Some Bayesian methods for inverse problems

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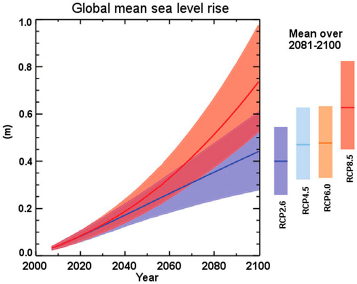
Motivation

Inverse problems & UQ in complex physical models



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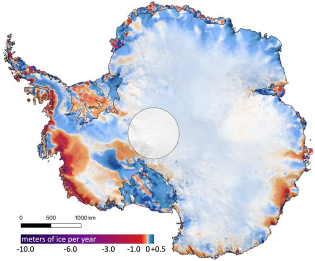

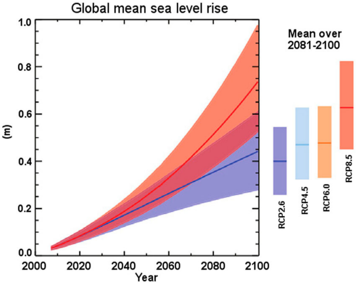

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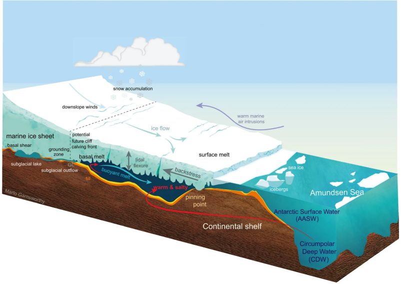

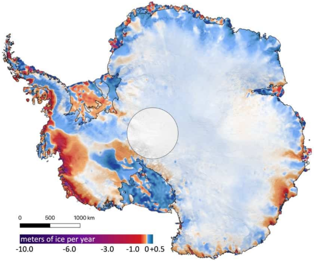

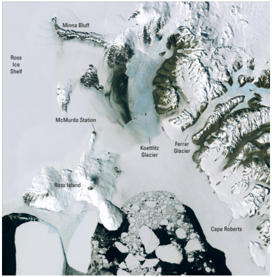

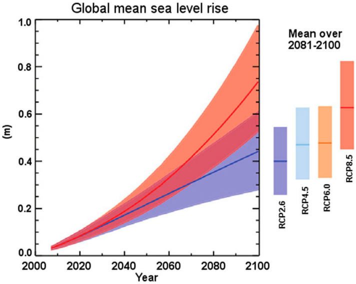

Images: BAS, NASA, B Smith et al/Science, IPCC

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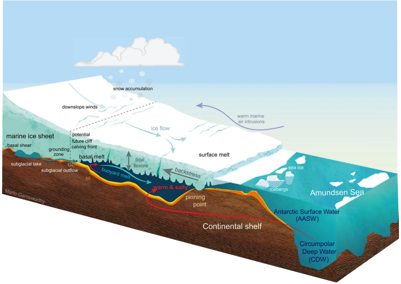

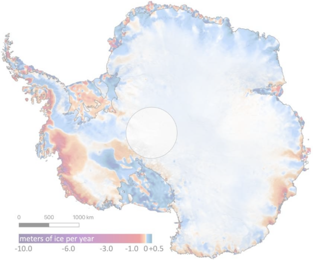

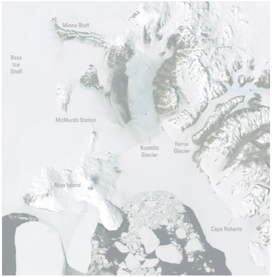

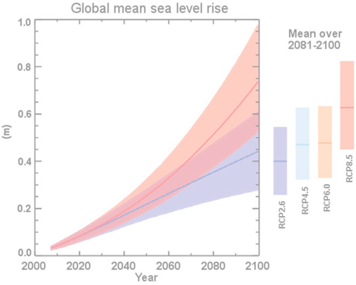

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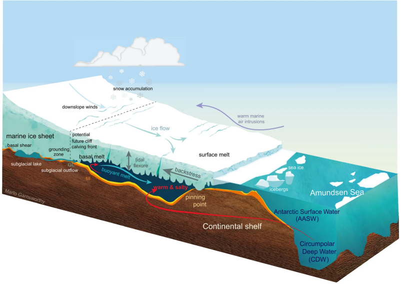

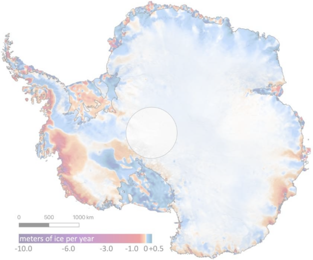

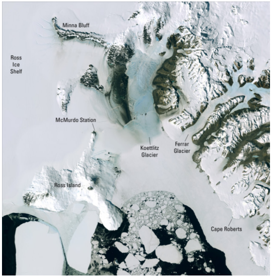

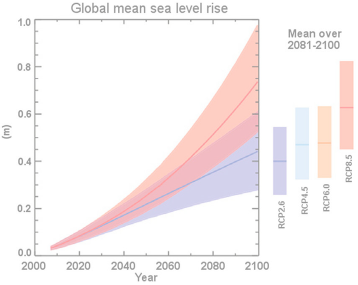

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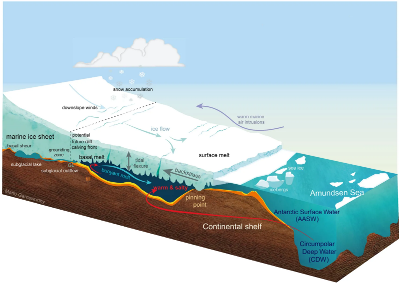

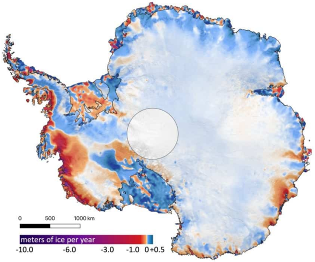

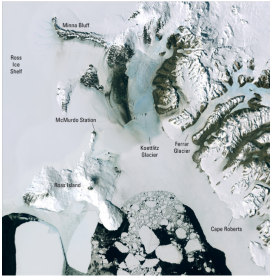

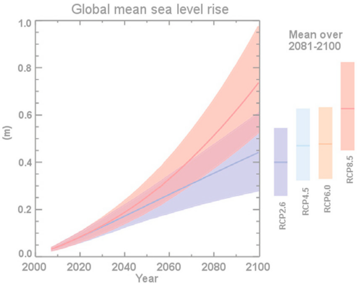

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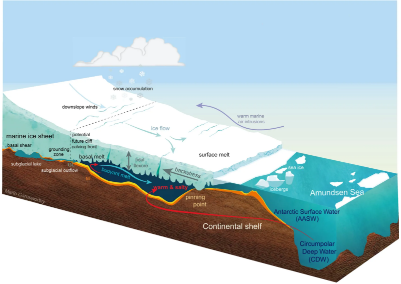

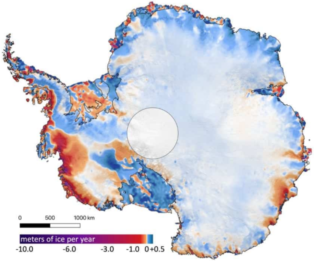

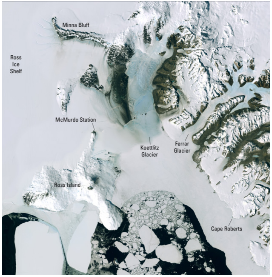

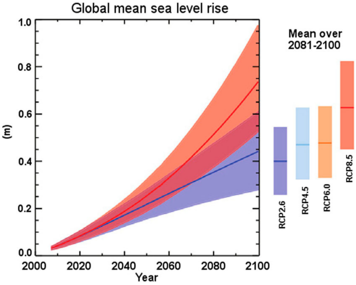

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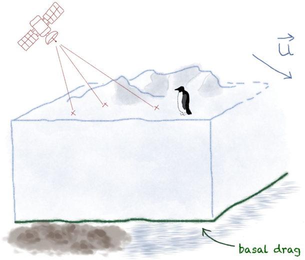
Inverse problems in ice sheet models



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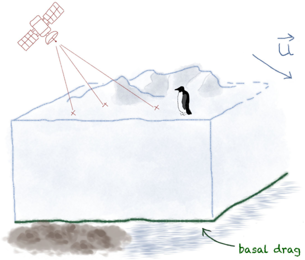

Stokes flow with Robin BC:

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\begin{aligned}
-\beta \Delta u + \nabla p &= f && \text{in } \mathcal{O} \\
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where f is forcing, β is viscocity, and θ is the unknown boundary parameter.

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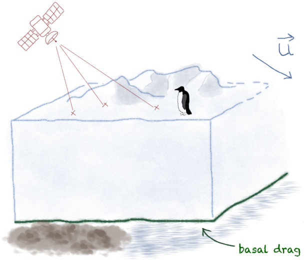

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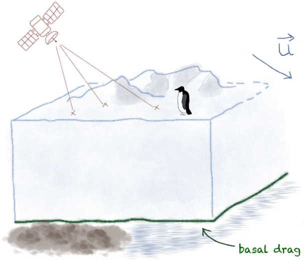

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$$Y_i = \mathcal{G}_\theta(X_i) + \varepsilon_i, \quad \varepsilon_i \overset{\text{iid}}{\sim} N(0,1)$$

with data $D_N = (Y_i, X_i)_{i=1}^N$.

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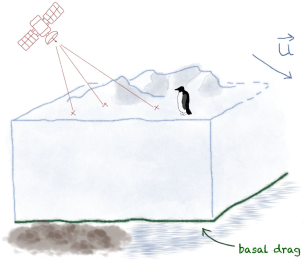

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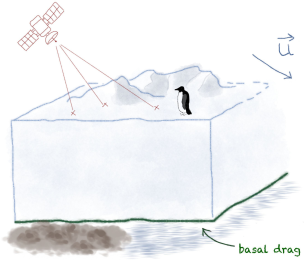

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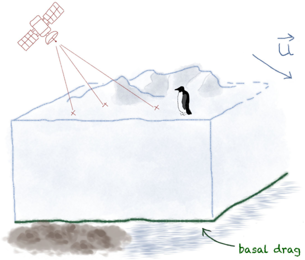

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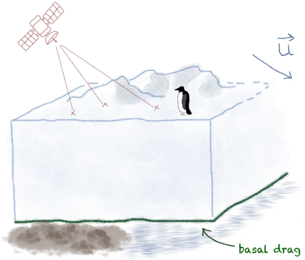

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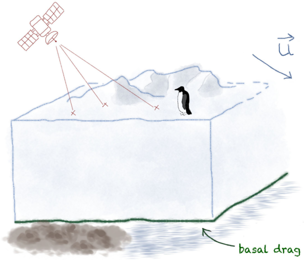

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- Solutions $\mathcal{G}_\theta$ for different θ available only through simulation.
- In general, D_N does not uniquely identify θ.
- $\mathcal{G}$ is 'smoothing'.

Ice sheet simulator / model / solver $G_\theta(x_i)$

$$G_\theta(x) = u$$

viscosity velocity

Observation model :

$$y_i = u_i + \varepsilon_i \qquad \varepsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma)$$

$$y_i = G_\theta(x_i) + \varepsilon_i$$

Regression problem!

Three Bayesian approaches

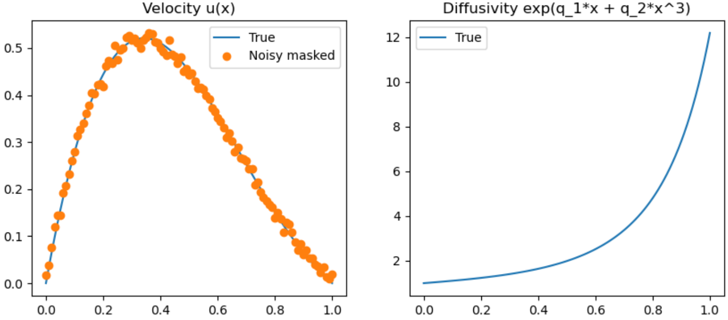
1. Bayesian approximate computation (ABC); likelihood-free inference
2. (Full) Bayesian inference
3. Bayesian optimisation (BO)

1D Poisson problem

Consider a simpler problem in 1D:

$$
\begin{aligned}
-\nabla \cdot \beta(x)\nabla u(x) &= f(x) && \text{in } \mathcal{O} \\
u(x) &= 0 && \text{on } \partial\mathcal{O}.
\end{aligned}
$$

Parameterise diffusivity as: $\beta(x) = exp(q_1 x + q_2 x^3)$.



I: Approximate Bayesian computation

for $i = 1$ **to** M **do**

Pick θ

Run $\mathcal{G}_\theta(X)$

Compare $u = \mathcal{G}_\theta(X_i)$ to observations y_i

Reject if error is large

end for

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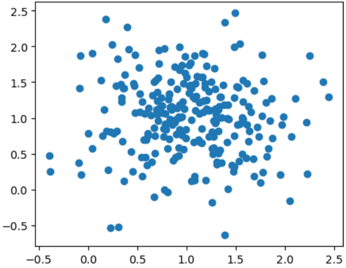

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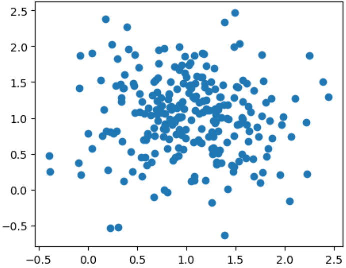
  Run $\mathcal{G}_\theta(X)$

  Compare $R = |\mathcal{G}_\theta(X_i) - y_i|^2$

  Accept if $R < e_{\text{threshold}}$

end for



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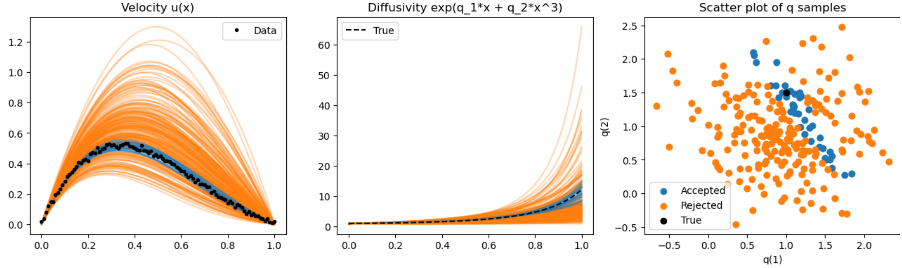
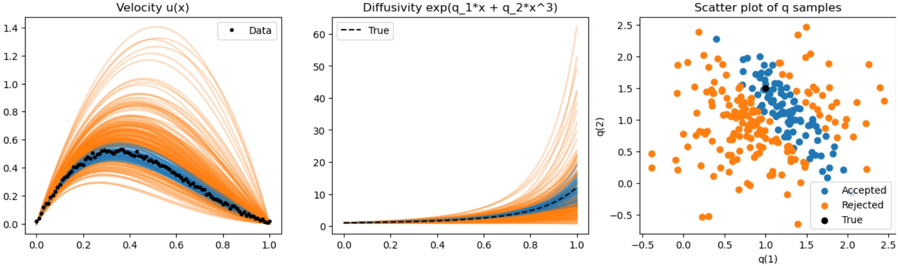


Approximate Bayesian computation: higher error threshold



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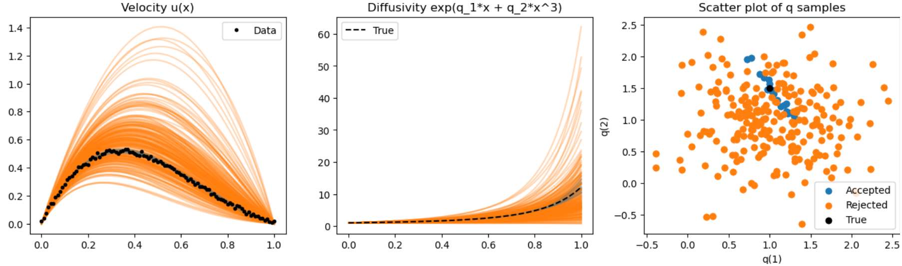
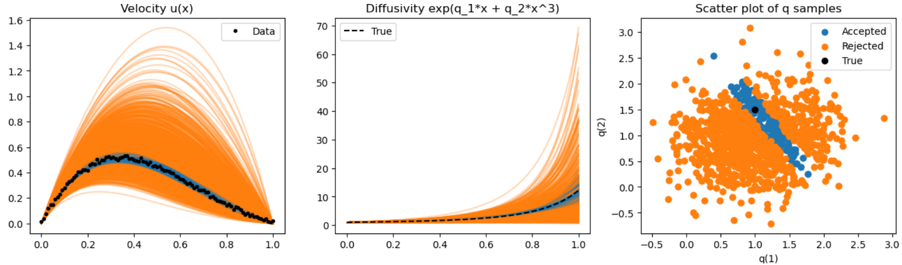


Approximate Bayesian computation: More samples



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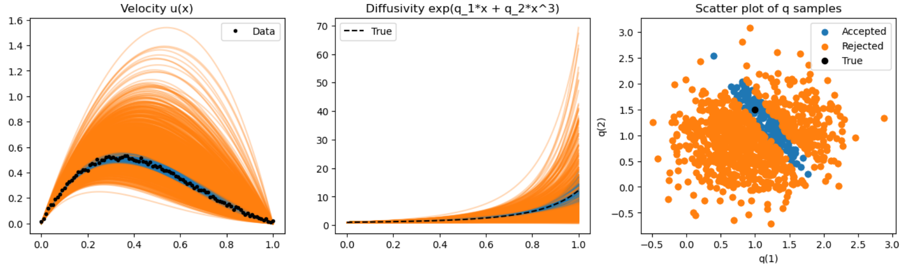

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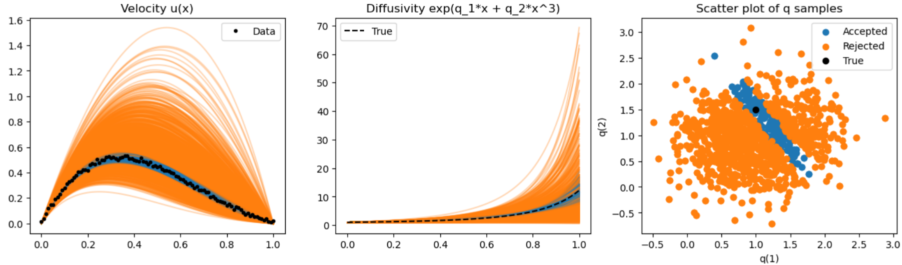

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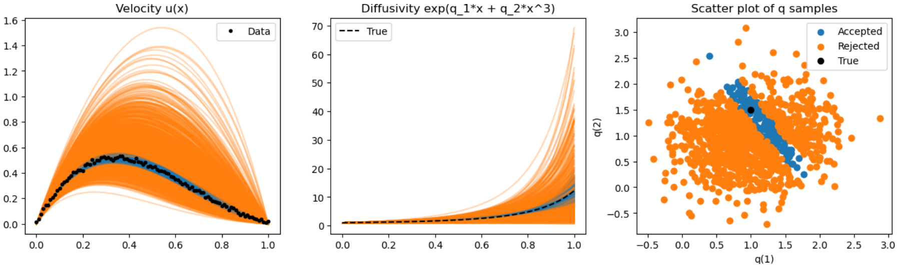

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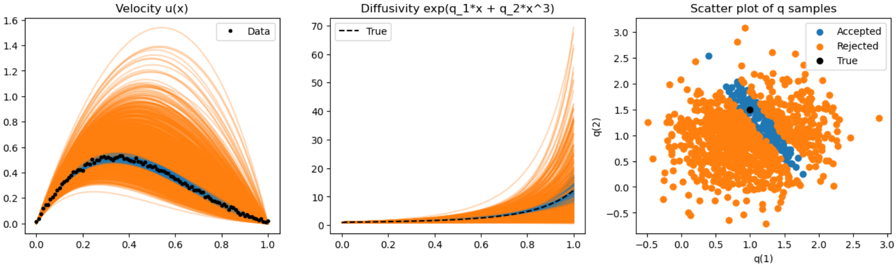

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- Approximately Bayesian

II: Bayesian inference

Recall Bayes' rule:

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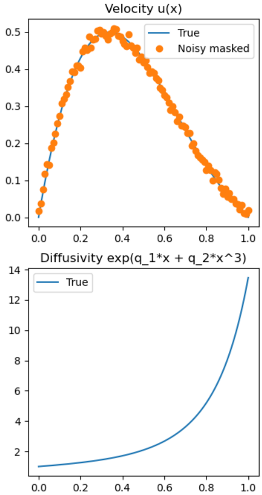
Inference in general:

- Sampling (Markov chain Monte Carlo)
- Variational inference

Metropolis-Hastings sampling for inference



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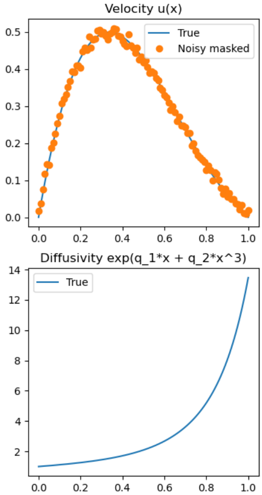
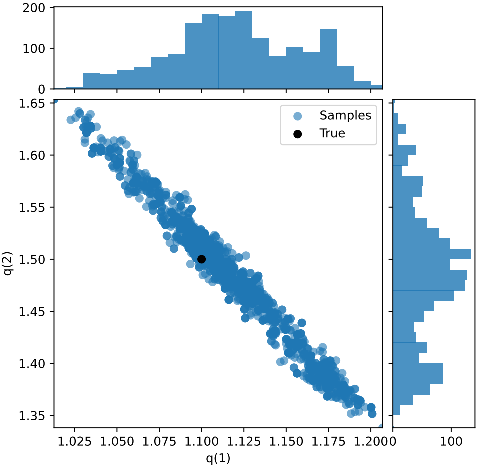
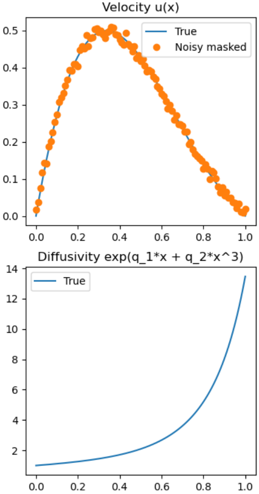
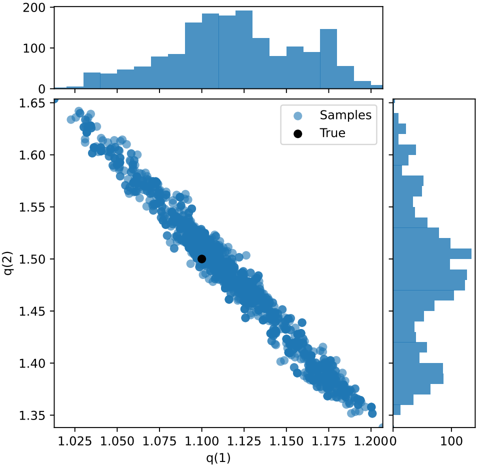




Metropolis-Hastings sampling for inference





Expensive!

III: Bayesian optimisation

- Global optimisation algorithm

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- Balances exploration and exploitation

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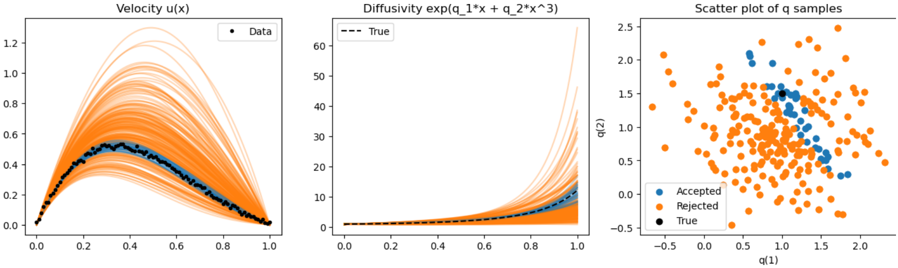
Recall, for approximate Bayesian computation:



III: Bayesian optimisation

Recall, we want θ that minimises the negative log-likelihood:

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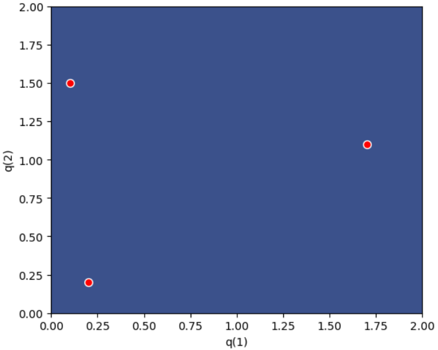

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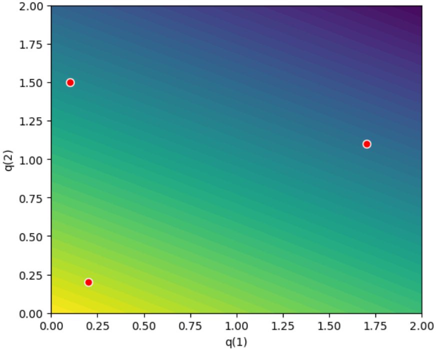

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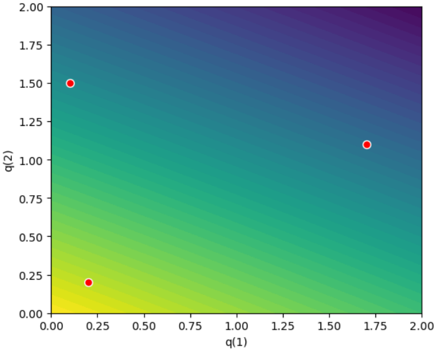

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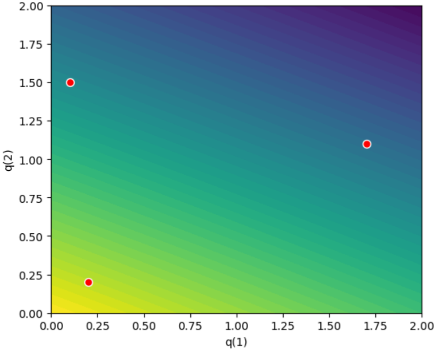

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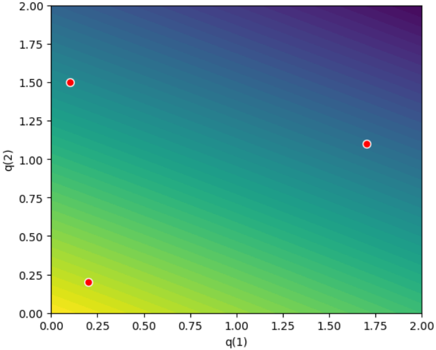

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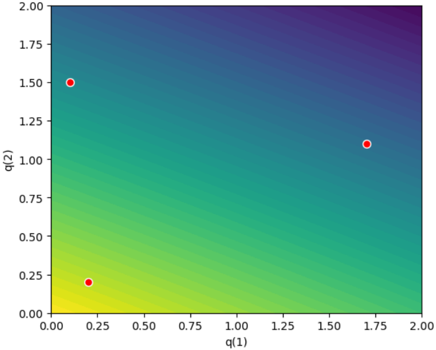

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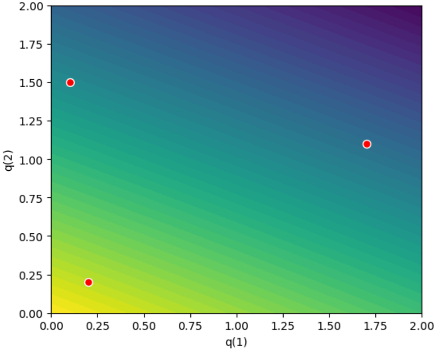

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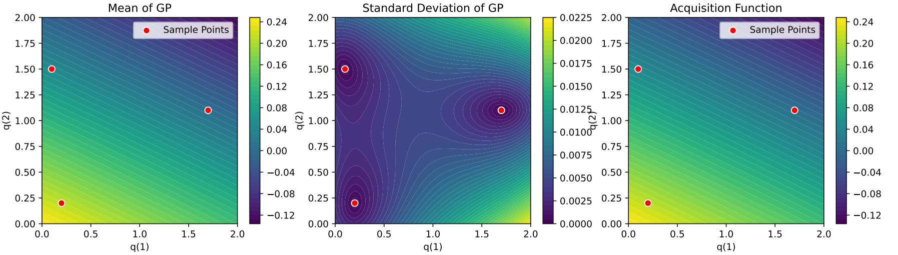
Gaussian process!

III: Bayesian optimisation: Exploitation



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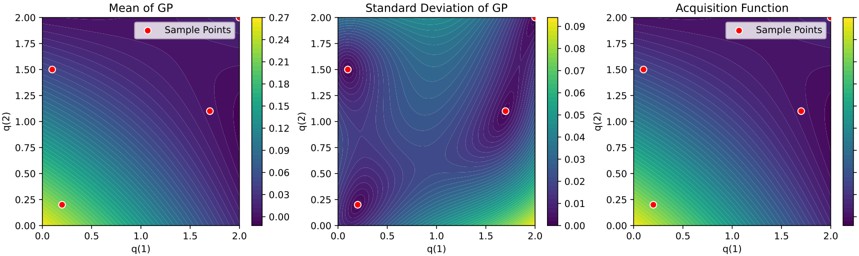

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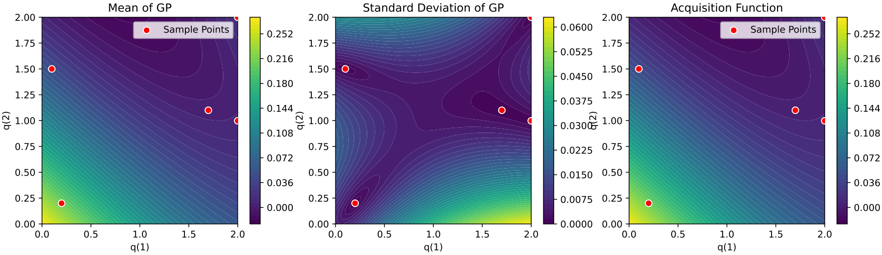

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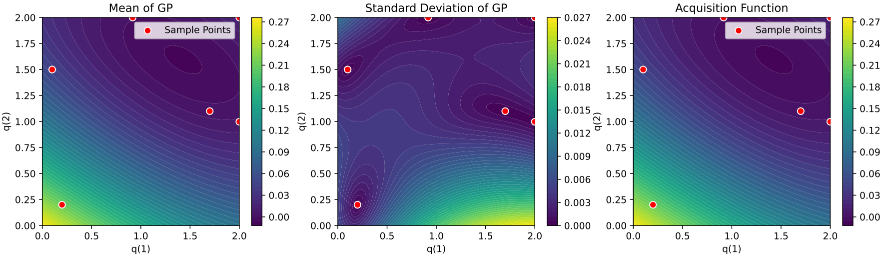

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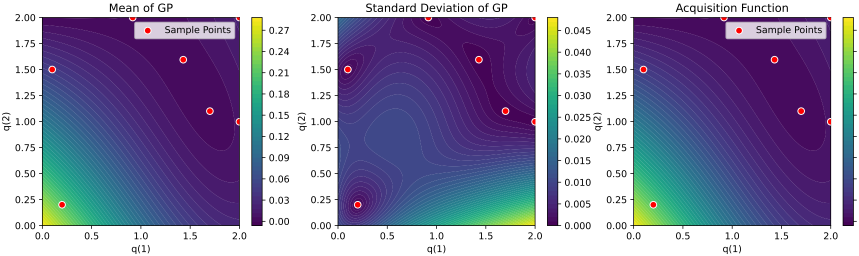

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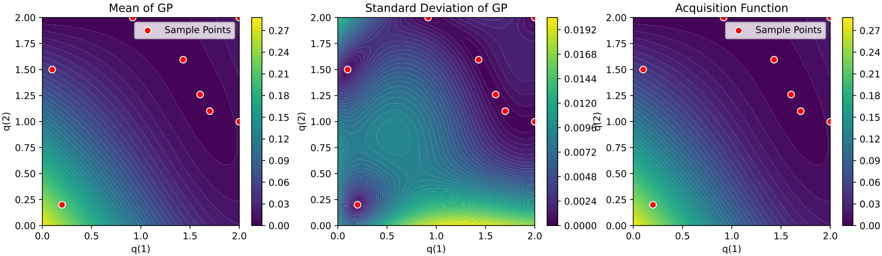

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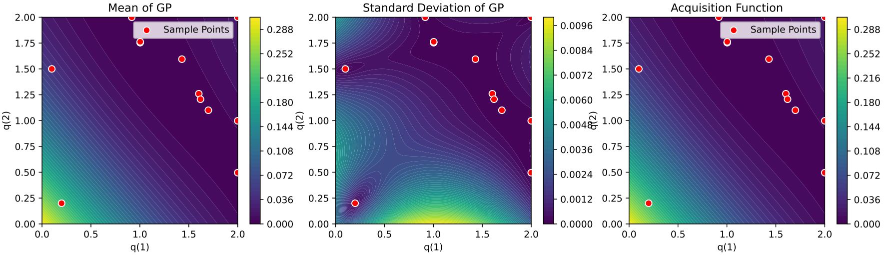

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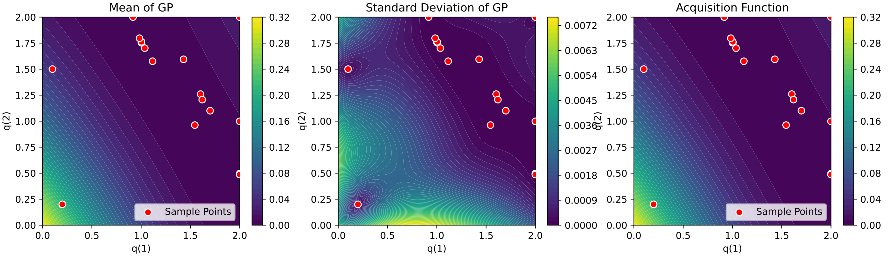

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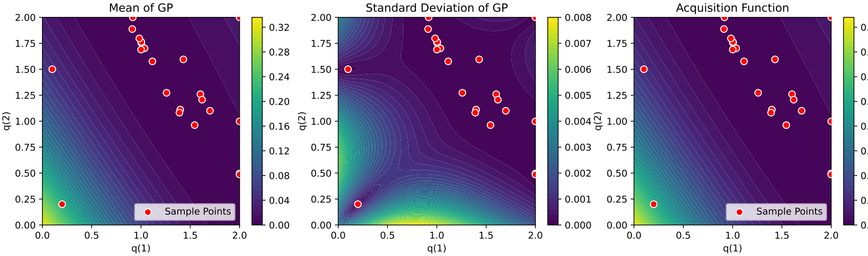

Exploitation

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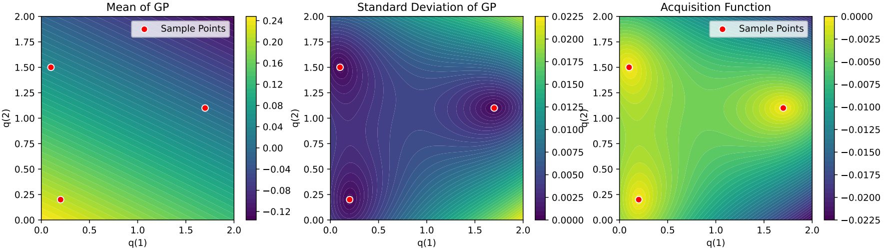

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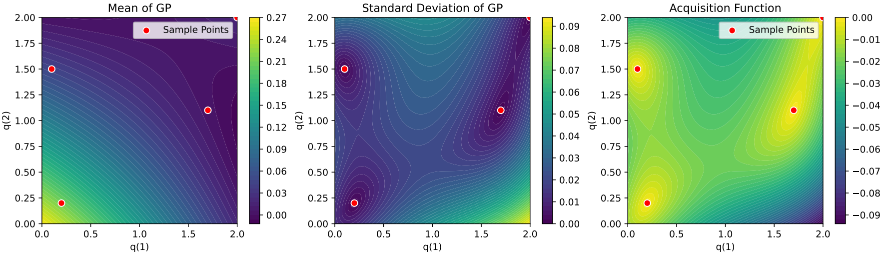

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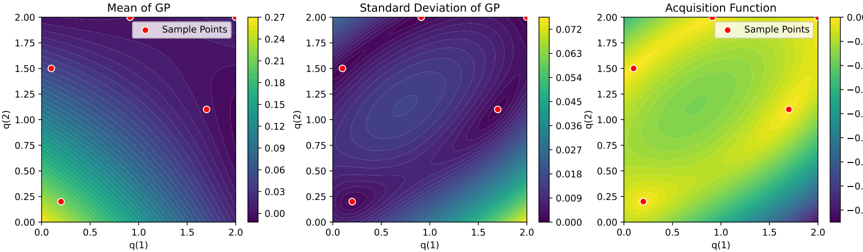

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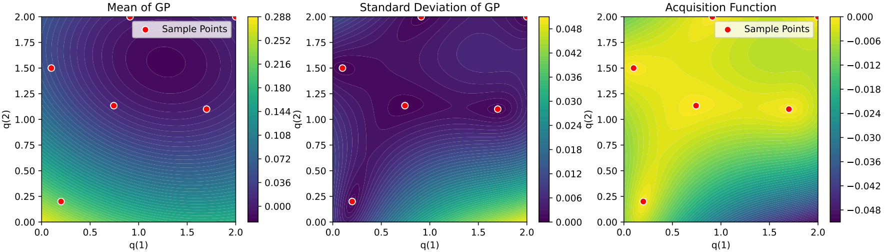

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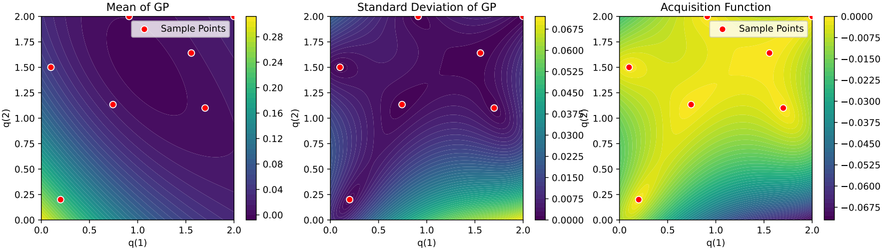

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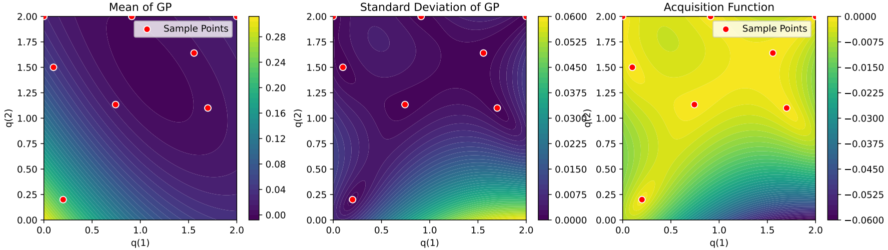

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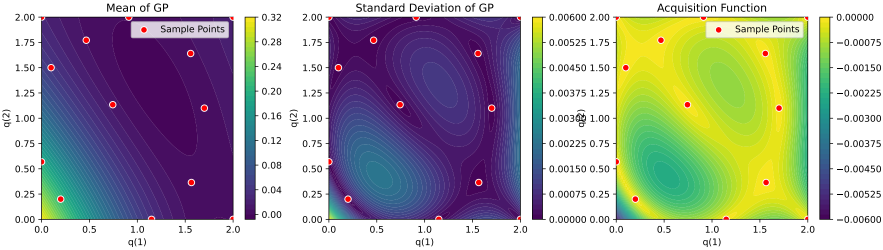

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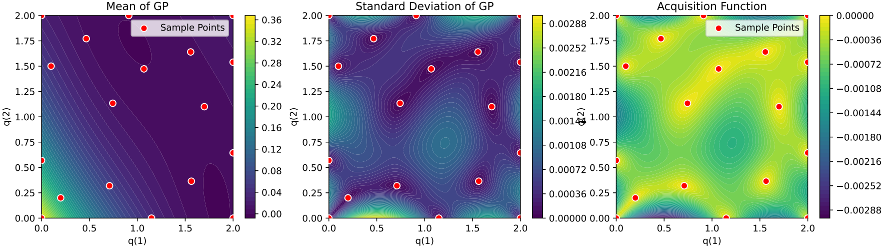

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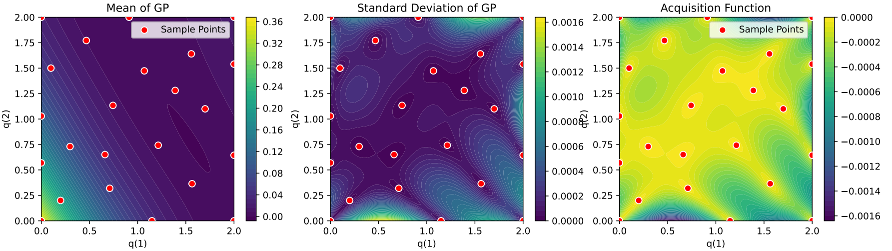

Exploration

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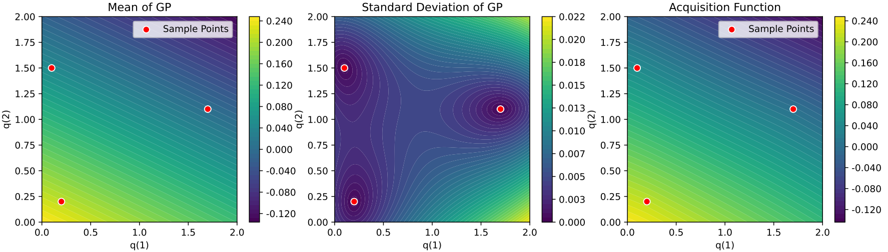

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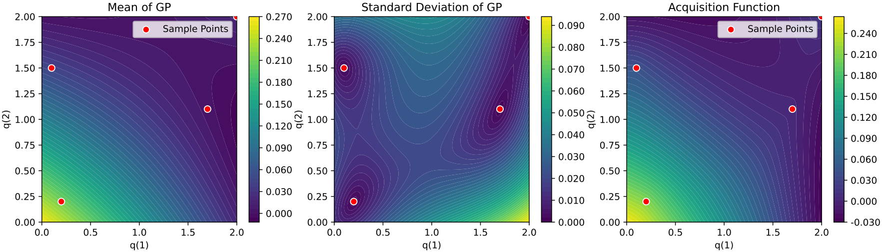

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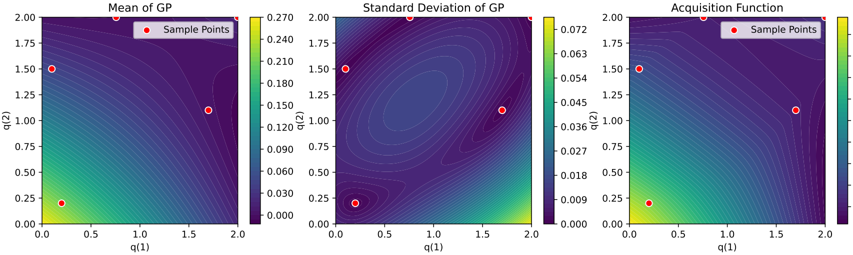

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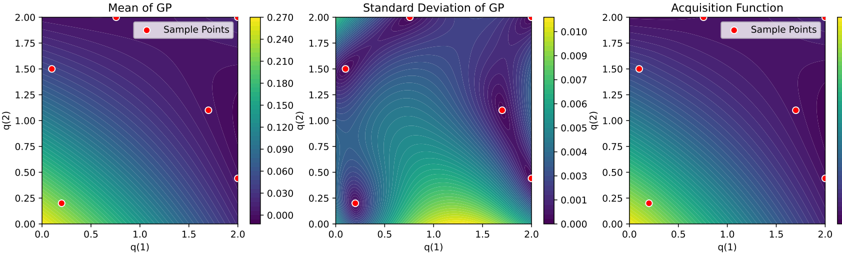

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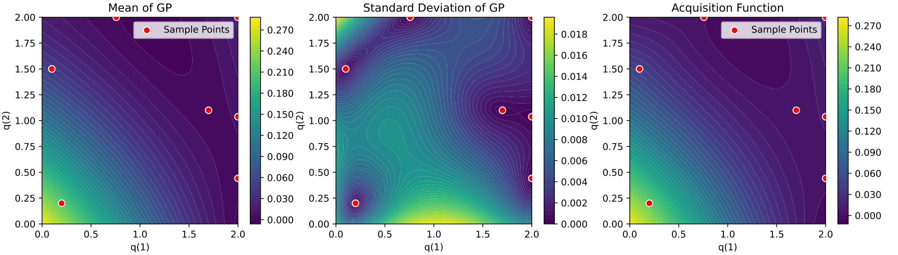

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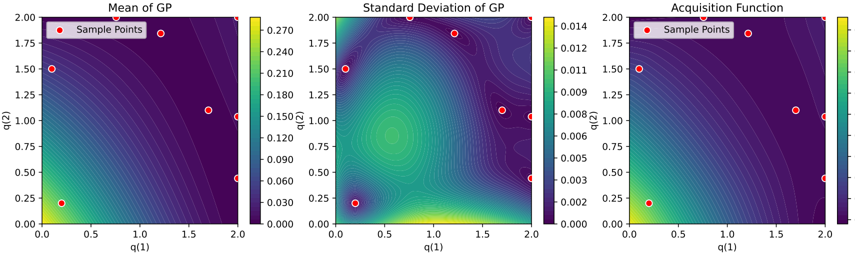

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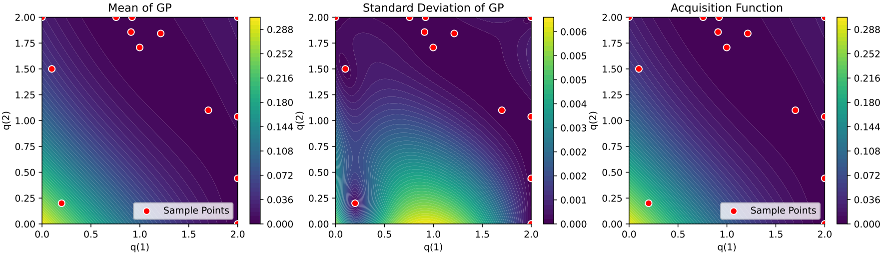

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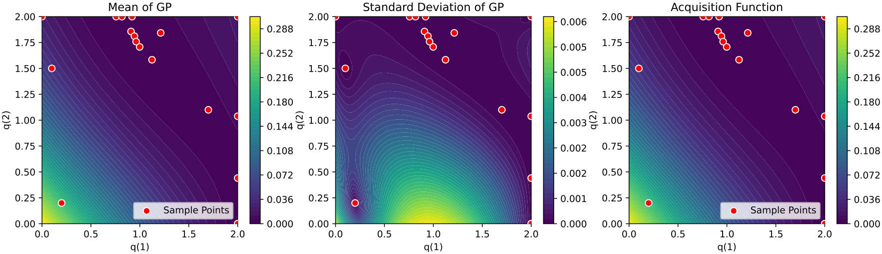

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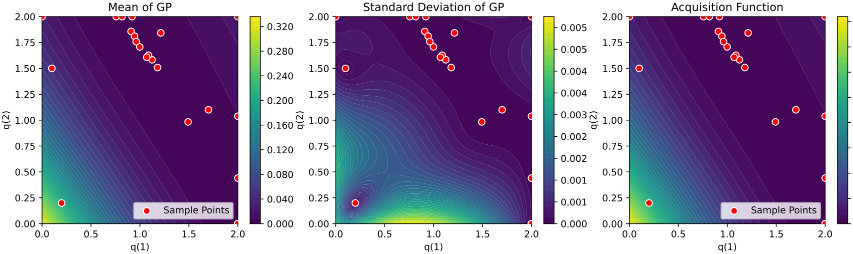

Exploration & exploitation

Three Bayesian approaches to inverse problems

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Open questions

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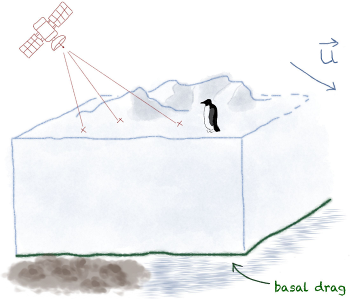

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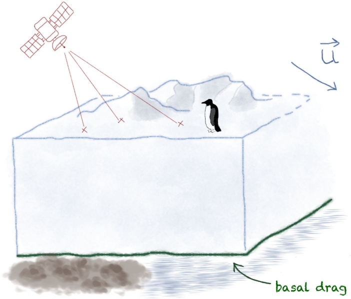

2. What are emulators and when to use them?

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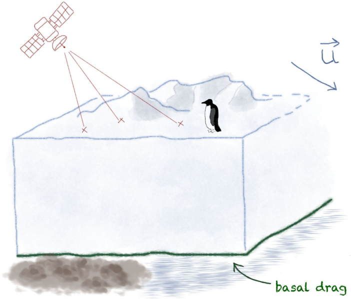

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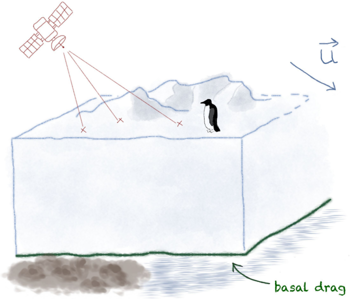

2. What are emulators and when to use them?
   - Unknown physics (parameterisations)
   - Plentiful data allows to replace the physical model

Thank you