Mathematical Methods for International Commerce

Week 9/1: Functions of Several Variables (5.1)

Iegor Vyshnevskyi, Ph.D.

Sogang University

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Agenda

1. Functions of Several Variables (5.1)

2. Class Activity

1. Functions of Several Variables (5.1)

Learning Objectives

- Use function notation $z = f(x, y)$
- Calculate **first-order partial derivatives**
- Calculate **second-order partial derivatives**
- Understand that $f_{xy} = f_{yx}$ (usually)
- Use the **small increments formula**
- Perform **implicit differentiation**

What Are Multivariable Functions?

A function with **two or more independent variables**.

Example:

Let $z = f(x, y) = 4x^2 + 3xy + y^2$

- z depends on both x and y
- This could model something like **profit**, **cost**, or **utility** depending on two goods or inputs

Multivariable Functions Graphically

A function of two variables can be visualized as a **surface** in 3D space.

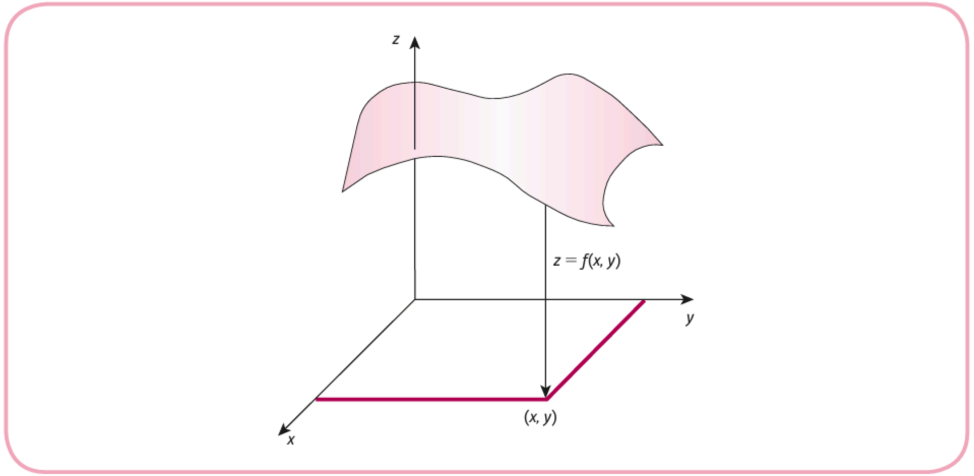


Figure 5.3

This graph shows how a function $z = f(x, y)$ maps a pair (x, y) into a surface value z.

It represents the output (e.g., cost, utility, profit) depending on two input variables.

The surface curves depending on how each variable changes, giving rise to partial derivatives.

First-Order Partial Derivatives

Partial derivative " means:

- **Partial**: only one variable is changing, the other is held constant
- **Derivative**: the rate of change of the function with respect to that variable
- **Take the derivative**: treat the other variable as a constant

Take the derivative **with respect to one variable**, holding the other constant.

Example:

Let $f(x, y) = 4x^2 + 3xy + y^2$

- $f_x = \frac{\partial f}{\partial x} = 8x + 3y$
- $f_y = \frac{\partial f}{\partial y} = 3x + 2y$

> **Interpretation**: how z changes when we slightly change x (or y) keeping the other constant.

Second-Order Partial Derivatives

Take the partial derivative **again**.

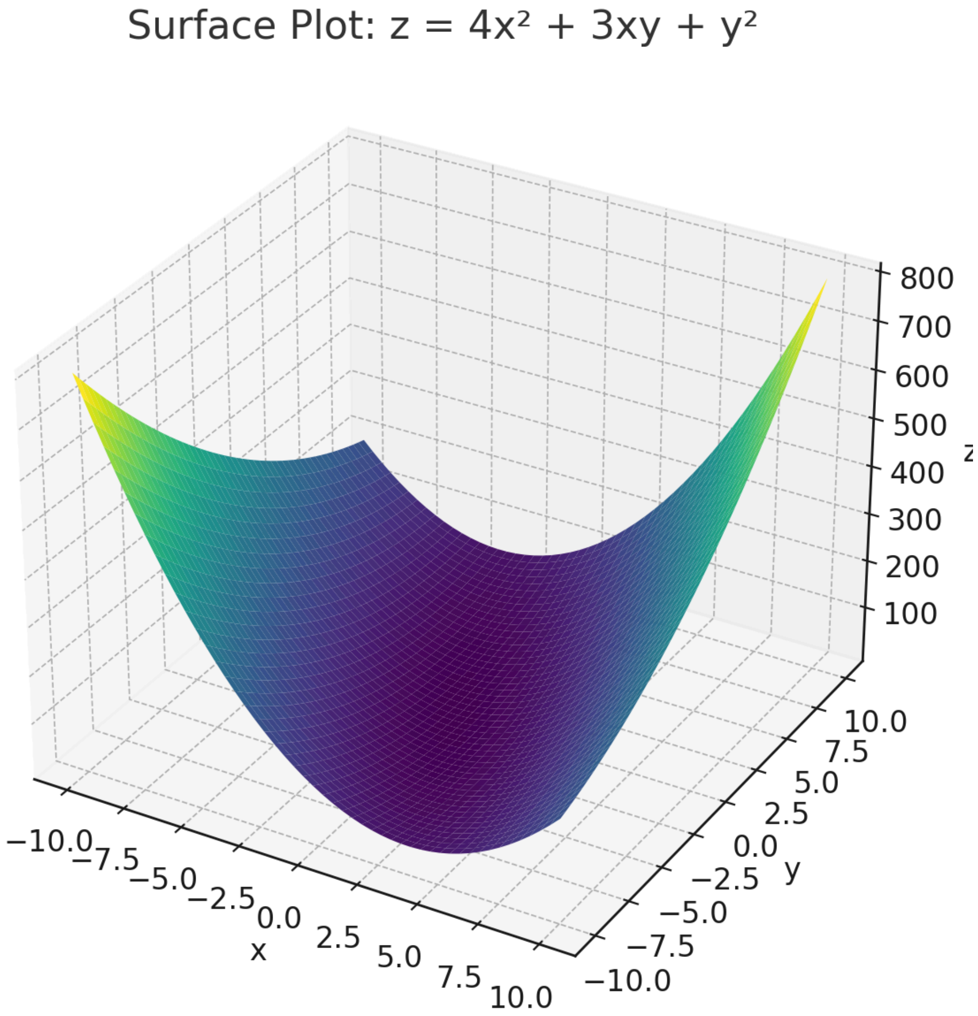
Using $f(x, y) = 4x^2 + 3xy + y^2$:

- $f_{xx} = \frac{\partial^2 f}{\partial x^2} = 8$
- $f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2$
- $f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 3$
- $f_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 3$

> $f_{xy} = f_{yx} \rightarrow$ **mixed partial derivatives are equal** (if f is smooth)

Surface Plot of a Multivariable Function

Function: (z = 4x^2 + 3xy + y^2)



Surface Plot of a Multivariable Function (continued)

Interpretation:

- This 3D plot shows how the output (z) varies with two input variables (x) and (y).
- The **curved surface** represents a **multivariable quadratic function**.
- As both (x) and (y) increase in magnitude, the value of (z) increases rapidly.
- The function is **convex**, indicating that it has a **minimum point** where the surface is lowest.
- Such plots are commonly used in **cost**, **utility**, and **production functions** in economics to visualize how two inputs interact.

Small Increments Formula

Increment is a small change in the variable. Let $f(x, y)$ be a function of two variables. If we have small changes Δx and Δy, we can estimate the change in z as:

$$\Delta z \approx f_x(x_0, y_0) \cdot \Delta x + f_y(x_0, y_0) \cdot \Delta y$$

Where:

- $f_x(x_0, y_0)$ is the partial derivative of f with respect to x at (x_0, y_0)
- $f_y(x_0, y_0)$ is the partial derivative of f with respect to y at (x_0, y_0)

Example:

Let $f(x, y) = x^2 + 3y^2$, find Δz when:

- $x = 1, y = 2$
- $\Delta x = 0.1, \Delta y = -0.2$

Partial derivatives:

- $f_x = 2x \Rightarrow f_x(1, 2) = 2$
- $f_y = 6y \Rightarrow f_y(1, 2) = 12$

$\Delta z \approx 2(0.1) + 12(-0.2) = 0.2 - 2.4 = \boxed{-2.2}$

Implicit Differentiation

When variables are related **implicitly** (not as $y = f(x)$), we still find $\frac{dy}{dx}$.

Example:

Let $x^2 + y^2 = 25$. Differentiate both sides:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0 \Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0$$

Solve:

$$\frac{dy}{dx} = -\frac{x}{y}$$

Practice Problems

1. Let $f(x, y) = 5x + xy^2 - 10$, and $g(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Evaluate:

   - (a) $f(0, 0)$
   - (b) $f(1, 2)$
   - (c) $f(2, 1)$
   - (d) $g(5, 6, 10)$
   - (e) $g(0, 0, 0)$
   - (f) $g(10, 5, 6)$

2. Find f_x, f_y, f_{xx}, f_{yy}, f_{xy} for $f(x, y) = x^2y + y^3$

3. Use small increments: $f(x, y) = 2x + y^2$, $x = 2$, $y = 1$, $\Delta x = 0.05$, $\Delta y = -0.1$

4. Implicit diff: Given $x^2 - xy + y^2 = 7$, find $\frac{dy}{dx}$

Summary

- Partial derivatives help analyze functions with multiple inputs
- Second-order and mixed derivatives are tools for optimization
- Small increments formula estimates change efficiently
- Implicit differentiation handles **non-solved** functions

In economics, these techniques are used in:

- **Cost functions** with multiple inputs
- **Utility and production functions**
- **Marginal analysis** in multivariate cases

2. Group Activity: Cost Function Strategy Game

Objective:

Use teamwork to analyze how changes in labor and capital affect total cost and marginal cost.

Instructions:

- Form **4 groups** of **4 students**.
- Each group gets:
  - A cost function: $C(L, K) = 20L + 30K + LK$
  - A table with sample values for L and K.
- Tasks:
  1. Calculate C_L and C_K (partial derivatives).
  2. Interpret their economic meaning.
  3. Discuss in your group: **How would increasing L while holding K constant affect costs?**
  4. Sketch a 3D cost surface or use a grid to show your interpretation.

Each group will present a 2-minute explanation of:

- Your calculated derivatives
- Your insights on labor and capital usage

Sample Table:

L (Labour)	K (Capital)	C(L, K)
1	1	51
2	1	72
1	2	82
2	2	104
3	3	177

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (May 2) Partial Elasticity and Marginal Functions (5.2)