Mathematical Methods for International Commerce

Week 7/2: The Derivative of the Exponential and Natural Log Functions (4.8)

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Agenda

1. The Derivative of the Exponential and Natural Log Functions (4.8)
2. Individual Activity: Solve the Problems!
3. Midterm Exam Review

1. The Derivative of the Exponential and Natural Log Functions (4.8)

Why It Matters: Economics & Finance Perspective

Exponential and logarithmic functions are foundational in economics and finance:

- They model **continuous growth and decay** – essential for understanding investment returns, population growth, inflation, and depreciation.
- The exponential function captures **compound interest**, while logarithms are key for solving equations involving **growth rates**.
- Derivatives of these functions help economists and analysts assess **rates of change** – such as marginal returns, discounting future values, or analyzing production over time.

Understanding how to **differentiate** exponential and logarithmic functions equips you with tools to interpret trends, optimize financial decisions, and make precise economic forecasts.

Learning Objectives

Section 4.8 – Derivatives of Exponential and Logarithmic Functions

- Differentiate the **exponential function**
- Differentiate the **natural logarithm function**
- Use **chain, product and quotient rules** with these functions
- Apply exponential models to **real-world economic problems**

Basic Rules of Differentiation (Exponential & Logarithmic Functions)

Exponential Function:

The exponential function grows **proportionally to its value**.

$$\frac{d}{dx}e^x = e^x$$

This is unique: the derivative of e^x is itself!

Natural Logarithm:

Logarithms are used to reverse exponential growth.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0$$

Basic Rules of Differentiation (Exponential & Logarithmic Functions) (cont'd)

Chain Rule:

Use when differentiating **functions within functions**.

- $$\frac{d}{dx}e^{f(x)} = f'(x) \cdot e^{f(x)}$$
- $$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$
- In fact, to be precise, we first differentiate the outer function, keeping the inner function untouched, then multiply by the derivative of the inner function.
- Although, mathematicaly, you will get same result, if you first differentiate the inner function and then multiply by the outer function.

These rules are essential in **economic models** involving growth, decay, and elasticity.

Examples: Basic Differentiation

1. Differentiate:

$$f(x) = e^{3x}$$

Solution:

$$f'(x) = 3e^{3x}$$

- Use chain rule: derivative of exponent times original exponential

2. Differentiate:

$$g(x) = \ln(5x^2 + 1)$$

Solution:

$$g'(x) = \frac{10x}{5x^2 + 1}$$

- Chain rule: differentiate inside function $5x^2 + 1 \to 10x$

Tip: Exponentials often model compounded growth, and logs appear in utility, elasticity, and returns.

Product & Quotient Rule in Action

Example 1: Product Rule

$$y = x^2 e^x$$

Apply:

$$\frac{d}{dx}(uv) = u'v + uv'$$

- $u = x^2, v = e^x$
- Derivative:

$$\frac{dy}{dx} = 2xe^x + x^2e^x = e^x(2x + x^2)$$

Use when both parts involve variables (common in cost/revenue products).

Product & Quotient Rule in Action (cont'd)

Example 2: Quotient Rule

$$y = \frac{\ln(x)}{x^2}$$

Apply:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

- $u = \ln(x), v = x^2$
- Derivative:

$$\frac{dy}{dx} = \frac{1/x \cdot x^2 - \ln(x) \cdot 2x}{x^4} = \frac{x - 2x\ln(x)}{x^4}$$

Use when differentiating ratios like **marginal utility/cost per unit**.

Application: Continuous Revenue Growth

Let the revenue function be:

$$R(t) = 5000e^{0.05t}$$

This implies revenue grows **at a continuous rate of 5%** per time unit.

Find the rate of change of revenue:

$$R'(t) = 5000 \cdot 0.05e^{0.05t} = 250e^{0.05t}$$

Interpretation:

- The **growth rate is proportional** to the current revenue.
- Common in modeling investment returns, inflation, or GDP.

Application: Elasticity of Growth

A common growth model:

$$Q(t) = Ae^{rt}$$

Where:

- $Q(t)$ is output (e.g., capital, population)
- A is initial value
- r is the growth rate
- t is time
- e is the base of natural logarithm

Application: Elasticity of Growth (cont'd)

Find:

- $$\frac{dQ}{dt} = rAe^{rt} = rQ(t)$$

- Elasticity of growth:

$$E = \frac{dQ/dt}{Q} = r$$

Interpretation: Elasticity is **constant** in exponential growth → % change in Q for 1% change in time.

Used in modeling:

- Population growth
- Compound interest
- Inflation and real returns

Example: Elasticity of Growth

Model:

$$Q(t) = 1000 \cdot e^{0.03t}$$

- Capital stock in billions
- Initial value (A = 1000)
- Growth rate (r = 0.03) (3% annually)

Step 1: Instantaneous Growth Rate

Differentiate:

$$\frac{dQ}{dt} = 1000 \cdot 0.03 \cdot e^{0.03t} = 0.03Q(t)$$

At (t = 5):

$$Q(5) = 1000 \cdot e^{0.15} \approx 1161.83 \left.\frac{dQ}{dt}\right|_{t=5} = 0.03 \cdot 1161.83 \approx \boxed{34.85}$$

Interpretation:

- At $t = 5$, capital is growing at a rate of **34.85 units per year**, or **3% of its size**, consistent with exponential growth.

Step 2: Elasticity of Growth

Elasticity Formula:

$$E = \frac{\frac{dQ}{dt}}{Q} = \frac{0.03Q}{Q} = \boxed{0.03}$$

Interpretation

- Elasticity of growth is **constant**: $E = 0.03$
- Output grows at a **constant rate of 3% per unit of time**
- Common in models of:
  - Population growth
  - Compound interest
  - Inflation-adjusted returns

Summary

- Exponential and log functions are common in **growth, interest, decay** models
- Chain rule is crucial when inside other functions
- Product and quotient rules still apply!

> Exponentials capture **compounding**; logs help **linearize** growth patterns.

2. Individual Activity: Solve the Problems!

Practice Problems

1. Differentiate:
   - (a) $f(x) = e^{2x^2}$
   - (b) $g(x) = \ln(x^2 + 1)$
2. Use product rule:
   - (a) $y = xe^{3x}$
3. Use quotient rule:
   - (a) $y = \frac{e^x}{x^3}$
4. Application:
   - Revenue grows as $R(t) = 12000e^{0.04t}$. Find $R'(t)$ and interpret.

3. Midterm Exam Review

Midterm Exam Review

Coverage:

- Basic Algebra & Solving Equations (1.1–1.4)
- Supply and Demand, Transposition, National Income (1.5–1.7)
- Quadratic Functions, Revenue & Profit (2.1–2.2)
- Indices, Logs, Exponentials (2.3–2.4)
- Percentages, Compound Interest (3.1–3.2)
- Geometric Series & Investment Appraisal (3.3–3.4)
- Derivatives & Marginal Functions (4.1–4.3)
- Chain/Product/Quotient Rules, Elasticity (4.4–4.5)
- Optimization (4.6–4.7), Derivatives of Exp/Log (4.8)

Also Review: Lecture slides, Homework #1, Quiz #1

What You Should Be Able to Do

- Simplify, factor, solve linear & quadratic equations
- Graph demand, supply, quadratic, exponential functions
- Interpret elasticity, marginal cost/revenue, and APL/MPL
- Apply formulas for compound interest, present value, annuities
- Differentiate power, exponential, log functions
- Use derivative tests to find and classify stationary points

Algebra & Quadratics

- Factor and solve:
  - $x^2 - 5x + 6 = 0$
  - $2x^2 - 3x - 2 = 0$
- Graph: $y = x^2 - 4x + 3$
- Use quadratic formula

Supply & Demand, Income Determination

- Sketch and solve:
  - $Q_d = 100 - 5P, Q_s = 20 + 3P$
- Find equilibrium
- National income:
  - $Y = C_0 + cY + I + G$, solve for Y

Logs, Indices, Exponentials

- Simplify:
  - $2^3 \times 2^2$
  - $\ln(e^3)$
- Solve: $3^x = 81 \Rightarrow x = 4$
- Evaluate: $(1.05)^5$

Finance Applications

- Calculate:
  - % increase/decrease
  - Compound value: $A = P(1+r)^n$
  - NPV: $\mathrm{NPV} = \sum \frac{C_t}{(1+r)^t} - C_0$
  - Loan instalment (annuity formula)

Derivatives & Optimization

- Power rule: $\frac{d}{dx}x^n$
- Product & chain rule, other rules
- Find:
    - Marginal Revenue/Cost
    - When MR = MC
    - Maximize profit using first and second derivative test

Elasticity

- Arc elasticity:

$$E_d = \frac{\Delta Q / \text{avg } Q}{\Delta P / \text{avg } P}$$

- Point elasticity:

$$E_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$$

- Know how elasticity affects revenue

Practice & Prep

Review your:

- Lecture slides and solved examples
- Homework #1
- Quiz #1

Follow exam instructions! No phones allowed. Only one A4 double-sided handwritten cheat sheet (formulas only) is permitted. The time limit will be strictly enforced.

Time to shine – you've got this!

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (April 23) No Class (Midterm Exam Week)
- (April 25) Mid term exam (in class):
  - Review all material from the beginning of the semester
  - Pay attention to the examples in the slides, HW #1, Quiz #1 and the exercises in the textbook
- (April 30) Functions of Several Variables (5.1)