Dynamic Programming:

Longest Common Subsequence

Longest Common Subsequence

- Given two sequences

  $X = \langle x_1, x_2, \ldots, x_m \rangle$

  $Y = \langle y_1, y_2, \ldots, y_n \rangle$

  A subsequence of a given sequence is just the given sequence with zero or more elements left out.

- A common subsequence $Z = \langle z_1, z_2, \ldots, z_k \rangle$ of X *and* Y
  - Z is a subsequence of both X and Y
- Example:

  X = A **B** **C** **B** D **A** B

  Y = **B** D **C** A **B** **A**

Goal: Find the Longest Common Subsequence (LCS)

An Impractical LCS Algorithm

- Brute-force algorithm: For every subsequence of x, check if it is a subsequence of y
  - How many subsequences of x are there?
  - What will be the running time of the brute-force algorithm?

- 2^m subsequences of x to check against n elements of y
  - Running time: $O(n\,2^m)$

Optimal Substructure Property of LCS

- The LCS problem has an *optimal substructure* property
  - solutions of subproblems are parts of the final solution
  - Subproblems: LCS of pairs of *prefixes* of X and Y
- An LCS of two sequences contains within it an LCS of prefixes of the two sequences.

- Given a sequence $X = \langle x_1, x_2, \ldots, x_m \rangle$, we define the ith prefix of X as $X_i = \langle x_1, x_2, \ldots, x_i \rangle$

  Example:

  X = ABCBDABBDCAB

  X_5 = ABCBD

  X_7 = ABCBDAB

Optimal Substructure Property of LCS

Theorem:

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be any sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

- If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
- If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
- If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

- *Proof ?*

- The Theorem tells us that an LCS of two sequences contains within it an LCS of prefixes of the two sequences.

  Thus the LCS problem has an optimal substructure property.

Overlapping Subproblem Property of LCS

Theorem:

Let $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$ be any sequences, and let $Z = \langle z_1, z_2, \ldots, z_k \rangle$ be any LCS of X and Y.

- If $x_m = y_n$, then $z_k = x_m = y_n$ and Z_{k-1} is an LCS of X_{m-1} and Y_{n-1}.
- If $x_m \neq y_n$, then $z_k \neq x_m$ implies that Z is an LCS of X_{m-1} and Y.
- If $x_m \neq y_n$, then $z_k \neq y_n$ implies that Z is an LCS of X and Y_{n-1}.

- The Theorem tells us that
  - To find an LCS of X and Y, we may need to find the LCSs of X and Y_{n-1} and of X_{m-1} and Y. But each of these subproblems has the subsubproblem of finding an LCS of X_{m-1} and Y_{n-1}.

  Thus the LCS problem has the overlapping subproblem property.

- The number of distinct subproblems: $O(mn)$.

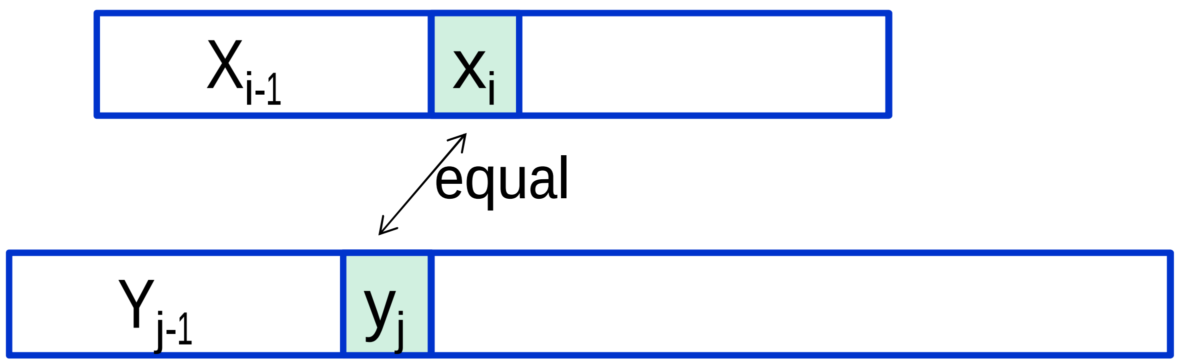
Develop a Recursive Solution

- Define $c[i, j]$ to be the length of an LCS of the sequences X_i and Y_j.
  - Goal: Find $c[m, n]$
  - Basis: $c[i, j] = 0$ if either $i = 0$ or $j = 0$
  - Recursion: How to define $c[i, j]$ recursively ?
- Finding an LCS of $X = \langle x_1, x_2, \ldots, x_m \rangle$ and $Y = \langle y_1, y_2, \ldots, y_n \rangle$
  - If $x_m = y_n$, then we must find an LCS of X_{m-1} and Y_{n-1}.
    - Appending $x_m = y_n$ to this LCS yields an LCS of X and Y.
  - If $x_m \neq y_n$, then we must solve two subproblems:
    - Finding an LCS of X_{m-1} and Y
    - Finding an LCS of X and Y_{n-1}
    - Whichever of these two LCSs is longer is an LCS of X and Y.
- The recursive formula is

$$c[i, j] = \begin{cases} c[i-1, j-1] + 1 & \text{if } i, j > 0 \text{ and } x[i] = y[j], \\ \max\{c[i, j-1], c[i-1, j]\} & \text{if } i, j > 0 \text{ and } x[i] \neq y[j] \end{cases}$$

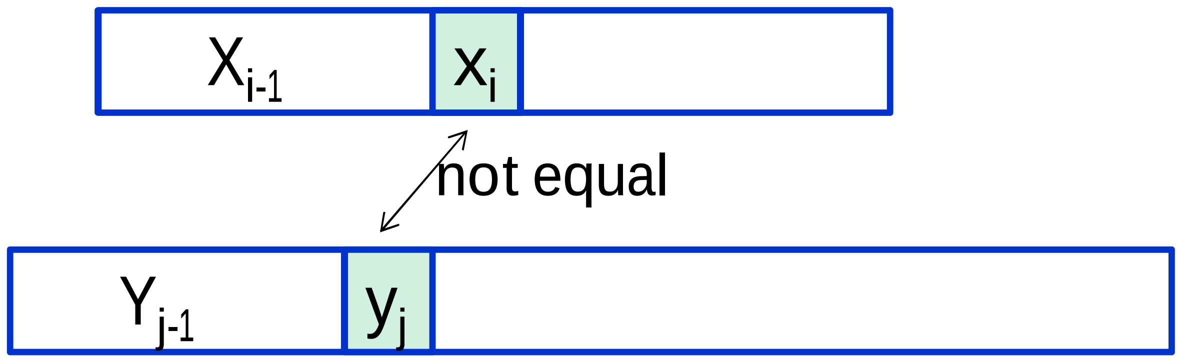
Develop a Recursive Solution

- Case 1: $x_i = y_j$
  - Recursively find LCS of X_{i-1} and Y_{j-1} and append x_i
  - So $c[i, j] = c[i-1, j-1] + 1$ if $i, j > 0$, and $x_i = y_j$

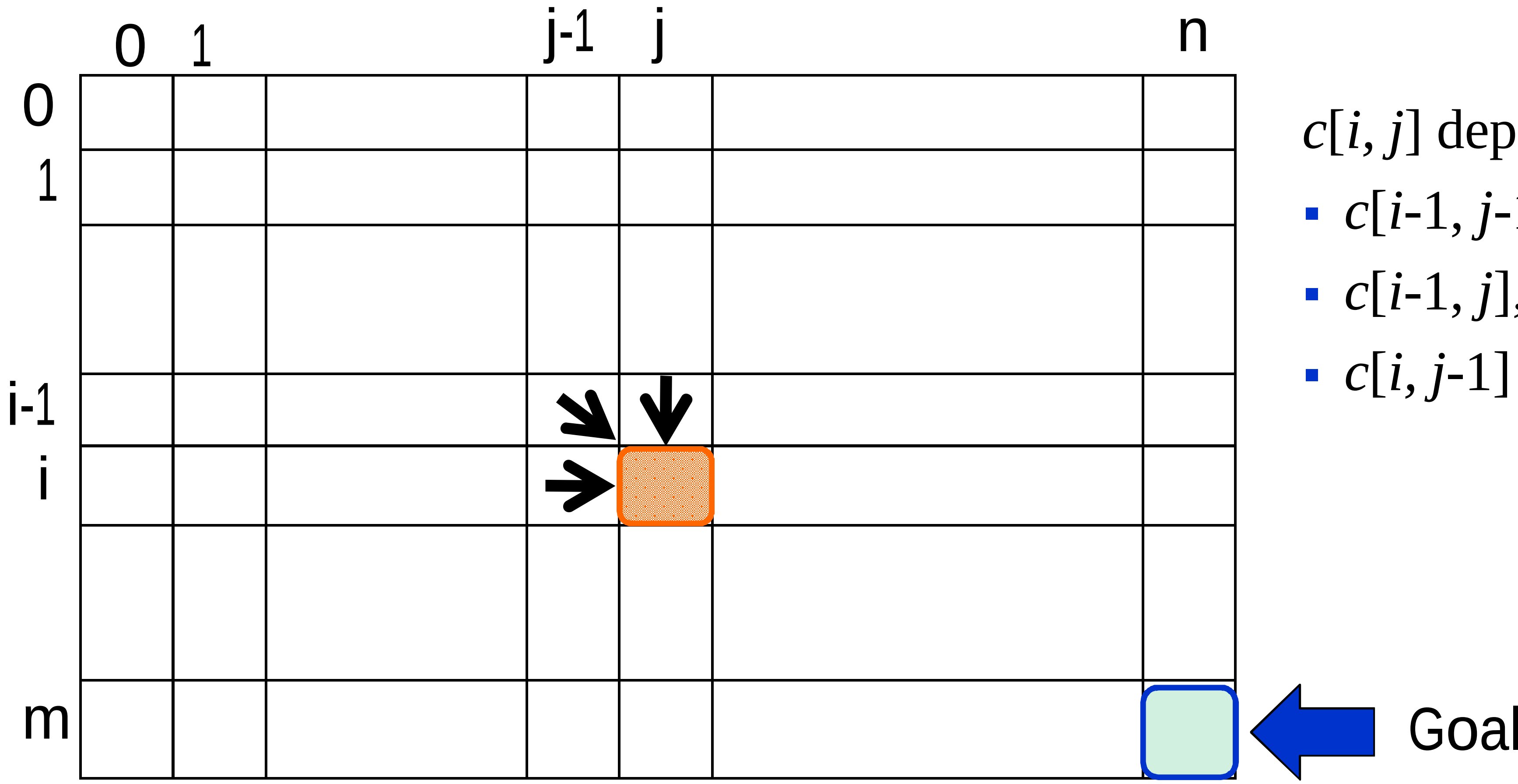


Develop a Recursive Solution

- Case 2: $x_i \neq y_j$
  - Recursively find LCS of X_{i-1} and Y_j
  - Recursively find LCS of X_i and Y_{j-1}
  - Take the longer one
  - So $c[i, j] = \max\{c[i, j-1], c[i-1, j]\}$ if $i, j > 0$, and $x_i \neq y_j$



Dependencies among Subproblems



$c[i, j]$ depends on

- $c[i\text{-}1, j\text{-}1]$,
- $c[i\text{-}1, j]$, and
- $c[i, j\text{-}1]$

- An order for solving the subproblems (*i.e.*, filling in the array) that respects the dependencies is row major order:
  - do the rows from top to bottom
  - inside each row, go from left to right

Develop a Recursive Solution

```
LCS-LENGTH(X, Y)
m ← length[X]
n ← length[Y]
for i ← 1 to m
        do c[i, 0] ← 0
for j ← 0 to n
        do c[0, j] ← 0
for i ← 1 to m
        do for j ← 1 to n
                do if x_i = y_j
                        then c[i, j] ← c[i − 1, j − 1] + 1
                             b[i, j] ← "↖"
                        else if c[i − 1, j] ≥ c[i, j − 1]
                                then c[i, j] ← c[i − 1, j]
                                     b[i, j] ← "↑"
                                else c[i, j] ← c[i, j − 1]
                                     b[i, j] ← "←"
return c and b
```

- The algorithm calculates the values of each entry of the array $c[m, n]$.
- Each $c[i, j]$ is calculated in constant time, and there are $m \cdot n$ elements in the array.
- So the running time is $O(m \cdot n)$.

LCS Example

We'll see how LCS algorithm works on the following example:

X = ABCG

Y = BDCAG

LCS(X, Y) = BCG

X = A **B** **C** **G**

Y = **B** D **C** A **G**

LCS Example

ABCG
BDCAG

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i						
1	**A**						
2	**B**						
3	**C**						
4	**G**						

X = ABCG; $m = |X| = 4$

Y = BDCAG; $n = |Y| = 5$

Allocate array: $c[5, 4]$

LCS Example

ABCG
BDCAG

i	j	0	1	2	3	4	5
		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**					
2	**B**	**0**					
3	**C**	**0**					
4	**G**	**0**					

for $i = 0$ to m $\quad c[i, 0] = 0$

for $j = 1$ to n $\quad c[0, j] = 0$

LCS Example

ABCG

BDCAG

		j	0	1	2	3	4	5
i		y_j		B	D	C	A	G
0	x_i		0	0	0	0	0	0
1	A		0	0				
2	B		0					
3	C		0					
4	G		0					

if ($x_i == y_j$)

c[i, j] = c[i-1, j-1] + 1

else c[i, j] = max(c[i-1, j], c[i, j-1])

LCS Example

ABCG
BDCAG

i	j	0	1	2	3	4	5
		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**		
2	**B**	**0**					
3	**C**	**0**					
4	**G**	**0**					

if ($x_i == y_j$)
$c[i, j] = c[i-1, j-1] + 1$
else $c[i, j] = \max(c[i-1, j], c[i, j-1])$

LCS Example

ABCG
BDCAG

i		j	0	1	2	3	4	5
		y_j		**B**	**D**	**C**	**A**	**G**
0	x_i		**0**	**0**	**0**	**0**	**0**	**0**
1	**A**		**0**	**0**	**0**	**0**	**1**	
2	**B**		**0**					
3	**C**		**0**					
4	**G**		**0**					

if ($x_i == y_j$)
  c[i, j] = c[i-1, j-1] + 1
else c[i, j] = max(c[i-1, j], c[i, j-1])

LCS Example

ABCG
BDCAG

		j	0	1	2	3	4	5
i			y_j	B	D	C	A	G
0	x_i		0	0	0	0	0	0
1	A		0	0	0	0	1 →	1
2	B		0					
3	C		0					
4	G		0					

if ($x_i == y_j$)
c[i, j] = c[i-1, j-1] + 1
else c[i, j] = max(c[i-1, j], c[i, j-1])

LCS Example

ABCG
BDCAG

i		j	0	1	2	3	4	5
		y_j		**B**	**D**	**C**	**A**	**G**
0	x_i		**0**	**0**	**0**	**0**	**0**	**0**
1	**A**		**0**	**0**	**0**	**0**	**1**	**1**
2	**B**		**0**	**1**				
3	**C**		**0**					
4	**G**		**0**					

if ($x_i == y_j$)
    $c[i, j] = c[i-1, j-1] + 1$
else $c[i, j] = \max(c[i-1, j], c[i, j-1])$

LCS Example

ABCG

BDCAG

		j	0	1	2	3	4	5
i			y_j	B	D	C	A	G
0	x_i		0	0	0	0	0	0
1	A		0	0	0	0	1	1
2	B		0	1	1	1	1	
3	C		0					
4	G		0					

if ($x_i == y_j$)

$c[i, j] = c[i-1, j-1] + 1$

else $c[i, j] = \max(c[i-1, j], c[i, j-1])$

LCS Example

ABCG
BDCAG

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**1**
3	**C**	**0**					
4	**G**	**0**					

if ($x_i == y_j$)
    c[i, j] = c[i-1, j-1] + 1
else c[i, j] = max(c[i-1, j], c[i, j-1])

LCS Example

ABCG
BDCAG

	j	0	1	2	3	4	5
i		y_j	B	D	C	A	G
0	x_i	0	0	0	0	0	0
1	A	0	0	0	0	1	1
2	B	0	1	1	1	1	1
3	C	0	1	1			
4	G	0					

if ($x_i == y_j$)
    c[i, j] = c[i-1, j-1] + 1
else c[i, j] = max(c[i-1, j], c[i, j-1])

LCS Example

ABCG
BDCAG

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**1**
3	**C**	**0**	**1**	**1**	**2**		
4	**G**	**0**					

if ($x_i == y_j$)
    c[i, j] = c[i-1, j-1] + 1
else c[i, j] = max(c[i-1, j], c[i, j-1])

LCS Example

ABCG
BDCAG

i		j	0	1	2	3	4	5
		y_j		**B**	**D**	**C**	**A**	**G**
0	x_i		**0**	**0**	**0**	**0**	**0**	**0**
1	**A**		**0**	**0**	**0**	**0**	**1**	**1**
2	**B**		**0**	**1**	**1**	**1**	**1**	**1**
3	**C**		**0**	**1**	**1**	**2**	**2**	**2**
4	**G**		**0**					

if ($x_i == y_j$)

$c[i, j] = c[i-1, j-1] + 1$

else $c[i, j] = \max(c[i-1, j], c[i, j-1])$

LCS Example

ABCG
BDCAG

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**1**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**G**	**0**	**1**				

if ($x_i == y_j$)
$c[i, j] = c[i-1, j-1] + 1$
else $c[i, j] = \max(c[i-1, j], c[i, j-1])$

LCS Example

ABCG
BDCAG

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**1**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**G**	**0**	**1**	**1**	**2**	**2**	

if ($x_i == y_j$)
c[i, j] = c[i-1, j-1] + 1
else c[i, j] = max(c[i-1, j], c[i, j-1])

LCS Example

ABCG
BDCAG

i		j	0	1	2	3	4	5
		y_j		**B**	**D**	**C**	**A**	**G**
0	x_i		**0**	**0**	**0**	**0**	**0**	**0**
1	**A**		**0**	**0**	**0**	**0**	**1**	**1**
2	**B**		**0**	**1**	**1**	**1**	**1**	**1**
3	**C**		**0**	**1**	**1**	**2**	**2**	**2**
4	**G**		**0**	**1**	**1**	**2**	**2**	**3**

if ($x_i == y_j$)
  $c[i, j] = c[i-1, j-1] + 1$
else $c[i, j] = \max(c[i-1, j], c[i, j-1])$

Another LCS Example

	j	0	1	2	3	4	5	6
i		y_j	B	D	C	A	B	A
0	x_i	0	0	0	0	0	0	0
1	A	0	↑ 0	↑ 0	↑ 0	↖ 1	← 1	↖ 1
2	B	0	↖ 1	← 1	← 1	↑ 1	↖ 2	← 2
3	C	0	↑ 1	↑ 1	↖ 2	← 2	↑ 2	↑ 2
4	B	0	↖ 1	↑ 1	↑ 2	↑ 2	↖ 3	← 3
5	D	0	↑ 1	↖ 2	↑ 2	↑ 2	↑ 3	↑ 3
6	A	0	↑ 1	↑ 2	↑ 2	↖ 3	↑ 3	↖ 4
7	B	0	↖ 1	↑ 2	↑ 2	↑ 3	↖ 4	↑ 4

How to Find Actual LCS

- So far, we have just found the *length* of LCS, but not LCS itself.
- We can modify this algorithm to make it output an LCS of X and Y.
- Each $c[i, j]$ depends on $c[i\text{-}1, j\text{-}1]$, or $c[i\text{-}1, j]$ and $c[i, j\text{-}1]$.
- For each $c[i, j]$ we can say how it was acquired.

2	2
2	3

For example, here
$c[i, j] = c[i\text{-}1, j\text{-}1] +1 = 2+1=3$

How to Find Actual LCS

- Remember that

$$c[i,j]=\begin{cases} c[i-1,j-1]+1 & \text{if } x[i]=y[j], \\ \max(c[i,j-1],c[i-1,j]) & \text{otherwise} \end{cases}$$

- So we can start from $c[m, n]$ and go backwards
- Whenever $c[i, j] = c[i\text{-}1, j\text{-}1]+1$, remember $x[i]$, because $x[i]$ is a part of LCS
- When i=0 or j=0 (*i.e.* we reached the beginning), output remembered letters in reverse order

Finding LCS: Example

i		j = 0	1	2	3	4	5
		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**G**	**0**	**1**	**1**	**2**	**2**	**3**

Finding LCS: Example

	j	0	1	2	3	4	5
i		y_j	**B**	**D**	**C**	**A**	**G**
0	x_i	**0**	**0**	**0**	**0**	**0**	**0**
1	**A**	**0**	**0**	**0**	**0**	**1**	**1**
2	**B**	**0**	**1**	**1**	**1**	**1**	**2**
3	**C**	**0**	**1**	**1**	**2**	**2**	**2**
4	**G**	**0**	**1**	**1**	**2**	**2**	**3**

LCS (reversed order): **G C B**

LCS (straight order): **B C G**

Finding LCS: Algorithm

```
PRINT-LCS(b, X, i, j)
if i = 0 or j = 0
    then return
if b[i, j] = "↖"
    then PRINT-LCS(b, X, i − 1, j − 1)
        print x_i
elseif b[i, j] = "↑"
    then PRINT-LCS(b, X, i − 1, j)
else PRINT-LCS(b, X, i, j − 1)
```

Trace backwards from $b[m, n]$