Mathematical Methods for International Commerce

Week 7/2: The Derivative of the Exponential and Natural Log Functions (4.8)

Iegor Vyshnevskyi, Ph.D.

Sogang University

April 18, 2025

Agenda

1. The Derivative of the Exponential and Natural Log Functions (4.8)
2. Individual Activity: Solve the Problems!
3. Midterm Exam Review

1. The Derivative of the Exponential and Natural Log Functions (4.8)

Why It Matters: Economics & Finance Perspective

Exponential and logarithmic functions are foundational in economics and finance:

- They model **continuous growth and decay** – essential for understanding investment returns, population growth, inflation, and depreciation.
- The exponential function captures **compound interest**, while logarithms are key for solving equations involving **growth rates**.
- Derivatives of these functions help economists and analysts assess **rates of change** – such as marginal returns, discounting future values, or analyzing production over time.

Understanding how to **differentiate** exponential and logarithmic functions equips you with tools to interpret trends, optimize financial decisions, and make precise economic forecasts.

Learning Objectives

Section 4.8 – Derivatives of Exponential and Logarithmic Functions

- Differentiate the **exponential function**
- Differentiate the **natural logarithm function**
- Use **chain, product and quotient rules** with these functions
- Apply exponential models to **real-world economic problems**

Basic Rules of Differentiation (Exponential & Logarithmic Functions)

Exponential Function:

The exponential function grows **proportionally to its value**.

$$\frac{d}{dx}e^x = e^x$$

This is unique: the derivative of e^x is itself!

Natural Logarithm:

Logarithms are used to reverse exponential growth.

$$\frac{d}{dx}\ln(x) = \frac{1}{x}, \quad x > 0$$

Basic Rules of Differentiation (Exponential & Logarithmic Functions) (cont'd)

Chain Rule:

Use when differentiating **functions within functions**.

- $$\frac{d}{dx}e^{f(x)} = f'(x) \cdot e^{f(x)}$$
- $$\frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

These rules are essential in **economic models** involving growth, decay, and elasticity.

Examples: Basic Differentiation

1. Differentiate:

$$f(x) = e^{3x}$$

Solution:

$$f'(x) = 3e^{3x}$$

- Use chain rule: derivative of exponent times original exponential

2. Differentiate:

$$g(x) = \ln(5x^2 + 1)$$

Solution:

$$g'(x) = \frac{10x}{5x^2 + 1}$$

- Chain rule: differentiate inside function $5x^2 + 1 \rightarrow 10x$

Tip: Exponentials often model compounded growth, and logs appear in utility, elasticity, and returns.

Product & Quotient Rule in Action

Example 1: Product Rule

$$y = x^2 e^x$$

Apply:

$$\frac{d}{dx}(uv) = u'v + uv'$$

- $u = x^2, v = e^x$
- Derivative:

$$\frac{dy}{dx} = 2xe^x + x^2e^x = e^x(2x + x^2)$$

Use when both parts involve variables (common in cost/revenue products).

Product & Quotient Rule in Action (cont'd)

Example 2: Quotient Rule

$$y = \frac{\ln(x)}{x^2}$$

Apply:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

- $u = \ln(x), v = x^2$
- Derivative:

$$\frac{dy}{dx} = \frac{1/x \cdot x^2 - \ln(x) \cdot 2x}{x^4} = \frac{x - 2x\ln(x)}{x^4}$$

Use when differentiating ratios like **marginal utility/cost per unit**.

Application: Continuous Revenue Growth

Let the revenue function be:

$$R(t) = 5000e^{0.05t}$$

This implies revenue grows **at a continuous rate of 5%** per time unit.

Find the rate of change of revenue:

$$R'(t) = 5000 \cdot 0.05e^{0.05t} = 250e^{0.05t}$$

Interpretation:

- The **growth rate is proportional** to the current revenue.
- Common in modeling investment returns, inflation, or GDP.

Application: Elasticity of Growth

A common growth model:

$$Q(t) = Ae^{rt}$$

Where:

- $Q(t)$ is output (e.g., capital, population)
- A is initial value
- r is the growth rate

Application: Elasticity of Growth (cont'd)

Find:

- $$\frac{dQ}{dt} = rAe^{rt} = rQ(t)$$

- Elasticity of growth:

$$E = \frac{dQ/dt}{Q} = r$$

Interpretation: Elasticity is **constant** in exponential growth → % change in Q for 1% change in time.

Used in modeling:

- Population growth
- Compound interest
- Inflation and real returns

Summary

- Exponential and log functions are common in **growth, interest, decay** models
- Chain rule is crucial when inside other functions
- Product and quotient rules still apply!

> Exponentials capture **compounding**; logs help **linearize** growth patterns.

2. Individual Activity: Solve the Problems!

Practice Problems

1. Differentiate:

    - (a) $f(x) = e^{2x^2}$
    - (b) $g(x) = \ln(x^2 + 1)$

2. Use product rule:

    - (a) $y = xe^{3x}$

3. Use quotient rule:

    - (a) $y = \frac{e^x}{x^3}$

4. Application:

    - Revenue grows as $R(t) = 12000e^{0.04t}$. Find $R'(t)$ and interpret.

3. Midterm Exam Review

Midterm Exam Review

Coverage:

- Basic Algebra & Solving Equations (1.1–1.4)
- Supply and Demand, Transposition, National Income (1.5–1.7)
- Quadratic Functions, Revenue & Profit (2.1–2.2)
- Indices, Logs, Exponentials (2.3–2.4)
- Percentages, Compound Interest (3.1–3.2)
- Geometric Series & Investment Appraisal (3.3–3.4)
- Derivatives & Marginal Functions (4.1–4.3)
- Chain/Product/Quotient Rules, Elasticity (4.4–4.5)
- Optimization (4.6–4.7), Derivatives of Exp/Log (4.8)

Also Review: Lecture slides, Homework #1, Quiz #1

What You Should Be Able to Do

- Simplify, factor, solve linear & quadratic equations
- Graph demand, supply, quadratic, exponential functions
- Interpret elasticity, marginal cost/revenue, and APL/MPL
- Apply formulas for compound interest, present value, annuities
- Differentiate power, exponential, log functions
- Use derivative tests to find and classify stationary points

Algebra & Quadratics

- Factor and solve:
  - $x^2 - 5x + 6 = 0$
  - $2x^2 - 3x - 2 = 0$
- Graph: $y = x^2 - 4x + 3$
- Use quadratic formula

Supply & Demand, Income Determination

- Sketch and solve:
  - $Q_d = 100 - 5P, Q_s = 20 + 3P$
- Find equilibrium
- National income:
  - $Y = C_0 + cY + I + G$, solve for Y

Logs, Indices, Exponentials

- Simplify:
  - $2^3 \times 2^2$
  - $\ln(e^3)$
- Solve: $3^x = 81 \Rightarrow x = 4$
- Evaluate: $(1.05)^5$

Finance Applications

- Calculate:
  - % increase/decrease
  - Compound value: $A = P(1 + r)^n$
  - NPV: $\mathrm{NPV} = \sum \frac{C_t}{(1+r)^t} - C_0$
  - Loan instalment (annuity formula)

Derivatives & Optimization

- Power rule: $\frac{d}{dx}x^n$
- Product & chain rule
- Find:
  - Marginal Revenue/Cost
  - When MR = MC
  - Maximize profit using first and second derivative test

Elasticity

- Arc elasticity: $E_d = \frac{\Delta Q / \text{avg } Q}{\Delta P / \text{avg } P}$
- Point elasticity: $E_d = \frac{dQ}{dP} \cdot \frac{P}{Q}$
- Know how elasticity affects revenue

Practice & Prep

Review your:

- Lecture slides and solved examples
- Homework #1
- Quiz #1

Follow exam instructions! No phones allowed. Only one A4 double-sided handwritten cheat sheet (formulas only) is permitted. The time limit will be strictly enforced.

Time to shine – you've got this!

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (April 23) No Class (Midterm Exam Week)
- (April 25) Mid term exam (in class):
  - Review all material from the beginning of the semester
  - Pay attention to the examples in the slides, HW #1, Quiz #1 and the exercises in the textbook