

- Dimensions of the result: $m \times p$

Basic Matrix Operations (continued)

Matrix Multiplication Advice

> Take the trouble to check before you begin that it is possible to form the matrix product and anticipate the order of the end result.

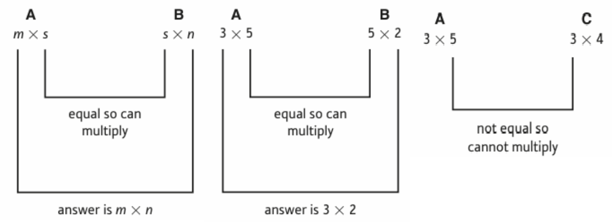
- Jot down the dimensions: inner numbers must match.
- The result's dimensions = outer numbers.

If:

$$A : 3 \times 5, \quad B : 5 \times 2, \quad C : 3 \times 4$$

Then:

- AB is possible → result is 3×2
- AC is **not** possible (inner numbers don't match)



Basic Matrix Operations (continued)

General Matrix Multiplication

If A is $m \times s$ and B is $s \times n$, then AB is $m \times n$. Element c_{ij} is:

$$c_{ij} = \mathrm{row}_i(A) \cdot \mathrm{col}_j(B)$$

Let:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 5 & 4 & 1 & 1 \end{bmatrix}$$

Check: A is 2×3, B is 3×4 ⇒ AB exists, size is 2×4

Basic Matrix Operations (continued)

Calculating Elements of AB

- $c_{11} = 2 \cdot 3 + 1 \cdot 1 + 0 \cdot 5 = 6 + 1 + 0 = 7$
- $c_{12} = 2 \cdot 1 + 1 \cdot 0 + 0 \cdot 4 = 2 + 0 + 0 = 2$
- $c_{13} = 2 \cdot 2 + 1 \cdot 1 + 0 \cdot 1 = 4 + 1 + 0 = 5$
- $c_{14} = 2 \cdot 1 + 1 \cdot 2 + 0 \cdot 1 = 2 + 2 + 0 = 4$

Continue for second row...

Full Product AB

$$AB = \begin{bmatrix} 7 & 2 & 5 & 4 \\ 23 & 17 & 6 & 5 \end{bmatrix}$$

Step-by-step matrix multiplication shows the power of matrix algebra in summarizing economic relationships.

Properties of Matrix Operations

- A matrix is a rectangular array of numbers, organized into rows and columns.
- The dimensions of a matrix are given as $m \times n$, where m is the number of rows and n is the number of columns.
- Each element in the matrix is indexed by its row and column position, denoted as a_{ij}.

Provided that the indicated sums and products make sense,

- $A + B = B + A$
- $A - A = 0$
- $A + 0 = A$
- $k(A + B) = kA + kB$
- $k(lA) = (kl)A$
- $A(B + C) = AB + AC$
- $(A + B)C = AC + BC$
- $A(BC) = (AB)C$

We also have the **non-property**:

- $AB \neq BA$

Matrix Representation of Systems

System of equations:

$$\begin{cases} 2x + 3y = 8 \\ 4x - y = 2 \end{cases} \quad \Rightarrow \quad AX = B$$

Where:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad B = \begin{bmatrix} 8 \\ 2 \end{bmatrix}$$

Identity Matrix

- Identity matrix I acts like 1 in multiplication:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad AI = IA = A$$

2. Matrix Inversion (7.2)

Matrix Inversion (2x2 Case)

Given:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

- The matrix A^{-1} is called the inverse of A, and it plays a role similar to the reciprocal of a number in arithmetic.
- Although the formula for A^{-1} may appear complex, its construction for a 2×2 matrix is straightforward and systematic.

If $\det(A) = ad - bc \neq 0$, the inverse is:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Note: $\det(A)$ is the determinant of A.

Nota bene

For any nonzero number x, its reciprocal is $1/x$.

- The reciprocal of 5 is 1/5.
- The reciprocal of 1/3 is 3 (because $1/3 \times 3 = 1$).
- Multiplying a number by its reciprocal always gives 1.

Solving Equations Using Inverses

From $AX = B$, multiply both sides by A^{-1}:

$$X = A^{-1}B$$

- This allows us to find the solution vector X directly.
- If A is invertible, we can solve systems of equations efficiently.

Example: Solving for Equilibrium Prices

We are given a system of equations:

$$-4P_1 + P_2 = -13$$
$$2P_1 - 5P_2 = -7$$

Express this system in matrix form and hence find the values of P1 and P2.

Step 1: Express in Matrix Form

Write the system as:

$$\begin{bmatrix} -4 & 1 \\ 2 & -5 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

Let:

$$A = \begin{bmatrix} -4 & 1 \\ 2 & -5 \end{bmatrix}$$

$$x = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$$

$$b = \begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

So the system becomes

$$Ax = b$$

Step 2: Find the Determinant of A

$$\det(A) = (-4)(-5) - (1)(2) = 20 - 2 = 18$$

Since

$$\det(A) \neq 0$$

, the matrix is invertible.

Step 3: Find the Inverse of A

Using the formula for a 2x2 inverse:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix} = \frac{1}{18} \begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}$$

Step 4: Solve for x

Multiply

$$A^{-1}$$

by

$$b$$

:

$$x = A^{-1}b = \frac{1}{18}\begin{bmatrix} -5 & -1 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} -13 \\ -7 \end{bmatrix}$$

Step 5: Matrix Multiplication

Compute:

$$P_1 = \frac{1}{18}((-5)(-13) + (-1)(-7)) = \frac{1}{18}(65 + 7) = \frac{72}{18} = 4$$
$$P_2 = \frac{1}{18}((-2)(-13) + (-4)(-7)) = \frac{1}{18}(26 + 28) = \frac{54}{18} = 3$$

Final Answer

$$P_1 = 4$$

$$P_2 = 3$$

These are the equilibrium prices.

3x3 Matrices, Determinants, and Cofactors

The concepts of determinant, inverse and identity matrices apply to 3x3 matrices as well.

The identity matrix:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Check that for any matrix A:

$$AI = A, \quad IA = A$$

To compute an inverse, we need **cofactors**:

- The cofactor A_{ij} is the determinant of the 2x2 matrix formed by removing row i and column j, multiplied by $(-1)^{i+j}$

3x3 Matrices, Determinants, and Cofactors (cont)

Example: For matrix A, to find A_{23}, delete row 2 and column 3 to get:

$$\text{Minor of } A_{23} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{bmatrix}$$

Cofactor:

$$A_{23} = (-1)^{2+3} \cdot (a_{11}a_{32} - a_{12}a_{31}) = -a_{11}a_{32} + a_{12}a_{31}$$

This sign pattern follows the "checkerboard" rule:

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

3x3 Matrices, Determinants, and Cofactors (cont)

3x3 Determinants and Inverses

We are now in a position to describe how to calculate the determinant and inverse of a 3x3 matrix.

To compute $\det(A)$:

- Multiply elements in any row/column by their cofactors
- The sum gives the determinant
- Same result regardless of row/column used — useful for checking!

If expanding along the first row:

$$\det(A) = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

Or down the second column:

$$\det(A) = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$$

This flexibility allows for easier calculations in practice.

3x3 Matrices, Determinants, and Cofactors (cont)

3x3 Inverse Matrix Structure

The inverse of the 3×3 matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is computed using cofactors and determinant.

1. Form the matrix of cofactors:

$$\operatorname{cof}(A) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

2. Take its transpose (adjoint):

$$\operatorname{adj}(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

3x3 Matrices, Determinants, and Cofactors (cont)

3x3 Inverse Matrix Structure (cont)

1. Multiply by $\frac{1}{\det(A)}$:

$$A^{-1} = \frac{1}{\det(A)} \cdot \operatorname{adj}(A)$$

❗ If $\det(A) = 0$, the matrix is **singular** and the inverse **does not exist**.

Advice: Check your result by confirming that:

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I$$

3x3 Matrices, Determinants, and Cofactors (cont)

Example: Inverse of a 3x3 Matrix

Given:

$$A = \begin{bmatrix} 2 & 4 & 1 \\ 4 & 3 & 7 \\ 2 & 1 & 3 \end{bmatrix}$$

Previously computed cofactors:

$$\begin{gathered} A_{11} = 2, \quad A_{12} = -11, \quad A_{13} = 25 \\ A_{21} = 2, \quad A_{22} = 4, \quad A_{23} = 6 \\ A_{31} = -2, \quad A_{32} = -10, \quad A_{33} = -10 \end{gathered}$$

$$\text{adjugate}(A) = \begin{bmatrix} 2 & 2 & -2 \\ -11 & 4 & -10 \\ 25 & 6 & -10 \end{bmatrix}, \quad \text{adjoint}(A) = \begin{bmatrix} 2 & 2 & -2 \\ -11 & 4 & -10 \\ 25 & 6 & -10 \end{bmatrix}^T = \begin{bmatrix} 2 & -11 & 25 \\ 2 & 4 & 6 \\ -2 & -10 & -10 \end{bmatrix}$$

3x3 Matrices, Determinants, and Cofactors (cont)

Example: Inverse of a 3x3 Matrix (cont)

Given $\det(A) = 10$, we compute:

$$A^{-1} = \frac{1}{10} \cdot \begin{bmatrix} 2 & -11 & 25 \\ 2 & 4 & 6 \\ -2 & -10 & -10 \end{bmatrix}$$

Verification:

$$A^{-1}A = I, \quad AA^{-1} = I \quad \checkmark$$

Practice Problems

- Given the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

  - Find A^T (transpose).
  - Calculate $2A$ (scalar multiplication).
  - Add A to itself.

- Multiply:

$$\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

- Find the inverse of:

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

- Solve for x and y:

$$\begin{cases} 3x + 2y = 10 \\ 4x - y = 5 \end{cases}$$

Practice Problems (continued)

- We are given a system of equations:

$$9P_1 + P_2 = 43$$
$$2P_1 + 7P_2 = 57$$

Express this system in matrix form and hence find the values of P1 and P2.

- Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Summary

- A matrix is a way to organize data into rows and columns.
- Matrix operations help simplify and solve systems of economic equations
- Inversion is crucial for solving systems when direct substitution isn't feasible
- Input-output analysis is a powerful economic application
- Linear algebra is a foundation of modern data modeling and optimization

3. Home work #2

Homework #2

- **Due Date:** June 13, 2025, before the start of class.
- **Submission Format:** Submit your solutions as a single PDF file via the Cyber Campus.

- **Instructions:**
  - Clearly show all steps and calculations.
  - Include explanations for your answers where applicable.
  - Ensure your submission is neat and well-organized.
  - Bring any questions to the office hours or email me.

- Work on your Home Assignment #2 (Jacques, 10th edition, Chapters 5-9):
  - Chapter 5.1: Exercise 5.1, Problem 5 (p. 411)
  - Chapter 5.2: Exercise 5.2, Problem 7 (p. 427)
  - Chapter 5.3: Exercise 5.3, Practice Problem inside the chapter (p. 432 only)
  - Chapter 5.4: Exercise 5.4, Problem 5 (p. 457)
  - Chapter 5.5: Exercise 5.5, Problems 6 and 7 (p. 467)
  - Chapter 5.6: Exercise 5.6, Problem 4 (p. 479)
  - Chapter 6.1: Exercise 6.1, Problem 3 (p. 506)
  - Chapter 6.2: Exercise 6.2, Problem 6 (p. 521)
  - Chapter 7.1: Exercise 7.1, Problem 5 (p. 553)
  - Chapter 7.2: Exercise 7.2, Problem 6 (p. 572)
  - Chapter 7.3: Exercise 7.3, Problem 5 (p. 584)
  - Chapter 8.1: Exercise 8.1, Problem 6 (p. 615)
  - Chapter 8.2: Exercise 8.2, Problem 3 (p. 626)
  - Chapter 9.1: Exercise 9.1, Problem 4 (p. 655)
  - Chapter 9.2: Exercise 9.2, Problem 3 (p. 670)

Good luck!

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (June 4) Cramer's Rule (7.3)