Mathematical Methods for International Commerce

Week 9/2: Partial Elasticity and Marginal Functions (5.2)

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Why It Matters in Economics & Business

In real economic environments, multiple inputs and variables affect production, utility, and costs.

Understanding **how sensitive an output is to one specific input** (holding others constant) is key to making efficient decisions.

- Partial elasticities tell us **how responsive** an outcome is to one variable
- Marginal rates (MRCS and MRTS) help understand **trade-offs** in consumption and production
- Euler’s theorem gives a neat characterization of **returns to scale** for homogeneous production functions

This is foundational for microeconomic theory, cost analysis, and optimization problems.

Let's begin!

Agenda

1. Functions of Several Variables (5.1)
2. Class Activity

1. Functions of Several Variables (5.1)

Learning Objectives

- Calculate **partial elasticities**
- Calculate **marginal utilities** and **marginal products**
- Calculate the **marginal rate of commodity substitution** (MRCS)
- Calculate the **marginal rate of technical substitution** (MRTS)
- Understand **Euler's theorem** for homogeneous functions

What is Partial Elasticity?

- **Partial elasticity** measures the percentage change in a function (e.g., output, utility) when **one variable changes**, holding the **other constant**.

Formula:

If $z = f(x, y)$, then:

$$E_x = \frac{\partial z}{\partial x} \cdot \frac{x}{z}, \quad E_y = \frac{\partial z}{\partial y} \cdot \frac{y}{z}$$

Example: Partial Elasticities

Let $Q = x^{0.5}y^{0.5}$

- $\frac{\partial Q}{\partial x} = 0.5x^{-0.5}y^{0.5}$
- $\frac{\partial Q}{\partial y} = 0.5x^{0.5}y^{-0.5}$

Then:

$$E_x = 0.5x^{-0.5}y^{0.5} \cdot \frac{x}{x^{0.5}y^{0.5}} = 0.5, \quad E_y = 0.5$$

Interpretation: **1% increase** in x or y increases Q by **0.5%**.

Marginal Utilities

If $U = f(x, y)$ is a utility function:

- $MU_x = \frac{\partial U}{\partial x}$
- $MU_y = \frac{\partial U}{\partial y}$

These represent the **extra utility** from consuming **one more unit** of good x or y.

Example: Marginal Utility

Let $U(x, y) = 2x + 3y$.

- $MU_x = 2$
- $MU_y = 3$

Interpretation: Utility increases by **2 units** for each extra unit of x, **3 units** for y.

Marginal Product

If $Q = f(L, K)$ is a production function:

- $MP_L = \frac{\partial Q}{\partial L}$
- $MP_K = \frac{\partial Q}{\partial K}$

These are used in **firm decisions** about labor and capital inputs.

Example: Marginal Product

Let $Q = 10L^{0.5}K^{0.5}$.

- $MP_L = 5L^{-0.5}K^{0.5}$
- $MP_K = 5L^{0.5}K^{-0.5}$

At $L = 4$, $K = 9$:

- $MP_L = 5 \cdot \frac{1}{2} \cdot 3 = 7.5$
- $MP_K = 5 \cdot 2 \cdot \frac{1}{3} = 3.33$

Marginal Rate of Commodity Substitution (MRCS)

Rate a consumer substitutes x for y **keeping utility constant**:

$$MRCS = \left| \frac{MU_x}{MU_y} \right|$$

Example: MRCS

Let $U(x, y) = x^{0.5}y^{0.5}$

- $MU_x = 0.5x^{-0.5}y^{0.5}$
- $MU_y = 0.5x^{0.5}y^{-0.5}$

Then:

$$MRCS = \left|\frac{MU_x}{MU_y}\right| = \left|\frac{y}{x}\right|$$

Interpretation: At $(x, y) = (2, 4)$, $MRCS = 2$. The consumer is willing to give up 2 units of x for 1 more unit of y.

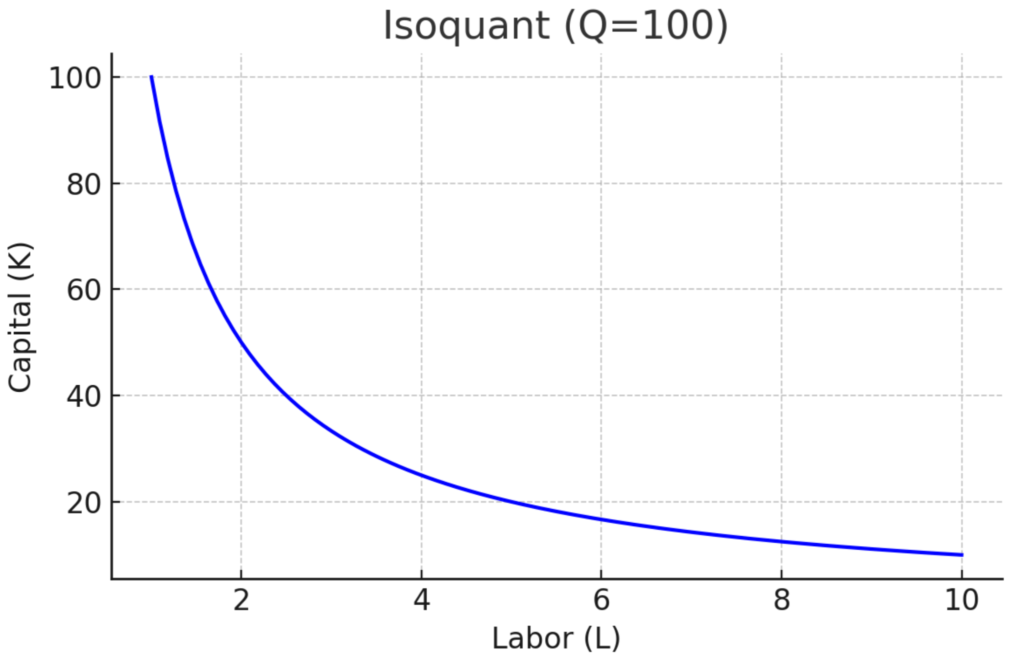
Marginal Rate of Technical Substitution (MRTS)

In production:

$$MRTS = \left| \frac{MP_L}{MP_K} \right|$$

This tells how much capital can be replaced by labor without changing output.

Visual: Isoquant and MRTS



It shows the **trade-off** between labor and capital while keeping output constant. The slope of the isoquant is the MRTS. The steeper the slope, the more labor can be substituted for capital.

Euler's Theorem

Euler's Theorem states that for a function $f(x, y)$ that is **homogeneous of degree n**:

- If $f(tx, ty) = t^n f(x, y)$ for all $t > 0$, then:

$$f(x, y) = \frac{\partial f}{\partial x} \cdot x + \frac{\partial f}{\partial y} \cdot y$$

Used in economic models with **returns to scale**.

Practice Problems

1. Let $Q = x^{0.6}y^{0.4}$. Find E_x, E_y
2. For $U = 3x + 4y$, compute MU_x, MU_y, and MRCS
3. Let $Q = L^{0.7}K^{0.3}$, compute MP_L, MP_K, and MRTS at $L = 2$, $K = 3$
4. Verify Euler's theorem for $f(x, y) = x^2 + y^2$

Problem # 4

Verify Euler's theorem for $f(x,y) = x^2 + y^2$

We'll verify Euler's Theorem for the function:

$$f(x,y) = x^2 + y^2$$

Step 1: Homogeneity Check

Substitute tx and ty into f:

$$f(tx,ty) = (tx)^2 + (ty)^2 = t^2x^2 + t^2y^2 = t^2(x^2 + y^2)$$

So $f(x,y)$ is homogeneous of **degree 2**.

Step 2: Euler's Theorem

Euler's Theorem states:

If f is homogeneous of degree n, then:

$$f(x,y) = x \cdot f_x + y \cdot f_y$$

For our case, $n = 2$.

Problem # 4 (continued)

Step 3: Find the Partial Derivatives

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$$

Substitute into Euler's identity:

$$x \cdot 2x + y \cdot 2y = 2x^2 + 2y^2 = 2(x^2 + y^2) = 2f(x, y)$$

Verified! Euler's Theorem holds.

Interpretation: Returns to Scale

Since the function is homogeneous of **degree 2**, it exhibits:

Increasing Returns to Scale

> Doubling both inputs will **quadruple** the output.

This concept is used widely in:

- Production theory
- Utility and cost functions
- Growth models

Summary

- Partial elasticities show **percentage responsiveness** to one variable
- Marginal utility/product: **sensitivity of outcome** to small changes
- MRCS and MRTS reflect **substitution** between inputs or goods
- Euler's theorem links **homogeneity and returns to scale**

2. Group Activity: Marginal Thinking in Real Life

Group Activity: Marginal Thinking in Real Life

Instructions

- Form **4 groups** of **4 students**.
- Each group receives a different scenario.
- Use concepts from **partial elasticity**, **marginal utility/product**, and **MRTS/MRCS** to answer.
- Prepare a **2-minute explanation**.

Group Scenarios

Group 1 – Production Line

A factory uses labor and capital to produce widgets:

- $Q = L^{0.6}K^{0.4}$
- Evaluate MP_L and MP_K at $L = 5, K = 5$
- What is the MRTS? What does it mean for the factory?

Group 2 – Consumer Behavior

A consumer has utility function $U(x, y) = 2x + 3y$

- Find MU_x, MU_y, and MRCS
- If the consumer gives up 1 unit of y, how much x do they need to maintain utility?

Group Scenarios (continued)

Group 3 – Elastic Demand

A firm's revenue depends on two prices:

- $R = p_1^{0.7} p_2^{0.3}$
- Compute partial elasticities with respect to p_1 and p_2
- Which price affects revenue more? How should the firm respond?

Group 4 – Policy Maker

A government economist analyzes GDP:

- $Y = C^{0.8} I^{0.2}$
- Compute E_C, E_I
- If investment falls, can consumption make up for it?

Debrief Questions (for all groups)

1. Which input had the largest marginal effect?
2. Were the substitution rates intuitive?
3. How can these results help in **decision-making**?

Any QUESTIONS?

Thank you for your attention!

Next Classes

- (May 7) Comparative Statics (5.3)