CSE 215: ASSIGNMENT ON PAPER REVIEW

Topic: Paper "Cop vs. Gambler"

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Abstract: This paper studies a variant of "Cop vs. Robber" problem related to discrete mathematics. This variant can be described as the Cop and the Gambler (or Robber) move simultaneously. Here the gambler can choose his position using a time-independent probability distribution and the cop can move at most 1 one edge out from his current location. The gambler tries to maximize the expected capture time and the cop tries the opposite. This paper shows that expected capture time for a connected n-vertex graph is exactly n with optimal play (if the probability distribution is known). It also analyzes the case of unknown probability distribution.

Introduction: This problem has some "ancestor" problems, like as "Cop vs. Robber" and "Hunter vs. Rabbit". But only this problem includes the idea of probability distribution or gambling. The capture time is defined as the number of moves including the capture moment. Such problem ideas are widely used in warfare. It is also used in designing anti-incursion program where the attackers' port-choice distribution can be known or unknown.

The paper is arranged sequentially to find the expected capture time for –

1. Cop vs. known Gambler on a tree
2. Cop vs. known Gambler on general graph
3. Cop vs. unknown Gambler on general graph.

1. Cop vs. Known Gambler on a tree:

This is the most core point, its idea is used in next two points too. It is proved that the cop can catch the gambler in O(n) expected time. To prove this, we can root the tree at node 1. Then the best way would be to travel from root to a node (to be decided) with a simple path.

Let P_v be probability of choosing node v at any moment and T_v is the expected time to catch from node v with optimal play. We assign m_v to each node defining the size of subtree. We also assign c_v the sum of probability of all nodes in subtree of node v. Now the claim turns to be the expected time T_v is at most $\frac{m_v}{c_v}$ (as the previous claim is equivalent to current claim for node 1). Using Bernoulli trial, expected time for any leaf *lf* is $T_{lf} = \frac{1}{P_{lf}} = \frac{m_{lf}}{c_{lf}}$, which becomes the base of inductional proof. Now, for any node u, if $T_u = \frac{1}{P_u} \leq \frac{m_u}{c_u}$, he can wait there and make no further move, supporting the claim. Otherwise, he can choose an incident node x of node u's subtree to achieve expected time $T_u \leq 1 + (1 - P_u)T_x$. And this turns out to be less than $\frac{m_u}{c_u}$ after some steps of calculation. Hence the proof stands for all nodes. This claim will be highly used afterwards cases.

2. Cop vs. known Gambler on general graph:

For this case, it claims the expected time is **exactly** n. It turns out to be surprising, but it can be proved with ease where both players move optimally. Firstly, it is true that the expected time is never greater than n. Because, the Cop can select a spanning tree of the graph and get the expected time at most n (shown in point 1). Now, the Gambler has to maximize it (or make it exactly n) by choosing an optimal probability distribution. Most convenient way is to choose $P_i = \frac{1}{n}$ for all possible i. With this way, Bernoulli trials with success probability $\frac{1}{n}$, making the expected time n irrespective of the Cop's strategy. Hence, the claim of point 2 is correct for optimal gameplay.

3. Cop vs. unknown Gambler on general graph:

This is more realistic case. When the Cop does not know the Gambler's probability to choose a node, he can achieve an expected time less than *2n* irrespective of the Gambler's move. To prove this, we again choose an arbitrary spanning tree of the given graph. The cop will perform a depth first search move by flipping coin at every round whether to go upward or downward. Here, once he moves upward from node *u* to node *v,* he will not move downward from node *v* to node *u* again. If he enters a leaf, he waits for an extra turn. In this order, every edge is visited at most 2 times, making the number of move to be at most $3n - 2$ (at most n-1 leaf is possible). For this specific movement, we count the probability to catch the Gambler.

The probability to fail to capture is $\prod_1^n (1 - P_i)^2$ since every node is visited at most 2 times. So probability to capture the Gambler is at most $1 - \prod_1^n (1 - P_i)^2$. This is maximized when $P_i = \frac{1}{n}$.

So, the probability gets bounded by $1 - \left(1 - \frac{1}{n}\right)^{2n} \geq 1 - \frac{1}{e^2}$, making $\frac{1}{1-\frac{1}{e^2}}$=1.157 rounds to be necessary to capture with this movement. As a result, the expect capture time is at most

$\frac{3n}{1-\frac{1}{e^2}} - \frac{3n}{2} \approx 2n.$

A lower bound for this movement can be found to be n, as the gambler can pick a vertex and hide in there. Hence, the value of expected time to capture lies between n and 2n. However, there can be a tighter bound.

Observation: This paper includes the proof and deep analysis of Cop vs. Gambler's expected time. It also cited some proofs from some other papers. Most notably, it conjectures that the tightest upper bound for 3rd case is $\frac{3n}{2}$, which is yet to prove.