The Islamic University of Gaza
Faculty of Engineering
Civil Engineering Department

Numerical Analysis
ECIV 3306

Chapter 21

Newton-Cotes Integration Formula

Numerical Differentiation and Integration

Part 6

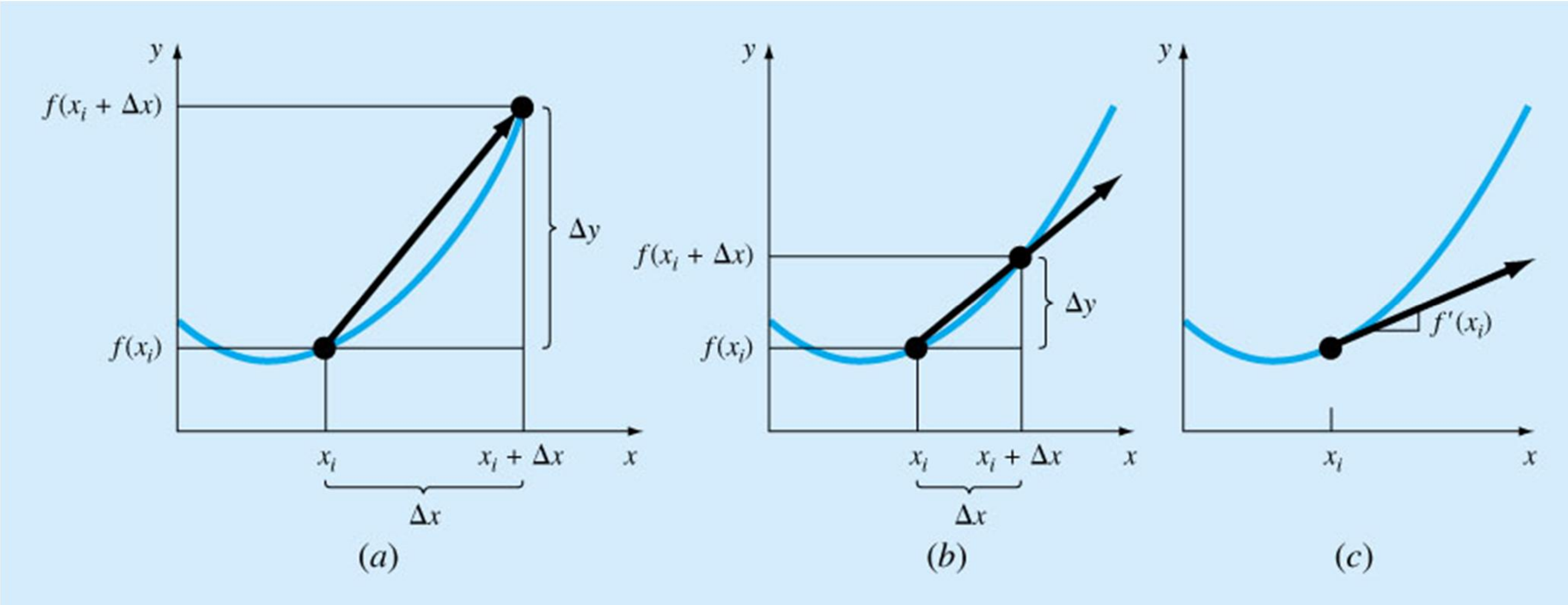
- Calculus is the mathematics of change. Because engineers must continuously deal with systems and processes that change, calculus is an essential tool of engineering.

- Standing in the heart of calculus are the mathematical concepts of *differentiation* and *integration*:

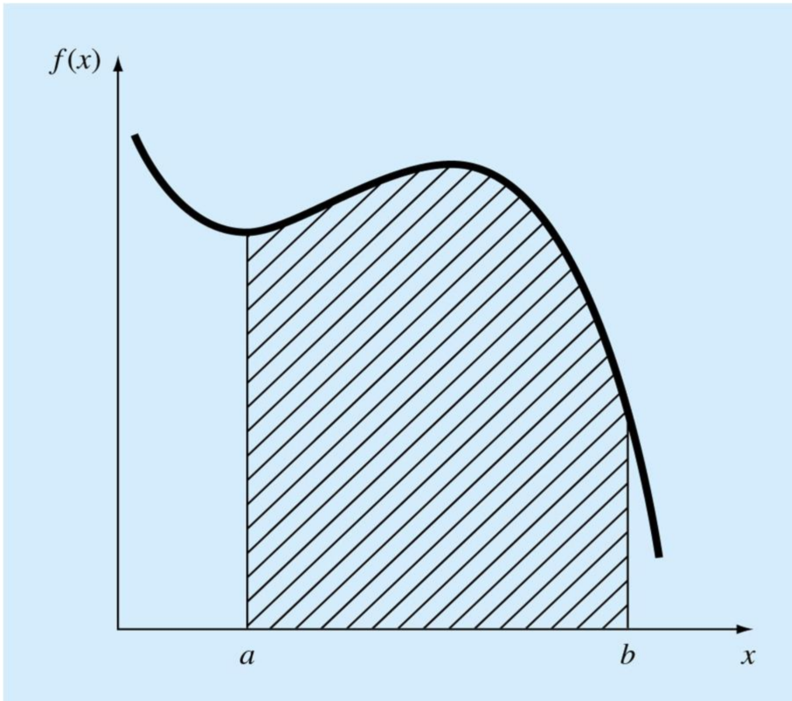
$$\frac{\Delta y}{\Delta x} = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x}$$

$$I = \int_a^b f(x)dx$$



y
$f(x_i + \Delta x)$
Δy
$f(x_i)$
x_i
$x_i + \Delta x$
x
Δx
(a)
y
$f(x_i + \Delta x)$
Δy
$f(x_i)$
x_i
$x_i + \Delta x$
x
Δx
(b)
y
$f'(x_i)$
x_i
x
(c)



$f(x)$
a
b
x

What is Integration?

- Integrate means “to bring together”, as parts, into a whole; to indicate total amount.

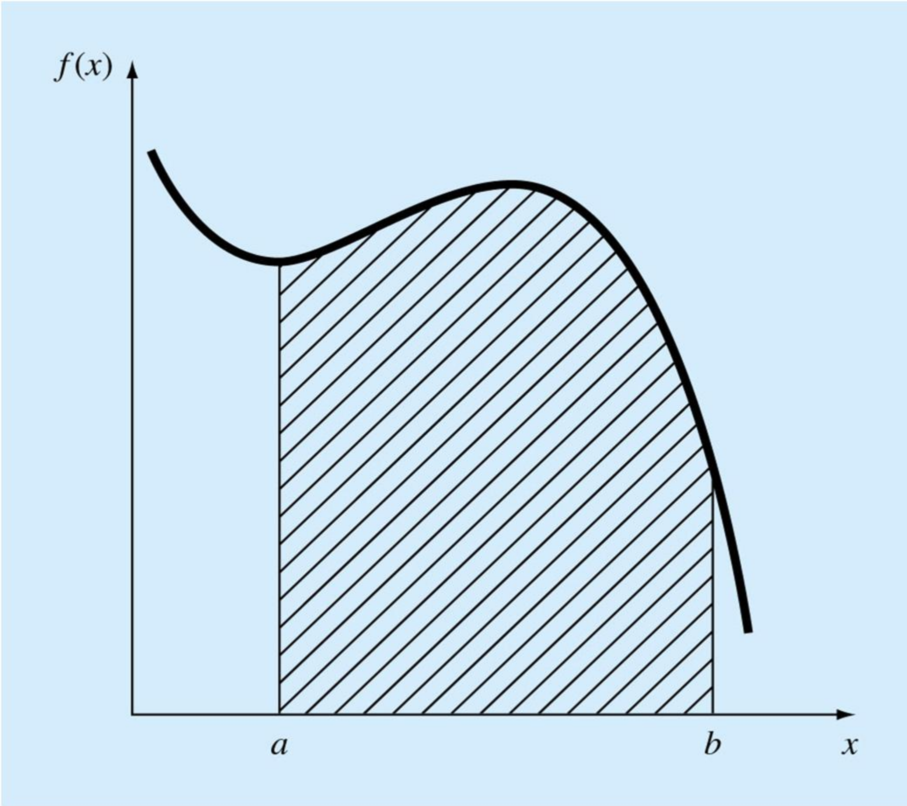
$$I = \int_a^b f(x).dx$$

- The above stands for integral of function *f(x)* with respect to the independent variable *x* between the limits x = a to x = b.

What is Integration?

- Graphically integration is simply to find the area under a certain curve between the 2 integration limits.

$$I = \int_a^b f(x).dx = A$$



Newton-Cotes integration Formulas

Introduction

- The *Newton-Cotes formulas* are the most common numerical integration methods.
- They are based on the strategy of replacing a complicated function with an approximating function that is easy to integrate.

$$I = \int_a^b f(x)dx \cong \int_a^b f_n(x)dx$$

$$f_n(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} + a_n x^n$$

1. Trapezoidal Rule

The trapezoidal rule uses a polynomial of the first degree to replace the function to be integrated.

$$I = \int_a^b f(x).dx \cong \int_a^b f_1(x).dx$$

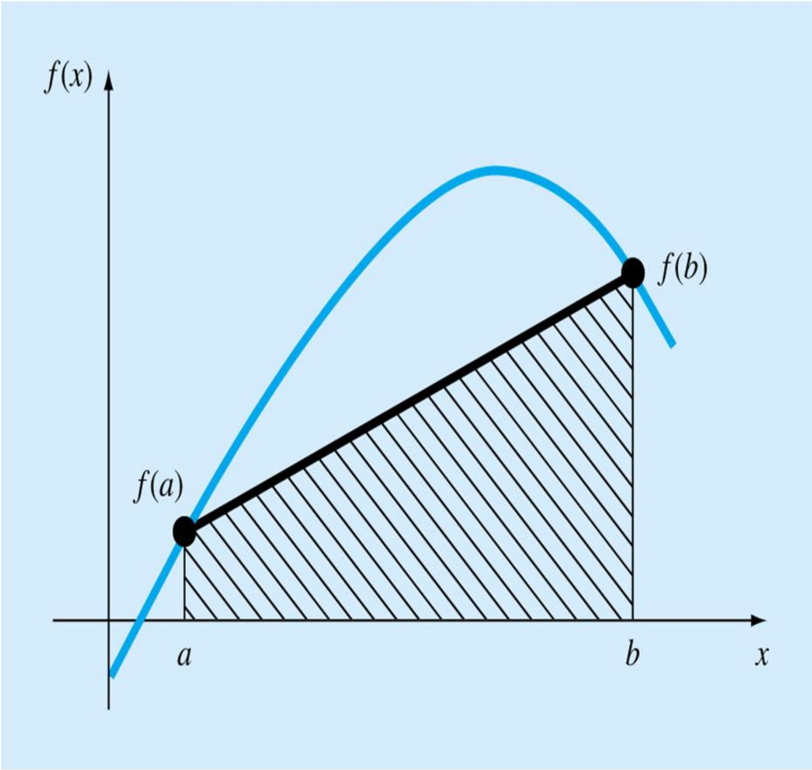
$$f_1(x) = a + \frac{f(b) - f(a)}{b-a}(x-a)$$

$$I = \int_a^b f(x).dx \cong \int_a^b f_1(x).dx$$

$$= \int_a^b \left\{ a + \frac{f(b) - f(a)}{b-a}(x-a) \right\}.dx$$

$$\boxed{I = (b-a)\frac{f(a) + f(b)}{2}}$$

Trapezoidal rule



Error of the Trapezoidal Rule

When we employ the integral under a straight line segment to approximate the integral under a curve, error may be:

$$E_t = -\frac{1}{12} f''(\xi)(b-a)^3$$

Where ξ lies somewhere in the interval from a to b.

Trapezoidal Rule

Example 21.1

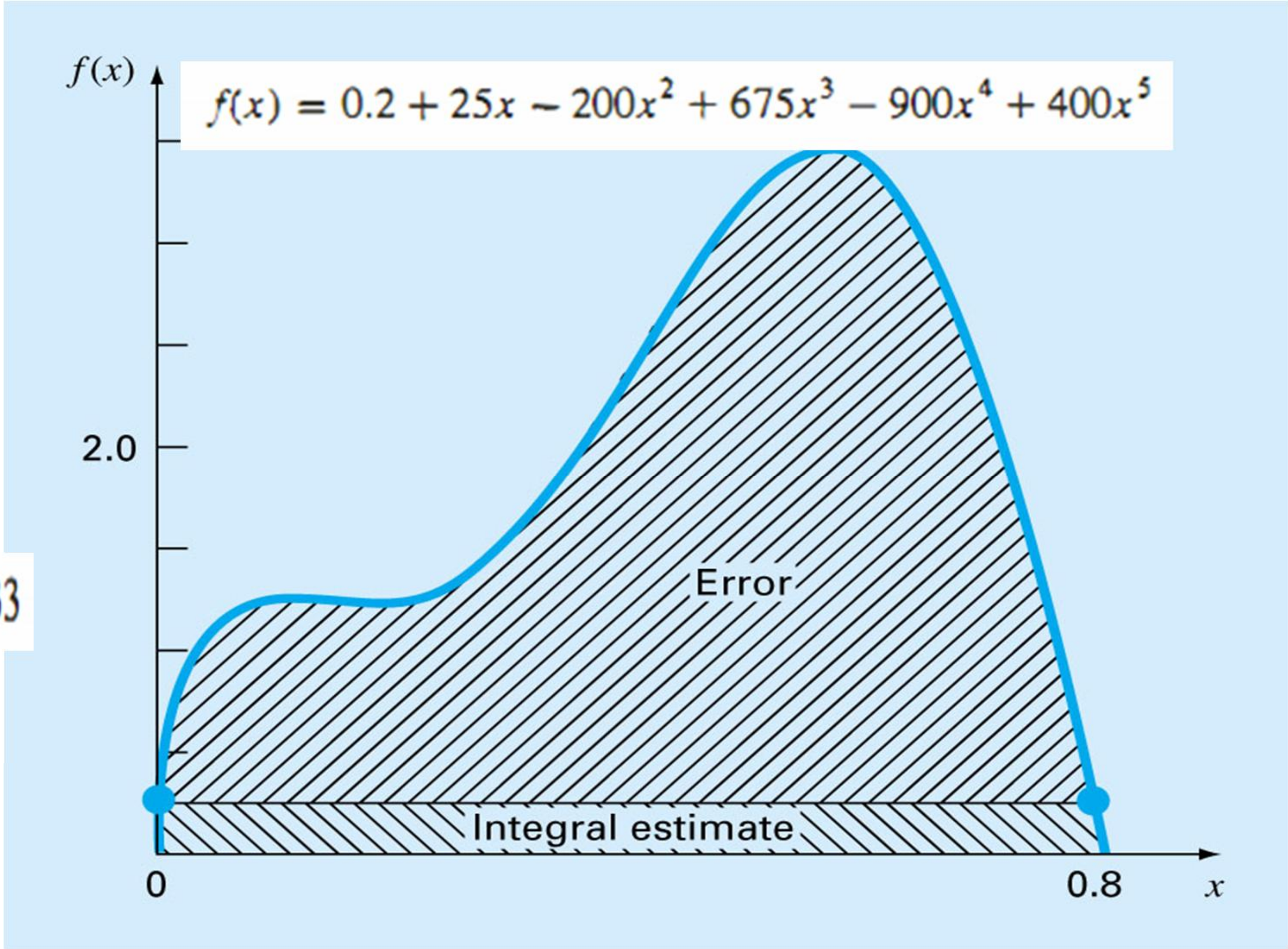
I= 1.640533

$$f(0) = 0.2$$
$$f(0.8) = 0.232$$

$$I \cong 0.8\frac{0.2 + 0.232}{2}$$
$$= 0.1728$$

$$E_t = 1.640533 - 0.1728 = 1.467733$$

$$\varepsilon_t = 89.5\%.$$

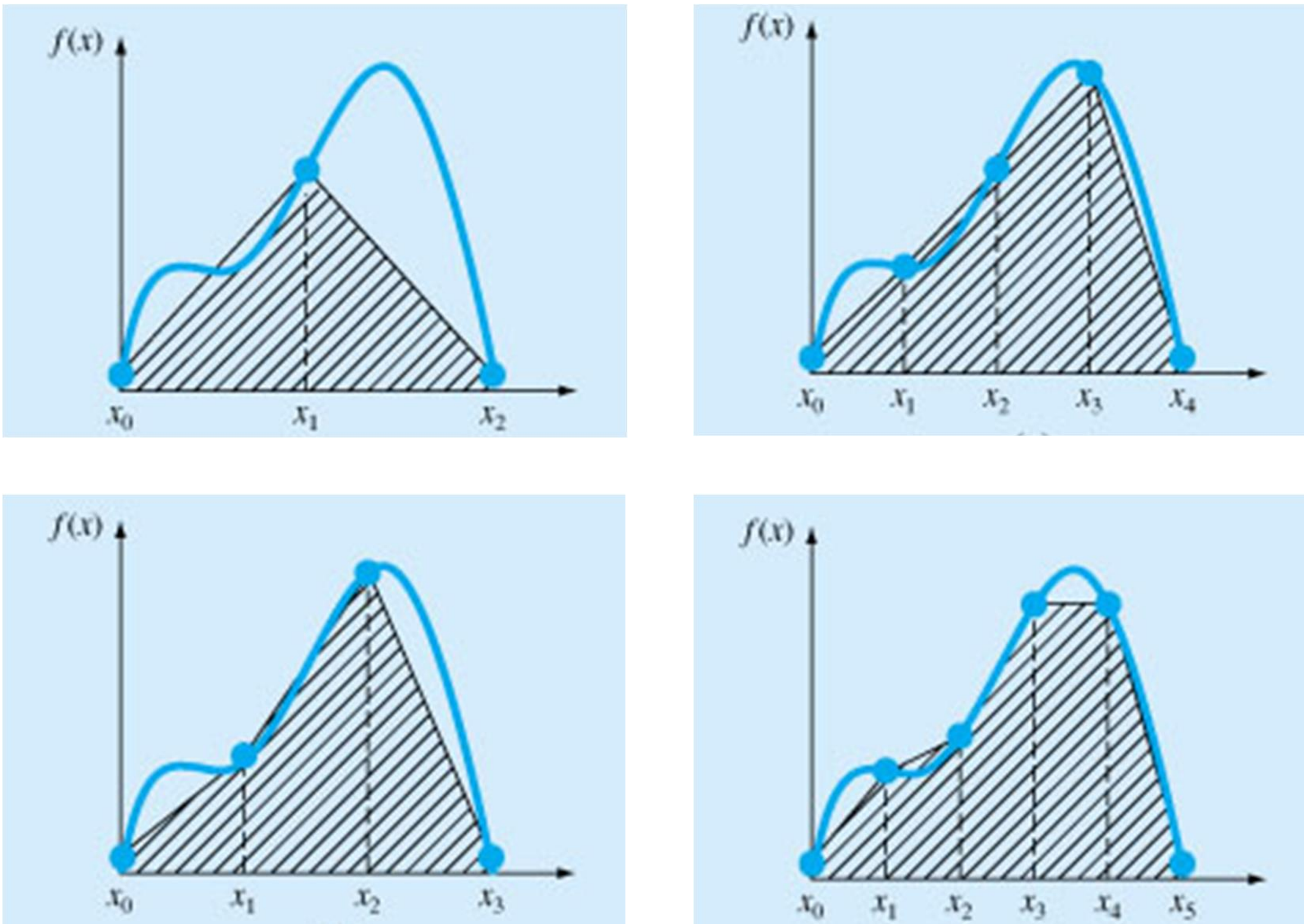


Multiple Trapezoidal Rule

- One way to improve the accuracy of the trapezoidal rule is to divide the integration interval from a to b into a number of segments and apply the method to each segment.

- The areas of individual segments can then be added to yield the integral for the entire interval.

Multiple Trapezoidal Rule



Multiple Trapezoidal Rule

$$h = \frac{b-a}{n} \qquad a = x_0 \quad b = x_n$$

$$I = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \cdots\cdots + \int_{x_{n-1}}^{x_n} f(x)dx$$

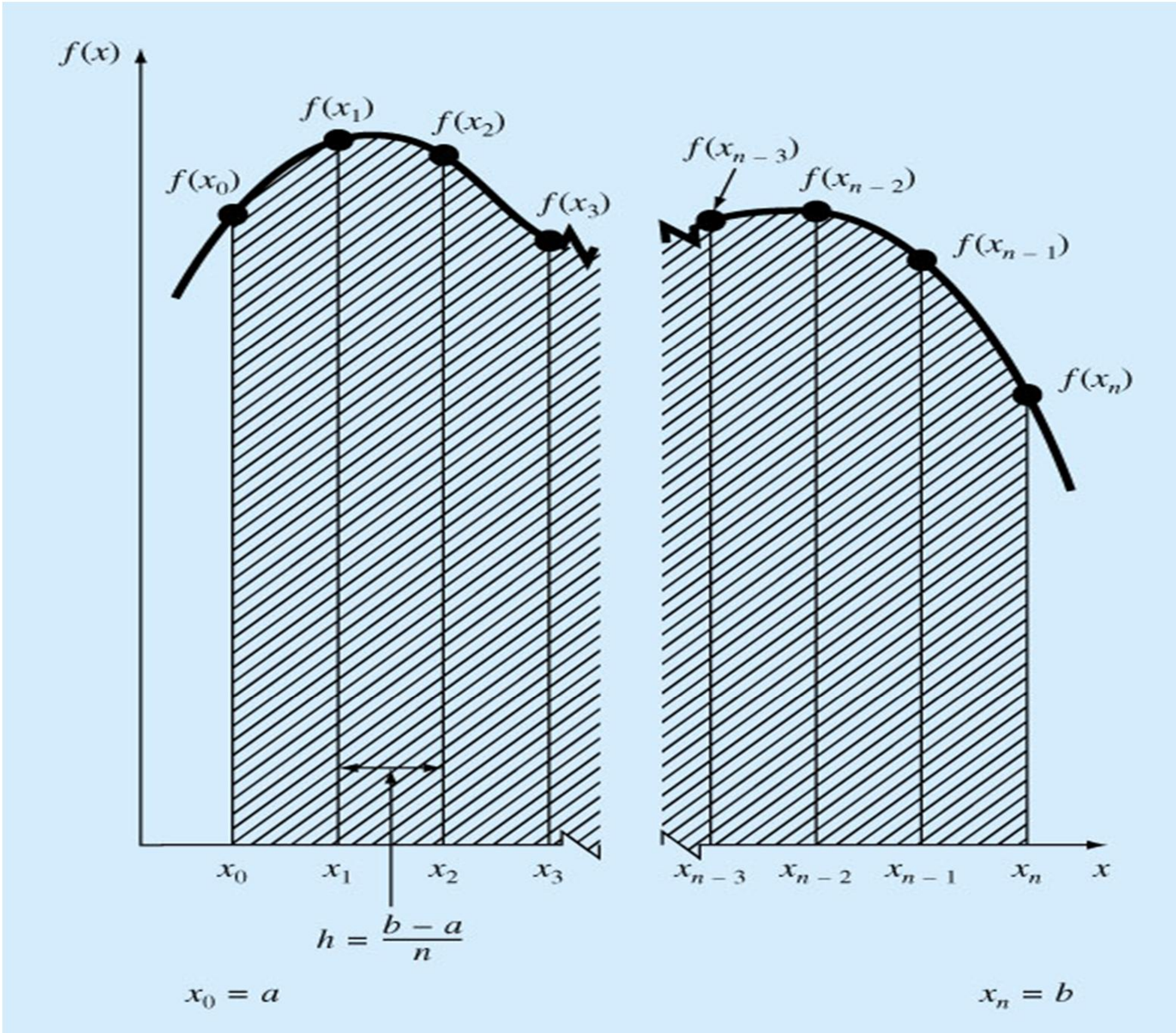
Substitute into the integrals for f(x) by $f_1(x)$ in each segment and integrate:

$$I = h\frac{f(x_0) + f(x_1)}{2} + h\frac{f(x_1) + f(x_2)}{2} + \cdots + h\frac{f(x_{n-1}) + f(x_n)}{2}$$

$$\boxed{I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]}$$

OR

$$\boxed{I = (b-a)\frac{f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n)}{2n}}$$



$f(x)$
$f(x_0)$
$f(x_1)$
$f(x_2)$
$f(x_3)$
$f(x_{n-3})$
$f(x_{n-2})$
$f(x_{n-1})$
$f(x_n)$
x_0
x_1
x_2
x_3
x_{n-3}
x_{n-2}
x_{n-1}
x_n
x
$h = \frac{b-a}{n}$
$x_0 = a$
$x_n = b$

Multiple Trapezoidal Rule

An error for multiple-application trapezoidal rule can be obtained by summing the individual errors for each segment:

$$\sum f''(\xi i) \cong n\bar{f}''$$

$$E_a = -\frac{(b-a)^3}{12n^2}\,\bar{f}''$$

Simpson’s Rules

More accurate estimate of an integral is obtained if a high-order polynomial is used to connect the points. The formulas that result from taking the integrals under such polynomials are called *Simpson’s Rules*.

Simpson’s Rules

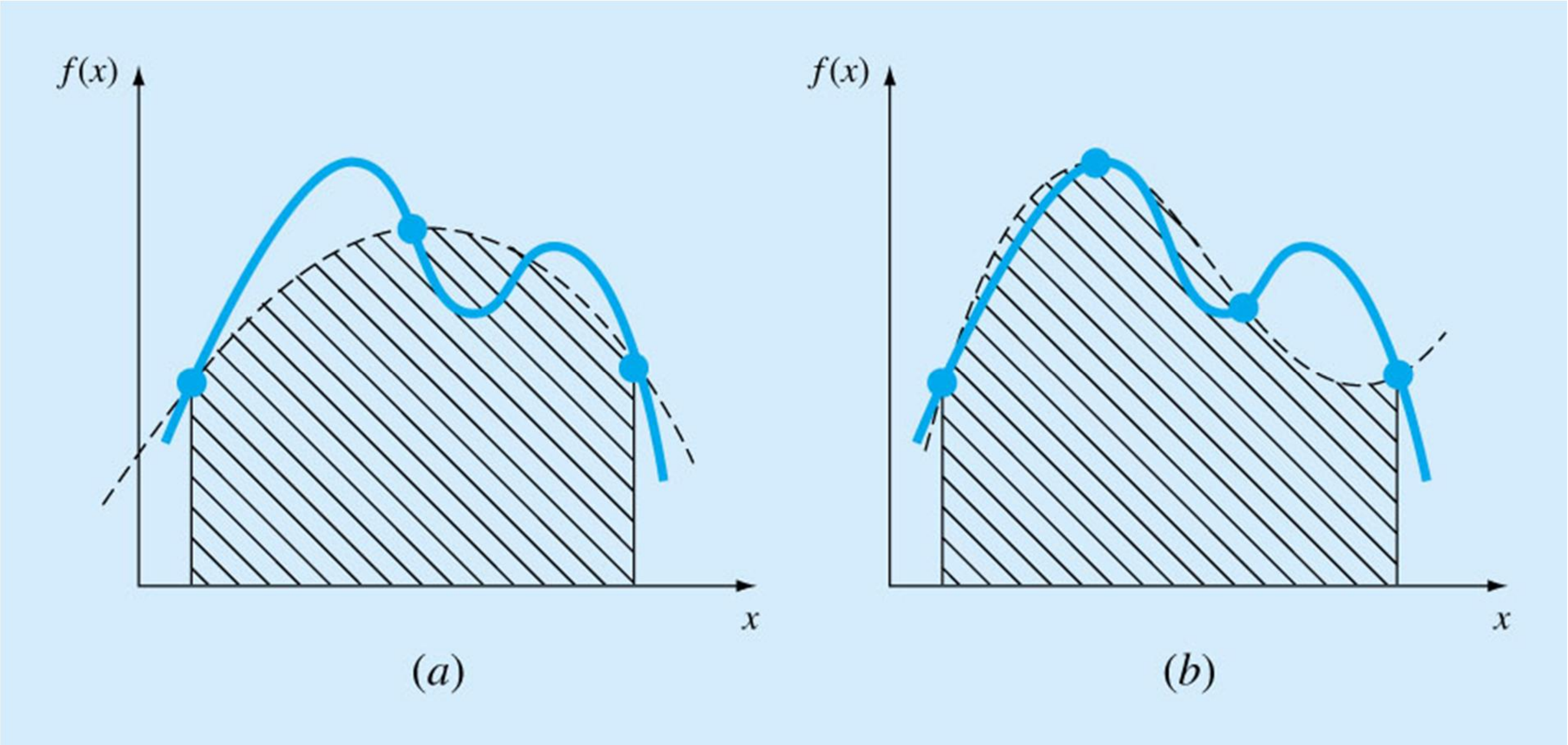
- Simpson’s 1/3 Rule

  Results when a second-order interpolating polynomial is used.

- Simpson’s 3/8 Rule

  Results when a third-order (cubic) interpolating polynomial is used.

Simpson's Rules



Simpson's 1/3 Rule

Simpson's 3/8 Rule

Simpson's 1/3 Rule

$$I = \int_a^b f(x)dx \cong \int_a^b f_2(x)dx$$

$$a = x_0 \quad b = x_2$$

$$I = \int_{x_0}^{x_2} \left[\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) \right] dx$$

$$\boxed{I \cong \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]}$$

OR

$$\boxed{I = (b-a)\frac{f(x_0) + 4f(x_1) + f(x_2)}{6}}$$

$$h = \frac{b-a}{2}$$

Simpson's 1/3 Rule

Simpson's 1/3 Rule

- Single segment application of Simpson's 1/3 rule has a truncation error of:

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\xi) \qquad a < \xi < b$$

- Simpson's 1/3 rule is more accurate than trapezoidal rule.

The Multiple-Application Simpson's 1/3 Rule

- Just as the trapezoidal rule, Simpson's rule can be improved by dividing the integration interval into a number of segments of equal width.

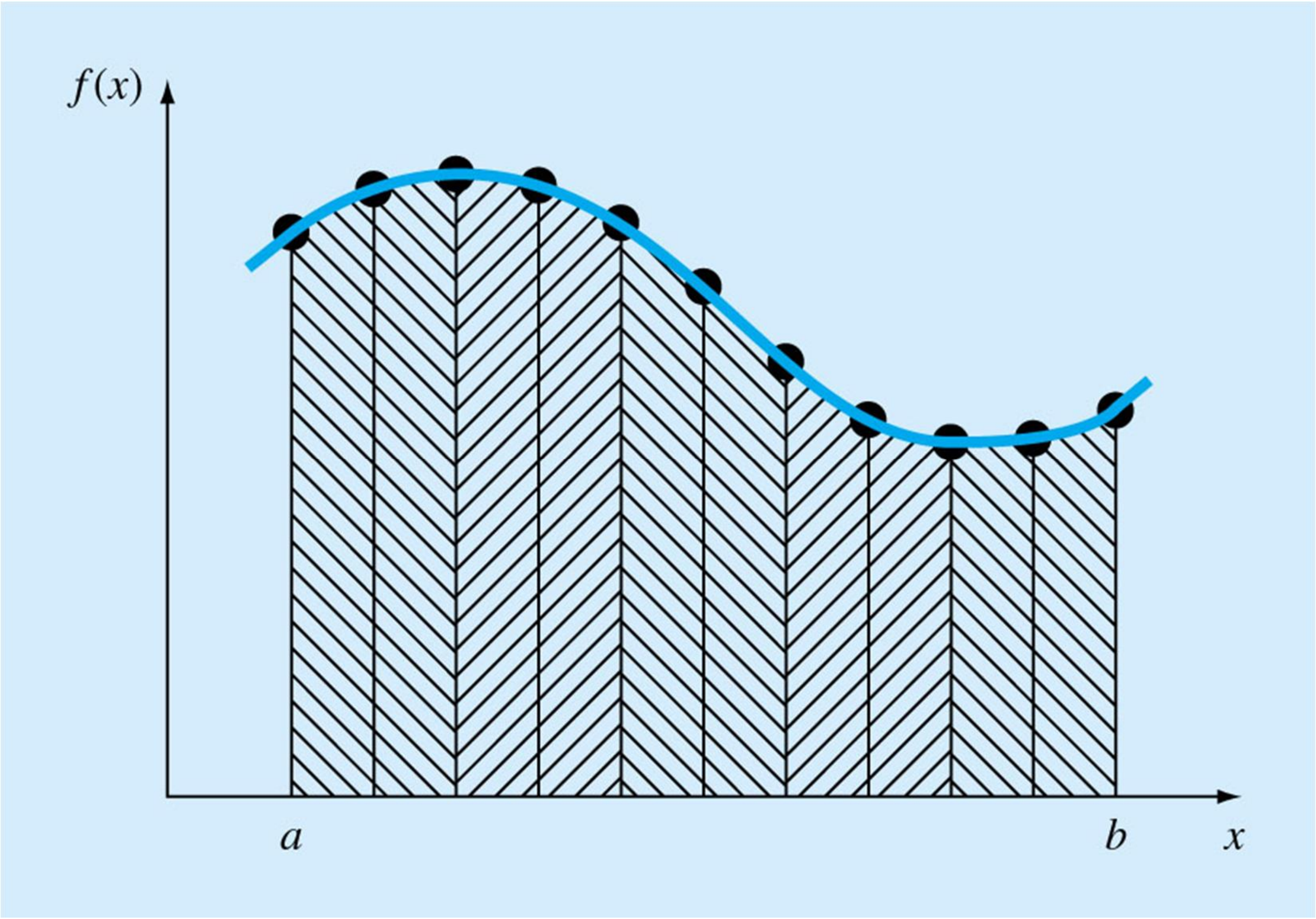
$$I \cong 2h\frac{f(x_o)+4f(x_1)+f(x_2)}{6}+2h\frac{f(x_2)+4f(x_3)+f(x_4)}{6}$$

$$+\dots\dots\dots+2h\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6} \qquad with \quad h=\frac{b-a}{n}$$

$$\cong (b-a)\frac{\left\{ f(x_o)+f(x_n)+4\sum_{i=1,3,5}^{n-1} f(x_i)+2\sum_{j=2,4,6}^{n-2} f(x_j)\right\}}{3n}$$

$$E_t=-\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}(\xi)$$

The Multiple-Application Simpson's 1/3 Rule



The Multiple-Application Simpson's 1/3 Rule

- *However*, it is limited to cases where values are equi-spaced.

- *Further*, it is limited to situations where there are an even number of segments and odd number of points

The Multiple-Application Simpson's 1/3 Rule

EXAMPLE 21.5 Multiple-Application Version of Simpson's 1/3 Rule

Problem Statement. Use Eq. (21.18) with $n = 4$ to estimate the integral of

$$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$$

from $a = 0$ to $b = 0.8$. Recall that the exact integral is 1.640533.

Solution. $n = 4$ $(h = 0.2)$:

$$f(0) = 0.2 \qquad f(0.2) = 1.288$$
$$f(0.4) = 2.456 \qquad f(0.6) = 3.464$$
$$f(0.8) = 0.232$$

From Eq. (21.18),

$$I = 0.8\,\frac{0.2 + 4(1.288 + 3.464) + 2(2.456) + 0.232}{12} = 1.623467$$

$$E_t = 1.640533 - 1.623467 = 0.017067 \qquad \varepsilon_t = 1.04\%$$

Simpson's 3/8 Rule

An odd-segment-even-point formula used in conjunction with the 1/3 rule to permit evaluation of both even and odd numbers of segments.

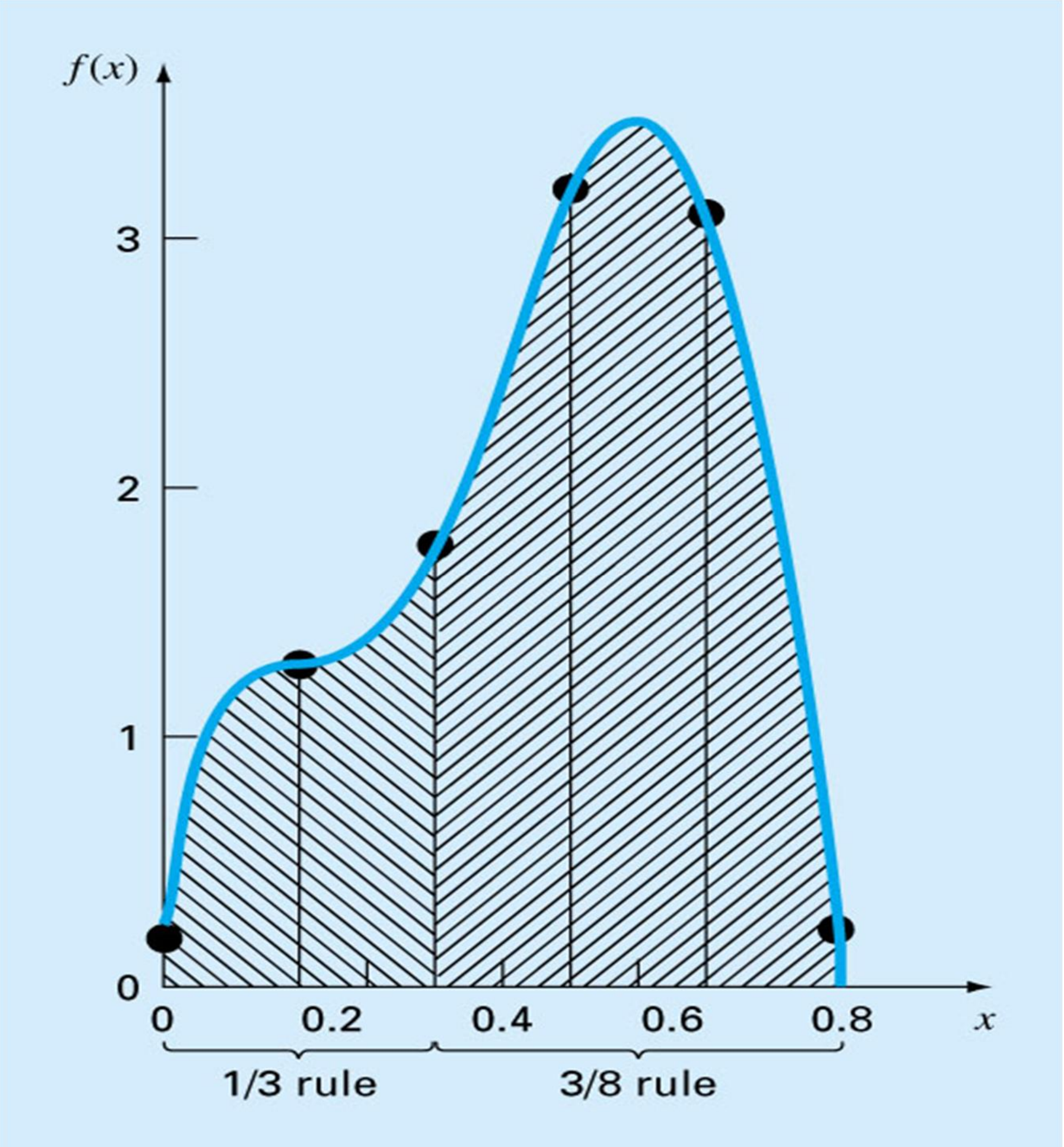
If there are 2 extra points between the integration limits a and b, then a 3rd degree polynomial can be used instead of the parabola to replace the function to be integrated:

$$I = \int_a^b f(x)dx \cong \int_a^b f_3(x)dx$$

$$\boxed{I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\right]}, \quad h = \frac{(b-a)}{3}$$

$$E_t = -\frac{(b-a)^5}{6480} f^{(4)}(\xi)$$

Simpson's 3/8 Rule



f(x)
3
2
1
0
0
0.2
0.4
0.6
0.8
x
1/3 rule
3/8 rule

Newton Cotes Integration-Example

Find the integral of:

$f(x) = 0.2 + 25x - 200x^2 + 675x^3 - 900x^4 + 400x^5$

Between the limits 0 to 0.8, $f(0) = 0.2$, $f(0.8) = 0.232$,

$I_{exact} = 1.640533$

1. The trapezoidal rule (ans. 0.1728)

$$I = (b-a)\frac{f(a)+f(b)}{2} \Rightarrow \quad I = (0.8-0)\frac{0.2+0.232}{2} = 0.1728$$

$$E_t = 1.640533 - 0.1728 = 1.467733 \quad \Rightarrow \quad \varepsilon_t = 89.5\%$$

$$f''(x) = -400 + 4050x - 10{,}800x^2 + 8000x^3$$

$$\bar{f}''(x) = \frac{\int_0^{0.8}(-400 + 4050x - 10{,}800x^2 + 8000x^3)dx}{0.8-0} = -60$$

$$E_a = -\frac{1}{12}(60)(0.8)^3 = 2.56$$

Newton Cotes Integration-Example

2. Multiple trapezoidal rule (n=4) (ans. 1.4848)

f(0)=0.2, *f(0.2)*=1.288, *f(0.4)*=2.456, *f(0.6)=3.464* ,f(0.8)=0.232

$$h = \frac{(b-a)}{4} = \frac{(0.8-0)}{4} = 0.2$$

$$I = \frac{h}{2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) \right]$$

$$= \frac{0.8}{2}\left[0.2 + 2(1.288 + 2.456 + 3.464) + 0.232\right] = 1.4848$$

Newton Cotes Integration-Example

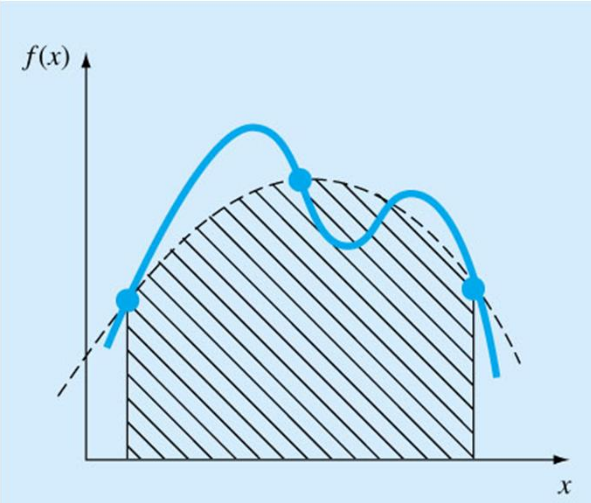
3. The Simpson 1/3 rule (ans. 1.367467)

f(0) = 0.2, *f(0.4)* = 0.2.456, *f(0.8)* = 0.232

$$h = \frac{b-a}{2} = \frac{0.8-0}{2} = 0.4$$

$$I = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right]$$

$$= \frac{0.4}{3}\left[0.2 + 4\times 2.456 + 0.232\right] = 1.367467$$



$$E_t = 1.640533 - 1.367467 = 0.2730667 \quad \Rightarrow \quad \varepsilon_t = 16.6\%$$

$$\bar{f}^{(4)}(x) = -2400$$

$$E_a = -\frac{(b-a)^5}{2880} f^{(4)}(\xi) = -\frac{(0.8-0)^5}{2880}(-2400) = 0.2730667$$

Newton Cotes Integration-Example

4. Multiple application of Simpson 1/3 rule (n=4)

   (ans. 1.623467).

f(0)=0.2, *f(0.2)*=1.288, *f(0.4)*=2.456, *f(0.6)=3.464* ,f(0.8)=0.232

$$h=\frac{(b-a)}{4}=\frac{(0.8-0)}{4}=0.2$$

$$I=\frac{h}{3}\left[f(x_0)+4\sum_{i=1,3,5}^{n-1}f(x_i)+2\sum_{i=2,4.6}^{n-2}f(x_i)+f(x_n)\right]$$

$$=\frac{0.2}{3}\left[0.2+4(1.288+3.464)+2(2.456)+0.232\right]=1.623467$$

$$E_t=1.640533-1.623467=0.017067 \quad\Rightarrow\quad \varepsilon_t=1.04\%$$

$$E_a=-\frac{(b-a)^5}{180n^4}\bar{f}^{(4)}(\xi)=-\frac{0.8^5}{180(4)^4}(-2400)=0.017067$$

Newton Cotes Integration-Example

5. The Simpson 3/8 rule (ans. 1.519170)

f(0)=0.2, *f(0.2667)*=1.432724, *f(0.5333)*=3.487177, f(0.8)=0.232

$$h = \frac{(b-a)}{3} = \frac{(0.8-0)}{3} = 0.2667$$

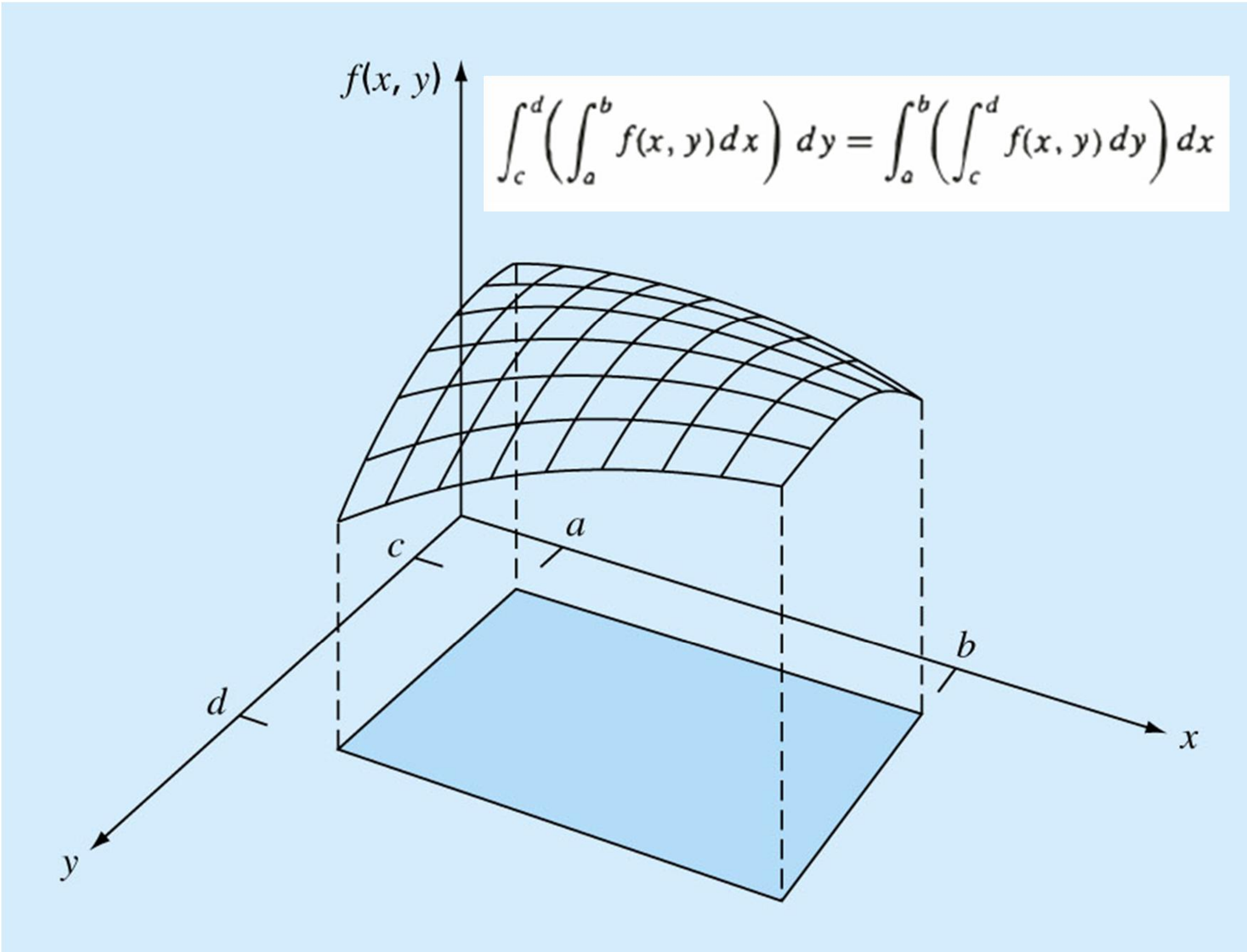
$$I = I \cong \frac{3h}{8}\left[f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) \right]$$

$$= \frac{0.8}{8}\left[0.2 + 3\times 1.432724 + 3\times 3.487177 + 0.232\right] = 1.519170$$

$$E_t = 1.640533 - 1.51917 = 0.121363 \quad \Rightarrow \quad \varepsilon_t = 7.4\%$$

$$E_a = -\frac{(b-a)5}{6480} f^{(4)}(\xi) = -\frac{0.8^5}{6480}(-2400) = 0.1213630$$

Double Integral



$$\int_c^d \left(\int_a^b f(x, y)\, dx \right) dy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

Double Integral-Example

Suppose that the temperature of a rectangular heated plate is described by the following function:

$$T(x, y) = 2xy + 2x - x^2 - 2y^2 + 72$$

If the plate is 8m long (x direction) and 6 m wide (y direction) compute the average temperature.

Solution

The function can also be evaluated analytically to yield a result of S8.66667.

Double Integral-Example

Suppose that the temperature of a rectangular heated plate is described by the following function:

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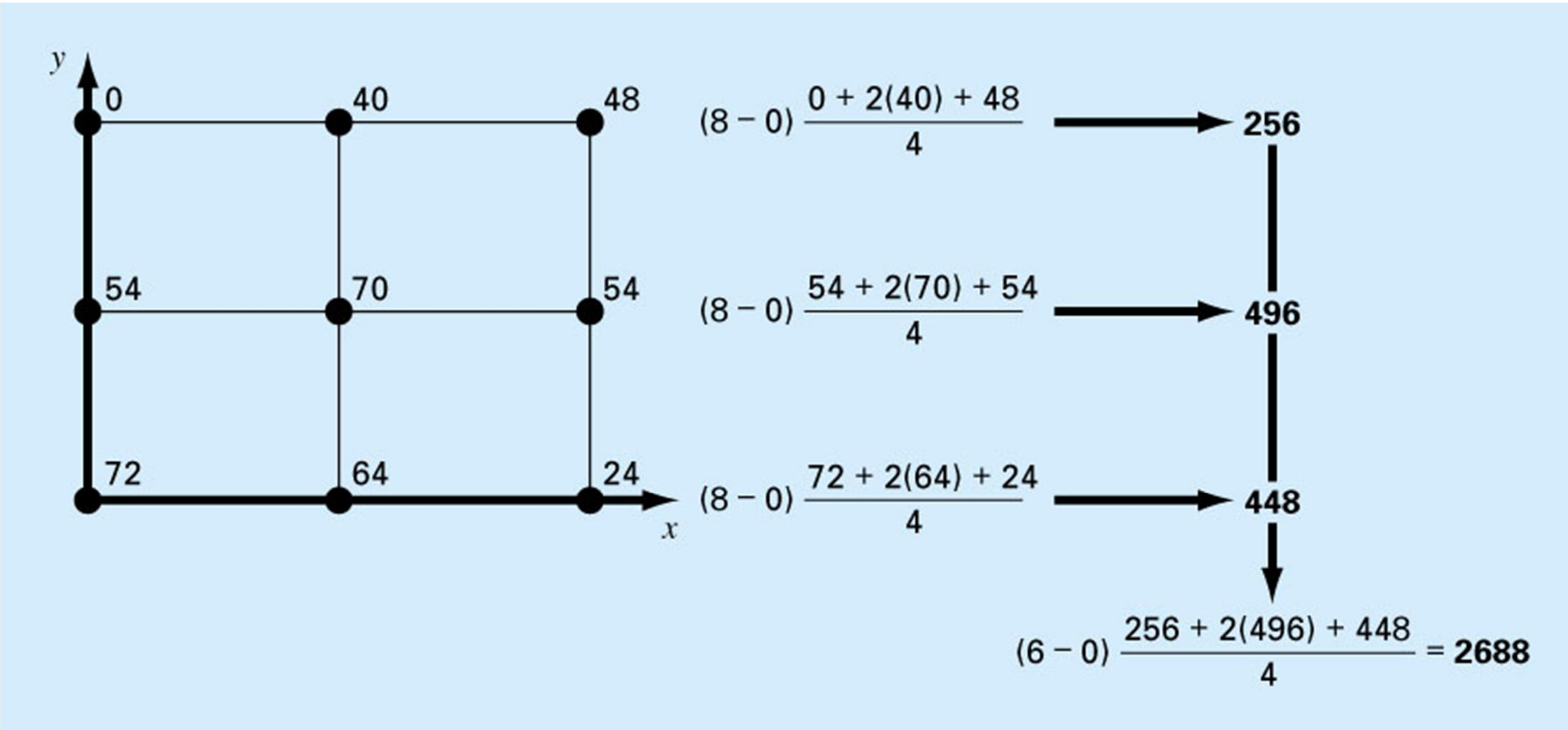
Solution

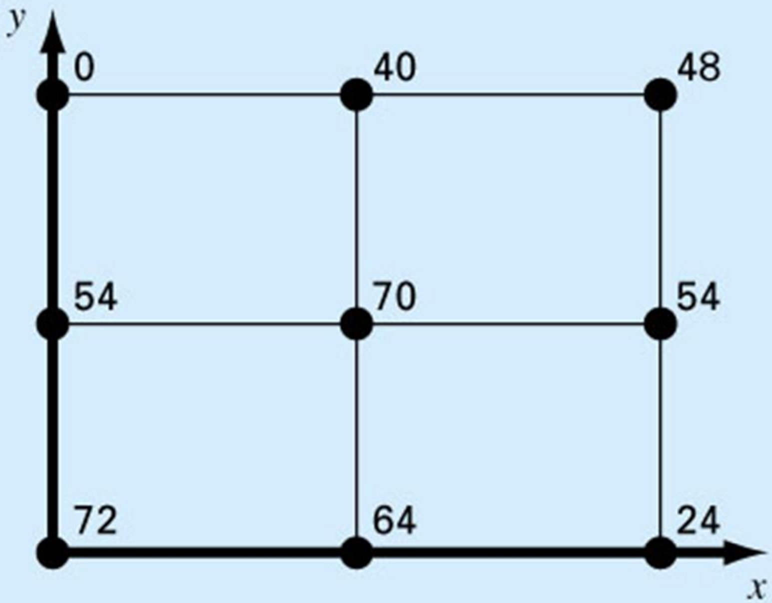
The function can also be evaluated analytically to yield a result of S8.66667.

Trapezoidal approximation



The average temperature = 2688/(6 x 8) = 56.

Simpson's 1/3 rule



$$\frac{(8-0)}{6}(0+4\times 40+48) \rightarrow 277.333$$

$$\frac{(8-0)}{6}(54+4\times 70+54) \rightarrow 517.333$$

$$\frac{(8-0)}{6}(72+4\times 64+24) \rightarrow 469.333$$

$$\frac{(6-0)}{6}(277.333+4\times 517.333+469.333) = 2816$$

The average temperature = 2816/(6 x 8) = 58.667.

H.W

Read Examples; 21.6 - 21.7

Solve problems; 21.3 - 21.10- 21.21