Chapter 14 Problem 6 Rice

Rice, John A. Mathematical Statistics and Data Analysis, Cengage

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08 September, 2017

Two objects of unknown weights w1 and w2 are weighed on an error-prone pan balance in the following way:

- (1) object 1 is weighed by itself, and the measurement is 3 g;
- (2) object 2 is weighed by itself, and the result is 3 g;
- (3) the difference of the weights (the weight of object 1 minus the weight of object 2) is measured by placing the objects in different pans, and the result is 1 g;
- (4) the sum of the weights is measured as 7g.

The problem is to estimate the true weights of the objects from these measurements.

- a. Set up a linear model, $Y = X\beta + \epsilon$.
- b. Find the least squares estimates of $w1$ and $w2$.
- c. Find the estimate of σ^2
- d. Find the estimated standard errors of the least squares estimates of part (b).
- e. Estimate w1-W2 and its standard error.
- f. Test the null hypothesis $H_0 : w1 = w2$

For notational convenience we denote $w_1 = \beta_1$ and $w_2 = \beta_2$

Also, we do not include an intercept in this model

Our model is

$$\mathbf{Y} = \mathbf{X}\,\boldsymbol{\beta} + \boldsymbol{\epsilon}$$

The response \$ $\mathbf{Y}$\$ is our measurements from the scale, the two column model matrix $\mathbf{X}$ describes the configuration of weights on the scale, and $\boldsymbol{\epsilon}$ is the error of the scale. For now we assume the errors are iid mean zero random variables, in physical systems we may want to make the additional assumption that $\epsilon_i \sim N(0, \sigma) \;\; \forall\, i$

We seek $\hat{\beta}$ such that

$$\mathbf{Y} = \mathbf{X}\,\hat{\boldsymbol{\beta}}$$

where

$$Y = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 7 \end{pmatrix}$$

and

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Since

$$\mathbf{X} \in \mathbb{R}^{4x2}$$

we know that

$$\mathbf{X}^\intercal \mathbf{X} \in \mathbb{R}^{2x2}$$

and we have hope that this can be easily calculated by hand.

$$\mathbf{X}^\intercal \mathbf{X} == \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

which gives

$$(\mathbf{X}^\intercal \mathbf{X})^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$$

Now

$$(\mathbf{X}^\intercal \mathbf{Y}) = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 3 \\ 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 11 \\ 9 \end{pmatrix}$$

Finally we have that

$$(\mathbf{X}^\intercal \mathbf{X})^{-1} \, \mathbf{X}^\intercal \mathbf{Y} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 11 \\ 9 \end{pmatrix} = \begin{pmatrix} \frac{11}{3} \\ \frac{9}{3} \end{pmatrix} = \hat{\boldsymbol{\beta}}$$

Calculate s^2

$$\hat{\boldsymbol{\epsilon}} = \mathbf{Y} - \mathbf{X} \, \hat{\boldsymbol{\beta}} == \begin{pmatrix} \frac{-2}{3} \\ 0 \\ \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

And

$$\frac{\hat{\boldsymbol{\epsilon}}^\intercal \hat{\boldsymbol{\epsilon}}}{n-p} = s^2$$

Where $n = 4$ and $p = 2$ in this case. Note if we had used an intercept we would have $p = 3$. Putting the values in we get that

$$s^2 = \frac{1}{3}$$

Calculate $se(\beta_i)$

Let

$$\mathbf{C} = (\mathbf{X}^\intercal \mathbf{X})^{-1}$$

Then

$$se(\beta_i) = s\sqrt{C_{i\,i}}$$

and putting our values in from above we have that

$$se(\beta_1) = se(\beta_2) = \frac{1}{\sqrt{3}}\,\frac{1}{\sqrt{3}} = \frac{1}{3}$$

###Estimate $\beta_1 - \beta 2$ and find the it's standard error

If $\mathbf{x}_0 \in \mathbb{R}^p$ is a vector of predictor variables then the prediction Y_0 is given by $\mu_0 = \mathbf{x}_0^\intercal \hat{\boldsymbol{\beta}}$. We saw that the covariance matrix of $\hat{\boldsymbol{\beta}}$ is

$$\Sigma_{\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}} = \sigma^2\,(\mathbf{X}^\intercal \mathbf{X})^{-1}$$

under the assumption that the errors are iid with constant variance. We can use this with the theorem about linear transforms of random vectors to get that

$$Var(\mu_0) = \mathbf{x}_0^\intercal\,\Sigma_{\hat{\boldsymbol{\beta}}\,\hat{\boldsymbol{\beta}}}\,\mathbf{x}_0$$

Now we're looking to estimate $\beta_1 - \beta_2$ so our predictor vectors is going to be

$$x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Putting all our values from above in to the expression for μ_0 and $Var(\mu_0)$ we get that

$$\mu_0 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{11}{3} \\ 3 \end{pmatrix} = \frac{2}{3}$$

$$se(\widehat{\beta_1 - \beta_2}) = \sqrt{Var(\mu_0)} = \sqrt{\begin{pmatrix} 1 & -1 \end{pmatrix} \frac{1}{3} \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}} = \frac{\sqrt{2}}{3}$$

Note that we've use the estimate s^2 for σ^2

Test $H_0 : \beta_1 = \beta_2$

If we adopt the additional assumption that the errors are normally distributed, we will have that the $(1-\alpha)100\%$ CI for $\hat{\mu_0}$ is

$$\hat{\mu_0} \pm t_{n-p}(\frac{\alpha}{2}) s_{\hat{\mu_0}}$$

```
mu0 <- 2/3
se_mu0 <- sqrt(2)/3
n = 4
p = 2
t_alpha <- qt(0.95, n - p)
leftCI <- mu0 - t_alpha * se_mu0
rightCI <- mu0 + t_alpha * se_mu0
pander(data.frame(left = leftCI, right = rightCI), caption = "95% CI")
```

Table 1: 95% CI

left	right
-0.7098	2.043

Since our null hypotheses H_0 is in the CI we do not have enough evidence to reject it.

Checking ealier calculations in R.

We calculated the solution by hand (attached below) and then checked portions of it in R.

```
data <- data.frame(X1 = c(1, 0, 1, 1), X2 = c(0, 1, -1, 1), Y = c(3, 3, 1, 7))


lm.fit.nointercept <- lm(Y ~ X1 + X2 - 1, data = data)

summary(lm.fit.nointercept)
```

```
##
## Call:
## lm(formula = Y ~ X1 + X2 - 1, data = data)
##
## Residuals:
##          1          2          3          4
## -6.667e-01  4.996e-16  3.333e-01  3.333e-01
##
## Coefficients:
##    Estimate Std. Error t value Pr(>|t|)
## X1   3.6667     0.3333      11  0.00816 **
## X2   3.0000     0.3333       9  0.01212 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5774 on 2 degrees of freedom
## Multiple R-squared:  0.9902, Adjusted R-squared:  0.9804
## F-statistic:   101 on 2 and 2 DF,  p-value: 0.009804
```

This agrees - up to numerical precision - with the exact calculations we did by hand for β, $se(\beta_1)$, $se(\beta_2)$ and s^2