Chain Rule

Given $f(z) = \log_e(1+z)$ where, $z = x^T x$, $x \in R^d$

Soln:

If, $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}$

Then, $x^T = \begin{bmatrix} x_1 & x_2 & \cdots & x_d \end{bmatrix}$

$x^T x = \begin{bmatrix} x_1^2 + x_2^2 + \cdots x_d^2 \end{bmatrix}$

Applying Chain Rule:

$$\frac{\partial f}{\partial x} = \frac{\partial f}{dz} \cdot \frac{dz}{dx}$$

$$= \frac{\partial}{\partial z}\left(\log(1+z)\right) \cdot \frac{\partial}{\partial x}(x^T x)$$

$$= \frac{1}{1+z} \cdot \frac{\partial}{\partial z}(z) \frac{\partial}{\partial x}(x_1^2 + x_2^2 + \cdots + x_d^2)$$

$$= \frac{1}{1+z}(2x_1 + 2x_2 + \cdots + 2x_d)$$

$$= \frac{2}{1+z} \sum_{i=1}^{d} x_i \quad \text{(Ans)}$$

(7) $f(z) = e^{-z/2}$; where $z = g(x)$, $\phi(y) = y^T S^{-1} y$, $y = h(x)$

$h(x) = x - \mu$

→ <u>Using Chain Rule:</u>

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

here, $$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(e^{-z/2}) = -\frac{e^{-\frac{z}{2}}}{2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(y^T S^{-1} y)$$

$$= \lim_{h \to 0} \frac{\phi(y+h) - \phi(y)}{h}$$

$$= \lim_{h \to 0} \frac{(y^T + h) S^{-1} (y+h) - y^T S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{y^T S^{-1} y + y^T S^{-1} h + h S^{-1} y + h^2 S^{-1} y}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(y^T S^{-1} + S^{-1} y + h S^{-1})}{\cancel{h}}$$

$$= \lim_{h \to 0} (y^T S^{-1} + S^{-1} y + h S^{-1}) = y^T S^{-1} + S^{-1} y.$$

$$\frac{\partial y}{\partial x} = \frac{\partial (x-\mu)}{\partial x} = 1.$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial x}$$

$$= -\frac{e^{-\frac{z}{2}}}{2}\left(y^T s^{-1} + s^{-1} y\right) \cdot 1.$$

$$= -\frac{e^{-z/2}}{2} \cdot \frac{1}{s}\left(y^T + y\right) \text{ (Ans.)}$$