Algorithms: Greedy Method

Minimum Spanning Tree

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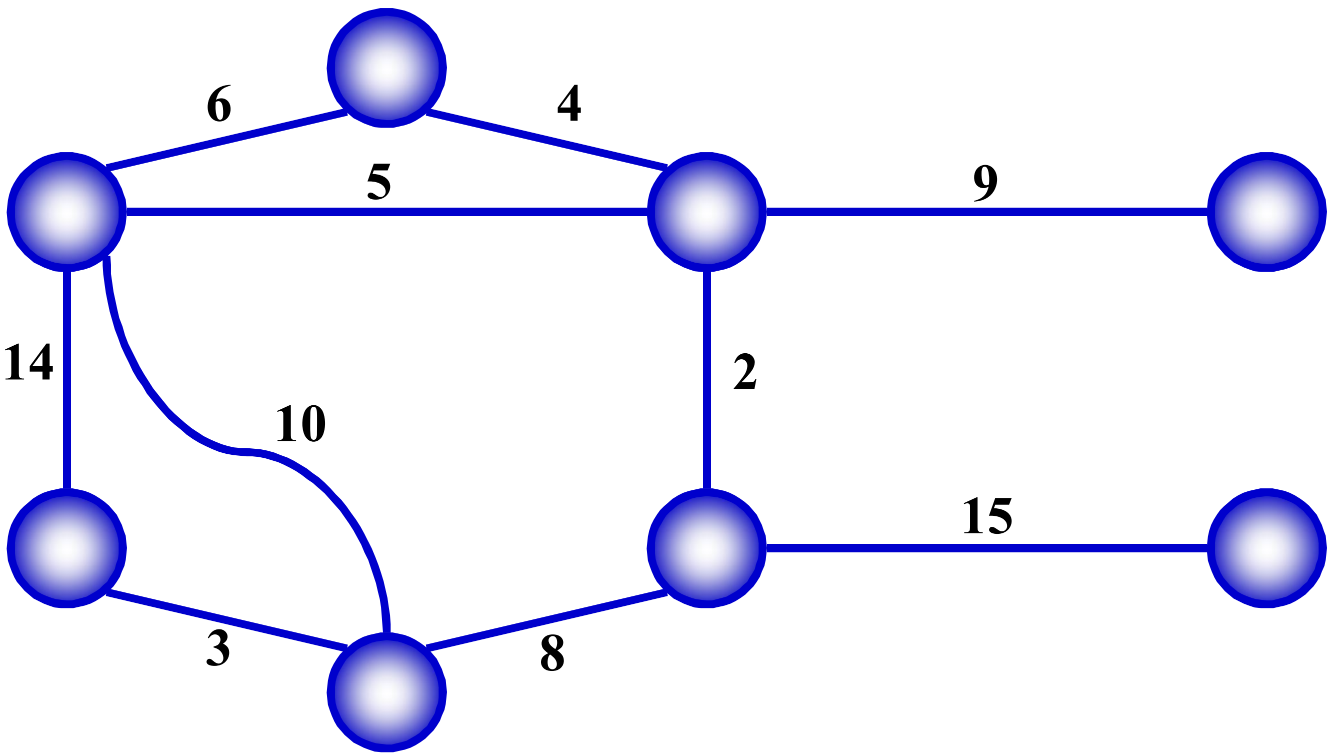
Greedy Algorithms: Principles

- A greedy algorithm works in phases.
- At each phase:
  - You take the best you can get right now, without regard for future consequences.
  - You hope that by choosing a local optimum at each step, you will end up at a global optimum.

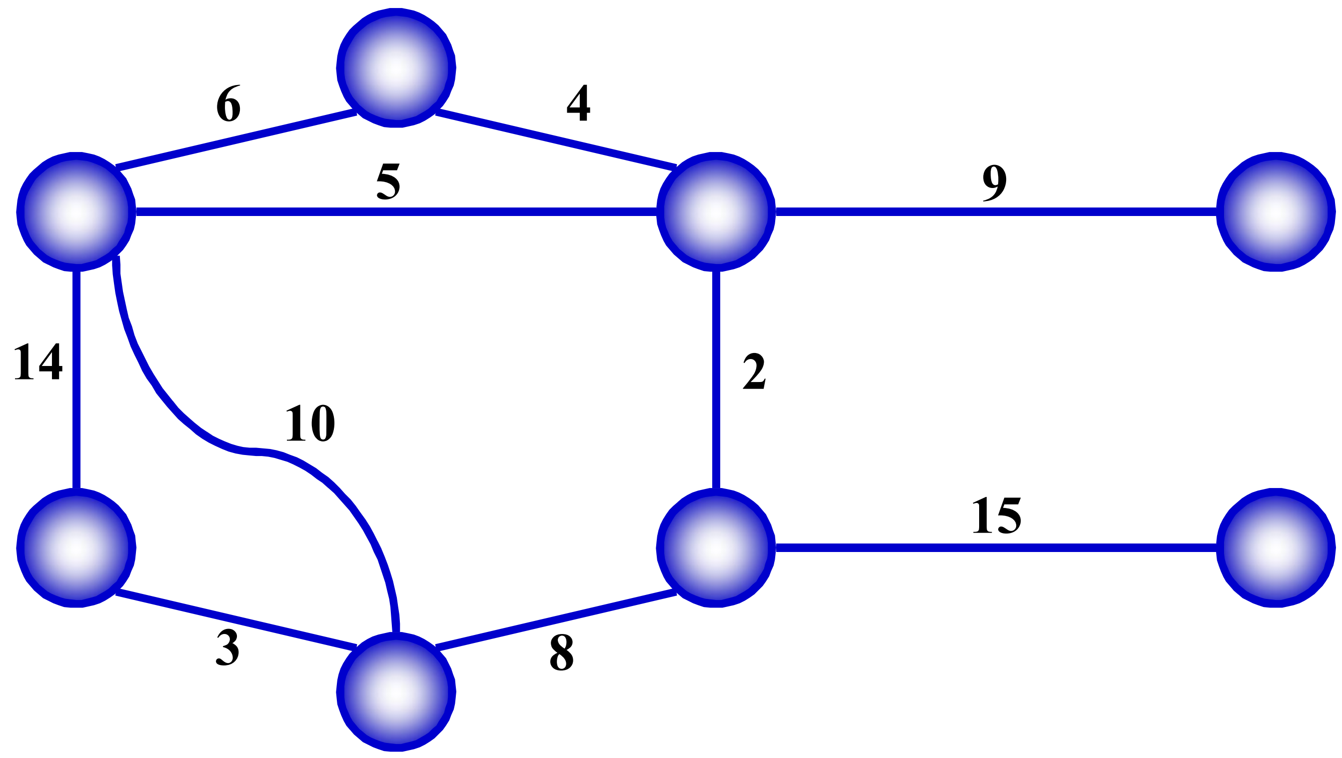
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph:



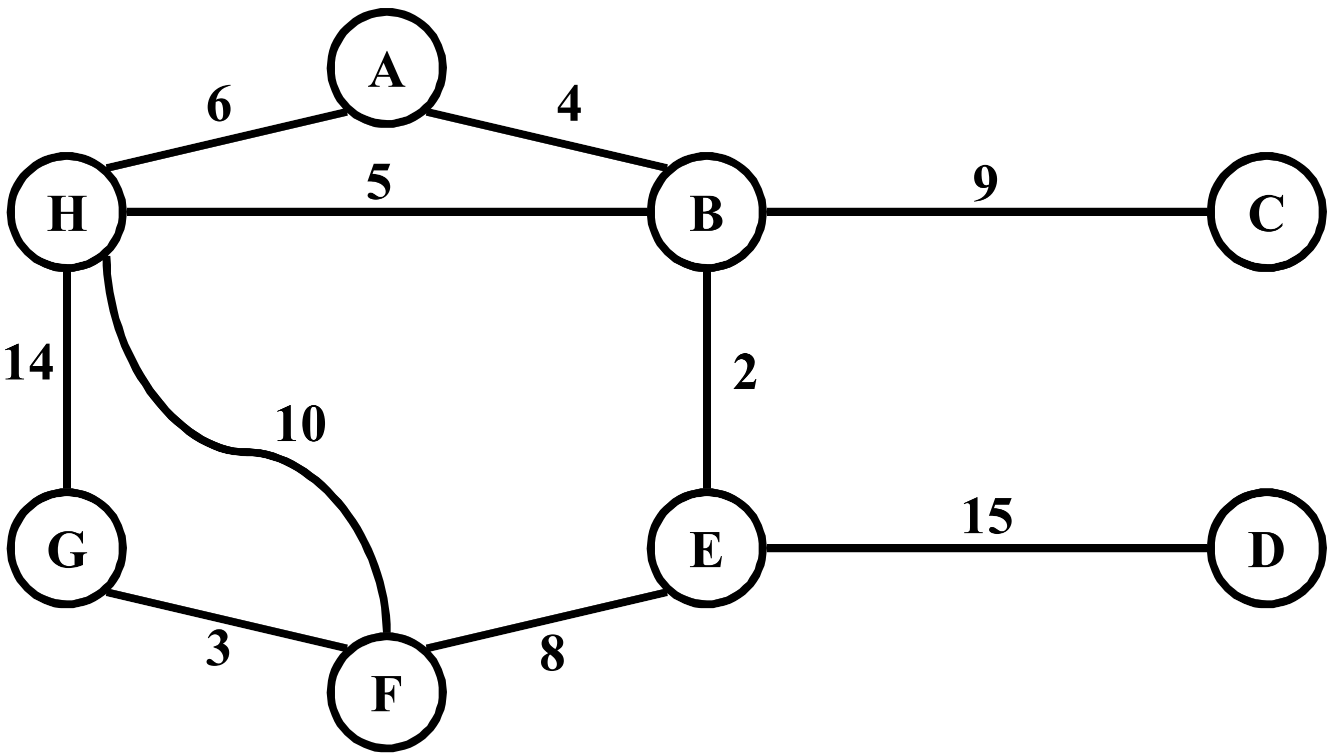
Minimum Spanning Tree

- Problem: given a connected, undirected, weighted graph, find a *spanning tree* using edges that minimize the total weight



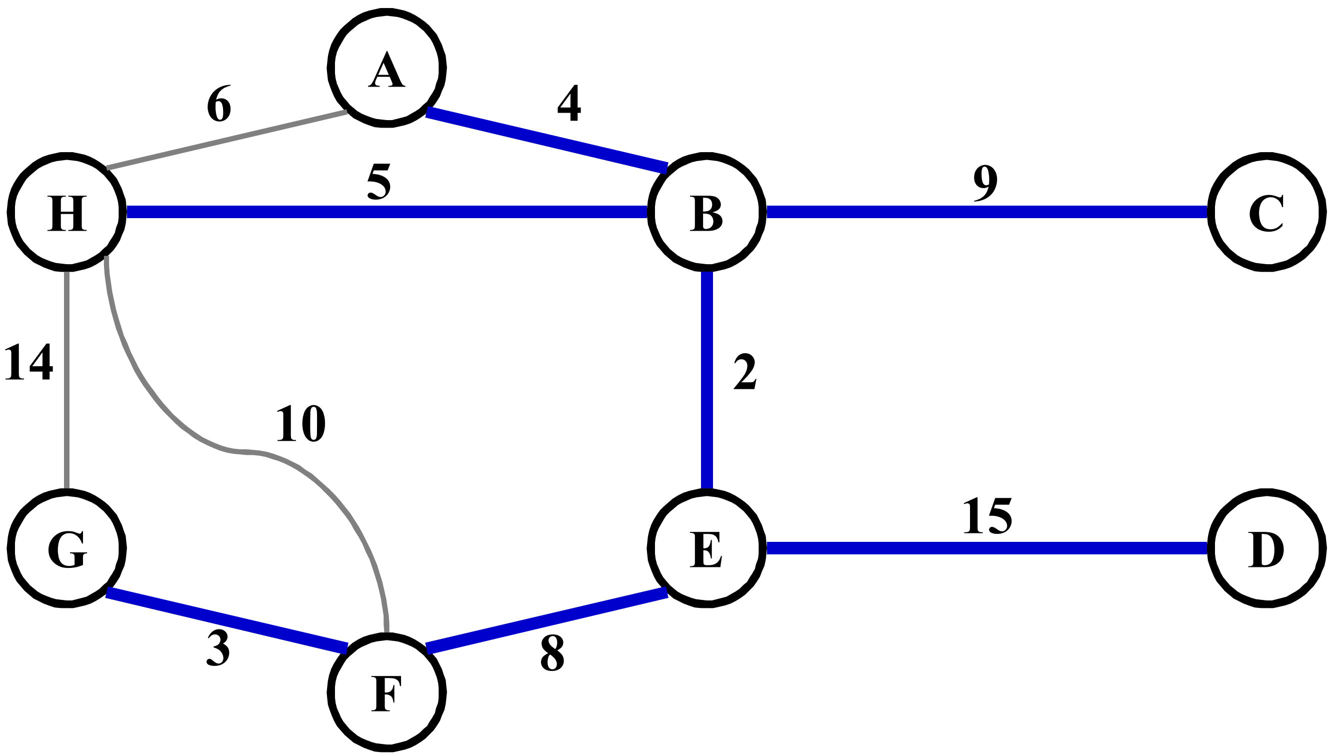
Minimum Spanning Tree

- *Which edges form the minimum spanning tree (MST) of the graph as shown below?*



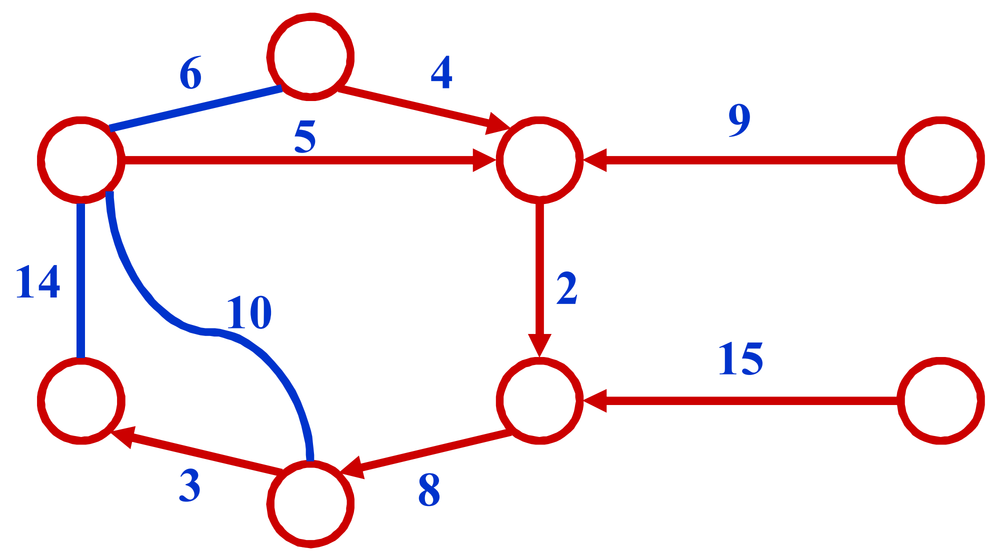
Minimum Spanning Tree

- Answer:



Minimum Spanning Tree

- MSTs satisfy the *optimal substructure* property: an optimal minimum spanning tree is composed of optimal minimum spanning subtrees
  - Let T be an MST of G with an edge (u, v) in the middle
  - Removing (u, v) partitions T into two trees T_1 and T_2
  - Claim: T_1 is an MST of $G_1 = (V_1, E_1)$, and T_2 is an MST of $G_2 = (V_2, E_2)$
    (*Do V_1 and V_2 share vertices? Why?*)
  - Proof: $w(T) = w(u,v) + w(T_1) + w(T_2)$
    (There can't be a better tree than T_1 or T_2. Then T would be suboptimal)



Prim's Algorithm

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        for each v ∈ Adj[u]
            if (v ∈ Q and w(u,v) < key[v])
                p[v] = u;
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```

This greedy algorithm was first developed in 1930 by Czech mathematician Vojtěch Jarník and later rediscovered and republished by computer scientists Robert C. Prim in 1957 and E. W. Dijkstra in 1959.
Therefore, it is also sometimes called the Jarník's algorithm, the Prim–Jarník algorithm, or the Prim–Dijkstra algorithm.

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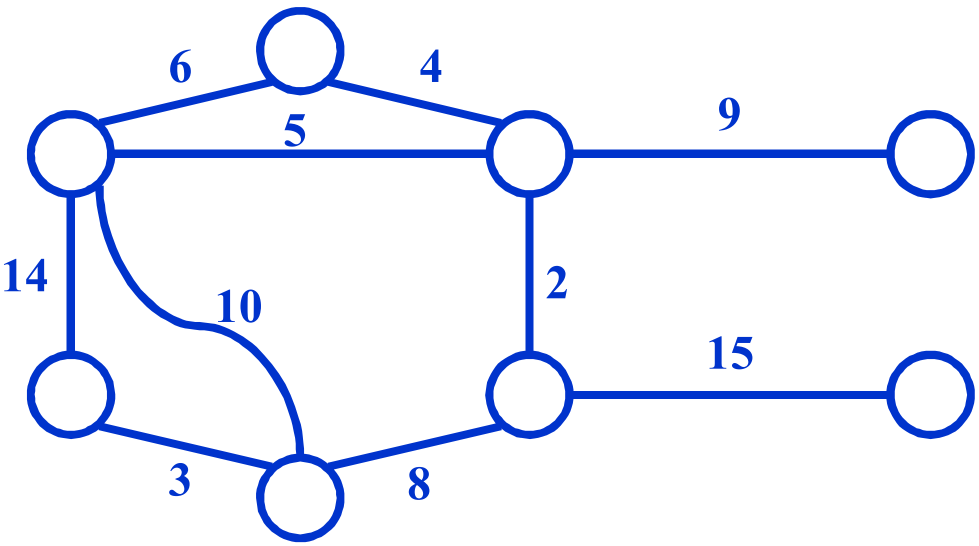

Run on example graph

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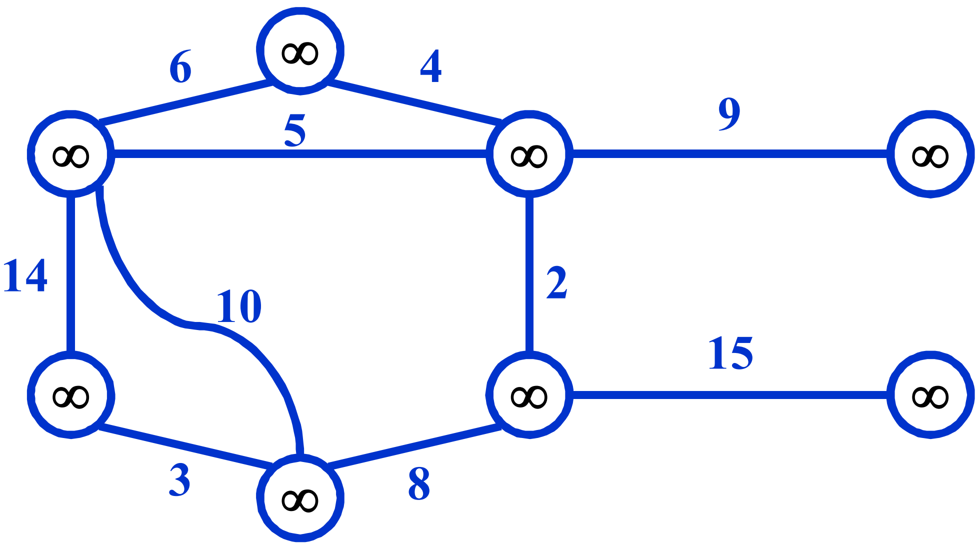

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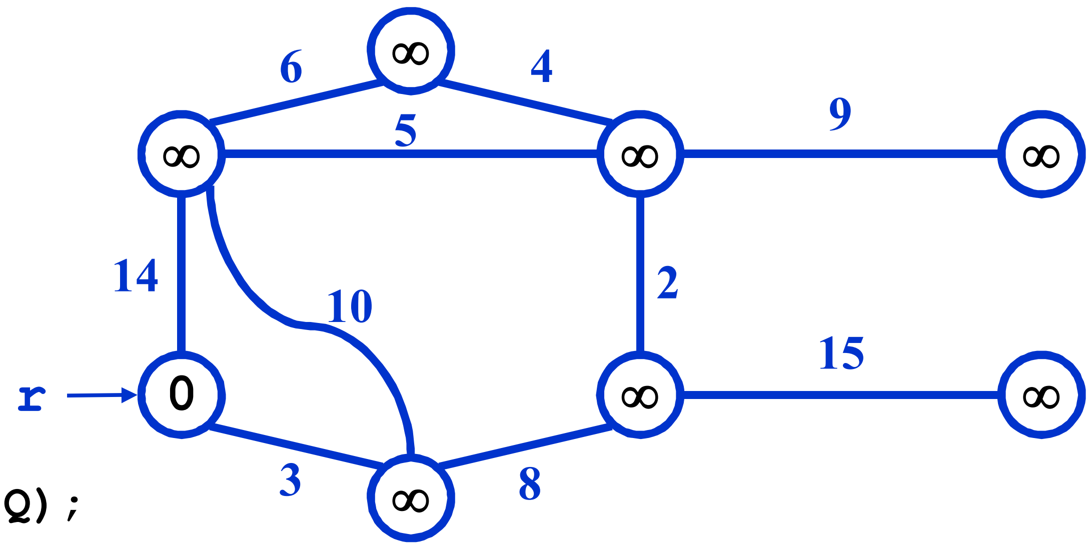

Pick a start vertex r

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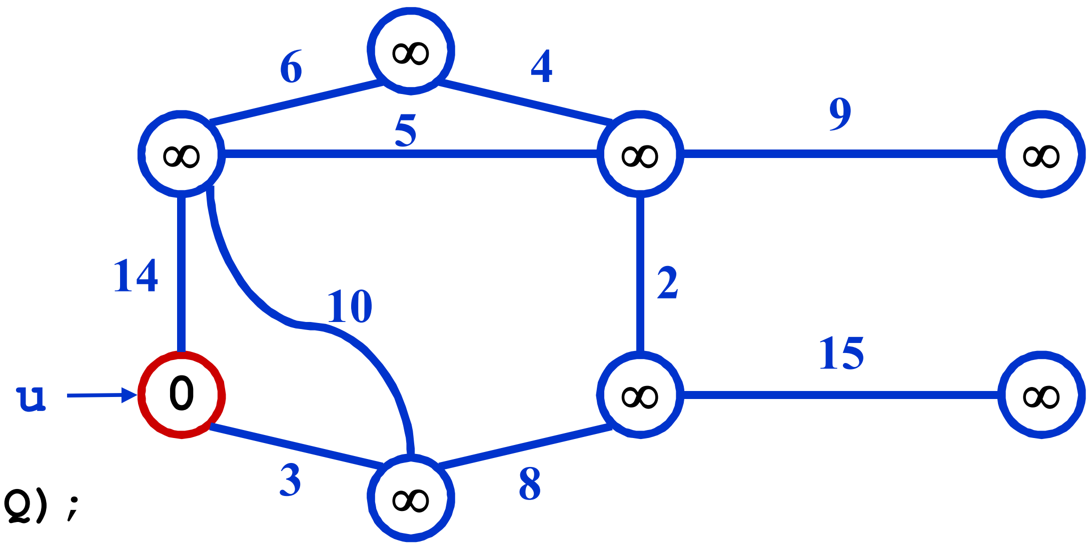

Red vertices have been removed from Q

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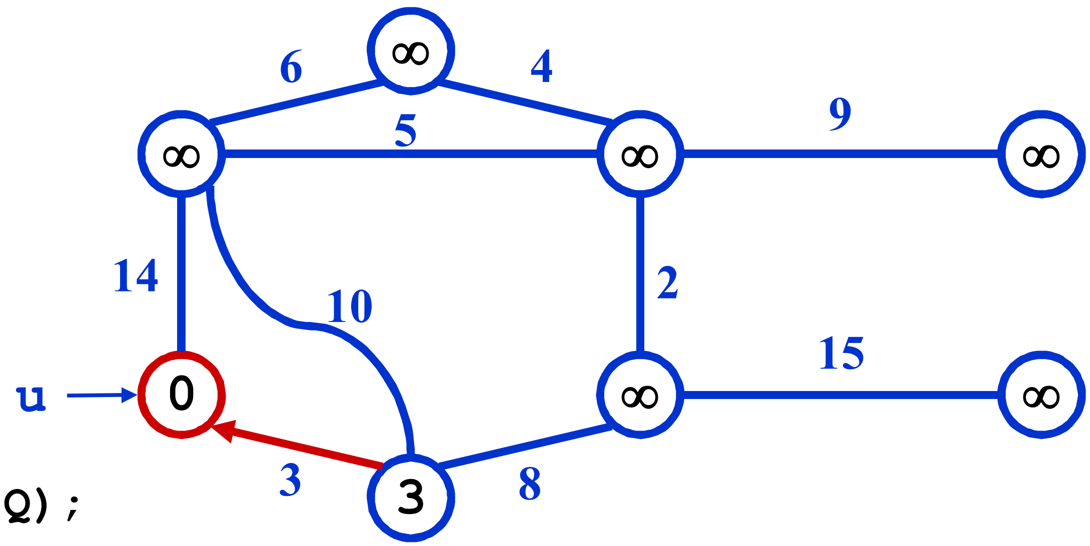

Red arrows indicate parent pointers

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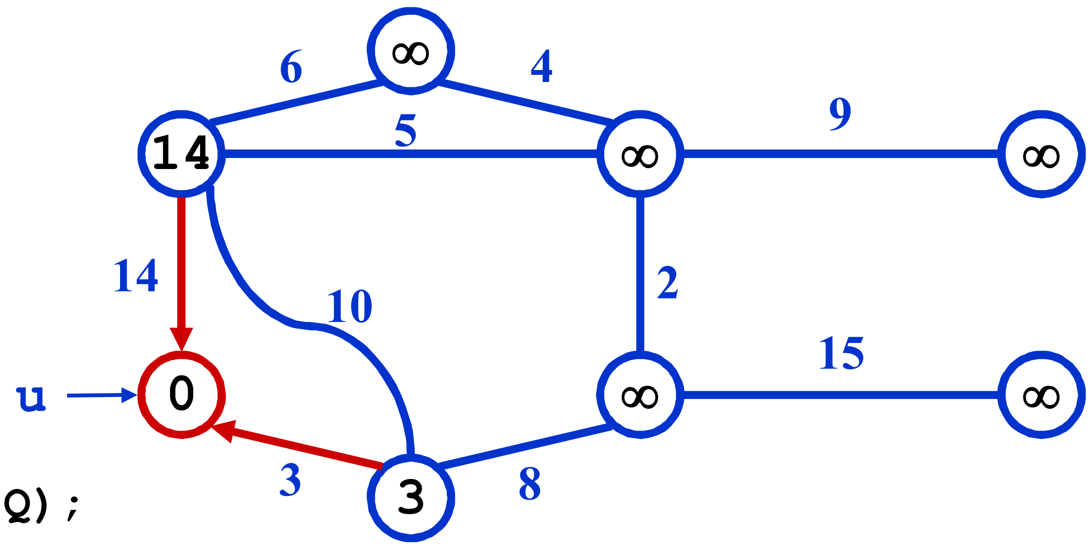

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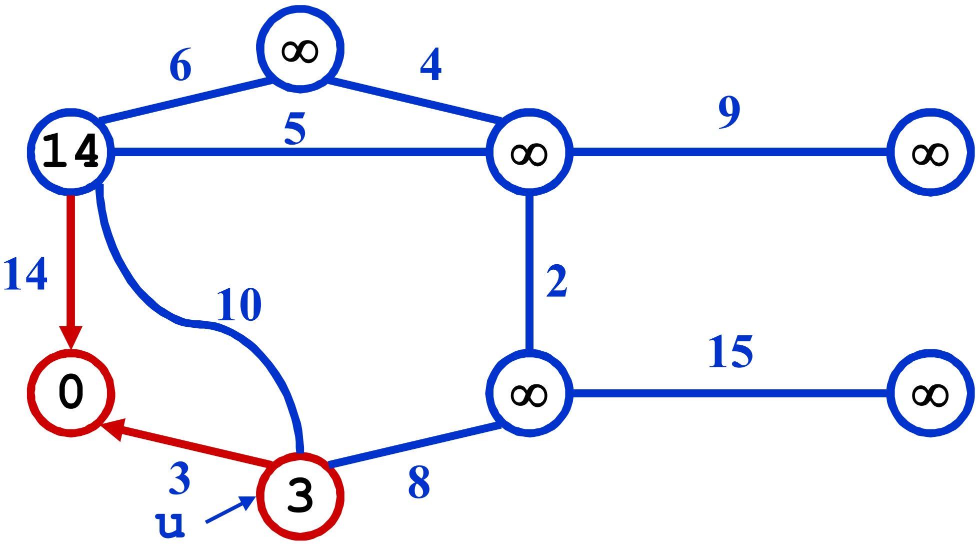

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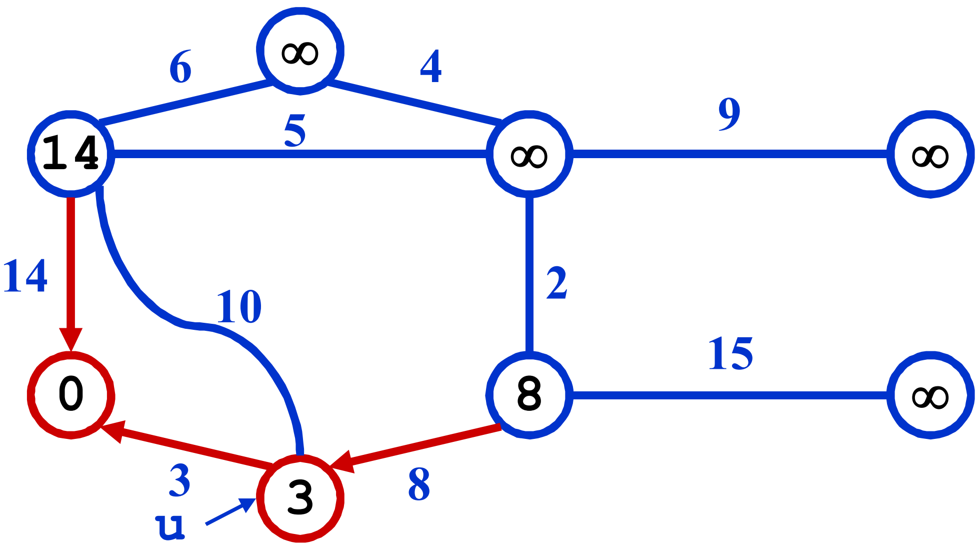

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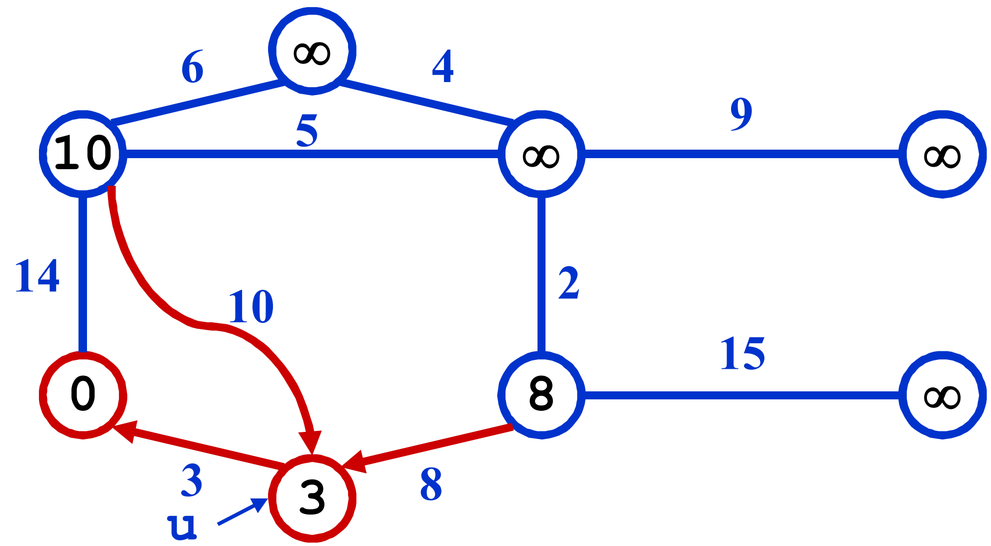

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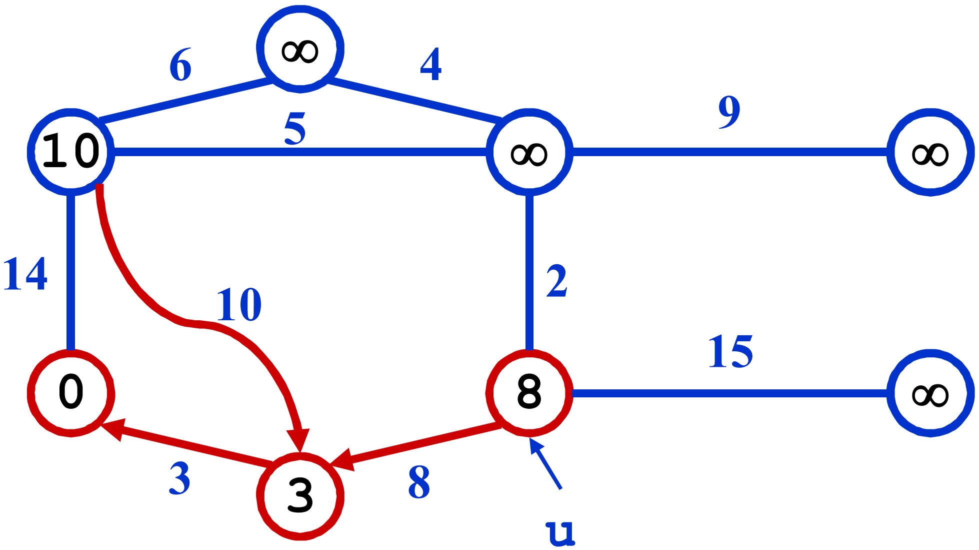

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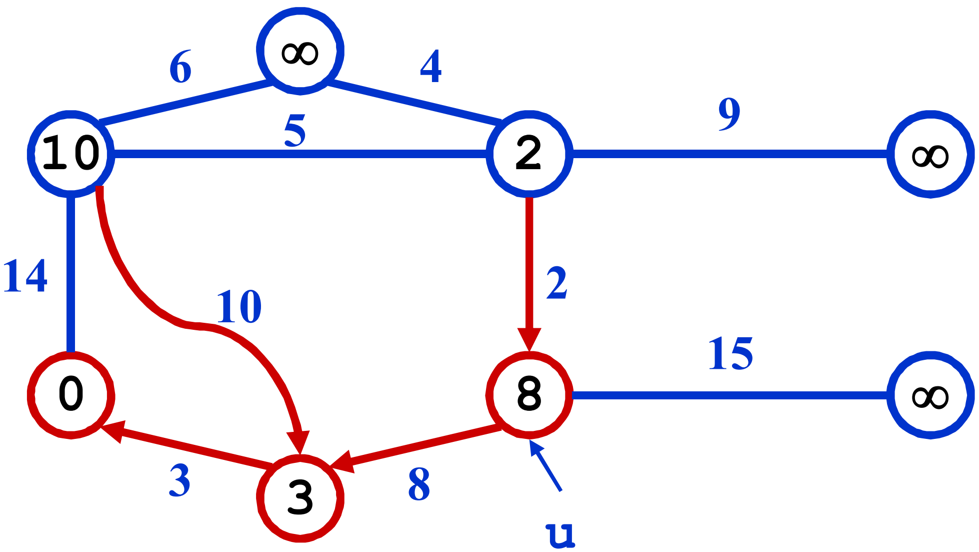

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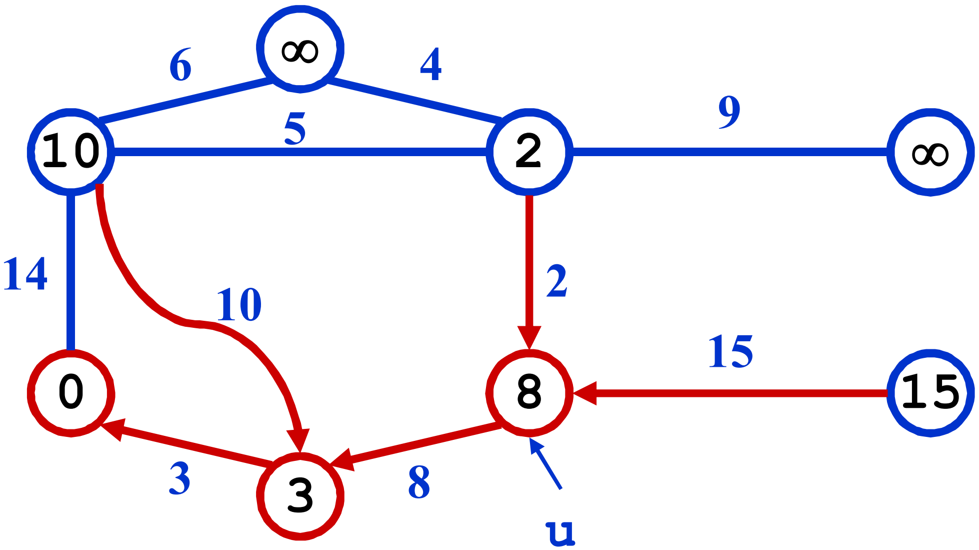

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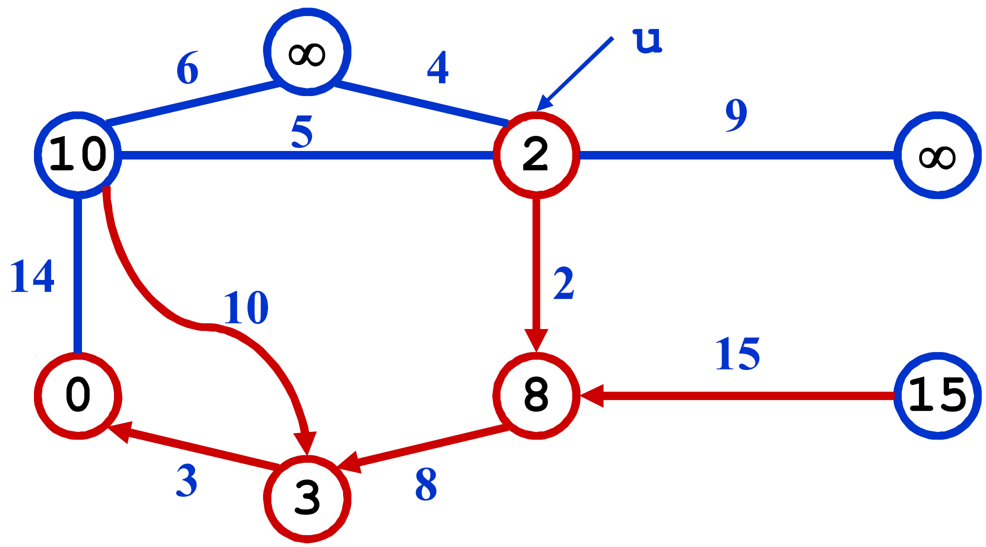

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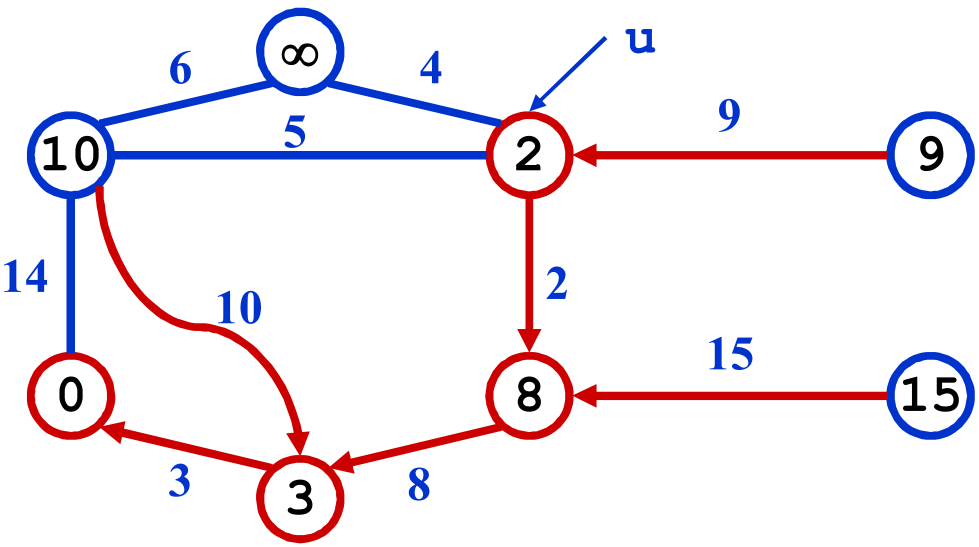

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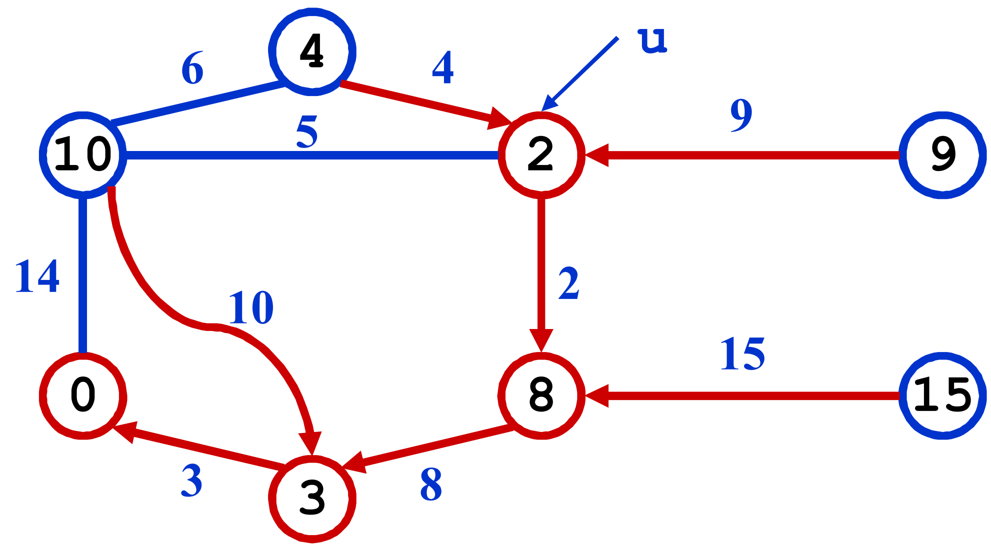

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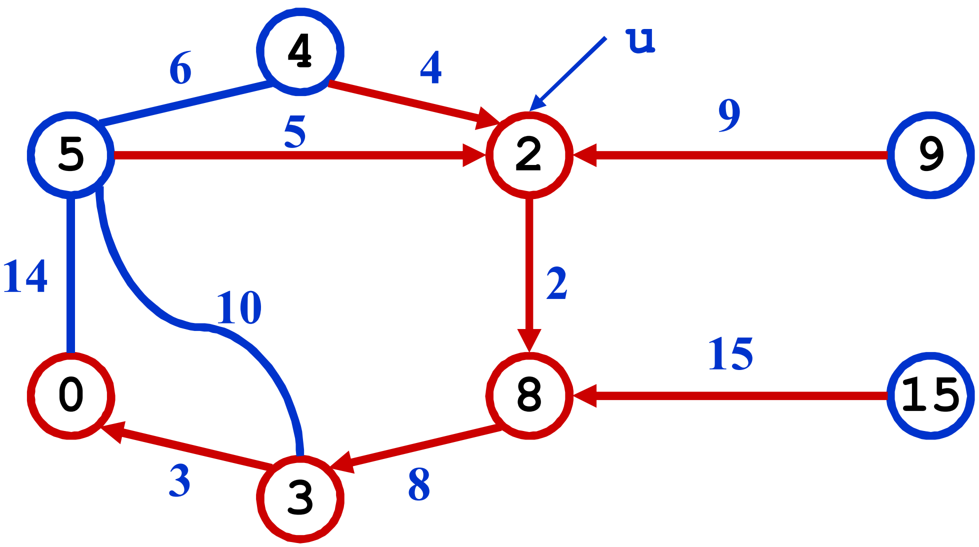

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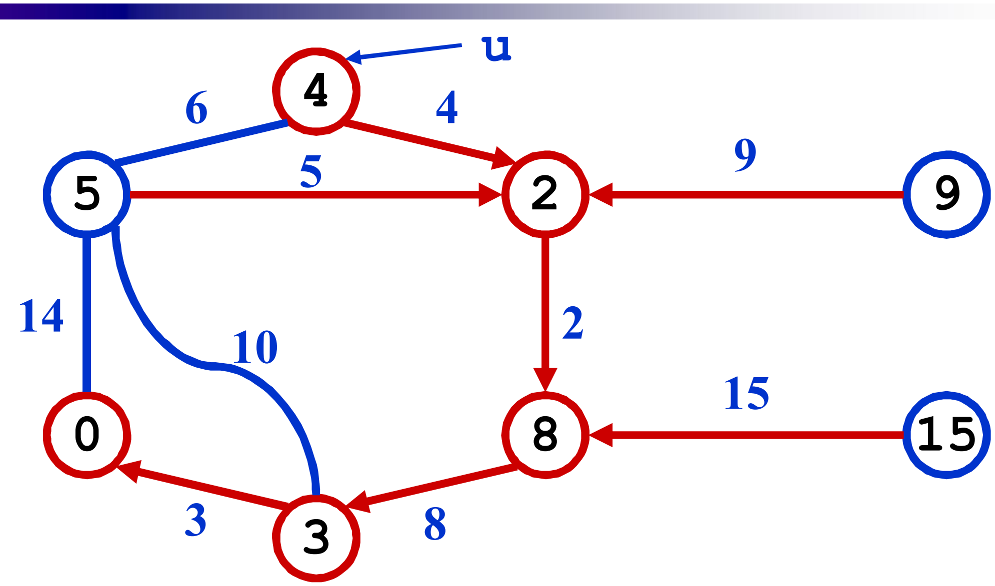

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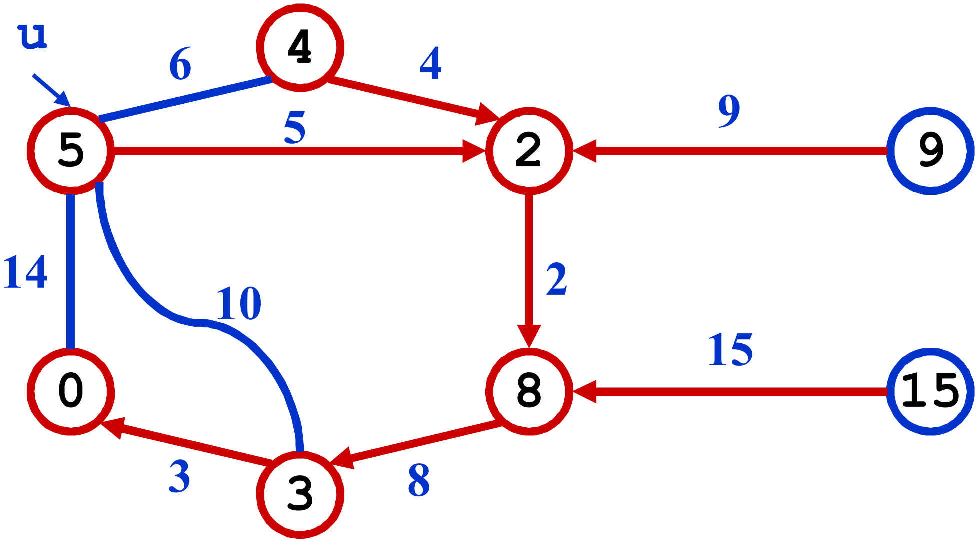

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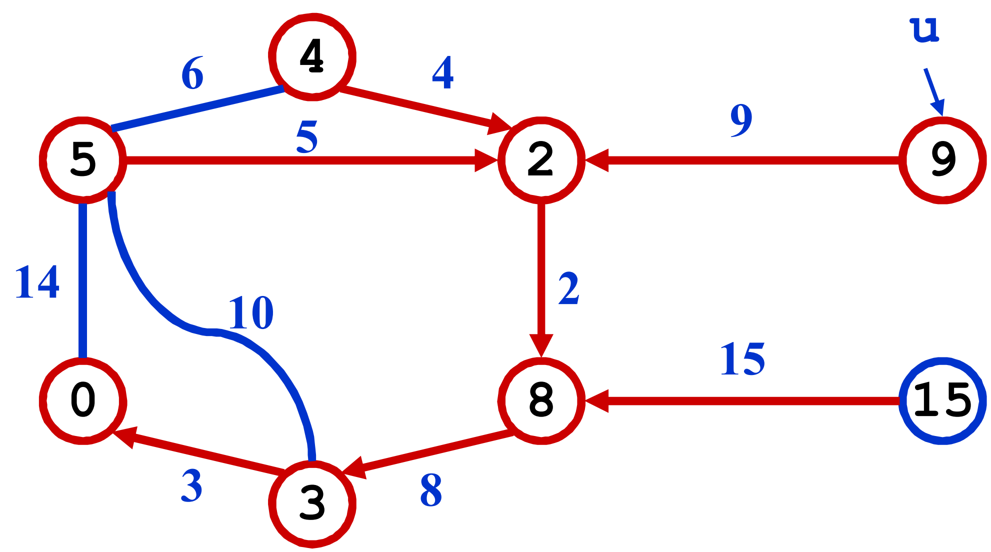

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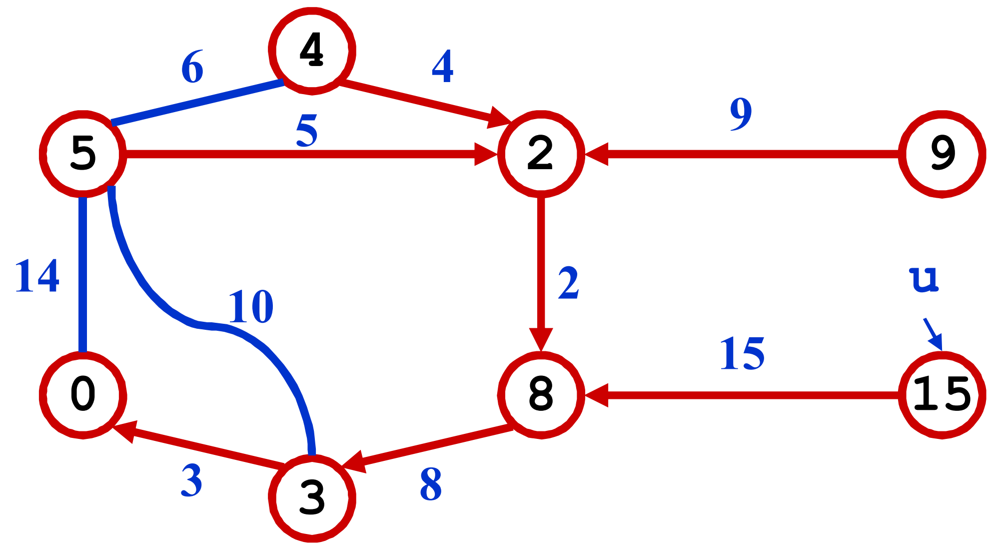

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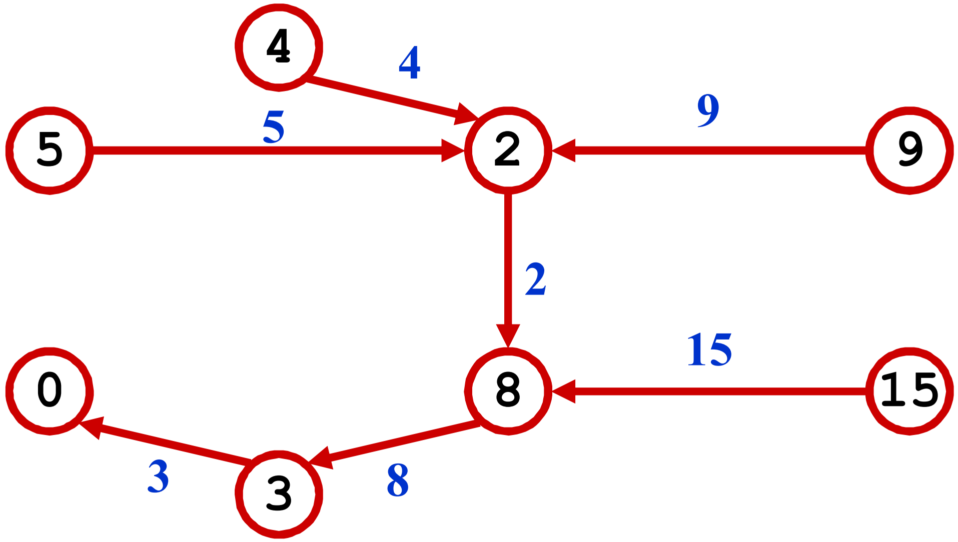

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What is the hidden cost in this code?

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How often is ExtractMin() called?

How often is DecreaseKey() called?

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What will be the running time?

Ans: Depends on queue

binary heap: $O(E \lg V)$

Fibonacci heap: $O(V \lg V + E)$

Disjoint-Set Union Problem

- Want a data structure to support disjoint sets
  - Collection of disjoint sets $\mathcal{S} = \bigcup_i \{S_i\}$, $S_i \cap S_j = \varnothing$
- Need to support following operations:
  - MakeSet(x): $\mathcal{S} = \mathcal{S} \cup \{\{x\}\}$
  - Union(S_i, S_j): $\mathcal{S} = \mathcal{S} - \{S_i, S_j\} \cup \{S_i \cup S_j\}$
  - FindSet(x): return $S_i \in \mathcal{S}$ such that $x \in S_i$
- Before discussing implementation details, we look at example application: MSTs

Kruskal's Algorithm

```
Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E into nondecreasing order by weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
            T = T U {{u,v}};
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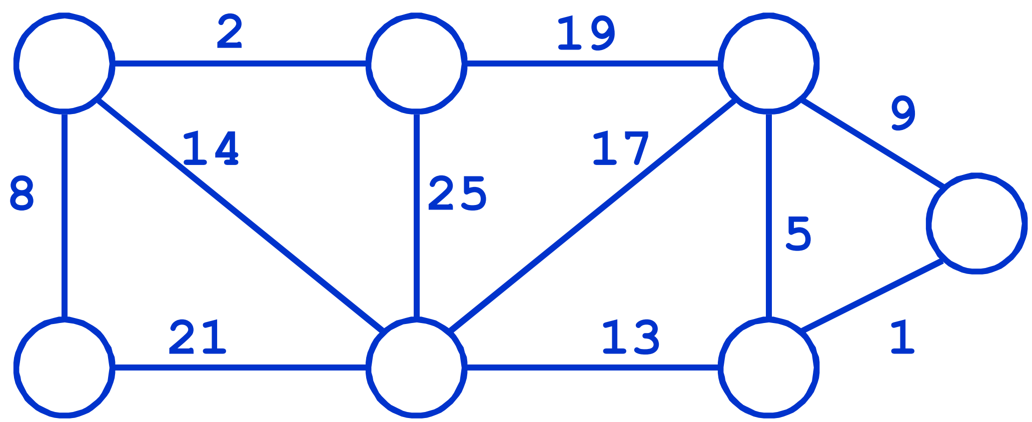
Kruskal's Algorithm

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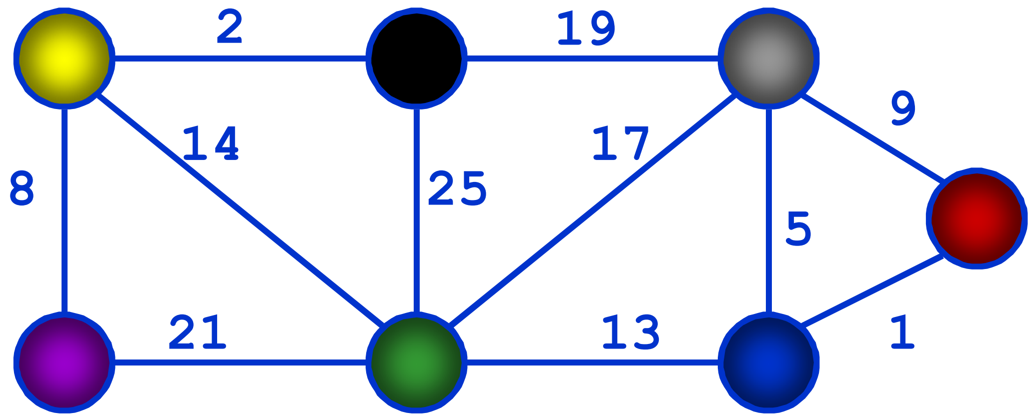


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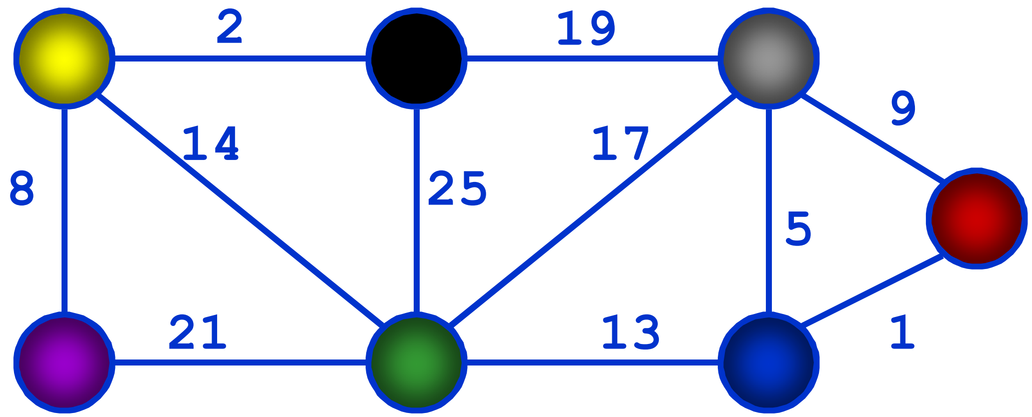


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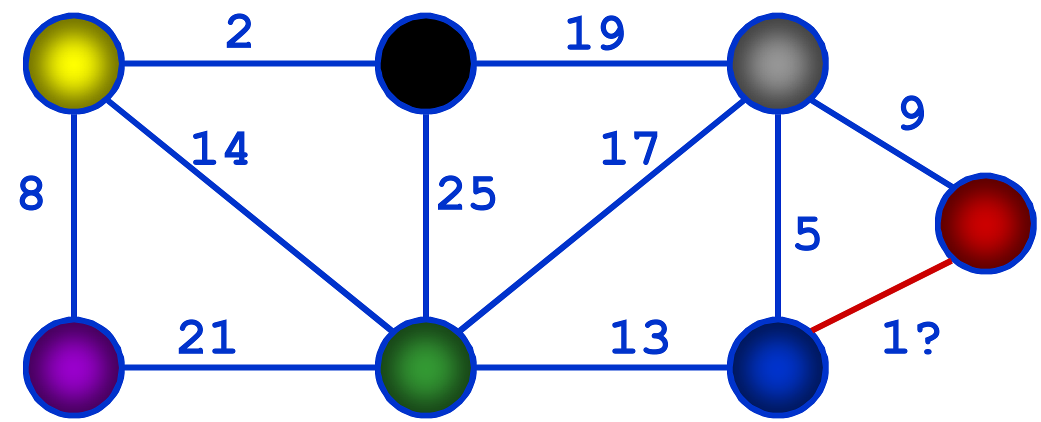
Kruskal's Algorithm

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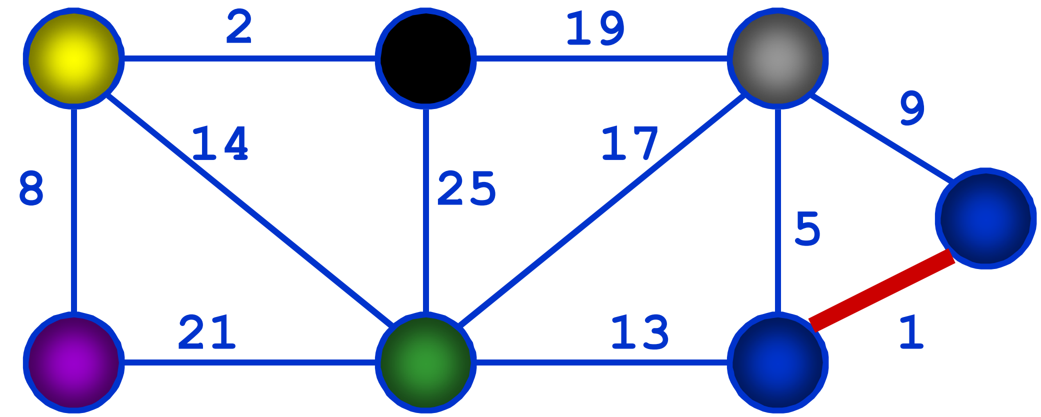


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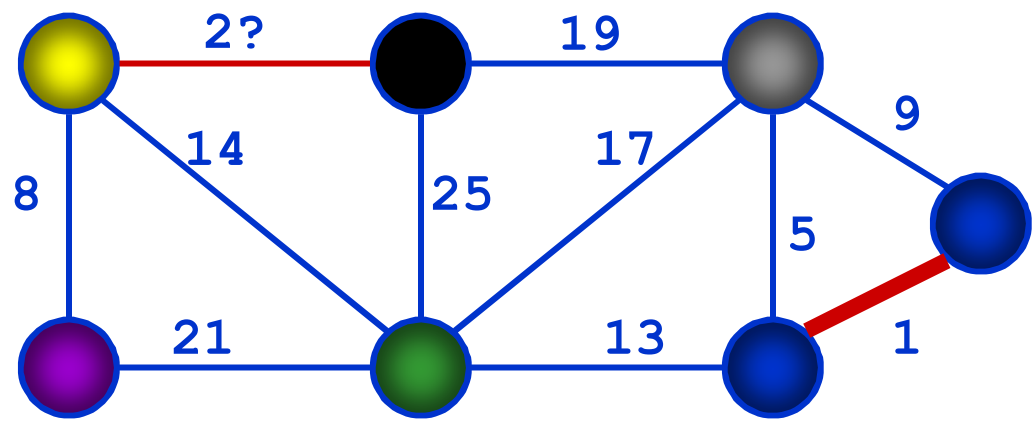

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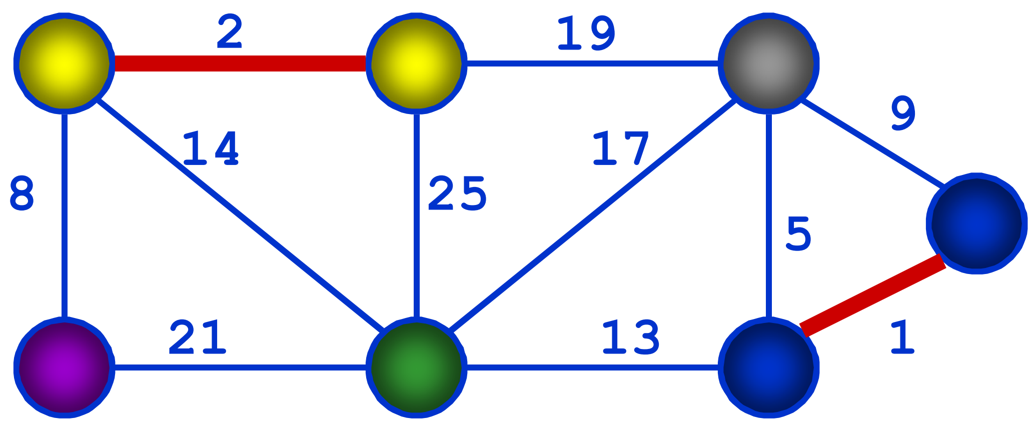


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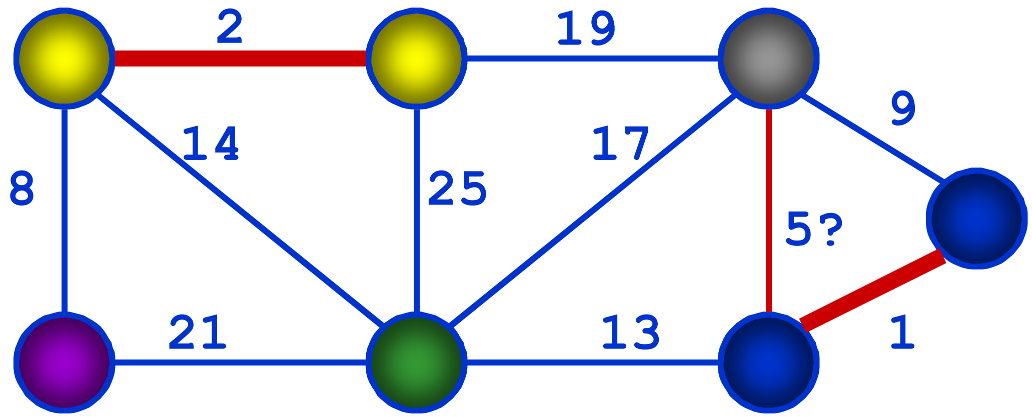


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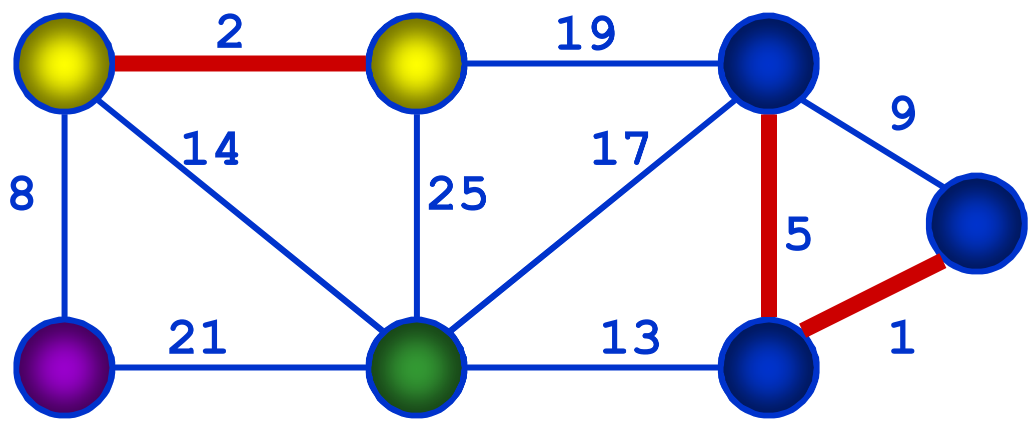


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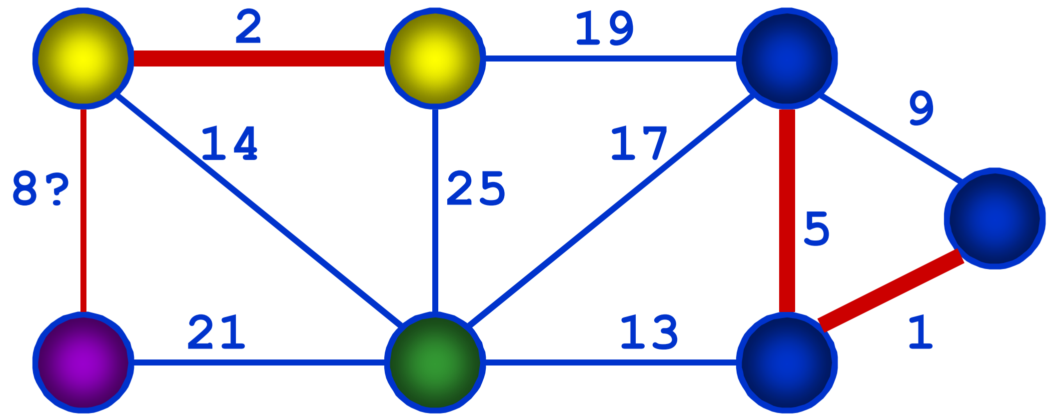
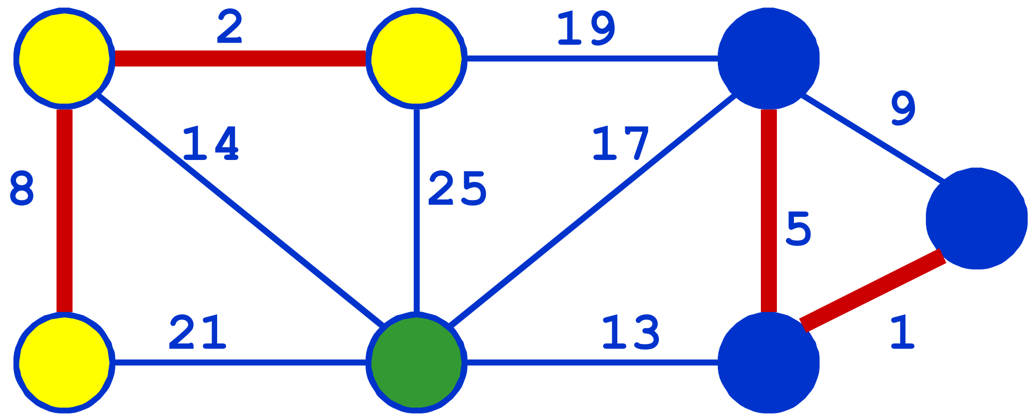


Kruskal’s Algorithm

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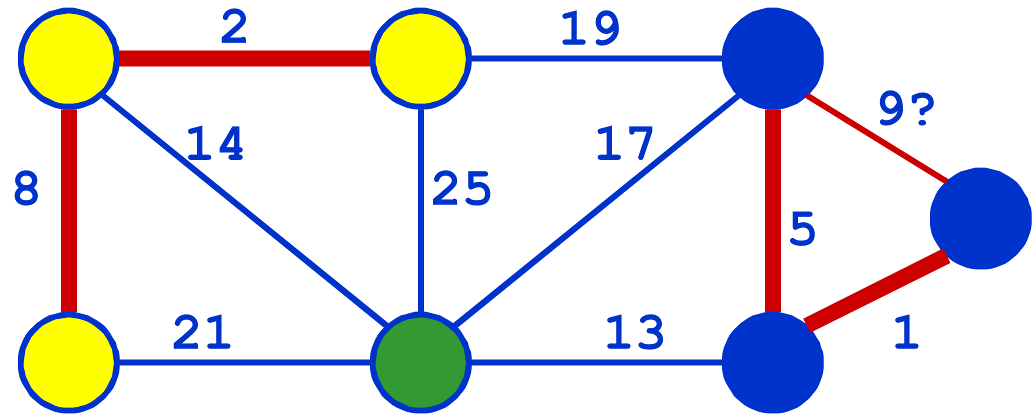


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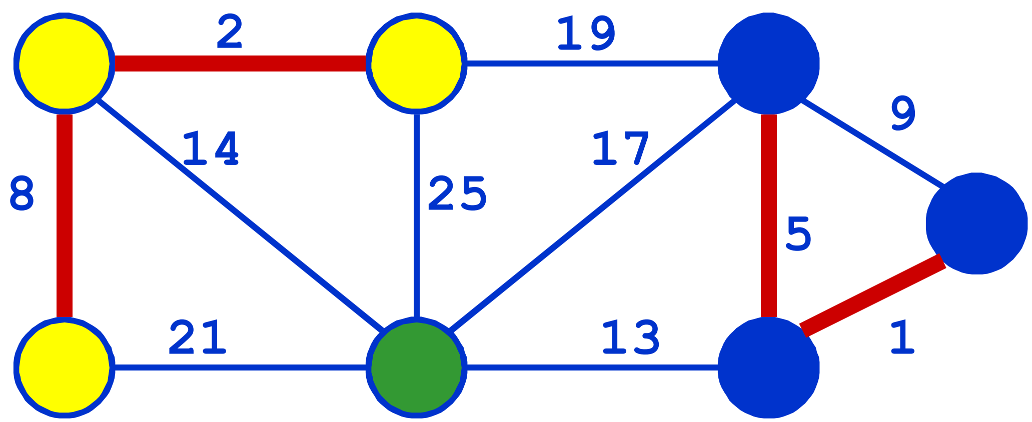


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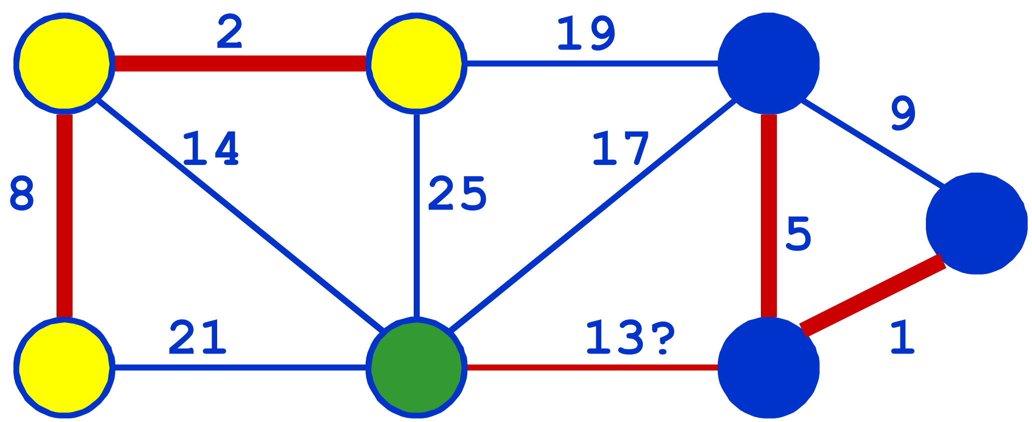


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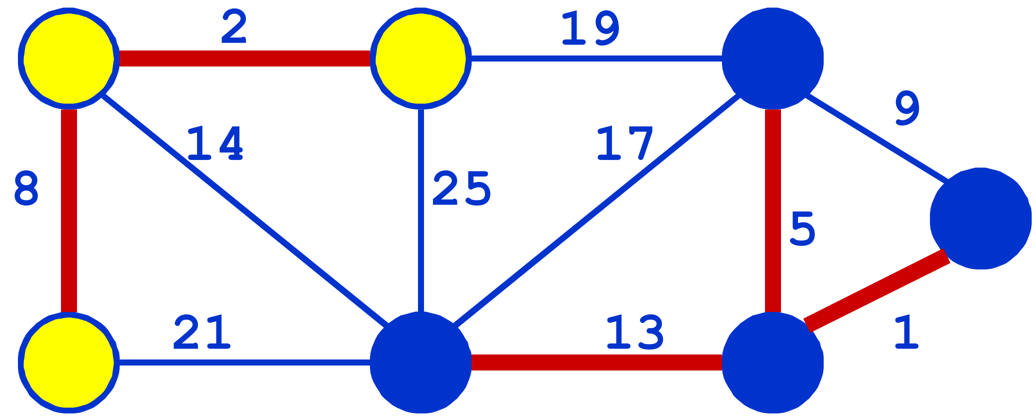
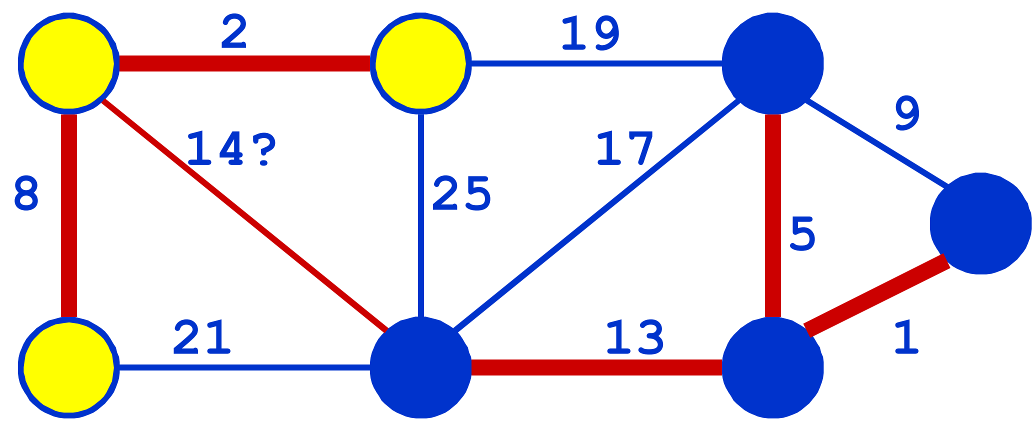


Kruskal's Algorithm

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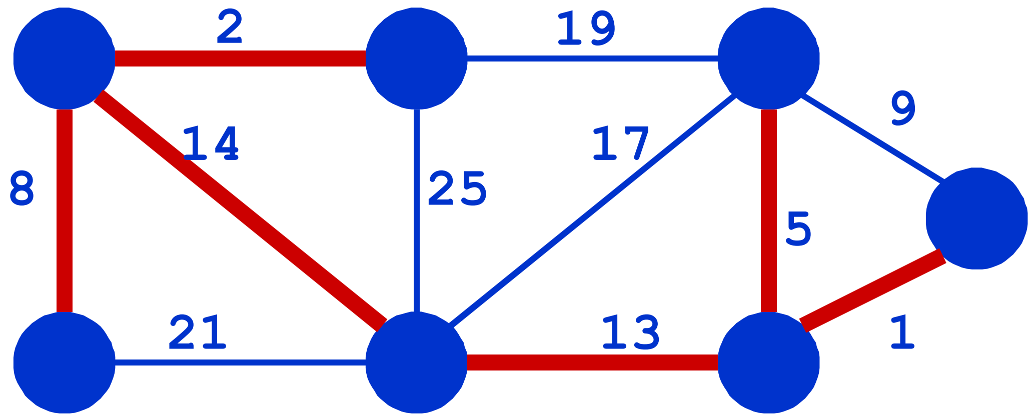


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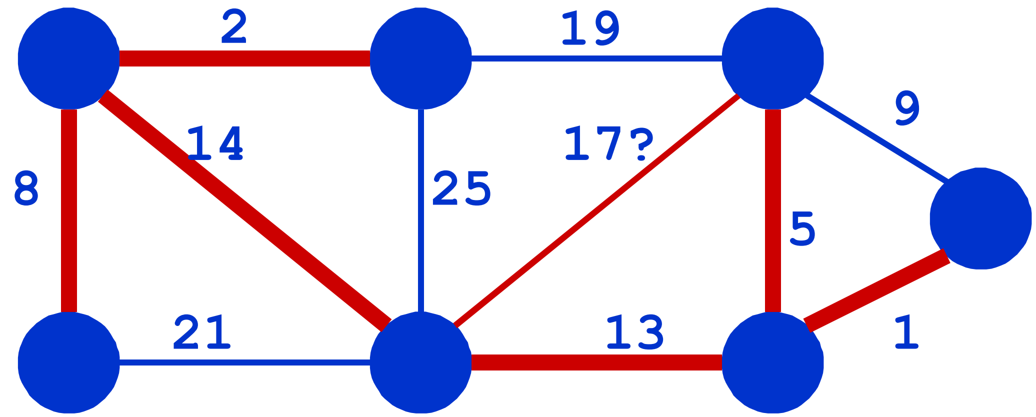

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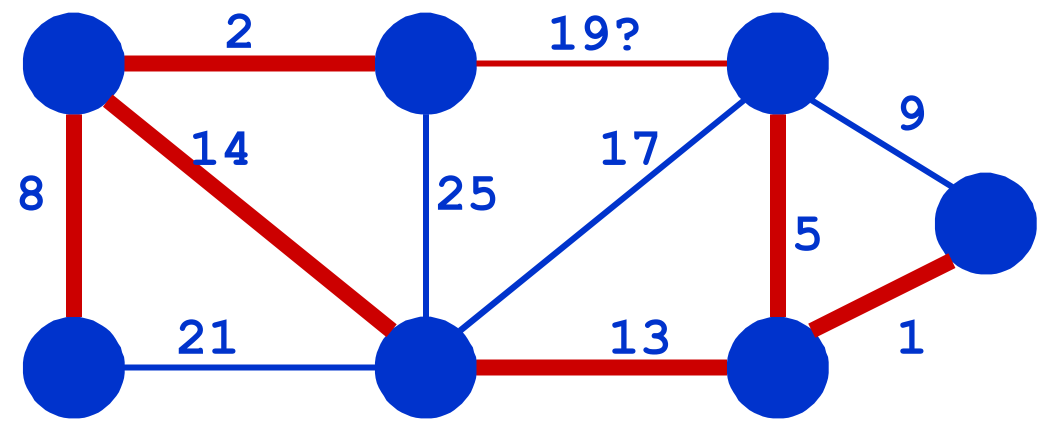


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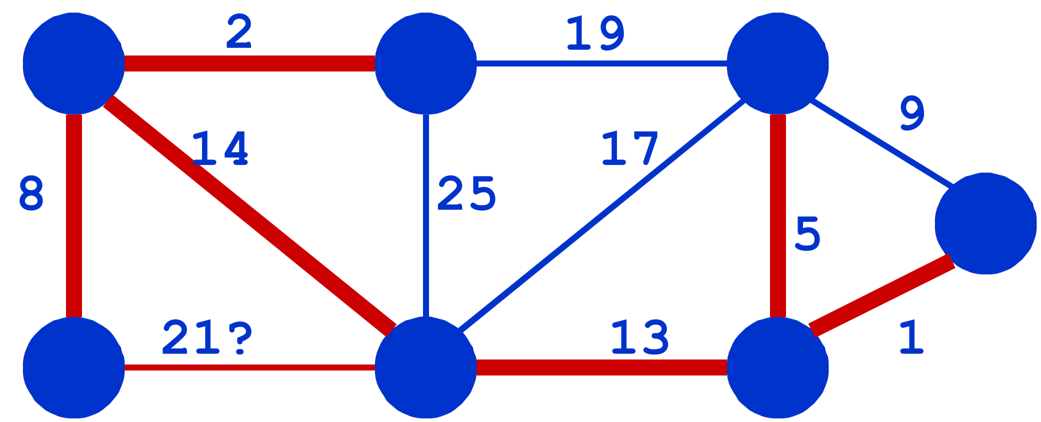


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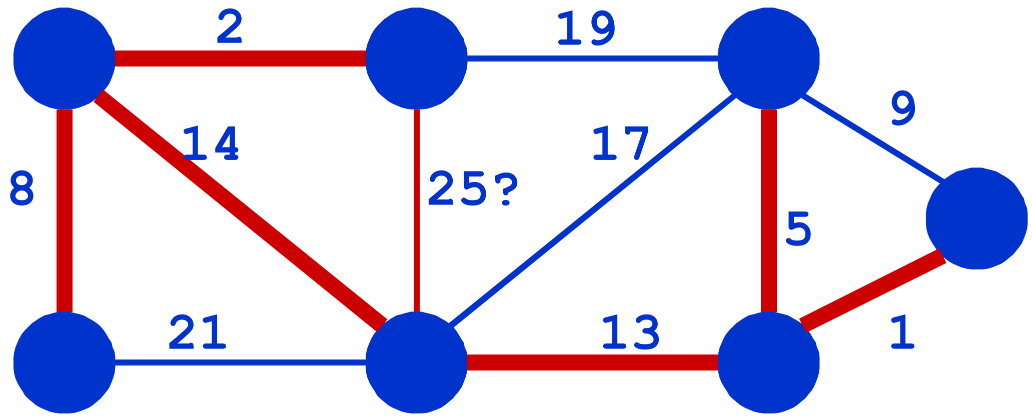

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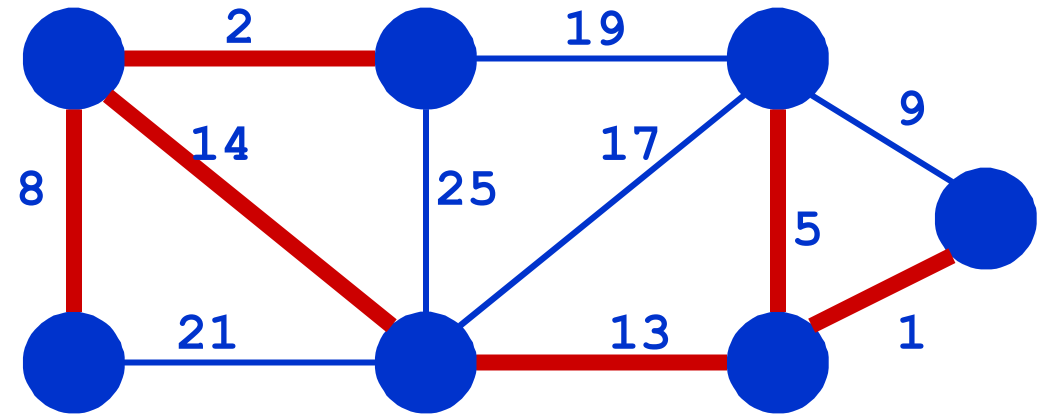


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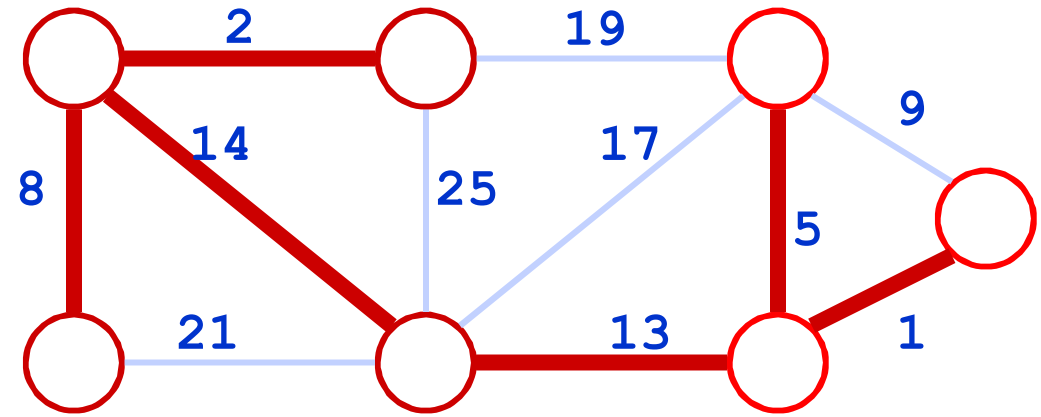


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Correctness of Kruskal's Algorithm

Theorem: In a connected weighted graph G, Kruskal's Algorithm constructs a minimum-weight spanning tree.

Proof: 1/3

- We show first that the algorithm produces a tree.

  We never choose an edge that completes a cycle.

  If the final graph has more than one component, then there is no edge joining two of them and G is not connected

  - Since G is connected, some such edge exists and we considered it.

  Thus the final graph is connected and acyclic, which makes it a tree.

Correctness of Kruskal's Algorithm

Proof: continue

- Let T be the resulting tree, and let T^* be a spanning tree of minimum weight.
- If $T=T^*$, we are done.
- If $T \neq T^*$, let e be the first edge chosen from T that is not in T^*. Adding e to T^* creates one cycle C. Since T has no cycle, C has an edge $e^* \notin E(T)$. Consider the spanning tree T^*+e-e^*.

The first edge chosen for T that is not in T^*

$$T: \quad \ldots\ldots\ldots\ldots\ldots \; e, \; \ldots\ldots\ldots\ldots$$

$$T^*: \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{Identical}} \; e^*, \ldots\ldots\ldots\ldots$$

Correctness of Kruskal's Algorithm

Proof: continue

- Since T^* contains e^* and all the edges of T chosen before e, both e^* and e are available when the algorithm chooses e, and hence $w(e) \le w(e^*)$.
- Thus $T^*+e\text{-}e^*$ is a spanning tree with weight at most T^* that agrees with T for a longer initial list of edges than T^* does.

$$\begin{array}{rl} T\!: & \ldots\ldots\ldots\ldots\ldots \; e,\; \ldots\ldots\ldots\ldots \\ T^*\!: & \underbrace{\ldots\ldots\ldots\ldots\ldots}_{\text{Identical}} \; e^*,\ldots\ldots\ldots\ldots \end{array}$$

$$\begin{array}{rl} T\!: & \ldots\ldots\ldots\ldots\ldots\ldots \; e,\; \ldots\ldots\ldots\ldots \\ T^*+e\text{-}e^*\!: & \underbrace{\ldots\ldots\ldots\ldots\ldots\ldots \; e}_{\text{Identical}},\; \ldots\ldots\ldots\ldots \end{array}$$

- ❑ Repeating this argument eventually yields a minimum-weight spanning tree that agrees completely with T.

Kruskal's Algorithm: Running Time

```
Kruskal()
{
    T = ∅;
    for each v ∈ V
        MakeSet(v);
    sort E by increasing edge weight w
    for each (u,v) ∈ E (in sorted order)
        if FindSet(u) ≠ FindSet(v)
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What will affect the running time?

Kruskal's Algorithm: Running Time

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What will affect the running time?

1 Sort

O(V) MakeSet() calls

O(E) FindSet() calls

O(V) Union() calls

(*Exactly how many Union()s?*)

Kruskal's Algorithm: Running Time

- To summarize:
  - Sort edges: O(E lg E)
  - O(V) MakeSet()'s
  - O(E) FindSet()'s
  - O(V) Union()'s
- Upshot:
  - Best disjoint-set operation algorithm makes above three operations to take O(E lg E) time.
  - Thus overall time is O(E lg E) = O(E lg V), since $|E| < |V|^2$