Mathematical Methods for International Commerce

Week 11/2: Lagrange Multipliers (5.6)

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Why It Matters in Economics & Finance

- **Resource Allocation:** Optimal use of limited resources
- **Cost Minimization:** Firms aim to minimize costs given production constraints
- **Utility Maximization:** Consumers maximize utility given budget constraints

Learning Objectives

- Apply the Lagrange multiplier method to solve constrained optimization problems.
- Interpret the economic meaning of the Lagrange multiplier as the marginal value of the constraint.
- Solve optimization problems involving Cobb-Douglas production functions with cost constraints.
- Demonstrate that at the optimal point, the ratio of marginal product to price is equal for all inputs.

What Are Lagrange Multipliers?

- Lagrange multipliers provide a way to find the **maximum or minimum** of a function subject to a constraint.
- They are particularly useful in economics for problems involving **cost minimization** or **utility maximization**.

- The method involves introducing a new variable (the Lagrange multiplier) to incorporate the constraint into the optimization problem.

- For a function $f(x, y)$ subject to a constraint $g(x, y) = c$, we define the Lagrangian as:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda \cdot (c - g(x, y))$$

- The Lagrange multiplier λ can be interpreted as the **rate of change of the objective function** with respect to the constraint.

Lagrange Multipliers - Example

Problem Statement

- Optimize the objective function:

$$f(x, y) = x^2 - 3xy + 12x$$

- Subject to the constraint:

$$2x + 3y = 6$$

Step 1: Define the Lagrangian

- The Lagrangian function is given by:

$$g(x, y, \lambda) = x^2 - 3xy + 12x + \lambda(6 - 2x - 3y)$$

Step 2: Derive First-Order Conditions

- Compute the partial derivatives of g:

$$\frac{\partial g}{\partial x} = 2x - 3y + 12 - 2\lambda$$

$$\frac{\partial g}{\partial y} = -3x - 3\lambda$$

$$\frac{\partial g}{\partial \lambda} = 6 - 2x - 3y$$

Set each of these derivatives to zero to form the system of equations.

Step 3: Form the System of Equations

1. $2x - 3y - 2\lambda = -12$
2. $-3x - 3\lambda = 0$
3. $2x + 3y = 6$

Step 4: Eliminate Variables

- Multiply (1) by 3 and (2) by 2, then add:

$$-9y - 12\lambda = -36$$

$$-6y - 2\lambda = -18$$

- Eliminate y:

Multiply (4) by 6 and (5) by 9, then subtract:

$$-54\lambda = -54 \implies \lambda = 1$$

Step 5: Substitute to Find y and x

- Substitute $\lambda = 1$ in (5):

$$-6y - 2(1) = -18$$

$$-6y = -16 \implies y = \frac{8}{3}$$

- Substitute $y = \frac{8}{3}$ and $\lambda = 1$ in (1):

$$2x - 3\left(\frac{8}{3}\right) - 2(1) = -12$$

$$2x - 8 - 2 = -12 \implies x = -1$$

Step 6: Calculate Optimal Value

- Optimal solution:
  - $x = -1, y = \frac{8}{3}, \lambda = 1$
- Calculate the objective function at the optimal point:

$$f(-1, \frac{8}{3}) = (-1)^2 - 3(-1)\left(\frac{8}{3}\right) + 12(-1)$$

$$= 1 + 8 - 12 = -3$$

- Optimal value of the objective function: **-3**

Step 7: Economic Interpretation

- The Lagrange multiplier λ represents the **marginal value of the constraint**.
- In this case, it indicates how much the objective function would change if the constraint were relaxed by one unit.
- A positive λ suggests that the constraint is binding, meaning that the optimal solution is constrained by the budget.
- A negative λ indicates that the constraint is not binding, and the optimal solution is not affected by the constraint.
- The optimal solution (x, y) represents the best allocation of resources given the constraint.
- The Lagrange multiplier λ provides insight into the trade-offs involved in the optimization problem, helping to understand the sensitivity of the objective function to changes in the constraint.
- In this case, the optimal solution suggests that the firm should allocate resources in a way that maximizes the objective function while adhering to the constraint.

Lagrange Multipliers: Profit Maximization

A monopolistic producer of two goods G_1 and G_2 has the following cost function:

$$TC = 10Q_1 + Q_1Q_2 + 10Q_2$$

Demand equations:

$$P_1 = 50 - Q_1 + Q_2$$

$$P_2 = 30 + 2Q_1 - Q_2$$

Objective: Maximize profit given the constraint that the total output is 15 units.

Step 1: Formulate the Lagrangian

Objective function (Profit):

$$\pi = TR - TC$$

Total Revenue:

$$TR_1 = (50 - Q_1 + Q_2)Q_1 = 50Q_1 - Q_1^2 + Q_1Q_2$$

$$TR_2 = (30 + 2Q_1 - Q_2)Q_2 = 30Q_2 + 2Q_1Q_2 - Q_2^2$$

Total Revenue:

$$TR = 50Q_1 - Q_1^2 + 3Q_1Q_2 + 30Q_2 - Q_2^2$$

Profit function:

$$\pi = 50Q_1 - Q_1^2 + 3Q_1Q_2 + 30Q_2 - Q_2^2 - (10Q_1 + Q_1Q_2 + 10Q_2)$$

Simplify:

$$\pi = 40Q_1 - Q_1^2 + 2Q_1Q_2 + 20Q_2 - Q_2^2$$

Step 2: First-Order Conditions

1. Derivative with respect to Q_1:

$$\frac{\partial L}{\partial Q_1} = 40 - 2Q_1 + 2Q_2 - \lambda = 0$$

1. Derivative with respect to Q_2:

$$\frac{\partial L}{\partial Q_2} = 20 + 2Q_1 - 2Q_2 - \lambda = 0$$

1. Derivative with respect to λ:

$$15 - Q_1 - Q_2 = 0$$

Step 3: Solving the System

Rewriting the equations:

1. $40 - 2Q_1 + 2Q_2 - \lambda = 0$
2. $20 + 2Q_1 - 2Q_2 - \lambda = 0$
3. $Q_1 + Q_2 = 15$

Adding the first two equations to eliminate λ:

$$60 - 4Q_1 = 0 \implies Q_1 = 10$$

Substituting in the constraint:

$$10 + Q_2 = 15 \implies Q_2 = 5$$

Optimal output levels: $Q_1 = 10$, $Q_2 = 5$

Step 4: Profit Calculation

Substitute $Q_1 = 10$, $Q_2 = 5$ in the profit function:

$$\pi = 40(10) - (10)^2 + 2(10)(5) + 20(5) - (5)^2$$

$$\pi = 400 - 100 + 100 + 100 - 25 = 475$$

Economic Interpretation

- The optimal profit is **475 units**.
- The Lagrange multiplier $\lambda = 30$ represents the **marginal increase in profit** for each additional unit of the constraint (production quota).
- If the quota increases by 1 unit, profit increases by approximately 30 units.

Your turn

Practice Problem 1: Maximizing $ 2x^2 - xy $

Use Lagrange multipliers to optimise

$$2x^2 - xy$$

subject to

$$x + y = 12$$

Give economic interpretation of the solution.

Practice Problem 2: Utility Maximization

A consumer's utility function is given by

$$U(x_1, x_2) = 2x_1x_2 + 3x_1$$

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where x_1 and x_2 denote the number of items of two goods G_1 and G_2 that are bought. Each item costs \$1 for G_1 and \$2 for G_2. Use Lagrange multipliers to find the maximum value of U if the consumer's income is \$83. Estimate the new optimal utility if the consumer's income rises by \$1.

2. Home work #2

Homework #2

- **Due Date:** June 13, 2025, before the start of class.
- **Submission Format:** Submit your solutions as a single PDF file via the Cyber Campus.

- **Instructions:**

  - Clearly show all steps and calculations.
  - Include explanations for your answers where applicable.
  - Ensure your submission is neat and well-organized.
  - Bring any questions to the office hours or email me.

- Work on your Home Assignment #2 (Jacques, 10th edition, Chapters 5-9):

  - Chapter 5.1: Exercise 5.1, Problem 5 (p. 411)
  - Chapter 5.2: Exercise 5.2, Problem 7 (p. 427)
  - Chapter 5.3: Exercise 5.3, Practice Problem inside the chapter (p. 432 only)
  - Chapter 5.4: Exercise 5.4, Problem 5 (p. 457)
  - Chapter 5.5: Exercise 5.5, Problems 6 and 7 (p. 467)
  - Chapter 5.6: Exercise 5.6, Problem 4 (p. 479)
  - Chapter 6.1: Exercise 6.1, Problem 3 (p. 506)
  - Chapter 6.2: Exercise 6.2, Problem 6 (p. 521)
  - Chapter 7.1: Exercise 7.1, Problem 5 (p. 553)
  - Chapter 7.2: Exercise 7.2, Problem 6 (p. 572)
  - Chapter 7.3: Exercise 7.3, Problem 5 (p. 584)
  - Chapter 8.1: Exercise 8.1, Problem 6 (p. 615)
  - Chapter 8.2: Exercise 8.2, Problem 3 (p. 626)
  - Chapter 9.1: Exercise 9.1, Problem 4 (p. 655)
  - Chapter 9.2: Exercise 9.2, Problem 3 (p. 670)

Good luck!

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (May 16) Lagrange Multipliers (5.6)
  - Home Work #2 announcement