Mathematical Methods for International Commerce

Week 5/1: The Derivative of Functions and Rules of Differentiation

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March 28, 2025

Why This Topic Matters

In economics and business, we often want to understand how **one variable responds to changes in another**. This is where **derivatives** come in:

- Derivatives measure **rates of change** — critical for cost, revenue, profit, and utility analysis.
- Used to find **slopes**, **marginal values**, **optimization**, and **elasticities**.
- Widely applied in **demand/supply**, **investment analysis**, and **business decision-making**.

Simply put, derivatives help us understand how small changes in one variable can lead to changes in another.

> "Derivatives are the language of marginal thinking in economics."

Section 4.1: The Derivative of a Function

What is a Derivative?

The **derivative** of a function at a point gives the **slope of the tangent line** — how much the function is changing at that point.

Notation:

- $f'(x)$
- $\frac{dy}{dx}$

Interpretation:

If $f(x)$ is output, then $f'(x)$ is the **marginal change** in output for a unit change in x.

Slope of a Function

The slope of a function at a point is the **instantaneous rate of change** of the function at that point.

- It tells us how steep the function is at that point.
- A **positive slope** means the function is increasing, while a **negative slope** means it is decreasing.
- A **zero slope** means the function is flat (no change).
- The slope can be thought of as the **rise over run**: how much the function rises (or falls) for a given change in x.
- For linear functions, the slope is constant and can be calculated using the formula:

$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

- The slope is **not constant** for non-linear functions, but we can approximate it using the derivative.

Slope from Two Points

For a straight line:

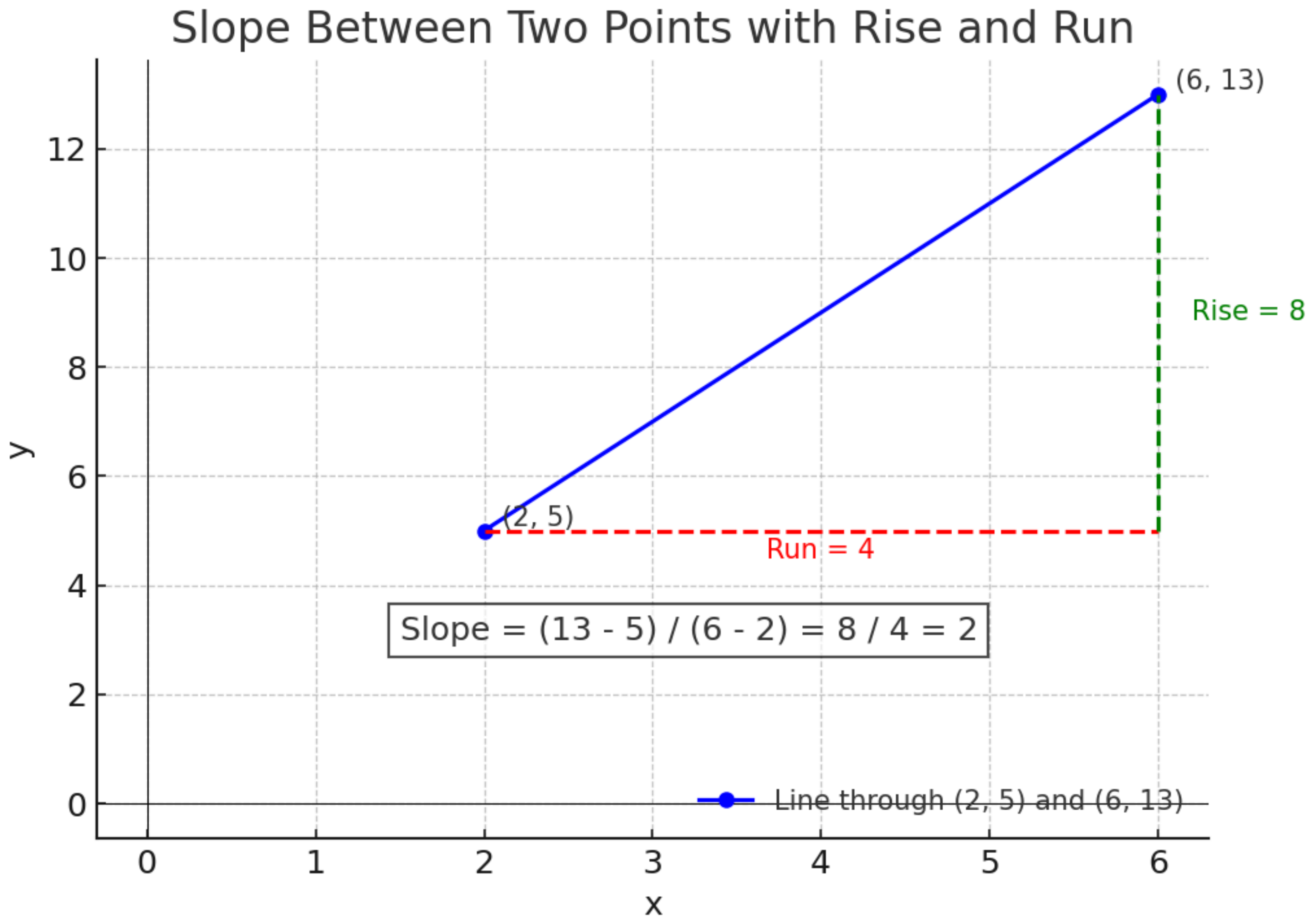
$$\text{slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example: Find the slope between (2, 5) and (6, 13):

$$\frac{13 - 5}{6 - 2} = \frac{8}{4} = 2$$

Interpretation: The function rises by 2 units for every 1 unit increase in x.

Illustration

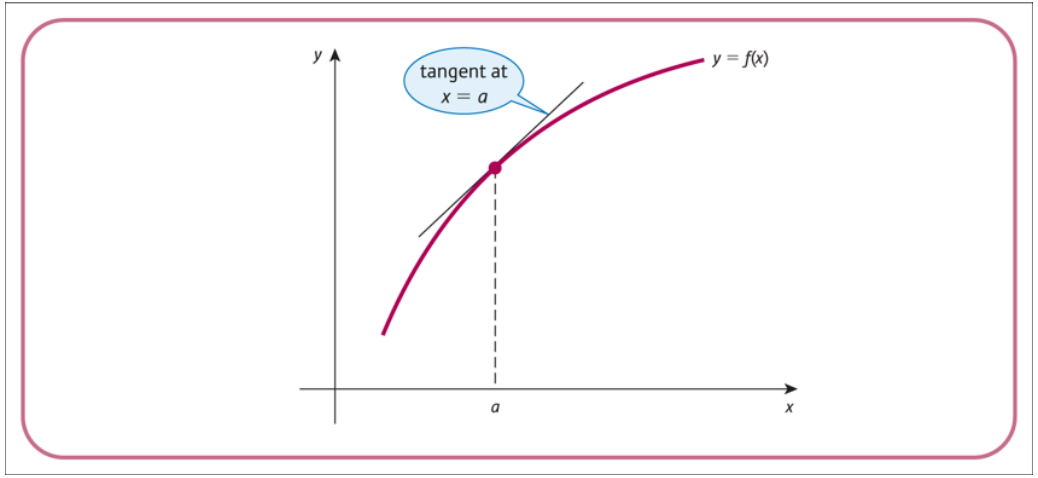


Visual Meaning of Derivative

To **estimate** the slope of a curve at a point:

- Draw the **tangent line** at that point.
- The slope of the tangent ≈ $f'(x)$

The steeper the curve, the larger the derivative.



Differentiate Power Functions

Power rule:

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

- Example:

$$\frac{d}{dx}(x^3) = 3x^2$$

  - Solution:

$$\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$$

Differentiate Power Functions (continued)

- Example:

$$\frac{d}{dx}(5x^4) = 20x^3$$

  - Solution:

$$\frac{d}{dx}(5x^4) = 5 \cdot 4x^{4-1} = 20x^3$$

- Example:

$$\frac{d}{dx}(\frac{1}{x}) = \frac{d}{dx}(x^{-1}) = -x^{-2}$$

  - Solution:

$$\frac{d}{dx}(x^{-1}) = -1 \cdot x^{-2} = -\frac{1}{x^2}$$

Your turn! Differentiate

- Find $\frac{d}{dx}(x^5)$
- Find $\frac{d}{dx}(x^{-3})$
- Differentiate $f(x) = 2x^3 + 3x^2 - 4x + 1$
- Differentiate $f(x) = 3x^2 + 4x - 5$

Section 4.2: Rules of Differentiation

Constant Rule

If c is a constant:

$$\frac{d}{dx}(c) = 0$$

- $\frac{d}{dx}(7) = 0$
- $\frac{d}{dx}(-12) = 0$
- Example:

$$\frac{d}{dx}(3) = 0$$

Constant Multiple Rule

$$\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$$

- $\frac{d}{dx}[5x^3] = 5 \cdot 3x^2 = 15x^2$
- Example:

$$\frac{d}{dx}[7x^2] = 7 \cdot 2x^{2-1} = 14x$$

Sum and Difference Rule

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$$

- Example:

$$\frac{d}{dx}(x^2 + 3x) = 2x + 3$$

- Example:

$$\frac{d}{dx}(x^3 - x^2) = 3x^2 - 2x$$

Second-Order Derivatives

The **second derivative** tells us how the **first derivative** (i.e., the slope) is changing. It gives insight into the **curvature** or **concavity** of the function:

$$\frac{d^2y}{dx^2} = f''(x)$$

Interpretation:

- If $f''(x) > 0$: the function is **concave upward** — the slope is increasing, like a smile
- If $f''(x) < 0$: the function is **concave downward** — the slope is decreasing, like a frown

Think of it as the **slope of the slope** — how fast the rate of change is changing.

Example:

$$\frac{d^2}{dx^2}(x^3) = \frac{d}{dx}(3x^2) = 6x$$

This means the concavity of x^3 depends on x: when $x > 0$, it's concave up; when $x < 0$, it's concave down.

Practice Problems

1. Find $\frac{d}{dx}(4x^3 - 2x + 7)$
2. Differentiate $5x^2 + 6x - 1$ and interpret the result.
3. Compute $\frac{d}{dx}(x^{-2})$
4. Find $\frac{d^2}{dx^2}(x^3 - 2x^2 + x)$
5. Given $f(x) = x^2 + 3x$, find and sketch $f'(x)$

Example from Finance: Marginal Cost and Profit

A company produces and sells a product. Its **cost** and **revenue** functions (in dollars) are given by:

$$C(x) = 200 + 10x + 0.5x^2 \quad \text{(Cost Function)}$$

$$R(x) = 40x \quad \text{(Revenue Function)}$$

The **profit function** is:

$$\Pi(x) = R(x) - C(x) = 40x - (200 + 10x + 0.5x^2) = -0.5x^2 + 30x - 200$$

Find:

1. The **marginal profit:** $\Pi'(x)$
2. The **rate at which marginal profit changes:** $\Pi''(x)$
3. The production level that **maximizes profit**.
4. The **maximum profit** at that level of production.

Example from Finance: Marginal Cost and Profit (continued)

Solution:

1. $\Pi'(x) = \frac{d}{dx}(-0.5x^2 + 30x - 200) = -x + 30$
2. $\Pi''(x) = \frac{d}{dx}(-x + 30) = -1$

- Since $\Pi''(x) < 0$, the profit function is **concave down**, meaning the maximum exists.

1. Set $\Pi'(x) = 0 \Rightarrow -x + 30 = 0 \Rightarrow x = 30$

So, the company **maximizes profit at 30 units of output**.

Answers

1. Marginal profit: $\Pi'(x) = -x + 30$
2. Rate of change of marginal profit: $\Pi''(x) = -1$ (constant, negative)
3. Production level that maximizes profit: $x = 30$
4. Maximum profit:
   $\Pi(30) = R(30) - C(30) = 40(30) - (200 + 10(30) + 0.5(30^2)) = 1200 - 200 - 300 - 450 = 25$

Your Turn! Marginal Profit Application

A startup produces custom-designed notebooks. Its **cost** and **revenue** functions are:

$$C(x) = 150 + 12x + 0.3x^2 \quad \text{(Cost Function)}$$

$$R(x) = 36x \quad \text{(Revenue Function)}$$

The **profit function** is:

$$\Pi(x) = R(x) - C(x)$$

Tasks for You

1. **Find the profit function** $\Pi(x)$.
2. **Differentiate** to get the **marginal profit function** $\Pi'(x)$.
3. **Find the second derivative** $\Pi''(x)$ and interpret it.
4. **Determine** the level of production that **maximizes profit**.
5. **Calculate the maximum profit** at that level.

- Try to solve step-by-step using the rules from today's lecture.
  We will discuss the solution together after 10 minutes!

Summary

- Derivatives = **instantaneous rate of change**
- Power rule and linearity make differentiation efficient
- Second derivatives describe curvature and behavior

Any QUESTIONS?

Thank you for your attention!

Next Class: Quiz #1

- Please review **Homework #1**, all **in-class examples**, and **relevant textbook problems**.
- Quiz will be **paper-based** and will last **70 - 80 minutes**.
- **No electronic devices allowed**, except for a **basic calculator**.