Mathematical Methods for International Commerce

Week 3/1: Quadratic Functions, and Revenue, Cost, and Profit

Iegor Vyshnevskyi, Ph.D.

Sogang University

March 19, 2025

What is a U-turn?

- **U-turn** is a maneuver used to reverse the direction of travel.
- **Quadratic functions** are like U

Why Study Quadratic Functions?

- **Appear in business & economics**: Cost, revenue, profit functions.
- **Essential for decision-making**: Finding maximum revenue, profit, and break-even points.
- **Graphical analysis**: Helps visualize relationships in markets.

At the end of this class, you will be able to:

- Solve quadratic equations using **factorization & quadratic formula**.
- Sketch **quadratic function graphs** using tables and key points.
- Solve **quadratic inequalities** with graphs & sign diagrams.
- Analyze **total revenue, cost, and profit** functions.
- Find **optimal output & break-even levels**.

Section 2.1: Quadratic Functions

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What is a Quadratic Function?

A **quadratic function** has the form:

$$f(x) = ax^2 + bx + c$$

where:

- a, b, c are constants.
- $a \neq 0$ (ensures it is a quadratic function).
- The graph is a **parabola** (U-shaped or inverted U).

Graph of Quadratic Functions

- **Vertex**: The turning point of the parabola.
- **Axis of symmetry**: The line that divides the parabola into two equal halves.
- **Intercepts**: Points where the parabola crosses the x- and y-axes.

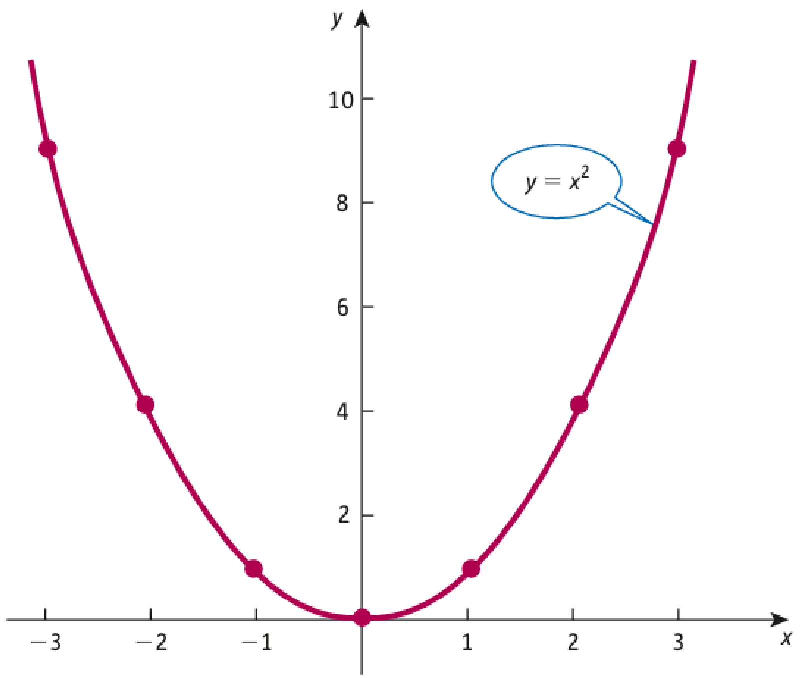


Figure 2.1

Examples of Quadratic Functions in Economics

1. **Cost Function:** $C(x) = 0.1x^2 + 10x + 100.$

2. **Revenue Function:** $R(x) = -0.2x^2 + 50x.$

3. **Profit Function:** $P(x) = -0.2x^2 + 50x - 100.$

Real-World Example: Loan Repayment Optimization

Imagine you **borrow money from a bank**. The **total repayment cost** depends on how much you borrow.
Banks often use **quadratic equations** to estimate **total repayment costs**.

Loan Cost Function:

$$C = 5000 + 300Q - 5Q^2$$

where:

- C = **Total cost of repayment** (in dollars)
- Q = **Loan amount borrowed** (in thousands of dollars)
- 5000 = Fixed bank fee
- $300Q$ = Interest cost per loan size
- $-5Q^2$ = Discount on large loans

Goal: Find the loan amount that minimizes total repayment cost.

Step 1: Understanding the Function

The equation:

$$C = 5000 + 300Q - 5Q^2$$

is a **quadratic function** because of the Q^2 term.

Since the **coefficient of Q^2 is negative** (-5), the graph is an **upside-down parabola** (meaning the function decreases after reaching a maximum point, indicating that there is an optimal loan amount where costs are minimized).

Why does this make sense?

- At **small loan amounts** (Q), total cost is high due to **fixed fees**.
- At **large loan amounts**, costs decrease because banks offer **discounts on large loans**.

Somewhere in between, the cost is minimized.

Step 2: Finding the Optimal Loan Amount

The **minimum cost** occurs at the **vertex** of the quadratic function.

Formula for the vertex of a quadratic equation $ax^2 + bx + c$ (the x-coordinate of the vertex):

$$Q^* = \frac{-b}{2a}$$

For our function:

$$C = -5Q^2 + 300Q + 5000$$

- (a = -5)
- (b = 300)

Using the vertex formula:

$$Q^* = \frac{-300}{2(-5)} = 30$$

Optimal Loan Amount: $Q^* = 30,000$ dollars.

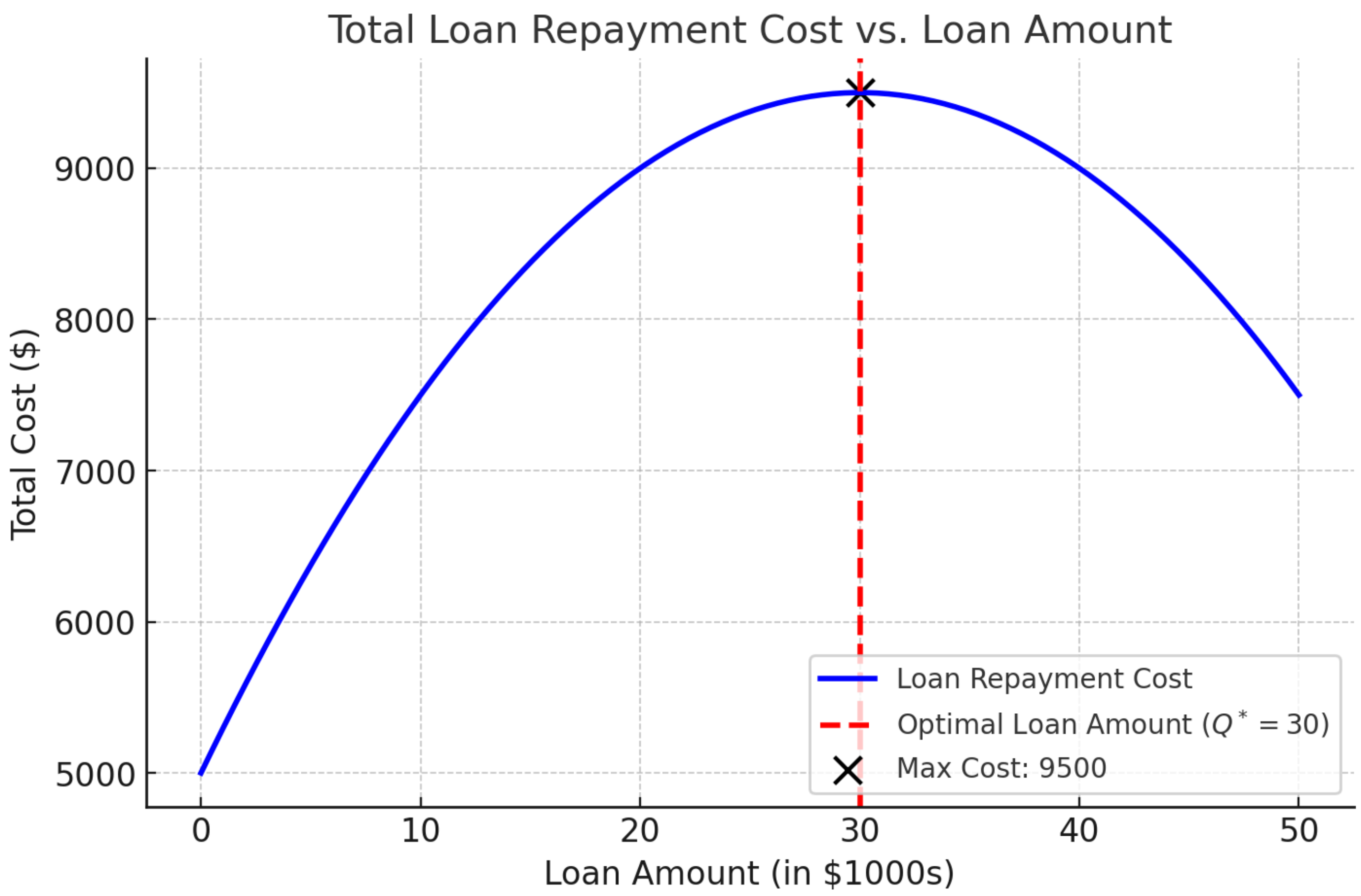
Step 3: Interpreting the Result

The **optimal loan amount** is $30,000.

- For loans less than **$30,000**: Repayment costs remain high due to fixed administrative fees.
- For loans greater than **$30,000**: Costs decrease as banks offer discounts on larger loans, reducing the overall repayment burden.

Step 4: Graphing the Function

Let's **graph the loan cost function** to visualize the relationship between loan amount and total cost.



Solving Quadratic Equations

1. Factorization Method

If a quadratic can be **factored**, we set each factor to zero.

Example:

$$x^2 - 5x + 6 = 0$$

Factorizing:

$$(x-2)(x-3) = 0$$

Setting each factor to zero:

$$\begin{aligned} x - 2 = 0 \quad &\Rightarrow \quad x = 2 \\ x - 3 = 0 \quad &\Rightarrow \quad x = 3 \end{aligned}$$

Solutions: $x = 2, x = 3$.

Solving Quadratic Equations (cont'd)

Your Turn: Solve the Quadratic Equation

$$x^2 - 7x + 10 = 0$$

Hint: think about two numbers that multiply to 10 and add up to 7.

$$2x^2 - 5x - 3 = 0$$

Hint: think about $(x - 3)$ as a factor.

$$3x^2 + 2x - 8 = 0$$

Hint: think about $(x + 2)$ as a factor.

Step 1: Factorize the quadratic equation.

Step 2: Set each factor to zero.

Step 3: Find the solutions.

Solving Quadratic Equations (cont'd)

2. Quadratic Formula

For any equation $ax^2 + bx + c = 0$, the **quadratic formula** is:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example: Solve

$$2x^2 - 3x - 2 = 0$$

.

Step 1: Recall the Quadratic Formula

where:

- (a = 2) (coefficient of x^2)
- (b = -3) (coefficient of x)
- (c = -2) (constant term)

Solving Quadratic Equations (cont'd)

2. Quadratic Formula (cont'd)

Step 2: Compute the Discriminant

The discriminant is:

$$D = b^2 - 4ac$$

Substituting the values:

$$D = (-3)^2 - 4(2)(-2)$$

$$D = 9 + 16 = 25$$

Since (D = 25) is **positive**, we get **two real solutions**.

- When $D < 0$, there are **no real solutions**.
- When $D = 0$, there is **one real solution**.

Solving Quadratic Equations (cont'd)

2. Quadratic Formula (cont'd)

Step 3: Solve for (x)

Substituting into the quadratic formula:

$$x = \frac{-(-3) \pm \sqrt{25}}{2(2)}$$

$$x = \frac{3 \pm 5}{4}$$

Splitting into two cases:

$$x_1 = \frac{3+5}{4} = \frac{8}{4} = 2$$

$$x_2 = \frac{3-5}{4} = \frac{-2}{4} = -\frac{1}{2}$$

Final Answers:

$$\boxed{x = 2, \quad x = -\frac{1}{2}}$$

Your Turn: Solve Quadratic Equations

Example Problems

1. $x^2 + 3x - 10 = 0$.
2. $2x^2 - 3x - 2 = 0$.
3. $3x^2 + 2x - 8 = 0$.

Step 1: Recall the **quadratic formula**.

Step 2: Compute the **discriminant**.

Step 3: Solve for (x).

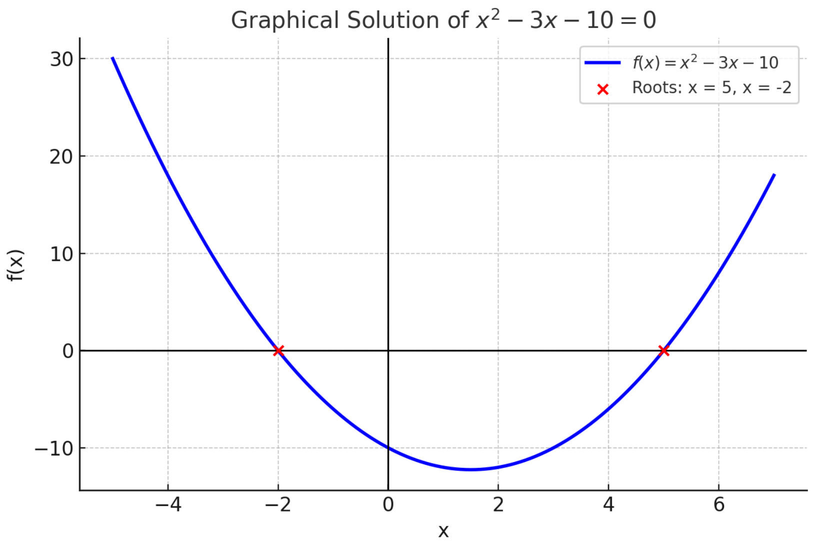
Remember to **check your answers**!

Solving Quadratic Equations (cont'd)

3. Graphical Solution

- **Graph the quadratic function.**
- **Find the x-intercepts** (where the function crosses the x-axis).
- **Solutions are the x-intercepts.**

Example: Solve $x^2 - 3x - 10 = 0$.



Solving Quadratic Equations (cont'd)

3. Graphical Solution (cont'd)

We analyze the quadratic function:

$$f(x) = x^2 - 5x + 6$$

What is a Sign Diagram?

A **sign diagram** helps us determine:

- **Where the function is positive or negative**
- **The intervals where the function is increasing or decreasing**
- **The critical points (x-intercepts)**

Step 1: Solve for Roots

Solving $x^2 - 5x + 6 = 0$ by factorization:

$$(x-2)(x-3) = 0$$

Thus, the **roots** (x-intercepts) are:

$$x = 2, \quad x = 3$$

Also, let's find vertex for better sketching:

$$x = \frac{-(-5)}{2(1)} = \frac{5}{2} = 2.5$$

$$f(2.5) = (2.5)^2 - 5(2.5) + 6 = 6.25 - 12.5 + 6 = -0.25 \quad \text{(lowest point)}$$

The graph is symmetric around $x = 2.5$.

Step 2: Constructing the Sign Diagram

Since $f(x) = x^2 - 5x + 6$ is a quadratic equation with **a positive leading coefficient**, the parabola opens **upward**.

We divide the number line into three **intervals**:

- Left of $x = 2$: Choose $x = 1$, substitute into $f(x)$:

$$f(1) = 1^2 - 5(1) + 6 = 1 - 5 + 6 = 2 \quad \text{(Positive)}$$

- Between $x = 2$ and $x = 3$: Choose $x = 2.5$, substitute:

$$f(2.5) = (2.5)^2 - 5(2.5) + 6 = 6.25 - 12.5 + 6 = -0.25 \quad \text{(Negative)}$$

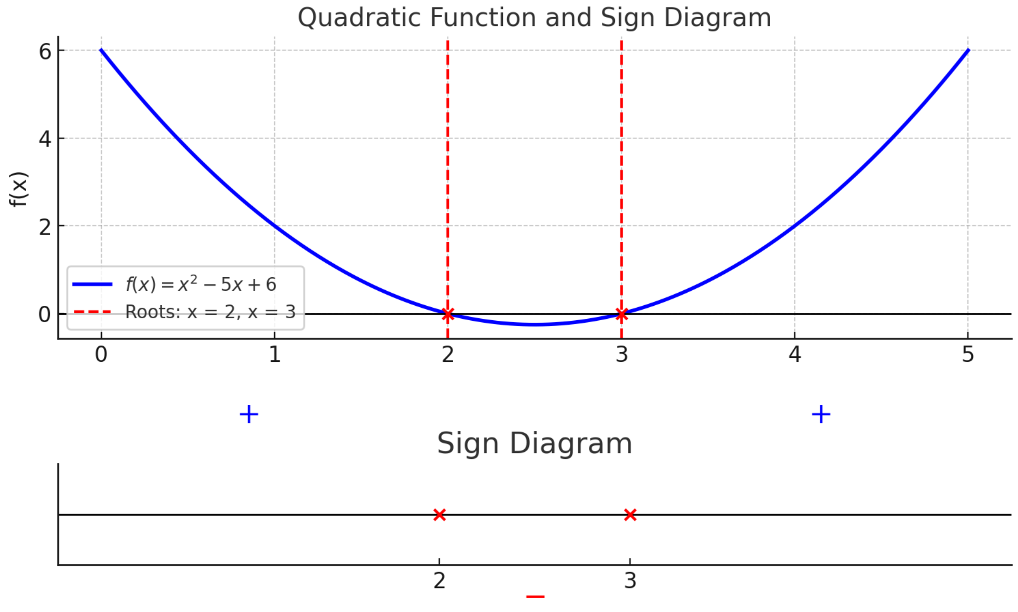
- Right of $x = 3$: Choose $x = 4$, substitute:

$$f(4) = 4^2 - 5(4) + 6 = 16 - 20 + 6 = 2 \quad \text{(Positive)}$$

Step 3: Interpretation

- For ($x < 2$): ($f(x) > 0$) → Function is positive 📈
- For ($2 < x < 3$): ($f(x) < 0$) → Function is negative 📉
- For ($x > 3$): ($f(x) > 0$) → Function is positive 📈
- At ($x = 2$) and ($x = 3$), ($f(x) = 0$) → These are the roots (x-intercepts).

Step 4: Graph & Sign Diagram



Interpretation of the Sign Diagram:

1. The function is **positive** for (x < 2) and (x > 3).
2. The function is **negative** between **(2 < x < 3)**.
3. The function **crosses the x-axis at (x = 2) and (x = 3)**.
4. The function **changes signs** at these points.

Your Turn: Graphical Solution (cont'd)

Example Problem

Quadratic Equation: $x^2 + 3x - 10 = 0$. Graph the quadratic function with sign diagram and find the solutions.

Step 1: Graph the function. Step 2: Find the x-intercepts. Step 3: Solutions are the x-intercepts.

Hint: Use the quadratic formula to verify your answers.

Section 2.2: Revenue, Cost, and Profit Functions

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What are Revenue, Cost, and Profit Functions?

- **Revenue Function:** $R(x) = p(x) \cdot x$.
- **Cost Function:** $C(x) = f(x) \cdot x + k$.
- **Profit Function:** $P(x) = R(x) - C(x)$.

where:

- x = Quantity of output.
- $p(x)$ = Price per unit.
- $f(x)$ = Fixed cost per unit.
- k = Fixed cost.

Total Revenue Function

The **total revenue** is the **product of price and quantity**:

$$R(x) = p(x) \cdot x$$

Example: If the price is $10 and you sell 100 units, the total revenue is $1000.

Total Cost Function

The **total cost** is the **sum of fixed and variable costs**:

$$C(x) = f(x) \cdot x + k$$

Example: If the fixed cost is \$1000, variable cost is \$5 per unit, and you produce 100 units, the total cost is \$1500.

Total Profit Function

The **total profit** is the **difference between total revenue and total cost**:

$$P(x) = R(x) - C(x)$$

Example: If total revenue is $1000 and total cost is $1500, the total profit is $500.

Break-Even Analysis

Break-even point is where **total revenue equals total cost**:

$$R(x) = C(x)$$

Example: If total revenue is $1000 and total cost is $1000, the break-even point is 100 units.

Understanding Break-Even Analysis

- At this point, **profit is zero**.
- Businesses use break-even analysis to determine the minimum sales required to **cover costs**.

Step 1: Defining the Functions

Given the **total revenue** and **total cost functions**, we define:

$$\begin{aligned} R(x) &= 20x \quad \text{(Total Revenue)} \\ C(x) &= 5x + 100 \quad \text{(Total Cost)} \\ P(x) &= R(x) - C(x) \quad \text{(Profit)} \end{aligned}$$

where:

- (x) = **Number of units sold.**
- (R(x)) = **Revenue from selling (x) units at $20 per unit.**
- (C(x)) = **Fixed cost of $100 plus $5 per unit produced.**
- (P(x)) = **Profit function (Revenue - Cost).**

Step 2: Finding the Break-Even Point

Break-even occurs when:

$$R(x) = C(x)$$

$$20x = 5x + 100$$

Solving for (x):

$$20x - 5x = 100$$

$$15x = 100$$

$$x = \frac{100}{15} = 6.67$$

Break-even point is at (x = 6.67) units.

Step 3: Computing the Profit

$$P(x) = R(x) - C(x)$$

Substituting (x = 6.67):

$$P(6.67) = 20(6.67) - (5(6.67) + 100)$$

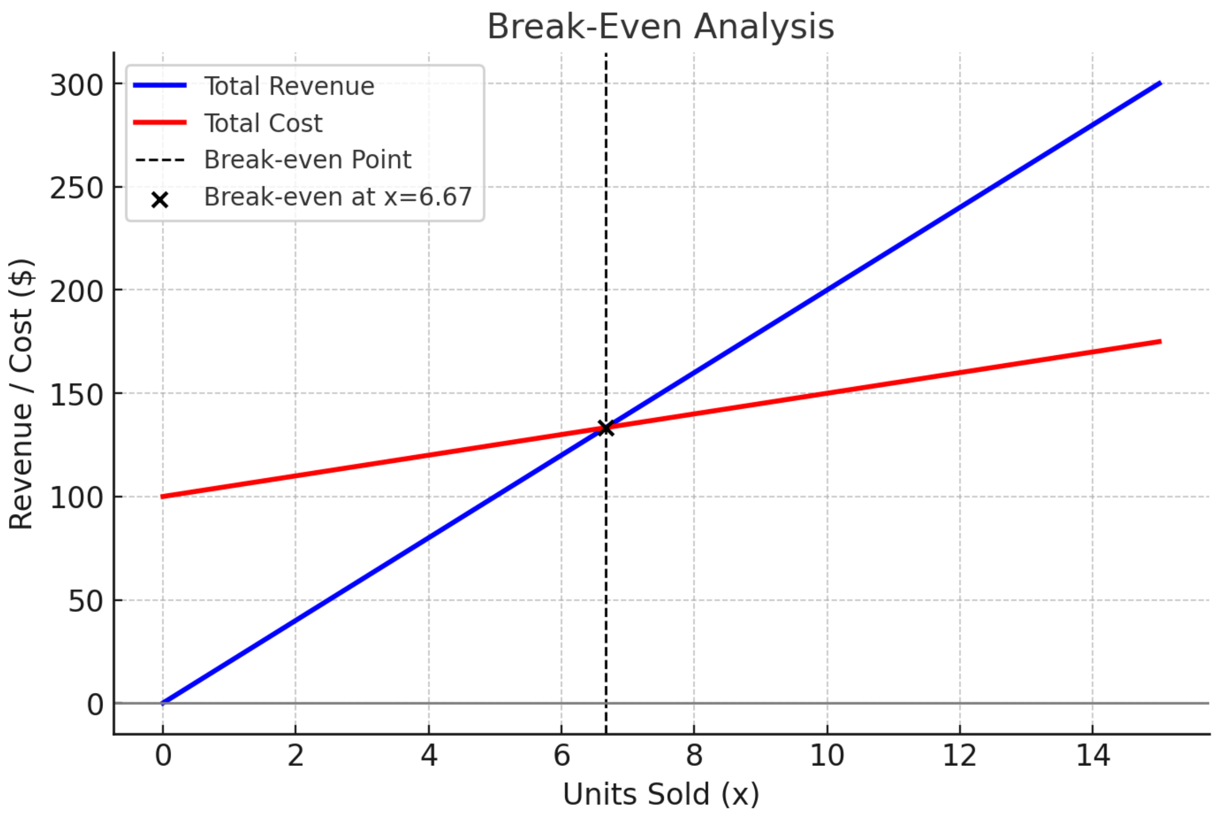
$$= 133.33 - (33.33 + 100)$$

$$= 133.33 - 133.33 = 0$$

Profit is zero at break-even point, as expected!

Step 4: Graphing the Functions

Let's **graph the revenue, cost, and profit functions** to visualize the relationships.



Your Turn: Break-Even Analysis

Example Problem

Total Revenue Function: $R(x) = 10x$. **Total Cost Function:** $C(x) = 5x + 50$. **Profit Function:** $P(x) = R(x) - C(x)$.

Step 1: Find the **break-even point**.

Step 2: Calculate the **profit** at the break-even point.

Step 3: Graph the **revenue, cost, and profit functions**.

Summary

1. **Quadratic functions** are essential in economics for analyzing cost, revenue, and profit functions.

2. **Solving quadratic equations** helps find optimal solutions in business and economics.

3. **Revenue, cost, and profit functions** are crucial for decision-making and break-even analysis.

4. **Graphical analysis** helps visualize relationships between variables.

Math is powerful—and fun!

Any QUESTIONS?

Thank you for your attention!

Next Class

- (Mar 21) Indices and Logarithms (2.3), Exponential and Natural Log Functions (2.4)