CSE 203

Solution to Class Work on Time complexity and Big-Oh notation

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Problem 1:

Assume that each of the expressions below gives the processing time $T(n)$ spent by an algorithm for solving a problem of size n. Select the dominant term(s) having the steepest increase in n and specify the lowest Big-Oh complexity of each algorithm.

Expression	Dominant term(s)	$O(\ldots)$
$5 + 0.001n^3 + 0.025n$		
$500n + 100n^{1.5} + 50n \log_{10} n$		
$0.3n + 5n^{1.5} + 2.5 \cdot n^{1.75}$		
$n^2 \log_2 n + n(\log_2 n)^2$		
$n \log_3 n + n \log_2 n$		
$3 \log_8 n + \log_2 \log_2 \log_2 n$		
$100n + 0.01n^2$		
$0.01n + 100n^2$		
$2n + n^{0.5} + 0.5n^{1.25}$		
$0.01n \log_2 n + n(\log_2 n)^2$		
$100n \log_3 n + n^3 + 100n$		
$0.003 \log_4 n + \log_2 \log_2 n$		

Ans:

Expression	Dominant term(s)	$O(\ldots)$
$5 + 0.001n^3 + 0.025n$	$0.001n^3$	$O(n^3)$
$500n + 100n^{1.5} + 50n \log_{10} n$	$100n^{1.5}$	$O(n^{1.5})$
$0.3n + 5n^{1.5} + 2.5 \cdot n^{1.75}$	$2.5n^{1.75}$	$O(n^{1.75})$
$n^2 \log_2 n + n(\log_2 n)^2$	$n^2 \log_2 n$	$O(n^2 \log n)$
$n \log_3 n + n \log_2 n$	$n \log_3 n$, $n \log_2 n$	$O(n \log n)$
$3 \log_8 n + \log_2 \log_2 \log_2 n$	$3 \log_8 n$	$O(\log n)$
$100n + 0.01n^2$	$0.01n^2$	$O(n^2)$
$0.01n + 100n^2$	$100n^2$	$O(n^2)$
$2n + n^{0.5} + 0.5n^{1.25}$	$0.5n^{1.25}$	$O(n^{1.25})$
$0.01n \log_2 n + n(\log_2 n)^2$	$n(\log_2 n)^2$	$O(n(\log n)^2)$
$100n \log_3 n + n^3 + 100n$	n^3	$O(n^3)$
$0.003 \log_4 n + \log_2 \log_2 n$	$0.003 \log_4 n$	$O(\log n)$

Problem 2:

The statements below show some features of "Big-Oh" notation for the functions $f \equiv f(n)$ and $g \equiv g(n)$. Determine whether each statement is TRUE or FALSE and correct the formula in the latter case.

Statement	Is it TRUE or FALSE?	If it is FALSE then write the correct formula
Rule of sums: $O(f+g) = O(f) + O(g)$		
Rule of products: $O(f \cdot g) = O(f) \cdot O(g)$		
Transitivity: if $g = O(f)$ and $h = O(f)$ then $g = O(h)$		
$5n + 8n^2 + 100n^3 = O(n^4)$		
$5n+8n^2+100n^3 = O(n^2 \log n)$		

Ans:

Statement	Is it TRUE or FALSE?	If it is FALSE then write the correct formula
Rule of sums: $O(f+g) = O(f) + O(g)$	FALSE	$O(f+g) =$ $\max\{O(f), O(g)\}$
Rule of products: $O(f \cdot g) = O(f) \cdot O(g)$	TRUE	
Transitivity: if $g = O(f)$ and $h = O(f)$ then $g = O(h)$	FALSE	if $g = O(f)$ and $f = O(h)$ then $g = O(h)$
$5n + 8n^2 + 100n^3 = O(n^4)$	TRUE	
$5n + 8n^2 + 100n^3 = O(n^2 \log n)$	FALSE	$5n + 8n^2 + 100n^3 = O(n^3)$

Problem 3:

Algorithms **A** and **B** spend exactly $T_{\mathrm{A}}(n) = 0.1n^2 \log_{10} n$ and $T_{\mathrm{B}}(n) = 2.5n^2$ microseconds, respectively, for a problem of size n. Choose the algorithm, which is better in the Big-Oh sense, and find out a problem size n_0 such that for any larger size $n > n_0$ the chosen algorithm outperforms the other. If your problems are of the size $n \leq 10^9$, which algorithm will you recommend to use?

Ans:

In the Big-Oh sense, the algorithm **B** is better. It outperforms the algorithm **A** when $T_{\mathrm{B}}(n) \leq T_{\mathrm{A}}(n)$, that is, when $2.5n^2 \leq 0.1n^2 \log_{10} n$. This inequality reduces to $\log_{10} n \geq 25$, or $n \geq n_0 = 10^{25}$. If $n \leq 10^9$, the algorithm of choice is **A**.

Problem 4:

Assume that the array a contains n values, that the method `randomValue` takes constant number c of computational steps to produce each output value, and that the method `goodSort` takes $n \log n$ computational steps to sort the array. Determine the Big-Oh complexity for the following fragments of code taking into account only the above computational steps:

```
for( i = 0; i < n; i++ ) {
  for( j = 0; j < n; j++ )
    a[ j ] = randomValue( i );
  goodSort( a );
}
```

Ans:

The inner loop has linear complexity cn, but the next called method is of higher complexity $n \log n$. Because the outer loop is linear in n, the overall complexity of this piece of code is $n^2 \log n$.

Problem 5:

Work out the computational complexity (in the "Big-Oh" sense) of the following piece of code and explain how you derived it using the basic features of the "Big-Oh" notation:

```
for( int bound = 1; bound <= n; bound *= 2 ) {
   for( int i = 0; i < bound; i++ ) {
      for( int j = 0; j < n; j += 2 ) {
         ... // constant number of operations
      }
      for( int j = 1; j < n; j *= 2 ) {
         ... // constant number of operations
      }
   }
}
```

Ans:

The first and second successive innermost loops have $O(n)$ and $O(\log n)$ complexity, respectively. Thus, the overall complexity of the innermost part is $O(n)$. The outermost and middle loops have complexity $O(\log n)$ and $O(n)$, so a straightforward (and valid) solution is that the overall complexity is $O(n^2 \log n)$.