Mathematical Methods for International Commerce

Week 12/2: Definite Integration (6.2)

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Why It Matters in Economics & Finance

- Definite integration is crucial for calculating areas under curves, total costs, revenues, and investment values.
- In economics, it is used to determine **consumer and producer surplus** and **capital accumulation**.
- In finance, it helps in calculating **present value of continuous revenue streams**.

Understanding Definite Integration

- Integration finds the **area under a curve** between two specific points.
- Notation: If $f(x)$ is continuous over $[a, b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where

- $f(x)$ is the function to be integrated,
- a and b are the limits of integration,
- $F(x)$ is the **antiderivative** of $f(x)$.
  - $F(b)$ and $F(a)$ are the values of the antiderivative at b and a, respectively.

Visualizing the Definite Integral

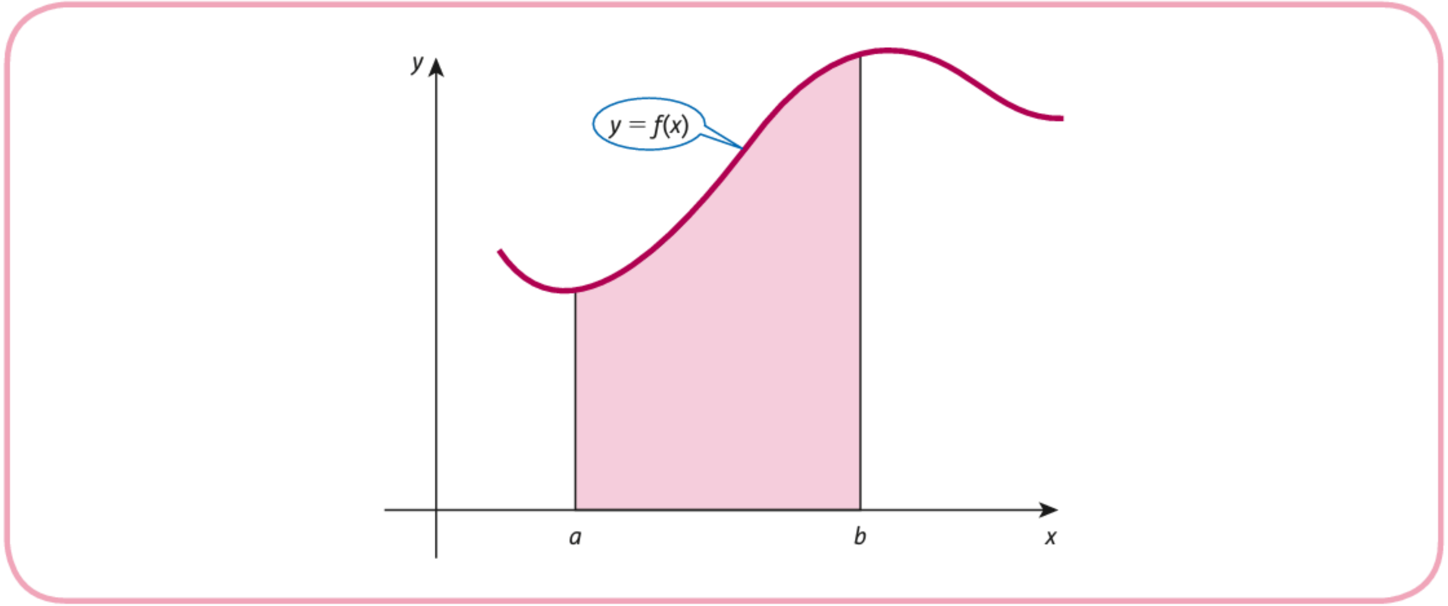

Figure 6.2

- This graph shows the general concept of finding the area under a curve between two points, a and b.
- The definite integral gives the **net area** between the curve and the x-axis over the interval $[a, b]$.
- The area can be positive or negative depending on the position of the curve relative to the x-axis.
- If the curve is above the x-axis, the area is positive; if below, the area is negative.

Understanding Definite Integration (cont)

- Example:

$$\int_0^3 (2x+1)\,dx = \left[x^2+x\right]_0^3 = (9+3)-(0+0) = 12$$

- Example:
  - Find the area under $y = x^2$ from $x = 1$ to $x = 2$.

$$\int_1^2 x^2\,dx = \left[\frac{x^3}{3}\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$$

Visualizing the Area Under a Curve

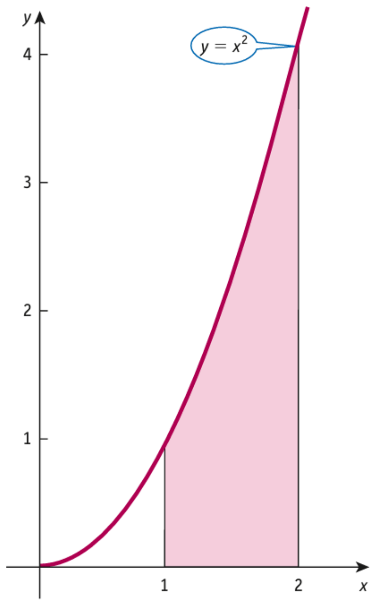

Figure 6.1

- This graph illustrates the area under the curve $y = x^2$ between $x = 1$ and $x = 2$.
- The area represents the **definite integral** value.

Definite Integration Examples

Example (a): Evaluating the Definite Integral

Evaluate the definite integral:

$$\int_2^6 3\,dx$$

Solution:

We integrate the constant function:

$$\int_2^6 3\,dx = [3x]_2^6 = 3(6) - 3(2) = 18 - 6 = 12$$

This can also be confirmed graphically by calculating the area of the rectangle (Figure 6.3):

- **Base:** $6 - 2 = 4$
- **Height:** 3
- **Area:** $4 \times 3 = 12$

Example (a): Evaluating the Definite Integral (cont)

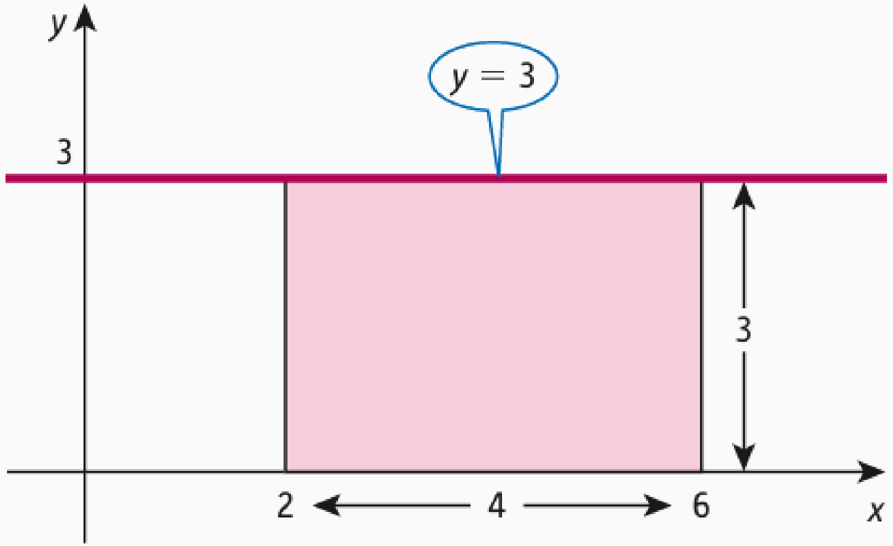

Figure 6.3

Example (b): Evaluating the Definite Integral

Evaluate the definite integral:

$$\int_0^2 (x+1)\,dx$$

Solution:

We integrate the linear function:

$$\int_0^2 (x+1)\,dx = \left[\frac{x^2}{2} + x\right]_0^2$$

Substituting the limits:

$$= \left(\frac{2^2}{2} + 2\right) - \left(\frac{0^2}{2} + 0\right)$$

$$= (2+2) - (0) = 4$$

Example (b): Evaluating the Definite Integral (cont)

Graphically, this is represented as a region under $y = x + 1$ (Figure 6.4a) and a one-half of the rectangle (Figure 6.4b):

- **Base:** 2
- **Height:** 4
- **Area:** $\frac{1}{2} \times 2 \times 4 = 4$

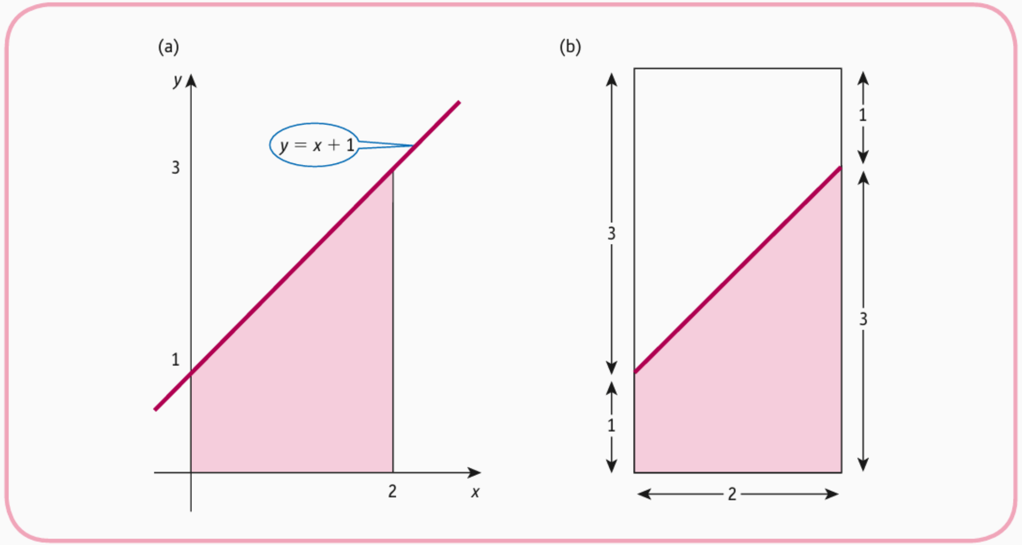

Figure 6.4

Area Under Demand Curve - Consumer Surplus

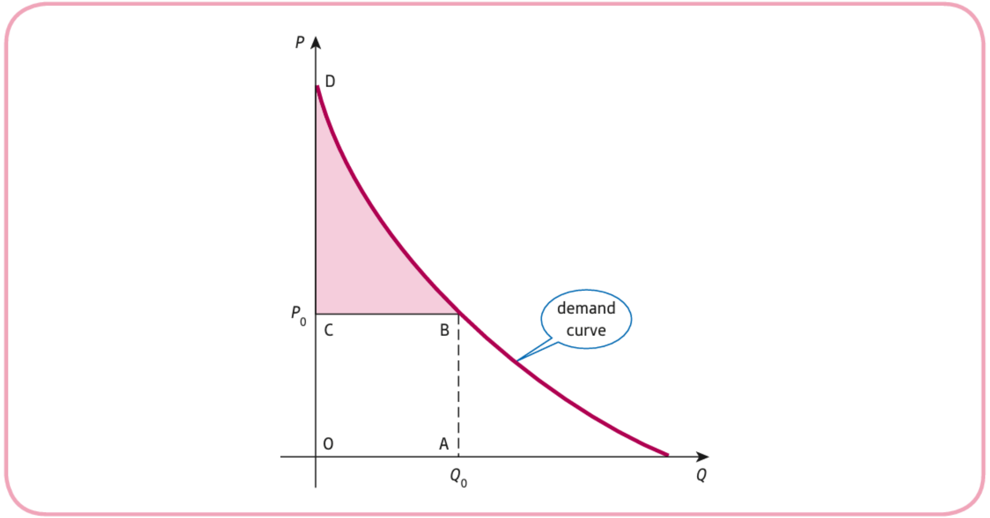

Figure 6.5

- Consumer surplus is the area under the demand curve and above the price line.
- It measures the **benefit consumers receive** when purchasing a product at a given price.

Area Under Demand Curve - Consumer Surplus (cont)

Problem Statement

- Find the consumer's surplus at $Q = 5$ for the demand function:

$$P = 30 - 4Q$$

Area Under Demand Curve - Consumer Surplus (cont)

Step 1: Define the Function

- The demand function is:

$$f(Q) = 30 - 4Q$$

- At $Q_0 = 5$, the price is:

$$P_0 = 30 - 4(5) = 10$$

Area Under Demand Curve - Consumer Surplus (cont)

Step 2: Consumer Surplus Formula

The formula for **consumer's surplus** is:

$$CS = \int_0^{Q_0} f(Q)\,dQ - Q_0 \times P_0$$

Substituting the given values:

$$CS = \int_0^{5} (30 - 4Q)\,dQ - 5 \times 10$$

Area Under Demand Curve - Consumer Surplus (cont)

Step 3: Solve the Integral with R

```
# Define the function
f <- function(Q) { 30 - 4 * Q }

# Integrate
cs_integral <- integrate(f, lower = 0, upper = 5)$value

# Calculate CS
P0 <- 10
Q0 <- 5
CS <- cs_integral - Q0 * P0

# Display result
CS
```

```
## [1] 50
```

Area Under Demand Curve - Consumer Surplus (cont)

Step 4: Mathematical Solution

1. Integrate:

$$\int_0^5 (30 - 4Q)\, dQ = [30Q - 2Q^2]_0^5$$

$$= (30 \times 5 - 2 \times 5^2) - (30 \times 0 - 2 \times 0^2)$$

$$= (150 - 50) - (0 - 0) = 100$$

1. Calculate CS:

$$CS = 100 - 5 \times 10 = 100 - 50 = 50$$

Answer: The consumer surplus is **50 units**.

Producer Surplus

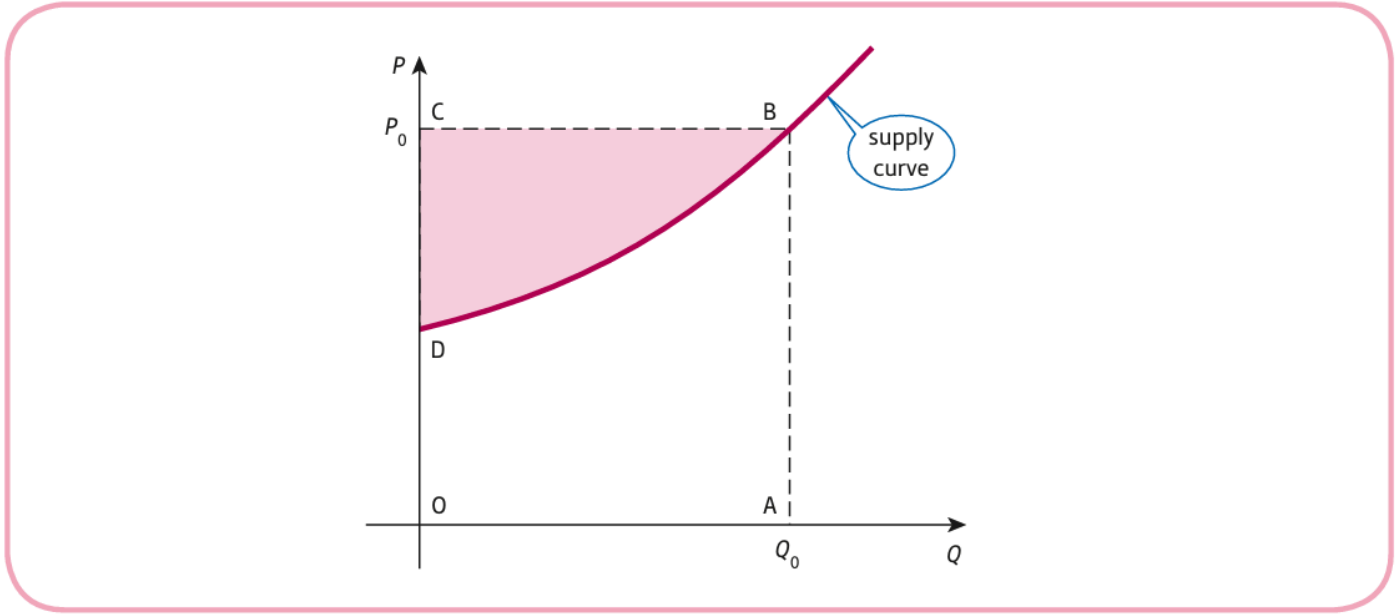

Figure 6.6

- Producer surplus is the area above the supply curve and below the price line.
- It represents the **benefit producers receive** by selling at a market price higher than the minimum they would accept.

Producer Surplus (cont)

Producer's Surplus - Example

Given:

- Demand function: $P = 35 - Q^2$
- Supply function: $P = 3 + Q^2$

Objective: Calculate the producer's surplus assuming pure competition.

Producer Surplus (cont)

Step 1: Find Equilibrium

- Set demand equal to supply:

$$35 - Q^2 = 3 + Q^2$$

$$35 - 2Q^2 = 3$$

$$-2Q^2 = -32$$

$$Q^2 = 16 \implies Q = 4$$

- Equilibrium price:

$$P_0 = 35 - (4)^2 = 19$$

Producer Surplus (cont)

Step 2: Define Producer Surplus

$$PS = Q_0 P_0 - \int_0^{Q_0} (3 + Q^2)\, dQ$$

Where:

- $Q_0 = 4$
- $P_0 = 19$

Producer Surplus (cont)

Step 3: Calculate Producer Surplus wih R

```r
Q0 <- 4
P0 <- 19

# Define the supply function
supply <- function(Q) {
  3 + Q^2
}

# Calculate integral
integral <- integrate(supply, lower = 0, upper = Q0)$value

# Calculate PS
PS <- Q0 * P0 - integral
PS
```

```
## [1] 42.66667
```

Producer Surplus (cont)

Step 4: Mathematical Solution

1. Calculate the integral:

$$\int_0^4 (3 + Q^2)\, dQ = [3Q + \frac{Q^3}{3}]_0^4$$

$$= (3 \times 4 + \frac{4^3}{3}) - (0)$$

$$= (12 + \frac{64}{3}) - 0$$

$$= \frac{36}{3} + \frac{64}{3} = \frac{100}{3}$$

Producer Surplus (cont)

Step 4: Mathematical Solution (cont)

1. Calculate PS:

$$PS = Q_0 P_0 - \int_0^{Q_0} (3 + Q^2)\, dQ$$

$$= 4 \times 19 - \frac{100}{3}$$

$$= 76 - \frac{100}{3}$$

$$= \frac{228}{3} - \frac{100}{3} = \frac{128}{3}$$

$$= 42.67$$

Producer Surplus (cont)

Step 5: Interpretation

- The producer's surplus is approximately 42.67.
- This represents the area above the supply curve and below the equilibrium price line up to the equilibrium quantity.

Net Investment and Capital Formation

Understanding Net Investment

- **Net Investment (I)** is defined as the rate of change of capital stock (K).
- The relationship is given by:

$$I(t) = \frac{dK}{dt}$$

- Here, $I(t)$ is the flow of money in dollars per year, and $K(t)$ is the accumulated capital.

Net Investment and Capital Formation (cont)

Capital Formation - Definite Integral

- To calculate the capital formation over a period from t_1 to t_2, we use:

$$\int_{t_1}^{t_2} I(t)\,dt$$

- If we are given the investment flow function, we integrate to find the capital stock.

Note: Capital accumulation is the total amount of capital accumulated over time, considering continuous inflows and outflows.

Net Investment and Capital Formation (cont)

Example - Capital Formation

Investment Flow Function:

$$I(t) = 9000\sqrt{t}$$

(a) Calculate the capital formation from the end of the first year to the end of the fourth year with R:

```r
# Integration for capital formation
capital_formation <- integrate(function(t) 9000 * sqrt(t), lower = 1, upper = 4)$value
capital_formation
```

```
## [1] 42000
```

Net Investment and Capital Formation (cont)

Example - Capital Formation (cont)

(b) Determine the number of years before capital stock exceeds $100,000 with R.

We need to solve:

$$\int_0^T 9000\sqrt{t}\,dt = 100000$$

```r
# Solving for T
solve_T <- function(T) {
  integral_value <- integrate(function(t) 9000 * sqrt(t), lower = 0, upper = T)$value
  return(integral_value - 100000)
}

T_value <- uniroot(solve_T, c(0, 10))$root
T_value
```

```
## [1] 6.524767
```

- The capital stock reaches $100,000 approximately **6.5 years** into the investment period.

Net Investment and Capital Formation (cont)

Example - Capital Formation (cont)

Mathematical Solution (a)

- The integral is:

$$\int_1^4 9000\sqrt{t}\,dt = 9000\left[\frac{2}{3}t^{3/2}\right]_1^4$$

- Evaluating the integral:

$$= 9000\left(\frac{2}{3}(4^{3/2}) - \frac{2}{3}(1^{3/2})\right)$$

$$= 9000\left(\frac{2}{3}(8) - \frac{2}{3}(1)\right)$$

$$= 9000\left(\frac{16}{3} - \frac{2}{3}\right) = 9000\left(\frac{14}{3}\right) = 42000$$

Answer: The capital formation from the end of the first year to the end of the fourth year is **$42,000**.

Net Investment and Capital Formation (cont)

Example - Capital Formation (cont)

Mathematical Solution (b)

- To find T such that:

$$\int_0^T 9000\sqrt{t}\,dt = 100000$$

- The integral is:

$$\int_0^T 9000\sqrt{t}\,dt = 9000\left[\frac{2}{3}t^{3/2}\right]_0^T$$

- Evaluating the integral:

$$= 9000\left(\frac{2}{3}T^{3/2} - 0\right)$$

$$= 6000T^{3/2}$$

Net Investment and Capital Formation (cont)

Example - Capital Formation (cont)

Mathematical Solution (b) (cont)

- Setting equal to $100,000$:

$$6000T^{3/2} = 100000$$

$$T^{3/2} = \frac{100000}{6000}$$

$$T^{3/2} = \frac{50}{3}$$

$$T = \left(\frac{50}{3}\right)^{2/3} \approx 6.5$$

Answer: The capital stock exceeds $100,000$ approximately **6.5 years** after the start of the investment.

Net Investment and Capital Formation (cont)

Interpretation and Insights

- Capital formation provides a measure of how investment over time accumulates into capital stock.
- Understanding the time it takes for investment to achieve certain capital levels helps in financial planning and forecasting.

Present Value of Continuous Revenue Stream

Problem Statement

- Calculate the present value of a continuous revenue stream of $1000 per year for 5 years, discounted at 9% annually.

- The present value is given by:

$$P = \int_0^5 1000e^{-0.09t}\,dt$$

Present Value of Continuous Revenue Stream (cont)

Step 1: Setup the Integral

- We need to evaluate the definite integral:

$$P = \int_0^5 1000e^{-0.09t}\, dt$$

- This integral represents the present value of a revenue stream discounted continuously.

Present Value of Continuous Revenue Stream (cont)

Step 2: Solve the Integral

- The integral is of the form:

$$\int e^{at}\,dt = \frac{1}{a}e^{at} + C$$

- Applying the integral:

$$\int 1000e^{-0.09t}\,dt = \frac{1000}{-0.09}e^{-0.09t}\Big|_0^5$$

Present Value of Continuous Revenue Stream (cont)

Step 3: Evaluate the Integral

- Evaluating at the bounds:

$$P = \frac{1000}{0.09}\left(e^{-0.09\times 5} - e^{0}\right)$$

$$= -11111.11\left(e^{-0.45} - 1\right)$$

Present Value of Continuous Revenue Stream (cont)

Step 4: Calculate the Present Value

- Approximating $e^{-0.45} \approx 0.6376$:

$$
\begin{aligned}
P &\approx -11111.11(0.6376 - 1) \\
&\approx -11111.11 \times -0.3624 \\
&\approx 4026.35
\end{aligned}
$$

Answer: The present value of the continuous revenue stream is **\$4026.35**.

Practice Problems

1. Evaluate the definite integral:
   - $\int_1^4 (3x^2 + 2)\, dx$
2. Calculate the consumer surplus given the demand curve $P = 50 - 2Q$ and market price $P = 30$.
3. Determine the present value of a continuous revenue stream given by $R(t) = 150e^{0.04t}$ over 3 years.

Summary

- Definite integration calculates areas under curves, total costs, revenues, and surplus values.
- It is applied in economic analysis to assess **consumer and producer surplus** and **capital accumulation**.
- Setting up and evaluating definite integrals is essential for economic applications.

2. Home work #2

Homework #2

- **Due Date:** June 13, 2025, before the start of class.
- **Submission Format:** Submit your solutions as a single PDF file via the Cyber Campus.

- **Instructions:**
  - Clearly show all steps and calculations.
  - Include explanations for your answers where applicable.
  - Ensure your submission is neat and well-organized.
  - Bring any questions to the office hours or email me.

- Work on your Home Assignment #2 (Jacques, 10th edition, Chapters 5-9):
  - Chapter 5.1: Exercise 5.1, Problem 5 (p. 411)
  - Chapter 5.2: Exercise 5.2, Problem 7 (p. 427)
  - Chapter 5.3: Exercise 5.3, Practice Problem inside the chapter (p. 432 only)
  - Chapter 5.4: Exercise 5.4, Problem 5 (p. 457)
  - Chapter 5.5: Exercise 5.5, Problems 6 and 7 (p. 467)
  - Chapter 5.6: Exercise 5.6, Problem 4 (p. 479)
  - Chapter 6.1: Exercise 6.1, Problem 3 (p. 506)
  - Chapter 6.2: Exercise 6.2, Problem 6 (p. 521)
  - Chapter 7.1: Exercise 7.1, Problem 5 (p. 553)
  - Chapter 7.2: Exercise 7.2, Problem 6 (p. 572)
  - Chapter 7.3: Exercise 7.3, Problem 5 (p. 584)
  - Chapter 8.1: Exercise 8.1, Problem 6 (p. 615)
  - Chapter 8.2: Exercise 8.2, Problem 3 (p. 626)
  - Chapter 9.1: Exercise 9.1, Problem 4 (p. 655)
  - Chapter 9.2: Exercise 9.2, Problem 3 (p. 670)

Good luck!

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (May 28) **Quiz 2**
- (May 30) Basic Matrix Operation (7.1) Matrix Inversion (7.2)