Mathematical Methods for International Commerce

Week 4/1: Percentages and Compound Interest

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Introduction

Mathematics of Finance is crucial for **investment decisions** and **economic analysis**.

- **Why do we study these?**
  - **Percentages** are used in **finance, economics, and business**.
  - **Compound interest** is crucial for **investment decisions**.
  - **Inflation rates** and **population growth** are expressed in percentages.

Percentages

Section 3.1: Percentages

What is a Percentage?

- A **percentage** is a fraction out of 100.
- Expressed as:

$$\text{Percentage} = \frac{\text{Part}}{\text{Whole}} \times 100$$

Example:

If a product costs $200 and increases to $240:

$$\text{Percentage Increase} = \frac{240 - 200}{200} \times 100 = 20\%$$

Percentage Increase and Decrease

- **Increase**: Multiply by $1 + \frac{r}{100}$
- **Decrease**: Multiply by $1 - \frac{r}{100}$

Example:

Price increases by 15%:

$$\text{New Price} = 100 \times (1 + 0.15) = 115$$

Bps and Percentage Points

- **Basis points (bps)** are used to express **small percentage changes**.
- **Percentage points** are used to express **absolute changes**.

Example:

- If the interest rate increases from 5% to 6%:
  - This is a **1% increase** or **100 bps increase**.
- If the interest rate increases from 5% to 7%:
  - This is a **2% increase** or **200 bps increase**.

Scale Factors

- A **scale factor** expresses the effect of a percentage change.
- For example, a 20% increase corresponds to a scale factor of 1.20.

Example:

- If a price decrease by 10%:
  - The scale factor is 0.90.
- If a price increases by 25%:
  - The scale factor is 1.25.

Overall Percentage Changes

To find total percentage change over multiple steps:

$$\text{Overall Scale Factor} = (1 + r_1)(1 + r_2)\cdots(1 + r_n)$$

Example:

Increase by 10%, then decrease by 5%:

$$1.10 \times 0.95 = 1.045 \Rightarrow 4.5\% \text{ total increase}$$

Index Numbers

- Index numbers show how a value changes relative to a base period.

$$\text{Index} = \frac{\text{Current Value}}{\text{Base Value}} \times 100$$

Example:

CPI increases from 100 to 108:

- This indicates an **8% increase in prices**.
- Calculated as:

$$\text{CPI} = \frac{108}{100} \times 100 = 108$$

Adjusting for Inflation

To find real value:

$$\text{Real Value} = \frac{\text{Nominal Value}}{\text{Price Index}} \times 100$$

PS: **CPI** is used as a **price index**. Real value is a value adjusted for inflation. Nominal value is the original value.

Example:

Nominal wage = $500, CPI = 125:

$$\text{Real Wage} = \frac{500}{125} \times 100 = 400$$

Calculating Index Numbers: Table 3.1 Example

Household Spending (in billions of dollars)

Year	Y1	Y2	Y3	Y4	Y5
Spending	686.9	697.2	723.7	716.6	734.5

- Table 3.1 shows the values of household spending (in billions of dollars) during a five year period.
- Calculate the index numbers when Year 2 is taken as the base year and give a brief interpretation.

Step-by-Step: Base Year = Y2 (697.2)

- **Formula**:

$$\text{Index Number} = \left(\frac{\text{Value in Year X}}{\text{Value in Base Year}}\right) \times 100$$

- **Y2** (Base Year):

$$\text{Index} = \left(\frac{697.2}{697.2}\right) \times 100 = 100$$

- **Y3**:

$$\text{Index} = \left(\frac{723.7}{697.2}\right) \times 100 \approx 103.8$$

- **Interpretation**:
  Spending in **Y3** was **103.8%** of that in Y2 → increase of **3.8%**

Continuing with Y4 and Y5

- **Y4**:

$$\text{Index} = \left(\frac{716.6}{697.2}\right) \times 100 \approx 102.8$$

→ Spending rose only **2.8%** from Y2, lower than Y3.

- **Y5**:

$$\text{Index} = \left(\frac{734.5}{697.2}\right) \times 100 \approx 105.4$$

- **Y1**:

$$\text{Index} = \left(\frac{686.9}{697.2}\right) \times 100 \approx 98.5$$

- **Interpretation**:
  Spending in **Y5** was **5.4%** higher than in Y2.

Final Index Table (Base Year = Y2)

Year	Y1	Y2	Y3	Y4	Y5
Index	98.5	100	103.8	102.8	105.4

Why Use Index Numbers?

- Simplifies comparison across time periods
- Common in **price indices**, **GDP**, **inflation tracking**
- Allows interpretation like:
  “Spending in Y5 is **5.4% higher** than in Y2”

Compound Interest

Section 3.2: Compound Interest

Simple vs Compound Interest

- **Simple Interest**: Earned only on the principal.
- **Compound Interest**: Earned on **principal and accumulated interest.**

Example:

- $1000 invested at 5% for 3 years:
  - **Simple Interest**: $1000 + $150 = $1150
  - **Compound Interest**: $1157.63
- Because:
  - $1000 + $1000(0.05) * 3 = 1150$
  - $1000(1 + 0.05)^3 = 1157.63$

Annual Compounding Formula

- **Annual Compounding** stands for interest *calculated once a year*.

$$A = P(1 + r)^t$$

where:

- A: final amount
- P: principal
- r: interest rate per year
- t: time in years

Example:

$P = 1000, r = 5\%, t = 3$:

$$A = 1000(1 + 0.05)^3 = 1157.63$$

Continuous Compounding Formula

- **Continuous Compounding** means that there is *no limit* to how often interest can compound.

$$A = Pe^{rt}$$

Example:

$P = 1000, r = 0.05, t = 3$:

$$A = 1000e^{0.15} \approx 1161.83$$

Constant Growth Rate Over Time

Example: Output Growth Over 5 Years

A firm wants to increase output from **50,000** to **60,000** over **5 years** at a **constant annual rate**.

We are solving for the **growth rate $ r \% $**.

Step-by-Step Solution

Let the scale factor be:

$$(1 + \frac{r}{100})^5$$

To reach 60,000 in 5 years:

$$50,000 \cdot \left(1 + \frac{r}{100}\right)^5 = 60,000$$

Divide both sides by 50,000:

$$\left(1 + \frac{r}{100}\right)^5 = 1.2$$

Taking the Fifth Root

Solve:

$$1 + \frac{r}{100} = (1.2)^{1/5}$$

Compute:

$$1 + \frac{r}{100} \approx 1.0371$$

Subtract 1:

$$\frac{r}{100} \approx 0.0371 \Rightarrow r \approx 3.71\%$$

Conclusion: The firm needs a constant **annual growth rate of approximately 3.7%** to reach 60,000 in 5 years.

Why This Matters in Economics?

- Helps forecast growth of **output**, **investment**, **GDP**, or **population**.
- Realistic use of **compound growth**, especially in planning or project evaluation.

Effective Annual Rate (EAR)

- **EAR** is the **true return on investment**. It accounts for compounding.

If interest is compounded more than once a year:

$$\text{EAR} = \left(1 + \frac{r}{n}\right)^n - 1$$

where n = number of compounding periods per year

Example:

Nominal rate 6% compounded quarterly:

$$\text{EAR} = \left(1 + \frac{0.06}{4}\right)^4 - 1 \approx 0.0614 = 6.14\%$$

It means that the effective annual rate is 6.14%.

Practice Problems

Percentages

1. A good's price rose from \$80 to \$92. What’s the percentage increase?
2. CPI rose from 120 to 132. What is the inflation rate?
3. If a product's price falls from \$100 to \$80, what is the percentage decrease?
4. If a country's GDP rises from \$500 to \$550, what is the percentage increase?

Compound Interest

1. Calculate the compound amount on \$2000 at 6% for 5 years.
2. Compare the amount earned with annual vs. continuous compounding.
3. If the nominal rate is 8% compounded monthly, what is the EAR?
4. If the nominal rate is 5% compounded daily, what is the EAR?
5. If the nominal rate is 4% compounded quarterly, what is the EAR?

Summary

- Percentage changes and index numbers help interpret price and value shifts.
- Compound interest is central to banking and investment decisions.
- EAR reveals the true return on investment.

Any QUESTIONS?

Thank you for your attention!

Next Class

- (Mar 28) Geometric Series (3.3), Investment Appraisal (3.4)