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TJCDCG^3 2020+1

Chiang Mai , Thailand

The 23rd Thailand-Japan Conference on Discrete and Computational Geometry, Graphs, and Games (TJCDCG^3 2020+1)

September 3 - 5, 2021

Organized by
Department of Mathematics
Faculty of Science, Chiang Mai University, Thailand

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The 23rd Thailand-Japan Conference on Discrete and Computational Geometry, Graphs, and Games (TJCDCG^3 2020+1)

September 3rd-5th, 2021 (GMT +7)

Department of Mathematics, Faculty of Science, Chiang Mai University

Friday September 3rd, 2021

Time	Activities		
09.00 – 09.25	Opening Ceremony • VDO Presentation (Sci CMU – Math CMU) • Opening Address by Prof. Torranin Chairuengsri, The Dean of the Faculty of Science, Chiang Mai University Prof. Jin Akiyama, The Chair of JCDCGGG steering committee • Brief Introduction of TJCDCGGG2020 by Supanut Chaidee, The Co-chair of TJCDCGGG2020 Assoc. Prof. Wanida Hemakul, The Chair of Program Committee		
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	Korina Ernjulie Manaloto* and Rovin Santos. *Prime labeling of trees using Eisenstein integers* [video presentation]	Joshua Ani, Josh Brunner*, Erik D. Demaine, Martin L. Demaine, Dylan Hendrickson, Victor Luo and Rachana Madhukara. *Orthogonal fold & cut*	
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Friday September 3rd, 2021

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	Zijian Xu and Vorapong Suppakitpaisarn*. *On the size of minimal separators for treedepth decomposition*	Takashi Yoshino*. *Renzuru tilings with asymmetric quadrilaterals*	Hironori Kiya* and Hirotaka Ono. *Multi-player open-hand BABANUKI* [video presentation]
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10.40 – 11.40	Invited Speaker 6: Prof. Jittat Fakcharoenphol *Approximation Schemes for Geometric NP-hard Problems: Geometry Meets Algorithms* Chair: Wanida Hemakul		
11.40 – 12.00	Closing Ceremony • Thank you speech by Nattakorn Sukantamala, The Head of the Department of Mathematics • Special issue information by Supanut Chaidee, The Co-chair of TJCDCGGG2020 • Closing speech by Assoc. Prof. Wanida Hemakul, The Chair of Program Committee Prof. Jin Akiyama, The Chair of JCDCGGG steering committee		

Abstract

(Invited Speakers)

Extended Abstract
The 23rd Thailand-Japan Conference on
Discrete and Computational Geometry, Graphs, and Games
Department of Mathematics, Chiang Mai University

Family Tree of Impossible Objects Created by Optical Illusions

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Abstract

This article classifies impossible objects created by Kokichi Sugihara into several categories and locates them in a form of a family tree according to the reason why they look impossible.

Keywords: optical illusion, impossible figure, impossible motion, ambiguous object, depth perception.

2010 MSC: Primary 00A66; Secondary 51A05.

1 Introduction

Impossible objects were first shown by 2D figures called impossible figures [2]. When we see those pictures, we have some ideas of 3D structures, but at the same time we feel they cannot physically exist. In this sense, impossible objects were imaginary structures. However, several trickes were found to construct real 3D structures whose appearances match impossible figures [1]. In particular, Sugihara proposed various real 3D objects that look impossible due to optical illusions [3, 4]. In this article, we define the term *impossible objects* as 3D physical objects whose behaviors look impossible, and classify Sugihara's impossible objects into a form of a family tree.

2 Impossible Objects and Their Family Tree

There are two main sources of 3D optical illusions. One is the freedom in the choice of the depth from a picture. Illusion occurs when the brain chooses a depth which is different from the real depth. The second is the freedom in the choice of the viewpoint from which we see a 2D image. Illusion occurs when we see the image from a viewpoint different from the center of projection. On the basis of these sources of illusions, we can classify the impossible objects.

Figure 1 shows a family tree of impossible objects. Here, the root is at the upper left corner and the branches extend downward and to the right. The rectangles show the freedom- and perception-related properties, and rounded boxes represent the classes of impossible objects. A tree branch indicates that a child node is created from the parent node, and the symbols by edges represent how the child nodes are created. The symbol "=" means that the same principle is applied to a different way of display. The symbol ">" means that the principle is strengthened by additional constraints. The symbol "+" means that two principles are combined to create a new class.

Figure 2 shows examples of objects belonging to each class in the tree.

Acknowledgments. This work is supported by JSPS KAKENHI Grant Nos. 21H03530 and 21K19801.

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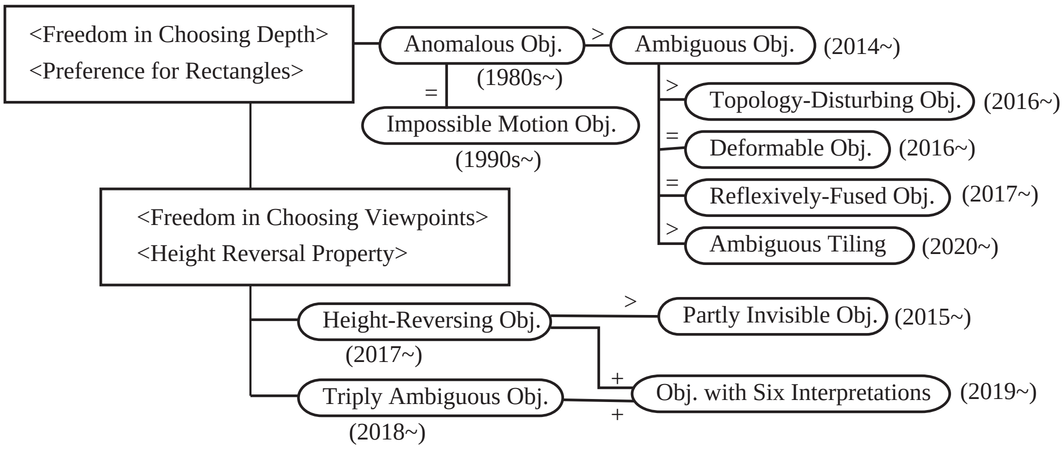

Figure 1: Family tree of impossible objects.

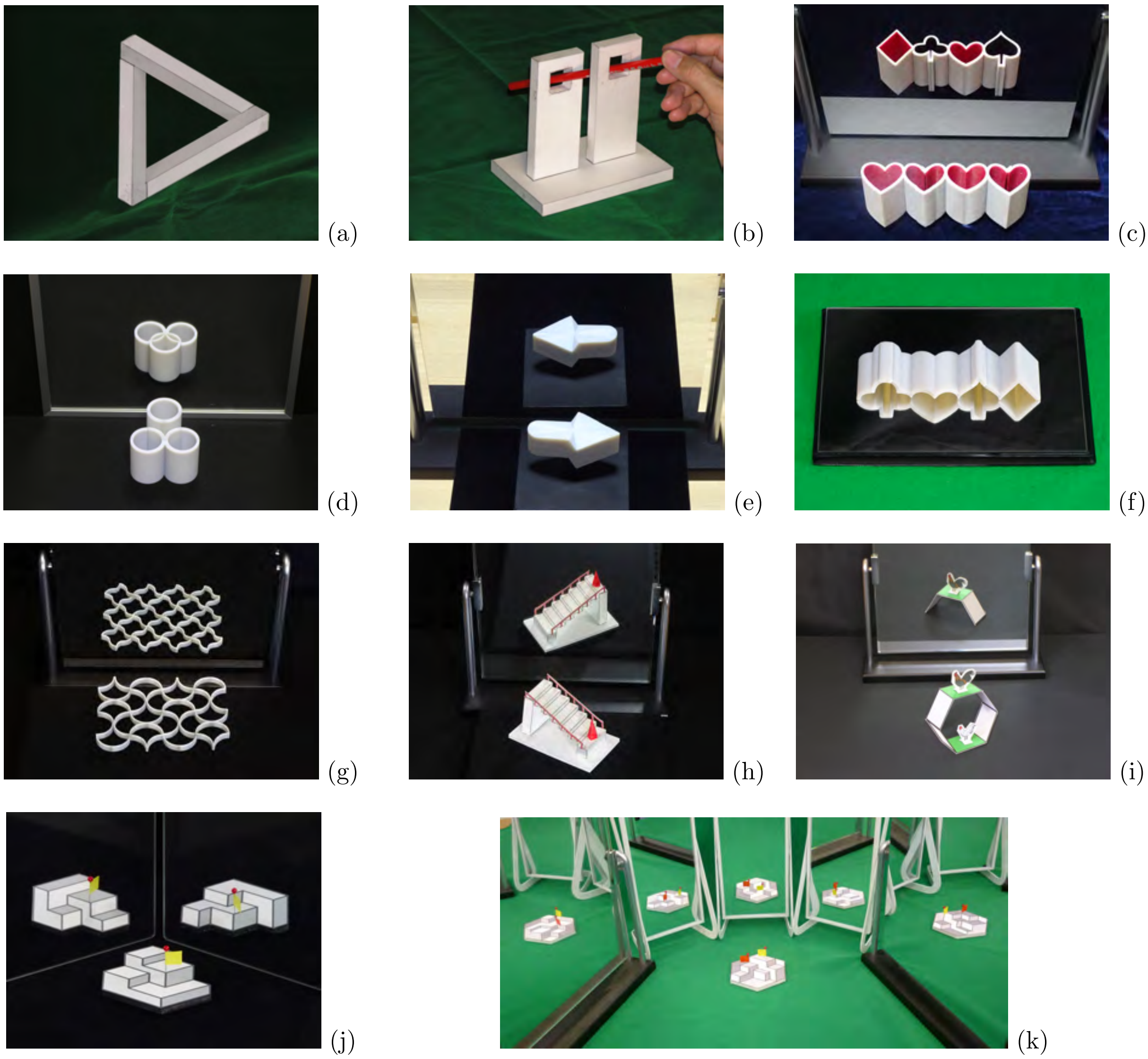

Figure 2: Examples of impossible objects located in the tree: (a) anomalous object; (b) impossible motion object; (c) ambiguous object; (d) topology-disturbing object; (e) deformable object; (f) reflexibly-fused object; (g) ambiguous tiling; (h) height-reversing object; (i) partly invisible object; (j) triply ambiguous obect; (k) object with six interpretations.

Extended Abstract
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Towards Dynamic Voronoi Diagrams

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Abstract

Voronoi Diagrams for a set P of points in the plane associate to each point $q \in P$ the region of the plane that is closer to q than to any other point of P. Together with their dual, Delaunay Graphs, they are some of the most relevant structures used to understand the distance or proximity for sets of points. Now what happens when the set P is modified?

It turns out Voronoi Diagrams are remarkably fragile, and the addition, deletion or perturbation of a single point can sometimes change a constant fraction of its features. This makes the goal of efficiently maintaining Voronoi Diagrams as point sets change or move, seemingly unattainable. But is it?

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Extended Abstract
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Understanding the Complexity of Motion Planning through Gadgets

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Abstract

Most motion planning problems — designing the route for one or more agents (robots, humans, cars, drones, etc.) through a changeable environment — are computationally difficult: NP-hard, PSPACE-hard, or worse. Such hardness proofs usually consist of several *gadgets* — local pieces of environment with limited agent interactions/traversals, some of which change local state, which in turn change available interactions/traversals — that can be pasted together into the overall reduction. In this talk, I'll describe our quest to characterize exactly which such gadgets suffice to prove different kinds of hardness, in our *motion-planning-through-gadgets framework* that has developed over the past few years [2–6]. This theory enables many hardness proofs, old and new, to be distilled down to a single diagram of a single gadget.

Keywords: gadgets, motion planning, computational complexity.

2010 MSC: Primary 68Q17; Secondary 68Q25.

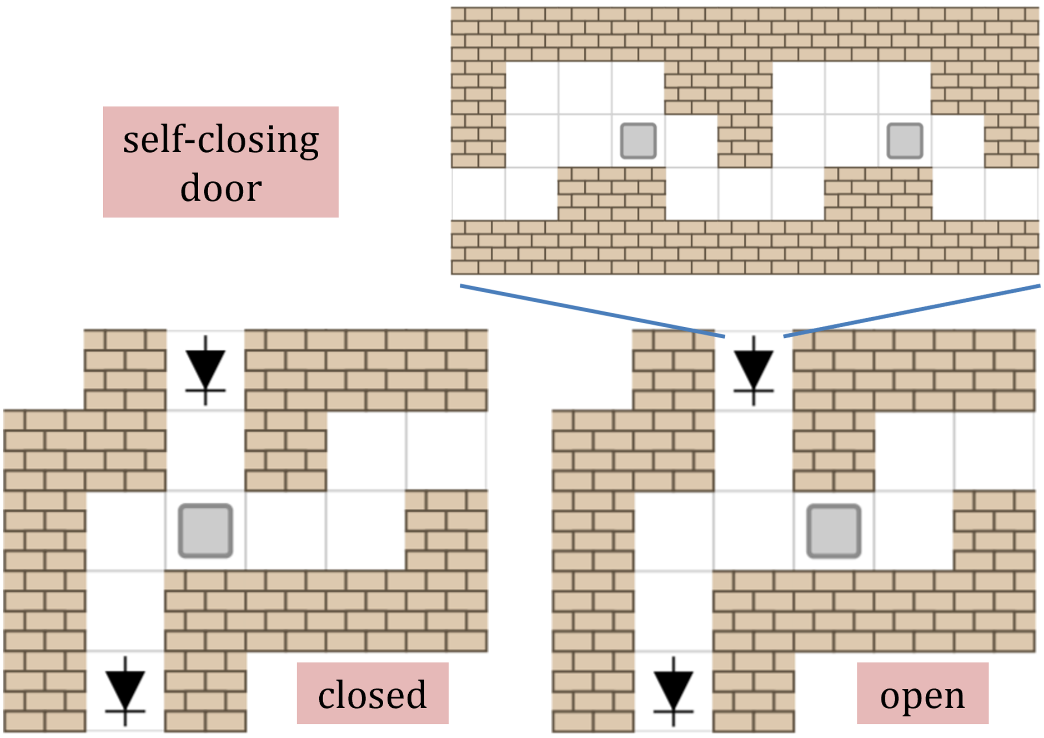

Figure 1: An example of a one-figure proof that Pull!-1F(G) (pulling blocks with forced pulling, fixed blocks, and optional gravity) is PSPACE-complete, from JCDCGGG 2019 [1]. This construction implements a self-closing door gadget from the framework of [2].

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Extended Abstract
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Decomposing complete multigraphs into stars of varying sizes

Rosalind A. Cameron[1] and **Daniel Horsley**[2]

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Abstract

An edge decomposition of a multigraph G is a set of subgraphs of G such that each edge of G occurs in precisely one of the subgraphs. In 1979 Tarsi showed that an edge decomposition of a complete multigraph into stars of size m exists whenever the obvious necessary conditions hold. In 1992 Lonc gave necessary and sufficient conditions for the existence of an edge decomposition of a (simple) complete graph into stars of sizes $m_1, \dots, m_t$. We are concerned with the common generalisation of these problems: when does a complete multigraph admit an edge decomposition into stars of sizes $m_1, \dots, m_t$? This problem exhibits more complicated and interesting behaviour than either of its specialisations. It turns out that the general problem is NP-complete, but that it becomes tractable if we place a strong enough upper bound on $\max(m_1, \dots, m_t)$. We are able to determine the upper bound at which this transition occurs. The basis of this result is a characterisation of when an arbitrary multigraph can be decomposed into stars of sizes $m_1, \dots, m_t$ with specified centres, which extends a result of Hoffman. A generalisation of Landau's theorem on tournaments also follows from this characterisation.

Keywords: edge decomposition, star, complete multigraph, NP-completeness, tournament.

2010 MSC: Primary 05C51; Secondary 68Q17.

1 Background

For a positive integer m, an m-star is a connected simple graph with m edges, all of which are incident with a single vertex which we call the *centre*. We denote the λ-fold complete multigraph on n vertices by λK_n. An *(edge) decomposition* of a multigraph G is a collection of subgraphs of G such that each edge of G occurs in precisely one subgraph of the decomposition.

In 1979 Tarsi [6] showed that some numerical necessary conditions for the existence of a decomposition of a complete multigraph into stars of a uniform specified size are also sufficient:

Theorem 1.1 (Tarsi [6]). *The complete λ-fold multigraph λK_n of order $n \geq 3$ has a decomposition into m-stars if and only if $\lambda\binom{n}{2} \equiv 0 \pmod{m}$ and*

$$m \leq \begin{cases} \frac{n}{2} & \text{if } \lambda = 1 \\ n-1 & \text{if } \lambda \text{ is even} \\ \frac{\lambda}{\lambda+1}(n-1) & \text{if } \lambda \geq 3 \text{ is odd.} \end{cases}$$

In 1992 Lonc [5] gave a simple numerical characterisation for the existence of a decomposition of a simple complete graph into stars of various specified sizes:

Theorem 1.2 (Lonc [5]). *The complete graph K_n has a decomposition into stars of sizes $m_1 \geq \cdots \geq m_t$ if and only if $\sum_{i=1}^{t} m_i = \binom{n}{2}$ and*

$$\sum_{i=1}^{k} m_i \leq \binom{n}{2} - \binom{n-k}{2} \qquad \text{for each } k \in \{1, \dots, n-1\}.$$

2 Our work

Our work [2] addresses the common generalisation of the problems solved by Theorems 1.1 and 1.2: when does a complete multigraph admit an edge decomposition into stars of sizes $m_1, \dots, m_t$? Perhaps surprisingly, this problem exhibits more complicated and interesting behaviour than either of its specialisations. It turns out that the general problem is NP-complete, but that it becomes tractable if we place a strong enough upper bound on $\max(m_1, \dots, m_t)$. In fact, we are able to determine the upper bound at which this transition occurs.

To be more precise, we first formalise the question as a family of decision problems, one for each positive integer λ and real number α such that $0 \le \alpha \le 1$:

(λ, α)-STAR DECOMP
Instance: Positive integers n and $m_1 \ge \dots \ge m_t$ such that $m_1 \le \alpha(n-1)$ and $m_1 + \dots + m_t = \lambda\binom{n}{2}$.
Question: Does λK_n have a decomposition into stars of sizes $m_1, \dots, m_t$?

Note that $m_1 + \dots + m_t = \lambda\binom{n}{2}$ and $m_1 \le n-1$ are obvious necessary conditions for the existence of the required decomposition. Then our main result is the following.

Theorem 2.1. *Let $\lambda \ge 2$ be an integer. Then (λ, α)-STAR DECOMP is NP-complete if $\alpha > \alpha'$, where*

$$\alpha' = \begin{cases} \frac{\lambda}{\lambda+1} & \text{if } \lambda \text{ is odd} \\ 1 - \frac{2}{\lambda}(3 - 2\sqrt{2}) & \text{if } \lambda \text{ is even.} \end{cases}$$

Furthermore, if $\alpha \le \alpha'$ then every instance of (λ, α)-STAR DECOMP is feasible and the required decompositions can be constructed in polynomial time.

The foundation of our proof of Theorem 2.1 is a characterisation of when an arbitrary multigraph can be decomposed into stars of sizes $m_1, \dots, m_t$ where the centre vertex of each star is specified. This extends a result of Hoffman [3]. Like Hoffman's result, our characterisation is proved using the max-flow min-cut theorem.

We also use our extension of Hoffman's result to obtain a generalisation of Landau's theorem on tournaments [4]. A λ*-fold tournament* is a λ-fold complete multigraph equipped with an orientation of its edges. A simple extension of Landau's result (see [1, Theorem 2.2.4], for example) characterises when there exists a λ-fold tournament of order n with a specified out-degree at each vertex. This characterisation is in terms of n easily computable inequalities. We achieve an analogous characterisation when, for each vertex, both the out-degree and a lower bound on the number of distinct out-neighbours are specified.

Acknowledgments. This work was supported by an AARMS Postdoctoral Fellowship and Australian Research Council grants DP150100506 and FT160100048.

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Extended Abstract
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Crossing Parallels

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Abstract

According to the crossing lemma of Ajtai, Chvátal, Newborn, Szemerédi and Leighton (1981), the crossing number of any graph with n vertices and e edges is at least $c\frac{e^3}{n^2}$, where c is an absolute constant. This result, which is tight up to the constant factor, has been successfully applied to a variety of problems in discrete and computational geometry, additive number theory, algebra, and elsewhere. The lemma easily extends to multigraphs: if the multiplicity of every edge is at most m, then the crossing number of the graph is at least $c\frac{e^3}{mn^2}$. However, much better bounds can be established for multigraphs in which no two parallel edges are homotopic. The latest results are joint with Fox and Suk.

Extended Abstract
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Approximation schemes for geometric NP-hard problems: geometry meets algorithms

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Abstract

In computer science, numerous optimization problems are NP-hard; thus, it is unlikely to find efficient algorithms for solving them exactly. However, many problems have natural instances on Euclidean planes, for example the Travelling Salesman Problem, the Minimum Steiner Tree Problem, covering problems in geometric graphs, and many clustering problems. Geometry gives structures to these instances, allowing computer scientists to devise efficient approximation schemes that find $(1+\epsilon)$-approximate solutions in polynomial time for any constant $\epsilon > 0$. In this survey, we outline many important ideas used in those algorithms, including dynamic programming and randomization.

Keywords: NP-hard problems, geometric instances, approximation schemes.

Abstract

(Contributed Talks)

Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

A k-ary Middle Levels Conjecture

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Abstract

We generalize the middle levels conjecture from binary to k-ary strings. Our conjecture is equivalent to star transposition graphs on uniform-frequency multiset permutations being Hamiltonian.

Keywords: Middle levels conjecture, Hamilton cycle, Gray codes, multiset permutations, star transpositions.

2010 MSC: Primary 05C45; Secondary 94C15, 90B99, 90C99.

1 The Middle Levels Conjecture and Theorem(s)

> *A felicitous but unproved conjecture may be of much more consequence for mathematics than the proof of many a respectable theorem.*
>
> — A. Selberg (Collected papers, vol. I (1989), p. 700)

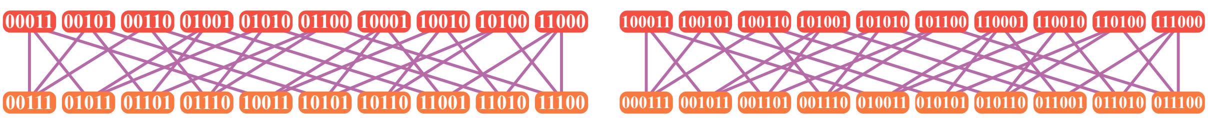
Let $\mathbb{B}_w(n)$ be the set of n-bit binary strings with *weight* w (i.e. with w 1's) and $\mathbb{B}_\ell^u(n) = \mathbb{B}_\ell(n) \cup \cdots \cup \mathbb{B}_u(n)$. The graph $\mathsf{G}_1(\mathbb{S})$ has vertex set $\mathbb{S}$ and edges between two vertices with Hamming distance one; see Fig. 1a. A widely renowned combinatorial problem involves this graph on the *middle levels strings* $\mathbb{B}_{f-1}^f(2f-1)$.

Conjecture 1.1 (Middle Levels*)**.** *The* middle levels graph $\mathsf{G}_1(\mathbb{B}_{f-1}^f(2f-1))$ *is Hamiltonian when* $f \geq 2$.

The conjecture was given a difficulty rating of 49/50 by Knuth in *The Art of Computer Programming* [4], and it was actively worked on for 30 years before a constructive solution by Mütze [7]. The middle levels conjecture has also provided the starting point for additional research in several directions.

(1) *Generalization.* The *central levels theorem*: $\mathsf{G}_1(\mathbb{B}_{f-d}^{f+d-1}(2f-1))$ is Hamiltonian when $d \geq 1$ [2].
(2) *Strengthening.* There are Hamilton cycles in $\mathsf{G}_1(\mathbb{B}_{f-1}^f(2f-1))$ with additional symmetries [6].
(3) *Variation.* De Bruijn sequences for middle and central level strings were constructed [8; 10; 12].

We provide a new generalized conjecture on k-ary strings, and some supporting computational evidence.



(a) Middle levels graph $\mathsf{G}_1(\mathbb{B}_2^3(5))$. Note: $\mathbb{B}_2^3(5) \simeq \mathbb{F}_2^3(2,5)$. (b) Star graph $\mathsf{G}^*(\mathbb{B}_3(6))$. Note: $\mathbb{B}_3(6) \simeq \mathbb{F}_3(2)$.

Figure 1: (a) Middle levels graph and (b) an isomorphic star graph. They are $k = 2$ and $f = 3$ in our conjectures.

2 Star Transpositions and Star Graphs

We start by recalling an equivalent formulation of the middle levels problem by Buck and Wiedemann [1]. Given a string $s_1 \ldots s_n$, a *star transposition* exchanges s_1 and s_t, thus creating $s_t s_2 \ldots s_{t-1} s_1 s_{t+1} \ldots s_n$. A *star (transposition) graph* $\mathsf{G}^*(\mathbb{S})$ has vertices $\mathbb{S}$, and edges joining pairs differing by a star transposition. Mütze's middle levels theorem can be restated using star graphs and *half-weight binary strings* $\mathbb{B}_f(2f)$.

Theorem 2.1 (Star Graph of Half-Weight Binary Strings [7])**.** $\mathsf{G}^*(\mathbb{B}_f(2f))$ *is Hamiltonian when* $f \geq 2$.

*The origin of the middle levels conjecture has been debated, with Havel [3], Dejter, Erdős, and Trotter being mentioned.

Proof. The middle levels graph $\mathsf{G}_1(\mathbb{B}^f_{f-1}(2f-1))$ is isomorphic to the star graph $\mathsf{G}^*(\mathbb{B}_f(2f))$. In particular, a suitable mapping adds prefix 1 to all $v \in \mathbb{B}_{f-1}(2f-1)$, and 0 to all $v \in \mathbb{B}_f(2f-1)$; see Fig. 1b. $\square$

Let $\mathbb{P}(k)$ be the permutations of $[k] = \{1, 2, \dots, k\}$ in one-line notation. The star graph on permutations is Hamiltonian [5] (even when restrictions are placed on the values that are transposed [11]).

Theorem 2.2 (Star Graph of Permutations [5])**.** $\mathsf{G}^*(\mathbb{P}(k))$ *is Hamiltonian when* $k \geq 3$.

Now we consider a generalization of Theorems 2.1 and 2.2. Let $\mathbb{F}_f(k)$ be the *uniform-frequency multiset permutations* with f copies of each symbol in $[k]$. For example, $\{1,1,2,2,3,3\}$ has $\frac{6!}{2!\cdot 2!\cdot 2!} = 90$ permutations, so $|\mathbb{F}_2(3)| = 90$. Permutations and half-weight binary strings are special cases: $\mathbb{P}(k) = \mathbb{F}_1(k)$ and $\mathbb{B}_f(2f) \simeq \mathbb{F}_f(2)$. We also refer to uniform-frequency multiset permutations as *uniform strings*.

Conjecture 2.3 (Star Graph of Uniform Strings)**.** $\mathsf{G}^*(\mathbb{F}_f(k))$ *is Hamiltonian when* $k \geq 2$ *and* $k+f \geq 4$.

Conjecture 2.3 is false without uniform frequency. For example, $\mathsf{G}^*(\{112, 121, 211\})$ isn't Hamiltonian. We verified the smallest open case, $k = 3$ and $f = 2$, by computer search, with a suitable order in Table 1.

112233	312123	113223	213321	212331	322131	123132	313122	213312	212313	322113	123213	313212	233211	132231
211233	132123	311223	123321	112332	232131	321132	133122	123312	112323	232113	321213	113232	323211	231231
121233	231123	131223	321321	211332	332121	231132	331122	321312	211323	332112	231213	311232	223311	321231
221133	321123	331221	231321	121332	233121	132132	131322	231312	121323	233112	132213	131232	322311	123231
122133	123123	133221	132321	221331	323121	312132	311322	132312	221313	323112	312213	331212	232311	213231
212133	213123	313221	312321	122331	223131	213132	113322	312312	122313	223113	213213	133212	332211	312231

Table 1: Hamilton cycle in the star graph of uniform strings $\mathsf{G}^*(\mathbb{F}_2(3))$ (or 3-ary middle levels graph $\mathsf{G}_1(\mathbb{F}^3_2(3,5))$).

3 A k-ary Middle Levels Conjecture

Now we restate Conjecture 2.3 using k-ary generalizations of middle level strings and graphs. Let $\mathbb{F}^u_\ell(k,n)$ be the set of *frequency range strings* of length n, in which each symbol in $[k]$ appears at least ℓ and at most u times. In particular, uniform strings are $\mathbb{F}^f_f(k,kf) = \mathbb{F}_f(k)$. By removing one symbol, we obtain the *k-ary middle levels strings* $\mathbb{F}^f_{f-1}(k,kf-1)$. These strings have f copies of each symbol in $[k]$, except any one which will occur $f-1$ times. This generalizes the binary middle levels strings, which have f 1's and $f-1$ 0's, or vice versa. The *k-ary middle levels graphs* are $\mathsf{G}_1(\mathbb{F}^f_{f-1}(k,kf-1))$; see Figure 1b.

Conjecture 3.1 (k-ary Middle Levels)**.** $\mathsf{G}_1(\mathbb{F}^f_{f-1}(k,kf-1))$ *is Hamiltonian when* $k \geq 2$ *and* $k+f \geq 4$.

Table 1 gives a solution for $k = 3$ and $f = 2$, but the first symbol in each string must be omitted. The original (binary) middle levels problem in Conjecture 1.1 is the $k = 2$ case of Conjecture 3.1. Adventurous readers may consider Conjecture 3.1 with (1) the central generalization, and/or (2) symmetric strengthening in Section 1; (3) the k-ary middle levels de Bruijn sequence variation was settled [9; 13].

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Every Tree is a Subtree of a Graceful Unicyclic Graph

Sethuraman G[1] and Murugan V[2]

Abstract

Acharya (1982) showed that every connected graph can be embedded in a graceful graph. The generalization of this result that any set of graphs can be packed into a graceful graph was proved by Sethuraman and Elumalai (2005). Sethuraman and Ragukumar (2015) have shown that, every tree can be embedded in a graceful tree. Inspired by these fundamental structural properties of graceful graphs, in this paper, we show that every tree can be embedded in a graceful unicyclic graph. More precisely, an algorithm is designed to achieve this embedding.

Keywords: Graceful tree, Graceful unicyclic graph, Graceful tree embedding, Graceful labeling, Graph labeling.

2010 MSC: 05C05, 05C78, 05C85.

1 Introduction

In 1963, Ringel posed his celebrated conjecture, popularly called Ringel Conjecture [9], which states that K_{2n+1}, the complete graph on $2n+1$ vertices can be decomposed into $2n+1$ isomorphic copies of a given tree with n edges. In [5], Kotzig independently conjectured the specialized version of the Ringel Conjecture that the complete graph K_{2n+1} can be cyclically decomposed into $2n+1$ copies of a given tree with n edges. In an attempt to solve both Ringel and Kotzig Conjectures, in 1967, Rosa, in his classical paper [10] introduced hierarchical series of labelings called σ, ρ, β and α labelings as a tool to settle both the Ringel and Kotzig Conjectures. Later, Golomb [4] called β-labeling as graceful labeling, and now this term is being widely used. A function f is called graceful labeling of a graph G with m edges, if f is an injective function from $V(G)$ to $\{0, 1, 2, \cdots, m\}$ such that, when every edge uv is assigned the edge label $|f(u) - f(v)|$, then the resulting edge labels are distinct. A graph which admits graceful labeling is called graceful graph.

Rosa [10] proved the following significant theorem which exhibits the utility of graceful labeling in achieving a cyclic decomposition of the complete graph K_{2n+1}, where $n \geq 1$.

Theorem 1.1. *[10] A graph G with n edges admits graceful labeling if and only if the complete graph K_{2n+1} can be cyclically decomposed into $2n+1$ copies G.*

From Theorem 1, one can observe that, if every tree T with n edges admits graceful labeling then the tree T naturally decomposes the complete graph K_{2n+1}. Theorem 1 led to a famous conjecture, called "Graceful Tree Conjecture": All trees are graceful. This conjecture remains open over five decades.

Although the structural characterization of graceful graphs appear to be one of the most difficult problems in graph theory, very few general structural properties have been established. Acharya [1] showed that every connected graph can be embedded in a graceful graph. In [6], Sethuraman and Elumalai generalized this result and showed that every set of graphs can be packed into a graceful graph. Sethuraman and Ragukumar [8] have also shown that every tree can be embedded in a graceful tree. Recently, [7] Sethuraman and Murugan have shown that every forest can be embedded in a graceful unicyclic graph. Inspired by these fundamental structural properties of graceful graphs, in this paper, we show that every tree can be embedded in a graceful unicyclic graph. More precisely, an Algorithm is presented to achieve this embedding. Our algorithm yields a class of graceful unicyclic graphs generated from an arbitrary tree. The embedding presented in this paper is totally different and more simpler than the embedding presented by Sethuraman and Murugan [7]. This result supports the well known Truszczynski's Conjecture [11], which claims that, except for the cycles C_{4m+1} and C_{4m+2}, all unicycle graphs are graceful.

2 Main Result

In this section, we present our Embedding Algorithm, which can embed a given arbitrary tree into a graceful unicyclic graph. More precisely, we construct a graceful unicylic graph from any given tree using the Embedding Algorithm. Some important procedures of the Embedding Algorithm is given below:

- Labeling the vertices of the given tree T after naming them in a particular order.
- Constructing an intermediate tree T^* from the tree T by sequentially adding some labelled pendant vertices.
- Constructing a unicyclic graph G from T^* by adding an edge between a chosen pair of non adjacent vertices in T^* such that it leads to a graceful labeling of G.

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Extended Abstract
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Graph of uv-paths in 2-connected graphs

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Abstract

For a 2-connected graph G we define the *uv path graph of* G as the graph $\mathcal{P}(G_{uv})$ whose vertices are the paths joining u and v in G, where two paths S and T are adjacent if T is obtained from S by replacing a subpath S_{xy} of S with an internally disjoint subpath T_{xy} of T. We prove that $\mathcal{P}(G_{uv})$ is always connected and give a necessary condition and a sufficient condition for connectedness in the case where adjacency between two uv paths S and T is restricted by a set of cycles $\mathcal{C}$ where the unique cycle contained in $S \cup T$ must lie.

Keywords: uv-paths, f-monotone path, Tree Graph.

2010 MSC: Primary 05C38.

For any vertices x, y of a path L, we denote by L_{xy} the subpath of L that joins x and y. Let G be a 2-connected graph and u and v be vertices of G. The *uv path graph of* G is the graph $\mathcal{P}(G_{uv})$ whose vertices are the paths joining u and v in G, where two paths S and T are adjacent if T is obtained from S by replacing a subpath S_{xy} of S with an internally disjoint subpath T_{xy} of T. The uv path graph $\mathcal{P}(G_{uv})$ is closely related to the graph $G(P, f)$ of f-monotone paths on a polytope P (see C. A. Athanasiadis *et al* [1, 2]), whose vertices are the f-monotone paths on P and where two paths S and T are adjacent if there is a 2-dimensional face F of P such that T is obtained from S by replacing an f-monotone subpath of S contained in F with the complementary f-monotone subpath of T contained in F. We show that the graphs $\mathcal{P}(G_{uv})$ are always connected as is the case for the graphs $G(P, f)$.

If S and T are adjacent paths in a uv-path graph $\mathcal{P}(G_{uv})$, then $S \cup T$ is a subgraph of G consisting of a unique cycle σ joined to u and v by disjoint paths P_u and P_v. Let $\mathcal{C}$ be a set of of cycles of G; the *uv-path graph of G defined by $\mathcal{C}$* is the spanning subgraph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ of $\mathcal{P}(G_{uv})$ where two paths S and T are adjacent if and only if the unique cycle σ which is contained in $S \cup T$ lies in $\mathcal{C}$. A graph $\mathcal{P}_{\mathcal{C}}(G_{uv})$ may be disconnected.

The uv path graph $\mathcal{P}(G_{uv})$ is also related to the well-known *tree graph* $\mathcal{T}(G)$ of a connected graph G, studied by R. L. Cummins [3], in which the vertices are the spanning trees of G and the edges correspond to pairs of trees S and R which are obtained from each other by a single edge exchange. As in the uv path graph, if two trees S and R are adjacent in $\mathcal{T}(G)$, then $S \cup R$ is a subgraph of G containing a unique cycle. X. Li *et al* [5] define, in an analogous way, a subgraph $\mathcal{T}_{\mathcal{C}}(G)$ of $\mathcal{T}(G)$ for a set of cycles $\mathcal{C}$ of G and give a necessary condition and a sufficient condition for $\mathcal{T}_{\mathcal{C}}(G)$ to be connected. We show that the same conditions apply to uv path graphs $\mathcal{P}_{\mathcal{C}}(G_{uv})$.

Similar results are obtained by A. P. Figueroa *et al* [4] with respect to the *perfect matching graph* $\mathcal{M}(G)$ of a graph G where the vertices are the perfect matchings of G and in which two matchings L and N are ajacent if their symmetric difference is a cycle of G. Again, if L and N are adjacent matchings in $\mathcal{M}(G)$, then $L \cup M$ contains a unique cycle of G.

Keywords: uv-paths, f-monotone path, Tree Graph.
2010 MSC: Primary 05C38.

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Extended Abstract
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Prime Labeling of Trees Using Eisenstein Integers

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Abstract

A tree on n vertices is said to admit a prime labeling if the vertices can be labeled with the first n natural numbers in such a way that two adjacent vertices have relatively prime labels. The notion of prime labeling has been extended to the Gaussian integers by Lehmann and Park (2016). We extend this further to the Eisenstein integers by defining an ordering for the Eisenstein integers lying in the sector $\left[0, \frac{\pi}{3}\right)$ of the complex plane. Using this ordering, we give a prime labeling for trees with $n \leq 100$ vertices.

Keywords: prime labeling, trees, graph theory, eisenstein integers, eisenstein primes.

2010 MSC: Primary 05C05; Secondary 05C78.

1 Introduction

A tree T on n vertices is said to admit a prime labeling if we can label its vertices using the first n natural numbers in such a way that adjacent vertices have relatively prime labels. In 2002, Pikhurko [4] showed that trees of at most 34 vertices admit a prime labeling under the set of natural numbers without utilizing computer search algorithms. The notion of prime labeling has been extended to the Gaussian integers by Lehmann and Park [3] in 2016 using a similar technique. In this paper, we try to achieve analogous results in prime labeling using the set of Eisenstein integers, that is, complex numbers of the form $a + b\omega$, where $\omega = e^{\frac{i2\pi}{3}}$ and $a, b \in \mathbb{Z}$.

2 Preliminaries

2.1 Diagonal Ordering of the Eisenstein Integers

To attain our goal, we begin by introducing the following ordering of the Eisenstein integers lying in the sector $[0, \frac{\pi}{3})$.

Definition 2.1 (Diagonal Ordering of Eisenstein Integers)**.** The nth Eisenstein integer γ_n is defined recursively as follows: $\gamma_1 = 1$, and if $\gamma_n = a + b\omega$ for $n \in \mathbb{N}$, we have

$$\gamma_{n+1} = \begin{cases} \gamma_n + 1, & \text{if} \quad a \equiv 1 \pmod 2 \text{ and } b = 0 \\ \gamma_n + \omega + 1, & \text{if} \quad a \equiv 0 \pmod 2 \text{ and } b = a - 1 \\ \gamma_n + \omega, & \text{if} \quad a \equiv 0 \pmod 2 \text{ and } b \neq a - 1 \\ \gamma_n - \omega, & \text{if} \quad a \equiv 1 \pmod 2 \text{ and } b \neq 0. \end{cases}$$

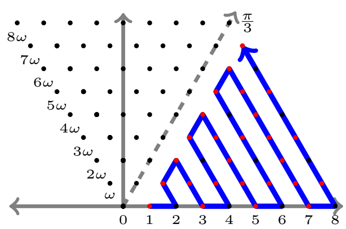

Figure 1: Diagonal Ordering of the Eisenstein integers.

Figure 1 provides a visual representation of the diagonal ordering of the Eisenstein integers. Some relevant properties of the diagonal ordering are listed in the following lemmas.

Lemma 2.2. *Eisenstein primes are not preceded by their multiples in the diagonal ordering.*

Lemma 2.3. *Let α be an Eisenstein integer and u be a unit of $\mathbb{Z}[\omega]$. Then α and $\alpha + u$ are relatively prime.*

As a consequence of Lemma 2.3, we conclude that consecutive Eisenstein integers, as well as odd Eisenstein integers two indices away from each other are relatively prime. Using the preceding properties and the lemmas presented in the next section, we may now give a prime labeling for trees using Eisenstein integers which we formally define in the following manner:

Definition 2.4. We say that *a tree* $T = (V, E)$ *on* n *vertices admits an Eisenstein prime labeling* if we can label its vertices with the first n Eisenstein integers in the diagonal ordering in such a way that $u, v \in V$ have relatively prime labels whenever $uv \in E$.

2.2 Prime Labeling Strategy

The following are the key lemmas in our labeling strategy using the diagonal ordering. The proof of Lemma 2.5 is analogous to the ordinary integer case, since the only properties needed to establish the proof are Lemma 2.2 and Lemma 2.3. Meanwhile, the proof of Lemma 2.6 can be found in [4].

Lemma 2.5. *Let* γ_p *be a prime Eisenstein integer and suppose that all trees of order* $p - 1$ *admit an Eisenstein prime labeling under the diagonal ordering. Then all trees of order* p, $p + 1$, *and* $p + 2$ *also admit an Eisenstein prime labeling under the diagonal ordering.*

Lemma 2.6 (Lemma 2, Pikhurko)**.** *Let* T *be a tree with at least 4 vertices. Then there exists a vertex* u *in* T *such that either*

1. *there is a subset* A *of* $T - u$ *with 3 vertices that can be represented as a union of (one or more) components of* $T - u$*; or*
2. u *has a neighbor* v *of degree* $k + 1 \geq 3$ *such that its other* k *neighbors* $\{v_1, v_3, \ldots, v_{2k-1}\}$ *each has degree 2 and are incident to endvertices* $\{v_2, v_4, \ldots, v_{2k}\}$ *correspondingly.*

Lemma 2.5 allows us to exclude from the labeling process trees of order p where γ_p is prime or those trees whose order are at most two more than p. For those trees whose orders are not accommodated by Lemma 2.5, we will use Lemma 2.6 which lets us isolate a branch of the tree that we can easily label with large integers in order to reduce the remainder into a smaller tree. The reduction of the tree into a smaller order will allow us to apply Lemma 2.5 and ensure that it can be labeled using the leftover Eisenstein integers. However, there is no guarantee that the remainder is still a tree. If such situation occurs, we can use the following result:

Lemma 2.7. *If any tree on* n *vertices admit an Eisenstein prime labeling, then any forest on* n *vertices also admit an Eisenstein prime labeling.*

3 Main Results

Theorem 3.1. *All trees of order* $n \leq 100$ *admit an Eisenstein prime labeling under the diagonal ordering.*

The proof of the theorem above is done inductively. We start by labeling the tree of order one with γ_1. We then apply Lemma 2.5 to exclude from the labeling process trees of order p, $p+1$ and $p+2$, where γ_p is prime. The leftover indices are labelled by applying Lemma 2.6. In this strategy, the labeling largely depends on the distance of the index of the largest prime from the number of vertices of the tree. We may break these cases into several lemmas, reapplying Lemma 2.6 if necessary. Lastly, we manually label the cases not covered by the previous process.

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Unfolding Orthotubes with a Dual Hamiltonian Path

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Abstract

An *orthotube* consists of orthogonal boxes (e.g., unit cubes) glued face-to-face to form a path. In 1998, Biedl et al. showed that every orthotube has a *grid unfolding*: cutting along edges of the boxes so that the surface unfolds into a connected planar shape without overlap. We give a new algorithmic grid unfolding of orthotubes with the additional property that the rectangular faces are attached in a single path — a Hamiltonian path on the rectangular faces of the orthotube surface.

Keywords: unfolding polyhedra, orthotubes, Hamiltonicity, algorithm.

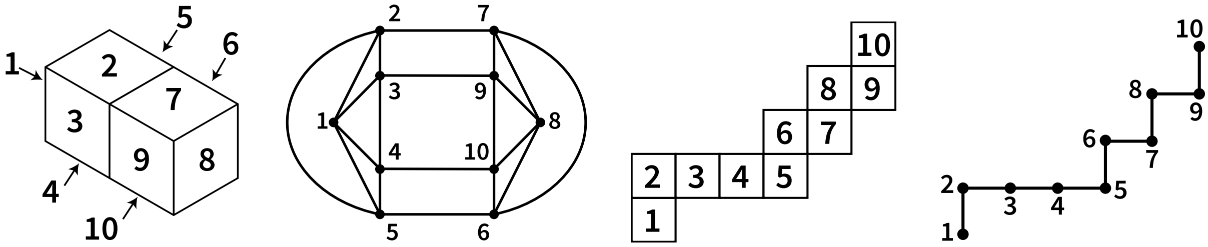
2010 MSC: Primary 68Q25; Secondary 52C99.

1 Introduction

Does every orthogonal polyhedron have a *grid unfolding*, that is, a cutting along edges of the induced grid (extending a plane through every face of the polyhedron) such that the remaining surface unfolds into a connected planar shape without overlap? This question remains unsolved over 20 years after this type of unfolding was introduced in 1998 [1]; see [9] for a survey and [3–5] for recent progress. This problem is in some sense the orthogonal nonconvex version of the older and more famous open problem of whether every convex polyhedron has an edge unfolding (cutting only along edges of the polyhedron) [6].

The first class of orthogonal polyhedra shown to have a grid unfolding is *orthotubes* [1], formed by gluing together a sequence of orthogonal boxes where every pair of consecutive boxes in the sequence share one face (and no other boxes share faces). Roughly speaking, this unfolding consists of a monotone dual-path of rectangular faces, with $O(1)$ rectangles attached above and below the path.

In this paper, we show that orthotubes have a grid unfolding with a stronger property we call *dual-Hamiltonicity*, where the unfolded shape consists of a single dual path of rectangular faces, as shown in Figure 1. More precisely, define the *face adjacency graph* to have a node for each rectangular face of a box, and connect two nodes by an edge whenever the corresponding rectangular faces share an edge. Then the unfolding is given by keeping attached the duals of edges that form a Hamiltonian path in the face adjacency graph. Implicitly, we take advantage of the fact that 4-connected planar graphs (which includes face adjacency graphs) have Hamiltonian cycles [2, 10].



(a) orthotube (b) face adjacency graph (c) dual-Hamiltonian unfolding (d) dual Hamiltonian path

Figure 1: An example of an orthotube, its face adjacency graph, dual-Hamiltonian unfolding corresponding to chain code $RSSLRLRL$, and the corresponding dual Hamiltonian path.

2 Unfolding

Our main result is the following:

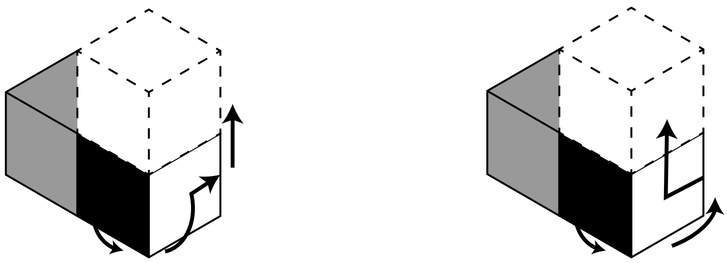
Theorem 2.1. *Given an orthotube, there is a Hamiltonian path of its face adjacency graph such that, if we follow the path, we get a non-overlapping unfolding.*

To prove the result, we use a "chain code" (similar to [7,8]) to represent a path on the surface of an orthotube. A *chain code* is an ordered sequence of the form $a_1a_2\ldots a_n$, where $a_i \in \{L, R, S\}$ represents an (intrinsic) left turn, right turn, or continuing straight to move from the ith face to the $(i+1)$st face.

For each chain code $c = a_1a_2\ldots a_n$, we define the cumulative quarter turning $\mathrm{qturn}(c)$ to be $\sum_{i=1}^{n} \mathrm{qturn}(a_i)$ where $\mathrm{qturn}(R)$, $\mathrm{qturn}(L)$, and $\mathrm{qturn}(S)$ are $+1$, -1, and 0 respectively. To guarantee that our unfolding does not overlap, we prove the invariant that the qturn of any prefix of the chain code is in $\{-1, 0, 1\}$, that is, the unfolding proceeds monotonically in one direction.

2.1 Algorithm

At a high level, the algorithm creates a chain code for unfolding an orthotube up to three boxes at a time, to maintain the stronger condition that the unfolding so far has qturn $= 0$. We divide into cases based on each box's relative position to the next one, two, and sometimes three or four boxes (if they exist). Figure 2 shows two cases.



(a) $RLRL$ (b) $RSLR$

Figure 2: Two possible chain codes, $RLRL$ and $RSLR$, to unfold the current box from the black face, when the next box (white) is on top. The first code preserves qturn, so we use it unless the next next box (not drawn) is attached on the face that we try to enter on the next box (white). The second code adds 1 to qturn, and we show how in all cases to add an additional box to restore qturn to 0.

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Extended Abstract
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Complexity of Simple Folding Orthogonal Crease Patterns

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Abstract

Continuing results from JCDCGGG 2016 and 2017, we solve several new cases of the *simple folding problem* — deciding which crease patterns can be folded flat by a sequence of (some model of) simple folds. We give new efficient algorithms for *mixed* crease patterns, where some creases are assigned mountain/valley while others are unassigned, for all 1D cases and for 2D rectangular paper with one-layer simple folds. By contrast, we show strong NP-completeness for mixed crease patterns on 2D rectangular paper with some-layers simple folds, complementing a previous result for all-layers simple folds. We also prove strong NP-completeness for finite simple folds (no matter the number of layers) of unassigned orthogonal crease patterns on arbitrary paper, complementing a previous result for assigned crease patterns, and contrasting with a previous positive result for infinite all-layers simple folds. In total, we obtain a characterization of polynomial vs. NP-hard for all cases — finite/infinite one/some/all-layers simple folds of assigned/unassigned/mixed orthogonal crease patterns on 1D/rectangular/arbitrary paper — except the unsolved case of infinite all-layers simple folds of assigned orthogonal crease patterns on arbitrary paper.

Keywords: computational origami, simple folding, NP-hardness, algorithms.

2010 MSC: Primary 68Q25; Secondary 52C99.

1 Introduction

In the well-studied *simple foldability problem* [1–3], we are given a *crease pattern* consisting of line-segment *creases*, possibly *assigned* mountain or valley, on a 2D region called the *piece of paper*, and are asked whether all of the creases can be folded via a sequence of "simple folds". Each *simple fold* folds some set of layers of the piece of paper around a single line by $\pm 180°$ (thus preserving flatness of the folding). In the *1D* variation, the piece of paper is a 1D line segment and the creases are points.

Many different models for simple folds and special cases for the simple foldability problem have been considered, depending on the following characteristics:

- How many layers of paper a simple fold can move at once. The most powerful model, *some-layers*, allows the top or bottom k layers to be folded for any k. Two more restrictive models are *one-layer*, which permits folding only a single layer at a time (modeling thick material); and *all-layers*, which requires folding all layers simultaneously.
- Whether simple folds can be along *finite* line segments (chords of the crease pattern), or must be along *infinite* lines (modeling a half-plane/large flipping tool).
- Whether some creases are assigned as needing to be folded mountain or valley. In the more common *assigned* and *unassigned* cases, all creases either have or lack an assignment, while in the *mixed* case [1], each crease may be mountain, valley, or unassigned.
- What shape the paper is allowed to have. In 1D, the only option is a line segment, while in 2D the two cases considered are rectangles and arbitrary polygons, though all NP-hardness results hold even for simple orthogonal polygons.

		1D	⊂	Rectangular	⊂	Arbitrary Paper		
						Infinite	&	Finite
One Layer	Assigned	poly	⇐	poly [3]		NP-comp. [2]		NP-comp. [2]
	∩	⇑		⇑		⇓		⇓
	Mixed	**poly** (R1)	⇐	**poly** (R1)		NP-comp.		NP-comp.
	∪	⇓		⇓		⇑		⇑
	Unassigned	poly	⇐	poly [3]		NP-comp. [2]		**NP-comp.** (R4)
Some Layers	Assigned	poly	⇐	poly [3]		NP-comp. [2]		NP-comp. [2]
	∩	⇑				⇓		⇓
	Mixed	**poly** (R1)		**NP-comp.** (R3)	⇒	NP-comp.	&	NP-comp.
	∪	⇓				⇑		⇑
	Unassigned	poly	⇐	poly [3]		NP-comp. [2]		**NP-comp.** (R4)
All Layers	Assigned	poly	⇐	poly [3]		OPEN		NP-comp. [2]
	∩	⇑						⇓
	Mixed	**poly** (R2)		NP-comp. [1]	⇒	NP-comp.	&	NP-comp.
	∪	⇓						⇑
	Unassigned	poly	⇐	poly [3]	⇐	poly [1]		**NP-comp.** (R4)

Table 1: Summary of past results (cited) and new results (bold, with result numbers) about simple folding orthogonal crease patterns. "NP-comp." denotes strong NP-completeness. "⊂" denotes containment between classes of instances. "⇒" denotes implications between results. "R" numbers refer to the results list in Section 2. Hardness results for arbitrary paper hold even for paper a simple orthogonal polygon.

We consider exclusively *orthogonal crease patterns*, which contain only vertical and horizontal creases. Prior work [2, 3] has also studied versions of the problem including diagonal creases.

Table 1 presents all known results according to these parameters, including both previously known results and new results from this paper (in bold). Note that, for orthogonal crease patterns, the finite vs. infinite simple fold distinction only plays a role with arbitrary paper: no distinction can be made in the 1D case, and equivalence for rectangular paper is given by [2, Theorem 8].

2 Results

Our new results can be summarized as follows:

1. **One- & some-layers mixed 1D; one-layer mixed rectangular.** We adapt arguments from [3], and provide a new characterization of when a 1D mixed crease pattern is flat-foldable. For one-layer rectangular, crossing creases can never be folded, so the problem reduces to 1D paper.
2. **All-layers mixed 1D.** We show that folding the superficially foldable (foldable ignoring assignments) crease nearest an end of the paper never makes the crease pattern unfoldable, giving an efficient greedy strategy for deciding simple foldability.
3. **Some-layers mixed rectangular.** We show that the NP-hardness reduction from [1] for all-layers mixed rectangular also works for some-layers.
4. **Unassigned arbitrary finite.** We modify the NP-hardness proof from [2] for assigned crease patterns to work for unassigned crease patterns, by adding a component that enforces the relevant aspects of the assignment. The original proof and our adaptation apply to any number of layers.

Notably, the only case that remains unsolved for orthogonal crease patterns is infinite all-layers simple folds of assigned crease patterns on arbitrary paper.

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Extended Abstract
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Multifold tiles of polyominoes and convex lattice polygons

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Abstract

A family of 2-dimensional shapes $\mathcal{T}$ is called a *tiling* if they (rotating and reflecting are allowed) cover the whole plane without gaps or overlaps, and if all shapes belonging to $\mathcal{T}$ are congruent each other, then the shape is called a *tile*. We study *k-fold tilings* which are extended to cover the plane with the thickness of k folds and *k-fold tiles* which belong to it. Intuitively it means that a family of 2-dimensional shapes covers the plane such that they overlap k times at any point in the plane. Since clearly a (1-fold) tile is a k-fold tile for any positive integer k, the subjects of our research are 2-dimensional shapes with property "not a tile, but a $k(\geq 2)$-fold tile." We call a plane shape satisfying this property a *nontrivial k-fold tile*. In this talk, we clarify some facts as follows: first, we show that for any integer $k \geq 2$, there exists a polyomino satisfying a property that "not a h-fold tile for any positive integer $h < k$, but a k-fold tile." We also find for any integer $k \geq 2$, polyominoes with the minimum number of cells among ones which are nontrivial k-fold tiles. Next, we prove that for any integer $k = 5$ or $k \geq 7$, there exists a convex unit-lattice polygon with an area of k that is a nontrivial k-fold tile.

Keywords: multiple tilings, k-fold tiles, polyominoes, convex polygons, thickness.

2010 MSC: Primary 52C20; Secondary 05B45, 05B50.

1 Introduction

A family of 2-dimensional shapes $\mathcal{T}$ is called a *tiling* if they (rotating and reflecting are allowed) cover the whole plane without gaps or overlaps. It is said that a tiling $\mathcal{T}$ is *monohedral* [3] if any two shapes belonging to $\mathcal{T}$ are congruent. If a tiling $\mathcal{T}$ is monohedral, the unique shape in the tiling is called a *tile*. In this article, when we simply say a "tiling," it means a monohedral tiling. We study (monohedral) *k-fold tilings* which are extended to cover the plane with the thickness of k folds and *k-fold tiles* which belong to it. More specifically, a family of 2-dimensional shapes $\mathcal{T}$ is called a k-fold tiling if any point in $\mathbb{R}^2$ which is not on the boundary of any shape in $\mathcal{T}$ is covered with exactly k shapes in $\mathcal{T}$, and the shape belonging to $\mathcal{T}$ is called a k-fold tile. Since we can obtain a k-fold tiling by piling up k sheets of a tiling, it is trivial to consider a k-fold tiling with (1-fold) tiles. Hence we are interested in 2-dimensional shapes with the property: not a tile, but a $k(\geq 2)$-fold tile. We call a 2-dimensional shape having this property a *nontrivial k-fold tile*.

If all shapes in k-fold tiling $\mathcal{T} = \{T_1, T_2, \dots\}$ are translates of T_1, then $\mathcal{T}$ is called a *k-fold translative tiling* [5, 6], and the shape belonging to $\mathcal{T}$ is called a *k-fold translative tile* [5, 6]. The origin of the study of multiple tilings is the one by Furtwängler [2] in 1936. He considered trivial multiple tilings as a generalization of Minkowski's conjecture [4]. As far as we know, nontrivial multiple tilings were first investigated by Bolle [1] who gave a necessary and sufficient conditions for a convex polygon to construct a multiple translative tiling with regularity (called a *multiple lattice tiling* [5, 6]). Recently, Yang and Zong [5] gave a characterization of all convex k-fold translative tiles for any $k = 2, 3, 4, 5$. Although there is various research on multiple translative tilings other than those mentioned above, there seems to be no existing research on multiple tilings allowing rotations and reflections. Therefore, the subjects of our research is such nontrivial multiple tilings. In this article, we mainly consider polyominoes: plane shapes formed by joining one or more congruent squares edge to edge, and convex lattice polygons as basic 2-dimensional shapes and present some properties.

2 Preliminaries

We introduce the following terms.

Definition 2.1. If a shape P is a k-fold tile, then k is a *tile-fold number* of P. The set of tile-fold numbers of P is denoted by $\mathrm{TFN}(P)$. If an integer k satisfies that $k \in \mathrm{TFN}(P)$ and $h \notin \mathrm{TFN}(P)$ for every positive integer $h < k$, then we call k the *minimum tile-fold number* of P, and it is denoted by $\tau^{\bullet}(P)$ [6].

Definition 2.2. Let k and h be positive integers. If an h-omino P is a nontrivial k-fold tile and there is no h'-omino that is a nontrivial k-fold tile for any positive integer $h' < h$, then h is called the *minimum size of nontrivial k-fold-tile polyomino* and P is called a *minimum-sized nontrivial k-fold-tile polyomino.*

3 Main Results

We show the following theorems.

Theorem 3.1. *For any integer $k \geq 2$, there exists a polyomino P that satisfies $\tau^{\bullet}(P) = k$.*

Theorem 3.2. *For any integer $k \geq 2$, the minimum size of nontrivial k-fold-tile polyomino is 7, and the heptominoes C7, F7, and X7 listed in Fig. 1 are all minimum-sized nontrivial k-fold-tile polyominoes. Furthermore, the heptomino G7 in Fig. 1 is also a minimum-sized nontrivial k-fold-tile polyomino for every $k \geq 2$ except for $k = 3, 5$.*

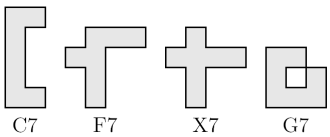

Figure 1: The heptominoes in Theorem 3.2

Theorem 3.3. *For any integer $k = 5$ or $k \geq 7$, there exists a convex unit-lattice polygon that satisfies*

(i) *a nontrivial k-fold tile and*

(ii) *the area is k.*

(iii) *it is a hexagon if $k = 5$ or 8; an octagon otherwise.*

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Extended Abstract
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Orthogonal Fold & Cut

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Abstract

We characterize the shapes that can be produced by "Orthogonal Fold & Cut": folding a rectangular sheet of paper on vertical and horizontal creases, and then making a single straight cut. We also solve a handful of simpler related problems: Orthogonal Fold & Punch, 1-Dimensional Fold & Cut, Signed 1-Dimensional Fold & Cut, and 1-Dimensional Interval Fold & Cut.

Keywords: computational origami, folding, cutting, orthogonal, characterization.

2010 MSC: Primary 68Q25; Secondary 52C99.

1 Introduction

Given a rectangular piece of paper marked with some line segments called *cuts*, *Fold & Cut* asks us to fold the paper into a flat configuration and then make a single infinite cut to cut exactly along the marked line segments. More formally, we wish to fold the paper such that the cuts all lie on the same line, and no other part of the paper is on the line. It turns out that Fold & Cut is solvable for any (finite) set of line segments [2, 3].

Orthogonal Fold & Cut asks the same question, but we are now restricted to *orthogonal* folds, meaning every crease must be parallel to an edge of the rectangle. Again we use only a single cut, which may be at any angle. It is easy to construct sets of cuts which cannot be obtained this way, but some complicated-looking shapes still can be. Figures 1 and 2 show example instances and their solutions.

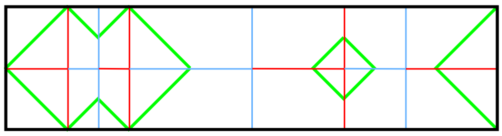

Figure 1: An example instance of Orthogonal Fold & Cut (bold green lines), and the crease pattern our algorithm generates (thin red mountains and blue valleys).

We characterize when Orthogonal Fold & Cut is solvable. In addition, we provide characterizations for several simpler related problems:

- Given a line segment of paper with marked *cut points*, *1-Dimensional Fold & Cut* asks us to fold the line segment and make a single (0-dimensional) cut that hits exactly the cut points. We do not allow folds at cut points.
- In *Signed 1-Dimensional Fold & Cut*, each cut point is given a sign, either positive or negative, and we are asked to have the paper oriented according to the sign at each cut point. Intuitively, the cut points are marked on only one side, and they need to all be face up.

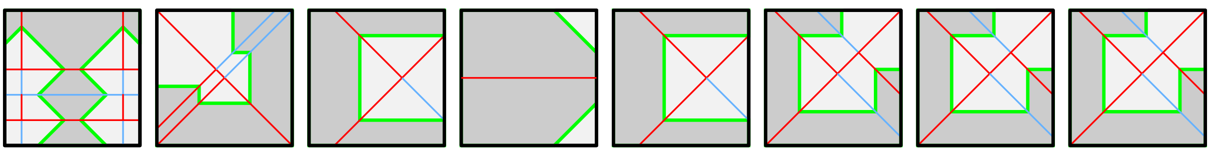

Figure 2: A few letters from our Orthogonal Fold & Cut mathematical font, in the notation of Figure 1.

- In *1-Dimensional Interval Fold & Cut*, we are asked to fold a line segment so that specified *cut intervals* lie on a common interval in the folded state, and no other part of the line segment lies on this interval. Additionally, the cut intervals contain some marked creases which are required to be folded, and no other folds can be made within any cut interval.
- In *Orthogonal Fold & Punch*, we are given a rectangular piece of paper marked with points called *holes*, and are asked to fold it flat using only orthogonal folds and punch out a single point to remove exactly the holes. The nonorthogonal Fold & Punch problem always has a solution [1].

2 Results

Our characterizations take the form of defining an easy-to-compute "canonical" crease pattern and showing that, if the problem has any solution, then the canonical one works.

Lemma 2.1. *1-Dimensional Fold & Cut is always solvable.*

Lemma 2.2. *Signed 1-Dimensional Fold & Cut is solvable if and only if the signs of cuts alternate.*

Refer to Figure 3. The canonical crease pattern for Signed and unsigned 1D Fold & Cut puts a crease point at the midpoint between each consecutive pair of cut points. For 1D Interval Fold & Cut, the canonical crease pattern has a crease at each required crease and at the midpoint between each pair of consecutive cut intervals.

Lemma 2.3. *An instance of 1-Dimensional Interval Fold & Cut is solvable if and only if the canonical crease pattern is a solution.*

Figure 3: The canonical solution for an instance of 1-Dimensional Fold & Cut, and the canonical solution for an instance of 1-Dimensional Interval Fold & Cut.

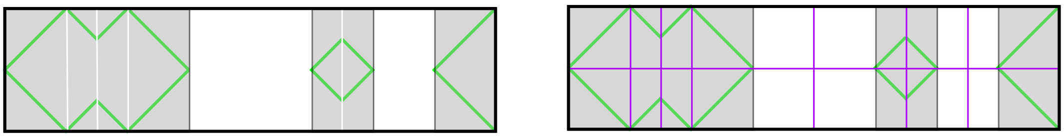

Figure 4: An instance of Orthogonal Fold & Cut with vertical bands shaded (left), and its (unsigned) canonical crease pattern (right).

For the main problem, Orthogonal Fold & Cut, we first note that the slopes of cut lines in a solvable instance must all be $\pm\alpha$ for a common α. The degenerate cases $\alpha = 0$ and ∞ are easy, and otherwise we can scale the paper so $\alpha = 1$, as in Figure 4. Next, local constraints imply that some intervals, which we call *bands*, must not have vertical creases, and that there are vertical creases at some particular positions. We call other intervals *stripes*, including zero-width stripes at positions required to have creases. Our main result characterizes the solvable instances of Orthogonal Fold & Cut in terms of stripes and bands, by factoring the instance into two othogonal instances of 1D Interval Fold & Cut.

Theorem 2.4. *Consider an instance of Orthogonal Fold & Cut on rectangular paper in which every crease has finite nonzero slope $\pm\alpha$. We call the crease pattern which puts one vertical (resp. horizontal) crease at the center of each vertical (resp. horizontal) stripe, including zero-width stripes, the* canonical *crease pattern. If the instance is solvable, then the canonical crease pattern is a solution.*

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

Redundancy of Linear Codes with Graph Constraints

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Abstract

In this paper we use the probabilistic method to obtain bounds for the redundancy of linear codes with graphical constraints and determine sufficient conditions for existence of such codes.

Keywords: Linear Codes, Graphical Constraints, Redundancy, Probabilistic Method.

2010 MSC: Primary 05C90; Secondary 05C50.

1 Introduction

Linear codes with graph based constraints arise often in applications like for example, low density parity check (LDPC) codes and expander codes [1,3] that requires that the bipartite graph representation satisfy an *expansion* property with respect to the left nodes. In this paper, we consider *generalized* constraints and use random bipartite graphs to obtain linear codes that attain the Gilbert-Varshamov redundancy bound in addition to satisfying the said constraints.

2 Linear Codes With Graph Constraints

Consider a random bipartite graph with left vertex set $X = \{a_1, a_2, \ldots, a_n\}$ and right vertex set $Y = \{b_1, b_2, \ldots, b_m\}$ obtained as follows. Let $\{X_{i,j}\}_{1 \leq i \leq n, 1 \leq j \leq m}$ be independent and identically distributed (i.i.d.) binary random variables with $\mathbb{P}(X_{1,1} = 1) = p = 1 - \mathbb{P}(X_{1,1} = 0)$ where $0 < p < \frac{1}{2}$ is a constant. Throughout, constants do not depend on n. An edge is present between vertices a_i and b_j if and only if $X_{i,j} = 1$. Let $m = n\epsilon$ for some constant $\epsilon > 0$ and let $G = G_n$ be the resulting random graph defined on the probability space $(\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. For simplicity we drop the subscript from $\mathbb{P}_n$ henceforth.

For $1 \leq j \leq m$ let $\mathcal{R}_j$ be the set of neighbours of the right vertex b_j and define the (random) code $\mathcal{C} = \mathcal{C}(G)$ as follows. A word $\mathbf{c} = (c_1, \ldots, c_n) \in \mathcal{C}$ if and only if $\sum_{i \in \mathcal{R}_j} c_i = 0 \mod 2$ for all $1 \leq j \leq m$. By construction (see [1]), the code $\mathcal{C}$ is linear (i.e. a vector subspace of $\{0,1\}^n$) with redundancy $1 - \frac{\log(\#\mathcal{C})}{n} \leq \frac{m}{n} = \epsilon$. We define the *distance* between $\mathbf{c} = (c_1, \ldots, c_n)$ and $\mathbf{d} = (d_1, \ldots, d_n)$ to be $d(\mathbf{c}, \mathbf{d}) := \sum_{i=1}^{n} \mathbb{1}(c_i \neq d_i)$, where $\mathbb{1}(.)$ refers to the indicator function. We also define the minimum distance $d_H(\mathcal{C})$ of the code $\mathcal{C}$ to be the minimum distance between any two codewords of $\mathcal{C}$ and set the relative distance of $\mathcal{C}$ to be $\delta_H(\mathcal{C}) := \frac{d_H(\mathcal{C}) - 1}{n}$.

An $n-$length *constraint* $\mathcal{E}_n$ on the random code $\mathcal{C}$ is an event in $\mathcal{F}_n$. For example, the event $\mathcal{H}_n$ that for each $2 \leq i \leq \sqrt{n}$ the right vertices b_{i-1} and b_{i+1} both have the left vertex a_i as a neighbour is an example of a constraint. We say that the random code $\mathcal{C}$ satisfies the constraint $\mathcal{E}_n$ if $G \in \mathcal{E}_n$. Letting $H(x) := -x \cdot \log x - (1-x) \cdot \log(1-x)$ be the (binary) entropy function, where logarithms are to the base two throughout and using the notation $o(1) \longrightarrow 0$ as $n \to \infty$, we have the following result.

Theorem 2.1. *Let $0 < \delta < \frac{1}{2}$ be any constant and let $\mathcal{E}_n, n \geq 1$ be constraints satisfying $\frac{\log \mathbb{P}(E_n)}{n} = o(1)$. For all n large, there exists a deterministic $n-$length linear code $\mathcal{D}_n$ with relative distance at least δ, redundancy at most $H(\delta) + o(1)$ and satisfying the constraint $\mathcal{E}_n$.*

Thus if the constraints are not too severe, then there are codes that achieve the Gilbert-Varshamov redundancy bound [2] of $H(\delta) + o(1)$ in addition to satisfying the said constraints. The event $\mathcal{H}_n$ described before occurs with probability at least $p^{2\sqrt{n}}$ and therefore satisfies the conditions of Theorem 2.1.

Proof of Theorem 2.1: The proof of Theorem 2.1 consists of two steps. In the first step, we show that the minimum distance of the random code $\mathcal{C}$ obtained above is at least $\delta n+1$ with high probability i.e., with probability converging to one as $n \to \infty$. We then incorporate the constraints into $\mathcal{C}$ to obtain the desired code. Throughout we let $\epsilon > H(\delta)$ be a constant and let $m = n\epsilon$ be the number of parity (right) nodes in the graph G so that the size of $\mathcal{C}$ is at least $2^{n(1-\epsilon)}$.

For a set $\mathcal{S} \subseteq \{1,2,\ldots,n\}$ we define the word $\mathbf{v}(\mathcal{S}) = (v_1,\ldots,v_n)$ satisfying $v_i = \begin{cases} 1 & \text{if } i \in \mathcal{S} \\ 0 & \text{otherwise.}\end{cases}$ Letting $\mathcal{S} = \{a_{l_1},\ldots,a_{l_g}\}$ be any set of left vertices we upper bound the probability that $\mathcal{S}$ would cause no parity check violations; i.e. we estimate the probability that the word $\mathbf{v}(\mathcal{S})$ belongs to the code $\mathcal{C}$. We consider two cases depending on whether $\#\mathcal{S} \leq t$ or not, for some integer constant $t \geq 1$ to be determined later.

Case I ($\#\mathcal{S} = g \leq t$): For $1 \leq i \leq g$ let $\mathcal{N}_i$ be the set of neighbours of the left vertex a_{l_i}. Let $\mathcal{M}_i := \mathcal{N}_i \setminus \left(\bigcup_{\substack{1\leq l\leq t\\ l\neq i}} \mathcal{N}_l\right)$ be the set of (unique) neighbours of vertex a_{l_i} not adjacent to any of the remaining vertices in $\mathcal{S}\setminus\{a_{l_i}\}$. Defining $Y_{i,j} := 1\!\!1(b_j \in \mathcal{M}_i)$ we then have that $\{Y_{i,j}\}_{1\leq j\leq m}$ are i.i.d. for any $1 \leq i \leq g$, with $\mathbb{P}(Y_{i,j}=1) = p(1-p)^{g-1} = 1-\mathbb{P}(Y_{i,j}=0)$. Thus $\mathbb{P}(\mathcal{M}_i = \emptyset) = \left(1-p(1-p)^{g-1}\right)^m \leq e^{-mp(1-p)^{g-1}}$ and consequently $\mathbb{P}\left(\bigcup_{1\leq i\leq g}\{\mathcal{M}_i=\emptyset\}\right) \leq ge^{-mp(1-p)^{g-1}}$. If the event $E(\mathcal{S}) := \bigcap_{1\leq i\leq g}\{\mathcal{M}_i \neq \emptyset\}$ occurs, then the word $\mathbf{v}(\mathcal{S}) \notin \mathcal{C}$. Therefore if the event $E_{low} := \bigcap_{\mathcal{S}} E(\mathcal{S})$ occurs where the intersection is with respect to all subsets $\mathcal{S} \subset \{1,2,\ldots,n\}$ of size $g \leq t$, then the minimum distance of any word in $\mathcal{C}$ from the all zeros codeword is at least $t+1$. Since $\mathcal{C}$ is linear this implies that the minimum distance of $\mathcal{C}$ is at least $t+1$.

If E^c_{low} is the complement of E_{low}, then $\mathbb{P}(E^c_{low}) \leq \sum_{g=1}^t g\binom{n}{g}e^{-mp(1-p)^{g-1}} \leq t^2\binom{n}{t}e^{-mp(1-p)^{t-1}}$ for $t < \frac{n}{2}$. Using $\binom{n}{t} \leq \left(\frac{ne}{t}\right)^t$, we get that $\mathbb{P}(E^c_{low}) \leq e^{-\Delta_0}$ where $\Delta_0 := mp(1-p)^{t-1} - t\log\left(\frac{ne}{t}\right) - 2\log t$. Since $m = \epsilon n$ and p and t are constants, we have that $\Delta_0 \geq \frac{m}{2}p(1-p)^{t-1} \geq 4C\cdot n$ for all n large and some constant $C > 0$.

Case II ($t+1 \leq \#\mathcal{S} \leq \delta n$): For a right vertex b_j, we recall that $\mathcal{R}_j$ is the random set of (left) neighbours of the vertex b_j. Define the event $F_j(\mathcal{S}) := \{\#(\mathcal{R}_j \cap \mathcal{S}) \text{ is odd}\}$. If $F_j(\mathcal{S})$ occurs, then the word $\mathbf{v}(\mathcal{S})$ would cause a parity check violation at the right vertex b_j. Therefore if $\bigcup_{1\leq j\leq m} F_j(\mathcal{S})$ occurs, then $\mathbf{v}(\mathcal{S}) \notin \mathcal{C}$. Define the event $E_{up} := \bigcap_{\mathcal{S}}\left(\bigcup_{1\leq j\leq m} F_j(\mathcal{S})\right)$ where the intersection is with respect to all sets $\mathcal{S}$ whose cardinality lies between $t+1$ and δn. Extending the above argument we see that if E_{up} occurs, then there is no word in $\mathcal{C}$ whose distance from the all zeros codeword lies between $t+1$ and δn. Combining, we have that if $E_{low} \cap E_{up}$ occurs, then the minimum distance of the code is at least $\delta n+1$.

For any right vertex b_j, the number of left neighbours $\#\mathcal{R}_j$ is Binomially distributed with parameters n and p. Therefore for a deterministic set $\mathcal{S}$ with $\#\mathcal{S} = g$, the cardinality of the random set $\#(\mathcal{R}_j \cap \mathcal{S})$ is Binomially distributed with parameters g and p. Therefore $\mathbb{P}(F^c_j(\mathcal{S})) = \sum_{\substack{0\leq k\leq g\\ k \text{ even}}}\binom{g}{k}p^k(1-p)^{g-k} = \frac{1}{2}+\frac{1}{2}(1-2p)^g$. Let $0 < \eta < \frac{1}{2}$ be a small constant. Using $g \geq t+1$ and choosing t sufficiently large, we then get that $\mathbb{P}\left(F^c_j(\mathcal{S})\right) \leq \frac{1}{2^{1-\eta}}$ and so $\mathbb{P}\left(\bigcap_{j=1}^m F^c_j(\mathcal{S})\right) \leq \left(\frac{1}{2^{1-\eta}}\right)^m = \frac{1}{2^{(1-\eta)\epsilon n}}$. Further, there are $\binom{n}{g}$ sets of cardinality g and so using standard Hamming ball estimates (Proposition 3.3.1 [2]), we get that $\mathbb{P}\left(E^c_{up}\right) \leq \left(\sum_{g=t+1}^{\delta n}\binom{n}{g}\right)\cdot\frac{1}{2^{(1-\eta)\epsilon n}} \leq \frac{1}{2^{\beta n}}$, where $\beta := (1-\eta)\epsilon - H(\delta)$. Since $\epsilon > H(\delta)$ strictly, we choose $\eta > 0$ small enough so that β is strictly positive. Fixing such an η, we define $E_{dist} := E_{low} \cap E_{up}$ and get that $\mathbb{P}(E_{dist}) \geq 1 - e^{-4Cn} - \frac{1}{2^{\beta n}} \geq 1 - e^{-3Cn}$ for all n large, where the constant $C > 0$ is as in the estimate for Δ_0 in Case I above.

Finally, to incorporate the constraints, we use the fact that $\mathcal{E}_n$ occurs with probability at least e^{-Cn} for all n large and so $\mathbb{P}(\mathcal{E}_n \cap E_{dist}) \geq e^{-Cn} - e^{-3Cn} > 0$. This implies that there exists an $n-$length linear code with relative distance at least δ, redundancy at most ϵ, and satisfying the constraint $\mathcal{E}_n$. $\square$

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Extended Abstract
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Random Derangement with Fixed Number of Cycles

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Abstract

We study the uniform sampling of permutations without fixed points, or derangements, with m cycles. Since the number of cycles in a random derangement tends to the standard distribution, using rejection sampling to randomly select derangements with the desired number of cycles can take a long time. We propose a method for generating a uniformly random n-derangement with m cycles in time $O(n^3)$ using dynamic programming.

Keywords: derangement, random generation, dynamic programming, polynomial time algorithm.

2010 MSC: Primary 05A05; Secondary 68R05, 90C39, 11B73.

1 Introduction

A derangement is a permutation without fixed points. A notation $\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{pmatrix}$ denotes the permutation; thus, a derangement requires $\forall i[i \neq \pi(i)]$. Also, n-permutation and n-derangement denote a permutation and a derangement over a set of n items respectively. Pierre Remond de Montmort proposed a problem for counting the number of distinct derangements in the first edition of his essay, then proved the result in the second edition, which Nicolaus I Bernoulli also derived about the same time [1]. The result size of n-derangements, in its simplified form, is $d(n) = n! \sum_{i=0}^{n} (-1)^i / i!$, using the inclusion-exclusion principle.

There are $d(n)/n! \approx 1/e$ chance that a random permutation will become a derangement. Thus, arises a simple rejection sampling algorithm that has an expectation of $O(n)$ number of randoms until a derangement is found. Recently, there have been efforts to efficiently generate a random derangement by reducing the number of tries [3–5].

A permutation may be viewed as a set of disjoint cycles, where each cycle is a list of elements after repeatedly applying the permutation until arriving at the initial element. A cycle of length k may be written as a k-cycle. For example, the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 2 & 6 & 4 & 5 \end{pmatrix}$ has three cycles. The first cycle is (1), which is 1-cycle, or a fixed point. Next is $(2, 3)$, a 2-cycle, also called a transposition. Lastly, $(6, 5, 4)$, in that order, is a 3-cycle.

Sattolo also gave an algorithm for generating a cyclic permutation, an n-derangement with exactly one n-cycle [7]. The number of such permutations are only $(n-1)!$, however.

A Stirling number of the first kind, denoted by $s(n, m)$, counts the number of distinct n-permutations with m cycles [6]. Besides, Flajolet and Soria have shown that the distribution of the number of cycles of derangements tends to the standard distribution [2]. Consequently a rejection sampling technique for finding derangement of a specific number of cycles is made difficult.

Employing a dynamic programming technique, we propose a method for generating a uniformly random n-derangement with m cycles in $O(n^3)$ time.

2 Method and Result

The outline for our algorithm works as follow, given n items and m cycles, we compute a 3-dimensional table. Then we uniformly sample an index i over possible number of distinct n-derangement with m cycles. We use the index i to reconstruct the type of the permutation first, then find the index cyclic permutation for cycle of each length.

Let *type* t of an n-permutation be an n-tuple of integers such that for $1 \leq b \leq n$, t_b counts the number of b-cycle of the permutation. For example, $\pi = \left(\begin{smallmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 1 & 3 & 5 & 8 & 7 & 6 \end{smallmatrix}\right)$ has two fixed points, one transposition, and a 4-cycle. Thus, the type of the permutation is $t = (2, 1, 0, 1, 0, 0, 0, 0)$. To simplify the notation, we may think of a type as having infinite trailing zero's. Addition of types is simply vector addition. The number of distinct n-permutations of type t are $\left(n!/\prod_b b^{t_b}(t_b)!\right)$.

We use a function $T(n, m)$ to compute a set of distinct types of n-derangement with m cycles, which can be described by this recurrence relation

$$\begin{aligned} T(n,1) &= \Big\{ \overbrace{(0, \cdots, 0, 0, 1)}^{n\text{-tuple}} \Big\}, \\ T(n,m) &= \Big\{ \underbrace{(0, \cdots, 0, 0, 1)}_{b\text{-tuple}} + t \;\Big|\; 2 \leq b \leq \left\lfloor \frac{n}{m} \right\rfloor, t \in T(n-b, m-1), a < b, t_a = 0 \Big\}. \end{aligned}$$

Using dynamic programming approach, we evaluate the recurrence into a 3-dimensional table A, where $A(n, m, r)$ counts distinct n-derangements with m cycles where the length of each cycle is no less than r. Each entry in A can be computed in $O(1)$ time. Note that $r \leq n$; thus, $O(n^2 m)$ time is required to compute the table.

To find a random n-derangement of m cycles, we uniformly sample an index $1 \leq i \leq A(n, m, 2)$. Next, we reconstruct the type of the permutation corresponding to the index i by backtracking the table in n steps. For each step b we considering every possibility of using any number of b-cycles. Thus, the complexity is $O(n^3)$ time for type reconstruction. Finally, we split the index i into subindexes, and find the indexed permutation for each cycle in the reconstructed type, which, for each set of b-cycle, can be done in $O(b^2)$ time. Hence, we obtain an algorithm with overall $O(n^3)$ time.

Theorem 2.1. *There exists an algorithm to generate a uniform random n-derangement with m cycles that has running time complexity $O(n^3)$.*

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Extended Abstract
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Multiple Hook Removing Game

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Abstract

The Hook Removing Game is a combinatorial game with positions represented by Young diagrams. It was invented in 1970 by Mikio Sato, who also found a formula for the $\mathcal{G}$-values* [2, 3]. Young diagrams also correspond to symmetric groups. More generally, there are several companion diagrams to Young diagrams, called d-complete posets, to which some Weyl groups correspond. In 2001, Noriaki Kawanaka showed that the Hook Removing Game using d-complete posets, called the Plain Game, can be written as a closed expression of the $\mathcal{G}$-values [1].

In this study, we investigated the properties of hooks in numbered rectangular Young diagrams and the $\mathcal{G}$-values of positions represented by such diagrams in the Multiple Hook Removing Game†. In particular, we obtained closed expressions for the $\mathcal{G}$-values in the case of the diagrams whose starting position is $1 \times n$, $2 \times n$, $n \times n$, and $n \times (n+1)$, respectively. Moreover, we have clarified the relationship between the $\mathcal{G}$-values in the case of the diagrams whose starting position is $m \times n$ and $m \times (n+1)$.

Keywords: Combinatorial Game, Hook Removing Game, $\mathcal{G}$-value, Young Diagram, Weyl Group.

2010 MSC: Primary 91A46; Secondary 05E10.

1 Introduction

Let $Y_{m,n}$ be a numbered rectangular Young diagram‡ with m rows and n columns.

Definition 1.1. For box (i, j) of Young diagram Y,

$$h_{Y(i,j)} := \{(i, j)\} \sqcup \{(i', j) \in Y \mid i' > i\} \sqcup \{(i, j') \in Y \mid j' > j\}$$

is called the hook corresponding to the box (i, j).

The rules of the Multiple Hook Removing Game are as follows:

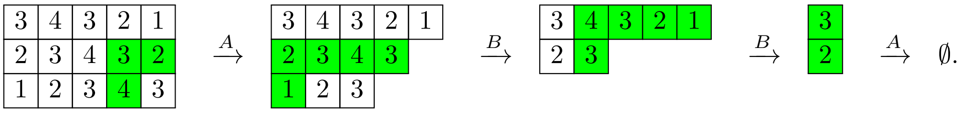
1. Given a numbered rectangular Young diagram, the two players alternate between the following operations:
   (a) Select a box and remove the corresponding hook. Let $C(h)$ be a multiset consisting of the numbers of the removed hook.
   (b) After that, if there is another hook that has the same number as $C(h)$, remove that hook. As long as such a hook exists, repeat this operation.
2. The last player to remove a hook wins.

Example 1.2. A and B are players, and A has the first move.

*The $\mathcal{G}$-value is a value defined for a position in the game, and this value can be used to determine whether the first player has a winning strategy or the second player has a winning strategy (See [4]).

†Based on Tada's work on a representation method using d-complete posets with special coloring for Weyl groups.

‡For $1 \leq i \leq m, 1 \leq j \leq n$, we define the number of box (i, j) as $\min(j - i + m, i - j + n)$.



At player B's turn, two hooks have been removed according to the game rule 1.(b).

Lemma 1.3. *In the Multiple Hook Removing Game, a player can only remove a hook at most* 2 *times.*

2 Main Results

2.1 In the case of $Y_{m,n}$ and $Y_{m,n+1}$

Theorem 2.1. *If $m+n$ is even and $m \leq n$, then the $\mathcal{G}$-value of a position in the Multiple Hook Removing Game whose starting position is $Y_{m,n}$ and the $\mathcal{G}$-value of a position in the Multiple Hook Removing Game whose starting position is $Y_{m,n+1}$ are equal.*

2.2 In the case of $Y_{1,n}$

Theorem 2.2. *In the Multiple Hook Removing Game, the $\mathcal{G}$-value of the starting position $Y_{1,n}$ (i.e., $\mathcal{G}(Y_{1,n})$) is as follows:*

$$\mathcal{G}(Y_{1,n}) = \begin{cases} n & (n \text{ is odd}), \\ n-1 & (n \text{ is even}). \end{cases}$$

2.3 In the case of $Y_{2,n}$

Theorem 2.3. *Let $\mathbb{Z}_{>0}$ be the set of all positive integers. In the Multiple Hook Removing Game, the $\mathcal{G}$-value of the starting position $Y_{2,n}$ (i.e., $\mathcal{G}(Y_{2,n})$) is as follows:*

$$\mathcal{G}(Y_{2,n}) = \begin{cases} 3 \ (n = 2, 3), \\ 2 \ (n = 2 + 8m, 3 + 8m \ (m \in \mathbb{Z}_{>0})), \\ 1 \ (\text{otherwise}). \end{cases}$$

2.4 In the case of $Y_{n,n}$ and $Y_{n,n+1}$

Theorem 2.4. *The Multiple Hook Removing Game in the numbered rectangular Young diagram $Y_{n,n+1}$ and the Hook Removing Game in the shifted Young diagram $(n, n-1, n-2, \dots, 2, 1)$ are isomorphic (i.e., the game trees are equal), and the $\mathcal{G}$-values are equal (even in the intermediate positions).*

Corollary 2.5. *Let $Y_{n,n}$ or $Y_{n,n+1}$ be the position in the Multiple Hook Removing Game and denote it by Z. Then we have*

$$\mathcal{G}(Z) = \bigoplus_{1 \leq k \leq n} k,$$

where $\bigoplus$ denotes the sum of all numbers in binary form without carrying[§]*.*

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[§] This notion is equivalent to "bitwise XOR".

Extended Abstract
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Complete Geometric Graphs with no Partition into Plane Spanning Trees

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Abstract

It is a long-standing conjecture that every complete geometric graph on $2n$ vertices can be partitioned into n plane spanning trees. Despite several approaches, it remained open to this day. In this note we provide a family of counterexamples, based on bumpy wheel sets.

Keywords: plane spanning trees, bumpy wheel sets.

2010 MSC: Primary 52C15; Secondary 05C05, 05C10.

1 Introduction

A *geometric graph* $G = (P, E)$ consists of a set of vertices P, which are points in the plane in general position, and a set of straight-line edges E. Hurtado et al. [4] introduced the famous and long-standing conjecture that the edges of any complete geometric graph on $2n$ vertices can be partitioned into n plane spanning trees (see also [1]). Despite a lot of effort, very little is known about this problem, namely only for P being in convex position [2] or in regular wheel configuration [6] an affirmative answer is known. Also, many closely related problems have been studied, e.g. in the context of packing plane spanning trees into complete graphs: Biniaz and García [3] showed that $\lfloor n/3 \rfloor$ plane spanning trees can always be packed into a complete geometric graph.

In this note, we provide a family of counterexamples to the conjecture above. The point sets we are using for our construction, so-called *bumpy wheel sets*, have been introduced in [5]. For positive parameters k and ℓ, a set P of $k \cdot \ell + 1$ points is in *bumpy wheel configuration* if $k \cdot \ell$ of its points are placed along a circle into k groups each containing ℓ points and the remaining point being the center of the circle. The points within each group are ε-close (see Figure 1 for an illustration). We denote the corresponding complete geometric graph by $BW_{k,\ell}$. Since we are only concerned with graphs having an even number of vertices, we consider $BW_{k,\ell}$ only for odd parameters k and ℓ. For $BW_{3,5}$ and $BW_{3,7}$ we show the following even stronger statement:

Theorem 1.1. *The edges of $BW_{3,5}$ cannot be partitioned into $\frac{n}{2} = 8$ color classes such that each color class is plane, contains exactly $n - 1 = 15$ edges, and is triangle free. $BW_{3,7}$ can't even be partitioned into $\frac{n}{2}$ plane subgraphs.*

In this note we will present a computer assisted proof based on an ILP formulation. On the other hand, we omit a pen and paper proof that $BW_{k,\ell}$ cannot be partitioned into plane spanning trees for any odd $k \geq 3$ and $\ell \geq 5$ (which is based on structural insights) from this version due to space constraints.

2 The ILP Model

In this section we give an overview of the computer assisted proof of Theorem 1.1. Given a geometric graph $G = (P, E)$ and a fixed number m of available colors as input, our ILP contains a binary variable $x_{e,c}$ for each edge-color combination, i.e. in our setting there are $\binom{n}{2} \cdot m$ variables. $x_{e,c}$ being 1 then corresponds to edge e receiving color c.

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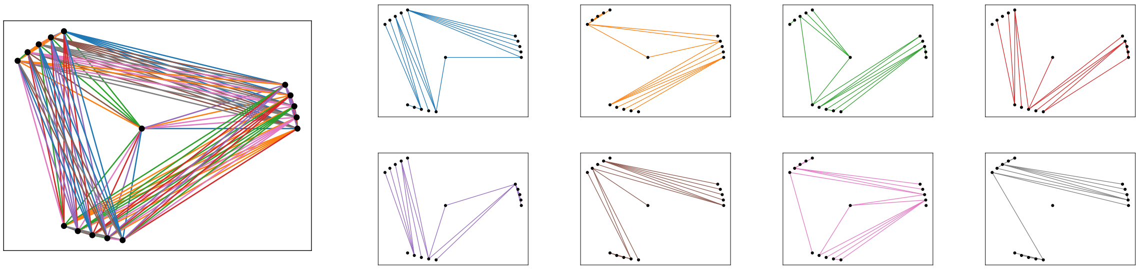

Figure 1: Partition of the bumpy wheel $BW_{3,5}$ into 8 plane subgraphs. A partition into plane spanning trees is not possible.

We need to implement the following constraints, enforcing that every edge receives exactly one color, crossing edges receive different colors, ensuring $n-1$ edges in each color class, and forbidding monochromatic triangles:

$$\sum_{c=1}^{m} x_{e,c} = 1 \qquad \forall e \in E \tag{2.1}$$

$$x_{e,c} + x_{f,c} \leq 1 \qquad \forall c \in \{1,\ldots,m\};\ \forall e,f \text{ crossing} \tag{2.2}$$

$$\sum_{e \in E} x_{e,c} = n-1 \qquad \forall c \in \{1,\ldots,m\} \tag{2.3}$$

$$x_{e,c} + x_{f,c} + x_{g,c} \leq 2 \qquad \text{for each triangle } e,f,g;\ \forall c \in \{1,\ldots,m\} \tag{2.4}$$

All experiments were run on an Intel Core i5, 1.6 GHz, 16 GB RAM running macOS Big Sur Version 11.4. All algorithms were implemented in Python 3.9.1, and for solving the ILP we used Gurobi Optimizer Version 9.1.2 with default settings.

For $BW_{3,5}$ and $m = 8$ as input, our program reports an infeasible ILP (taking less than a minute). Furthermore, for $BW_{3,7}$ and $m = 11$ as input our program reports an infeasible ILP even if omitting the constraints (2.3) and (2.4) (taking roughly 5h). Figure 1 shows a partition of $BW_{3,5}$ into plane subgraphs found by the program, if omiting the triangle constraint (2.4).

3 Outlook

A natural generalization would be to partition complete geometric graphs into k-planar or k-quasiplanar subgraphs, which is part of our currently ongoing work on this project.

Acknowledgments. We are particularly grateful to Patrick Schnider for suggesting the bumpy wheel sets. We also thank the organizers and participants of the 5^{th} DACH Workshop on Arrangements and Drawings, that took place in March 2021 (online) and was funded by Deutsche Forschungsgemeinschaft (DFG), the Austrian Science Fund (FWF) and the Swiss National Science Foundation (SNSF).

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Extended Abstract
The 23rd Thailand-Japan Conference on
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On the Size of Minimal Separators for Treedepth Decomposition

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Abstract

The treedepth of a graph is a parameter that describes how it is close to a star graph. Recently, many parameterized algorithms have been developed for solving various problems on graphs with small treedepth. Since those algorithms are based on treedepth decompositions of input graphs, many solvers are developed to find optimal treedepth decompositions. Many practical solvers enumerate minimal separators to construct a treedepth decomposition in a top-down way. The bottleneck of these algorithms is often the enumeration of minimal separators. Therefore, to speed up the computation, it is essential to reduce the number of minimal separator candidates that are used for an optimal solution. We show that, to obtain an optimal treedepth decomposition, it is sufficient to consider only minimal separators of size at most 2tw, where tw denotes the treewidth of the input graph. We also show that this upper bound is tight up to a constant.

Keywords: Graph Algorithms, Treedepth, Minimal Vertex Separator, Treewidth.

2010 MSC: Primary 68R10; Secondary 05C85, 05C40, 68W40.

1 Introduction

Treedepth [9] is a graph parameter that has recently received attentions [7, 8]. Recently, treedepth has been well studied for parameterized algorithms. Given a treedepth decomposition of height k, Iwata et al. [8] gave an $O(km)$ algorithm for Maximum Matching and $O(k(m + n \log n))$ algorithms for several problems including Weighted Matching and Negative Cycle Detection. Hegerfeld and Kratsch [7] developed single-exponential FPT algorithms for connectivity problems including Connected Vertex Cover and Steiner Tree by extending the Cut&Count technique for treewidth by Fomin et al. [5].

To make use of those parameterized algorithms, we need a treedepth decomposition of a small height. The treedepth is defined as the minimum height of all treedepth decompositions. Computing treedepth is NP-hard even for chordal graphs [3]. However, the appearance of the formula is simple, and it has been re-discovered as different parameters in many fields. Denote the set of connected components of a graph G by $\mathcal{C}(G)$. The treedepth of a graph G, denoted as $\mathrm{td}(G)$, can be computed as follows [9]:

$$\mathrm{td}(G) = \begin{cases} \max\limits_{G' \in \mathcal{C}(G)} \mathrm{td}(G') & \text{(if } G \text{ is not connected)} \\ 1 & (|V| = 1) \\ 1 + \min\limits_{v \in V} \mathrm{td}(G \backslash v) & \text{(otherwise),} \end{cases} \tag{1.1}$$

where $G \backslash v$ denotes the graph obtained from G by deleting vertex v. The naive dynamic programming based on Equation (1.1) yields an algorithm of $O^*(2^n)$, where O^* notation denotes the complexity hiding the polynomial factor. Speeding up the computation of treedepth is essential for those parameterized algorithms. Fomin et al. [4] modified the dynamic programming of Equation (1.1) and obtained an exact exponential algorithm of $O^*(1.9602^n)$ time. Reidl et al. [10] proposed a parameterized algorithm which decides whether the treedepth is at most k in $2^{O(k^2)} \cdot n$ time.

2 Top-down Algorithm and its Bottleneck

In addition to the algorithms with theoretical guarantee mentioned in the previous paragraph, the algorithms that work practically fast for real-world graphs are also studied. Two major approaches used for the practical algorithms were the top-down algorithm by recursively choosing minimal separators (e.g. [1]) and the bottom-up positive-instance dynamic programming (e.g. [11]). In the top-down algorithm, the computation is based on the following formula [2]:

$$\mathrm{td}(G) = \begin{cases} \max_{G' \in \mathcal{C}(G)} \mathrm{td}(G') & \text{(if } G \text{ is not connected)} \\ |V| & \text{(if } G \text{ is complete)} \\ |S| + \min_{S \in \Delta} \mathrm{td}(G \backslash S) & \text{(otherwise)}, \end{cases} \tag{2.1}$$

where Δ is the set of all minimal separators of G and $G \backslash S$ denotes the graph obtained from G by deleting vertex set S. Note that when G is complete, $\Delta = \emptyset$. For a connected, non-complete graph G, a minimal separator S is called *optimal top separator* if $\mathrm{td}(G) = |S| + \mathrm{td}(G \backslash S)$. The bottleneck of the computation by Equation 2.1 is the enumeration of minimal separator (i.e., the computation of Δ), because the number of minimal separators is usually quite large and it can be exponential [6]. To speed up the algorithm, it is promising to reduce the number of candidate minimal separators that can be an optimal top separator. Experimentally, it is known that the size of some optimal top separator can be very small. In order to design a practically fast exact algorithm for treedepth, it has been an important challenge to analyze the size of optimal top separators.

3 Main Results

In this paper, we show that we can obtain treedepth decomposition by considering only minimal separators of size at most 2tw, where tw denotes the treewidth of the input graph. We can replace the set Δ at Equation 2.1 by the set $\Delta' = \{S \in \Delta \mid |S| \leq 2\mathrm{tw}\}$ and still obtain $\mathrm{td}(G)$. Moreover, we show that this upper bound is tight up to a constant.

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Hamiltonian Decompositions of Complete 4-partite 3-uniform Hypergraphs

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Abstract

Based on the definition of Hamiltonian cycles given by Katona and Kierstead, we provide a construction of Hamiltonian decompositions of complete 4-partite 3-uniform hypergraphs $K^{(3)}_{4(2m)}$, where $2m$ is the size of each partite set.

Keywords: Hamiltonian decomposition, uniform hypergraph, multipartite hypergraph.

2010 MSC: Primary 05C65.

1 Introduction

A *Hamiltonian decomposition* of a hypergraph is a partition of its hyperedge set into mutually disjoint Hamiltonian cycles. The definition of a Hamiltonian cycle can be extended to hypergraphs in various ways. The definition in this paper based on a Hamitonicity of cycles for k-uniform hypergraphs $\mathcal{H}(V, E)$ of order n given by Katona and Kierstead [5]: a *Hamiltonian cycle* of $\mathcal{H}$ is a cyclic ordering $C = (v_1\ v_2\ \dots\ v_n)$ of all n elements of V such that k consecutive vertices form a hyperedge in E.

The existence problem of a Hamiltonian decomposition for the complete 3-uniform hypergraph $K^{(3)}_n$ has been studied by many authors with a variety of construction techniques. However, the result was established only for some n where $n \le 46$ or $n = 2^m$ for $m \ge 2$ (see [1,3,6,7]). The existence problem was also studied for complete multipartite k-uniform hypergraphs, defined as follows :

Definition 1.1. A *complete multipartite k-uniform hypergraph* $K^{(k)}_{n_1,n_2,\dots,n_t}$ or $K^{(k)}_{n_1,n_2,\dots,n_t}(V_1, V_2, \dots, V_t)$ is a hypergraph with vertex set $V = V_1 \cup V_2 \cup \dots \cup V_t$ where $|V_i| = n_i$ for all $i \in \{1, 2, \dots, t\}$, and

$$E(\mathcal{H}) = \{e : e \subset V, |e| = k \text{ and } |e \cap V_i| < k \text{ for } i \in \{1, 2, \dots, t\}\}.$$

In particular, if $n_i = n$ for all $i \in \{1, 2, \dots, t\}$, then $K^{(k)}_{\underbrace{n, n, \dots n}_{t}}$ is denoted by $K^{(k)}_{t(n)}$.

In literature, the problem of Hamiltonian decompositions of $K^{(k)}_{t(n)}$ has been investigated only for the case $k = 3$ and $t = 2$ and 3. Wang and Jirimutu [4] studied on $K^{(3)}_{n,n}$ when n is a prime number in 2001. Later on, Xu and Wang [7] provided a complete study for all $n \ge 2$ in 2002. The study for complete tripartite 3-uniform hypergraphs, $K^{(3)}_{n,n,n}$, was also completed by Boonklurb *et al.* [2] in 2015. Continuing along this line, we are interested in the case $t = 4$ and n is any even positive integer. In other words, we provide a construction of Hamiltonian decompositions of $K^{(3)}_{4(2m)}$ for all positive integer m.

2 Main Results

We first classify hyperedges of $K^{(3)}_{4(2m)}$ into two types. Let e be a hyperedge of $K^{(3)}_{4(2m)}$, if e contains at most one vertex from each partite set, e is then called a hyperedge of *Type* 1, otherwise (that is e contains two vertices from a partite set) e is called a hyperedge of *Type* 2. Let $\mathcal{T}_i(K^{(3)}_{4(2m)})$ be the subhypergraph of $K^{(3)}_{4(2m)}$ which consists of all hyperedges of Type i for $i \in \{1, 2\}$.

Let $V_i = \{a_1^i, a_2^i, \ldots, a_{2m}^i\}$ for $i \in \{1, 2, 3, 4\}$ be partite sets of $K_{4(2m)}^{(3)}(V_1, V_2, V_3, V_4)$. We will give a construction of $K_{4(2m)}^{(3)}$ by decomposing $\mathcal{T}_1(K_{4(2m)}^{(3)})$ and $\mathcal{T}_2(K_{4(2m)}^{(3)})$ into Hamiltonian cycles separately. In our construction, we write cycle C as $(P_1\ P_2\ \ldots\ P_s)$ if vertices along the cycle C are partitioned into *paths* P_j (a sequence of vertices) along this cycle.

To decompose $\mathcal{T}_1(K_{4(2m)}^{(3)})$, we establish a stronger result by providing a Hamiltonian decomposition of the subhypergraph $\mathcal{T}_1(K_{4(n)}^{(3)})$ for all positive integers n. The construction uses a 1-factorization $\mathscr{F}$ of the complete bipartite graph $K_{n,n}$.

Theorem 2.1. *The hypergraph $\mathcal{T}_1(K_{4(n)}^{(3)})$ has a Hamiltonian decomposition for all positive integer n.*

On the other hand, to decompose $\mathcal{T}_2(K_{4(2m)}^{(3)})$, we use a 1-factorization $\mathscr{F}$ of K_{2m} on the vertex set $\{1, 2, \ldots, 2m\}$ and a collection $\mathscr{D}$ of three permutations of $\{1, 2, 3, 4\}$ with a certain property. We then construct a collection of Hamiltonian cycles, each of which is composed of two paths of order $4m$, $\mathscr{C} = \{C_t(D, F) = (P_1^t\ P_2^t) : t \in \{0, 1, \ldots, n-1\}, D \in \mathscr{D}, F \in \mathscr{F}\}$. The construction is separated into two cases depending on the parity of m.

Theorem 2.2. *The subhypergraph $\mathcal{T}_2(K_{4(2m)}^{(3)})$ has a Hamiltonian decomposition when m is odd.*

Theorem 2.3. *The subhypergraph $\mathcal{T}_2(K_{4(2m)}^{(3)})$ has a Hamiltonian decomposition when m is even.*

Example 2.4. *An illustration of $C_0(D, F)$ which are in the construction for $\mathcal{T}_2(K_{4(2m)}^{(3)})$ when $m = 5$, $D = (1, 3, 4, 2)$ and $F = \{\{j, f(j)\} : j \in \{1, 2, 3, 4, 5\}\}$ is a 1-factor of K_{10} (the vertex set of K_{10} is relabeled as $\{1, 2, 3, 4, 5, f(1), f(2), f(3), f(4)f(5)\}$).*

In Figure 1, for $x \in \{1, 2, 3, 4\}$, each vertex a_ℓ^x in the partite set V_x in the cycle $C_0(D, F)$ is represented by its subscript ℓ. Moreover, the solid lines indicate two consecutive vertices in the same path, and the dash lines indicate two consecutive vertices from different paths.

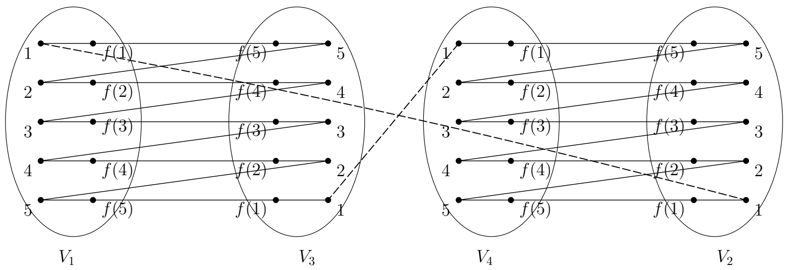

Figure 1: $C_0(D, F)$ of $\mathcal{T}_2(K_{4(10)}^{(3)})$.

Although the construction in Theorem 2.3 is a bit more complicated, the proof of the Hamiltonicity of cycles and the decomposition of $\mathscr{C}$ uses a similar approach as in Theorem 2.2.

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Extended Abstract
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On the Edge Irregularity Strength of Some Disjoint Union Graphs

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Abstract

Let $G = (V(G), E(G))$ be a graph and k be a positive integer. A vertex k-labeling $f : V(G) \to \{1, 2, \ldots, k\}$ is called an edge irregular labeling if there are no two edges with the same weight, where the weight of an edge uv is $f(u)+f(v)$. The edge irregularity strength of G, denoted by $es(G)$, is the minimum k such that G has an edge irregular k-labeling. This labelings were introduced by Ahmad, Al-Mushayt, and Bača in 2014. In this paper, we determine the edge irregularity strength of disjoin union of stars, disjoint union of cycles, and disjoint union of paths.

Keywords: cycle, disjoint union edge irregular labeling, edge irregularity strength, path, star.

2010 MSC: 05C78.

1 Introduction

Graph labeling is a topic in graphs that is growing quite rapidly. Thehe are different types of graph labeling. One of the type of graph labeling is edge irregular labeling. An edge irregular labeling is a kind of vertex labeling, which is a labeling that label only vertices in a graph. The edge irregular labeling was introduced by Ahmad et al. [1] In 2014. The definition of edge irregular labeling is given the the definition below.

Definition 1.1. For an integer k, a total labeling $f : V(G) \to \{1, 2, \ldots, k\}$ is called an edge irregular k-labeling of G if every two distinct edges e_1 and e_2 in E satisfy $w_f(e_1) \neq w_f(e_2)$, where $w_f(e_1 = uv) = f(u) + f(v)$.

Definition 1.2. The minimum k for which a graph G has an edge irregular k-labeling, denoted by $es(G)$, is called the edge irregularity strength of G.

2 Preliminaries

In a paper published in 2014, Martin Baca and Ali Ahmad got a lower bound of the edge irregularity strength of arbitrary graph as follows.

Theorem 2.1. *Theorem 1. [1] Let $G = (V, E)$ be a graph with the maximum degree Δ, then*

$$es(G) \geq \max\left\{\left\lceil \frac{|E(G)|+1}{2} \right\rceil, \Delta(G)\right\}$$

.

In 2020, Ahmad et al performed a computer based experiment dealing with the edge irregularity strength of complete bipartite graphs [2]. In 2017, Ahmad et al gave the edge irregularity strengths of some chain graphs and the join of two graphs. They also introduced a conjecture and open problems for researchers for further research. [3] Besides that, the edge irregularity strength of graphs are given by

Imran et al. in [4]. In the paper, they determined the egde irregularity strength of caterpillars, n-star graphs, kite graphs, cycle chains and friendship graphs. In [5], Tarawneh et al determined he edge irregularity strength of corona product of cycle with isolated vertices. Furthermore, In [6], Tarawneh et al gave the exact value of edge irregularity strength for triangular grid graph , zigzag graph and Cartesian product of P_n, P_m, and P_2.

3 Main Results

Theorem 3.1. *Let S_n be star with $n+1$ vertices and mS_n be m copies of S_n. Then for $m \geq 1$ and $n \geq 1$,*

$$es(mS_n) = \left\lceil \frac{mn+1}{2} \right\rceil.$$

Theorem 3.2. *Let P_n be path with n vertices and mP_n be m copies of P_n. Then for $m \geq 2$ and $n \geq 2$,*

$$es(mP_n) = \left\lceil \frac{mn-m+1}{2} \right\rceil.$$

Theorem 3.3. *Let C_n be cycle with n vertices and mC_n be m copies of C_n. Then for $m \geq 3$ and $n \geq 3$,*

$$es(mP_n) = \left\lceil \frac{mn+1}{2} \right\rceil.$$

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Extended Abstract
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Tiling the Plane Connectively with Wang Tiles

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Abstract

Inspired by the Nintendo game Zelda, the problem of tessellating the plane by translated copies from a set of Wang tiles is generalized to the connected tiling problem. We solve the connected tiling problem for the case that there is only one door color and one wall color.

Keywords: Wang tile, tiling, connectivity, complexity, Zelda.

2010 MSC: Primary 52C20; Secondary 68Q17.

1 Introduction

The motive of this study comes from two sources. The first source is original Domino Problem posed by Wang in 1961 [1]. A *Wang tile* is a unit square with each edge assigned a color. Given a finite set of Wang tiles, we would like to tile the plane with translated copies of them such that the common edge of each pair of neighboring Wang tiles must have the same color. This is known as Wang's *Domino Problem*. This problem has been shown to be undecidable by Berger [2].

The second source is the video game Zelda published by Nintendo. One of the most successful title of the Zelda series is *The Legend of Zelda: Link's Awakening* which was released for the handheld console GameBoy in 1993 and for GameBoy Color in 1998. In 2019, the game was remade and published for the Switch console 26 years after its first release. Besides the huge improvement in graphics, the Switch version is almost identical to the GameBoy version in which the player walks in different 2D dungeons to solve puzzles. One major new features added to the Switch version is a mini-game which allow the gamers to create their own dungeons by putting together square rooms (See Figure 1).

Figure 1: Screenshots of the game Zelda

If we regard the doors of the rooms as one color, and the walls of the rooms another color, the Zelda's dungeon-making mini-game is just a special case of Wang's problem. The Zelda's dungeon-making puzzle and Wang's problem differ in two points: (1) the Zelda's dungeon need not fill up the entire plane; (2) the Zelda's dungeon must be connected, that is, the player must be able to walk from one room to any other rooms of the dungeon through doors.

This inspires us to propose the following problem, which incorporates connectivity into the original Wang's problem.

Problem 1.1 (Connected Tiling Problem)**.** Given two finite sets of colors D (the set of *door* colors) and W (the set of *wall* colors), and a finite set of Wang tiles. Each side of a Wang tile is asigned to one of the colors from either D or W. Can we tile the whole plane by translated copies from this set of Wang tiles such that the tiling is *connected* (*i.e.* walk from one tile to another only through sides with colors in D)?

If $W = \emptyset$, then any tiling must be connected, therefore in this case the problem is the same as the original Wang's problem. So the connected tiling problem includes the original Wang's tiling problem as a special case. As a result, the general connected tiling problem must also be undecidable. But the problem can be solvable if we limit the number of colors. In this paper, we study the case that $|D| = |W| = 1$. We show that in this case the problem is solvable by giving the complete list of sets of Wang tiles which can tile the plane connectively.

2 The Main Result

In this section, we only consider the connected tiling problem with $|D| = |W| = 1$. So the set of all possible 16 Wang tiles are illustrated in Figure 2, with the color in D represented by blue lines.

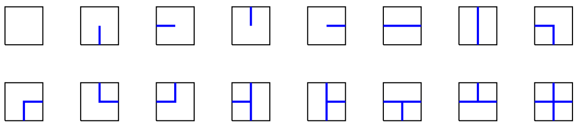

Figure 2: Wang tiles with $|D| = |W| = 1$

There are $2^{16} - 1 = 65535$ different nonempty subsets of these Wang tiles. Our main result is the following theorem which gives a complete classification of the subsets that can tile the plane connectively.

Theorem 2.1. *For the connected tiling problem with $|D| = |W| = 1$, a set of Wang tiles can tile the plane connectively by translated copies if and only if it is one of the following sets:*

1. *sets containing the* 4*-door tile;*
2. *sets containing a pair of opposite* 3*-door tiles;*
3. *sets containing a pair of adjacent* 3*-door tiles, and the two straight* 2*-door tiles;*
4. *sets containing a pair of adjacent* 3*-door tiles, a straight* 2*-door tile and a complement parallel* 1*-door tile;*
5. *sets containing a pair of adjacent* 3*-door tiles, a complement turing* 2*-door tile and two complement* 1*-door tile;*
6. *sets containing one* 3*-door tile and two complement adjacent* 2*-door tiles;*
7. *sets containing the four turing* 2*-door tiles and one more straight* 2*-door tile;*
8. *sets containing the four turing* 2*-door tiles and one* 1*-door tile.*

Sketch of Proof. We prove the theorem on a case-by-case study. For the cases that there exist connected tilings, we will give detailed illustratations of the tilings. For cases without connected tilings, we will show the impossibility by using techniques such as parity argument. Due to page limit, more definitions and details of the proof will be provided in the conference presentation and the full paper. ☐

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Generating Frames via Discretized Substitution Tilings

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Abstract

This work aims to aperiodically discretize Gabor frames for $L^2(\mathbb{R}^d)$. To do so, discrete sets are obtained from aperiodic tilings of $\mathbb{R}^{2d}$ called Pisot family substitution tilings, and the specific class of primitive substitution tiling systems. The resulting aperiodic sets namely model sets and discretized primitive tilings satisfying certain hole conditions generate Gabor frames for $L^2(\mathbb{R}^d)$ with appropriate window functions.

Keywords: Pisot family substitution tiling, aperiodic model set, primitive substitution tiling system, discretized primitive tiling, Gabor frame.

2010 MSC: Primary 52C23, 42C15; Secondary 52C22, 46B15.

1 Introduction

Gabor frames $\mathcal{G}(f, \Lambda)$ based on some window function $f \in L^2(\mathbb{R}^d)$ and some subset $\Lambda \subset \mathbb{R}^{2d}$ are essential tools for signal analysis and quantization. Most research on Gabor frames focuses on finding appropriate windows, or on periodic discretization via lattices. This study aims to extend such results by providing an aperiodic discretization based on aperiodic tilings of $\mathbb{R}^{2d}$ namely Pisot family substitution tilings and primitive substitution tiling systems.

2 Preliminaries

2.1 Model Sets and Discretized Primitive Tilings

A **Pisot family substitution tiling** ϱ is a primitive substitution tiling on $\mathbb{R}^d$ with expansion map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ whose eigenvalues form a Pisot family [1]. If ϱ has finite local complexity (FLC), that is, the tiling has finitely many R-patches $B_R(\varrho)$ up to translations, then a **cut-and-project scheme** associated with ϱ may be formulated. This produces a **model set** $\Lambda(\Gamma) \subset \mathbb{R}^d$ which is an aperiodic Delone set on $\mathbb{R}^d$ with FLC. In this case, we say that ϱ admits a **PST model set**. Define the **hull** $\Omega(\Lambda(\Gamma))$ of a PST model set $\Lambda(\Gamma)$ as the set of all Delone sets Λ' locally isomorphic to $\Lambda(\Gamma)$, which can also be characterized as $\Omega(\Lambda(\Gamma)) = \overline{\{t + \Lambda(\Gamma) : t \in \mathbb{R}^d\}}$ [3].

2.2 Discretized Primitive Tilings

A special family of PST model sets may be directly derived from primitive substitution tiling systems defined as follows. Let $\mathcal{P} = \{P_1, \ldots, P_N\}$ be a set of tiles in $\mathbb{R}^d$ which are complements of quasilattices $\Lambda_1, \ldots, \Lambda_N$ in $\mathbb{R}^d$ such that P_i faces P_j for all $1 \le i, j \le N$. A **primitive substitution rule** ω with scaling factor $\eta > 1$ on $\mathcal{P}$ is a rule which associates with $P_i \in \mathcal{P}$ a patch $\omega(P_i)$ with support ηP_i, whose tiles are translates of elements of $\mathcal{P}$. Furthermore, there exists $m \in \mathbb{N}$ for which $\omega^m(P_i)$ contains a translate of P_j for every $1 \le i, j \le N$ [2]. In this case, $(\mathcal{P}, \omega)$ is called a **primitive substitution tiling system (PSTS)**. Additional assumptions listed in Remark 2.1 ensures the existence of a unique **primitive tiling space** we denote as $\Omega^{(\mathcal{P},\omega)}$ containing every tiling obtainable from the PSTS.

Remark 2.1. Assume that all PSTSs $(\mathcal{P}, \omega)$ have FLC and $\omega : \Omega^{(\mathcal{P},\omega)} \to \Omega^{(\mathcal{P},\omega)}$ is one-to-one so that $\Omega^{(\mathcal{P},\omega)}$ is compact, $\omega : \Omega^{(\mathcal{P},\omega)} \to \Omega^{(\mathcal{P},\omega)}$ is a homeomorphism, and ω is locally invertible. This implies that $\Omega^{(\mathcal{P},\omega)}$ is unique, compact, and contains no periodic tilings [2].

Every primitive tiling $\mathcal{T}$ based on a PSTS $(\mathcal{P},\omega)$ satisfying Remark 2.1 corresponds to an aperiodic discrete set $\Xi(\mathcal{T})$ consisting of vertices of the tiles of $\mathcal{T}$ by considering its control points [1]. In this case, we call $\Xi(\mathcal{T})$ a **discretized primitive tiling**, and define

$$\Omega_{\Xi}^{(\mathcal{P},\omega)} = \left\{\Xi(\mathcal{T}) : \mathcal{T} \in \Omega^{(\mathcal{P},\omega)}\right\}$$

to be the discretized primitive tiling space associated with $(\mathcal{P},\omega)$. In fact, $\Omega_{\Xi}^{(\mathcal{P},\omega)}$ is equal to the hull of any discretized primitive tiling based on $(\mathcal{P},\omega)$.

The **hole** of any $\Lambda \subset \mathbb{R}^d$ is given by $\rho(\Lambda) = \sup_{x\in\mathbb{R}^d} \inf_{\lambda\in\Lambda} d(\lambda, x)$. Given $\Lambda \subset \mathbb{R}^d$ with FLC and any fixed $\delta > 0$, there exist finitely many disjoint translates $\{t_i + \Lambda\}_{i=1}^n$ of Λ such that $\rho(\bigcup_{i=1}^n (t_i + \Lambda)) < \epsilon$ [3]. In this case, we call $\{t_i\}_{i=1}^n$ a set of **δ-translates** of Λ.

2.3 Gabor Frames

The Gabor transform π of any $h \in L^2(\mathbb{R}^d)$ with respect to $x = (x_1, x_2) \in \mathbb{R}^d \times \mathbb{R}^d \cong \mathbb{R}^{2d}$ is defined by

$$\pi(x)h(w) = e^{2\pi i x_2\cdot w} h(w - x_1) \text{ for any } w \in \mathbb{R}.$$

For any function f with $||f||_{L^2} = 1$ in the modulation space $\mathcal{M}^1(\mathbb{R}^d) = \left\{f \in L^2(\mathbb{R}^d) : ||\mathcal{V}_f f||_{L^1} < \infty\right\}$ where $\mathcal{V}_f h(x) := \int_{\mathbb{R}^d} h(w)\overline{f(w - x_1)} e^{-2\pi i x_2\cdot w} dw = \langle h, \pi(x)f\rangle$, a $\delta > 0$ can always be chosen so that

$$\alpha_\delta(f) = \sup_{\substack{x,y\in\mathbb{R}^{2d} \\ |x-y|\le\delta}} ||\pi(x)f - \pi(y)f||_{\mathcal{M}^1} = \sup_{\substack{x,y\in\mathbb{R}^{2d} \\ |x-y|\le\delta}} ||\mathcal{V}_f(\pi(x)f - \pi(y)f)||_{L^1} < 1.$$

With these, the **Gabor frame** for $L^2(\mathbb{R}^d)$ generated by $\{f_i\}_{i=1}^n \subset L^2(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^{2d}$ is the collection

$$\mathcal{G}\left(\{f_i\}_{i=1}^n, \Lambda\right) = \{\pi(\lambda)f_i : 1 \le i \le n, \lambda \in \Lambda\}.$$

3 Main Results

The following are the main results of the study characterizing model sets and discretized primitive tilings generating Gabor frames.

Theorem 3.1. *Let $\delta > 0$. Suppose $\Lambda(\Gamma) \subset \mathbb{R}^{2d}$ is a PST model set in $\mathbb{R}^{2d}$ with FLC, and $\{t_i = (t_{i_1}, t_{i_2})\}_{i=1}^N$ for $t_{i_1}, t_{i_2} \in \mathbb{R}^d$ be a set of δ-translates of $\Lambda(\Gamma)$ so that $\widetilde{\Lambda} = \bigcup_{i=1}^n (t_i + \Lambda(\Gamma))$ satisfies $\rho\left(\widetilde{\Lambda}\right) < \delta$. Then every Λ' in the hull $\Omega(\Lambda(\Omega))$ of $\Lambda(\Gamma)$ generates the Gabor frames $\mathcal{G}(f, \widetilde{\Lambda})$ and $\mathcal{G}(\pi(t_1)f, \dots \pi(t_n)f, \Lambda')$ for $L^2(\mathbb{R}^d)$, for any $f \in \mathcal{M}^1(\mathbb{R}^d)$ with $\alpha_\delta(f) < 1$.*

Theorem 3.2. *Let $\delta > 0$ and $(\mathcal{P},\omega)$ be a primitive substitution tiling system in $\mathbb{R}^{2d}$ satisfying* Remark 2.1 *with $\max_{P\in\mathcal{P}} \left\{\max_{p\in P_\Lambda} d\left(\mathfrak{c}_P, p\right)\right\} = \delta$. Then every $\Lambda \in \Omega_{\Xi}^{(\mathcal{P},\omega)}$ generates a Gabor frame $\mathcal{G}\left(f, \Lambda\right)$ with any window $f \in \mathcal{M}^1(\mathbb{R}^d)$ satisfying $\alpha_{\delta'}(f) < 1$ for some $\delta' \ge \delta$.*

Theorem 3.3. *Let $\delta > 0$. Suppose $f \in \mathcal{M}^1(\mathbb{R}^d)$, and $\delta > 0$ with which $\alpha_\delta(f) < 1$. If $\Lambda \subset \mathbb{R}^{2d}$ is a discretized primitive tiling with $\rho(\Lambda) = \varepsilon$, then $\mathcal{G}(f, \Lambda)$ is a Gabor frame if and only if $\varepsilon \le \frac{\delta}{|\Lambda\cap B_\varepsilon(\mathbf{0})|^{1/2d}}$.*

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Extended Abstract
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Tiling of the Sphere by Congruent Polygons

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Abstract

In an edge-to-edge tilings of the sphere by congruent polygons, the polygon is triangle, quadrilateral, or pentagon. The classification of triangular tilings was completed Ueno and Agaoka in 2002. We almost completed the classification of pentagonal tilings, and have started the classification of quadrilateral tiling. We survey these results, with emphasis on pentagonal tilings.

Keywords: spherical tiling, platonic solid, classification.

2010 MSC: Primary 05B45; Secondary 52B10.

1 Triangular Tilings

The classification of edge-to-edge tilings of the sphere by congruent triangles was started by Sommerville [7], and completed by Ueno and Agaoka [8]. The classification can be divided into two categories:

- Construction from Platonic solids, and their modifications.
- Construction of the earth map kind, and their modifications.

The tilings in the first category have fixed number of tiles, and the tilings in the second category allow arbitrary number of "timezones".

2 Pentagonal Tilings

The pentagonal tilings has the following possible edge length combinations: $a^2b^2c, a^3bc, a^3b^2, a^4b, a^5$. Here is the classification:

1. a^2b^2c [9,11]: Three families of pentagonal subdivisions of the Platonic solids. Each family has two free parameters.
2. a^3bc [9]: Two double pentagonal subdivisions of the Platonic solids. Each has unique pentagon.
3. a^3b^2 [10]: (1) Five families of pentagonal subdivisions of the Platonic solids, each having one free parameter; (2) For three special cases of (1), each with unique pentagon, we have ten flip modifications of the pentagonal subdivision tilings.
4. a^4b [4,5]: If the number of distinct angles is not four, then the tilings are (1) Three families of pentagonal subdivisions of the Platonic solids, each with one free parameter; (2) Three families of earth map tilings, and their flip modifications.
5. a^5 [2]: (1) Three pentagonal subdivision tilings, each with unique pentagon; (2) Four earth map tilings, each with unique pentagon; (3) Flip modification of one tilings from (2).

The only incomplete case is a^4b with four distinct angles.

The classification shows that the tilings can also be divided into two categories, similar to the triangular case.

3 Quadrilateral Tilings

The classification can be divided into three kinds.

1. Kite, dart and rhombi [6]: These tilings can be reduced to triangular tilings. The classification follows from [8].

2. Edge combination a^2bc: Completed by a team led by Erxiao Wang.

3. Edge combination a^3b: Only some preliminary results are obtained.

The quadrilateral tilings is the most difficult, and the difficulty is concentrated in the third kind.

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Extended Abstract
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Renzuru Tilings with Asymmetric Quadrilaterals

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Abstract

Renzuru, conjoined origami cranes, is one field of origamis. Most figures of Renzuru are based on tilings with squares. However, the recent development of origami theory demonstrates that the origami cranes, constituents of Renzuru, can be folded from asymmetric quadrilaterals with the inscribed circle. In this paper, we consider the extensions of the Renzuru by less symmetrical quadrilaterals than squares. We focus our consideration on the tilings with one shape in this study. Therefore, we consider two cases: (a) the periodic tilings with congruent quadrilaterals and (b) the tilings with similar quadrilaterals. For the first case, we list ten types of tilings. Some tilings make possible the 3-folded or 6-folded Renzurus without torsions. For the second case, the removal of the central part is necessary to fold Renzuru.

Keywords: Renzuru, origami cranes, tiling.

2010 MSC: Primary 52C20; Secondary 05B45, 51M04, 51M05, 01A27.

1 Introduction

We consider the variations of tilings for Renzuru (連鶴), conjoined origami cranes. The traditional Renzuru were summarized in the famous book titled "The Secret Heritage of How to Fold Thousands of Cranes" (秘伝千羽鶴折形) published in 1797 [1]. The traditional origami crane is folded from a square sheet so the traditional Renzuru is also folded from the sheet consisting of squares. An example of Renzuru is shown in Fig. 1. The Renzuru was folded from a square sheet, and the sheet consisted of 3×3 smaller squares and was cut on the edges of small squares except for the narrow areas around the vertices.

On the other hand, recent researches revealed that the origami cranes could be folded from non-square paper [2, 3]. The result brings us the possibility of folding new types of Renzuru. In this paper, we construct a list of quadrilateral tilings that enable us to fold Renzuru.

2 Method

We defined the Renzuru tilings as satisfying both the two following conditions: one is that "origami cranes could be folded ideally from all the constituted quadrilaterals", and the other is that "no vertex

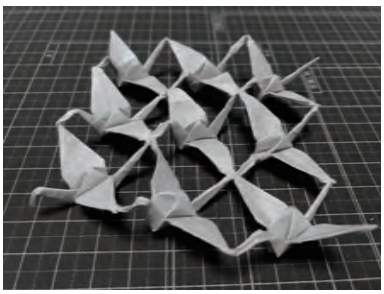

Figure 1: An example of Renzuru (conjoined origami cranes).

was on the edge of other quadrilaterals". The second condition was necessary to connect all origami cranes. According to Justin [2] and Kawasaki [3], the necessary and sufficient conditions for a sheet shape of origami cranes are that the quadrilateral having an inscribed circle. This condition is equivalent to "the sums of opposite edges of the quadrilateral are equal", so we used this condition as the first one in the definition hereafter.

In this study, we focus our consideration on the tilings with one quadrilateral shape. It can be classified into to cases; one is the tilings with congruent quadrilaterals and the other is the tilings with similar (but not congruent) quadrilaterals. In the first case, we used the complete list of the periodic tilings with congruent convex quadrilaterals made by Grünbaum and Shephard [4]. According to the list, the variation of the tilings is up to fifty-six. We examined all the items in the list if they satisfied the required conditions. We referred to the tilings satisfying the Renzuru conditions as periodic Renzuru tilings. In the second case, we considered the divisions based on logarithmic spirals. We constructed the division by using the points on the spirals as $(e^{bt_n}\cos t_n, e^{bt_n}\sin t_n)$ with $t_n = 2\pi n/k$. For each value of $k(=3,4,5,6)$, we determined the value of b satisfying the condition for origami cranes.

3 Main Results

For the first case, the tilings with congruent quadrilaterals, we found ten types of periodic Renzuru tilings. Figure 2 shows all the tilings. Eight tilings, from A to H, consist of the vertices of the degree of four. On the other hand, the rest tilings, I and J, include the vertices of degrees of three and six. They are new types of conjoined origami cranes having 3-fold and 6-fold symmetries.

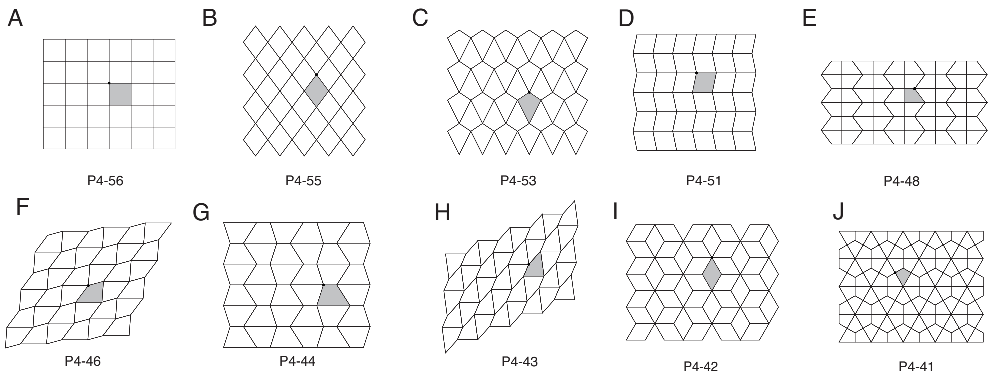

Figure 2: Periodic Renzuru tilings.

For the second case, we obtained four types of divisions as shown in Fig. 3. The figures correspond to the cases of different values of k = 3, 4, 5, and 6 (from left to right). For all the cases, the removals of the central part were necessary because these parts did not satisfy the condition for origami cranes.

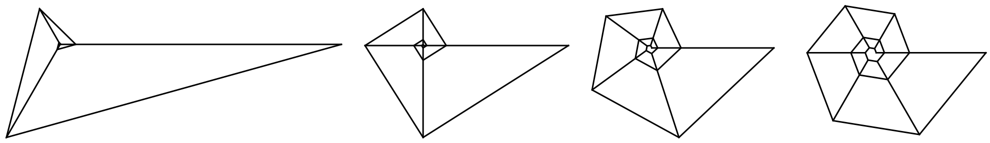

Figure 3: Spiral Divisions.

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Moving gold sand game

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Abstract

The paper deals with an algorithmic problem concerning combinatorial game theory. Here we introduce and analyze a continuous generalization of Chip Game from [2]. The general Chip game was introduced by Aslam and Dhagat in [1] and models, in particular, on-line and list on-line colorings of graphs and hypergraphs.

Keywords: on-line coloring, list on-line coloring, property B, Chip game, proper coloring, hypergraph coloring.

2010 MSC: Primary xxXxx; Secondary xxXxx, xxXxx, xxXxx.

1 Introduction

During the last two decades the algorithmic aspect of classic Erdős coloring problems [3] have received new attention. For instance, property B was formulated in on-line version [1]. In this games one player wishes to color all the elements so that the resulting coloring, for example, is proper and the other player wishes to prevent this. Another motivation of this research is finding fast 2-coloring algorithms.

1.1 Chip game

There are actually a whole family of Chip games, first defined by Spenser [4]. Now we start with two very similar versions of Chip game, which we called *Chip Game*$_1(N, k)$ and *Chip Game*$_2(N, k)$. Both versions were created to model hypergraph and graph on-line coloring.

Chip Game$_\mathbf{1}(\mathbf{N}, \mathbf{k})$: There is a board with positions $-k, \dots, -1, 0, 1, \dots, k$. In the starting arrangement only 0 position is occupied by N chips. Two players (Pusher and Remover) are playing. In each round Pusher splits chips in each position into standing and running parts. Then Remover chooses one of two paths (positive or negative) and removes all running parts (except running part in 0 position) on the chosen path. Then Pusher moves all remaining running parts one step from 0 (towards $-k$ if Remover removed positive path and towards k, otherwise).Then a new turn begins. The game ends when either there is a chip on the k or $-k$ position (Pusher wins) or all chips are removed (Remover wins).

In 1993 Aslam and Dhagat in [1] proved that

Proposition 1.1. *Pusher has a winning strategy in Chip Game*$_1(k \cdot \phi^{2k}, k)$, *where* ϕ *is a golden ratio and Remover has a winning strategy in Chip Game*$_1(2^{k-1}, k)$.

Chip Game$_\mathbf{2}(\mathbf{N}, \mathbf{k})$: There is a board with positions $-k, \dots, -1, 0, 1, \dots, k$. In the starting arrangement only k and $-k$ positions are occupied, each of them by exactly N chips. Two players (Pusher and Remover) are playing. In each round Pusher splits chips in each position into standing and running parts. Then Remover chooses one of two paths (positive or negative) and removes all running parts on the chosen path. Then Pusher moves all remaining running parts one step forward (towards 0). Then a new turn begins. The game ends when either there is a chip on the 0 position (Pusher wins) or all chips are removed (Remover wins).

In 2015 Duraj, Gutowski and Kozik in [2] proved that

Proposition 1.2. *Pusher has a winning strategy in Chip Game*$_2(8 \cdot 2^k, k)$ *and Remover has a winning strategy in Chip Game*$_2(2^{k-1}, k)$.

Note that Proposition 1.2 gives much better bound (up to a multiplicative constant) for Pusher's strategy than does Proposition 1.1 with $k \cdot \phi^{2k} \approx k(2.62)^k$.

2 Main Results

Consider the following game:

Game$_1$(X, k): in each integer coordinate (we call it cell) i of the coordinate line OX lies a certain amount x_i of gold sand. Two players (Pusher and Remover) are playing. In each round Pusher splits gold sand in each cell into two parts, standing and running (divides as he wants). Then Remover (knowing how Pusher shared and knowing the amount of gold sand in each cell) chooses one of two directions of the coordinate line OX (positive or negative) and removes all running parts on the chosen direction (road). Then Pusher moves all remaining running parts one step forward (towards 0). Moreover, all sand from cell 0 instantly becomes Pusher's win.

The question: for a given arrangement $X = (x_{-k}, x_{-k+1}, \ldots, x_k)$ find the supremum of Pusher's win.

Theorem 2.1. *Consider the function $g : [0;1] \mapsto \mathbb{R}_+$, defined by $g(p) = x_0 + x_1 p + x_2 p^2 + \ldots + p^k x^k + x_{-1}(1-p) + x_{-2}(1-p)^2 + \ldots + x_{-k}(1-p)^k$. Then the supremum of Pusher's win is the minimum of $g(p)$ on $[0, 1]$.*

In case $x = (x_{-k} = 2^{k-1}, 0, \ldots, 0, x_k = 2^{k-1})$, i.e. in the continuous generalisation of *Chip Game*$_2(2^{k-1}, k)$, our Theorem 2.1 says that supremum of Pusher's win is equal to 1.

Theorem 2.2. *Consider on-line version of Chip Game$_2(N, k)$. Then supremum of Pusher's win is again minimum of $g(p)$ on $[0, 1]$.*

3 Related work: chip game application

3.1 On-line property B

The on-line version of Erdős-Hajnal problem was firstly introduced by Aslam and Dhagat in [1]. For the sake of brevity, we describe it in slightly different form (just for r colors).

Game$_4$(k, N, r) is parametrized by three numbers: the cardinality of edges k, the number of colors r and the number of edges N. Two players (Presenter and Colorer) are playing. In each round Presenter presents one vertex and declares in which edges it is contained. He cannot add vertices to edges which already contain k vertices. Then Colorer chooses any of r colors to the presented vertex. Then a new turn begins. The game ends when either there is a monochromatic edge (Presenter wins) or all N edges contain k presented vertices each and no monochromatic edge (Colorer wins).

Let $m_{ol}(k, r)$ denote the minimum N such that Presenter has a winning strategy in Game$_4(N, k, r)$. Clearly, $m_{\mathrm{ol}}(k, r) \leqslant m(k, r)$. From Duraj, Gutowski and Kozik's work [2] follows that

$$m_{ol}(k, 2) \leqslant 16 \cdot 2^k.$$

As an application of above results in [1] considered a Committee Meeting Problem: there are n committee members and s committees drawn from these members. At a conference there will be two general meetings held in parallel. Is there a way to assign each member to one of these two meetings so that each committee has at least one member at each meeting? Further, is it possible to make these assignments on-line?

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Multi-Player Open-Hand Babanuki

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Abstract

Babanuki is a card game played by two or more players. It is played in a hidden way, and thus it is rather a mind game than a mathematical game. In this paper, we propose an open-hand Babanuki, which is a perfect information variant, in order to investigate its mathematical nature. By this change, we have a trivial winning strategy for 2-player case, but it is not obvious for 3 or more player's cases. We find a necessary and sufficient condition of the existence of the winning strategy for the 3-player case. In the 4-player case, we find that an endless-loop phenomenon, so-called "repetition draw", occurs by all the players taking their own best strategies.

Keywords: multi-player combinatorial game, perfect information, winning strategy, repetition draw.

2010 MSC: Primary 91A46; Secondary , 91A06.

1 Introduction

Babanuki is a popular game with playing cards in Japan. The rule is quite simple. In the setup phase, all the cards including one joker are (evenly) distributed to the players. If a player has two cards with an identical number in their hand, they discard them. Define a cyclic order of the players 0, 1, 2, ... (imagine a clock), and the game starts with player 1. Player 1 arbitrarily draws a card from the cards of Player 0, which are put face down. A drawn card is added to the hand of Player 1, and if it forms a pair, they discard them as the setup phase. In the next turn, Player 2 similarly draws a card from the cards of Player 1 and discards cards if possible; the game continues according to the cyclic (i.e., clock-wise) order. Note that the joker is never discarded since it is single. A player that gets to have no card leaves the game, and the order of leaving is the ranking of the game. This is the basic rule of Babanuki . Many similar games are played all over the world. For example, it is similar to card game "Old maid", "Black Peter", "Vieux Garçon", "Le Pouilleux" or "Ekae" [1].

In this paper, we analyze Babanuki as a multi-player card-discarding game with perfect information. More precisely, we newly introduce "open-hand Babanuki " based on the ordinary one. Here, "open-hand" means that players play Babanuki with cards put face up. Note that a p-player perfect information game with $p \geq 3$ does not always have an optimal strategy, though any two-player perfect information game has an optimal strategy by the so-called Zermelo's theorem (see [2], for example). In fact, the optimal strategy for the 2-player open-hand Babanuki is obvious; the player who does not have the joker can always win just by drawing a card other than the joker. Thus, we focus on the cases with 3 or more players.

Our results are summarized as follows: (1) We find a necessary and sufficient condition of the existence of the winning strategy for the 3-player case. (2) In the 4-player case, we find that an endless-loop phenomenon, so-called "repetition draw", occurs by all the players taking their own best strategies. In the remainder of the paper, we give a formal definition of open-hand Babanuki and a little more detailed results in Sections 2 and 3, respectively.

2 Formal Model

We give the formal model of open-hand Babanuki . We use labels $P_0, P_1, \ldots, P_{p-1}$ ($p \geq 2$) for p players. These labels are ordered in the cyclic manner, that is, $(P_0, P_1, \ldots, P_{p-1}, P_0, \ldots)$. We emphasize that each P_i is not a name but a label for a player. As explained later, the label of a player may change during a game, but we simply say "player P_i" if no confusion arises, otherwise stated as "the player of P_i". Note that p may decrease during a game. The set of card faces is $[n] \cup [\bar{n}] \cup \{\iota\}$ for a positive integer n, where $[n] = \{1, 2, \ldots, n\}$, $[\bar{n}] = \{\bar{1}, \bar{2}, \ldots, \bar{n}\}$ and ι is the joker. Notice that $\iota \notin [n] \cup [\bar{n}]$. Here, card x and $\bar{x}$ form a pair, and $\bar{\bar{x}} = x$ holds. For player P_i, let H_i denote the set of cards of P_i at the moment. If a player P_i has cards x and $\bar{x}$ during a game, they can discard them; H_i is updated as $H_i := H_i \setminus \{x, \bar{x}\}$. A game proceeds as follows:

(Setup phase) Distribute all the cards to players, which defines H_i for $i = 0, \ldots, p-1$. While some H_i contains both x and $\bar{x}$ for some $x \in [n]$, update $H_i := H_i \setminus \{x, \bar{x}\}$. Let $i := 1$ to represent turn, and $k := 1$ for ranking.

(Drawing phase) Player P_i draws an arbitrary card x from H_{i-1} (H_{p-1} for $i = 0$), that is, $H_{i-1} := H_{i-1} \setminus \{x\}$. If H_i contains $\bar{x}$, they are discarded (i.e., $H_i := H_i \setminus \{\bar{x}\}$). Otherwise, just add x to H_i (i.e., $H_i := H_i \cup \{x\}$).

1. If $H_{i-1} = \emptyset$, the player of P_{i-1} leaves the game and is ranked as k-th. If $i \neq p$, reorder the labels of players as $P_{i-1+j} := P_{i+j}$ for $j = 0, 1, \ldots, p-i-2$. Update $p := p-1$ and $k := k+1$.
2. If $H_i = \emptyset$, the player of P_i is ranked as k-th leaves the game. If $i < p-1$, reorder the labels of players as $P_{i+j} := P_{i+j+1}$ for $j = 0, 1, \ldots, p-i-2$. Update $p := p-1$ and $k := k+1$.
3. Let $i := i+1$ for $i+1 < p$ and $i := 0$ for $i+1 = p$. If $p = 1$, the game ends. Otherwise, go back to the beginning of the drawing phase.

We assume that a smaller rank is better for every player. The result of a game depends on the initial $H_0, H_1, \ldots, H_{p-1}$ and the actions of all the players. The actions that player P_i can take are just to select a card to draw from H_{i+1} at Drawing phases. We define an *optimal strategy* of a player as a strategy under which they can achieve the best rank among the achievable ranks. A player taking an optimal strategy is called *rational*. In particular, we call a *winning strategy* a strategy by which a player can get the first rank. We call a player having a winning strategy a *definite winner*. As mentioned in Introduction, there is a game that has no definite winner for p-player open-hand BABANUKI for $p \geq 3$, whereas 2-player one always has.

3 Results

We can construct an example that no player is a definite winner for 3-player open-hand BABANUKI . Showing the non-existence of a winning strategy is not obvious, since a case analysis requires to check an exponential size of possible cases. For 3-player open-hand BABANUKI, we find an explicit necessary and sufficient condition for the existence of a winning strategy, though we omit the detail due to the space limitation. It yields the following theorem.

Theorem 3.1. In 3-player open-hand BABANUKI, we can compute the definite winner $O(n)$ time if exists.

Since the configurations of p-player open-hand BABANUKI include the configurations of $(p-1)$-player open-hand BABANUKI, the situation becomes more complicated for more players. Indeed, there is a game where every player is never ranked due to an endless loop phenomenon for 4-player open-hand BABANUKI .

Lemma 3.2. Suppose that players $P_0, \ldots, P_3$ are rational and have hands such that $H_1 = H_3$ with $|H_1| = 2$, $H_0 = H_3 \cup \{\iota\}$ with $|H_3| = 2$, and the turn starts with P_1. Then, the optimal strategy of P_1 is to draw ι of H_0.

By this lemma, P_1's action yields a new configuration that is essentially same as the original configuration by symmetry, which leads an endless loop.

Theorem 3.3. In 4-player open-hand BABANUKI, there is a game that never ends by every rational player drawing the joker in their turn.

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Extended Abstract
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Playing impartial games on a simplicial complex as extension of emperor sum

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Abstract

In this study, we consider impartial games on a simplicial complex. On each vertex of given simplicial complex, there is a position of an impartial game. A player, in their turn, choose a face of the simplicial complex and for each position on each vertex of the face, the player can make arbitrary many moves. Moreover, the player can make at most one move for each position on each vertex not of the face. We show how we can characterize the $\mathcal{P}$-positions of this game by using $\mathcal{P}$-position length. This result can be considered as an extension of the theory of emperor sum.

Keywords: Combinatorial Game Theory, impartial game, $\mathcal{P}$-position length, Emperor Sum.

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1 Introduction

Combinatorial Game Theory studies two-player perfect information games with no chance moves. *Impartial games* are games in which both players have the same set of options in each position. In this study, we consider that games are under *normal play* convention, that is, the player who makes the last move is the winner. We also assume that games are *loopfree*, i.e., a position occurs at most once in a play. Under these conventions, exactly one of the both players has a winning strategy at any given position. We say a position in which the next (resp. previous) player has a winning strategy is an $\mathcal{N}$*-position* (resp. $\mathcal{P}$*-position*). An *option* of a position g is a position after one move from g, and a *strict follower* of g is a position after arbitrary many moves from g.

Disjunctive sum of games is one of the most popular concept in Combinatorial Game Theory. For any positions g and h, disjunctive sum of impartial games $g + h$ is the game whose options are $g' + h$ and $g + h'$, where g' and h' ranges all options of g and h, respectively. That is, in disjunctive sum of games, a player chooses one component and make a move.

On the other hand, in this study, we consider the case that the player chooses multiple components and makes multiple moves under certain conditions.

1.1 Early Results

Nim is one of the most famous impartial games. In this game, there are several heaps of stones. A player chooses a heap and removes any positive number of stones from the heap. As we consider that games are under normal play convention, the winner of nim is the player who removes the last stone. Bouton proved that a position in nim is a $\mathcal{P}$-position if and only if the bitwise XOR of the numbers of stones in the heaps is zero [1].

Let V be a finite set of vertices. A *simplicial complex* Δ on V is a subset of $2^{|V|}$ such that for any element $v \in V$, $\{v\} \in \Delta$ and if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$. An element of Δ is called a *face* of Δ. Ehrenborg and Steingrímsson studied nim on simplicial complexes [3]. Let Δ be a simplicial complex on a finite set V. Each vertex has some stones. In nim on Δ, a player, in their turn, chooses a non-empty face F in Δ and arbitrary removes any positive numbers of stones from all vertices in F. The original nim, Moore's nim [4] and circular nim [2] can be considered as special cases of this game. Ehrenborg and Steingrímsson had not characterized $\mathcal{P}$-positions of nim on Δ with no restriction, but found good constructions and characterized $\mathcal{P}$-positions for certain cases.

We consider to expand Ehrenborg and Steingrímsson's study to general impartial games.

Definition 1.1. Let Δ be a simplicial complex on a finite set $V = (v_1, v_2, \ldots, v_n)$ and $G = (g_1, g_2, \ldots, g_n)$ be a collection of positions of impartial games.

Simplicial emperor sum of G on Δ, which is denoted by $\Delta(G)$, is a position such that on each vertex v_i, there is a position of an impartial game g_i and a player, in their turn, chooses a face F in Δ and make arbitrary many moves for positions on all vertices in F. In addition, for each position on the vertex not in F, the player makes at most one move.

To characterize $\mathcal{P}$-positions of this ruleset, we use *$\mathcal{P}$-position length*.

Definition 1.2. For any position g, the $\mathcal{P}$-position length of g is

$$\mathrm{Pl}(g) = \begin{cases} 0, & \text{If } g \text{ is a terminal position.} \\ \max(\{\mathrm{Pl}(g') : g' \text{ is a } \mathcal{P}\text{-position and a strict follower of } g\}) + 1, & \text{Othewise.} \end{cases}$$

$\mathcal{P}$-position length is used to characterize $\mathcal{P}$-positions of *emperor sum* of games [5].

In emperor sum of games, a player selects one component and makes arbitrarily many moves. For every other component, the player moves at most once. A position in emperor sum is a $\mathcal{P}$-position if and only if every component is an $\mathcal{P}$-position and the bitwise XOR of $\mathcal{P}$-position lengths of the components is zero. Note that emperor sum is a special case of simplicial emperor sum on Δ such that every face of Δ has single vertex.

2 Main Result

In this section, we discuss the way to determine which player has a winning strategy in simplicial emperor sum. Let Δ be a simplicial complex on a set $V = (v_1, v_2, \ldots, v_n)$ and P be the set of $\mathcal{P}$-positions of NIM on Δ.

The following Lemmas are trivial from the definition of NIM on a simplicial complex.

Lemma 2.1. *Let $A = (a_1, a_2, \ldots, a_n) \in P$. For a position $A' = (a'_1, a'_2, \ldots, a'_n)$, if there is a face F such that for any $v_i \in F, a'_i < a_i$ and for any $v_i \notin F, a'_i = a_i$, then A' is an $\mathcal{N}$-position.*

Lemma 2.2. *For any $B = (b_1, b_2, \ldots, b_n) \notin P$, there is a position $B' = (b'_1, b'_2, \ldots, b'_n) \in P$ such that a face F satisfies for any $v_i \in F, b'_i < b_i$ and for any $v_i \notin F, b'_i = b_i$.*

Theorem 2.3. *For a collection of positions of impartial games $G = (g_1, g_2, \ldots, g_n)$, $\Delta(G)$ is a $\mathcal{P}$-position if and only if $(\mathrm{Pl}(g_1), \mathrm{Pl}(g_2), \ldots, \mathrm{Pl}(g_n)) \in P$ and for any i, g_i is a $\mathcal{P}$-position.*

Sketch of the proof. Let X and Y be collections of positions of impartial games such that $X = \{(g_1, g_2, \ldots, g_n) : (\mathrm{Pl}(g_1), \mathrm{Pl}(g_2), \ldots, \mathrm{Pl}(g_n)) \in P$ and for any i, g_i is a $\mathcal{P}$-position.$\}$ and $Y = \{(g_1, g_2, \ldots, g_n) : (g_1, g_2, \ldots, g_n) \notin X\}$.

It is enough for proving this theorem to show following two claims.

Claim 1. For any $G \in X$, every G' such that $\Delta(G')$ is an option of $\Delta(G)$ satisfies $G' \in Y$.

Claim 2. For any $H \in Y$, there is a collection of positions of impartial games H' such that $H' \in X$ and $\Delta(H')$ is an option of $\Delta(H)$.

Claim 1 and Claim 2 can be proved by using Lemma 2.1 and 2.2, respectively.

□

As emperor sum is a special case of simplicial emperor sum, this result can be considered as a generalization of the theory of emperor sum of games.

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Extended Abstract
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Losing Positions of Splythoff and 2-Splythoff encoded in the Tribonacci word

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Abstract

We analyze 2-Splythoff, a generalization of Fokkink and Rust's game of Splythoff, and show how its P-positions are encoded in the Tribonacci word. Splythoff is itself a modification of Wythoff, allowing an extra move called a *split*. The table of letter positions of the Tribonacci word yields a mex rule for 2-Splythoff that generalizes the result of Fokkink and Rust. We then show that this generalized mex rule can be used to generate a new family of infinite words on a finite alphabet.

Keywords: combinatorial games, tribonacci word, integer sequence.

2010 MSC: 91A46; 37B10

1 Introduction

Wythoff's game is defined as follows. Two piles of counters are placed on the table. Two players alternately either take an arbitrary number of counters from a single pile or an equal arbitrary number from both piles; these moves are called *singles* or *doubles*, respectively. The player who takes the last counter, or counters, wins. Alternatively, the first player who cannot make a move loses.

To transform it into Splythoff, we allow for the option of a *split*. If a player performs a double which leaves one of the piles empty, they may redistribute the remaining tokens to split that pile into two. For example, from position $(4, 7)$, it is possible to reach the position $(1, 2)$; take 4 counters from each pile, then split the remaining counters into piles of size 1 and 2 [1]. To generalize to k-Splythoff, we loosen the restrictions on a double move. A player may take x counters from one pile and y counters from the other if and only if $|x - y| < k$. The conditions for a split do not change.

Since k-Splythoff is finite and deterministic, the winner or loser from a given starting position should be a foregone conclusion, assuming both players are behaving optimally, i.e. if it is possible for a player to win, they always win. An N-position or winning position is one from which the current turn player wins. A P-position or losing position is one from which the previous turn player wins.

The terminal position of $(0, 0)$ is defined to be a P-position by the game rules. From there, the N and P-positions can be recursively defined. A position is an N-position if there exists a move that leads to a P-position. A position is a P-position if all moves from it lead to N-positions.

2 Preliminaries

We can enumerate the P-positions of Splythoff as follows. Let A and B be natural number sequences such that (a_i, b_i) are the P-positions of Splythoff, where $A < B$ (i.e., the smaller pile goes in A)*. We will similarly enumerate the P-positions of 2-Splythoff using the sequences $\hat{A}$ and $\hat{B}$. We can then similarly enumerate the P-positions of 2-Splythoff using the sequences $\hat{A}$ and $\hat{B}$.

Fokkink and Rust showed that the P-positions of Splythoff are encoded in the *Tribonacci word*. Consider words over the finite alphabet $\{0, 1, 2\}$. The *Tribonacci substitution* τ sends $\texttt{0} \mapsto \texttt{01}$, $\texttt{1} \mapsto \texttt{02}$, and $\texttt{2} \mapsto \texttt{0}$. Repeated iterations of τ on $\texttt{0}$ produces the sequence $\texttt{0} \mapsto \texttt{01} \mapsto \texttt{0102} \mapsto \texttt{0102010} \mapsto \dots$, which then yields the infinite *Tribonacci word* $\mathbf{t} = \texttt{0102010010201}\dots$ in the limit.

*Following the convention of Fokkink and Rust, we denote infinite sequences by uppercase letters such as A, and denote their entries by $a_1, a_2, a_3, \dots$, using the corresponding lowercase letter

	1	2	3	4	5	6	7	
A	1	3	4	6	7	9	10	...
B	2	5	8	11	13	16	19	...

Figure 1: The first 7 terms of A and B.

	1	2	3	4	5	6	7	
$\hat{A}$	1	2	4	5	6	7	9	...
$\hat{B}$	3	8	12	17	20	25	29	...

Figure 2: The first 7 terms of $\hat{A}$ and $\hat{B}$.

From this, we let X^0, X^1 and X^2 be the natural number sequences that enumerate the (1-indexed) positions which contain $\mathtt{0}, \mathtt{1}$, and $\mathtt{2}$, respectively, i.e. $i \in X^j$ if $\mathbf{t}_i = j$. This defines the *positions table* of the Tribonacci word. Define the sequences $\Delta^0 = X^1 - X^0$ and $\Delta^1 = X^2 - X^1$ which comprise the *difference table* of the Tribonacci word. Finally, let $d\Delta = \Delta^1 - \Delta^0$ and $s\Delta = \Delta^0 + \Delta^1$ be the *double-difference* and *sum-difference* sequences of the Tribonacci word.

	1	2	3	4	5	6	7	8	
X^0	1	3	5	7	8	10	12	14	...
X^1	2	6	9	13	15	19	22	26	...
X^2	4	11	17	24	28	35	41	48	...
Δ^0	1	3	4	6	7	9	10	...	
Δ^1	2	5	8	11	13	16	19	...	
$d\Delta$	1	2	4	5	6	7	9	...	
$s\Delta$	3	8	12	17	20	25	29	...	

Figure 3: The first 8 terms of the positions table of the Tribonacci word. The difference table is also shown, along with the double-difference and sum-difference sequences.

Fokkink and Rust showed the following connection between the Tribonacci word and Splythoff.

Theorem 2.1 (Fokkink and Rust)**.** *The ith P-position in Splythoff, (a_i, b_i), is given by $a_i = (\delta^0)_i$ and $b_i = (\delta^1)_i$, the rows of the difference table of the Tribonacci word [1].*

3 Main Results

Fokkink and Rust leave as an open question whether such a relationship exists for the general k-Splythoff. Our main result is the following theorem for 2-Splythoff.

Theorem 3.1. *The ith P-position in 2-Splythoff $(\hat{a}_i, \hat{b}_i)$ is given by $\hat{a}_i = d\delta_i$ and $\hat{b}_i = s\delta_i$, the double-difference and sum-difference sequences of the Tribonacci word.*

We prove this by establishing a mex recurrence relationship between the sequences $\hat{A}$ and $\hat{B}$, and then using properties of the Tribonacci word to show that this recurrence relationship is also satisfied by $d\Delta$ and $s\Delta$. This mex rule in 2-Splythoff is a natural generalization of a similar mex rule that appears in Splythoff. Unfortunately, this pattern does not continue for k-Splythoff with $k > 2$; numerical evidence leads us to conjecture that no substitution rule exists whose fixed point encodes the winning positions of k-Splythoff for $k > 2$. Nonetheless, the generalized mex rule can be used to generate a new family of infinite words, whose properties we also study in and of themselves.

Acknowledgments. The author would like to thank Dr Eden Miro, his undergraduate thesis advisor, for her support and patience in the completion of this paper.

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Linear-Time Rectilinear Drawings of Triconnected Subcubic Planar Graphs with Orthogonally Convex Faces

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Abstract

A graph is called planar if it admits a planar drawing on the plane, i.e., no two edges create a crossing except possibly at their common endpoint. In a rectilinear drawing Γ of a planar graph, each vertex is drawn as a point and each edge is drawn as either horizontal or vertical line segment. A face in Γ is called *orthogonally convex* if every horizontal or vertical line segment connecting two points within the face does not intersect any other face. We examine the decision problem that takes a planar graph as an input and seeks for a rectilinear drawing where the faces are drawn as orthogonally convex polygons. A linear-time algorithm for this problem is known for biconnected planar graphs, but the algorithm relies on complex data structures and linear-time planarity testing, which are challenging to implement. In this paper, we give a necessary and sufficient condition for a triconnected subcubic planar graph to admit such a drawing, and design a linear-time algorithm to check the condition and compute a desired drawing, if it exists. We also show that if a triconnected subcubic planar graph G admits a rectilinear drawing, then it must also admit a rectilinear drawing with orthogonally convex faces.

Keywords: Graph drawing, Rectilinear Drawing, Orthogonally convex face, Subcubic graph.

2020 MSC: Primary 68R10; Secondary 05C85, 52C30, 52C35.

1 Introduction

Rectilinear drawing is a popular graph drawing model that finds applications in VLSI floorplanning [4]. A rich body of research has examined this drawing style in the literature [4]. Initial works focused on plane graphs, i.e., a planar embedding of the graph is given as an input where the output must respect the given embedding. Rahman et al. gave a necessary and sufficient condition for a plane graph to have a rectilinear drawing and an algorithm for finding a drawing if it exists [5] and developed a linear-time algorithm based on this condition. Later, Rahman et al. gave a linear-time algorithm for triconnected subcubic planar graph [3]. Here a *subcubic planar graph* is a planar graph where the degree of every vertex is at most three, and a graph is *triconnected* if the removal of any pair of vertices does not disconnect the graph. In addition to the rectilinear constraints, researchers also focused on restricting the shapes of the faces [1, 2]. Chang and Yen gave a linear-time algorithm that can decide whether a plane graph admits a rectilinear drawing with orthogonally convex faces, and compute such a drawing if it exists [2]. Hasan and Rahman presented an algorithmic outline to extend the idea for biconnected planar graphs, but the technique is quite involved as it is based on linear-time planarity testing [4].

In this paper we give a simpler linear-time algorithm for computing a rectilinear drawing of a triconnected subcubic planar graph with orthogonally convex faces (if it exists). Our algorithm is based on a simple necessary and sufficient condition. We also prove that if a triconnected subcubic planar graph G admits a rectilinear drawing, then it must also admit a rectilinear drawing with orthogonally convex faces.

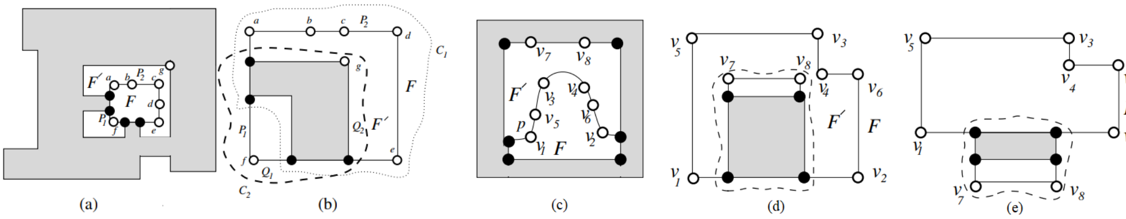

Figure 1: (a)-(b) Illustration of Condition (iv) of Theorem 2.1: 2-vertices are drawn by white circles. (c)-(e) Illustration of Theorem 2.3, where a rectilinear drawing can be used to find a face that can be used as an outerface to compute a rectilinear drawing with orthogonally convex faces.

2 Rectilinear Drawings with Orthogonally Convex Faces

Preliminaries: For the basic graph theoretic terminologies we refer the readers to [4]. Let Γ be an arbitrary planar embedding of a planar graph. A *k-vertex* is a vertex of degree k. A *subdivision* of an edge (p, q) in Γ is an operation that removes the edge (p, q) and adds a path $(p, v_1, v_2, \ldots, q)$ between p and q. We refer to the path $(p, v_1, v_2, \ldots, q)$ as a *chain*. An edge which is incident to exactly one vertex of a cycle C and is located outside of C is called a *leg* of C. The vertex of C to which a leg is incident is called a *leg-vertex* of C. A cycle C in Γ is called a *k-legged cycle* of Γ if C has exactly k legs and there is no edge which joins two vertices on C and is located outside C. Symmetrically, we can define *hand, hand-vertex, and k-handed* cycle where the hands lie interior to C. If a 3-legged/handed cycle C contains one or more 2-vertices, then C is called *good*, or else C is called *bad*.

Main Results: The main results are outlined in Theorems 2.1–2.3, but the proofs are omitted.

Theorem 2.1. *Let G be a triconnected subcubic planar graph, and let Γ be an arbitrary plane embedding of G. Then G has a rectilinear drawing with orthogonally convex faces if and only if Γ has a face F satisfying the following conditions (i)–(iv): (i) There are at least four 2-vertices on F, (ii) F is contained in the inner subgraph $\Gamma_I(C)$ for any bad 3-legged cycle C in Γ, (iii) F is contained in the outer subgraph $\Gamma_O(C)$ for any bad 3-handed cycle C in Γ, and (iv) there are at least two chains on the facial cycle for F in Γ. If there are exactly two chains P_1 and P_2 on F, and any of them, say P_1, contains exactly one vertex, then the other face F' that contains P_2, has at least one 2-vertex which is not on P_2 (Figure 1(a)-(b)).*

Traversing the contours of all faces in Γ, one can check in linear time whether G satisfies the conditions $(i) - (iv)$ of Theorem 2.1. If G satisfies the conditions $(i) - (iv)$ of Theorem 2.1 then G has a planar embedding Γ' which satisfies the conditions by Chang and Yen [2], and hence using the drawing algorithm of Chang and Yen [2], one can also find a rectilinear drawing with orthogonally convex faces of the planar embedding Γ' of G in linear time. Thus the following theorem holds.

Theorem 2.2. *Let G be a triconnected subcubic planar graph. Then we determine in linear time whether G has a rectilinear drawing with orthogonally convex faces or not, and find such a drawing, if it exists.*

Given a rectilinear drawing of G with a face F satisfying the condition by [3], we show a face F' exists that satisfies the conditions of Theorem 2.1 (Figure 1(c)-(e)). Hence we have the following theorem.

Theorem 2.3. *Let G be a planar triconnected subcubic graph. Then G has a rectilinear drawing with orthogonally convex faces if and only if G has a rectilinear drawing.*

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Extended Abstract
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A complete combinatorial characterization of greedy-drawable trees

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Abstract

A (Euclidean) greedy drawing of a graph is a drawing in which, for any two vertices s, t, there is a neighbor vertex of s that is closer to t than s in the Euclidean distance. Graph classes that admit greedy drawings have been actively studied and greedy-drawable trees were characterized in terms of non-linear inequalities by Nöllenburg and Prutkin [4]. Using the characterization, they gave a linear-time recognition algorithm of greedy-drawable trees of maximum degree ≤ 4. However, a combinatorial characterization of greedy-drawable trees of maximum degree 5 is not known. In this paper, we give a combinatorial characterization of greedy-drawable trees of maximum degree 5, which leads to a complete combinatorial characterization of greedy drawable trees.

Keywords: graph drawing, greedy routing, non-linear inequalities

2010 MSC: 68R10, 65G30

1 Introduction

The greedy routing is a routing protocol where a message is always forwarded to the neighbor closest to the destination. In the greedy routing, a message delivery might fail if the message is forwarded to a node with no neighbor closer to the destination. A graph drawing where the greedy routing is guaranteed to succeed is called a *greedy drawing*. That is, a (Euclidean) greedy drawing of a graph is a drawing in which, for any two vertices s, t, there is a neighbor vertex of s that is closer to t than s in the Euclidean distance.

Graph classes that admit greedy drawings have been actively studied since Papadimitriou and Ratajczak presented a conjecture that every 3-connected planar graph admits a greedy drawing. Leighton and Moitra [3] and Angelini et al. [2] independently proved the conjecture of Papadimitriou and Ratajczak. Leighton and Moitra [3] also investigated a condition that a binary tree does not admit a greedy drawing. Later, Nöllenburg and Prutkin [4] characterized greedy drawable trees in terms of non-linear inequalities. Using the characterization, they gave a linear-time recognition algorithm of greedy-drawable trees of maximum degree 4 and an explicit description of greedy-drawable binary trees. However, a combinatorial characterization of greedy-drawable trees of maximum degree 5 was not known.

In this paper, we give a combinatorial characterization (an explicit description) of greedy-drawable trees of maximum degree 5. Since a careful analysis on the recognition algorithm by Nöllenburg and Prutkin [4] easily leads to a combinatorial characterization of greedy drawable trees of maximum degree ≤ 4 and the maximum degree of a greedy drawable tree is less than 6, the result immediately leads to a complete combinatorial characterization of greedy drawable trees.

2 Preliminaries

Let T be a tree. A drawing Γ of T is called a *straight-line drawing* if every node is represented as a point in the plane and every edge as the line segment between its endpoints. We hereafter assume that all drawings are *plane straight-line drawings*, that is, straight-line drawings with no edge crossing. The drawing Γ is said to be *greedy* if for every pair of vertices $s, t (s \neq t)$ there exists a neighbor vertex u of s with $d(u,t) < d(s,t)$, where d is a distance function. In this paper, we consider the case where d is the Euclidean distance. If T has a greedy drawing, we say that T is *greedy-drawable*.

Let T^u_{uv} be the subtree of T that contains u obtained by deleting an edge uv and let h^u_{uv} be the open half-plane bounded by $axis(uv)$ that contains u, where $axis(uv)$ is the perpendicular bisector of uv. Angelini et al. [1] showed that a drawing Γ is greedy if and only if every subtree T^u_{uv} is contained in h^u_{uv} in Γ.

Let r be a vertex of T and let $v_0, \dots, v_{d-1}$ be the neighbors of r. Then, we consider a rooted tree $T_i = (V_i, E_i)$ with the root r defined by $T^{v_i}_{rv_i} + rv_i$ $(i = 0, \dots, d-1)$. We let $polytope(T_i) := \bigcap\{h^w_{uw} \mid uw \in E_i, uw \neq rv_i, d_T(w,r) < d_T(u,r)\}$, where d_T is the graph distance and let $\widetilde{T}$ be the subtree induced by the vertices $r, v_0, \dots, v_{d-1}$. Assume that we have a greedy drawing of each T_i and want to construct a greedy drawing of T by glueing the drawings. Then, the task is to construct a drawing in which $\widetilde{T}$ is drawn greedily and each T_j $(j \neq i)$ is contained in $polytope(T_i)$. If $polytope(T_i)$ is unbounded, we extend the two unbounded edges to the intersection of them and define the *open angle* of T_i, denoted by $\angle T_i$, as the open cone bounded by the resulted rays. (The original definition in [4] is slightly different; but this definition works for our purpose). The angle of the cone is denoted by $|\angle T_i|$. Nöllenburg and Prutkin [4] proved a remarkable lemma, called shrinking lemma, that states that if T has a greedy drawing in which each T_i has an open angle, then there is a greedy drawing of T in which each T_i is drawn infinitesimally small, keeping the size of each open angle. Based on this lemma, they proved that T has a greedy drawing with $|\angle T_i| < \varphi_i$ $(i = 0, \dots, d-1)$ if and only if there is a permutation τ on $\{0, \dots, d-1\}$ such that the following inequality system has a solution. For $i = 0, \ldots, d-1$,

$$\begin{aligned}
&0 \le \alpha_i, \beta_i, \gamma_i \le 180, \\
&\beta_i < \alpha_i,\ \gamma_i < \alpha_i, \\
&\beta_i + \gamma_{i+1} < \varphi_{\tau(i)}\ (i \bmod d), \\
&\alpha_i + \beta_i + \gamma_i = 180,\ \alpha_0 + \cdots + \alpha_{d-1} = 360, \\
&\sin(\beta_0) \cdot \cdots \cdot \sin(\beta_{d-1}) = \sin(\gamma_0) \cdot \cdots \cdot \sin(\gamma_{d-1}).
\end{aligned} \tag{2.1}$$

This inequality system describes the condition that $\widetilde{T}$ is drawn greedily and each open angle $\angle T_i$ contains the subtrees T_j $(j \neq i)$. We do not explain further details here; see [4] for more details.

3 Rooted trees that can be drawn with open angles

Nöllenburg and Prutkin [4] introduced a procedure to compute the tight upper bound on the maximum possible open angle of a rooted tree. By analyzing their results carefully, we give an explicit description of the rooted trees that can be drawn with an open angle.

To describe the result, we introduce some definitions. Let P be a path. A tree T is called a degree-k path associated to P if T is constructed by attaching at most $k-1$ leaves to each terminal vertex of P and at most $k-2$ leaves to each internal vertex of P. The number of degree-k vertices is called the *weight* of T. A leaf of T that is adjacent to one of the terminal vertices of P is called an *end leaf* of T. We denote by $|\angle T|_*$ the tight upper bound on the maximum possible open angles of T. If T can be drawn with an open angle $\varphi - \epsilon$ for any $\epsilon > 0$, but not φ, we write $|\angle T|_* = \varphi^-$

Proposition 3.1. *Let T be a rooted tree with root r and assume that T contains no degree-2 vertex. Then, T has a greedy drawing with an open angle if and only if T satisfies one of the following conditions.*

- *(A) T is single edge and r is one of the end vertices. In this case, we have $|\angle T|_* = 180°$ and call T the type-A tree.*
- *(B) T is a degree-3 path of weight $n \ge 1$ and r is an end leaf of T. In this case, we have $|\angle T|_* = (90° + 60° \times \frac{1}{2^n})^-$ and call T a type-B_n tree. See Figure 1.*
- *(C) T is a degree-4 path of weight $n \ge 1$ and r is an end leaf of T. In this case, we have $|\angle T|_* = (120° \times \frac{1}{2^n})^-$ and call T a type-C_n tree. See Figure 2.*
- *(D) T is obtained by attaching a degree-4 path of weight $n \ge 1$ and two degree-3 paths of weight $k, l \ge 1$ and a leaf to single vertex v. r is one of the end leaves of the attached degree-4 path that is farthest from v (with respect to the graph distance). In this case, we have $|\angle T|_* = (45° \times \frac{1}{2^k} + 30° \times \frac{1}{2^l})^- \times \frac{1}{2^{n-1}}$ and call T a type-$D_{k,l,n}$ tree. See Figure 3.*
- *(E) T is obtained by attaching a degree-4 path of weight $n \ge 1$ and two degree-3 paths of weight $k, l \ge 1$ to single vertex v. r is one of the end leaves of the attached degree-4 path that is farthest from v (with respect to the graph distance). In this case, we have $|\angle T|_* = (60° \times \frac{1}{2^k} + 60° \times \frac{1}{2^l})^- \times \frac{1}{2^n}$ and call T a type-$E_{k,l,n}$ tree. See Figure 4.*
- *(F) T is obtained by attaching three degree-3 paths to single vertex v. r is one of the end leaves of the attached degree-3 paths that is not adjacent to v. Denoting by k and r the weights of the two attached degree-3 paths that do not contain r, we call T a type-$F_{k,l}$ tree. In this case, we have $|\angle T|_* = 60° \times (\frac{1}{2^k} + \frac{1}{2^l})^-$. See Figure 5.*

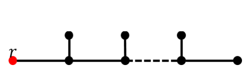

Figure 1: type-B_n tree

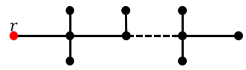

Figure 2: type-C_n tree

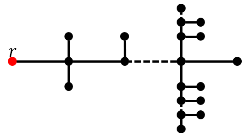

Figure 3: type-$D_{k,l,n}$ tree

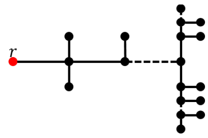

Figure 4: type-$E_{k,l,n}$ tree

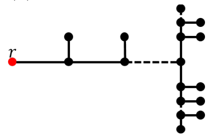

Figure 5: type-$F_{k,l,n}$ tree

4 Main result

In this section, we describe a combinatorial characterization (an explicit description) of greedy drawable trees of maximum degree 5.

Let T be a tree of maximum degree 5. Since removing degree-2 vertices does not change greedy-drawability [4], we assume that T does not contain a degree-2 vertex. Also, since T cannot contain two degree-5 vertices if T is greedy-drawable, we assume that there is only one degree-5 vertex r. Let $v_0, \ldots, v_4$ be the neighbors of r and consider the rooted tree $T_i := T^{v_i}_{rv_i} + rv_i$ $(i = 0, \ldots, 4)$ as in Section 2. If T is drawn greedily, each T_i must be drawn with an open angle [4]. Thus we can decide greedy drawability of T by checking feasibitliy of the inequality system (2.1) in Section 2, setting $\varphi_0 = |\angle T_0|_*, \ldots, \varphi_4 = |\angle T_4|_*$. By checking feasibility of the inequality system for each possible combination $(T_0, \ldots, T_4)$ (each T_i must satisfy one of the conditions in Proposition 3.1) carefully, we can prove the following theorem.

Theorem 4.1. *Let T be a tree of maximum degree 5 with no degree-2 vertex. Then, T is greedy-drawable if and only if T is a subgraph of a tree obtained by attaching trees $T_0, \ldots, T_4$ described in Table 1 to single vertex.*

T_0	T_1	T_2	T_3	T_4
A	A	A	B_1	$D_{k,l,n}$
A	A	A	B_n	$D_{1,3,1}, C_2, F_{2,2}, E_{1,1,1}, F_{1,l}$
A	A	B_1	B_1	$D_{k,l,n}$
A	A	B_1	B_2	$E_{1,l,1}, F_{2,l}, D_{1,1,2}, D_{2,2,1}, D_{1,l,1}$
A	A	B_1	B_3	$D_{1,l,1}, C_2, F_{2,2}, E_{1,1,1}, F_{1,l}$
A	A	B_1	B_n	$D_{1,3,1}, C_2, F_{2,2}, E_{1,1,1}, F_{1,l}$
A	A	B_2	B_2	$F_{1,l}, D_{1,1,1}$
A	A	B_2	B_n	$F_{1,2}, C_1$
A	A	B_n	B_n	$C_1, F_{1,1}$
A	B_1	B_1	B_1	$F_{1,4}, D_{1,1,1}$
A	B_1	B_1	B_2	$F_{1,2}, C_1$
A	B_1	B_1	B_n	$C_1, F_{1,1}$
A	B_1	B_2	B_4	$C_1, F_{1,1}$
A	B_1	B_n	B_n	B_n
A	B_2	B_3	B_n	B_n
A	B_2	B_3	B_4	B_n
A	B_2	B_4	B_5	B_5
A	B_3	B_3	B_3	B_n
B_1	B_1	B_1	B_n	B_n
B_1	B_1	B_2	B_2	B_n

Table 1: Subtrees of maximal greedy-drawable trees of maximum degree 5

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Extended Abstract
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Number of Go positions on a connected graph

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Abstract

Let $L(\mathcal{B})$ be the number of legal positions of the game Go played on a graph board $\mathcal{B}$. When $|\mathcal{B}|$ goes to infinity, we determine an interval of length $10^{-5.9\times 10^{10}}$ which contains the lower limit of $\sqrt[|V(\mathcal{B})|]{L(\mathcal{B})}$ and we find that $\sqrt[|V(\mathcal{B})|]{L(\mathcal{B})}$ has upper limit 3. We also raise a heuristic algorithm that calculates the exact value of the lower limit. When $\mathcal{B}$ is a tree, we propose an algorithm to compute $L(\mathcal{B})$ in $O(|V(\mathcal{B})|^2 \log |V(B)|)$ time.

Keywords: graph Go, tree Go, position number.

2010 MSC: Primary 91A46; Secondary 05A99, 05C30.

1 Introduction

A *graph* G is a pair of two sets $V(G)$ and $E(G)$ such that $E(G) \subseteq \binom{V(G)}{2}$. We call $V(G)$ and $E(G)$ the vertex set and the edge set of G respectively. For every $S \subseteq V(G)$, $G[S]$ refers to the graph with vertex set S and edge set $\binom{S}{2} \cap E(G)$. If a graph $\mathcal{B}$ is connected, we name it as a board. A *position* f of a board $\mathcal{B}$ is a mapping from $V(\mathcal{B})$ to $\{0,1,2\}$. In a position f, we name $f^{-1}(0)$, $f^{-1}(1)$ and $f^{-1}(2)$ as the uncolored vertices, the black stones and the white stones, respectively. Two stones $x, y \in V(\mathcal{B})$ are *solidly connected* in a position f if there exists $i \in \{1,2\}$ such that x, y belong to the same connected component of $\mathcal{B}[f^{-1}(i)]$. A stone $x \in V(\mathcal{B})$ is *free* in a position f if it is solidly connected to a stone adjacent to an uncolored vertex. A position of a board is *legal* if all stones on the board are free. We define $L(\mathcal{B})$ to be the number of legal positions on the board, and let $\mathcal{L}(\mathcal{B}) := \sqrt[|V(\mathcal{B})|]{L(\mathcal{B})}$. An $m \times n$ gird board is the Cartesian product of two paths with number of points m and n respectively. We use $L(m,n)$ for the number of legal positions of the $m \times n$ grid board.

Go is a game between two players played on a 19×19 grid board, where the two players alternatively transform the current legal position on the board to another legal position according to some rule. The number of legal positions plays an important role in judging the difficulty of a game.

The mathematical analysis of Go game and its variants attracts the attention of the researchers of algorithmic game theory [2,3]. Flammenkamp was the first to find the simulation results with the Monte Carlo method back in 1992. However, it was not until 2016 that Tromp finally found the exact value of the possible legal positions of the original game Go [4,5]. Tromp and Farnebäck [5] also considered legal positions on an $n \times m$ grid board and proved that

$$\lim_{n,m\to\infty} \sqrt[mn]{L(m,n)} = 2.9757\cdots.$$

Meanwhile, Farr [1] proposed the concept of Go polynomial to study the possibility of a randomly placed board being legal. As a special case of both rectangle grid boards and tree boards, Weninger and Hayward [6] studied the positional linear Go where the rectangle grid board is of size $1 \times n$. They tried to solve the game for small n and found some properties on the strategies of the players.

We consider arbitrary boards, and we are interested in the growth rate of the number of legal positions of boards when the number of vertices of this board tends to infinity.

Theorem 1.1.
- *It holds that*

$$\lim_{n\to\infty} \max_{|V(\mathcal{B})|=n} \mathcal{L}(\mathcal{B}) = 3.$$

- *Let* $l_3 = 2.5512093917689424830741873156479710179 48\cdots$. *Then it holds that*

$$\liminf_{n\to\infty} \min_{|V(\mathcal{B})|=n} \mathcal{L}(\mathcal{B}) \in (l_3, l_3 + 1.6\times 10^{-59099333243}).$$

2 Estimating the number of Go positions

A *tree* is a connected acyclic graph. A *rooted tree* is a pair (T, R_T) where T is a tree and R_T is a specified vertex of the tree known as the *root*. A *leaf* is a vertex with no children. The *height* $\operatorname{Ht} T$ of a rooted tree T is the length of the longest path to a leaf from the root. We use $\mathscr{RT}_h$ for the set of all rooted trees with height h.

Let T be a rooted tree with its root R_T. We use $L_0(T)$ for the number of positions of T where the root R_T has an unfree black stone and all stones are free except those that are solidly connected to the root R_T. Let us define

$$L_L(T) := L(T) + L_0(T),$$
$$L_U(T) := L(T) + 2L_0(T).$$

For $h \in \mathbb{N}$, let $l_h = \inf_{T\in\mathscr{RT}_h} \sqrt[|V(T)|]{L_L(T)}$ and $u_h = \inf_{T\in\mathscr{RT}_h} \sqrt[|V(T)|]{L_U(T)}$.

We need to establish the next two technical results for the proof of our theorem.

Proposition 2.1. *It holds for all $h \in \mathbb{N}$ that $l_h < l_{h+1} < u_{h+1} < u_h$. Moreover, it holds*

$$\liminf_{N\to\infty} \min_{|V(T)|=n} \mathcal{L}(T) = \lim_{h\to\infty} u_h = \lim_{h\to\infty} l_h.$$

Proposition 2.2. *Let $\mathscr{X}_h := \arg\min_{T\in\mathscr{RT}_h} \sqrt[|V(T)|]{L_L(T)}$. Then $\mathscr{X}_h$ is nonempty for all $h \in \mathbb{N}$. When $1 \le h \le 3$, $\mathscr{X}_h$ contains exactly one element $\mathcal{X}_h$; Moreover, $|V(\mathcal{X}_1)| = 2$, $|V(\mathcal{X}_2)| = 58$, $|V(\mathcal{X}_3)| = 4807557017275$, and*

$$\mathcal{X}_h = \arg\min_{T\in\mathscr{RT}_h} \sqrt[|V(T)|]{L_U(T)}$$

for $h = 2, 3$.

Proof of Theorem 1.1. Let K_n be the complete graph on n vertices. We can check that $L(K_n) = 3^n - 1$. Obviously, $L(T) < 3^{|V(T)|}$ for all boards T. Henceforth, we obtain

$$\lim_{n\to\infty} \max_{|V(T)|=n} \mathcal{L}(T) = \lim_{n\to\infty} \sqrt[n]{3^n - 1} = 3.$$

From Proposition 2.1 it follows

$$l_3 < \liminf_{|V(T)|\to\infty} \mathcal{L}(T) < u_3.$$

With Proposition 2.2 we can calculate l_3 and u_3 and find that $u_3 - l_3 < 1.6 \times 10^{-59099333243}$, and hence we see that

$$\liminf_{n\to\infty} \min_{|V(T)|=n} \mathcal{L}(T) \in (l_3, l_3 + 1.6 \times 10^{-59099333243}). \qquad \square$$

3 Algorithms

This part is omitted due to space constraint.

Acknowledgments. This work is supported by NSFC 11971305.

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

The maximum numbers of the rigid faces and edges in continuous flattening processes of a polyhedron

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Abstract

There are many ways to continuously flatten polyhedra. We focus on the maximum number of the rigid faces, as well as the rigid edges, in continuous flattening processes for a given polyhedron. Some upper bounds of the maximum numbers are given for general polyhedra, and then for a regular polyhedron, we provide a continuous flattening process in which the number of rigid faces (edges) attains the maximum, or gives a lower bound expected as the maximum.

Keywords: polyhedron, continuous flattening, rigid face, rigid edge.

2010 MSC: Primary 52B05; Secondary 52C25.

1 Introduction

We use the terminology *polyhedron* for a polyhedral surface in $\mathbb{R}^3$ that is permitted to touch itself but without self-intersect. *Flat folding* of a polyhedron refers to its folding by creases, without self-crossing, into a multi-layered flat folded state with a finite number of creases. *Continuous flattening* of a polyhedron is flat folding by a continuous process. There are many ways to continuously flatten polyhedra in the literature.

In applications of flattening techniques of polyhedra to other fields, it may be better to make the rigid area on its surface as large as possible. The authors in [3, 4] have discussed the numbers of rigid faces and rigid edges during the flattening processes for some specified polyhedra.

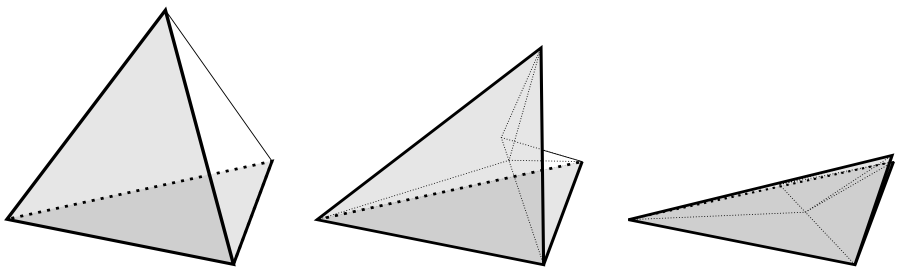

Figure 1: A flattening process of a regular tetrahedron with two rigid faces (grey faces) and five rigid edges (bold edges).

In this talk, we focus on the maximum numbers of rigid faces and rigid edges in flattening processes for a given polyhedron P.

2 Preliminaries

First, we define the two numbers $N_f(P)$ and $N_e(P)$.

Definition 2.1. Let P be a given polyhedron and $\mathcal{S}$ be the set of all continuous flattening processes of P. Further let $n_f(P,\phi)$ and $n_e(P,\phi)$ be the numbers of rigid faces and rigid edges, respectively, in the flattening process ϕ of P. Then for P, the maximum numbers of rigid faces and rigid edges are defined by

$$N_f(P) = \max\{n_f(P,\phi) \mid \phi \in \mathcal{S}\}$$

and

$$N_e(P) = \max\{n_e(P,\phi) \mid \phi \in \mathcal{S}\},$$

respectively.

Figure 1 shows an example of a continuous flattening process of a regular tetrahedron P. The example follows that $N_f(P) \geq 2$ and $N_e(P) \geq 5$. On the other hand, the bellows theorem in [1] gives some upper bounds. Furthermore, by considering some properties of the continuous flattening processes for a given polyhedron P, we can obtain the following upper bounds of the numbers $N_f(P)$ and $N_e(P)$.

Lemma 2.2. *Let P be a polyhedron. Let $f(P)$ and $e(P)$ be the numbers of faces and edges of P, respectively. Then we have the following.*

(1) *$N_f(P) \leq f(P) - 1$ and $N_e(P) \leq e(P)$ hold.*

(2) *If all the faces of P are triangular, then*

$$N_f(P) \leq f(P) - 2,\ N_e(P) \leq e(P) - 1;$$

moreover, if P has $v'(P)$ vertices at each of which the total angle of adjacent faces is smaller than 2π and can not be divided in halves by any subset of the faces, then

$$N_f(P) \leq f(P) - 2\left\lceil \frac{v'(P)}{4} \right\rceil + \varepsilon,\ N_e(P) \leq e(P) - \left\lceil \frac{v'(P)}{4} \right\rceil,$$

where $\varepsilon = 1$ when $v'(P)$ is $1 \bmod 4$, and 0 otherwise.

3 Main Results

It seems difficult to determine the values of $N_f(P)$ and $N_e(P)$ for a polyhedron in general. By investigating various flattening methods as well as applying results in the literature, the numbers of $N_f(P)$ and $N_e(P)$ can be determined or restricted for regular polyhedra.

Theorem 3.1.

(1) *For a regular tetrahedron P, $N_f(P) = 2$ and $N_e(P) = 5$.*

(2) *For a regular octahedron P, $N_f(P) = 4$ and $N_e(P) = 10$.*

(3) *For a cube P, $N_f(P) = 3,\ 11 \leq N_e(P) \leq 12$.*

(4) *For a regular dodecahedron P, $2 \leq N_f(P) \leq 6,\ 25 \leq N_e(P) \leq 30$.*

(5) *For a regular icosahedron P, $5 \leq N_f(P) \leq 14,\ 21 \leq N_e(P) \leq 27$.*

We think it is most likely that $N_e(P) = 11$ for a cube P and $N_e(P) = 25$ for a regular dodecahedron P, but we have not yet proved these.

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Extended Abstract
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Minimizing Visible Edges in Polyhedra

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Abstract

We prove that, given a polyhedron $\mathcal{P}$ in $\mathbb{R}^3$, every point in $\mathbb{R}^3$ that does not see any vertex of $\mathcal{P}$ must see eight or more edges of $\mathcal{P}$; this bound is tight.

Keywords: Edge Guards, Polyhedron, Visibility Graphs.

2010 MSC: Primary 52; Secondary 52B05, 52B10.

1 Introduction

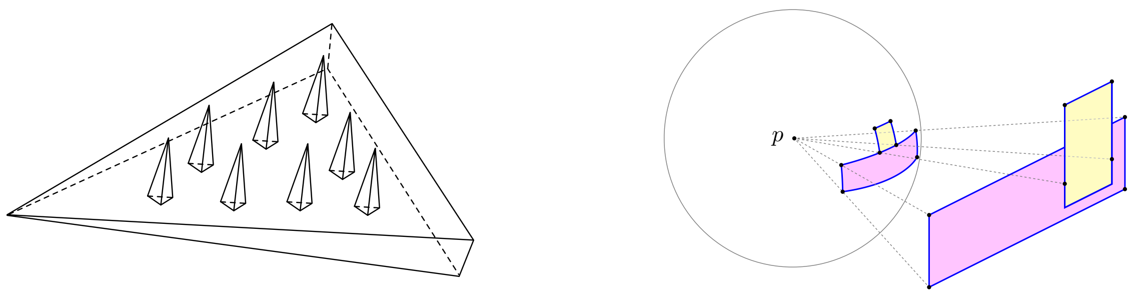
A pivotal result in the study of art gallery or illumination problems is that every simple polygon with n vertices can be illuminated with at most $\lfloor \frac{n}{3} \rfloor$ point lights placed at its vertices. If a simple polygon is orthogonal (i.e., its edges are parallel to the x- or y-axis), $\lfloor \frac{n}{4} \rfloor$ vertices suffice; see [3, 4].

Not much is known about art gallery problems in $\mathbb{R}^3$. We say that a point p in a polyhedron $\mathcal{P}$ *sees* a point q if the line segment pq is completely contained in $\mathcal{P}$. It is known that the vertices of a polyhedron $\mathcal{P}$ do not necessarily see all points in the interior of $\mathcal{P}$ (see Figure 2a for an example). Upper and lower bounds on the number of edges needed to illuminate a polyhedron in $\mathbb{R}^3$ are given in [1, 2, 5].

In this paper we show the following:

Theorem 1.1. *Let $\mathcal{P}$ be a polyhedron in $\mathbb{R}^3$. Any point $p \in \mathbb{R}^3$ that sees no vertices of $\mathcal{P}$ sees at least eight of its edges. Moreover, any point $p \in \mathbb{R}^3$ sees at least six edges of $\mathcal{P}$. These bounds are tight.*

In the *brush* polyhedron depicted in Figure 1a, any interior point close to the tip of a tetrahedral "spike" sees exactly six edges. It is not hard to prove that any point $p \in \mathbb{R}^3$ sees at least six edges of *any* polyhedron. We remark that this also holds if p is in the exterior of the polyhedron. Indeed, for an edge to be visible to p, it is sufficient that p sees one of its endpoints.



(a) A brush

(b) Constructing a Spherical Occlusion Diagram

Figure 1: Visibility in $\mathbb{R}^3$

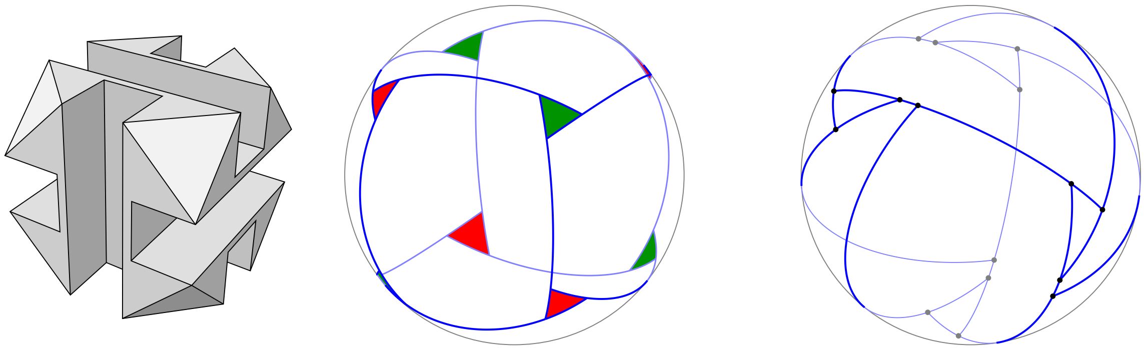
When $\mathcal{P}$ is an orthogonal polyhedron, i.e., when all its faces have normal vectors parallel to one of the coordinate axes, any point in the interior of $\mathcal{P}$ sees at least twelve edges (while any exterior point sees at least eight edges). In fact, take the three planes through an internal point p with normal vectors parallel to the coordinate axes; each of these planes intersects at least four edges visible to p. The bound is tight; it is achieved, for example, in a cube or in the polyhedron in Figure 2a.

2 Spherical Occlusion Diagrams

Proving that, for an arbitrary polyhedron in $\mathbb{R}^3$, any point that sees no vertices can always sees at least eight edges is tricky. To prove our results we study *Spherical Occlusion Diagrams* (SOD), introduced in [6]. SODs arise from the visibility region of a point p that does not see any vertex of a polyhedron $\mathcal{P}$. Specifically, a SOD is the geometric figure obtained by orthographically projecting all visible edge sub-segments onto a small sphere centered at p; see Figure 1b. What is obtained is a spherical (hence planar) arrangement of noncrossing arcs of great circle, and the contact graph of these arcs is a simple planar directed graph where each vertex has outdegree 2. Figure 2a shows the SOD corresponding to the point p at the center of the polyhedron on the left.

Hence, a SOD is a set of noncrossing arcs of great circle on a sphere satisfying the following properties, which are called *diagram axioms* in [6]: (a) if two arcs intersect, one "feeds into" the other; (b) each arc feeds into two arcs; (c) all arcs that feed into the same arc reach it from the same side. These axioms easily imply some properties of SODs:

Theorem 2.1. *The following statements hold for every SOD. (1) No two arcs in a SOD feed into each other. (2) Each arc in a SOD is shorter than a great semicircle. (3) Every SOD is connected. (4) A SOD with n arcs partitions the sphere into $n+2$ spherically convex regions.*



(a) The SOD corresponding to the point p at the center of the polyhedron on the left; the colors of the swirls indicate the bipartition of the swirl graph

(b) A SOD with eight arcs and four swirls

Figure 2: Some Spherical Occlusion Diagrams

A crucial structure in a SOD is a *swirl*, defined as a cycle of arcs such that each arc feeds into the next going clockwise or counterclockwise. In the SOD in Figure 2a, the clockwise swirls are colored green, and the counterclockwise ones are red. The *swirl graph* of a SOD is the undirected multigraph on the set of swirls such that, for each arc shared by two swirls, there is an edge in the swirl graph.

Theorem 2.2. *The swirl graph of a SOD is a simple planar bipartite graph with nonempty partite sets; moreover, every SOD has at least four swirls.*

As a consequence we have the following corollary, which in turn implies our main result, Theorem 1.1:

Corollary 2.3. *Any SOD has at least eight arcs.*

Figure 2b shows an example of a SOD with exactly eight arcs and four swirls.

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

Flat folding problem with parallel creases with mountain-valley assignment on a convex polygonal piece of paper

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Abstract

Deciding flat foldability of a rectangular piece of paper with a given mountain-valley pattern is known to be NP-complete. One special case known to be solvable in linear time is when the creases are parallel to each other and to one side of the rectangular piece of paper; this case reduces to a purely one-dimensional folding problem. Recently, this case was generalized to a parallelogram of paper: if the piece of paper is a parallelogram and creases are parallel to each other and to one side of the paper, then it can be solved in linear-time. In this talk, we try to extend the result to a more general shape of papers. That is, this talk shows that under the restriction of that all creases are parallel to each other, if the shape of the paper is any convex polygon, the flat-foldability problem with mountain-valley assignment can be solved in linear-time.

Keywords: origami, crease, flat folding, convex polygon, algorithm

1 Introduction

A classic problem in computational origami is flat foldability: given a crease pattern drawn on a piece of paper, can the paper be folded flat (into the plane) so as to have creases (folds by 180°) exactly as specified by the crease pattern? In the variant considered here, we are also given a mountain-valley assignment, that is, a specification of whether each crease should be folded in one direction (+180° or mountain) or the other (−180° or valley); together, the crease pattern and mountain-valley assignment constitute a mountain-valley pattern. Both versions of the flat foldability problem — given a mountain-valley pattern or just a crease pattern — are known to be NP-complete, even when the crease pattern consists of horizontal, vertical, and diagonal creases on a rectangle of paper [1]. However, in the special case where the paper is a rectangle (intuitively, a long narrow strip) and all creases are perpendicular to two of the sides as shown in Fig. 1, the piece of paper effectively becomes a one-dimensional segment and flat foldability becomes tractable. Indeed, every crease pattern is flat foldable with the alternating mountain-valley assignment, and flat foldability of a specific mountain-valley pattern can be decided in linear time [2,4]. We call this problem one-dimensional flat foldability with mountain-valley assignment (1DFF-MV for short). Recently a more-general case of flat foldability with parallel creases on parallelogram paper was solved: The all creases are parallel to each other and to one side of the paper, as shown in Fig. 2. It was shown that this problem can be solved in linear time by extending the algorithm for 1DFFMV [3]. In this talk, we present that the algorithm can be extended to general convex polygons of paper.

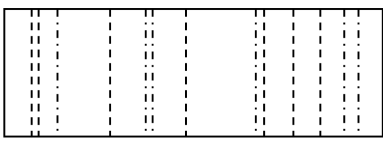

Figure 1: An instance of 1DFFMV [2]

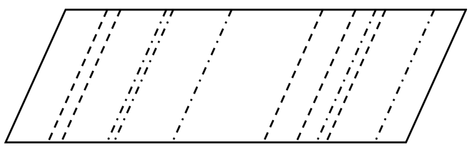

Figure 2: An instance of the problem solved by [3]

2 Main Results

We show the following theorem.

Theorem 2.1. *The flat folding problem with mountain-valley assignment in which the shape of the piece of paper is a general convex polygon and all the creases are parallel to each other, can be solved in linear-time.*

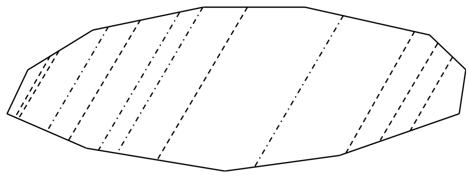

Figure 3: An instance of the flat folding problem with parallel creases with mountain-valley asignment on a convex piece of paper

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Max-norm analogs of Euclidean Ramsey theorems

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Abstract

We find several analogs of classical Euclidean Ramsey results for spaces with the max-norm. More concretely, we prove that, given a finite set S, the chromatic number of $\mathbb{R}^n$ with forbidden S exponentially tends to infinity. Moreover, we find the base of this exponent for a large class of forbidden sets S and relate this to a problem of finding the optimal covering of $\mathbb{Z}$ by translated copies of a given set. We also find an infinite set S such that the chromatic number of $\mathbb{R}^n$ with forbidden S tends to infinity.

Keywords: Ramsey theory, chromatic number, coverings.

2021 MSC: Primary 05D10; Secondary 05D40.

1 Introduction

There are several ways to generalize the well-known problem of Nelson about finding the chromatic number $\chi(\mathbb{R}^n)$ of the n-dimensional Euclidean space. *Euclidean Ramsey theory* that lies at the junction of geometry and Ramsey theory studies the following one. Given a subset $S \subset \mathbb{R}^d$ (with induced metric), find the *chromatic number* $\chi(\mathbb{R}^n; S)$, i.e., the minimum number of colors needed to color all points of the Euclidean space $\mathbb{R}^n$ with no monochromatic isometric copy $S' \subset \mathbb{R}^n$ of S. A set S is called *Ramsey* if $\chi(\mathbb{R}^n; S) \to \infty$ as $n \to \infty$.

A systematic study of such questions begins with the papers [1], [2] of Erdős, Graham, Montgomery, Rothschild, Spencer, and Straus. They showed that $\chi(\mathbb{R}^n; S) = 2$ for all $n \in \mathbb{N}$ and all infinite $S \subset \mathbb{R}^n$. Moreover, they proved that each Ramsey set must also be *spherical*.

Relatively few sets are known to be Ramsey. Kříž [5] showed that each 'fairly symmetric' set is Ramsey, e.g., the set of vertices of each regular polytope. Frankl and Rödl [4] proved that $\chi(\mathbb{R}^n; S)$ grows exponentially in case S is a set of vertices of a box or a simplex. The best known lower and upper bounds on $\chi(\mathbb{R}^n; S)$ for these S are relatively far from each other. For example, in the simplest two cases, when S is a pair of points (clearly, in this case $\chi(\mathbb{R}^n; S) = \chi(\mathbb{R}^n)$) or S is a set of vertices of an equilateral triangle Δ it is only known that $(1.239... + o(1))^n \leq \chi(\mathbb{R}^n) \leq (3 + o(1))^n$ and $(1.01446... + o(1))^n \leq \chi(\mathbb{R}^n; \Delta) \leq (2.732... + o(1))^n$, as $n \to \infty$. The lower and upper bounds on $\chi(\mathbb{R}^n)$ are due to Raigorodskii [10] and Larman and Rogers [7] respectively. The lower and upper bounds on $\chi(\mathbb{R}^n; \Delta)$ are due to Naslund [8] (see also the paper [11] of the third author) and Prosanov [9] respectively.

In the present paper we give the analogs of all these statements for the n-dimensional spaces $\mathbb{R}^n_\infty$ with the max-norm, defined for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as $\|\mathbf{x} - \mathbf{y}\|_\infty = \max_{1 \leq i \leq n} \{|x_i - y_i|\}$.

2 Main Results

In the upcoming paper [3] we prove that studying the values $\chi(\mathbb{R}^n_\infty; S)$ for infinite S becomes more difficult in comparison with the Euclidean case, where two colors are always enough for a proper coloring.

Theorem 2.1. *For all $n \in \mathbb{N}$ and for all infinite $S \subset \mathbb{R}^n_\infty$, we have $\chi(\mathbb{R}^n_\infty; S) \leq n+1$. Moreover, for all $n \in \mathbb{N}$, there is an infinite $S \subset \mathbb{R}^n_\infty$ for which the last bound is tight.*

Theorem 2.2. *There is an infinite $S \subset \mathbb{R}$ such that $\chi(\mathbb{R}^n_\infty; S) \geq \log_3 n$ for all $n \in \mathbb{N}$.*

The last two authors [6] recently showed that all finite sets are Ramsey in the max-norm. Moreover, for all $d \in \mathbb{N}$ and for all finite $S \subset \mathbb{R}^d_\infty$, there is a constant $\chi(S) > 1$ such that $\chi(\mathbb{R}^n_\infty; S) > (\chi(S) + o(1))^n$ as $n \to \infty$. In the upcoming paper of all three authors [3] we find the optimal value $\chi(S)$ for several families of forbidden sets S. The boxes are among the most interesting of them.

Theorem 2.3. *Let $S \subset \mathbb{R}^d_\infty$ be a set of vertices of a box. Then $\chi(\mathbb{R}^n_\infty; S) = (2 + o(1))^n$ as $n \to \infty$.*

We also solve this problem for one-dimensional sets, but the precise statement requires some additional notation. Let $S \subset \mathbb{Z}$ be a finite set. Then we call a subset $A \subset \mathbb{Z}$ to be an *S-covering* if $S + A = \mathbb{Z}$, where $S + A = \{s + a : s \in S, a \in A\}$. Similarly, a set $A \subset \mathbb{Z}$ is called *$(S, -S)$-covering* if both sets $S + A$ and $-S + A$ coincide with $\mathbb{Z}$. We define the values $d_c(\mathbb{Z}; S)$ and $d_c(\mathbb{Z}; S, -S)$ as the minimum possible lower density of S-covering and $(S, -S)$-covering sets respectively. The following theorem relates these notions to the asymptotically tight bound on $\chi(\mathbb{R}^n_\infty; S)$.

Theorem 2.4. *Let $S \subset \mathbb{Z}$ be a finite set. Then $\chi(\mathbb{R}^n_\infty; S) = (1 - d_c(\mathbb{Z}; S, -S) + o(1))^{-n}$ as $n \to \infty$.*

Although the problem of finding the explicit expression for $d_c(\mathbb{Z}; S)$ and $d_c(\mathbb{Z}; S, -S)$ may be solved for any fixed S by exhaustive search, it is probably very hard in general. In case $|S| = 2$ it is almost obvious that $d_c(\mathbb{Z}; S) = d_c(\mathbb{Z}; S, -S) = \frac{1}{2}$. Schmidt and Tuller [12] stated a conjecture concerning the values of these functions for $|S| = 3$ (depending on S). We confirm their conjecture.

Theorem 2.5. *Let λ_1, λ_2 be two coprime positive integers and $S = \{0, \lambda_1, \lambda_1 + \lambda_2\}$. Then*

$$d_c(\mathbb{Z}; S) = d_c(\mathbb{Z}; S, -S) = \min\left(\frac{\lceil \frac{1}{3}(\lambda_1 + 2\lambda_2)\rceil}{\lambda_1 + 2\lambda_2}, \frac{\lceil \frac{1}{3}(2\lambda_1 + \lambda_2)\rceil}{2\lambda_1 + \lambda_2}\right).$$

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

Celeste is PSPACE-hard

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Abstract

We investigate the complexity of the platform video game Celeste. We prove that navigating Celeste is PSPACE-hard in four different ways, corresponding to different subsets of the game mechanics.

Keywords: video games, hardness, PSPACE.

2010 MSC: 68Q17.

1 Introduction

*Celeste** is a 2D platform video game released in 2018 by Extremely OK Games. It won the Best Independent Game and Games for Impact awards at The Game Awards 2018 and sold over a million copies [6]. In Celeste, the player controls a single character, Madeline, who must navigate various hazards along her journey. We consider the following decision problem about Celeste:

Definition 1.1 (CELESTE)**.** Given a Celeste level, is it possible for Madeline to traverse from a designated start location to a designated end location?

We amend a previous result of Ahmed et al. [1] which attempted to show that CELESTE is NP-complete. We give four proofs that CELESTE is instead PSPACE-hard, each using a different set of game mechanics. All of these proofs involve constructing a polynomial-time reduction to CELESTE from a motion-planning problem through a planar network of doors [3]. We make use of both *open-close-traverse* doors, as introduced in [2, 4, 5] and shown not to need crossovers in [3], and *self-closing doors*, as introduced in [3]. In all but one case we additionally show containment in PSPACE.

2 Main Results

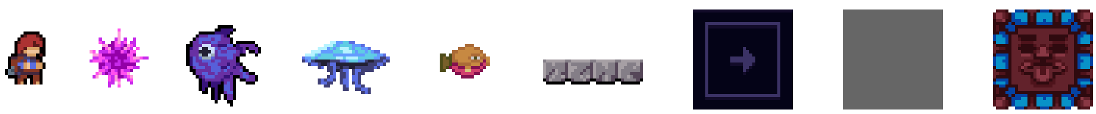

Figure 1: Celeste entities mentioned in our results. From left to right: Madeline (the player, who can walk, jump, climb, and dash), Spinner (spikes), Seeker (enemies), Jellyfish (reducing Madeline's fall speed), Pufferfish (can be bounced on), Move block (can slide in one direction until collision), Barrier (obstacle for enemies), Kevin block (can be moved in all directions, and reverses their path after collision).

Theorem 2.1. CELESTE *with spinners, seekers, barriers, and move blocks is PSPACE-complete.*

*https://exok.com/games/celeste/. Celeste and its sprites are the properties of Extremely OK Games. Sprites are used here under Fair Use for the educational purpose of illustrating mathematical theorems.

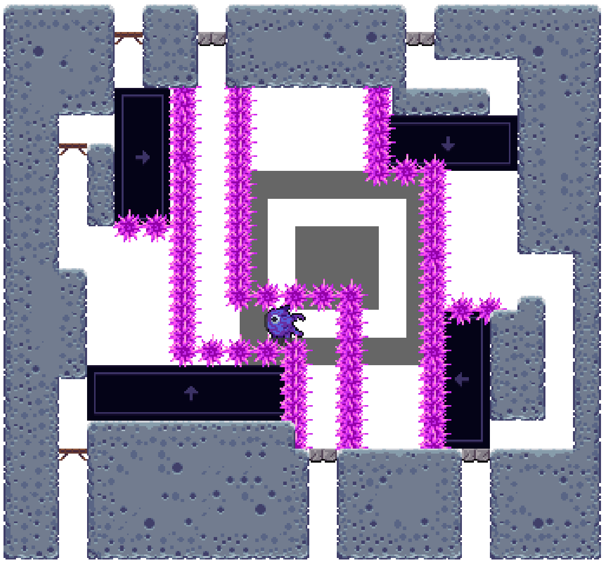

Figure 2: Part of the proof of Theorem 2.1: an open-close-traverse door constructed with spinners, a seeker, barriers, and move blocks. Currently in the "closed" state.

See Figure 2 for the main gadget in the proof of Theorem 2.1.

Theorem 2.2. Celeste *with spinners, jellyfish, and barriers is PSPACE-complete.*

Theorem 2.3. Celeste *with spinners and pufferfish is PSPACE-complete.*

Theorem 2.4. Celeste *with spinners and Kevin blocks is PSPACE-hard.*

Other subsets of mechanics remain to be studied. In particular, it would be interesting to find subsets that are harder than PSPACE (e.g., undecidable), NP-complete, or (nontrivially) polynomial time.

Acknowledgments. This work was initiated during extended problem solving sessions with the participants of the MIT class on Algorithmic Lower Bounds: Fun with Hardness Proofs (6.892) taught by Erik Demaine in Spring 2019. We thank the other participants for their insights and contributions. In particular, we thank Dylan Hendrickson and Josh Brunner for helping simplify the proof of Theorem 2.4, and we thank Michael Coulombe and Jayson Lynch for reading and suggesting revisions to drafts of the paper.

Images of sprites and gadgets were composed and tested using the fan-made level editor Ahorn†. We thank Cruor, Vexatos, and Ahorn's other contributors for creating this excellent tool. We additionally thank the Celeste speedrunning, modding, and Tool-Assisted Speedrunning community for extensively researching Celeste's mechanics.

Finally, we thank Extremely OK Games for producing Celeste, a difficult and wonderful experience in many ways.

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† https://github.com/CelestialCartographers/Ahorn

Extended Abstract
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Department of Mathematics, Chiang Mai University

NP-Completeness of Peg Solitaire on Graphs

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Abstract

Peg solitaire is a single-player table game. The goal is to remove all but one peg from the gameboard. Peg solitaire is generalized to be played on arbitrary graphs. A graph is called solvable if there exists some vertex s such that the graph can be solved starting with s as the initial hole. We prove that it is NP-complete to decide the solvability of a graph.

Keywords: peg solitaire, game, graph, NP-completeness.

2010 MSC: Primary 68Q17; Secondary 91A43, 05C57.

1 Introduction

Peg solitaire is a single-player table game which has been played for centuries. The goal of the game is to remove all but one peg from the gameboard. Peg solitaire usually starts from a board with only one initial hole (a vertex without a peg). If there are three adjacent vertices x, y and z in a row or a column and pegs are placed on x and y, and not on z, a peg on x can jump over the peg on y and be placed on z. The peg on y is removed. The problem to decide if the player can solve the given situation on peg solitaire is known to be NP-complete [1].

Since Beeler and Hoilman generalized peg solitaire to be played on graphs in [2], solvability of peg solitaire on several graphs has been studied (e.g. [3, 4]). A graph is called solvable if there exists some vertex s such that the graph can be solved starting with s as the initial hole. In this paper, we prove that it is NP-complete to decide the solvability of a graph.

2 NP-Completeness

First, we show the NP-completeness of the problem to decide the solvability when an initial hole is given together with a graph. After that, we show the NP-completeness when the initial hole can be chosen arbitrarily.

Theorem 2.1. *Given a graph and an initial hole, it is NP-complete to decide if the graph is solvable with respect to the initial hole.*

Proof. NP-hardness is proved by reduction from the Hamiltonian circuit problem for planar directed graphs with degree three [5]. Let G be the given directed graph and H be the graph obtained by the reduction. For clarity, we use terms 'node' and 'arc' for G, and terms 'vertex' and 'edge' for H. We can assume that a node is either with indegree two and outegree one, or with indegree one and outdegree two. A node gadget representing a node with indegree one is shown in Fig.1. A node gadget with indegree two is the same as the graph except that the direction of the corresponding arcs are reversed. Node gadgets are directly connected by an edge if an arc exists between two nodes. The initial hole is vertex v_9 of an arbitrary gadget representing a node with indegree one.

To solve H when G has a Hamiltonian circuit, visit the node gadgets in the order they appear in the circuit (or in the reversed order). Pegs on the corresponding circuit in H can be removed with moves such as $u_{11} \to u_{10} \to u_9$, $u_8 \to u_9 \to u_{10}$, $u_{13} \to u_{12} \to u_{11}$, $u_{10} \to u_{11} \to u_{12}$, $\cdots$. During the process to

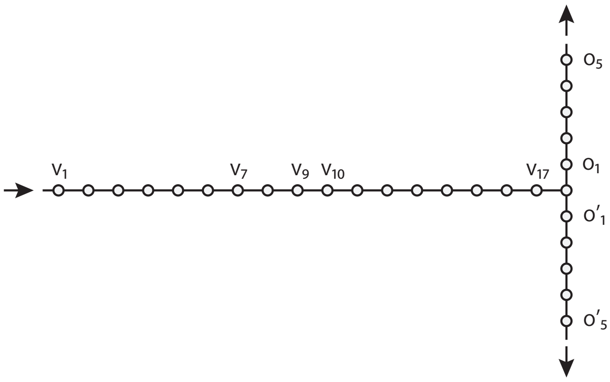

Figure 1: A node gadget with indegree one.

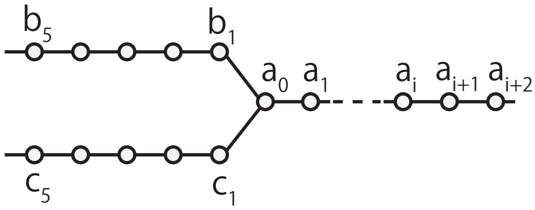

Figure 2: Part of a gadget.

remove pegs on the corresponding circuit in H, five pegs that are not on the circuit can also be removed in each gadget.

Next, consider the case that G does not have a Hamiltonian circuit. Let Fig.2 be a part of the graph obtained by the reduction. Among the vertices shown in the figure, only vertices a_i, a_{i+1} and a_{i+2} $(i \geq 2)$ are holes. To move a peg to a_{i+1}, it is necessary to remove pegs from $a_0, \cdots a_i$. Thus, it is not possible to move a peg to a_{i+2} from the left.

It can be checked that, during the process to remove pegs in a node gadget, at most five pegs that are not on the path can be removed, that is, the pegs on the other gadgets cannot be removed. Therefore, if a series of moves enters a node gadget from an incoming (resp. outgoing) arc, it must go out from an outgoing (resp. incoming) arc. We can also see that it is not possible to fork the process of removing pegs into two directions. □

Theorem 2.2. *Given a graph, it is NP-complete to decide if the graph is solvable.*

Proof. The reduction is obtained with a small modification of the one in Theorem 2.1. We only add eight vertices to v_7 and v_{10} as a path, respectively, on a node gadget with indegree one. If G has a Hamiltonian circuit, the reduced graph can be solved by choosing an appropriate vertex on the path from v_{10} as the initial hole. If the initial hole is not on the path, at least one peg remains on the path. Thus, to solve the graph, the initial hole is on one of the added paths and the last peg remains on the other added path. □

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Prime-Graceful Number

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Abstract

A graph G with n vertices and m edges, is said to be k-prime-graceful, if there is an injection $\psi : V(G) \to \{1, 2, ..., k\min\{m, n\} + 1\}$, where $\gcd(\psi(u), \psi(v)) = 1$ for all $e = \{u, v\} \in E(G)$ and the induced function $\psi^* : E(G) \to \{1, 2, ..., m\}$ defined as $\psi^*(e) = |\psi(u) - \psi(v)|$ is injective. The prime-graceful number of graph G is the minimum k such that G is k-prime-graceful.

In this research, we introduce k-prime-graceful labeling and the prime-graceful number, then show that prime-graceful numbers of star $K_{1,n}$, bistar $B_{n,n}$, bistar $B_{n,p-2}$, where p is an odd prime, complete bipartite graph $K_{2,n}$, triangular book graph $B_n^{(3)}$ and some spiders are equal to 1.

Keywords: graph labeling, prime-graceful labeling, k-prime-graceful, prime-graceful number.

2010 MSC: Primary 05C78; Secondary 05C30, 05C90.

1 Introduction

Graph labeling is one of the active fields in graph theory. There are numerous applications of graph labeling, including graph decomposition, coding for radar and missile guidance, X-ray crystallographic analysis, designing communications networks addressing, determining the optimal circuit layout, etc. More applications and details can be found in References [1–3].

A labeling of a graph is an assignment of labels to vertices and/or edges of the graph. The concept of graph labeling was first introduced by Rosa [7] in 1967. Since then, hundreds of graph labelings have been studied. Gallian has made a thorough survey on those labelings and gather them in a dynamic survey of graph labeling [4]. The term graceful labeling was first mentioned by Golomb [5] in 1972. While the term prime labeling was introduced by Tout, Dabboucy, and Howalla [9] in 1982.

In this research, we introduce the k-prime-graceful labeling which is a combination of graceful labeling and prime labeling, then define the prime-graceful number.

2 Preliminaries

The notion of prime graceful graph was first studied by Selvarajan and Subramoniam [8], where we renamed their work as 2-prime-graceful labeling. Later, Pavithra and Mary [6] extended the idea to 3-prime graceful labeling. Here are our generalize versions.

Definition 2.1. For a positive number k, a *k-prime-graceful labeling* of a graph $G = (V, E)$ with n vertices and m edges is an injective function $\psi : V(G) \to \{1, 2, \ldots, k\min\{n, m\} + 1\}$ with the following property: for any edge $e = \{u, v\} \in E(G)$,

1. the value $\gcd(\psi(u), \psi(v)) = 1$,
2. the induced function $\psi^* : E(G) \to \{1, 2, \ldots, m\}$, defined as $\psi^*(e) = |\psi(u) - \psi(v)|$, is injective.

A graph is called *k-prime-graceful* if it has a k-prime-graceful labeling.

Note that, when $k = 1$, we omit the k and call them *prime-graceful labeling* and *prime-graceful graph.*

It is interesting to find the smallest number k of each graph. Here we give the notion of the prime-graceful number.

Definition 2.2. The prime-graceful number of G, denoted $\xi(G)$, is the minimum k such that G is k-prime-graceful.

Thus, we can conclude from prior works that the prime-graceful number of path P_n, cycle C_n, star $K_{1,n}$, friendship graph F_n, bistar $B_{n,n}$, $C_4 \cup P_n$, $K_{m,2}$ and $K_{m,2} \cup P_n$ are at most 2. While the prime-graceful number of fan graph F_n, wheel graph W_n, helm graph H_n, gear graph G_n, flower graph Fl_n, sunflower graph SF_n, closed helm graph CH_n, and web graph Wb_n are at most 3.

3 Main Results

In this section, we show that star $K_{1,n}$, bistar $B_{n,n}$, bistar $B_{n,p-2}$, where p is an odd prime, complete bipartite graph $K_{2,n}$, triangular book graph $B_n^{(3)}$ and some spiders have prime-graceful number equal to 1.

Theorem 3.1. *Stars $K_{1,n}$ are prime-graceful.*

Corollary 3.2. *For any star $K_{1,n}$, $\xi(K_{1,n}) = 1$.*

Theorem 3.3. *For $n \in \mathbb{N}$, the bistar graph $B_{n,n}$ is prime-graceful.*

Theorem 3.4. *For $n \in \mathbb{N}$ and a prime $p > 2$, the bistar graph $B_{n,p-2}$ is prime-graceful.*

Corollary 3.5. *For any bistar $K_{n,m}$, $\xi(K_{1,n}) = 1$ if $m = n$ or $m + 2$ is an odd prime.*

Theorem 3.6. *For $n \in \mathbb{N}$ where $n > 2$, the spider of n legs where all of its legs have lengths two is prime-graceful if and only if $2n + 1$ or $2n + 3$ is prime.*

Theorem 3.7. *For $n \in \mathbb{N}$, the complete bipartite graph $K_{2,n}$ is prime-graceful.*

Corollary 3.8. *For any complete bipartite graph $K_{2,n}$, $\xi(K_{2,n}) = 1$.*

Theorem 3.9. *For $n > 1$, the triangular book graph $B_n^{(3)}$ is prime-graceful if and only if $n+2$ is an odd prime or $n + 1$ is a power of two.*

Corollary 3.10. *For any triangular book graph $B_n^{(3)}$ with $n > 1$, $\xi(B_n^{(3)}) = 1$ if and only if $n + 2$ is an odd prime or $n + 1$ is a power of two.*

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Rainbow Connection Number of Dutch Windmill Graph

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Abstract

The rainbow connection number of G, denoted $rc(G)$, is defined as the minimum number of coloring edges in G where every two vertices u and v in $V(G)$ is connected by at least one path for which no edge are colored the same. In this paper, we determine the rainbow connection number of the dutch windmill graph $D_n^{(m)}$. We also describe a method of coloring the edges in $D_n^{(m)}$ as a rainbow-connected graph.

Keywords: Rainbow connection number, dutch windmill graph, coloring.

2010 MSC: 05C15, 05C38, 05C40.

1 Introduction

Let G be a simple, finite, undirected and connected graph with a vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ and an edge set $E(G) = \{e_1, e_2, ..., e_m\}$. The rainbow connection number concept was introduced by Chartrand et al. [1] in 2008. A coloring of the edge of G, denoted c, is a function from its edge set to the set of the positive integer k, that is $c : E(G) \to \{1, 2, ..., k\}$ such that the adjacent edges can be colored the same. We call a $u - v$ path P in G as a *rainbow path*, if all edges in path P receive distinct colors. The graph G is called *rainbow-connected*, if there exists a rainbow $u - v$ path for every two vertices u and v of G. The *rainbow connection number* of G, denoted $rc(G)$, is defined as the minimum number of colors of rainbow-connected graph.

In 2004, Yixiao Liu and Zhiping Wang [2] announced the rainbow connection number of a windmill graph. In this paper, we shall consider the rainbow connection number of the ducth windmill graphs. The dutch windmill graph, denoted $D_n^{(m)}$, is defined for $m \geq 2$ and $n \geq 3$ by joining m copies of the cycle graph C_n with a common vertex. The order of $D_n^{(m)}$ is $|V(D_n^{(m)})| = m(n-1)+1$ and its size is $|E(D_n^{(m)})| = mn$.

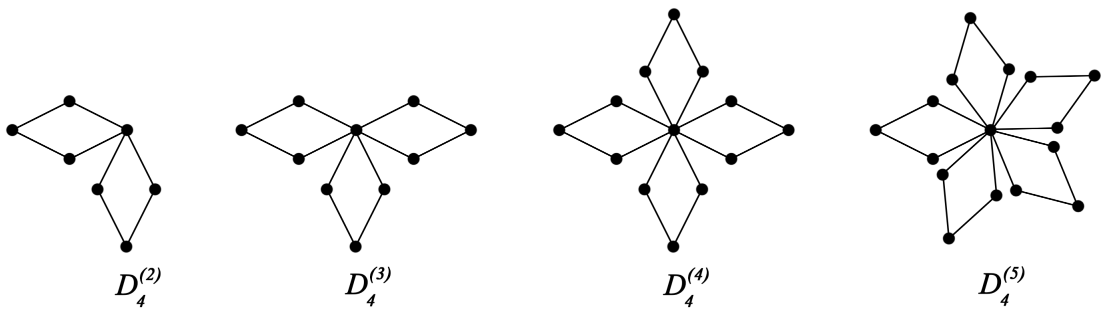

Figure 1: $D_4^{(m)}$ for $m = 2, 3, 4, 5$

2 Preliminaries

2.1 Distance and Diameter

For a connected graph G, the distance $d(u,v)$ between two vertices u and v is the minimum of lengths of the $u-v$ paths of G. The diameter $diam(G)$ of G is the maximum of distance between any pair of vertices. That is,

$$diam(G) = \max\{d(u,v)|u,v \in V(G)\}.$$

If G is a nontrivial connected graph with size $|E(G)|$ and its diameter $diam(G)$, then

$$diam(G) \leq rc(G) \leq |E(G)| \tag{2.1}$$

Lemma 2.1. *For each integer $m \geq 2$ and $n \geq 3$, the diameter of the ducth windmill graph $D_n^{(m)}$ is*

$$diam(D_n^{(m)}) = 2\left\lfloor \frac{n}{2} \right\rfloor. \tag{2.2}$$

2.2 Labeling vertices of $D_n^{(m)}$

In this section, we introduce a vertex labeled of $D_n^{(m)}$ for $m \geq 2$ and $n \geq 3$. $D_n^{(m)}$ consists of m copies of C_n with a common vertex. Given a graph vertex set $V(D_n^{(m)}) = \{v_0, v_1, v_2, ..., v_{m(n-1)}\}$, the common vertex of $D_n^{(m)}$ is defined by v_0. We define each copy of C_n as C_n^j where $j = 1,2,3,...,m$ and the labeling vertices in C_n^j where $j = 1,2,...,m$ to be $v_{i+(j-1)(n-1)}$ for $i = 1,2,...,n-1$.

3 Main Results

Theorem 3.1. *For each integer $m \geq 2$ and $n \geq 3$, the rainbow connection number of the ducth windmill graph $D_n^{(m)}$ is*

$$rc(D_n^{(m)}) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor + (m-2)(\left\lfloor \frac{n}{2} \right\rfloor - 1). \tag{3.1}$$

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Extended Abstract
The 23rd Thailand-Japan Conference on
Discrete and Computational Geometry, Graphs, and Games
Department of Mathematics, Chiang Mai University

Rainbow Connection Numbers of s-Overlapping r-Uniform Hypertrees

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Abstract

Let $\mathcal{H} = (X(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ be a non-trivial connected hypergraph. For some $k \in \mathbb{N}$, an edge k-coloring of $\mathcal{H}$ is a function $c : \mathcal{E}(\mathcal{H}) \to \{1, 2, \ldots, k\}$. A u-v path in $\mathcal{H}$ is said a rainbow, if its edges have distinct colors. An edge coloring of $\mathcal{H}$ is said rainbow connected, if for any two vertices u and v in $X(\mathcal{H})$, there exists a rainbow u-v path. A rainbow connected k-coloring of $\mathcal{H}$ is a rainbow connected coloring of $\mathcal{H}$ by using k colors. The rainbow connection number of $\mathcal{H}$, denoted by $rc(\mathcal{H})$, is the smallest natural number k such that there exists a rainbow connected k-coloring of $\mathcal{H}$.

In this talk, we determine the stricts upper and lower bounds for rainbow connection number of an s-overlapping r-uniform hypertree. For $r \geq 2$, $r > s \geq 1$, and $t \geq 1$, an s-overlapping r-uniform hypertree is an r-uniform connected hypertree with t edges and every pair of adjacent edges intersects in precisely s vertices.

Keywords: edge k-coloring, r-uniform hypergraph, rainbow connection number.

2010 MSC: Primary 05C15; Secondary 05C65.

1 Introduction

For $r \geq 2$, $r > s \geq 1$, and $t \geq 1$, an *s-overlapping r-uniform hypergraph*, denoted by $\mathcal{H}^r_{s,t}$, is an r-uniform connected hypergraph with t edges and every pair of adjacent edges intersects in precisely s vertices. The two extreme case of $s = 1$ and $s = r - 1$ are reffered to as *loose* and *tight* hypergraph, respectively. An *s-overlapping r-uniform hypertree* is an s-overlapping r-uniform connected hypergraph which not contain a cycle. In this talk, we shall consider hypergraphs that are finite, undirected, and simple (without multiple edges, included edge, or singleton). The terminology and notation used refers to Berge (1989) [1] and Voloshin (2009) [5].

In 2008, Chartrand et al. [4] introduced the concept of rainbow connection of a graph. Later, this concept was adopted by Carpentier et al. [3] to extend the notion of rainbow connection of a hypergraph. Then Carpentier et al. have implemented it on an r-uniform cycle hypergraph and an r-uniform complete multipartite hypergraph. In 2019, Budden et al. [2] studied the concept of rainbow connection on an r-uniform minimally connected hypergraph. In this talk, we consider the rainbow connection number of s-overlapping r-uniform hypertrees. In particular, we determine the rainbow connection number of an s-overlapping r-uniform interval hypergraph and an s-overlapping r-uniform hyperstar.

2 Rainbow Connections of Hypergraphs

Let $\mathcal{H} = (X(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ be a non-trivial connected hypergraph. For some $k \in \mathbb{N}$, an edge k-coloring of $\mathcal{H}$ is a function $c : \mathcal{E}(\mathcal{H}) \to \{1, 2, \ldots, k\}$. A u-v path in $\mathcal{H}$ is said a *rainbow*, if its edges have distinct colors. An edge coloring of $\mathcal{H}$ is said *rainbow connected*, if for any two vertices u and v in $X(\mathcal{H})$, there exists a rainbow u-v path. A *rainbow connected k-coloring* of $\mathcal{H}$ is a rainbow connected coloring of $\mathcal{H}$ by using k colors. The *rainbow connection number* of $\mathcal{H}$, denoted by $rc(\mathcal{H})$, is the smallest natural number k such that there exists a rainbow connected k-coloring of $\mathcal{H}$. As the rainbow connection numbers of a graph, the rainbow connection numbers of a hypergraph satisfies $rc(\mathcal{H}) \geq diam(\mathcal{H})$.

3 Main Results

3.1 Rainbow Connection Number of s-Overlapping r-Uniform Interval Hypergraphs

For $r \geq 2$, $r > s \geq 1$, and $t \geq 1$, an *s-overlapping r-uniform interval hypergraph* $\mathcal{P}^r_{(s,t)}$ is an r-uniform connected interval hypergraph with t edges and every pair of adjacent edges intersects in precisely s vertices. Therefore, an s-overlapping r-uniform interval hypergraph has the vertex set $X(\mathcal{P}^r_{(s,t)}) = \{x_1, \ldots, x_n\}$ with $n = (t-1)(r-s) + r$ and the edge set $\mathcal{E}(\mathcal{P}^r_{(s,t)}) = \{E_1, E_2, \ldots, E_t\}$ with $E_i = \{x_{(i-1)(r-s)+1}, x_{(i-1)(r-s)+2}, \ldots, x_{(i-1)(r-s)+r}\}$, $1 \leq i \leq t$.

Theorem 3.1. *Let $r \geq 2$, $r > s \geq 1$ with $r = a \bmod (r-s)$ for some $a \in \mathbb{N}$, and $t \geq 1$, then the rainbow connection number of an s-overlapping r-uniform interval hypergraph is $rc(\mathcal{P}^r_{(s,t)}) = \lceil \frac{t(r-s)+s-a}{r-a} \rceil$.*

For $s = r-1$, an s-overlapping r-uniform interval hypergraph is referred to as a tight r-uniform interval hypergraph with order n, denoted by $\mathcal{P}^r_n$. Thus, we have the following.

Corollary 3.2. *For $n > r \geq 2$, the rainbow connection number of a tight r-uniform interval hypergraph with order n is $rc(\mathcal{P}^r_n) = \lceil \frac{n-1}{r-1} \rceil$.*

3.2 Rainbow Connection Number of s-Overlapping r-Uniform Hyperstars

For $r \geq 2$, $r > s \geq 1$, and $t \geq 1$, an *s-overlapping r-uniform hyperstar* $\mathcal{S}^r_{(s,t)}$ is an r-uniform connected hyperstar with t edges and every pair of adjacent edges intersects in precisely s vertices. Therefore, an s-overlapping r-uniform hyperstar have the vertex set $X(\mathcal{S}^r_{(s,t)}) = \{x_1, \ldots, x_n\}$ with $n = t(r-s) + s$ and the edge set $\mathcal{E}(\mathcal{S}^r_{(s,t)}) = \{E_1, E_2, \ldots, E_t\}$ with

$$E_i = \{x_1, x_2, \ldots, x_s\} \cup \{x_{i(r-s)+s}, x_{i(r-s)+(s-1)}, x_{i(r-s)+(s-2)}, \ldots, x_{i(r-s)+(s-(r-s-1))}\},\ 1 \leq i \leq t.$$

Theorem 3.3. *Let $r \geq 2$, $r > s \geq 1$, and $t \geq 1$, then the rainbow connection number of an s-overlapping r-uniform hyperstar is $rc(\mathcal{S}^r_{(s,t)}) = t$.*

For $s = r-1$, an s-overlapping r-uniform hyperstar is referred to as a tight r-uniform hyperstar with order n, denoted by $\mathcal{S}^r_n$. Thus, we have the following.

Corollary 3.4. *For $n > r \geq 2$, the rainbow connection number of a tight r-uniform hyperstar with order n is $rc(\mathcal{S}^r_n) = n - r + 1$.*

3.3 Rainbow Connection Number of s-Overlapping r-Uniform Hypertrees

The following theorem gives the lower and upper bounds for an s-overlapping r-uniform hypertree. Both bounds are strict. An example of a hypergraph class that satisfies the lower bound is an s-overlapping r-uniform interval hypergraph. An example of a hypergraph class that satisfies the upper bound is an s-overlapping r-uniform hyperstar.

Theorem 3.5. *Let $r \geq 2$, $r > s \geq 1$ with $r = a \bmod (r-s)$ for some $a \in \mathbb{N}$, and $t \geq 1$, then the rainbow connection number of an s-overlapping r-uniform hypertree is*

$$\lceil \frac{t(r-s)+s-a}{r-a} \rceil \leq rc(\mathcal{T}^r_{(s,t)}) \leq t.$$

Moreover, the lower and upper bounds are sharp.

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Extended Abstract
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Distance to antipode of semi-regular polytope, measured by edges of equal length

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By following the theorem of symmetry, we can tell the number of reflecting hyperplanes of regular and semi-regular polytopes. This remarkable theorem is derived from the characteristic equation of group theory and expressed as Poincare polynomials, defined as a product of $\Phi_{m+1}(x) = (1-x^{m+1})/(1-x)$. Vertices of semi-regular polytopes locate inside the fundamental domain in primitive or boundary in non-primitive. Speaking on the primitive class, the distance from a vertex to antipode via edges (edge-distance) is equivalent to the number of reflecting hyperplanes. However, on the non-primitive class, the edge-distance to antipode remains unsolved. In this talk, we will give an algorithm for edge-distance to antipode, not by method of computer searching. This algorithm is based on Wythoffian geometry and feasible both for primitive and for non-primitive.

Our setting problem is oriented toward the edge-distance connecting North Pole and South Pole of a given semi-regular polytope. It is a matter for argument to avoid an exhaustive searching using a computer whether the exact distance or its upper bound is feasible or not. This problem may seem to be an eye-measurable property in 3D, but, in a hyperspace, the projection of polytopes failed to be black out and it is impossible to trace the edges to the antipode. The polytope shown in Fig. 1 belongs to the primitive class of $H_4 = [3, 3, 5]$. Even though blacken out, the group-theoretical enumeration is possible and the answer is 60, measured by edges of equal length.

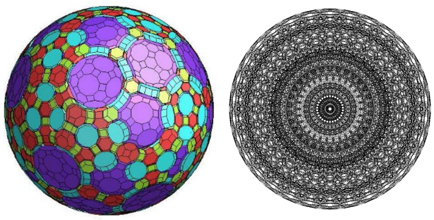

Fig. 1

$$\det \begin{bmatrix} -x & 1/2 & 0 & 0 \\ 1/2 & -x & 1/2 & 0 \\ 0 & 1/2 & -x & \tau/2 \\ 0 & 0 & \tau/2 & -x \end{bmatrix} = 0$$

Table 1

Algorithm for primitives

The edge-distance of primitive is equivalent to the number of reflecting hyperplanes. Because, it is impossible to arrive at antipode without across every reflecting hyperplane. Here, a set of exponents is denoted by $m = \{m_i\}$ and Poincare polynomials are defined as a product of $\Phi_{m+1}(x) = (1-x^{m+1})/(1-x)$. Edge-distance to antipode is given as a sum of exponents, i.e., $\sum m_i$. Number of vertices is given as a product of exponents $+\,1$, i.e., $\prod(m_i + 1)$. We can consider the exponents as invariants of symmetry and Poincare polynomial as a kind of factorization of discrete space.

An example of $A_3 = [3, 3]$

Truncated octahedron is the primitive of A3 and its generator is written as Poincare polynomial, $G(x) = 1 + 3x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6 = (1+x)(1+x+x^2)(1+x+x^2+x^3) = (1-x^2)/(1-x) \cdot (1-x^3)/(1-x) \cdot (1-x^4)/(1-x) = \Phi_2(x)\Phi_3(x)\Phi_4(x) = \prod \Phi_j(x)$. The characteristic equation has resulted in the second kind of Chebyshev polynomial, $U_3(x) = 8x^3 - 4x = 0$, $m = \{m_i\} = \{1, 2, 3\}$. Note that

$\{m_i+1\} = \{2,3,4\}$ reveals double, triple and quadruple rotational symmetry of truncated octahedron. $\Phi_2(x)\Phi_3(x)\Phi_4(x)$ implies factorization of discrete space.
$\sum m_i = 1+2+3 = 6$ $(= n(n+1)/2$ in $A_n)$, $\prod(m_i+1) = 2\times 3\times 4 = 24$ $(= (n+1)!$ in $A_n)$.

An example of $B_6 = [3,3,3,3,4]$
The characteristic equation has resulted in the first kind of Chebyshev polynomial, $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1 = 0$, $m = \{m_i\} = \{1,3,5,7,9,11\}$. $G(x) = \Phi_2(x)\Phi_4(x)\Phi_6(x)\Phi_8(x)\Phi_{10}(x)\Phi_{12}(x)$.
$\sum m_i = 1+3+5+7+9+11 = 36$ $(= n^2$ in $B_n)$, $\prod(m_i+1) = 2\times4\times6\times8\times10\times12 = 46080$ $(= 2n\cdot n!$ in $B_n)$.

An example of $H_4 = [3,3,5]$ (Fig. 1/Table 1)
The characteristic equation has resulted in linear combination of Chebyshev polynomials, $2\tau^2 T_4(x) - U_4(x) = 0$, $\tau = (1+\sqrt{5})/2$, $m = \{m_i\} = \{1,11,19,29\}$. $G(x) = \Phi_2(x)\Phi_{12}(x)\Phi_{20}(x)\Phi_{30}(x)$. $\sum m_i = 1+11+19+29 = 60$, $\prod(m_i+1) = 2\times 12\times 20\times 30 = 14400$.

Algorithm for non-primitives
For the non-primitive class, the situation is very different. Such a powerful calculation for primitives is not available. Even in difficulties, half of the edges are visible and half are invisible. Complemented by Wythoffian geometry, we can tell the edge-distance interpolated for the invisible half.

Upper Bound Theorem: Let P_n, P_{n-1} and π_n be a semi-regular polytope with Schlaefli-Wythoff code $\{p_1,p_2,\cdots,p_{n-1}\}(q_0,q_1,\cdots,q_{n-1})$, $\{p_2,p_3,\cdots,p_{n-1}\}(q_1,q_2,\cdots,q_{n-1})$ and $\{p_1,p_2,\cdots,p_{n-1}\}(1,1,\cdots,1)$, respectively. π_n stands for the primitive and $e(P_n)$, $e(P_{n-1})$ and $e(\pi_n)$ stand for the edge-distance between North Pole and South Pole. For arbitrary P_n with shape vector $\boldsymbol{q} = (q_0,q_1,\cdots,q_{n-1})$, the vectors $\boldsymbol{v} = (v_0,v_1,\cdots,v_{n-1})$ and $\boldsymbol{w} = (w_0,w_1,\cdots,w_{n-1})$ are determined corresponding to $[p_1,p_2,\cdots,p_{n-1}]$. Then,

$$e(P_n) \le \boldsymbol{q}\cdot\boldsymbol{v} + e(P_{n-1}) \le \boldsymbol{q}\cdot\boldsymbol{w} \le e(\pi_n).$$

An example of lattice n-polytope
When P_n belongs to $B_n = [3,3,\cdots,3,4]$, $e(\pi_n) = n^2$ and $e(P_n)$ is estimated arithmetically as follows; $e(P_n) \le (q_0,q_1,\cdots,q_{n-1})\cdot(2,2,2,\cdots,2,1) + e(P_{n-1}) \le (q_0,q_1,\cdots,q_{n-1})\cdot(2,4,6,\cdots,2(n-1),n) \le n^2$.
In the case of lattice n-polytope with integer vertex coordinates, edge-distance is exactly given $n(n-1)$ by a recursive algorithm starting from lattice 3-polyhedron $\{3,4\}(1,1,0)$ (i.e., $6 = 2+4$) and stepping up to higher dimension. Occasionally, the upper bound vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ give us an exact distance.
$\{3,3,4\}(1,1,1,0) : \boldsymbol{q}\cdot\boldsymbol{v} = (1,1,1,0)\cdot(2,2,2,1) = 2+2+2+0 = 6,\ 6+6 = 12$
$\{3,3,3,4\}(1,1,1,1,0) : \boldsymbol{q}\cdot\boldsymbol{v} = (1,1,1,1,0)\cdot(2,2,2,2,1) = 2+2+2+2+0 = 8, 12+8 = 20$
$\{3,3,3,3,4\}(1,1,1,1,1,0) : \boldsymbol{q}\cdot\boldsymbol{v} = (1,1,1,1,1,0)\cdot(2,2,2,2,2,1) = 2+2+2+2+2+0 = 10, 20+10 = 30$
$\{3,3,4\}(1,1,1,0) : \boldsymbol{q}\cdot\boldsymbol{w} = (1,1,1,0)\cdot(2,4,6,4) = 2+4+6+0 = 12 \le 16$
$\{3,3,3,4\}(1,1,1,1,0) : \boldsymbol{q}\cdot\boldsymbol{w} = (1,1,1,1,0)\cdot(2,4,6,8,5) = 2+4+6+8+0 = 20 \le 25$
$\{3,3,3,3,4\}(1,1,1,1,1,0) : \boldsymbol{q}\cdot\boldsymbol{w} = (1,1,1,1,1,0)\cdot(2,4,6,8,10,6) = 2+4+6+8+10+0 = 30 \le 36$

Conclusion
The setting-up problem has been completed by the above-mentioned algorithm based on Wythoffian geometry, not only for primitive class but also for non-primitive class of semi-regular polytopes.

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Extended Abstract
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Möbius Flowers and Buds

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Abstract

If you bisect conjoined two Möbius bands along each centerline, some of them end in interlocking hearts, and the others end in two separate hearts. What is the cause of this difference? We would like to dig deeper to unravel this "Möbius Love-Fate problem" by using the concept of knot theory.

Keywords: Möbius bands, linking number, knot theory

2010 MSC: 97A20

1 Introduction

Let us prepare an elongated rectangle paper. By gluing together the two ends of it, we obtain a band called **an annulus** as shown in the upper right figure in Fig.1. If we give one end a half-twist (180 degree rotation) and glue together the two ends of it, we obtain a twisted band called **a Möbius band** as shown in the lower right figure in Fig.1. The first thing we notice is that a Möbius band is a non-orientable surface with only one side and only one boundary. Cutting a Möbius band along the centerline yields one long band with two full-twists. Note that the length of the resulting band is twice as long as the original Möbius band. We then bisect conjoined two Möbius bands along each centerline, some of them end in interlocking hearts, and the others end in two separate hearts. The question now arises: What brings the difference between the two? In this article, we would like to reveal this "Möbius Love-Fate problem" and generalize the number of conjoined Möbius bands to an arbitrary natural number greater than 2.

2 Preliminaries

There exist two types of Möbius bands depending on the direction of the twist. Here, we provide a method **Δ-check** with which we can distinguish them simply. First, we collapse the band into a triangle shape called **a Möbius delta (Δ)** as shown in the right figure in Fig. 2. There are three sides of the triangle in layers: the topmost layer (both ends of it are visible), the middle layer (only one end of it is visible), and the bottom layer (both ends of it are invisible). Label the topmost layer as I, the middle layer as II, and the bottom layer as III.

A Möbius band is called **Type** α if the direction of I, II, and III is clockwise. On the other hand, a Möbius band is called **Type** β if the direction of I, II and III is counter-clockwise. Note that the direction of I, II, and III, in both types, remains unchanged even if we flip or rotate a Möbius delta.

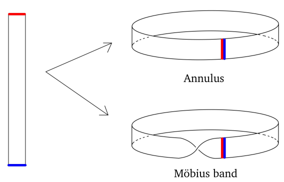

Fig. 1: Annulus and Möbius band

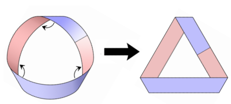

Fig. 2: Δ-check

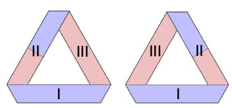

Fig. 3: Type α and Type β

In order to consider conjoined N Möbius bands for $N \geq 2$, we introduce the following definitions.

- Let R_i an elongated rectangle with four vertices A_i, B_i, A_{i+N}, and B_{i+N} $(i = 0, \cdots, N-1)$ (labeled clockwisely) as shown in Fig. 4. Note that l_i is its centerline. Arrange R_i $(i = 0, \cdots, N-1)$ such that the clockwisely-rotated l_0 around its center by $\frac{i\pi}{N}$ coincides with l_i and glue overlapped parts of R_i as shown in Fig. 5. We call the obtained object **an N-star**.

- Let us glue together the two ends of R_i of an N-star for each $i \in \{0, 1, \cdots, N-1\}$ and make N annuli $\mathcal{A}_i$ so that $\mathcal{A}_{i+1}$ goes over $\mathcal{A}_i$. We call the obtained object **an N-bud**(Fig. 6). Note that A_i is attached to B_{i+N} and B_i is attached to A_{i+N}.

- Let us glue together the two ends of R_i of an N-star for each $i \in \{0, 1, \cdots, N-1\}$ and make N Möbius bands $\mathcal{M}_i$ so that $\mathcal{M}_{i+1}$ goes over $\mathcal{M}_i$. We call the obtained object **an N-flower**(Fig. 7). Note that A_i is attached to A_{i+N} and B_i is attached to B_{i+N}. Let the **flower type** $\mathcal{S}$ be a sequence of the type of Möbius bands $\mathcal{M}_i$ for $i \in \{0, 1, \cdots, N-1\}$. (e.g., $\mathcal{S} = \alpha \cdots \alpha$ or $\alpha\beta\alpha \cdots \alpha$)

Note that if we bisect all the Möbius bands $\mathcal{M}_i$ $(i = 0, \cdots, N-1)$ in N-flower, N-flower ends in N (heart-shaped) bands whether they are interlocked or separated. Let P_i denote one of these bands which includes A_i, A_{i+N}, B_{i-1+N}, and B_{i-1} and call it **a petal** for each $i \in \{0, 1, \cdots, N-1\}$. We note that the subscripts are congruent modulo $2N$.

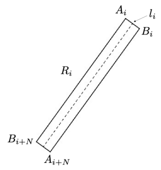

Fig. 4: Rectangle R_i

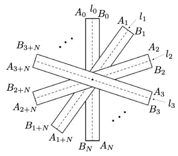
Fig. 5: N-star

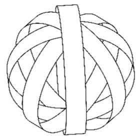
Fig. 6: 4-bud

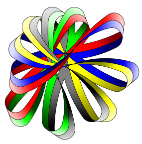
Fig. 7: 5-flower

3 Main Results

Theorem 3.1. *1. (Möbius Love-Fate Theorem) Let $N = 2$. If the flower type $\mathcal{S} = \alpha\alpha$ or $\beta\beta$, two petals P_0 and P_1 are separated. If the flower type $\mathcal{S} = \alpha\beta$ or $\beta\alpha$, they are interlocked. (Fig. 8)*

2. (N-flower Theorem) Let N be an integer greater than 2. If $\mathcal{M}_i$ $(i \in \{0, \cdots, N-1\})$ is type α, petals P_i and P_{i+1} are interlocked. If $\mathcal{M}_i$ $(i \in \{0, \cdots, N-1\})$ is type β, petals P_i and P_{i+1} are separated. P_i and P_j are separated if $|i-j| > 1$. Especially, a heart ring is obtained from the flower type $\mathcal{S} = \alpha\alpha \cdots \alpha$, while, a heart chain is obtained from the flower type $\mathcal{S} = \alpha\alpha \cdots \alpha\beta$. If the flower type is $\mathcal{S} = \beta\beta \cdots \beta$, the mutually-separated N hearts are obtained. (Fig. 9)

Note that the Olympic logo-like heart chain will appear from the 5-flower of type $\beta\alpha\alpha\alpha\alpha$. (Fig. 10)


Fig. 8: Möbius Love-Fate

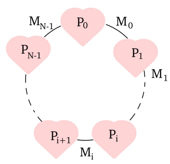

Fig. 9: N petals

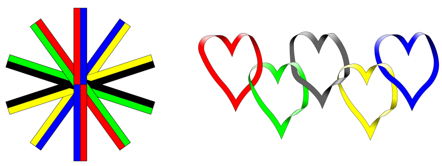
Fig. 10: Olympic logo-like heart chain

Theorem 3.2. *1. Let N be an even number. If we bisect each annulus $\mathcal{A}_i$ $(i = 0, \cdots, N-1)$ of a N-bud along its center line, a long band, whose length is $2N$ times as long as the original annulus, is obtained.(Fig. 11) Especially, a square frame is obtained from a 2-bud.(Fig. 12)*

2. Let N be an odd number. If we bisect each annulus $\mathcal{A}_i$ $(i = 0, \cdots, N-1)$ of a N-bud along its center line, the interlocked two long bands, whose lengths are both N times as long as the original annulus, are obtained.(Fig. 13)

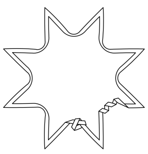
Fig. 11: A long band from 4-bud

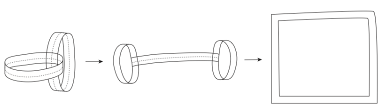
Fig. 12: A square frame from 2-bud

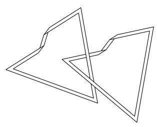
Fig. 13: Long bands from 3-bud

Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

The Conditions for Fitting a Square in a Trapezoid

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Abstract

We propose the necessary and sufficient conditions which fit a square in a trapezoid when the trapezoid is cut from a triangle parallel to its base. The conditions are presented in the form of the largest fitting square on each trapezoid side when the original triangle is right and acute triangle such that the sum of base angle is greater than $\pi/2$.

Keywords: largest inscribed square, polygon containment, trapezoid.

2010 MSC: Primary 52Cxx; Secondary 52C15.

1 Introduction

Assume that a polygon is given, the problem to find the necessary and sufficient conditions which other polygon fits in it were widely studied as shown in the previous studies. Wetzel [7] compiled the fitting and covering problems, and many variations were investigated such as [6] and [5]. In the computational geometry viewpoints, these problems were considered as the polygon containment problem as compiled in [3]. Agarwal et al. [1] proposed an algorithm to find the largest inscribed equilateral triangles and squares in convex and arbitrary polygons.

In this study, we propose the necessary and sufficient conditions when a square is fit in a trapezoid by cutting a triangle parallel to the base of a triangle. The condition as presented in [6] will be applied to each case of trapezoid obtained from cutting off the triangle.

2 Preliminaries

Assume that a trapezoid $\mathcal{T}$ has the height d, the length of parallel edges d_1, d_2 such that $d_1 < d_2$, and its base angle are α, β. Remark that a trapezoid obtaining by cutting a triangle parallel to its base can be separate into 4 types in Figure 1. In this study, we focus on the case $\alpha = \pi/2, \beta < \pi/2$ and $\alpha < \pi/2, \beta < \pi/2$ such that $\alpha + \beta \geq \pi/2$.

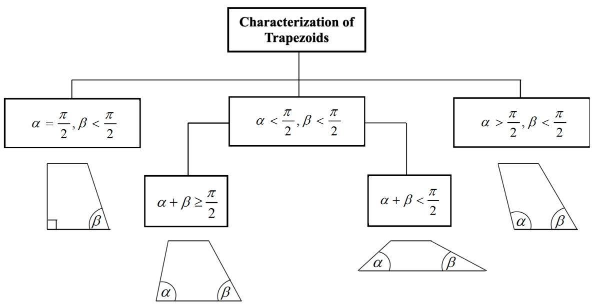

Figure 1: characterization of trapezoids with their parameters

Theorem 2.1 considers the properties of the largest square in a convex polygon P. When the convex polygon P is a triangle, the basic conditions were proposed by Brune [2] in Theorem 2.2. Wetzel finally proposed the necessary and sufficient conditions in [6]. The theorems based on the acute and right angle triangle are mentioned as follows.

Theorem 2.1. *[1] A maximum area square inscribed in a convex polygon P must satisfy at least one of the following two conditions: At least one corner is coincident with a vertex of P, or all four corners lie on the interior of edges of P.*

Theorem 2.2. *[2] A triangle T is given. If T is acute, the largest inscribed square has a side on the shortest of T; if T is right, the inscribed square on the two legs is larger square with a side on the hypotenuse; and if T is obtuse, there is only one inscribed square, which is ipsofacto largest.*

Theorem 2.3. *[6] Let ABC be an acute or right triangle such that c is a shortest side of T. Then a square with side s fits in T if and only if $s \leq 2\Delta c/(c^2 + 2\Delta)$, where Δ is the area of T.*

3 Main Results

When a trapezoid $\mathcal{T}$ is cut from the right triangle, the conditions can be separated into two parts: the largest square satisfying the condition of [6] in the case of a right triangle, and the cut below the condition of [6]. If $\mathcal{T}$ is derived from the right angle T or acute isosceles, we obtain the following condition.

Theorem 3.1. *Let x be the length of a square S. Then S can be fit in $\mathcal{T}$ iff $x \leq \dfrac{d + d_1 \tan\beta}{1 + \tan\beta}$ when $d_1 < d$, and $x \leq d$ when $d \leq d_1$.*

When a triangle ABC is an acute triangle such that a is the smallest length, Theorem 2.2 implies that there is a length σ which indicates the condition when we cut the triangle as shown in Figure 2.

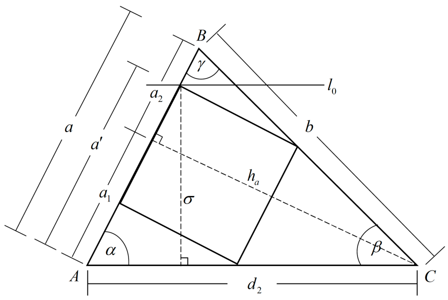

Figure 2: Orientation of largest inscribed square in a triangle ABC

Theorem 3.2. *Let x be the length of the inscribed square in an acute triangle T, and x_I be the length of a square S. Then S can be fit in $\mathcal{T}$ if it satisfies the following conditions.*

1. *When $d > x\sin\alpha + x\cos\alpha$, $x_I \leq \dfrac{d_2 \sin\alpha(\cot\alpha - \cot(\alpha+\beta))}{1 + \cot\alpha - \cot(\alpha+\beta)}$.*
2. *When $d \leq x\sin\alpha + x\cos\alpha$, we consider the following cases:*
   (a) *when $d_1 < d\left[\left(\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta}\right)\left(\dfrac{1}{\sin\alpha+\cos\alpha} - 1\right) + \dfrac{1}{\sin\alpha+\cos\alpha}\right]$,*
   $$x_I \leq \frac{d\tan\alpha + d\tan\beta + d_1\tan\alpha\tan\beta}{\tan\alpha + \tan\beta + \tan\alpha\tan\beta};$$
   (b) *when $d_1 \geq d\left[\left(\dfrac{1}{\tan\alpha} + \dfrac{1}{\tan\beta}\right)\left(\dfrac{1}{\sin\alpha+\cos\alpha} - 1\right) + \dfrac{1}{\sin\alpha+\cos\alpha}\right]$, $x_I \leq d\sqrt{\dfrac{\left(\frac{1-\tan\alpha}{1+\tan\alpha}\right)^2 + 1}{2}}$;*
   (c) *when $d \leq d_1$, $x_I \leq d$.*

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Previous Player's Positions of Impartial Three-Dimensional Chocolate-Bar Games

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Abstract

The authors presented research on three-dimensional chocolate games that are variants of CHOMP. A three-dimensional chocolate bar is a three-dimensional array of cubes in which a bitter cubic box exists in some parts of the bar. Two players take turns and cut the bar horizontally or vertically along the grooves. The player who manages to leave the opponent with a single bitter block is the winner. The authors studied the $\mathcal{P}$-positions of chocolate games, where $\mathcal{P}$-positions are positions of the game from which the previous player (the player who will play after the next player) can force a win, provided he/she plays correctly at every stage. The set of $\mathcal{P}$-positions is important because it provides the winning strategy of the game. The authors discovered some sufficient conditions for the chocolate bars by which a position $\{p, q, r\}$ is a $\mathcal{P}$-position if and only if $(p-1) \oplus (q-1) \oplus (r-1)$, where p, q, and r are the length, height, and width of the chocolate bar, respectively.

Keywords: Combinatorial game, chocolate game, previous player's position, CHOMP

2010 MSC: Primary 91A46; Secondary 97A20.

1 Introduction

In this study, the authors present a three-dimensional chocolate game that consists of a three-dimensional array of cubes in which a bitter cubic box printed in black exists in some parts of the bar. The shape of chocolate bar is defined by 1.1 using the coordinate system that shows the size of the chocolate bar in Fig. 2. This game is played under the rule in Definition 1.1. Fig. 1presents three methods of cutting a three-dimensional chocolate bar. This is a consequence of the research presented in [1] and [2].

Definition 1.1. (i) Let $x, y, z \in Z_{\geq 0}$ and $f(x, z)$ be a monotonically increasing function with respect to x, z. The three-dimensional chocolate bar comprises a set of $1 \times 1 \times 1$ boxes. For $u, w \in Z_{\geq 0}$, such that $u \leq x$ and $w \leq z$, the height of the column at position (u, w) is $y+1$, where $y \leq f(u, w)$. In other words, the maximum value of the height is always equal to or under $f(u, w)+1$, and hence the maximum value depends on u and w. A unique bitter box printed in black is placed at position $(0, 0)$. We denote this chocolate bar as $CB(f, x, y, z)$.
Note that $x+1, y+1$, and $z+1$ are the length, height, and width of the bar, respectively.
(ii) Each player, in turn, cuts the bar horizontally or vertically along the grooves and eats the broken piece. The player who manages to leave the opponent with a single bitter block (black block) is the winner.

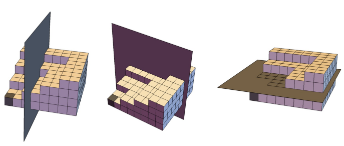

Figure 1.

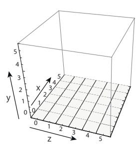

Figure 2.

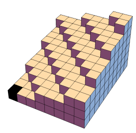

Figure 3.

The chocolate bar in Fig. 3 is $CB(f, 9, 5, 7)$, where $f(x, z) = \lfloor \frac{x+z}{3} \rfloor$.

Definition 1.2. Let x and y be nonnegative integers. We express x and y in base 2 as $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0,1\}$ and $n \in N$. We define nim-sum $x \oplus y$ as $x \oplus y = \sum\limits_{i=0}^{n} w_i 2^i$, where $w_i = x_i + y_i \pmod 2$.

For completeness, we briefly review some of the necessary concepts of combinatorial game theory; refer to [3] for more details. As chocolate bar games are impartial games without draws, only two outcome classes are possible.

Definition 1.3. (a) A position is called a $\mathcal{P}$-*position*, if it is a winning position for the previous player (the player who just moved), as long as he/she plays correctly at every stage.
(b) A position is called an $\mathcal{N}$-*position*, if it is a winning position for the next player, as long as he/she plays correctly at every stage.

Definition 1.4. (i) For any position $\mathbf{p}$ of game $\mathbf{G}$, there is a set of positions that can be reached by precisely one move in $\mathbf{G}$, which we denote as $move(\mathbf{p})$.
(ii) The *minimum excluded value* (mex) of a set S of nonnegative integers is the least non-negative integer that is not in S.
(iii) Each position $\mathbf{p}$ of an impartial game has an associated Grundy number, and we denote it as $\mathcal{G}(\mathbf{p})$. The Grundy number is found recursively: $\mathcal{G}(\mathbf{p}) = mex(\{\mathcal{G}(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\})$.

Theorem 1.5. *For any position* $\mathbf{p}$ *of the game,* $\mathcal{G}(\mathbf{p}) = 0$ *if and only if* $\mathbf{p}$ *is a* $\mathcal{P}$*-position.*

2 Results

One of the most important topics of research on combinatorial games is to obtain a closed formula for Grundy numbers. The authors studied the necessary and sufficient conditions for the shape of chocolate bars in which the Grundy number of chocolate bar is equal to $(p-1) \oplus (q-1) \oplus (r-1)$, where p, q, and r are the length, height, and width of the chocolate bar, respectively. They published their results on two-dimensional chocolate bar in [2], and currently, they have already submitted the result in a three-dimensional case to a journal.

Suppose that the Grundy number of the chocolate bar is $(p-1) \oplus (q-1) \oplus (r-1)$, where p, q, and r are the length, height, and width of the chocolate bar, respectively. Then, by Theorem 1.5, a chocolate bar is a $\mathcal{P}$-position if and only if $(p-1) \oplus (q-1) \oplus (r-1) = 0$.

Therefore, it is natural to find the condition under which a three-dimensional chocolate bar with length p, height q, and width r is $\mathcal{P}$-position if and only if $(p-1) \oplus (q-1) \oplus (r-1) = 0$, although it is difficult because there are many varieties of chocolate shapes that satisfy this condition. However, the authors discovered some sufficient conditions.

Theorem 2.1. *Let* $f(x,z) = \lfloor \frac{x+z}{k} \rfloor$ *for* $k = 4m+3$*. Then chocolate bar* $CB(f,x,y,z)$ *is a* $\mathcal{P}$*-position if and only if* $x \oplus y \oplus z = 0$*.*

Theorem 2.2. *Let* $f(x,z) = \lfloor \frac{x+z}{k} \rfloor$ *for* $k = 8m+4$ *with* $m \in Z_{\geq 0}$*. Then, under the condition that* $x, y, z \leq 12m+7$*, chocolate bar* $CB(f,x,y,z)$ *is a* $\mathcal{P}$*-position if and only if* $x \oplus y \oplus z = 0$*.*

The authors also obtained the following two conjectures that are derived by calculation of computer algebra system Mathematica.

Conjecture 2.3. *Let* $f(x,z) = \lfloor \frac{x+z}{k} \rfloor$ *for* $k = 4m+1$*. Therefore, chocolate bar* $CB(f,x,y,z)$ *is a* $\mathcal{P}$*-position if and only if* $(x+1) \oplus y \oplus (z+1) = 0$*.*

Conjecture 2.4. *Let* $f(x,z) = \lfloor \frac{x+z}{k} \rfloor$ *for* $k = 16m+8$*. Then, under the condition that* $x, y, z \leq 56m+31$*, chocolate bar* $CB(f,x,y,z)$ *is a* $\mathcal{P}$*-position if and only if* $x \oplus y \oplus z = 0$*.*

If we compare these theorems and conjectures, we find that the case of $f(x,z) = \lfloor \frac{x+z}{k} \rfloor$ with an odd number k is different from the case with an even number k. In the case of an even number k, there is a restriction on the size of the chocolate bar.

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Chocolate Games and Restricted Nim

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Abstract

In this study, we present the results of research on rectangular chocolate games with a restriction on the size of the chocolate piece to be eaten. The first result involves a closed formula for calculating the previous player's winning position. In this research, we used the results of our previous research on Max Nim presented at $JCDCG^3$ 2018. The second result is the research on rectangular chocolate bars with some parts are missing. The values of the Grundy number $\mathcal{G}(x, y)$ depend on the location and size of the missing parts.

Keywords: Combinatorial game, chocolate game, previous player's position, restricted nim

2010 MSC: Primary 91A46; Secondary 97A20.

1 Introduction

We herein studied a chocolate game with a restriction on the size of the chocolate piece to be eaten and presented closed formulas for the previous player's winning position. In this research, we used the results of their previous research on Max Nim presented in [1]. Let $Z_{\geq 0}$ be the set of non-negative numbers.

Definition 1.1. Let $g(n) \in Z_{\geq 0}$ be a monotonically increasing function defined on $Z_{\geq 0}$.
(i) We obtain the coordinate system shown in Fig.2. In this example, we describe this chocolate as $\{5, 3\}$.
(ii) A chocolate bar is a rectangular array of squares, and there is a bitter square at the bottom of the bar. Two players take turns. When a player's turn arrives, she or he must break the chocolate bar along any one of the horizontal or vertical lines into two pieces and eat the piece without the bitter part. As for the chocolate with the size of $\{x, y\}$, when one of the players eats the piece without a bitter part, what remains is $\{x - t, y\}$-chocolate with $t \leq g(x)$ or $\{x, y - t\}$-chocolate with $t \leq g(y)$.

There are different types of chocolate bars (see Fig.1). The authors have studied chocolate bars with some squares removed without any restriction on the size of the chocolate bar to be eaten, and they published their research in [2].

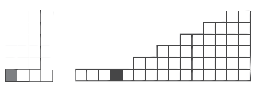

Figure 1.

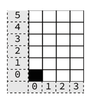

Figure 2.

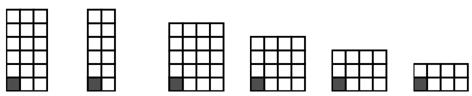

Figure 3.

We assume that $g(n) = \lceil \frac{n}{d} \rceil$ with a real number d such that $1 < d$. For example, if $d = 1.5$, then by (ii) of Definition 1.1, you get one of the chocolates shown in Fig. 3 when you start with the one in Fig. 2.

Definition 1.2. (a) $\mathcal{P}$-positions, that is, the winning positions for the previous player (the player who just moved), as long as they play correctly.
(b) $\mathcal{N}$-positions, that is, the winning positions for the next player, as long as they play correctly.

Definition 1.3. Let x, $y \in Z_{\geq 0}$. We represent $x = \sum_{i=0}^{n} x_i 2^i$ and $y = \sum_{i=0}^{n} y_i 2^i$ with $x_i, y_i \in \{0, 1\}$. We define the nim-sum $x \oplus y$ by $x \oplus y = \sum\limits_{i=0}^{n} w_i 2^i$, where $w_i = x_i + y_i$ $(mod\ 2)$.

Definition 1.4. (i) For any position $\mathbf{p}$ of a game $\mathbf{G}$, there is a set of positions that can be reached by making precisely one move in $\mathbf{G}$; this is denoted by $move(\mathbf{p})$. In the game of Definition 1.1,
$move(\{x,y\}) = \{\{x-t,y\} : t \leq g(x)\} \cup \{\{x,y-t\} : t \leq g(y)\}$.
(ii) The *minimum excluded value* (mex) of a set S of non-negative integers is the least non-negative integer not in S.
(iii) Let $\mathbf{p}$ be the position of an impartial game. The associated Grundy number is denoted by $G(\mathbf{p})$ and is recursively defined by $G(\mathbf{p}) = mex\{G(\mathbf{h}) : \mathbf{h} \in move(\mathbf{p})\}$.

Theorem 1.5. *Let $\mathcal{G}$ be the Grundy number. Then, a position $\mathbf{p}$ is a $\mathcal{P}$-position if and only if $\mathcal{G}(\mathbf{p}) = 0$.*

2 Main Results

We define a restricted nim.

Definition 2.1. Suppose there is a pile of n stones, and two players take turns to remove stones from the pile. In each turn, if the number of stones is m, the player is allowed to remove at least one and at most $g(m)$ stones, where $g(m) = \lceil \frac{m}{d} \rceil$ with a real number d such that $1 < d$. The player who removes the last stone or stones is the winner.

We let $\mathcal{G}_1(x)$ be the Grundy number of the restricted nim of Definition 2.1 when there are x-stones, and $\mathcal{G}_2(x,y)$ is the number of chocolate games in Definition 1.1 when the size of the chocolate is $\{x,y\}$.

Theorem 2.2. *Suppose that $1 < d$. Then, there exists a non-negative integer valued function h_d with a closed formula such that $\mathcal{G}_1(h_d(n)) = \mathcal{G}_1(n)$ for $n \in Z_{\geq 0}$.*

Theorem 2.3. $\mathcal{G}_2(x,y) = \mathcal{G}_1(x) \oplus \mathcal{G}_1(x)$.

This is clear because the game in Definition 1.1 is the sum of the two games in Definition 2.1.

Theorem 2.4. *Then, the set of $\mathcal{P}$-positions is $\{\{h_d^p(n), h_d^q(n)\} : n \in Z_{\geq 0}$ and $p,q \in N\}$.*

This theorem follows from theorems 1.5 2.2 and 2.3.

Next, we study a chocolate game with the condition that the piece to be eaten should be equal to or smaller than the remaining piece, and chocolates are any one of the chocolates in Fig. 4. Here, the chocolates are without bitter parts, except for the fifth one in Fig. 4.

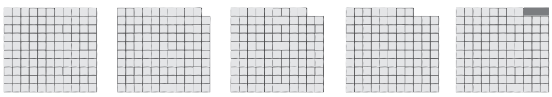

Figure 4. A chocolate without missing parts, a chocolate with one missing part, a chocolate with two missing parts, a chocolate with three missing parts, and a chocolate with three bitter parts.

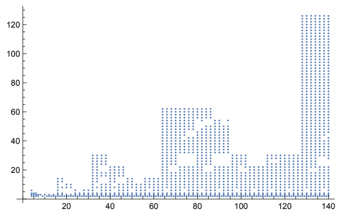

Figure 5. fractal pattern described in conjecture 2.6.

Let $\mathcal{G}_k(x,y)$ be the Grundy number of rectangular chocolates with k-missing parts for $k = 0,1,2,3$, where x,y are the width and height of the chocolate.

Theorem 2.5. *We have the following (i) and (ii).*
(i) $\mathcal{G}_1(x,y) = \mathcal{G}_0(x,y)$.
(ii) $\mathcal{G}_2(x,y) = \mathcal{G}_0(x,y)$ *if* $y \geq 3$ *and* $(x,y) \neq (3,3)$.

The following conjecture is based on the calculation by computer algebra system Mathematica.

Conjecture 2.6. (i) *$\mathcal{G}_3(x,y) = \mathcal{G}_0(x,y)$ for (x,y), such that $y \geq x$, and the set $\{(x,y) : \mathcal{G}_3(x,y) \neq \mathcal{G}_0(x,y)\}$ has a fractal structure, as shown in the graph in Fig. 5.*
(ii) *Grundy numbers of the fifth chocolate game in Fig. 4 are similar to Grundy numbers of the first chocolate game.*

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Permutation-Generated Maps between Dyck Paths

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Abstract

In 2003, Deutsch and Elizalde defined a family of bijections between the set of Dyck paths to itself which is induced by some particular permutations. In this paper, we extend the construction of the maps by allowing the permutation to be arbitrary and explore some consequences. Of particular interest, this leads to a reformulation of an existing combinatorial identity, a characterisation of bijection-generating permutations, and an equidistribution between some statistics of Dyck paths to the height statistics.

Keywords: Dyck paths, bijection on Dyck paths, statistic on Dyck paths.

2010 MSC: Primary 05A05; Secondary 05A18, 05A19.

1 Introduction

Bijective method is one of the classic tools in enumerative combinatorics. One enumeration problem can be translated to another, often in different discrete structure. In particular, we put a certain interest on the bijections between Dyck paths defined by Deutsch and Elizalde [1] having application in enumerating Dyck paths with some specific statistics as well as pattern-avoiding permutations restricted to several rules.

In this work, we muse on a natural generalisation of the maps defined by Deutsch and Elizalde by letting the maps be generated by arbitrary permutations. Motivated by the success of Deutsch and Elizalde's bijections, this generalisation, as we shall see, gives rise to new family of bijections which might be useful in the combinatorics of Dyck paths.

2 Preliminaries

2.1 Dyck Paths

A **Dyck path** of semilength n is a lattice path between $(0,0)$ and $(2n,0)$ consisting of up-steps $(1,1)$ and down-steps $(1,-1)$, represented by $\mathtt{u}$ and $\mathtt{d}$ consecutively, which never goes below the x-axis. Step k of a Dyck path P is denoted by $P_k \in \{\mathtt{u}, \mathtt{d}\}$. Let $\mathcal{D}_n$ denote the set of all Dyck paths of semilength n.

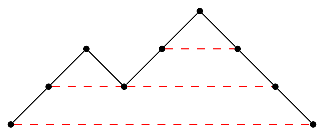

Figure 1: A Dyck path $P = \mathtt{uuduuddd}$, $P_2 = \mathtt{u}$, and $P_7 = \mathtt{d}$. The dashed lines denote the tunnel pairings.

For any Dyck path D, define a **tunnel** of D to be the horizontal segment between two lattice points of D that intersects D only in those two points and always stays below D. If $D \in \mathcal{D}_n$, then D has exactly n tunnels and each tunnel can be associated with a pair of (k,l) such that step k and step l are connected by a tunnel. The tunnel pairs of D can be encoded as a permutation τ_D, called the **tunneling** of D, where $\tau_D(k) = \ell$ if and only if (k,ℓ) is a tunnel pair.

2.2 Permutation Generated Maps between Dyck Paths

Definition 2.1. Given a permutation $\sigma \in S_{2n}$ and a Dyck path D, the σ-path of D, denoted by $\sigma(D)$, is a lattice path constructed by the following algorithm: at iteration $k \in [2n]$,

$$\sigma(D)_k = \begin{cases} \mathtt{u} & \text{if } \tau_D(\sigma_k) \notin \{\sigma_1, \sigma_2, \ldots, \sigma_k\}, \\ \mathtt{d} & \text{otherwise.} \end{cases}$$

The map $D \mapsto \sigma(D)$ is denoted by $\sigma(\cdot)$.

The original map defined by Deutsch and Elizalde in [1, Section 3] takes a particular choice of permutation σ which goes in zig-zag, that is

$$\sigma_k = \begin{cases} \frac{k+1}{2} & \text{if } k \text{ is odd,} \\ 2n + 1 - \frac{k}{2} & \text{if } k \text{ is even.} \end{cases}$$

3 Main Results

By noticing that some different permutations could generate an exact same map, a certain equivalence relation related to the permutation-generated maps can be utilised to rediscover an existing combinatorial identity connecting a certain statistic of Dyck paths and the double factorial term, as presented below.

Proposition 3.1 (Proposition 1 of [2]). *Let P be a Dyck path of size n and h_{u_k} be the height of step $u_k \in \mathbf{U}_P$. We have that*

$$\sum_{P \in \mathcal{D}_n} \prod_{k=1}^{n} h_{u_k}(P) = (2n-1)!!.$$

The following theorem characterises permutations which generate bijections. A permutation $\sigma \in S_{2n}$ is called a **circularly connected permutation** (CCP) if in a cycle graph C_n whose vertices are labelled from $\{1, 2, \ldots, 2n\}$, the subgraph induced by $\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$ is always connected for all $k \in [2n]$.

Theorem 3.2. *$\sigma(\cdot)$ is a bijection if and only if σ is a CCP.*

A particular application of the permutation-generated bijections is to show the equidistribution of certain statistics of Dyck paths depicted as the following theorem. Suppose that P is a Dyck path. Let $u_{a,k}(P)$ denote the number of steps in $U_{a,k} = \{P_a, P_{a+1}, \ldots, P_{a+k-1}\}$ (with the indices arithmetics is done under modulo $2n$) whose tunnel pair is not in $U_{a,k}$. Define $u^{(a)}_{\max}(P) = \max\limits_{k \in [2n]} u_{a,k}(P)$. Define also $h(P)$ to be the height of the highest peak of P.

Theorem 3.3. *Let $a, k \in [2n]$ be arbitrary. We have that for any $\ell \in [n]$,*

$$\Big|\{P \in \mathcal{D}_n : u_{a,k}(P) = \ell\}\Big| = \Big|\{P \in \mathcal{D}_n : h_k(P) = \ell\}\Big|,$$

as well as

$$\Big|\{P \in \mathcal{D}_n : u^{(a)}_{\max}(P) = \ell\}\Big| = \Big|\{P \in \mathcal{D}_n : h(P) = \ell\}\Big|.$$

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Extended Abstract
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Dynamic Independent Set of Squares

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Abstract

We present fully dynamic approximation algorithms for maintaining the maximum independent set of axis-aligned squares and hypercubes. It is known that a maximum independent set of a collection of n intervals can be found in $O(n \log n)$ time, while it is NP-hard already for a set of unit squares. The problem is inapproximable on many important graph families, however, admits a PTAS for a set of arbitrary pseudo-disks. A fundamental question in computational geometry is whether it is possible to maintain an approximate maximum independent set for a set of dynamic geometric objects, in truly sublinear time per insertion or deletion. In this work, we answer this question in the affirmative for squares and hypercubes. Our main result is a data structure for maintaining an expected constant factor approximate maximum independent set of axis-aligned squares in the plane, with polylogarithmic amortized update time. Moreover, our approach generalizes to d-dimensional hypercubes, providing a $O(4^d)$-approximation with polylogarithmic update time. These are the first fully dynamic approximation algorithms for any set of dynamic arbitrary size squares and hypercubes; previous results required bounded size ratios to obtain polylogarithmic update time. It is known that our results for squares (and hypercubes) cannot be improved to a $(1+\varepsilon)$-approximation with the same update time.

Keywords: Geometric independent sets, dynamic algorithms.

2010 MSC: Computational geometry; Dynamic graph algorithms.

1 Introduction

In the maximum independent set (MIS) problem, we are given a graph $G = (V, E)$ and the objective to find a subset $I \subseteq V$ of maximum cardinality, such that no two vertices in I are adjacent. MIS is a fundamental graph problem with a wide range of applications, and is one of Karp's 21 classic NP-complete problem. The problem is also known to be hard to approximate on general graphs: no polynomial time algorithm can achieve an approximation factor $n^{1-\varepsilon}$, for any constant $\varepsilon > 0$, unless P=NP.

Geometric Independent Set. In this work, we focus on geometric setting, called *geometric independent sets*. In this setting, we are given a set S of geometric objects, and the graph G is their intersection graph, where each vertex corresponds to an object, and two vertices form an edge if and only if the corresponding objects intersect. Independent sets of geometric objects in the plane such as axis-aligned squares or rectangles have been extensively studied due to their various applications in e.g., VLSI design, map labeling and data mining. However, even the case of computing a MIS of axis-aligned unit squares is known to be NP-complete. On the positive side, several geometric cases including pseudo-disks, admit a polynomial time approximation scheme (PTAS); see [2]. Recently, Gálvez et al. [3] obtained a constant-factor approximation algorithm for the MIS problem on axis-parallel rectangles in the plane.

Dynamic Geometric Independent Set. In the dynamic version of the geometric independent Set problem, objects are inserted and deleted over time. The goal is to achieve (almost) the same approximation ratio as in the offline (static) case while keeping the update time, i.e., the running time required to compute the new solution after insertion/deletion, as small as possible. Henzinger et al. [1] studied the geometric independent set problem for intervals, hypercubes and hyperrectangles. They obtained several results, many of which extend to the substantially more general weighted independent set problem where objects have weights. However, to obtain the upper bounds, they considered the special case where all objects are located in $[0, N]^d$ and have minimum length edge of 1, hence also bounded size ratio of N, and presented dynamic algorithms with approximation ratio $(1 + \varepsilon)$ for intervals and $(1 + \varepsilon) \cdot 2^d$ for d-dimensional hypercubes with update time polylog(n, N). We note that in general, N might be quite large such as exponential in n or even unbounded, thus those bounds are not truly sublinear in n in the general case. In [4], we showed that it possible to maintain a $(1 + \varepsilon)$-approximate independent set of intervals under insertions and deletions of intervals, in $O(\frac{\log n}{\varepsilon})$ worst-case update time per insertion and $O(\frac{\log n}{\varepsilon^2})$ per deletion, where $\varepsilon > 0$ is any positive constant and n is the total number of intervals. This is the first algorithm yielding such a guarantee in the comparison model, in which the only operations allowed on the input are comparisons between endpoints of intervals. Later, Compton et al. [5] improved this result by designing an $(1 + \varepsilon)$-approximate dynamic algorithm which supports both insertions and deletions in time $O(\frac{\log n}{\varepsilon})$.

1.1 Our Contribution

In this work, we consider the problem of maintaining dynamically an independent set of squares and hypercubes. In [1], Henzinger et al. established lower bounds that implies that no PTAS can be achieved in polylogarithmic time. Hence we turn our attention to a fully dynamic constant factor approximation algorithm. We prove the following theorem.

Theorem 1.1. *There exist algorithms for maintaining an expected $O(1)$-approximate independent set of axis-aligned squares in the plane under insertions and deletions of squares, in $O(\log^5 n)$ amortized time per update, where n is the total number of squares.*

In what follows, we provide a very high-level description. The detailed proof can be found in the long version of the paper [4].

High-level description. Our approach to maintain an approximate MIS of squares is divided into two parts: static construction and dynamization. We define a randomly scaled and shifted infinite quadtree, and associate each square with the smallest enclosing node of the quadtree. We call the squares that intersect the center of their quadtree node *centered* and discard all noncentered squares. Nodes of the quadtree associated with squares are called the *marked* nodes of the infinite quadtree, and let the quadtree Q be the union of all the marked nodes and their ancestors. Note that multiple squares may be associated with one quadtree node. The size of Q is unbounded but the total number of marked nodes is $O(n)$. We show that by losing a factor of 16 in expectation, thus we can restrict our attention to centered squares, thus we can indeed discard all non-centered squares. Then, we reduce the case of squares to intervals using the compressed random quadtree and decompose it carefully into relevant paths, and show that using an optimal solution for intervals we can obtain a $O(1)$-approximate solution for squares in polylogarithmic time. However, to make this structure dynamic, more involved work is needed due to the changing nature of the quadtree as well as to incorporate an approach used for approximate dynamic independent intervals.

Hypercubes and Beyond We show that our approach for squares naturally extends to axis-aligned hypercubes in d dimensions, providing a $O(4^d)$-approximate independent set in $O(2^d \log^{2d+1} n)$ time.

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Extended Abstract
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Γ-Induced-Paired Dominating Graphs of Cycles

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Abstract

A set D of vertices of a graph G is an induced-paired dominating set if the subgraph of G induced by D contains only nonadjacent edges and every vertex of G is adjacent to some vertex in D. The set D is minimal if every proper subset of D is not an induced-paired dominating set. The upper induced-paired domination number of G, denoted by $\Gamma_{ip}(G)$, is the maximum cardinality of a minimal induced-paired dominating set of G. An $\Gamma_{ip}(G)$-set is a minimal induced-paired dominating set of cardinality $\Gamma_{ip}(G)$. The Γ-induced-paired dominating graph of a graph G, denoted by $IPD_\Gamma(G)$, is the graph whose vertex set is the set of all $\Gamma_{ip}(G)$-sets. Two $\Gamma_{ip}(G)$-sets D_1 and D_2 are adjacent in $IPD_\Gamma(G)$ if $D_2 = D_1 \backslash \{v_1\} \cup \{v_2\}$ for some $v_1 \in D_1$ and $v_2 \notin D_1$. In this paper, the Γ-induced-paired dominating graphs are determined for all cycles.

Keywords: Induced-paired dominating graph, Induced-paired dominating set, Induced-paired domination number.

2010 MSC: Primary 05C69; Secondary 05C38, 05C60, 05C76.

1 Introduction

Let G be a graph. Wongsriya et al. [3] defined the Γ-total dominating graph of G, and the authors presented the Γ-total dominating graphs of some paths. Eakawinrujee and Trakultraipruk [1] defined the Γ-paired dominating graph of G, and they presented the Γ-paired dominating graphs of some paths. In [2], we introduced the Γ-induced-paired dominating graph of G, and we presented the Γ-induced-paired dominating graphs of all paths. In this paper, we present the Γ-induced-paired dominating graphs of all cycles.

2 Main Results

Let $C_n = v_0v_1v_2 \cdots v_{n-1}v_0$ be a cycle. First, we give the upper-induced paired domination number of a cycle. Since at most two vertices of any three consecutive vertices of a cycle can be in an induced-paired dominating set, we get the following lemma.

Lemma 2.1. *Let n be a positive integer such that $n \geq 3$ and $n \neq 5$. Then $\Gamma_{ip}(C_n) = 2\left\lfloor \frac{n}{3} \right\rfloor$.*

Obviously, $IPD_\Gamma(C_3)$ is the cycle with three vertices, and $IPD_\Gamma(C_5)$ is an empty graph. Next, we determine the Γ-induced-paired dominating graphs of the other cycles.

Theorem 2.2. *Let $k \geq 2$ be an integer. Then $IPD_\Gamma(C_{3k}) \cong 3K_1$.*

For example, the graph $IPD_\Gamma(C_6)$ is shown in Figure 1.

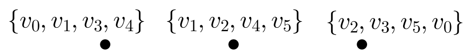

Figure 1: The Γ-induced-paired dominating graph of C_6

Theorem 2.3. *Let k be a positive integer. Then $IPD_\Gamma(C_{3k+1}) \cong C_{3k+1}$.*

For example, the graph $IPD_\Gamma(C_4)$ is shown in Figure 2.

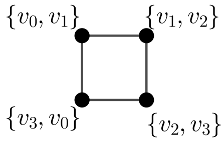

Figure 2: The Γ-induced-paired dominating graph of C_4

Let k be a positive integer. The *stairgrid* of size k, denoted by S_k, is the subgraph of the Cartesian product of the paths $u_1u_2\cdots u_k$ and $v_1v_2\cdots v_k$ induced by $\{(u_i, v_j) : 1 \le i \le j \le k\}$. Let $S_k^{(1)}, S_k^{(2)}, S_k^{(3)}$, and $S_k^{(4)}$ be stairgrids of size k, and $C_{x,y}^{(i)}$ the vertex at position (x, y) of $S_k^{(i)}$. The **spiral-wristband** of size k, denoted by SW_k, is the graph obtained from the stairgrids $S_k^{(1)}, S_k^{(2)}, S_k^{(3)}$, and $S_k^{(4)}$ by joining the vertex $C_{x,k}^{(i)}$ to $C_{1,x-1}^{(i+1)}$ for $2 \le x \le k$ and $1 \le i \le 3$, and the vertices $C_{x-1,y-1}^{(1)}$ and $C_{x,y}^{(4)}$ are the same for $2 \le x \le y \le k$.

Theorem 2.4. *Let $k \ge 2$ be a positive integer. Then $IPD_\Gamma(C_{3k+2}) \cong SW_{k-1}$.*

For example, the graph $IPD_\Gamma(C_{11})$ is shown in Figure 3.

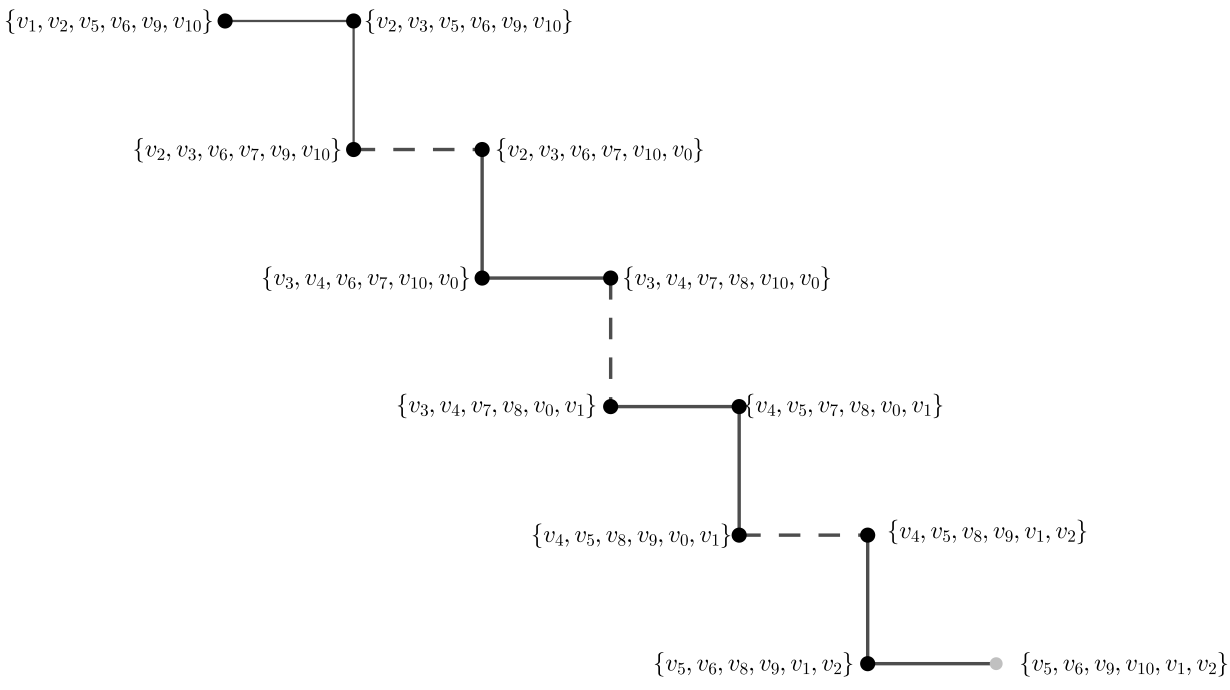

Figure 3: The Γ-induced-paired dominating graph of C_{11}

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Extended Abstract
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Biased Domination Game

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Abstract

We introduce an extended version of a domination game on a graph, called a biased domination game, in which Dominator and Staller play more than one move in each turn. We show some relations of biased game domination numbers for different biases.

Keywords: biased domination game, biased game domination number, maximal move, minimal move.

2010 MSC: Primary 91A46; Secondary 05C57, 91A43.

1 Introduction

A domination game was introduced in [1] as a game of two players, called *Dominator* and *Staller*, on a graph. The players take turns to perform a move by choosing a vertex in the graph. Vertices in the closed neighborhood of a chosen vertex are said to be *dominated*. A move u is legal if it creates at least one new dominated vertex. In other words, its closed neighborhood $N[u]$ is not contained in the closed neighborhood of vertices set which have been chosen before. That is, for a sequence of previously picked moves $u_1, u_2, u_3, \dots, u_{n-1}$, the player can pick a move u_n if and only if $N[u_n] \nsubseteq \bigcup_{i=1}^{n-1} N[u_i]$. The game is ended when all vertices in the graph are dominated. Dominator tries to end the game as soon as possible, while Staller tries to prolong the game. In the domination game, if Dominator starts the game, this game is said to be *Game* 1. Otherwise, it is said to be *Game* 2. If both of players play optimally in a domination game on a graph G, the number of moves when the game is ended is called the *domination numbers*, denoted by $\gamma_g(G)$ and $\gamma_g'(G)$ in a Game 1 and Game 2, respectively.

In this work, we introduce a variation of a domination game called a (δ, σ)*-biased domination game* or simply called a (δ, σ)*-biased game* where Dominator and Staller must pick δ and σ moves for each turn (except the last turn of a game), respectively. The (δ, σ)-biased game domination numbers are denoted by $\gamma_{(\delta,\sigma)}(G)$ for Game 1 and $\gamma'_{(\delta,\sigma)}(G)$ for Game 2. The main goal of this work is to compare the biased game domination numbers $\gamma_{(\delta,\sigma)}(G)$ and $\gamma_{(\delta+1,\sigma)}(G)$. Similarly, we compare the biased game domination numbers $\gamma_{(\delta,\sigma)}(G)$ and $\gamma_{(\delta,\sigma+1)}(G)$.

2 Preliminaries

First, we introduced some definitions and properties in domination games and biased domination game.

Definition 2.1 ([1])**.** Let G be a graph and S be a subset of $V(G)$. A *partially dominated graph* $G|S$ is a graph G in which all vertices in S are already dominated.

Theorem 2.2 (Continuation Principle [1])**.** *Let G be a graph and $A, B \subseteq V(G)$. If $A \subseteq B$, then $\gamma_g(G|A) \geq \gamma_g(G|B)$ and $\gamma_g'(G|A) \geq \gamma_g'(G|B)$.*

The proof of Theorem 2.2 also holds for a biased domination game.

Theorem 2.3 (Continuation Principle of the Biased Domination Game)**.** *Let G be a graph and $A, B \subseteq V(G)$. If $A \subseteq B$, then $\gamma_{(\delta,\sigma)}(G|A) \geq \gamma_{(\delta,\sigma)}(G|B)$.*

In 2010, Bresăr, Klavžar and Rall considered a domination game such that Dominator (resp. Staller) is allowed, but not obligated, to skip exactly one move in the game. That is, there is at most one turn such that Dominator (resp. Staller) may decide to pass. After the game end, the number of moves in a game where both players are playing optimally, is denoted by $\gamma_g^{dp}(G)$ (resp. $\gamma_g^{sp}(G)$). We call this game the *Dominator-pass game* (resp. *Staller-pass game*).

We define the pass game for a biased domination game as follows.

Definition 2.4 ([1]). In a (δ, σ)-game on a graph G, if Staller is allowed to pass the moves (except the first move of each turn) at most n moves, then we define such game as an *n-Staller-pass (δ, σ)-game* or *$sp(n)$-(δ, σ)-game*. The numbers of moves in an $sp(n)$-(δ, σ)-game when both players play optimally are denoted by $\gamma_{sp(n),(\delta,\sigma)}(G)$ in game 1 and $\gamma'_{sp(n),(\delta,\sigma)}(G)$ in game 2. Similar notation, $dp(n)$, is used for n-Dominator-pass games.

We define two special types of moves in a biased domination game.

Definition 2.5. We say that a move is *minimal* if it dominates exactly one new vertex.

Definition 2.6. We say that a move u is *maximal* if it dominates at least one new vertex w such that if u' is another move dominating w then all new vertices dominated by u' are also dominated by u. In other words, u is always a better move among all moves dominating w.

If A is the set of all dominated vertices by previous moves, we have

$$N[u'] \setminus N[A] \subseteq N[u] \setminus N[A] \tag{2.1}$$

for all legal moves u' dominating w.

3 Main Results

We get the following results similar to [1].

Theorem 3.1. *For any graph G and $i \geq 0$, if Staller can always make a minimal move, then*

$$\gamma_{sp(i+1),(\delta,\sigma)}(G) \leq \gamma_{sp(i),(\delta,\sigma)}(G)$$

and

$$\gamma_{(\delta,j)}(G) \leq \gamma_{(\delta,\sigma)}(G)$$

for all $j \leq \sigma$.

Theorem 3.2. *For any graph G and $i \geq 0$, if Dominator can always make a maximal move (except the first move), then*

$$\gamma_{dp(i+1),(\delta,\sigma)}(G) \geq \gamma_{dp(i),(\delta,\sigma)}(G)$$

and

$$\gamma_{(j,\sigma)}(G) \geq \gamma_{(\delta,\sigma)}(G)$$

for all $j \leq \delta$.

These results also hold for Game 2. Moreover, we can applied them to [2] and [3].

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

TJCDCG[3] 2020+1 Chiang Mai , Thailand

Unidirectional Monotonic Paths through Specified Points in Labeled Point Sets in Convex Position

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Abstract

Let P be a set of n points in convex position that are labeled with n distinct real values. Then for any point $p \in P$, there is a unidirectional non-crossing path connecting at least $\sqrt{n}$ points of P, containing p, along which the labels of points monotonically increase (or decrease). In other words, for any circular permutation with n distinct real-valued terms and for any term k in it, there exists a monotone subsequence of length at least $\sqrt{n}$ that contains k.

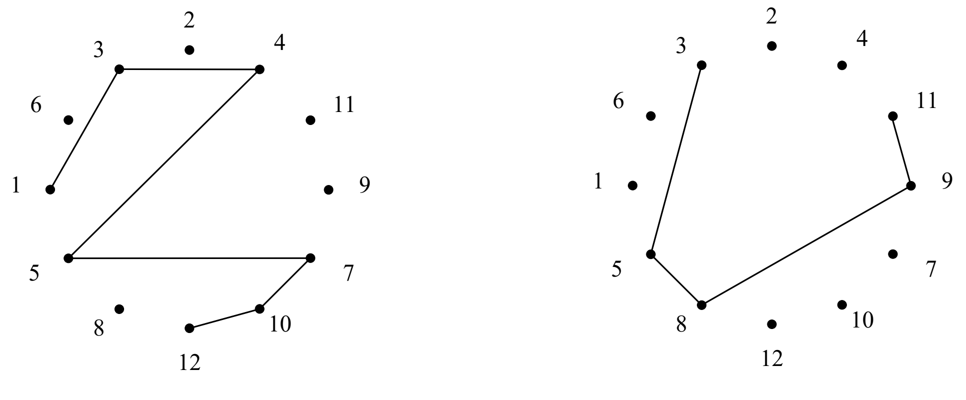
Keywords: labeled point set, convex position, monotonic path.

2010 MSC: Primary 52C10; Secondary 11B99.

1 Introduction

Let P be a set of points in the plane. We denote by $\mathrm{conv}(P)$ the convex hull of P. P is said to be *in convex position* if its elements are the vertices of $\mathrm{conv}(P)$. P is called a *labeled point set* if a number, called a *label*, is assigned to each point. The label of point p will be denoted by $l(p)$. All point sets P considered here are labeled and in convex position, and the labels of the elements of P are all different real numbers. $|P|$ will be denoted by n.

A polygonal line connecting k points $p_1, \ldots, p_k$ of P is called a *monotonic path of length k* if the sequence $l(p_1), l(p_2), \ldots, l(p_k)$ is monotonically increasing or decreasing (Figure 1(a); note that the length is *not* defined as the number of edges, but the number of vertices). Also, a polygonal line is called a *unidirectional path* if it connects points that appear in the same order along the perimeter of $\mathrm{conv}(P)$ (so, it has no self-intersection; Figure 1(b)).



(a) (b)

Figure 1: (a) A (non-crossing) monotonic path and (b) a unidirectional monotonic path.

In 1935, Erdős and Szekeres [2] showed that any sequence of $(l-1)(m-1)+1$ distinct real numbers contains either a monotonically increasing subsequence of length at least l, or a monotonically decreasing subsequence of length at least m (these bounds are tight). From this, it follows that any sequence of n distinct real numbers contains a monotonically increasing or decreasing subsequence of length at least $\sqrt{n}$.

The problem of finding non-crossing monotonic long paths in labeled point sets is a natural variation of the theorem, and this problem was first studied by Czyzowicz et al. [3]. They proved that any labeled point set P in *convex* position contains a non-crossing monotonic path of length at least $\sqrt{2n}$. This bound was improved by Sakai and Urrutia [4] to $\sqrt{3n-3/4}-1/2$ (they conjecture that there is a non-crossing monotonic path of length at least $2\sqrt{n}-1$) by giving a simple proof for a result by Chung [1] concerning the length of a longest unimodal subsequence in a sequence. Sakai and Urrutia [5] also proved that for any labeled point set P in convex position and for any element $p \in P$, there is a non-crossing monotonic path of length at least $\sqrt{2(n-1)}$ which passes through p.

2 Results

In this talk, we show the following result on unidirectional monotonic paths:

Theorem 2.1. *Let P be a set of n labeled points in convex position. Then for any point $p \in P$, there is a unidirectional monotonic path of length at least $\sqrt{n}$ that passes through p. (This bound is tight.)*

This implies that for any circular permutation with n distinct real-valued terms and for any term k in it, there exists a monotone subsequence of length at least $\sqrt{n}$ that contains k.

We also show some results on *disjoint* non-crossing monotonic paths (not necessarily unidirectional) for labeled point sets in convex position.

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Extended Abstract
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Logical Matrix Representations in Map Folding

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Abstract

Logical matrices are binary matrices often used to represent relations. In the map folding problem, each folded state corresponds to a unique partial order on the set of squares and thus could be described with a logical matrix. On the applicational level, such representations provide us an intuitive recognition of map folding. For instance, using the logical matrices, we can give a precise mathematical description of a folding process, such that some problems (ex., how to check self-penetration; how to deduce a folding process according to a given order of the squares) in map folding can be solved in a mathematical and natural manner. Using this representation, we can establish the correspondence between folded states and logical matrices on a category level. The constraint, which makes the correspondence (functor) one-to-one, helps us to embed the category of partly folded states to a generally used category, the category of relations **Rel**. Such analysis would develop a deep mathematical understanding of map folding.

Keywords: map folding, logical matrix representation, category of relations.

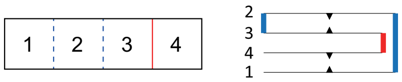
2010 MSC: Primary xxXxx; Secondary xxXxx, xxXxx, xxXxx.

1 Introduction

In this extended abstract, we first introduce a *logical matrix representation* for *map foldings*, i.e., foldings from a rectangular piece of paper embedded with a grid pattern to the shape of a unit square [1]. A logical matrix is a binary matrix. The set of the logical matrices form a semiring under the addition and multiplication operations on the semiring $(\Omega = \{0, 1\}, +, \cdot)$. They are naturally associated with the category of relations **Rel**. We also propose a natural transformation from the category of partly folded states of a map to **Rel**. As far as we have surveyed, there is so far no research associating foldings in origami to concrete categories, like the category of relations (**Rel**) and the category of finite Hilbert spaces (**FHilb**). In fact, not only in the field of origami, these two categories, **Rel** and **FHilb** have not been really valued until the contemporary development of quantum computing and its concise expressions using the category language. This extended abstract gives the first step in the direction of making correspondence between foldings and important categories in computational science subjects. There is the potential possibility that according to the category correspondence, the flat-folding of a piece of paper may have something to do with quantum computing.

In contemporary mathematics, the logical matrices are put together with some particular categories. New discoveries show that both the nondeterministic classical computing, handled within the category of relations, **Rel**, and the quantum computing, handled within the category of Hilbert spaces, **Hilb**, have their corresponding matrix representations with similar matrix operations and algebraic structures. **Rel** corresponds to the logical matrices while **FHilb** corresponds to complex matrices [2]. Let us align them together: logical matrices vs complex matrices, **Rel** vs **FHilb**, and **map folding** vs **flat-folding**. In each pair, the former is a degenerated case of the latter. In this extended abstract, we managed to associate map folding with logical matrices and **Rel**. This result makes it natural to use general research methods of **Rel** to study map folding. For example, the logical matrix representation brings some new and simple solutions to some map folding related problems.

Moreover, it is hopeful of associating flat-folding with complex matrices and **FHilb** in the future.



(1-1) (1-2)

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Figure 1: (1-1) and (1-2) illustrate a map of size 1×4 and one of its possible final state; the right side is its adjacency representation A and entire representation E

2 Logical Matrix Representation

An overlapping order of a map is a poset where the partial order means for the "up-down adjacent relation of squares". The logical matrix recording the adjacent relations is called the **adjacency representation**. The transitive closure of this partial order involves all the up-and-down relations (not necessarily adjacent) of all the squares. Correspondingly, the transitive closure through of the adjacency representation is called the **entire representation**. For example, a possible final state for the map in Figure 1(1-1), illustrated in (1-2), correspond to the poset P_1: $1 < 4 < 3 < 2$. We denote it by $(1, 2, 3, 4, <)$. The adjacency representation A and the entire representation E of P_1 are illustrated on the right side of Figure 1.

3 Main results

We have two main results. First, as aforementioned, we proposed the logical matrix representation of map folding and then established a one-to-one correspondence between the category of partly flat-folded states and the category of logical matrices. It means that the category of partly flat-folded states could be viewed as a sub-category of **Rel**. We use *simple folds* as the morphisms. A simple fold refers to a fold on some continuous uppermost or lowermost layers along a single crease, which does not cause any change on the layers except for the neighborhoods of the crease. The flow of our work is as follows.

Step 1. Define a logical matrix representation to represent every partly flat-folded state and the final flatly folded state of a map.

Step 2. Give a method to detect the self-penetrations by detecting the numbers of 0s and 1s in submatrices of a given logical matrix. This step is necessary because we need to clarify which logical matrix is a representation of a partly flat-folded state to build the correspondence from the category of partly flat-folded states to **Rel** by an intermediate of the category of logical matrix representations. The remaining logical matrices are taken as the objects in the category of logical matrix representations.

Step 3. Define the morphisms in the category of partly flat-folded states as simple folds and define the morphisms in the category of logical matrix representations using the addition operation on the logical matrix semiring, which is the same as the one on the semiring $(\Omega = \{0, 1\}, +, \cdot)$.

Step 4. Establish a one-to-one correspondence between the category of partly flat-folded states and the category of logical matrix representations. The correspondence is then composited with the monomorphic natural transformation from the category of logical matrix representations to **Rel**. Thus, finally we know that the category of partly flat-folded states is isomorphic to a sub-category of **Rel**.

Second, using the logical matrix representations, we can solve the following two problems: 1. A problem that asks whether a given order of n squares of a map of size $1 \times n$ is foldable by a sequence of simple folds or not. We also obtain one legal sequence if it exists. 2. A problem that asks whether there exist self-penetrations in the given order.

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Feedback game on Eulerian graphs

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Abstract

In this paper, we introduce a two-player impartial game on graphs, called *feedback game*, which is a variant of generalized geography. The feedback game can be regarded as undirected edge geography with an additional rule that the first player who goes back to the starting vertex wins the game. We consider feedback game on an Eulerian graph since the game ends only by going back to the starting vertex. We first show that it is PSPACE-complete in general to determine the winner of the feedback game on Eulerian graphs even if its maximum degree is at most 4. In the latter half of the paper, we discuss the feedback game on two subclasses of Eulerian graphs, i.e., triangular grid graphs and toroidal grid graphs.

Keywords: Feedback game, Edge geography, Eulerian graph, Triangular grid graph, Toroidal grid graph.

2010 MSC: Primary 05C07; Secondary 05C45, 91A43, 91A46.

1 Introduction

All graphs considered in this paper are finite, loopless and undirected unless otherwise mentioned. A graph G is *Eulerian* if each vertex of G has even degree. For other basic terminology in graph theory, we refer to [6].

In combinatorial game theory, impartial games have been well studied for a long time, where a game is *impartial* if the allowable moves depend only on the position and not on which of the two players is currently moving. So far, many interesting impartial games have been found; e.g., Nim [4], Kayles [8] and Poset game [14]. The most famous result in this area is the Sprague-Grundy theorem [11,15] stating that every impartial game (under the normal play convention) is equivalent to a one-heap game of Nim. There are also many interesting games played on graphs as for example; Vertex Nim [7], Ramsey game [9] and Voronoi game [16]. For more details and other topics, we refer the reader to survey several books and articles [1–3, 5].

One of the most popular impartial games on graphs is generalized geography. *Generalized geography* is a two-player game played on a directed graph D whose vertices are words and $xy \in A(D)$ if and only if the end character of a word x is the first one of y, where $A(D)$ is the set of arcs of D. For example, if x is "Japan" and y is "Netherlands", then $xy \in A(D)$ but $yx \notin A(D)$. In this setting, the game begins from some starting word and both players alternately extend a directed path using unused words. The first player unable to extend the directed path loses. It is PSPACE-complete to determine the winner of generalized geography [12]. Moreover, several variants of generalized geography have been considered, e.g., planar generalized geography [12], edge geography [13] and undirected geography [10]. It is also known that for each of above variants is PSPACE-complete to determine which player wins the game except undirected vertex geography; we can determine the winner in polynomial time.

In this paper, we consider a new impartial game on a graph, called *feedback game*, which is a variant of undirected edge geography.

Definition 1.1 (Feedback game)**.** There are two players; Alice and Bob, starting with Alice. For a given connected graph G with a starting vertex s, a token is put on s. They alternately move the token on a vertex u to a neighbor v of u and then delete an edge uv. The first player who moves the token back to s or to an isolated vertex (after removing an edge used by the last move) wins the game.

In this paper, we investigate feedback game on Eulerian graphs. Note that if a given connected graph G is Eulerian, then the game does not end until the token goes back to the starting vertex s, and further observe that Bob always wins feedback game on any connected bipartite Eulerian graph (cf. [10]): Let G be a connected bipartite Eulerian graph, and so, all vertices of G are properly colored by two colors, black and white. Without loss of generality, we may suppose that the starting vertex is colored by black. Throughout the game on G, a token is always moved to a white (resp., black) vertex by Alice (resp., Bob). Thus Bob necessarily wins the game.

2 Main results

Here main results are in this paper. Though definitions of technical terms are omitted, one can see them on ArXiV version*. The conjecture still unsolved. We hope one enjoy solving this open problem.

Theorem 2.1. *It is* PSPACE-*complete to determine whether there exists a winning strategy for the first player in feedback game, even if the given graph is Eulerian.*

Theorem 2.2. *If $n \neq 2^m - 3$ with $m \geq 2$, then Bob wins the game on the triangular grid graph T_n with a starting vertex v_0^0.*

Lemma 2.3. *There is no even kernel graph of T_n that does not include v_0^0 when $n > 1$.*

Theorem 2.4. *Alice wins feedback game on T_1 and T_5 with a starting vertex v_0^0.*

Theorem 2.5. *If $n = 2^m - 3$ with $m \geq 2$, then the triangular grid graph T_n has no even kernel graph.*

Theorem 2.6. *If $\gcd(m, n) = c > 1$, then Bob can win feedback game on $m \times n$ toroidal grid graph.*

Theorem 2.7. *If $m = 2$ or $m = 3$, and $\gcd(m, n) = 1$, then Alice can win feedback game on $m \times n$ toroidal grid graph.*

Theorem 2.8. *If $\gcd(m, n) = 1$, then there exists no even kernel graph of $m \times n$ toroidal grid graph.*

Conjecture 2.9. *Alice can win feedback game on $m \times n$ toroidal grid graph. if $\gcd(m, n) = 1$.*

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*https://arxiv.org/abs/2002.09570

Extended Abstract
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Solving teleportation mazes with limited visibility

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Abstract

We consider what we denote the Embedded Agent Limited Visibility Teleportion Maze (EALVTM) problem of efficiently utilizing a player-controlled agent, with distance-limited vision, to discover and subsequently navigate to a target of interest by moving along the edges of a rectangular or triangular mesh. Here, we prove an NP-hardness result for the optimization version of EALVTM where the objective is to minimize the total number of edge-wise hops, determine a set of cases where the problem of finding an object on a given mesh becomes Fixed-Parameter Tractable (FPT), and show how recent advances in low-distortion stochastic embeddings of higher-genus graphs in the plane [3,4] can be used to extend existing planar graph coverage and path search algorithms to treat the EALVTM problem on meshes embedded in surfaces having bounded orientable or non-orientable genus.

Keywords: Coverage Path Planning, Graph Coverage, Path Search, A-star, D-star.

2010 MSC: Primary 05C90; Secondary 05C85, 05C57, 05C10.

1 Extended Abstract

We introduce the Embedded Agent Limited Visibility Teleportion Maze (EALVTM) problem of efficiently solving the type of *teleportation maze* that has been part of the aesthetic of adventure and role-playing games since late 1970's titles such as William Crowther's *Colossal Cave Adventure* and Infocom's *Zork*. Here, controlling a single agent with distance-limited vision, our objective is to discover and subsequently navigate to a target, with the fewest possible number of steps, by walking along the edges of a hole-containing rectangular or triangular mesh embedded in an orientable or non-orientable genus surface. In this context, we assume either a specified or randomized initial placement of the agent, either no or only partial *a priori* knowledge concerning the agent's initial surroundings, and time and space computational resources at most polynomial in the size of the environment.

For visual intuition, in Figures 1(a-c) we show screenshots of the teleportion-based "player's maze" in Chris Avellone and Black Isle Studio's *Planescape: Torment*, and in Figure 1(d) we show a screenshot of the teleportion maze in Kikiyama's *Yume Nikki*. Concerning the *Planescape: Torment* "player's maze", treating pairs of teleportation portals as handles on a manifold with embedded directed paths of sufficient length that they cannot be seen through without being traversed, we can determine that the "Nameless One" protagonist is on a surface of genus at least $g \geq 5$ (see the remark in the Figure 1(b) caption concerning the maze's true topology). Assuming that an exit portal or item of interest can be placed arbitrarily, we can observe that EALVTM is at least as hard as a search variant of the Coverage Path Planning (CPP) [2] or milling problem [1]. Furthermore, we can observe that once an exit portal or object of interest is at least seen across a barrier – chasms being the vehicle for this in both the *Planescape: Torment* "player's maze" and the *Yume Nikki* teleportation maze (see Figure 1(c,d)) – there remains the non-trivial task of reaching the objective, which may be no easier than discovering it.

Letting n be the total number of nodes in a mesh for a given instance of EALVTM, defining a "step" of the agent as a move from one node to another along a mesh edge, and affording a budget of αn total steps, we prove in part that, unless $P = NP$, no polynomial time algorithm exists for the EALVTM subproblem of discovering a randomly placed exit portal or object with an average probability $\geq \left(\frac{\alpha}{2} + \epsilon\right)$

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for $\epsilon > 0$ and $\forall \alpha \in (0, 2]$. In particular, we show that this result holds even in the case of planar undirected teleportation mazes on both rectangular and triangular grids with holes, fixed or random initial agent placement, and additionally determine a set of cases under which this search subproblem becomes Fixed-Parameter Tractable (FPT). In terms of algorithms, we show that recent advances in low-distortion stochastic embeddings of higher-genus graphs on the plane [3, 4] can be used to adapt existing plane-based algorithms for the CPP problem to higher genus surfaces. We likewise describe how the heuristic for the Dynamic A* (D*) algorithm [5] and some of its well-known variants can be adapted to higher genus surfaces to solve the final navigation component of EALVTM once an exit or object of interest is discovered.

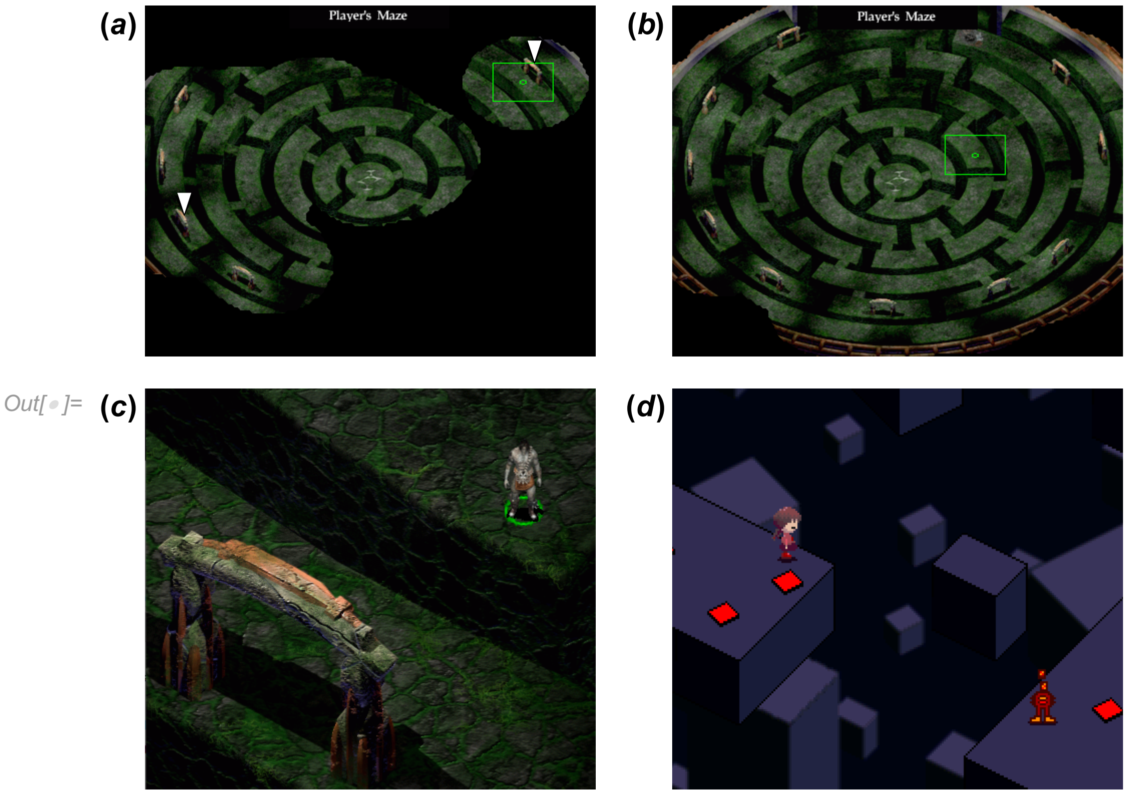

Figure 1: Author-taken screenshots of teleportation puzzles from (a-c) Chris Avellone and Black Isle Studio's *Planescape: Torment*, and from (d) Kikiyama's *Yume Nikki*: **(a)** map of the "player's maze" in *Planescape: Torment* with black "fog-of-war" covering unexplored regions, where the player character positioned at the green dot has just traversed a teleportation portal bridging the pair of gates indicated by the (author added) white downward arrows (note that portals cannot be seen through without being traversed); **(b)** nearly fully explored map of the "player's maze" (due to teleporter-induced state changes, the true maze consists of 12 copies of this planar surface embedded closely enough together in $\mathbb{R}^n$ that the overlapping areas on all planes can be viewed simultaneously by the agent); **(c)** the "Nameless One" protagonist ("Adahn") looking across a chasm at the exit teleportation portal; **(d)** the *Yume Nikki* protagonist Madotsuki in a teleportation maze – where the red tiles serve as portals between disconnected floating blocks – peering across a gap at a block inhabited by the character "Warpo".

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Extended Abstract
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Dominect: A Simple yet Deep 2-Player Board Game

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Abstract

In this work we introduce the perfect information 2-player game *Dominect*, which has recently been invented by two of the authors. Despite being a game with quite simple rules, Dominect reveals a high depth of complexity. We report on first results concerning the development of winning strategies, as well as a PSPACE-hardness result for deciding whether a given game position is a winning position.

Keywords: Connection game, 2-player game, PSPACE-hardness.

2010 MSC: Primary 91A46; Secondary 91A05.

1 Introduction and Game Rules

Dominect is a finite, deterministic perfect information 2-player game developed by Oswin Aichholzer and Maarten Löffler during the 33^{rd} Bellairs Winter Workshop on Computational Geometry in 2018. It belongs to the class of *connection games*, where a typical goal is to form a path connecting two opposite sides of the game board [5]. Prominent members of this class are Hex (aka Con-tac-tix), TwixT, and Tak.

The Rules of Dominect. *Dominect is a two-player game that is played on a rectangular board of $m \times n$ squares, where the board size $m, n \geq 3$ can vary. The name Dominect comes from Domino and Connect, where a domino is 2 squares long and 1 square wide and can be placed horizontally or vertically on empty neighboring squares of the board. The goal is to connect opposite sides of the board by placing dominoes of the player's color (red or blue) before the opponent manages to connect the other two board sides by dominoes of their color. Dominoes are connected if they are adjacent either vertical, horizontal or diagonal, that is, touching at a corner is sufficient. Starting with red, the two players alternatingly place one domino of their color at a time (from an unlimited supply) until one of the players wins or until no more domino can be placed – a draw.*

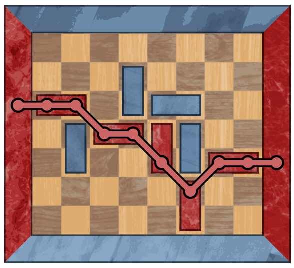

Figure 1: A connecting chain for red.

A detailed official book of rules with several illustrations is available on the game's homepage at https://dominect.ist.tugraz.at where also some variations of the game can be found.

Dominect is in several ways different from other popular connection games like Hex [3]: For example, a draw is possible for Dominect and also a red connecting chain could cross a blue connecting chain, that is, both players could simultaneously connect their board sides on the same board. *Strategy stealing*, an approach used to show that the second player cannot force a win for many games [1, 2], cannot be applied for Dominect. The reason is that each placed domino blocks several other potential domino placements (not only one). So giving one move away by placing a domino somewhere, as used in strategy stealing, can be a major obstacle and change the whole strategy.

2 Winning Strategies and PSPACE-Hardness

It might seem natural to play on quadratic boards (that is, $m = n$). But it turns out that, at least for small sizes, an $n \times (n-1)$ board is more balanced, where the first player has to play the distance n.

Theorem 2.1. *For all boards of size $m \times n$ with $m \leq 9$ or $n \leq 8$, except $1 \times 1, 3 \times 1, 4 \times 1$, one of the players can force a win (assuming that they both play perfectly). Table 1 lists who can force the win.*

Proof ideas. We assume w.l.o.g. that the red player just made a move (and has not yet won). We want to know for the resulting game position D if the red player can force a win, regardless of what the next moves of the blue player are. If yes, we call D a *winning position* (for the red player). Let $C_L = C_1$ to $C_R = C_k$ be the $k \geq 2$ red connected components (sets of red dominoes and board sides which are connected) in D, where C_L and C_R contain the left and right side of the board, respectively. Starting from D, a *bridge-move* is a red move that connects two components $C_i \neq C_j$. Two red moves are *independent* if they cannot be blocked simultaneously by a single blue domino, that is, if no covered square of one move is identical to or sharing an edge with a covered square of the other move.

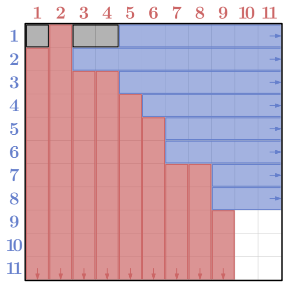

Table 1: Depiction of the results of Theorem 2.1.

Hence, if we can find 2ℓ pairwise independent bridge-moves in D and pair them in a way that after performing either move per pair, the red player has connected C_L and C_R, and if the blue player cannot connect in at most ℓ moves, then D is a winning position for the red player. The results of Theorem 2.1 can be obtained by using this observation in combination with a careful analysis of different starting moves (see Figure 2 for the 7×7 board). ☐

In the full version of this work, we also show that given a game position D and a set of pairwise independent bridge-moves in D, one can efficiently compute whether D is a winning position in the above sense. This can also be used to develop an algorithm that identifies game positions from which one player can force a win even though D is not a winning position via pairwise independent bridge-moves. However, while a positive answer of this algorithm guarantees that a player can force a win, a negative answer does not imply that the player cannot force a win. We next show that this situation is not avoidable, as deciding whether or not a player can force a win from a given game position is computationally hard.

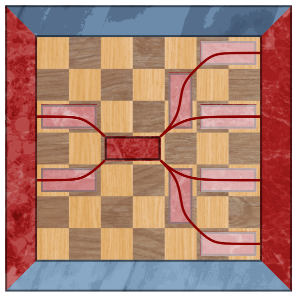

Figure 2: A 7×7 board winning strategy.

Theorem 2.2. *Deciding whether a given position in the generalized game on an $n \times n$ board is a winning position is PSPACE-complete.*

Proof sketch. To show this, we reduce from the generalized version of Hex, which is PSPACE-complete [4]. In the reduction each Hex cell corresponds to a rectangular gadget of dominoes in which exactly one more domino can be placed (see Figures 3, gadgets marked in black). Placing hexagonal stones in the Hex grid creates essentially the same connections as placing dominoes of the same colors in the corresponding gadgets. ☐

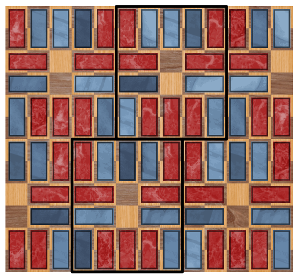

Figure 3: Two rows of three Hex cells each.

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

Rainbow and Properly Colored Spanning Trees in Edge-Colored Bipartite Graphs

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We deal with an edge-colored graph G, which have neither loops nor multiple edges. For an edge e of G, let $col(e)$ denote the color of e. For a vertex v of G, the *color degree* of v, denoted by $\deg_G^c(v)$, is the number of colors appeared in the edges incident with v. The *minimum color degree* of G is defined by

$$\delta^{col}(G) = \min\{\deg_G^c(v) : v \in V(G)\}.$$

An edge-colored graph G is called *rainbow* or *heterochromatic* if all the edges of G have distinct colors. Moreover, G is called *properly colored* if no two adjacent edges of G have the same color. Recently the following results were obtained.

Theorem 1 (Cheng, Kano and Wang [2]). *Let G be an edge-colored connected graph. Then the following statements hold.*

(i) If $\delta^{col}(G) \geq \frac{|G|}{2}$, and if for each color c, the set of edges colored with c forms a subgraph of order at most $\frac{|G|}{2}+1$, then G has a rainbow spanning tree.

(ii) If $\delta^{col}(G) \geq \frac{|G|}{2}$, and if for each color c, the set of edges colored with c forms a star, then G has a rainbow spanning tree.

(iii) If $\delta^{col}(G) \geq \frac{|G|}{2}$, then G has a properly colored spanning tree.

(iv) The condition $\delta^{col}(G) \geq \frac{|G|}{2}$ in the above statements (i), (ii) and (iii) are sharp.

Notice that the above two statements (ii) and (iii) are easily proved by using the statement (i). In this paper, we show that the minimum color degree condition given in Theorem 1 can be weakened for edge-colored bipartite graphs as the following theorem.

Theorem 2 ([4]). *Let G be an edge-colored connected bipartite graph. Then the following statements hold.*

(i) Assume that $\delta^{col}(G) \geq \frac{|G|}{3}$, for every color c, the set of edges colored with c forms a subgraph of order at most $\frac{|G|}{3}+1$, and that there are at least $|G|-1$ colors appearing in the edges of G. Then G has a rainbow spanning tree.

(ii) If $\delta^{col}(G) \geq \frac{|G|}{3}+1$, and if for each color c, the set of edges colored with c forms a star, then G has a rainbow spanning tree.

(iii) If $\delta^{col}(G) \geq \frac{|G|}{3}+1$, then G has a properly colored spanning tree.

(iv) The condition $\delta^{col}(G) \geq \frac{|G|}{3}$ in the above statements (i), and the condition $\delta^{col}(G) \geq \frac{|G|}{3}+1$ in (ii) and (iii) are sharp. Moreover, the condition that there are at least $|G|-1$ colors appearing in the edges of G in (i) is necessary.

New results on rainbow and properly colored spanning trees are found in [3], and many related results on spanning trees of graphs including the following theorems can be found in the book [1] and a survey [5].

It is known that some sufficient conditions for the usual graph (i.e., a non edge-colored graph) to have a spanning tree with prescribed property can be weakened for bipartite graphs. In order to explain some of them, we need some definitions and notation. Let $k \geq 2$ be an integer. Then a tree T is called

*This work was supported by JSPS KAKENHI Grant Number 19K03597.

an *k-ended tree* if it has at most k-leaves, and T is called a *k-tree* if the maximum degree of T is at most k. Furthermore define

$$\sigma_k(G) = \min\Big\{\sum_{v\in S}\deg_G(v) : S \text{ is an independent set of } k \text{ vertices of } G\Big\}.$$

Then it is known that for an integer $k \geq 2$, if a connected graph G_1 satisfies $\sigma_2(G_1) \geq |G_1| - k + 1$, then G_1 has a spanning k-ended tree. On the other hand, it is shown that if a connected bipartite graph $G_2(X,Y)$ with $|X| \leq |Y| \leq |X|+k-1$ satisfies $\sigma_2(G_2) \geq \frac{|G_2|-k}{2}+1$, then G_2 has a spanning k-ended tree. Moreover, it is shown that for an integer $k \geq 2$, if a connected graph G_3 satisfies $\sigma_k(G_3) \geq |G_3|-1$, then G_3 has a spanning k-tree. On the other hand, it is shown that if a connected bipartite graph $G_4(X,Y)$ with $|X| \leq |Y| \leq (k-1)|X|+1$ satisfies $\sigma_k(G_4) \geq |Y|$, then G_4 has a spanning k-tree. Thus we expect that Theorem 1 might be weakened for edge-colored bipartite graphs as above, and our theorems show that this expectation is true.

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Extended Abstract
The 23rd Thailand-Japan Conference on
Discrete and Computational Geometry, Graphs, and Games
Department of Mathematics, Chiang Mai University

The Set Chromatic Number of the Middle Graph of Extended Stars

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Abstract

Suppose G is a simple, undirected, finite, nontrivial, and connected graph and that $c : V(G) \to \mathbb{N}$ of G is a vertex coloring, not necessarily proper, of G. As introduced by Chartrand et al., c is called a set coloring of G if $NC(u) \neq NC(v)$ for every pair of adjacent vertices u and v; here, $NC(x)$ denotes the set of colors of all the neighbors of the vertex x. Moreover, the set chromatic number of G, denoted by $\chi_s(G)$, is the fewest number of colors that can be used to construct a set coloring of G. On the other hand, the middle graph of G is defined as the intersection graph of $V' \cup E(G)$, where V' is the collection of all singletons each containing a vertex of G. In this paper, we determine the set chromatic number of the middle graph of extended stars and other families of trees.

Keywords: neighbor-distinguishing coloring, set coloring, middle graph, stars

2010 MSC: Primary 05C15; Secondary 05C05, 05C76.

1 Introduction

All the graphs to be considered in this work are simple, undirected, finite, nontrivial, and connected. As coined in [7], a vertex coloring or edge coloring of a graph is *neighbor-distinguishing* if it induces a vertex labelling such that any two adjacent vertices receive distinct labels. Naturally, the classical proper vertex coloring is neighbor-dsitinguishing.

Among several neighbor-disinguishing colorings that have been introduced (such as those in [2], [3], [5], [6], [7], [8]), the topic of this work is set coloring, which was introduced in [4]. Given a graph G and $c : V(G) \to \mathbb{N}$ a vertex coloring of G, we have the following definitions:

- The *neighborhood color set* of a vertex v is defined as the set $NC(v) = \{c(w) : vw \in E(G)\}$.
- The coloring c is called a *set coloring* of G if $NC(v) \neq NC(w)$ for any two adjacent vertices v and w. If, in addition, $|c(V(G))| = k$ (i.e. c uses k colors), then we call c a set k-coloring of G.
- The set chromatic number of G is denoted by $\chi_s(G)$ and is defined as the smallest integer k for which G has a set k-coloring.

Set colorings have been studied in relation to different graph operations involving different graph families. For instance, there have been studies in relation to the join [10, 16], corona and vertex/edge deletions [4], and comb product [10]. Meanwhile, similar studies have been carried out involving other neighbor-distinguishing colorings such as the ones in [1], [11], [12], [14], [15], and [17].

Previously, in [9], the authors studied set colorings in relation to a graph operation called middle graph. Introduced by Hamada and Yoshimura [13], the *middle graph* $M(G)$ of a graph G is defined to be the intersection graph of $V' \cup E(G)$, where V' is the set of all singletons each containing a vertex of G. Alternatively, we may think of $M(G)$ as the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices u and v are adjacent if and only if

- u and v are adjacent edges in G; or
- $u \in V(G)$, $v \in E(G)$, and u is incident to v in G.

The results in [9] include bounds for $\chi_s(M(G))$ as well as the set chromatic number of the middle graph of stars, paths, cycles, and tadpoles. This work aims to continue [9] by considering the middle graph of other families of trees.

2 Main Results

Let m, n be positive integers. We define the *extended star* graph $T_{m,n}$ to be the graph obtained by connecting n pendant vertices to each pendant vertex of the star graph $K_{1,m}$. Such graphs have been used, for example, in [18]. We now present one of the main problems of our work:

Problem 2.1. Construct optimal set colorings and determine the set chromatic number of the middle graph of extended star graphs.

We introduce the following: Let m, n, D, ℓ be positive integers where $m \geq n+1 \geq 2$. We denote by $q_{m,n}(D,\ell)$, or $q(D,\ell)$ if there is no ambiguity on the values of m and n, the smallest positive integer k that satisfies $\sum_{i=0}^{r} C(k-\ell, i) \geq m - \ell + D$, where $r := \min\{k-\ell, n+1\}$. Our main result on the middle graph of extended stars is as follows:

Theorem 2.2. *Let $T_{m,n}$ be an extended star graph. Then*

$$\chi_s(M(T_{m,n})) = \begin{cases} n+1, & n \geq m \geq 4, \\ \min\{\max\{n+2, q_{1,2}\}, \max\{n+1, q_{3,3}\}\}, & m \geq n+1 \geq 2. \end{cases}$$

Aside from extended stars, our work also includes the determination of the set chromatic number of the middle graph of other families of trees such as banana graphs and some caterpillars.

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Extended Abstract
The 23rd Thailand-Japan Conference on
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Department of Mathematics, Chiang Mai University

On rainbow mean colorings of brooms and double brooms

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Abstract

Let G be a simple connected graph and c an edge coloring that uses positive integers. For $v \in V(G)$, let $\mathrm{cm}(v)$ be the average of the colors of the edges incident to v. If $\mathrm{cm}(v)$ is an integer for each $v \in V(G)$ and $\mathrm{cm}(v) \neq \mathrm{cm}(u)$ for any $u \neq v \in V(G)$, then c is called a **rainbow mean coloring**. In such a coloring c, let $\mathrm{rm}(c)$ be the maximum value of $\mathrm{cm}(v)$ for $v \in V(G)$. The **rainbow mean index** of G, denoted by $\mathrm{rm}(G)$, is the minimum value of $\mathrm{rm}(c)$ among all rainbow mean colorings c of G. A rainbow mean coloring c of a graph G is called **optimal** if $\mathrm{rm}(c) = \mathrm{rm}(G)$, and G is said to be **Type 1** if $\mathrm{rm}(G) = |V(G)|$. In this paper, we determine the rainbow mean index of families of trees that are called brooms and double brooms. We show that aside from P_4 and the stars with even order, all other brooms are Type 1. We also show that all double brooms that have handles longer than 3 are Type 1. We also determine an optimal rainbow mean coloring for such brooms and double brooms.

Keywords: rainbow mean coloring, rainbow mean index, brooms, double brooms.

2010 MSC: 05C15, 05C78

1 Introduction and Preliminaries

Let G be a simple connected graph and c an edge coloring that uses positive integers. For each $v \in V(G)$, the **chromatic mean** of v denoted by $\mathrm{cm}_c(v)$, or simply $\mathrm{cm}(v)$, is the average of the colors of the edges incident to v. If $\mathrm{cm}(v)$ is an integer for each $v \in V(G)$ and distinct vertices have distinct chromatic means, then c is called a **rainbow mean coloring**. In such a coloring c, the maximum chromatic mean of a vertex is called the **rainbow mean index** of c and is denoted by $\mathrm{rm}(c)$. On the other hand, the **rainbow mean index** of G, denoted by $\mathrm{rm}(G)$, is the minimum value of $\mathrm{rm}(c)$ among all rainbow mean colorings c of G.

The rainbow mean coloring of a graph is an example of a vertex-distinguishing coloring which has received increased attention in the last few decades [2]. Chartrand et.al. [1] introduced the rainbow mean coloring in 2019. They determined the rainbow mean index of paths, cycles, complete graphs, and stars. Meanwhile, Hallas et.al. [3] determined the rainbow mean index of several bipartite graphs including prisms, hypercubes, and complete bipartite graphs.

Let $n \geq 3$ and k be positive integers, $n > k$, P_{n-k} a path of order $n-k$ with terminal vertex u and $K_{1,k}$ the star with central vertex v. Then the graph of order n obtained from P_{n-k} and $K_{1,k}$ by identifying the vertices u and v is called a **broom** and is denoted by $B(n,k)$. We call the path P_{n-k} (as a subgraph of $B(n,k)$) the **handle** of the broom, with initial vertex the **tip** of the broom. If $k = 1$ then the broom is a path, while if $n - k = 1$ or 2, it is a star.

Now, let n, k_1, and k_2 be positive integers where $n \geq k_1 + k_2 + 2$, $k_1 \geq 2$, $k_2 \geq 2$. Let $P_{n-k_1-k_2}$ be a path with order $n-k_1-k_2$ and terminal vertices u and v. If we identify u with the central vertex of K_{1,k_1} and v with the central vertex of K_{1,k_2}, then the graph of order n obtained is called a **double broom** and is denoted by $DB(n, k_1, k_2)$. We will also call the path $P_{n-k_1-k_2}$ (as a subgraph) the **handle** of the double broom.

For convenience, we call a rainbow mean coloring c of a graph G **optimal** if $\mathrm{rm}(c) = \mathrm{rm}(G)$. Following the terminology in [3], we say G is **Type 1** if $\mathrm{rm}(G) = n$ where n is the order of G.

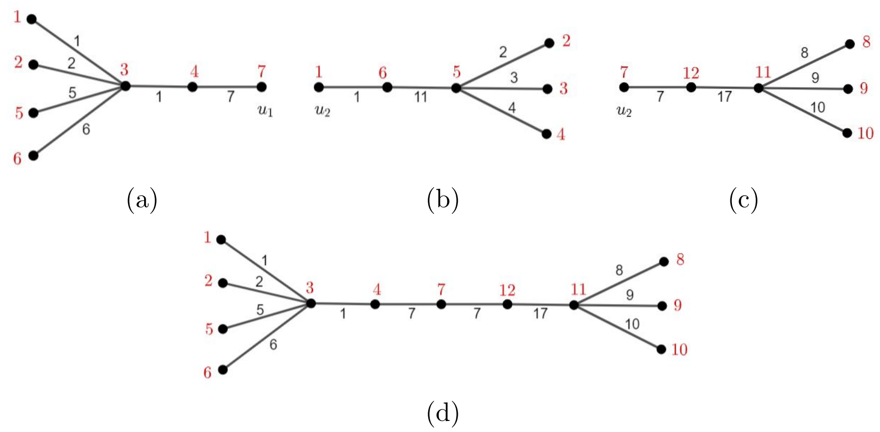

Figure 1: The double room $DB(12,4,3)$ with an optimal rainbow mean coloring.

2 Main Results

Lemma 2.1. *Let k any positive integer and G a connected graph with rainbow mean coloring c. Let c' be an edge coloring of G defined by $c'(e) = c(e) + k$, for each edge $e \in E(G)$. Then c' is a rainbow mean coloring of G. Moreover, $rm(c') = rm(c) + k$.*

Lemma 2.2. *Let G_1 and G_2 be connected graphs of order n_1 and n_2, $rm(G_1) = r_1$, $rm(G_2) = r_2$, and with optimal rainbow mean colorings c_1 and c_2, respectively. Suppose $u_i \in V(G_i)$, for $i = 1, 2$, with $cm_1(u_1) = r_1$ and $cm_2(u_2) = 1$. If G is the graph obtained from G_1 and G_2 by identifying vertices u_1 and u_2, then $rm(G) \leq r_1 + r_2 - 1$.*

Theorem 2.3. *For any positive integers $n \geq 3$ and k with $1 \leq k \leq n-1$, $rm(B(n,k)) = n$ except for the cases that $B(n,k) = P_4$ and $B(n,k) = K_{1,n-1}$ where n is even.*

Theorem 2.4. *Let n, k_1, and k_2 be positive integers where $k_1 \geq 2$, $k_2 \geq 2$, and $n - k_1 - k_2 \geq 5$. Then $rm(DB(n, k_1, k_2)) = n$.*

We illustrate the proof of Theorem 2.4 on $DB(12,4,3)$.

1. By Theorem 2.3, $\text{rm}(B(7,4)) = 7$ and $\text{rm}(B(6,3)) = 6$.

2. In the proof of Theorem 2.3, it can be shown that $B(7,4)$ has an optimal rainbow mean coloring c_1 where the tip (u_1) has chromatic mean 7 (Figure 1a).

3. It can also be shown that the broom $B(6,3)$ has an optimal rainbow mean coloring c_2 where the tip (u_2) has chromatic mean 1 (Figure 1b).

4. We can then apply Lemma 2.2 on the brooms $B(7,4)$ and $B(6,3)$ together with the colorings c_1 and c_2, to get the double broom $DB(12,4,3)$ by identifying vertices u_1 and u_2. By Lemma 2.2, $\text{rm}(DB(12,4,3)) \leq 7 + 6 - 1 = 12$. Since $\text{rm}(DB(12,4,3)) \geq 12$, it now follows that $\text{rm}(DB(12,4,3)) = 12$. Note that an optimal rainbow mean coloring for the double broom is obtained by first increasing the labels of c_2 for $B(6,3)$ by 6 (Figure 1c) before identifying u_1 and u_2.

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Fractional Graph Capacity

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Abstract

In this paper we study a generalization of graph capacity which we call as the fractional graph capacity and use the probabilistic method to determine bounds on the fractional capacity for arbitrary graphs.

Keywords: Fractional Graph Capacity, Bounds, Probabilistic Method.

2010 MSC: Primary 05C90; Secondary 05C50.

1 Introduction

The capacity of a graph was introduced by Shannon to determine the limits of error-free communication across channels with an underlying confusion graph. Lovász [4] obtained the expression for capacity of the cycle on 5 vertices using "umbrella" projection methods and since then, many bounds as well as variants of the capacity have been studied (see, for example, [5] and [1]). In this paper, we introduce the notion of fractional capacity of a graph and use the probabilistic method to obtain bounds in terms of the graph degree parameters.

2 Fractional Graph Capacity

Let $G = (V, E)$ be any connected graph containing $\#V = n \geq 3$ vertices and for a vertex u, let $\mathcal{N}_G[u]$ be the set of all neighbours of u, including u. The graph distance between any two nodes u and v is the number of edges in the shortest path between u and v. A set $\mathcal{F} \subset V$ of vertices is said to be stable if no two vertices in $\mathcal{F}$ are adjacent to each other in G. We denote $\alpha(G)$ to be the independence number, i.e. the maximum size of a stable set in G. For integer $r \geq 1$ let $G(r)$ be the r^{th} strong graph product of G obtained as follows: The graph $G(r)$ has vertex set V^r and two vertices $\mathbf{u} = (u_1, \ldots, u_r)$ and $\mathbf{v} = (v_1, \ldots, v_r)$ are adjacent if and only if $u_i \in \mathcal{N}_G[v_i]$ for each $1 \leq i \leq r$. For an integer $1 \leq k \leq r$, we now define the subgraph $G(r, k) \subseteq G(r)$ as follows. Two vertices $\mathbf{v}, \mathbf{u} \in V^r$ are adjacent in $G(r, k)$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are adjacent in $G(r)$ and differ in at most k entries i.e., $\sum_{i=1}^{r} \mathbb{1}(u_i \neq v_i) \leq k$, where $\mathbb{1}(.)$ refers to the indicator function. We have the following definition.

Definition 2.1. For a real number $0 < \gamma \leq 1$ we define the $\gamma-$fractional capacity of G to be

$$\Theta_\gamma(G) := \sup_{r \geq \frac{1}{\gamma}} \left(\alpha(G(r, \gamma r))\right)^{\frac{1}{r}}. \tag{2.1}$$

For $\gamma = 1$, the term $\Theta_1(G) =: \Theta(G)$ is the graph capacity as defined in Shannon (1956). For differentiation, we refer to $\Theta(G)$ as the *full* graph capacity and $\Theta_\gamma(G)$ as the *fractional* graph capacity.

In the context of codes, each vertex of $G(r)$ is a codeword of length r and the term γ represents the maximum fraction of symbols that undergo corruption when passed through a channel with confusion graph G. The quantity $(\Theta_\gamma(G))^r$ is then the maximum size of a code from $G(r)$ that allows for error free communication. For $0 \leq x \leq 1$ we let $H(x) := -x \cdot \log x - (1 - x) \cdot \log(1 - x)$ be the entropy function, where logarithms are to the base two throughout and have the following bounds on the fractional graph capacity.

Theorem 2.2. *Let G be a connected graph on n vertices and let d_{av} and Δ be the average and maximum vertex degree of G, respectively. For any $0 < \gamma \leq 1$ we have that*

$$n \cdot \max(f(\gamma, d_{av}), f(\gamma, \Delta), f(\gamma, n-1)) \leq \Theta_\gamma(G) \leq n \cdot \left(\frac{\Theta(G)}{n}\right)^\gamma \tag{2.2}$$

where $f(\gamma, x) := \begin{cases} \left(2^{H(\gamma)} \cdot x^\gamma\right)^{-1} & \text{for } 0 < \gamma < \frac{x}{x+1} \\ (x+1)^{-1} & \text{for } \frac{x}{x+1} \leq \gamma \leq 1. \end{cases}$

Setting $\gamma = 1$ in the lower bound of (2.2), we get that $\Theta(G) \geq \frac{n}{d_{av}+1}$ which is also obtained via the Turán's bound. For example the graph $G = C_5$, the cycle on 5 vertices, has $d_{av} = \Delta = 2$ and it is well-known that $\Theta(G) = \sqrt{5}$ [4]. Therefore setting $\gamma = \frac{1}{2}$, we get from (2.2) that $\frac{5}{2\sqrt{2}} \leq \Theta_{\frac{1}{2}}(C_5) \leq \frac{5}{\sqrt[4]{5}}$.

We prove the lower bound in (2.2) using the probabilistic method and a maximal stable set argument (the Gilbert-Varshamov argument [3]) and prove the upper bound in (2.2) using a recursion estimate similar to the Singleton argument [3].

Proof of the lower bound in (2.2): For a uniformly random vector $\mathbf{v} = (v_1, \ldots, v_r) \in V^r$, let $\mathcal{B}_\gamma(\mathbf{v})$ be the set of all vertices adjacent to $\mathbf{v}$ in the graph $G(r, \gamma r)$. The vertices $\{v_j\}_{1 \leq j \leq n}$ are mutually independent and uniformly distributed in V and so the expected degree of v_j is d_{av}. Consequently, the expected size of $\mathcal{B}_\gamma(\mathbf{v})$ is $\mathbb{E}\#\mathcal{B}_\gamma(\mathbf{v}) = \sum_{k=0}^{\gamma r} \binom{r}{k} d_{av}^k$.

For $0 < \gamma < 1 - \frac{1}{d_{av}+1}$, we use the Hamming ball estimate (Proposition 3.3.1, [2] to get that $\mathbb{E}\#\mathcal{B}_\gamma(\mathbf{v}) = \sum_{k=0}^{\gamma r} \binom{r}{k} d_{av}^k \leq n^{\theta r}$ where $\theta = \frac{H(\gamma) + \gamma \log d_{av}}{\log n}$ satisfies $0 < \theta < 1$. For $0 < \epsilon = \epsilon(r) < 1$ to be determined later, we let $\mathcal{A}(\epsilon) := \{\mathbf{u} \in V^r : \#\mathcal{B}_\gamma(\mathbf{u}) \leq n^{r(\theta+\epsilon)}\}$ and obtain from Markov inequality that $\#\mathcal{A}(\epsilon) \geq n^r\left(1 - n^{-r\epsilon}\right)$. Let $\mathcal{D} := \{\mathbf{w}_1, \ldots, \mathbf{w}_M\} \subseteq \mathcal{A}(\epsilon)$ be a stable set of maximum size in $G(r, \gamma r)$. By the maximality, we must have that the union $\bigcup_{i=1}^M \mathcal{B}_\gamma(\mathbf{w}_i) = \mathcal{A}(\epsilon)$ and so

$$n^r\left(1 - n^{-r\epsilon}\right) \leq \#\mathcal{A}(\epsilon) \leq \sum_{i=1}^M \#\mathcal{B}_\gamma(\mathbf{w}_i) \leq M \cdot n^{r(\theta+\epsilon)}.$$

Thus $M \geq n^{(1-\theta-\epsilon)r}(1 - n^{-r\epsilon})$ and choosing $\epsilon = \frac{1}{\sqrt{r}}$, taking r^{th} roots and allowing $r \to \infty$, we get that $\Theta_\gamma(G) \geq n f(\gamma, d_{av})$.

For $1 - \frac{1}{d_{av}+1} \leq \gamma \leq 1$, we use the inequality $\mathbb{E}\#\mathcal{B}_\gamma(\mathbf{v}) \leq \sum_{k=0}^{r} \binom{r}{k} d_{av}^k = (d_{av}+1)^r$ and the maximality argument as before, to get that $\Theta_\gamma(G) \geq n f(\gamma, d_{av})$. To see that $\Theta_\gamma(G) \geq n f(\gamma, \Delta)$, we use a similar argument as above along with the estimate that for any deterministic vector $\mathbf{u} \in V^r$ the ball size $\#\mathcal{B}_\gamma(\mathbf{u}) \leq \sum_{k=0}^{\gamma r} \binom{r}{k} \Delta^r$. Finally, a similar argument also shows that $\Theta_\gamma(G) \geq n f(\gamma, L-1)$ and this completes the proof of the lower bound in (2.2). $\square$

Proof of the upper bound in (2.2): Let $d = \gamma r$ and let $\mathcal{C} \subset G(n)$ be a stable set of maximum size in $G(r, d)$ and set $A(r, d) := \#\mathcal{C}$. We first derive a recursive estimate involving $A(r, d)$. For a vertex $v \in G$ let $\mathcal{C}(v)$ be the set of all vertices in $\mathcal{C}$ whose last entry is v. By the pigeonhole principle, there exists a $v_0 \in G$ such that $\#\mathcal{C}(v_0) \geq \frac{A(r,d)}{n}$. We remove the last entry in each vertex of $\mathcal{C}(v_0)$ and call the resulting set as $\mathcal{D}(v_0) \subset G(r-1)$. The set $\mathcal{D}(v_0)$ is also stable in $G(r-1, d)$ and this implies that $A(r, d) \leq n \cdot A(r-1, d)$. Continuing iteratively we get $A(r, d) \leq n^{r-d} \cdot A(d, d)$. Taking r^{th} roots and using the fact that $\sup_{r \geq \frac{1}{\gamma}} (A(d, d))^{\frac{1}{r}} = \sup_{d \geq 1} (A(d, d))^{\gamma d} = (\Theta(G))^\gamma$, we get the upper bound in (2.2). $\square$

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Extended Abstract
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Unit Tilt Row Relocation in a Square

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Abstract

Relocation in the square is the most basic open question related to the tilt model with unit moves under global uniform movement. It is open even with limited directions. We prove that deciding if a piece can move to the bottom row with only two directions is in P and gives the minimum column.

Keywords: tilt model, robot motion planning, relocation, uniform control, external forces.

2010 MSC: Primary 52C99; Secondary 68R05, 52B20, 68W01.

1 Introduction

Tilt assembly with uniform global instructions is currently a popular area of research since individually controlling nanobots may be infeasible or impractical at certain scales. One of the most basic open questions in the single step tilt model with uniform global movements is whether a tile can be relocated to specific location on the board (Figure 1a) even without any blocking geometry. We solve a simpler version of this problem asking whether, with only two directions, a tile can reach the bottom row.

Given a tilt system with an $n \times n$ board $\mathbb{B}$, a set of tiles $\mathbb{T} = \{t_1, \ldots t_m\}$ where each $t_l = (i, j)$ s.t. $1 \leq l \leq m$, $1 \leq i \leq n$, and $1 \leq j \leq n$. Given $t_i, t_j \in \mathbb{T}$, $t_i \neq t_j$ if $i \neq j$. For shorthand, we use $t_{i,j}$ for $t_l = (i, j)$. For just the row or column, we may use t_{l_r} and t_{l_c}. See [1–3] for model and related work.

Definition 1.1 (First Row Relocation)**.** Given a specific tile to relocate t_R at location $(r, c) = (t_{R_r}, t_{R_c})$, the first row relocation problem asks whether a series of transformations can translate t_R s.t. $t_{R_r} = 1$.

2 High Level Overview

In order for tile t_R to reach the bottom row, there must not be any other tiles beneath it. We first find the empty column needed, and then, through a process termed *knitting*, we attempt to move the empty column beneath t_R. We also label the board with sections (Figure 1a). See Figure 1b for examples of the empty column. We then label the rows with *counts* of tiles that may affect relocation (Figure 1c).

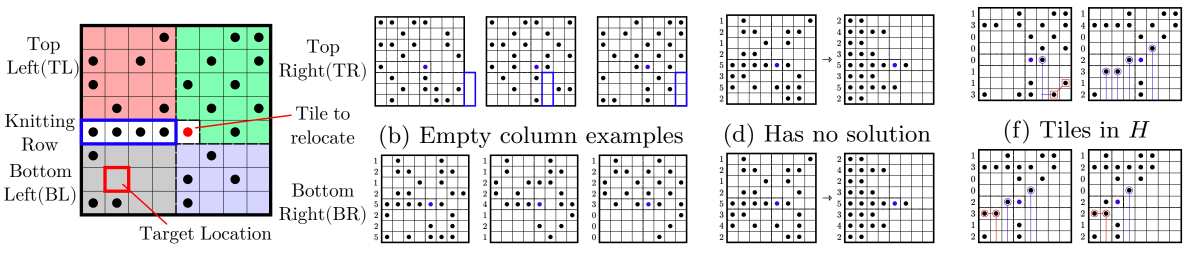
Definition 2.1 (Empty Column E_c)**.** Given a tilt system board configuration $\mathbb{S} = (\mathbb{B}, \mathbb{T})$ and tile $t_R = (r, c)$ where $t_R \in \mathbb{T}$, define $E_c = \min\{k : c \leq k \leq n+1 \text{ s.t. } |\{t_{i,j} : t_{i,j} \in \mathbb{T}, i < r, j = k\}| = 0\}$. If all columns in the BR section have tiles, $E_c = n + 1$ and enters the board after a west, $\langle W \rangle$, movement.

Definition 2.2 (Knitting)**.** The row between the BL and TL section is the knitting row. Knitting is the act of performing $\langle W \rangle$ movements when every position of the knitting area is occupied by a tile. Thus, t_R maintains its position.

Definition 2.3 (Counts)**.** Given a tilt system board configuration $\mathbb{S} = (\mathbb{B}, \mathbb{T})$, a tile $t_R \in \mathbb{T}$ with location (r, c), and the target empty column E_c. Define the count of a row k as follows.

$$Count(k, t_{r,c}) = \begin{cases} |\{t_{i,j} : t_{i,j} \in \mathbb{T}, i = k, 1 \leq j < c\}|, & \text{if } k > r \text{ (rows above } t_R) \\ |\{t_{i,j} : t_{i,j} \in \mathbb{T}, i = k, 1 \leq j \leq c\}|, & \text{if } k = r \text{ (} t_R \text{ row)}, \\ |\{t_{i,j} : t_{i,j} \in \mathbb{T}, i = k, 1 \leq j < E_c\}|, & \text{if } k < r \text{ (rows below } t_R) \end{cases}$$

This basic framework leads to two important lemmas (2.5 2.6). The algorithm will ensure that one of these is eventually met. The basic idea is Algorithm 1.



(a) Board Sections (b) Empty column examples (c) Example counts (d) Has no solution (e) Knitting solution (f) Tiles in H (g) Reducing tiles

Figure 1: (a) The board is an $n \times n$ square divided into 6 sections based on the tile to relocate. The four large sections are the Top Left (TL), Top Right (TR), Bottom Left (BL), and Bottom Right (BR) sections. There is also the knitting area (outlined in blue) and the tile to relocate (red dot). When we move to general relocation, there will also be a target spot (red square). (f) Tiles that enter the BR section after D movements are included in N_R. If the empty column is occupied by a tile after D movement (i.e., it becomes non-empty), then every tile between the empty column and the next empty column (shown in red) are also added to N_R. Reducing the number of tiles in H is done by $\langle W \rangle$ movement. (g) Tiles in H that lead towards another tile beneath it are removed from H.

2.1 Changing the Knitting Row

If neither condition in the lemmas is satisfied, we cannot knit with the tiles in the knitting row. Define the *candidate row* as the closest row in the TL section with a count higher than the knitting row.

Definition 2.4 (Candidate Row)**.** Given a tilt system board configuration $\mathbb{S} = (\mathbb{B}, \mathbb{T})$ and a tile $t_R \in \mathbb{T}$ at location (r, c). The knitting row may contain up to $c - 1$ tiles. The closest row (fewest south, $\langle S \rangle$, movements) in the TL section with a count higher than the knitting row is the candidate row.

Lemma 2.5. *[Existence] If the count of the knitting area is greater than the counts of the lower section, then first row relocation is possible.*

Lemma 2.6. *[Nonexistence] If there exist a count in the bottom sections that is larger than every count in the TL section and knitting area, then first row relocation is impossible.*

Result: First Row Relocation
while *Lemmas 2.5, 2.6 are unsatisfied* **do**
  Determine E_c and counts for each row;
  if *Lemma 2.5 is satisfied* **then** accept;
  if *Lemma 2.6 is satisfied* **then** reject;
  Set candidate row CR;
  Move CR to be the new knitting row;
Algorithm 1: High Level Idea

If we make any movements, we may introduce new tiles that must be considered. We look at new tiles that may enter the BL section (N_L) and tiles that may enter in the BR section (N_R).

We include the set of tiles in TR that are at most D distance (with $\langle S \rangle$ moves) from the BR that increase the counts of the bottom rows (Figure 1f). The BR and TR sections may change if t_R moves. If E_c has tiles after D movements, then we change E_c to the correct column. All tiles that enter the last row after D movements to the left of the new empty column are included in N_R. Let H be the set of all tiles in either the TL or BL section that are within D $\langle S \rangle$ movements from the first row. With $\langle W \rangle$ moves, we may be able to reduce H (Figure 1g). When first row relocation is possible, the sequence is $W^u S^v W^x S^y$ and u, v, x, y must be determined, but are bounded by n. There are $\leq c - 2$ possible candidate rows, and at most $n - r$ moves to bring any CR to the knitting row.

Theorem 2.7. *First row relocation of tile $t_{r,c} \in \mathbb{T}$ on an $n \times n$ board can be solved in $\mathcal{O}(cn + c|\mathbb{T}|)$ time.*

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Extended Abstract
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Discrete Quantum Systems via Tight Group Frames and their Geometrization

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Abstract

This work uses finite tight group frames to implement a Groenewold-Moyal or star-product formalism of quantum mechanics on finite-dimensional Hilbert space. Tight frames originated in signal analysis via wavelets and Gabor systems and in quantum mechanics via the discretization of coherent states. Geometrization of the star-product by contravariant tensors is deduced from the Wigner transform of quantum operators arising from frame quantization.

Keywords: Tight frames, discrete quantum mechanics, star-product, phase space quantum mechanics, geometrization.

2010 MSC: Primary 42C15, 81R05; Secondary 81S30, 81R60.

1 Introduction

Let $\mathcal{H}$ be a Hilbert space. A frame in $\mathcal{H}$ is a family $\{\psi_x : x \in X\} \subset \mathcal{H}$ parametrized by a discrete space X satisfying the following condition: $A||\phi||^2 \leq \sum_{x\in X} |\langle \phi, \psi_x\rangle|^2 \leq B||\phi||^2$, for some constants $0 < A \leq B$. If $A = B$, the frame is called a tight frame. In this latter case, a sort of Fourier expansion is provided by tight frames: $\varphi = \frac{1}{A}\sum_{x\in X}\langle\psi_x, \varphi\rangle\psi_x$. The difference with bases is that frames do not necessarily consist of linearly independent vectors. It is to this aspect that frames owe its utility in many areas of science and engineering [2,4].

This paper has two objectives, to use frames in the star-product formalism of discrete quantum systems and to propose a way of geometrizing the main objects of the formalism, given by the star-product, the Lie bracket and the Jordan bracket. It is notable that geometrization of quantum systems is a long standing program of mathematical physics, one of whose aim is the unification of quantum mechanics and gravity.

2 Preliminaries

2.1 Phase Space Representation of Quantum Mechanics

The dynamics in classical mechanics is given by Hamilton's Equation $\frac{df}{dt} = \{f, H\}$ in terms of the Poisson bracket on the space of functions on M, and the energy H of the system. On the other hand, quantum dynamics is given by Heisenberg's equation $\frac{d\widehat{A}}{dt} = \frac{\hbar}{i}[\widehat{H}, \widehat{A}]$, in terms of the energy $\widehat{H}$ of the system and commutator $[\cdot,\cdot]$. The similarity of the dynamics suggests that a basic task of quantum mechanics is to find a quantization mapping $\mathcal{Q} : f \mapsto \widehat{A}_f$ from classical observables to self-adjoint operators. However, the works of H. Weyl, E. Wigner, H. Groenewold, E. Moyal and others suggest another point of view, that of quantum mechanics in phase space [5]. The fact that there is a one-to-one correspondence between the Weyl transform of phase space functions and the Wigner transform of Hilbert-Schmidt operators allows for the representation of quantum operators by functions on classical phase space, given by $\mathcal{W}_{\widehat{A}}(q,p) = \frac{1}{h}\int_{-\infty}^{\infty} e^{-ipy/\hbar}\langle q - y/2|\widehat{A}|q + y/2\rangle dy$. To the composition $\widehat{A}\circ\widehat{B}$ corresponds a noncommutative and associative product on phase space, making it possible to implement an equivalent quantum theory on phase space, which is most suitable for the analysis of the classical to quantum correspondence. Moreover, a remarkable parallel with the methods of time-frequency representation of signal analysis shows the great utility of these ideas.

2.2 Coherent States and Frames

Coherent states are quantum states nearest in behavior to classical states via minimization of the Heisenberg Uncertainty Principle. The most basic, but also the most important, coherent states are $\widehat{T}(z)\psi_0 = e^{\overline{z}\widehat{a}-z\widehat{a}^\dagger}\psi_0, z \in \mathbb{C}$, associated to the quantum harmonic oscillator, introduced by E. Schrödinger in 1926. Coherent state quantization and frame quantization are the first applications of these objects, due to F. Berezin and to J. von Neumann.

Nonorthogonal expansions of square-integrable functions on the plane may be obtained by choosing an appropriate discrete subset Ω of the plane and forming the set of vectors $\widehat{T}(q_m, p_n)\psi_0$, where $z_{mn} = q_m + ip_n$ and ψ_0 is a Gaussian function. This set is nonorthogonal but has the property $f = \sum_{m,n\in\Omega}\langle f, \widehat{T}(q_m,p_n)\psi_0\rangle \widehat{T}(q_m,p_n)\psi_0$, for square integrable function f on the plane (called Gabor expansion). This collection of vectors is called a **frame** in Hilbert space. The consideration of frames in finite-dimensional Hilbert spaces is mainly due to numerical implementation of the theory in various applications, such as quantum optics, quantum information, and signal analysis and data processing [2,4].

3 Main Results

An important class of discrete and finite tight frames are those that arise from unitary group actions [4]. Let G be a finite group. A frame $(\varphi_g)_{g\in G}$ for $\mathcal{H}$ is a group frame or G-frame if there exists a unitary representation $U : G \longrightarrow \mathcal{U}(\mathcal{H})$ such that $g\varphi_h := U(g)\varphi_h = \varphi_{gh}$ for all $g, h \in G$. We will consider only unitary irreducible representations (unirreps).

1. *Frame Quantization.* Fix a unirrep $(U, \mathcal{H})$ of the finite group G. Consider the following set $\mathcal{A} = \{|U(x)\rangle : x \in G\}$ of finite tight group frames on the space $\mathcal{B}(\mathcal{H})$ of linear operators on $\mathcal{H}$. Here $|U(x)\rangle = U(x)I \in \mathcal{B}(\mathcal{H})$ and I is the identity operator. We have the following frame quantization $F \mapsto \sum_{x,y\in G} F(x,y)|U(x)\rangle\langle U(y)|$ where $F : G\times G \longrightarrow \mathbb{C}$.

2. *Star-product Formula.* Consider the restriction $L^2(G\times G)$ onto $L^2(G)$ given by $f(x, y_0) \mapsto f(x)$. Conversely, if $f \in L^2(G)$, a function $\widetilde{f} \in L^2(G\times G)$ is obtained by $\widetilde{f}(x, y_0) = f(x)$. Here one must fix y_0 at the beginning, say $y_0 = e$, the identity of the group. Then, the quantization $f \mapsto \widehat{A}_f = \sum_{x\in G} f(x)|U(x)\rangle\langle U(y)|$ provides the following star-product of functions $\widehat{A}_{f\star g} = \widehat{A}_f \circ \widehat{A}_g$. The bijection between operators and phase space functions gives the formula for the star-product, known from the literature, $f \star g(z) = \sum_{x,y\in G} f(x)g(y)Tr[U(xyz^{-1})]$ [1].

3. *Geometrization.* Let $U(x) = (u_{ij}(x))$ be the matrix elements of a unirrep U. From the Peter-Weyl Theorem, the functions $u_{ij} : G \longrightarrow \mathbb{C}$ form an orthonormal basis for the restriction of the regular representation to the space equivalent to the unirrep. Then the star-product $\star = \star_\lambda$ has the following geometrization by contravariant tensors, from which follows also the geometrization of the Lie and Jordan brackets [3]:

$$\star = q_{kn}\left[\sum_j \left(\frac{\partial}{\partial q_{nj}} \otimes \frac{\partial}{\partial q_{jk}}\right) - \left(\frac{\partial}{\partial p_{nj}} \otimes \frac{\partial}{\partial p_{jk}}\right)\right] - p_{kn}\left[\sum_j \left(\frac{\partial}{\partial q_{nj}} \otimes \frac{\partial}{\partial p_{jk}}\right) - \left(\frac{\partial}{\partial q_{nj}} \otimes \frac{\partial}{\partial p_{jk}}\right)\right]$$
$$+ip_{kn}\left[\sum_j \left(\frac{\partial}{\partial q_{nj}} \otimes \frac{\partial}{\partial q_{jk}}\right) - \left(\frac{\partial}{\partial p_{nj}} \otimes \frac{\partial}{\partial p_{jk}}\right)\right] + iq_{kn}\left[\sum_j \left(\frac{\partial}{\partial q_{nj}} \otimes \frac{\partial}{\partial p_{jk}}\right) - \left(\frac{\partial}{\partial q_{nj}} \otimes \frac{\partial}{\partial p_{jk}}\right)\right].$$

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Extended Abstract
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Department of Mathematics, Chiang Mai University

Direction-Critical Configurations in Noncentral-General Position

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Abstract

In 1982, Ungar proved that the connecting lines of a set of $2n$ points in the plane, not all collinear, determine at least $2n$ slopes. Sets achieving this minimum are called *direction-critical* (or *slope-critical*). By using allowable sequences, Ungar actually proved this result for *generalized configurations of points* (where points can be connected by pseudolines instead of straight lines). In 1986, Jamison proved that any direction-critical configuration of points in general position is an affine transformation of the vertices of a regular polygon. We study direction-critical centrally symmetric generalized configurations of points that are in *noncentral-general position* (where only the connecting (pseudo)lines not through the center of symmetry can pass through more than two points). We prove that, up to equivalence, the numbers of points in the (pseudo)lines through the center of symmetry completely determine a centrally symmetric direction-critical generalized configuration of points in noncentral-general position, and a configuration exists for any such set of numbers. We also show that infinitely many of this direction-critical generalized configurations of points are not stretchable.

Keywords: slope-critical, generalized configuration of points, pseudoline arrangement, allowable sequence, centrally symmetric.

2010 MSC: Primary 52C30; Secondary 05C10, 05C35, 52C10.

1 Introduction

In 1970, Scott [6] proposed the problem of finding the least number of directions (slopes) determined by a set of n points in the plane, not all collinear. He conjectured that the minimum number of different slopes determined by the connecting lines of a set of n points in the plane is $2\lfloor n/2 \rfloor$. This conjecture was settled by Ungar [7] in 1982 using allowable sequences. An *allowable sequence* of n points is a doubly-infinite periodic sequence of permutations $\Pi = \{\pi_i\}_{i=-\infty}^{\infty}$ of the set of *points* $\{1, 2, 3, \ldots, n\}$, satisfying the following properties: (1) By relabeling the points, if necessary, it can be assumed that $\pi_0 = 123 \ldots n$. (2) There is $h \in \mathbf{Z}^+$ such that π_{i+h} is the reversal of π_i for every $i \in \mathbf{Z}$. Then Π has period $2h$ and $\{\pi_0, \pi_1, \ldots, \pi_h\}$ is a *halfperiod* of Π (of *length* h). (3) π_{i+1} is obtained from π_i by the reversal of one or more disjoint substrings, each involving consecutive elements of π_i. These reversals are called *switches*. (4) Any pair of points switches exactly once within a halfperiod.

Allowable sequences were used by Goodman and Pollack to approach combinatorial problems of sets of points in the 1980s [1, 2]. Any set P of n points in the plane determines an allowable sequence $\Pi(P)$, called its *circular sequence*. The permutations of $\Pi(P)$ are determined by the projections of P onto a line that rotates around a circle enclosing P. So every set of points corresponds to an allowable sequence but not every allowable sequence is the circular sequence of a set of points. In fact, Goodman and Pollack [3] showed that up to combinatorial equivalence, there is a one-to-one correspondence between the set of allowable sequences and the set of *generalized configurations* of points, that is, a set of points together with a pseudoline arrangement so that every pair of points is contained in a unique pseudoline. So Ungar's proof actually extends the problem of finding the minimum number of directions from point-sets to generalized configurations of points. In this new setting, all switches occurring between consecutive permutations of an allowable sequence correspond to pseudolines determining the same direction. So the number of directions determined by an allowable sequence Π is the length $h(\Pi)$ of its halfperiod. Ungar proved that $h(\Pi) \geq 2\lfloor n/2 \rfloor$. Indeed, the number of directions determined by a set of points P is the same as the length of a halfperiod of its circular sequence $\Pi(P)$. The configurations/allowable sequences achieving this minimum are called *direction-critical* or *slope-critical*.

We consider centrally symmetric point-sets and generalized configurations of points. They correspond to allowable sequences whose permutations are centrally symmetric. We only study allowable sequences with an even number of points as every odd centrally symmetric allowable sequence is obtained from an even one by adding its center of symmetry. The switches reversing a centered substring of a permutation are called *crossing switches*. They correspond to connecting lines passing through the center of symmetry of a point-set, which are called *central connecting lines*. The number of points on each side of the center of symmetry along a central connecting line, or equivalently, half of the length of the corresponding crossing switch, is called the *distance* of the crossing switch. It follows from Ungar's proof that the sum of the distances $d_1, d_2, \ldots, d_t$ of all crossing switches in a half period of a direction-critical centrally symmetric allowable sequence on $2n$ points is exactly n. The ordered partition $n = d_1 + d_2 + \cdots + d_t$ is called *the crossing distance partition* of the allowable sequence, where the terms go with the order in which the corresponding switches occur within a half period. Jamison [5] proved that all slope-critical configurations of $2n$ points in general position are affine transformations of the regular $2n$-gon. We relax the general position assumption by allowing connecting lines through the center of symmetry to pass through more than two points, but all other connecting lines pass through exactly two points. In the setting of allowable sequences, this means that all switches, except for the crossing switches, are transpositions of two points. We say that these sets are in *noncentral-general position*.

The following problem is considered: For which positive integer vectors $(d_1, d_2, \ldots, d_t)$ there are direction-critical centrally symmetric point-sets or generalized configurations of points in noncentral-general position with crossing distance partition $n = d_1 + d_2 + \cdots + d_t$?

2 Main Results

Theorem 2.1. *Let $d = (d_1, d_2, \ldots, d_t)$ be a vector with positive integer entries. Then, up to combinatorial equivalence, there is a unique direction-critical centrally symmetric allowable sequence $\Pi(d)$ in noncentral-general position with crossing distance partition $n = d_1 + d_2 + \cdots + d_t$.*

Some of these allowable sequences are known to be the circular sequence of a point-set, i.e., *geometrically realizable* or *stretchable*. Moreover, due to the uniqueness in Theorem 2.1, the corresponding point-sets are unique up to affine transformations, as stated in the following corollaries.

Corollary 2.2. *Let $d = (1, 1, \ldots, 1) \in \mathbf{Z}^n$. The allowable sequence $\Pi(d)$ with crossing distance partition $n = 1 + \cdots + 1$ is uniquely geometrically realized, up to affine equivalence, by the regular $2n$-gon.*

Corollary 2.3. *Let $d = (d_1, d_2)$. The allowable sequence $\Pi(d)$ with crossing distance partition $n = d_1 + d_2$ is uniquely geometrically realized, up to affine equivalence, by the exponential cross $EX_\lambda(d_1 - 1, d_2 - 1)$ (without its center) as described by Jamison in [4], or by the bipencil when $d_1 = 1$ or $d_2 = 1$.*

However, the allowable sequence $\Pi(d)$ in Theorem 2.1 is not geometrically realizable for infinitely many vectors d.

Theorem 2.4. *Let $d = (d_1, d_2, \ldots, d_t)$ be a vector with positive integer entries. Then the allowable sequence $\Pi(d)$ is not geometrically realizable whenever (1) at least 3 entries of d are greater than or equal to 3, or (2) at least 4 entries of d are greater than 1.*

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Extended Abstract
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Parallel Curves Detection Method Based on Hough Transform

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Abstract

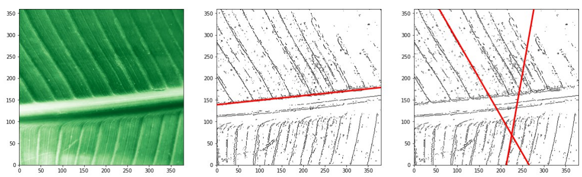
For recognizing straight lines and circles, the Hough transform is a widely used approach. Recently, an extension was published that allows us to detect a larger class of algebraic curves. The image with parallel curves is the subject of this study. An incorrect curve may be detected using the standard approach. We modify the accumulator function and introduce the concept of parallel curves into the detection method to tackle this problem. Synthetic and real-world images are used to demonstrate the effectiveness of this new methodology.

Keywords: Parallel curves, Algebraic plane curves, Pattern recognition, Hough transform

2010 MSC: Primary 14Q05; Secondary 68U10, 14H50.

1 Introduction

The Hough transform, defined in [1], considers all conceivable curves and counts how many points on the image space each curve contains. The counted value for each curve is referred to as an accumulator function. As a detected curve, the highest value of the accumulator function will be suggested. We focus on images with parallel curve. The issue is if any points from different curves are accidentally joined, our detection may fail. Because the accumulator function will reach a peak at that accidental curve if this case occurs. Take the following figure as an example if we aim to extract the direction of parallel veins in the image of a banana leaf. The Hough transform misled us to the midrib because there are more points on that line than on each parallel vein. To avoid this kind of incorrect detection, we will add a parallel assumption to our modified approach.



2 Preliminaries

Definitions and notation of *family of curve* and the *Hough transform* follow [1–3].

Definition 2.1. Consider a set $\mathcal{L} \subset \{1, \dots, t\}$. We say that a curve C_λ is $\mathcal{L}-$parallel to a curve $C_{\lambda'}$ if $\lambda_i = \lambda'_i$ for all $i \notin \mathcal{L}$.

We denote a family of all curve in $\mathcal{F}$ that are $\mathcal{L}-$parallel to a curve $C_{\lambda^{(0)}}$ by $\mathcal{F}_{\mathcal{L},\lambda^{(0)}}$. A family $\mathcal{F}_{\mathcal{L},\lambda^{(0)}}$ of curves in $\mathbb{R}^n$ is said to be a cover if $\bigcup_{\mathcal{C}_\lambda \in \mathcal{F}_{\mathcal{L},\lambda^{(0)}}} \mathcal{C}_\lambda = \mathbb{R}^n$, and said to be a partition if it is a cover and for all distinct $\mathcal{C}_\lambda, \mathcal{C}_{\lambda'} \in \mathcal{F}_{\mathcal{L},\lambda^{(0)}}$, we have $\mathcal{C}_\lambda \cap \mathcal{C}_{\lambda'} = \varnothing$ or $\mathcal{C}_\lambda = \mathcal{C}_{\lambda'}$. Take the family $\mathcal{F}_1$ of straight lines defined by $f_{(r,\theta)}(x, y) = r - x\cos\theta - y\sin\theta$ as an example. Let $(r_0, \theta_0) \in \mathbb{R}_0^+ \times [0, 2\pi)$. For $\mathcal{L} = \{1\}$, a subfamily $\mathcal{F}_{\mathcal{L},(r_0,\theta_0)}$ is the set of all straight lines such that the angle between it and the x-axis is equal to θ_0. This subfamily is a partition. For $\mathcal{L} = \{2\}$, a subfamily $\mathcal{F}_{\mathcal{L},(r_0,\theta_0)}$ is the set of all straight lines such that the distance between it and the origin is equal to r_0. This subfamily is not a cover. Another example is the family $\mathcal{F}_2$ of circles defined by $f_{(h,k,r)}(x, y) = r - (x - h)^2 - (y - k)^2$. Let $(h_0, k_0, r_0) \in \mathbb{R}^2 \times \mathbb{R}_0^+$.

For $\mathcal{L} = \{3\}$, a subfamily $\mathcal{C}_{\mathcal{L},(h_0,k_0,r_0)}$ is the set of all circles with fixed center at (h_0, k_0). This subfamily is a partition. For $\mathcal{L} = \{1, 2\}$, a subfamily $\mathcal{F}_{\mathcal{L},(h_0,k_0,r_0)}$ is the set of all circles with fixed radius r_0. This subfamily is a cover but not a partition.

3 Main Results

After observing the value of the accumulator function, we found that if there are parallel curves in the image, then the accumulator function in that direction will contain peaks at each position of the curves. On the other hand, there would have only one very high rise on the accidentally lined up direction. The following theorem is the conceptual idea for our proposed parallel curves recognition method.

Theorem 3.1. *Let $\mathcal{F}$ be a family of irreducible curves parametrized by t dimensional tuple. For $\mathcal{L} \subset \{1, ..., t\}$, then the following hold.*

1. *Fix $\lambda^{(0)}$. For each λ such that $C_\lambda \in \mathcal{F}_{\mathcal{L},\lambda^{(0)}}$, the Hough transform $\Gamma_P(\lambda)$ of the pairs (C_λ, P) when P varies on C_λ all pass through a point on set $E_{\mathcal{L},\lambda^{(0)}} = \{\Lambda \in \mathbb{R}^t : \Lambda_i = \lambda_i^{(0)}$ for all $i \notin \mathcal{L}\}$.*
2. *Fix $\lambda^{(0)}$. If the Hough transform $\Gamma_P(\lambda)$ of the pairs (C_λ, P) when P varies on curves C_λ and the Hough transform $\Gamma_{P'}(\lambda)$ of the pairs $(C_{\lambda'}, P')$ when P' varies on curves $C_{\lambda'}$ in $\mathcal{F}$ are all intersected at two points in the set $E_{\mathcal{L},\lambda^{(0)}}$, then C_λ is $\mathcal{L}$−parallel to $C_{\lambda'}$.*

The prior procedure of preventing this false detection is to integrate over all directions in $\mathcal{L}$, that is

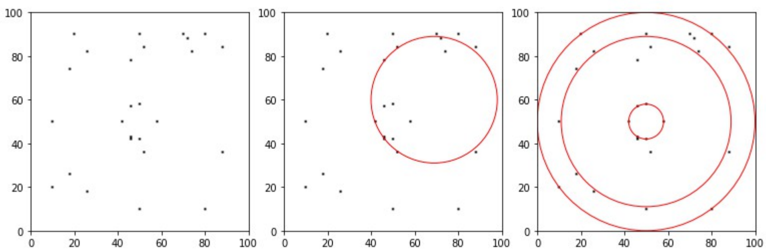
$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \mathcal{H}_{\mathcal{F}}|_{E_{\mathcal{L},\lambda^{(0)}}}(\Lambda) d\Lambda_1 \dots d\Lambda_\ell \tag{3.1}$$

where $\mathcal{H}_{\mathcal{F}}$ is a accumulator function. But if $\mathcal{F}_{\mathcal{L},\lambda^{(0)}}$ is a partition, we can show the integration 3.1 is a constant. As a result, we shall first apply some function F to the integrand rather than integrating directly. The function should be increasing. Otherwise, the order of the accumulator function will be lost. Moreover, we should not use linear or affine linear because the integral will remain constant. We recommend using a simple function such as square, soft thresholding, or hard thresholding function. Our proposed method is, instead of seeking for the maximum of the accumulator function, we seek for the peak of

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} F\big(\mathcal{H}_{\mathcal{F}}|_{E_{\mathcal{L},\lambda^{(0)}}}(\Lambda)\big) d\Lambda_1 \dots d\Lambda_\ell \tag{3.2}$$

where F is a function that we mention. That will provide us the direction of the parallel curves. After that, we project the accumulator function on $\mathbb{R}^\ell$. Then consider peaks of it to obtain the positions on those parallel curves.

We apply the proposed method to several synthetic inputs. Consider the image of circles of different sizes with the same center, which are parallel with respect to $\mathcal{L} = \{3\}$, together with another circle that contains more points. By using the usual method, the circle with more points will be detected as in the middle image. On the contrary, our proposed method with the square function will detect parallel circles as in the right one.



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Extended Abstract
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Department of Mathematics, Chiang Mai University

Computing the Probability of Striking a Battleship

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Abstract

A set of n non-overlapping rectangular 'battleships' with unit width are placed on an $H\times W$ 2D square grid uniformly at random. Given the state of an in-progress game, and an initial assumption of uniform probability over all legal configurations of the ships, our goal is to determine the probability that each square contains a ship. We describe and implement a practical solution to this problem, and extend our implementation to run on GPU.

Keywords: battleship, graph, clique, probability, GPU.

2010 MSC: Primary 68W05; Secondary 91A60, 05C85, 68W10.

1 Introduction

Battleship is a game played between two people. Each player starts by placing 'ships' into hidden locations on a 2D square grid unique to their own ships, and then players take turns attempting to 'attack' locations on the others' grid.

In [4], Fiat and Shamir describe an overarching strategy that bounds the worst-case number of attacks required, in the case of one ship. In this paper, we attempt to evaluate local choices within a game of battleship. Our goal is to quickly compute the exact probability of striking a ship with an attack, for each possible next choice of square point to attack. A method for computing or estimating these probabilities has previously been explored [3], but it is much slower than the techniques described here.

1.1 Probability Queries

We consider only one side in the game of battleship. A defending player places n unit-width ships $S=\{s_1,\ldots,s_n\}$ with lengths $L=\{l_1,\ldots,l_n\}$ onto an $H\times W$ square grid, such that no two ships occupy the same grid square. Note that we allow the ships to touch, but we will not make special use of this fact. An attacking player is allowed to perform queries, called **attack**s on squares, with at most one query per square. Each query on a square r has three possible responses.

If the attack **missed**, there is no ship at the sqaure.

If the attack **hit**, there is a ship at the square, and some part of the ship remains undiscovered.

If the attack **sunk**, there is a ship at the square, and all parts of the ship have now been discovered. The specific ship itself is identified in the query response, which we say has been sunk at square r.

The attacking player wishes to minimize the number of queries needed to sink all ships.

In this work we attempt to practically solve the following problem: Assume the ships were placed using a uniform distribution over all possible legal configurations of ship placements. Given a series of queries made by the attacker so far, and their responses, determine the probability that each square of the grid contains a ship.

2 Cliques and the Placement Graph

For a set of ships L and a grid size $H\times W$, we construct **placement graph** G_1 with the following vertices and edges: For each ship s, and each possible placement p of s onto the grid (a location and orientation), create a vertex $v_{s,p}$. Create an edge for each pair of vertices $v_{s,p},v_{s',p'}$ with $s\neq s'$ where s placed at p would not intersect with s' placed at p'. The configurations of all ships correspond to n-cliques in G which cover all squares of the grid that have received hit responses.

Then, given a list of queries and responses so far, we can construct the **valid placement subgraph** G_2 which is an induced subgraph of G_1 with the vertices $v_{s,p}$ such that:

- No square r that has received a miss response is inside the placement p of s.
- If a ship s has already been sunk at a square r, then $v_{s,p}$ covers only r and squares which received hit responses prior to the sunk response of r.

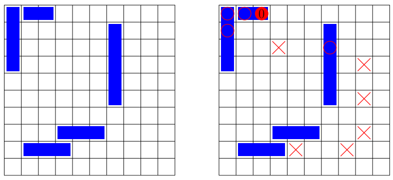

Figure 1: Left: A configuration of ships on the 10×10 grid. Right: The same configuration after some attacks. The 6 $\times$ symbols are misses. The 4 empty circles are hits. The filled circle with a 0 inside is a sink of ship 0. Note that the order of the attacks is not specified here.

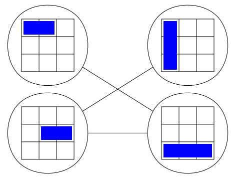

Figure 2: For a 3×3 board with $L=\{2,3\}$, the above diagram represents a small subgraph of G_1.

Using the valid placement subgraph, we can compute our probabilities with a simple algorithm. Let $h(v_{s,p})$ be the number of hit squares s touches, and let H be the total number of hit squares. We iterate through all n-cliques C of G_2. If $\sum_{v\in C} h(v)=H$, then C corresponds to a valid configuration of all ships given all the information we have. For each vertex $v_{s,p}$, we let $f_{v_{s,p}}$ denote the number of valid configurations containing the ship s in placement p. Finally, the total number of valid configurations covering a square r is the sum of $f_{v_{s,p}}$ over all placements p of a ship s that overlap with r. This value is sufficient to compute the exact probability that r appears in a configuration, by dividing by the total number of valid configurations.

3 Clique Iteration and Performance Engineering

The primary performance bottleneck of this approach is the iteration through all n-cliques. Bron and Kerbosch describe a method for iterating through all maximal cliques of general graphs [2]. However, for our n-partite graph, this can be done much more quickly with a simple recursive approach: Iterate through all placements p_1 of s_1. Inside the iteration of p_1, iterate through all placements p_2 of s_2 which do not conflict with the current placement of s_1. Inside the iteration of p_2, iterate through all placements p_3 of s_3 which do not conflict with the current placements of s_2 or s_1. And so on. This pattern can be implemented as a simple recursive function.

An implementation of our simple recursive approach can be refined for much better real-time performance: For each ship s_i, a set V_i of placements of s_i which do not intersect with the current placements of ships $s_1,s_2,\dots,s_{i-1}$ can be maintained. Furthermore, these sets can be updated quickly too: For each pair of ships s_i and s_j with $i<j$, and every pair of placement p_i of s_i, we can compute a set $B_{i,p_i,j}$ of placements of j which do not overlap p_i. Note that all such sets can be computed prior to the n-clique iteration, and that in practice the computation of these sets is negligible. When choosing a placement p_i for a ship s_i, each set V_j for $j>i$ can now be updated by performing the operation $V_j \leftarrow V_j \cap B_{i,p_i,j}$ By storing each V_i as a bitset, this operation can be performed extremely fast in practice.

This algorithm can also be adapted for a GPU: Pre-compute the first few levels of recursion on CPU, and compute the remaining levels in parallel on GPU.

4 Experimental Results

We evaluated our implementation on the version of the game with $n=5$, $L=\{5,4,3,3,2\}$, and $H\times W=10\times 10$. The input with the most valid configurations is the empty board, so it provides an upper bound on performance. We tested on an AMD Ryzen 5 1400 CPU and an NVIDIA RTX 3080 GPU. Our basic single-threaded CPU implementation was able to solve the empty board in 146.37s. Our parallelized implementation using hipSYCL [1] completed in 5.70s on GPU, and in 36.36s using 8 CPU threads.

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Extended Abstract
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The Legend of Zelda: The Complexity of Mechanics

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Abstract

We analyze some of the many game mechanics available to Link in the classic Legend of Zelda series of video games. In each case, we prove that the generalized game with that mechanic is polynomial, NP-complete, NP-hard and in PSPACE, or PSPACE-complete. In the process we give an overview of many of the hardness proof techniques developed for video games over the past decade: the motion-planning-through-gadgets framework, the planar doors framework, the doors-and-buttons framework, the "Nintendo" platform game / SAT framework, and the collectible tokens and toll roads / Hamiltonicity framework.

Keywords: video games, hardness, NP, PSPACE.

2010 MSC: Primary 68Q17; Secondary 68Q25.

1 Introduction

The Legend of Zelda* action–adventure video game series consists of 19 main games developed by Nintendo (sometimes jointly with Capcom), starting with the famous 1986 original which sold over 6.5 million copies [6], and most recently with Breath of the Wild which was a launch title for Nintendo Switch (and is arguably what made the Switch an early success). In each game, the elf protagonist Link explores a world with enemies and obstacles that can be overcome only by specific collectible items and abilities. Starting with nothing, Link must successively search for items that unlock new areas with further items, until he reaches and defeats a final boss enemy Ganon.

Across the 35-year history of the series, many different mechanics have been introduced, leading to a varied landscape of computational complexity problems to study: what is the difficulty of completing a generalized Zelda game with specific sets of items, abilities, and obstacles? Reviewing the two Zelda wikis† and playing the games ourselves, we have identified over 80 unique items with unique mechanics, throughout the 19 games in the Zelda franchise.

In tribute to the fun and challenge of the Zelda series, we propose a long-term undertaking where the video-game-complexity community throughly catalogs these mechanics and analyzes which combinations lead to polynomial vs. NP-hard computational problems. Toward this goal, we analyze in this paper the complexity of several new combinations of various items. Table 1 summarizes our results, along with previously known results about Legend of Zelda [1, 3].

Our new results also serve to highlight different techniques for proving polynomial/NP algorithms, NP-hardness, and PSPACE-hardness of video games involving the control of a single agent/avatar. For algorithms, we see the powerful approach of dynamic programming combined with the technique of shortcutting. One major category is Hamiltonian Path inspired reductions, often simplified with Viglietta's Metatheorem 2 [7] concerning collectible items and toll roads. Next is the Nintendo-style SAT

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†https://zelda.fandom.com/wiki/Category:Items and https://www.zeldadungeon.net/wiki/Category:Items

Game Mechanics	Games with Mechanics	Result	Ref
Hookshot, Pots, Pits	ALTTP, LA, PH, ALBW	$\in$ P	**new**
Hookshot, Pots, Pits, Keys	ALTTP, LA, PH, ALBW	NP-c.	**new**
Switch Hook, Diamond Blocks, Pits	OoA	$\in$ P	**new**
Crystal Switches, Raised Barriers	ALTTP, LA, OoA, OoS, PH, ALBW	$\in$ P	**new**
Roc's Feather, Pegasus Seeds	OoA, OoS, MM	NP-c.	**new**
Bombs, Renewing Cracked Walls	OoT, MM, OoA, OoS, TMC, ST	NP-c.	**new**
Ice Arrows, Water	MM	NP-c.	**new**
Fairies, Bottles, Unavoidable Damage Region	ALTTP, LA, OoT, MM, OoA, OoS, FS, TWW, FSA, TMC, TP, PH, ST, SS, ALBW, BotW	NP-c.	**new**
Magic Armor, Unavoidable Damage Region	ALTTP, OoT, TWW, TP	NP-c.	**new**
Bow or Bombs, and Crystal Switches for Raised Barriers	ALTTP, LA, OoA, OoS, PH	NP-c.	**new**
Colored-tile floor puzzles	LA, OoA, TMC	NP-c	**new**
Kodongos, low walls, sword	ALTTP	NP-hard	**new**
Buzz Blobs, Master Sword	ALTTP, LA, OoA, OoS, TMC, ALBW, TFH	NP-hard	**new**
Decayed Guardians, Bombs	BotW	NP-hard	**new**
Magnetic gloves, metal orbs, ledges, jump platforms	OoS	PSPACE-c.	**new**
Cane of Pacci, ground holes, ledges, tunnels	TMC	FPT in duration PSPACE-c.	**new** **new**
Magnesis Rune, metal platforms	BotW	PSPACE-c.	**new**
Statues, Pressure Plates, Doors	ALTTP, OoT, MM, OoA, OoS, FS, TWW, FSA, TMC, TP, PH, ST, SS, ALBW	PSPACE-c.	**new**
Ancient Orbs, Pedestals, Doors	BotW	PSPACE-c.	**new**
Minecarts	OoA, OoS, TMC	PSPACE-c.	**new**
Once-Pushable Blocks	Zelda I, LA, OoA, OoS, TMC	NP-c.	[1]
Once-Pushable/Pullable Blocks, hookshot, chests, pits, tunnels	ALTTP, LA, OoT, MM, TWW, ALBW	NP-c.	[1]
Keys, Doors, Ledges	AoL, ALTTP, LA, OoT, MM, OoA, OoS, FS, TWW, TMC, TP, PH, ST, SS, ALBW	NP-c.	[1]
Once-Pushable Blocks, Ice	OoT, MM, OoS, TMC, TP, ST	PSPACE-c.	[1]
Buttons, Doors, Teleporters, Pits, Crystal Switches	ALTTP, ALBW	PSPACE-c.	[1]
Spinners	OoA, OoS	PSPACE-c.	[3]

Table 1: Summary of new and past results about complexity of various Zelda mechanics.

reduction from [1] which later acted as inspiration for the door-opening gadgets in [4]. For PSPACE-hardness, we use the door-and-button framework of Forišek [5] and Viglietta [7]. Finally, we use the door gadget from [1] which, along with the other previous work, inspired the gadgets framework for the complexity of motion-planning problems [2–4] which we also use. As a secondary goal, we hope that this paper offers a nice sampling of proof techniques showing the hardness infrastructure that have been built up in recent years.

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Extended Abstract
The 23rd Thailand-Japan Conference on
Discrete and Computational Geometry, Graphs, and Games
Department of Mathematics, Chiang Mai University

Super Catalan Numbers, Chromogeometry, and Fourier Summations over Finite Fields

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Abstract

In this talk, we give an algebraic interpretation of the super Catalan numbers

$$S(m,n) = \frac{(2m)!(2n)!}{m!n!(m+n)!}$$

in the context of integrating polynomials over unit circles over finite field. We work in three metrical geometries: the Euclidean geometry and two relativistic geometries and show that the three geometries are related to each other. This is the phenomenon of chromogeometry.

Keywords: Super Catalan numbers, chromogeometry, Fourier summations, discrete geometry.

2010 MSC: Primary 43A70, 11T06, 51E26.

1 Introduction

The Super Catalan numbers are family of integers which, up to date, have no general combinatorial interpretations although there are some combinatorial interpretations for $S(m,n)$ for a particular value of m and n (see [3–6]). In this talk, we are able to give an algebraic interpretation of the super Catalan numbers through its closely related family of rational numbers

$$\Omega(m,n) = \frac{S(m,n)}{4^{m+n}} = \frac{(2m)!(2n)!}{4^{m+n}m!n!(m+n)!}$$

which we will call the **circular super Catalan numbers**.

We shall see that $\Omega(m,n)$ has a central algebraic interpretation in the context of integration or summation theory of polynomials over unit circles over finite fields of odd characteristic. Their role extends even to summations over suitably defined unit circles in relativistic geometries.

The context of summing polynomials over circles in the plane over different kinds of metrical geometries is the three-fold symmetry of chromogeometry [2] which is an outgrowth of rational trigonometry [1], and brings together Euclidean planar geometry with quadratic form $x^2 + y^2$ (blue) with two different forms of relativistic Einstein-Minkowski geometries with quadratic forms $x^2 - y^2$ (red) and xy (green). The relations and indeed symmetries between these three geometries play a crucial role in also our derivations, as formulas for summations over circles in both Euclidean and relativistic geometries are closely related, and connections between them provide a powerful tool for explicit evaluations.

2 Preliminaries

Consider the finite field $\mathbb{F}_q$ with $q = p^r$ elements for some prime $p > 2$ and $r \in \mathbb{N}$ and define

$$\Omega_p(m,n) = \Omega(m,n) \mod p.$$

We note that $\Omega_p(m,n)$ is well defined since $S(m,n)$ is always an integer and the denominator 4^{m+n} is not a multiple of p.

Introduce three different metrical geometries: Euclidean (blue) and relativistic (red and green) geometry with respective associated quadratic forms $x^2 + y^2$, $x^2 - y^2$, and xy. The blue unit circle $S_{b,q}$ is defined as the set of $(x, y) \in \mathbb{F}_q^2$ such that $x^2 + y^2 = 1$ with the red unit circle $S_{r,q}$ and green unit circle $S_{g,q}$ defined similarly. The subscript b, r, g stand for blue, red, and green respectively. For brevity, we use the subscript c to indicate an arbitrary color.

There is also a dichotomy of whether -1 is a square or not in $\mathbb{F}_q$ and we note that distinction through the Jacobi symbol which is a generalization of the Legendre symbol. For any odd prime p, the Legendre symbol is defined as

$$\left(\frac{-1}{p}\right) := \begin{cases} 1 & \text{if } -1 \text{ is a square in } \mathbb{F}_q, \\ -1 & \text{if } -1 \text{ is not a square in } \mathbb{F}_q \end{cases}$$

and thus for any power of prime $q = p^r$ we define the Jacobi symbol $\left(\frac{-1}{q}\right) := \left(\frac{-1}{p}\right)^r$.

Define the $\mathbb{F}_q$-valued summation functionals $\psi_{c,q}$ on the space of polynomials in two variables α and β to be the linear functional whose values on a monomial $\alpha^k \beta^l$ for $k, l \in \mathbb{N}$ are given by

$$\psi_{g,q} = -\sum_{(x,y)\in S_{g,q}} x^k y^l, \quad \psi_{r,q} = -\sum_{(x,y)\in S_{r,q}} x^k y^l, \quad \psi_{b,q} = -\left(\frac{-1}{q}\right) \sum_{(x,y)\in S_{b,q}} x^k y^l.$$

We will show that these functionals satisfy the following three conditions.

1. **(Normalization)** For the polynomial $\mathbf{1}$ which is constant equal to 1, we have $\psi_{c,q}(\mathbf{1}) = 1_{\mathbb{F}}$.
2. **(Locality)** For any polynomial f that restricts to zero on $S_{c,q}$, we have that $\psi_{c,q}(f) = 0$.
3. **(Invariance)** $\psi_{c,q}$ is invariant under any linear transformation that preserves the bilinear form of the associated geometry c.

We call any linear functional that satisfies the three conditions above **circular integral functional**.

3 Main Results

We state our main results without proof.

Theorem 3.1. *The linear functionals $\psi_{c,q}$ is the unique circular integral functional in each geometry.*

We present a simplified and restricted version of the formulas for $\psi_{c,q}$ involving the circular super Catalan numbers in the red and blue geometry, whose derivation stems from the result in the green geometry.

Theorem 3.2. *Let a finite field $\mathbb{F}_q$ be given where $q = p^r$.*

1. *In the red geometry, for $0 \le k + l < q - 1$,*

$$\psi_{r,q}\left(\alpha^k \beta^l\right) = \begin{cases} (-1)^n \Omega_p(m, n) & \text{if } k = 2m \text{ and } l = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

2. *In the blue geometry, for $0 \le k + l < q - 1$,*

$$\psi_{r,q}\left(\alpha^k \beta^l\right) = \begin{cases} \Omega_p(m, n) & \text{if } k = 2m \text{ and } l = 2n, \\ 0 & \text{otherwise.} \end{cases}$$

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Extended Abstract
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Snakes and Ladders with Large Spinners under an Alternative Winning Rule

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Abstract

Snakes and Ladders is a game that players move from square to square to reach the last one. On each turn, one player spins a spinner that gives a number $m \in \{1, 2, \ldots, r\}$ with equal probability, and moves forward for m squares. If a player spins a number that exceeds the number of the remaining squares, the player cannot move in that turn. However, under an alternative winning rule in Thailand, the player keeps going until reaching the last square and moves backward with the remaining number.

We focus on the expected number of turns until a player exactly reaches the last square of an n-square board when the value of r is close to n. If there are no ladders leading to the last square, the cases $r = n$ and $r = n - 1$ give the same expected number n, which is different from the case when $r = n - 2$ whose expected number is less than n if there are no snakes sending a player back to square 1. We provide an approach to calculate the expected numbers for the case when there is a single ladder leading to the last square for $r = n$ and $r = n - 1$.

Keywords: board game, expected number, snakes and ladders.

2010 MSC: Primary 60C05.

1 Introduction

Snakes and Ladders or *Chutes and Ladders* is a game that players take turns to walk from square to square following the number from a spinner that provides a number in $\{1, 2, \ldots, r\}$ with equal probability. This number r is called the *range* of the spinner. The winner is the player who exactly stops at the last square. Moreover, there are ladders which lead the player who stops at the lower square up to the upper one. On the other hand, there are snakes which drop the player who stops at the upper square down to the lower one. If the player is so close to the last square that the number gotten from the spinner is more than the number of the remaining squares, that player cannot walk and has to wait until the next turn. However, in Thailand, the winning rule is different. If this situation occurs, that player has to walk forward until reaching the last square, and walk backward from the last square with the number of steps left.

The expected number of turns for a player to win this classical game is a topic investigated by many mathematicians. Most of them follow the regular winning rule and apply Markov chain model (See eg. [1, 3, 4]). Among these works, the study by Connors and Glass [2] considered the case when the ranges of spinners are large, and dealt with the problem in a more combinatorial way. Whether using the combinatorial approach as in [2] or Markov chain techniques, the alternative rule in Thailand makes it more complicated to calculate the expected number. Therefore, we focus on the expected number of spins to win the game using large spinners under this alternative winning rule.

There are some properties that hold for most of the boards played in these days, and we assume these following properties throughout this study.

(i) Each snake's head is in the upper square and its tail is in the lower one.
(ii) There are no snake's heads in the last square.
(iii) There are no snakes or ladders before square 1.
(iv) In a square, there exists at most one of these: a snake's head, a snake's tail or an end of a ladder.

2 Main Results

Let E denote the expected number of spins to win the game on an n-square board under a given condition. The first case in our study focuses on the boards with no ladders leading to the last square. To find the expected number E in this case, we consider the case that there are no snakes or ladders using the following lemma.

Theorem 2.1. [5] *Consider a sequence of independent trials that each of them has two outcomes, success with probability p and failure with probability $1-p$. The expected number of trials until the first success is $\frac{1}{p}$.*

Since adding any snakes or ladders does not affect the probability of exactly reaching the last square in each trial, as long as no ladders leading to the last square, the results for this specific case lead to these theorems.

Theorem 2.2. *For an n-square board in the first case, if $r = n$ or $r = n-1$, then $E = n$.*

Theorem 2.3. *For an n-square board in the first case, if $r = n-2$ and there are no snakes sending a player back to square 1, then $E < n$.*

As for the second case when there is a single ladder leading to the last square, we let E_i be the expected number of spins to win the game when we are at square i and C be the coefficient matrix of the system $C\mathbf{E} = \mathbf{1}_{n-2}$ where $\mathbf{1}_{n-2}$ is the $(n-2)$-dimension vector whose components are all 1 and $\mathbf{E} = \begin{bmatrix} E_1 & \cdots & E_{k-1} & E_{k+1} & \cdots & E_{n-1} \end{bmatrix}^T$.

For $r = n$ and $r = n-1$, we show that for each case, C is invertible and can be used to calculate $E_1, \cdots, E_{k-1}, E_{k+1}, \cdots, E_{n-1}$. Finally, we can compute the expected number E. We summarize our works for the n-square boards in this case as follows.

Theorem 2.4. *If $r = n$, then $E = \dfrac{1}{n}\displaystyle\sum_{\substack{i=1\\ i\neq k}}^{n-1} E_i$ where each E_i can be obtained from $\mathbf{E} = C^{-1}\mathbf{1}_{n-2}$.*

Theorem 2.5. *If $r = n-1$, then $E = \dfrac{1}{n-1}\displaystyle\sum_{\substack{i=1\\ i\neq k}}^{n-1} E_i$ where each E_i can be obtained from $\mathbf{E} = C^{-1}\mathbf{1}_{n-2}$.*

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