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Reversibility and Surjectivity Problems of Cellular Automata

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The problem of deciding if a given cellular automaton (CA) is reversible (or, equivalently, if its global transition function is injective) is called the reversibility problem of CA. In this article we show that the reversibility problem is undecidable in case of two-dimensional CA. We also prove that the corresponding surjectivity problem—the problem of deciding if the global function is surjective—is undecidable for two-dimensional CA. Both problems are known to be decidable in case of one-dimensional CA. The proofs of the theorems are based on reductions from the well-known tiling problem of the plane, known also as the domino problem.

1. INTRODUCTION

Cellular automata are dynamical systems that have been extensively studied as discrete models for natural systems that are describable as large collections of simple objects locally interacting with each other. A d-dimensional cellular automaton consists of an infinite d-dimensional array of identical cells. Each cell is always in one state from a finite state set. The cells alter their states synchronously on discrete time steps according to a local rule. The rule gives the new state of each cell as a function of the old states of some nearby cells, its neighbors. The array is homogeneous so that all its cells operate under the same local rule. The states of all the cells in the array are described by a configuration. A configuration can be considered as the state of the whole array. The local rule of the automaton specifies a global function that tells how each configuration is changed in one time step.

The study of cellular automata was initiated in the late forties by John von Neumann. He introduced cellular automata as possible universal computing devices capable of mechanically reproducing themselves. Since then cellular automata have gained popularity as models for massively parallel computations. A particularly elegant cellular automaton, christened *Life*, was invented by John Conway in 1970.

Cellular automata provide simple models of complex natural systems encountered in physics, biology, and other fields. Like natural systems they consist of large numbers of simple basic components that together produce the complex behavior

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of the system. The numerous systems that have been modeled using cellular automata include lattice gases, spin systems, crystal growth processes, reaction-diffusion systems, etc. Without doubt the availability of high speed hardware implementations of cellular automata has greatly increased their popularity in simulating various physical systems. The inherent locality and parallelism of cellular automata make such implementations very natural (see [15] for more information).

A basic feature of microscopic mechanisms of physics is reversibility. It is possible for cellular automata to capture this important characteristic without sacrificing other essential properties like computational universality. A cellular automaton rule is called reversible if there exists another rule, called the inverse rule, that makes the automaton retrace its computation steps backwards in time. The earlier configurations are uniquely determined by the present one and no information is lost during the computation. A cellular automaton defined by a reversible local rule is called reversible. It is known that a cellular automaton is reversible if its global function is one-to-one [12].

It is a natural question to ask what kind of local rules are reversible. No general characterizations of reversible rules are known. In fact, there does not exist any algorithm that would decide of a given two-dimensional local rule whether it is reversible or not. This fact is the main result proved in this article. However, if one restricts oneself to one-dimensional rules then such an algorithm can be designed [1].

Another possible property of a cellular automaton is surjectivity, that is, the surjectivity of its global function. In a surjective cellular automaton every configuration can occur arbitrarily late during computations. It is known that an automaton is surjective if and only if the restriction of its global function to finite configurations is injective. This is the Garden of Eden theorem proved by Moore [9] and Myhill [11]. A configuration is called finite if it has only finitely many cells in states different from one specific quiescent state. One can show, using a similar method as in connection with the reversibility, the algorithmic undecidability of the problem of tesing whether a given cellular automaton is surjective. In the one-dimensional case the problem is, however, decidable [1].

The article is organized as follows. First we present formal definitions of cellular automata and the tiling problem of the plane on which our proofs are heavily based. Then the proof for the undecidability of the reversibility problem is given in Section 3. The idea of the proof is very simple, and it is described in Subsection 3.1. In the proof one specific tile set having the so-called plane-filling property is needed. To complete the proof one tile set with this property is constructed in Subsections 3.2, 3.3, and 3.4. In Section 4 surjective cellular automata are considered. The undecidability of the surjectivity problem is proved. Section 5 contains some concluding remarks and consequences of the results.

2. Definitions

In this section the essential concepts of the article are formally defined. They include cellular automata, tilings of the plane, and the tiling problem and, finally, directed tiles and the paths they define on the plane.

2.1. *Cellular Automata*

Formally, a *cellular automaton* (CA) is a quadruple

$$\mathscr{A} = (d, S, N, f),$$

where d is a positive integer indicating the *dimension* of $\mathscr{A}$, S is a finite *state set*, N is a *neighborhood vector*,

$$N = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n),$$

of n different elements of $\mathbb{Z}^d$ and f is the *local rule* of the CA presented as a function from S^n into S. The cells are laid on an infinite d-dimensional array and their positions are indexed by $\mathbb{Z}^d$, the set of d-tuples of integers. The *neighbors* of a cell situated in $\bar{x} \in \mathbb{Z}^d$ are the cells in positions

$$\bar{x} + \bar{x}_i \qquad \text{for} \quad i = 1, 2, ..., n.$$

The local rule f gives the new state of a cell from the old states of its neighbors.

A *configuration* of a CA $\mathscr{A} = (d, S, N, f)$ is a function

$$c: \mathbb{Z}^d \rightarrow S$$

that assigns states to all cells. Let $\mathscr{C}$ denote the set of all configurations. The local rule f determines the *global function*

$$G_f: \mathscr{C} \rightarrow \mathscr{C}$$

that describes the dynamics of the CA. At each time step a configuration c is transformed into a new configuration $G_f(c)$, where

$$G_f(c)(\bar{x}) = f(c(\bar{x} + \bar{x}_1), c(\bar{x} + \bar{x}_2), ..., c(\bar{x} + \bar{x}_n))$$

for all $\bar{x}$ in $\mathbb{Z}^d$.

Sometimes a *quiescent state* q in S is distinguished. The quiescent state must have the property

$$f(q, q, ..., q) = q.$$

A configuration is called *finite* if it has only finitely many cells in non-quiescent states. Let $\mathscr{C}_F$ denote the set of finite configurations. It follows from the special property of the quiescent state that a finite configuration remains finite in the evolution of the CA. Let G_f^F denote the restriction of the global function G_f to the set of finite configurations.

In this work mainly two-dimensional cellular automata are considered. In this case the cells are laid on the plane. The following two-dimensional neighborhood vector will be frequently used:

$$N_M = [(-1, -1), (-1, 0), (-1, 1), (0, -1), \\ (0, 0), (0, 1), (1, -1), (1, 0), (1, 1)].$$

This defines the *Moore neighborhood*—a cell is situated in the center of a square of 3×3 neighboring cells (see Fig. 1a). The eight directions of the compass are used when we refer to the eight surrounding cells.

Another widely used neighborhood is the *von Neumann neighborhood* defined by the vector

$$N_{vN} = [(0, 0), (1, 0), (0, 1), (-1, 0), (0, -1)].$$

In the von Neumann neighborhood each cell has five neighbors: the cell itself and the four adjacent cells (see Fig. 1b).

A CA is called *injective* if its global function is one-to-one. Similarly, a CA is called *surjective* if its global function is surjective.

EXAMPLE. Let

$$N = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$$

be any neighborhood vector of a d-dimensional CA,

$$S = S_1 \times S_2 \times \cdots \times S_n$$

a cartesian product of n finite sets and

$$\varphi: S \rightarrow S$$

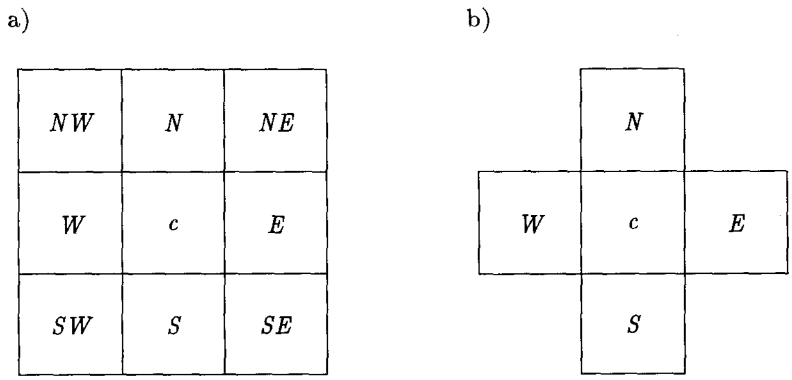

FIG. 1. The (a) Moore and (b) von Neumann neighborhoods of cell c.

a bijective function. Let

$$\pi_i\colon S_1 \times S_2 \times \cdots \times S_n \to S_i$$

denote the ith projection of S. Let f be a function from S^n into S defined by

$$f(s_1, s_2, ..., s_n) = \varphi(\pi_1(s_1), \pi_2(s_2), ..., \pi_n(s_n)).$$

It is easy to see that the d-dimensional cellular automaton $\mathscr{A} = (d, S, N, f)$ is injective. This follows from the bijective of φ—if you change its arguments then its value is changed as well. Define then another function g from S^n into S by

$$g(s_1, s_2, ..., s_n) = (\pi_1(\varphi^{-1}(s_1)), \pi_2(\varphi^{-1}(s_2)), ..., \pi_n(\varphi^{-1}(s_n)))$$

and let N^{-1} be the neighborhood obtained from N by changing the signs of all of its coordinates. Then the CA $\mathscr{B} = (d, S, N^{-1}, g)$ is injective as well and, moreover, it is the inverse automaton of $\mathscr{A}$. This means that $G_g = G_f^{-1}$.

The injective CA of the previous example turned out to be *reversible*. A cellular automaton is called reversible if there exists another CA, so-called *inverse automaton*, whose global function is the inverse of the global function of the original CA. Already in 1972 Richardson [12] showed that the phenomenon in our example was no coincidence: He proved that a cellular automaton is injective if and only if it is reversible.

2.2. *The Tiling Problem*

The *tiling problem* will play an essential role in the undecidability proofs of this article. In the tiling problem we are given a finite set of unit squares with colored edges, the tiles, placed with their edges horizontal and vertical. We have infinitely many copies of all these tiles and we want to tile the entire plane using the copies, without rotating any of them. The tiles are placed on the unit square regions of the Euclidean plane bordered by the lines $y = n$ and $x = m$ for all integers n and m. In a valid tiling the abutting edges of the tiles must have the same color. The tiling problem consists of deciding whether the plane can be tiled with a given collection of tiles. The tiling problem was proved undecidable by Berger [2]. A simplified proof was given later by Robinson [13].

An easy application of König's infinity lemma shows that if one can tile arbitrarily large squares with a given tile set, then one can tile the whole plane as well.

We do not need to restrict ourselves to tiles with colors. The notion can be generalized as follows. Let T be any finite set, $N = (\bar{x}_1, \bar{x}_2, ..., \bar{x}_n)$ a two-dimensional neighborhood vector defined similarly as in the preceding section and $R \subseteq T^n$ an n-ary relation on T. The triple $\mathscr{T} = (T, N, R)$ is called a tile set. The elements of T are the tiles that are placed on the plane. A tiling

$$\psi\colon \mathbb{Z}^2 \to T$$

of the plane is valid at the tile situated in $\bar{x} \in \mathbb{Z}^2$ iff

$$(\psi(\bar{x}+\bar{x}_1), \psi(\bar{x}+\bar{x}_2), ..., \psi(\bar{x}+\bar{x}_n)) \in R.$$

The tiling is valid iff it is valid at every tile on the plane; that is, the relation above holds for each $\bar{x} \in \mathbb{Z}^2$. Naturally the tilings with colored tiles can be thought as special cases of these more general tilings: choose $N = [(0, 0), (1, 0), (0, 1)]$ and define $(t_1, t_2, t_3) \in R$ if and only if the left edge of t_2 and the lower edge of t_3 have the same colors as the right edge of t_1 and the upper edge of t_1, respectively.

The following proposition expresses Berger's undecidability result.

PROPOSITION 1. *It is undecidable whether a given tile set $\mathcal{T} = (T, N_M, R)$ with the Moore neighborhood can be used to form a valid tiling of the plane.*

The proposition is not stated in the strongest possible form since the Moore neighborhood could be replaced with a smaller one. However, this formulation is most appropriate for our purposes.

2.3. *Directed Tiles and the Plane-Filling Property*

In the article, also *directed tiles* are needed. Directed tiles are, like ordinary tiles, unit squares that are placed on the plane. Valid tilings are again defined locally using a relation (specified, for example, by colors on the edges of tiles). In addition, the directed tiles have one direction from the set

$$\{N, E, S, W\}$$

attached to every tile. The direction refers to one of the four neighboring tiles lying to the north, east, south, or west. The directions attached to tiles in T are given by the direction function

$$d: T \to \{N, E, S, W\}.$$

So, a set of directed tiles is a quadruple $\mathcal{D} = (D, N, R, d)$, where (D, N, R) is a tile set as defined in the previous section, and d is the direction function.

The directions define paths through the tiles on the plane in a natural way. The direction of each tile tells which way a path coming to the tile proceeds. The next tile on the path is the adjacent tile pointed by the direction. Every tile on the plane starts a path that follows the directions. The path does not necessarily visit infinitely many tiles—it may come back to an earlier tile causing the path to fall into a loop.

Let $\mathcal{D} = (D, N, R, d)$ be a set of directed tiles. The set $\mathcal{D}$ is said to have the *plane-filling property* if it satisfies the following properties. First, it is required that there exists a valid tiling of the plane with the tiles. Second, the essential requirement is that on every tiling of the plane with the tiles, valid or not, the directions can define only two different types of paths: Either there is a tile on the path where the tiling is incorrect, or otherwise the path visits all tiles of arbitrarily large squares. So if the tiling property is not violated on the path then for each n there must be an $n \times n$ square each tile of which is on the path. Especially the directions cannot define any loop without a violation of the tiling property somewhere along the path.

The second requirement can be illustrated as follows. Consider a simple device that operates on the plane which is tiled with the directed tiles of $\mathscr{D}$. The device checks the tiling at the tile it is currently standing on. If the tiling is proper, then the device goes to the neighboring tile that is pointed by the direction of its current tile. The operation is repeated on the new tile. If, on the other hand, the tiling property is violated at the tile, then the device halts indicating that it has found an error.

If the tile set $\mathscr{D}$ has the plane-filling property then, no matter how the plane is tiled with the tiles, and no matter at which tile the device is started, there are just two possible ways the device can operate. Either it halts because it finds a tiling error, or it visits all tiles of arbitrarily large squares.

Note that the following property is weaker than the plane filling property: On every valid tiling of the plane the paths defined by the directions cover arbitrarily large squares. The plane filling property requires more—the complete tiling is not necessarily valid, because there can be tiling errors outside the path, but still the path is required to cover arbitrarily large squares. This complicates the construction of directed tiles with the plane-filling property.

If $\mathscr{D}$ admits periodic tilings then it cannot satisfy the plane-filling property. To see this, assume that $\psi\colon \mathbb{Z}^2 \to D$ is a valid periodic tiling and $n, m > 0$ integers satisfying $\psi(x+n, y) = \psi(x, y)$ and $\psi(x, y+m) = \psi(x, y)$ for all $x, y \in \mathbb{Z}$. In other words, n and m are the horizontal and vertical periods, respectively. On every path that follows the directions on the tiling ψ there are integers t and δ, $1 \leqslant t < t+\delta \leqslant mn+1$, such that if the tth tile on the path is in position (x_1, y_1) and the $(t+\delta)$th tile is in position (x_2, y_2), then $x_1 \equiv x_2 \pmod{n}$ and $y_1 \equiv y_2 \pmod{m}$. Denote $x_2 = x_1 + an$ and $y_2 = y_1 + bm$. Because of the periodicity of the tiling the $(t+k\delta)$th tile on the path is in position $(x_1 + kan, y_1 + kbm)$ for all $k \geqslant 0$. Consequently the path either forms a closed loop (if $a = b = 0$), or it essentially follows the straight line that goes through points (x_1, y_1) and (x_2, y_2) (the maximum distance of the path from the line is bounded by a constant). In neither case can the path cover arbitrarily large squares.

Let us briefly consider possible shapes of the paths defined by sets of directed tiles satisfying the plane filling property. It is easy to see that the path the directions define cannot contain arbitrarily long straight lines, because in this case there would exist also an infinitely long, correctly tiled straight path. More generally, for every $n > 1$ there must exist an integer N such that any finite path of length N that follows the directions and does not contain a tiling error has to visit all the tiles on a square consisting of $n \times n$ tiles. Namely, if there would exist arbitrarily long paths that do not cover any $n \times n$ square, then there would exist such infinite paths as well. This can easily be proved using a compact topology defined on the set of tilings (see, for example, [4] for the definition of the topology). This fact contradicts the plane-filling property. Consequently, the plane-filling curve cannot have the shape of a simple spiral, for example.

In Sections 3.2, 3.3, and 3.4 a specific set of directed tiles that possesses the plane-filling property is explicitly constructed, showing that such sets exist. The set is essential in the proof of the undecidability of the reversibility problem. As noted above, the tile set must be aperiodic. Several aperiodic tile sets are known [6]. The tiles used below resemble the ones used by Robinson in [13]. The paths that their directions define will have the self-similar shape of the so-called Hilbert curve [5].

3. Reversible CA

A CA is called reversible if there exists another CA that computes the inverse process. A CA is reversible iff it is injective, as Richardson proved in 1972 [12]. In this section we show that the reversibility of CA is an undecidable property.

3.1. *The Reversibility Problem*

The reversibility problem of cellular automata asks whether a given CA is reversible (or, equivalently, injective). This is a naturally arising problem, considering the importance of the notion of reversibility in physical systems. The problem has been extensively studied. Already in 1972 Amoroso and Patt gave an algorithm that decides whether a given one-dimensional local rule defines a reversible CA or not [1]. But the two- and higher-dimensional cases remained open. In the following we are going to show that in the case of two-dimensional CA that use the Moore neighborhood the reversibility problem is undecidable. In the last section we show this result can be extended for other neighborhoods like the von Neumann neighborhood.

The proof is based on a reduction from the tiling problem explained in Section 2.2. Also a specific set $\mathscr{D} = (D, N_M, R_0, d)$ of directed tiles satisfying the plane-filling property is needed (see Section 2.3). Such a set will be constructed later, which proves

PROPOSITION 2. *There exists a tile set with the plane-filling property. The tile set uses the Moore neighborhood* N_M.

Once we have the tile set $\mathscr{D}$ the reduction of the tiling problem to the reversibility problem of two-dimensional CA is very simple. All the complicated details of the reduction are contained in the construction of the tile set with the plane-filling property.

THEOREM 1. *Ii is undecidable whether a given two-dimensional cellular automaton with the Moore neighborhood is reversible.*

Proof. Let $\mathscr{T} = (T, N_M, R)$ be any ordinary tile set using the Moore neighborhood, that is, a set of tiles without directions, and let $\mathscr{D} = (D, N_M, R_0, d)$ be a set of directed tiles satisfying the plane-filling property. Let us construct a two-dimensional cellular automaton

$$\mathscr{A}_{\mathscr{T}} = (2, D \times T \times \{0, 1\}, N_M, f_{\mathscr{T}}),$$

such that $\mathscr{A}_{\mathscr{T}}$ is not injective if and only if the tile set $\mathscr{T}$ can be used to tile the plane.

The automaton $\mathscr{A}_{\mathscr{T}}$ uses the Moore neighborhood. Its states contain two tile components, one from D and one from T. In addition there is a bit, 0 or 1, attached to each state. The local rule $f_{\mathscr{T}}$ may change only the bits. The tile components are never changed. At each cell the tilings with both the D- and T-components are checked. If there is a tiling error in either of the components then the state of the cell is not altered. If the tilings are correct then the bit component is changed by performing the exclusive or (=xor) operation with the bit that is attached to the cell pointed by the direction of the D-component. The xor operation is the same as addition of the bits modulo 2.

Suppose that there exists a tiling of the plane with the set $\mathscr{T}$. Construct two configurations c_0 and c_1 of $\mathscr{A}_{\mathscr{T}}$ as follows: The tile components of c_0 and c_1 constitute the same legal tilings of the plane with the tiles of $\mathscr{D}$ and $\mathscr{T}$. In c_0 all bits are 0 while in c_1 they equal 1. Because the tilings are correct everywhere each bit is changed using the xor operation with the next bit on the path. This means that in both configurations c_0 and c_1 the bits are all changed to 0. So in $\mathscr{A}_{\mathscr{T}}$ there are two different configurations c_0 and c_1 that are turned into the same configuration in one time step and $\mathscr{A}_{\mathscr{T}}$ cannot be injective.

Conversely, suppose that $\mathscr{A}_{\mathscr{T}}$ is not injective. Let c_0 and c_1 be two different configurations that $\mathscr{A}_{\mathscr{T}}$ turns into the same configuration in one time step. The tile components of c_0 and c_1 must coincide. Consider a cell whose bit component is different in c_0 and c_1. The tilings must be correct at the cell, and the bits are different also in the cell pointed by the direction of the D-component. The reasoning can be repeated for this next cell. By continuing through succeeding cells it is concluded that the tiling properties may not be violated at any of them. Since $\mathscr{D}$ has the plane-filling property this path goes through arbitrarily large squares. So arbitrarily large squares can be tiled using the tiles of $\mathscr{T}$. This means that the whole plane can be tiled.

We showed that there is a valid tiling of the plane using the tile set $\mathscr{T}$ if and only if the cellular automaton $\mathscr{A}_{\mathscr{T}}$ is not injective, or equivalently, not reversible. If there were an algorithm for solving the reversibility problem, then this algorithm applied to $\mathscr{A}_{\mathscr{T}}$ would solve the tiling problem. This is not possible because the tiling problem is undecidable. ∎

In the following sections a set of directed tiles satisfying the plane-filling property will be constructed.

3.2. *The Basic Tiles*

To complete the proof of Theorem 1 we need to present $\mathscr{D} = (D, N_M, R_0, d)$, a set of directed tiles that satisfies the plane-filling property. In this section tiles without directions are defined. The directions are added in Section 3.3. The tiles we use resemble those presented by Robinson [13]. He used them as an example of tiles that permit only non-periodic tilings. The tiles are built in such a way that in all valid tilings the tiles form special squares constructed recursively from four smaller squares. On such squares the directions will define paths that have the shape of the well-known Hilbert curve. The Hilbert curve consists of four smaller Hilbert curves through the four quadrants of the square that are connected to each other as detailed in Section 3.3. Originally the Hilbert curve was used for showing that the unit square is a continuous image of the unit segment [5]. Similar plane-filling curves are called more generally Peano curves.

We do not use colors on the edges to define the relation R_0 that tells which tilings are valid. Instead we use labeled arrows. On a valid tiling each arrow head must meet an arrow tail in the neighboring tile with the same label. This is attained if the relation R_0 is defined so that the tiling is valid in a tile, if inside the 3×3 block that forms the Moore neighborhood of the tile, all arrow heads meet matching arrow tails and vice versa. The labels we use are *NE*, *NW*, *SE*, and *SW*, referring to the four cornerwise directions. Also we separate single and double arrows. First the arrows will be only horizontal and vertical. Later we introduce some diagonal arrows as well.

There are five different types of tiles represented in Fig. 2: single and double crosses and single, double, and mixed arms.

There is a *single* and *double cross* (Figs. 2a and 2b) for each one of the four labels. In the crosses all the four arrow heads have the same label. This label is called the label of the cross. The single crosses are special in the sense that they occur only in those positions of the plane where both coordinates are odd, and the tiles in these odd–odd positions must always be single crosses. This can be accomplished using parity tiles as in [13], but perhaps it is more convenient in our case to think that the final tiles will be composed of 2×2 blocks of the tiles considered here. Then we can restrict ourselves to blocks having exactly one single cross located in the upper right corner.

A *single arm* (Fig. 2c) consists of a single arrow leading through the tile, called the *principal arrow*, and two incoming *side arrows* at right angles to the principal

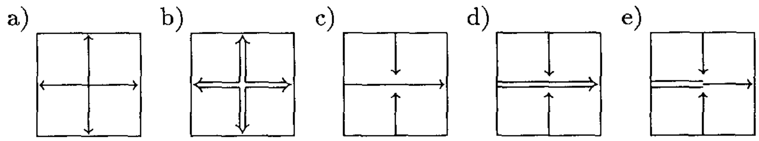

FIG. 2. The five types of tiles: (a) single cross; (b) double cross; (c) single arm; (d) double arm; (e) mixed arm.

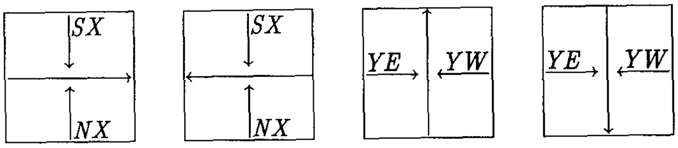

FIG. 3. The possible labels of the two side arrows on a single arm: $X \in \{E, W\}$ and $Y \in \{N, S\}$.

arrow. The arm may be rotated so that the principal arrow may point at any direction. This is called the direction of the arm. The label of the principal arrow may be any of the four labels, but the labels on the two side arrows are restricted. The possible labels are determined by the direction of the arm (see Fig. 3).

In case of a horizontal arm we have two possibilities: either the upper side arrow is labeled with SE and the lower one with NE, or the upper arrow is labeled with SW and the lower one with NW. If the arm is vertical then the possibilities are as follows: the left arrow NE and the right arrow NW, or the left arrow SE and the right one SW.

A *double arm* (Fig. 2d) is similar to a single arm. The only difference is that the principal arrow is double. This arrow may point at any direction and its label may be any of the four possible labels. The restrictions on the labels of the two side arrows are the same as with the single arm (Fig. 3).

In a *mixed arm* (Fig. 2e) the principal arrow changes within the tile. Its tail is double but the head is single. Its label does not change. The restrictions on the labels of the two side arrows are different from those with the single and double arms. They are shown in Fig. 4. Note that the direction of the mixed arm uniquely determines the labels of the two side arrows.

Let us now study tilings permitted by these five types of tiles. Remember, however, that the tiles are not final—some diagonal arrows will be added later. Also the directions need to be defined.

For each natural number n, four $(2^n - 1)$-XY-squares are defined recursively, one for each label XY ($XY = NE$, NW, SE or SW). The number $(2^n - 1)$ denotes the length of the sides of the squares. A single cross labeled XY forms the 1-XY-square. The $(2^{n+1} - 1)$-XY-square consists of four $(2^n - 1)$-squares separated by a double cross labeled by XY, called the central cross of the square, and rows of arms leading radiately out from the center (see Fig. 5). The arms are double near the central cross but in the halfway mixed arms make them single.

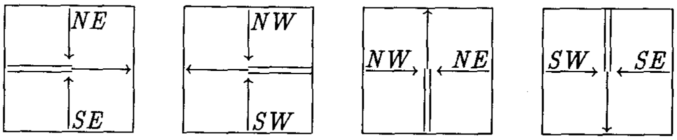

FIG. 4. The possible labels of the two side arrows on a mixed arm.

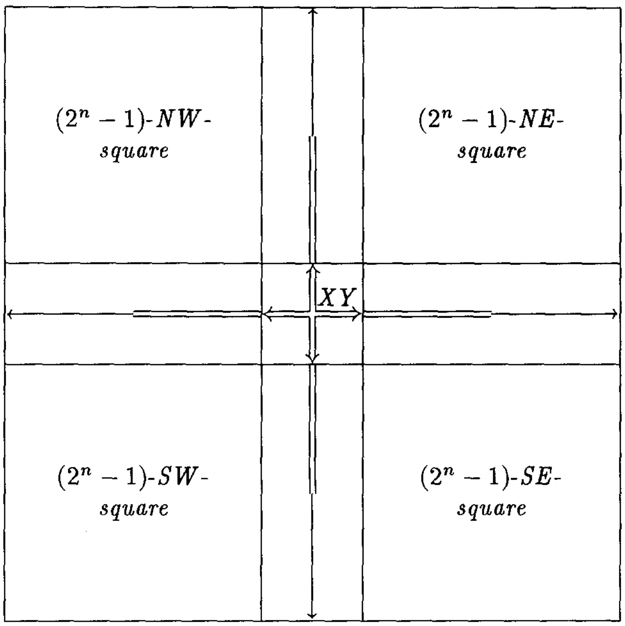

FIG. 5. Constructing the $(2^{n+1}-1)$-XY-square.

From now on, when we talk about (2^n-1)-squares we mean these special types of squares. The label of the central cross gives the name to the whole square. Now it is obvious why the labels are named after the four cornerwise directions: The label of a cross tells where the square, whose center the cross is, is supposed to be located inside the bigger square. In Fig. 6 there is an example of a seven-square.

Note that on the border of a square all the tiles have single arrow heads pointing outwards. The labels of these arrows on the northern border are NW or NE, except in the middle of the border where the label is the same as in the central cross. Similarly the arrows on the western, southern and eastern border are labeled XW, SX, and XE, respectively.

The tiling on the $(2^{n+1}-1)$-square is correct—also the restrictions we made for the labels of the two side arrows on arms are satisfied. Note especially how the different labels on the arrows in the middle of the borders of the (2^n-1)-squares make the mixed arms possible.

The following lemma proves that two different (2^n-1)-squares cannot overlap each other.

LEMMA 1. *Let S_1 and S_2 be two (2^n-1)-squares (with the same n) on the plane and suppose that there is a single cross on the plane that belongs to both S_1 and S_2. Then the squares S_1 and S_2 coincide.*

Proof. We prove the lemma using induction on n. If n is one then S_1 and S_2 consist of a single cross alone and the claim is trivially true.

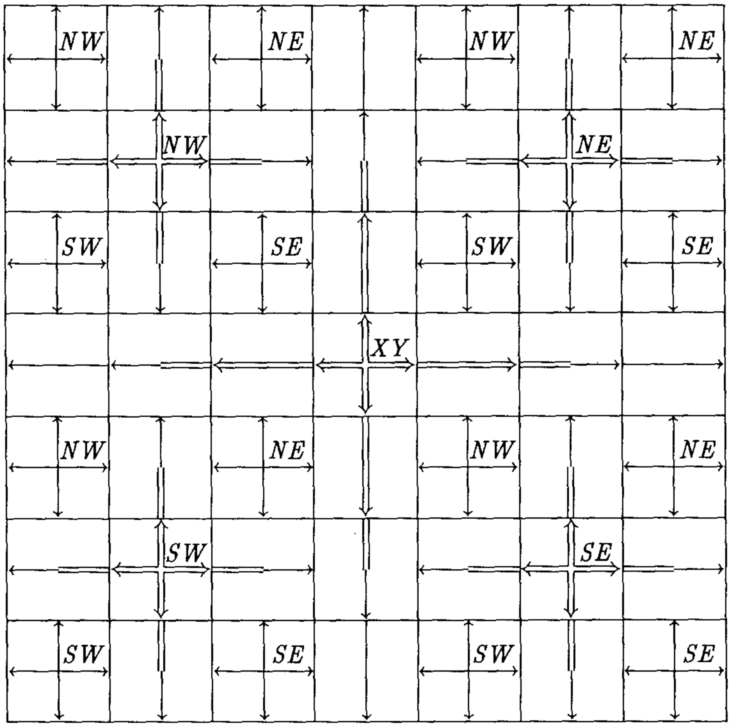

FIG. 6. The 7-XY-square with the labels on the crosses. Only the principal arrows of the arms are drawn.

Suppose then that the lemma has been proved for (2^n-1)-squares and let S_1 and S_2 be two $(2^{n+1}-1)$-squares sharing a common single cross. The single cross has to belong to one of the four (2^n-1)-squares that S_1 is composed of, and also to one of the (2^n-1)-squares that S_2 is composed of. According to the inductive hypothesis, S_1 and S_2 contain a common (2^n-1)-square. This square (its label and position) determines uniquely the position of the central cross of the $(2^{n+1}-1)$-square it belongs to. This means that the central crosses of S_1 and S_2 coincide. ∎

Let us now complete the basic tiles by adding some diagonal arrows to the tiles. Each tile has exactly one diagonal arrow from the upper left corner to the lower right corner and one diagonal arrow from the upper right corner to the lower left corner. The purpose of the diagonal arrows is to guarantee that the vertical and horizontal arms alternate on each diagonal row of tiles on the plane. There may be any number of crosses between the arms, but the next arm after a horizontal arm must always be vertical and vice versa. It is not difficult to prove that this is always the case in the (2^n-1)-squares (see Lemma 2 below).

The diagonal arrows are labeled either *Ver* or *Hor*. The head and the tail of the arrow may have different labels. In a valid tiling the arrow head and tail that meet must have the same label.

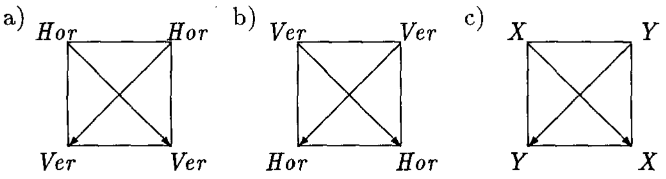

FIG. 7. The diagonal arrows on (a) horizontal arms, (b) vertical arms and (c) crosses ($X, Y \in \{Hor, Ver\}$).

On each horizontal arm the diagonal arrows have the label *Hor* on their tail and *Ver* on their head (see Fig. 7). On vertical arms the labels are the opposite: *Ver* on the tail and *Hor* on the head. The diagonal arrows of the crosses have always the same label on their tail and head. So, for each cross there are four possibilities to choose the labels of the diagonal arrows: Both of them may be labeled either *Hor* or *Ver*.

The diagonal arrows force the horizontal and vertical arms to alternate on each diagonal row of tiles. The tiling remains valid inside $(2^n - 1)$-squares also after the diagonal arrows are added.

LEMMA 2. *The $(2^n - 1)$-squares can be given diagonal arrows in such a way that the tiling remains valid.*

Proof. Using induction on n we show that the diagonal arrows can be added to the tiles of a $(2^n - 1)$-square so that the tiling is valid and

(1) the incoming diagonal arrows on the left and right edges of the square are labeled *hor* and on the upper edge with *ver*, and

(2) the outgoing diagonal arrows on the left and right edge of the square are labeled *ver* and on the lower edge with *hor*.

The arrows entering and leaving the square in the corners may have either label.

The case $n = 1$ is trivially true. In this case the square consists of a single cross whose both diagonal arrows are in the corners.

Assume that the claim has been proved for $(2^n - 1)$-squares and let S be a $(2^{n+1} - 1)$-square. Let S_1, S_2, S_3, and S_4 denote the $(2^n - 1)$-squares situated in the *NW*-, *NE*-, *SW*-, and *SE*-quadrants of S, respectively. Consider any diagonal row of tiles in S. We have several cases depending on the position of the row. We consider only one possibility here—the other cases are similar. Assume that the diagonal row enters S in the upper half of the left edge of S. First the row goes through S_1 from the left edge to the lower edge. The next tile on the row is a horizontal arm situated in between the squares S_1 and S_3. Then the row goes through S_3 from its upper edge to its right edge. Next it goes through a vertical arm which is between S_3 and S_4. Finally the row enters S_4 from its left edge and comes out from the bottom edge of S_4.

If the diagonal arrows on the squares $S_1, \ldots, S_4$ are labeled so that (1) and (2) are satisfied, then the labels of the diagonal arrows match each other on the row described above. Also (1) and (2) are satisfied on S. The other diagonal rows of tiles through S are similar. ∎

In the following technical lemma we consider a $(2^n - 1)$-square on the plane. We assume that the tiling property is not violated in any of its tiles, which means, by definition, that also in the tiles immediately outside the square each arrow head meets an arrow tail with the same label and vice versa. (Remember that a tiling is correct in a tile, if the arrows match in the Moore neighborhood of the tile.) Then the tiles immediately outside the square—the tiles in the Moore neighborhood of the square—must be the "correct" ones. By the "correct" tiles we mean the kind of tiles that allow the tiling to be extended into a $(2^{n+1} - 1)$-square containing the original $(2^n - 1)$-square. The lemma is proved for the SW-squares, but the other cases are symmetrical.

LEMMA 3. *Let S be a $(2^n - 1)$-SW-square on the plane. Suppose that the tiling is correct in every tile of the square. Then the tile just outside the upper right corner of the square (tile A in Fig.* 8) *is a double cross. The tiles below (in the area F in Fig.* 8) *and on the left side (in the area G) of that tile are double arms. In the midway (tiles B and C) there are mixed arms and after them (in areas H and I) the arms are single. Both rows of the arms end in the corners of the square: the horizontal row in the vertical arm situated in the tile E of Fig.* 8 *and the vertical row in the horizontal arm of the tile D.*

Proof. Suppose that the tile A of Fig. 8 is not a cross but an arm. Then the tiles below it (in the area F) should be arms pointing upwards, or the tiles on its left side (area G) are arms pointing to the right. Without loss of generality assume the first.

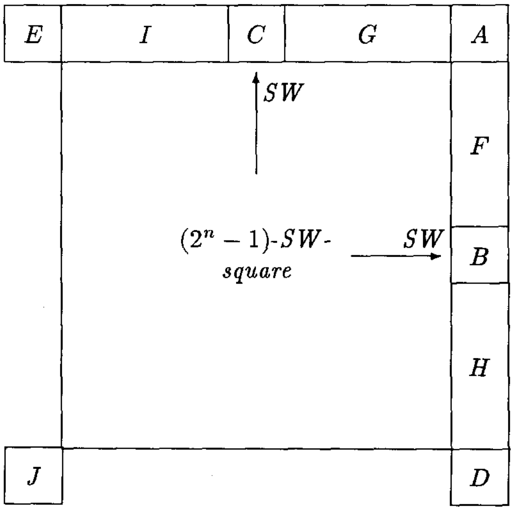

FIG. 8. The SW-square used in Lemma 3.

Then the tile B should be an arm pointing upwards. This is impossible, because the arrow coming to the tile B from the left has the label SW (see the restrictions for the labels of the side arrows in Fig. 3 and 4). We conclude that the tile A is a double cross.

The tiles below A, that is, the tiles in the area F must be double arms pointing downwards. This is because the double and mixed arms are the only tiles with an incoming double arrow. None of the tiles in F can be a mixed arm since a downwards pointing mixed arm has a side arrow with the label SW on its left edge (see the restrictions in Fig. 4), while the arrows entering the tiles of F from the left side have labels NE and SE. For the same reason the tile B has to be a mixed arm pointing downwards.

Below, in the area H, the tiles must be single arms pointing downwards. Note that the diagonal arrows guarantee that the tiles in H must be vertical, not horizontal, arms. This follows from the fact that the diagonal arrows entering the area H from the (2^n-1)-square are labeled with *ver* (cf. the proof of Lemma 2).

In a similar way we conclude that the tiles in G, C, and I have to be double, mixed, and single arms, respectively, pointing to the left.

Finally, consider the tiles D and E of Fig. 8. They must both be arms. They cannot both be horizontal or vertical, because they are on the same diagonal row with only crosses between them (remember the diagonal arrows). We want to show that D is horizontal and E is vertical. Assume the opposite. If D is vertical then the tiles on its left side have to be single arms pointing to the right. All the downward arrows on the lower border of the square have labels SW or SE, so that none of the arms on the left side of D can be mixed. We conclude that the tile J must have a single outgoing arrow on its right edge. Similarly, it must also have a single outgoing arrow on its upper edge. This means that the tile J has to be a single cross, which is not possible, because the single crosses are allowed only in the odd–odd positions of the plane. We conclude that the tile D is a horizontal arm and E a vertical arm, as required. ∎

3.3. *The Directions*

Next we add the directions to the basic tiles. Let us first see what kind of paths we want the directions to define. The paths are constructed recursively through squares consisting of $2^n \times 2^n$ tiles, for all $n > 0$. For every n there will be four different paths (all obtainable from each other through rotation and/or flipping) through the square of $2^n \times 2^n$ tiles. One of the paths starts from the lower right corner of the square, visits all the tiles of the square, and ends up at the lower left corner of the square. This path is called the A_{2^n}-path. The B_{2^n}-path starts from the upper left corner of the square and ends in the upper right corner, the C_{2^n}-path goes from the lower right corner to the upper rigt corner, and the D_{2^n}-path goes from the upper left corner to the lower left corner. In Fig. 9 are the shortest non-trivial examples, A_2, B_2, C_2, and D_2.

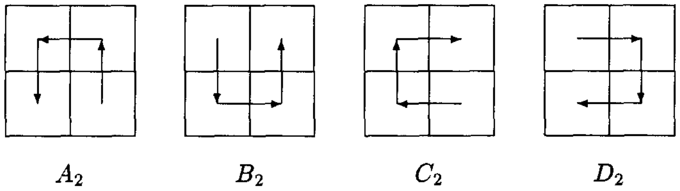

FIG. 9. The four different paths through the squares of 2×2 tiles.

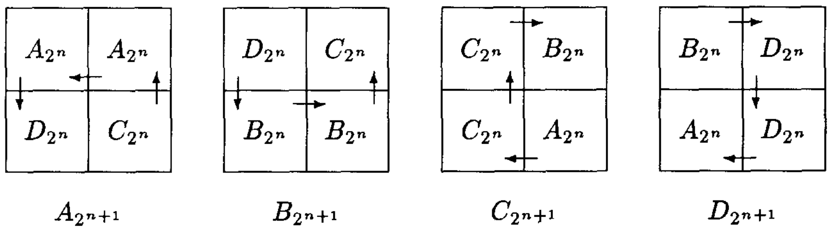

FIG. 10. Constructing paths through squares of $2^{n+1} \times 2^{n+1}$ tiles.

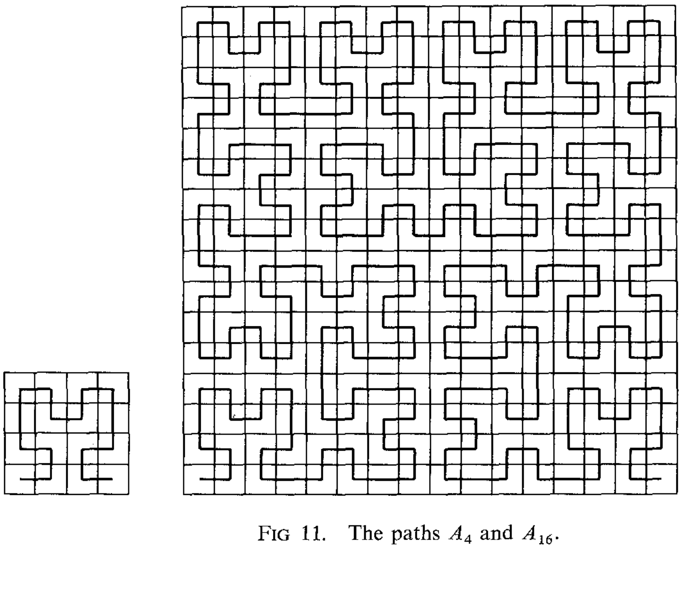

FIG 11. The paths A_4 and A_{16}.

The paths through the squares of $2^{n+1} \times 2^{n+1}$ tiles are defined recursively using the paths through the squares of $2^n \times 2^n$ tiles. The path $A_{2^{n+1}}$ is obtained by combining the paths C_{2^n}, A_{2^n}, A_{2^n}, and D_{2^n} as shown in Fig. 10. The path starts in the lower right corner and goes through the lower right quadrant of the square following the C_{2^n}-path. From the last tile the path continues upwards, where an A_{2^n}-path through the upper right quadrant starts. From the last tile on this quadrant the path continues to the left. The path follows an A_{2^n}-path through the upper left quadrant, and finally a D_{2^n}-path through the lower left quadrant, ending up in the lower left corner of the square.

The B-, C-, and D-paths are composed in a similar way from the shorter paths as depicted in Fig. 10. It is obvious from the way that the paths are constructed that all of them visit all the tiles of the square. As an example, the paths A_4 and A_{16} are shown in Fig. 11.

Let us see next how we can decide into which direction the path should continue from an arbitrary tile on the path. Consider a square of $2^n \times 2^n$ tiles through which we want to define the A_{2^n}-path. Let us call such a square an A-square. As we know the A-square is divided into four quadrants as depicted in Fig. 10: two A-squares, one D-square, and one C-square. These four squares are further divided into four quadrants each. This process can be continued until squares consisting of one tile are reached. Each tile in the $2^n \times 2^n$ square belongs to $n+1$ different A-, B-, C-, or D-squares of different sizes: squares consisting of 1×1, 2×2, 4×4, ..., $2^n \times 2^n$ tiles.

In Fig. 10 one can see that the path continues upwards from a tile which is situated in the upper right corner of the lower right quadrant of any A-square. Such a tile is, namely, necessarily the last tile on the C-path through the lower right quadrant of the A-square. Similarly, the path continues to the left from a tile situated in the lower left corner of the upper right quadrant of an A-square, and downwards from the lower left corner of the upper left quadrant of an A-square. These three rules guarantee that an A-path is composed from the four shorter paths in the correct way.

Similar rules based on Fig. 10 can be made for the connecting tiles of the B-, C-, and D-paths as well. Altogether we have 12 rules. The rules are summarized in Table I.

TABLE I

The Rules Describing the Path Direction from an Arbitrary Tile on an A-, B-, C-, or D-Path

The path continues from a tile	
Upwards,	if it is situated in the upper right corner of either the lower left quadrant of a C-square or the lower right quadrant of an A- or B-square.
To the right,	if it is situated in the upper right corner of either the lower left quadrant of a B-square or the upper left quadrant of a C- or D-square.
Downwards,	if it is situated in the lower left corner of either the upper right quadrant of a D-square or the upper left quadrant of an A- or B-square.
To the left,	if it is situated in the lower left corner of either the upper right quadrant of an A-square or the lower right quadrant of a C- or D-square.

Note that the rules in Table I are enough to uniquely determine the direction in which the path continues from each tile (except the last tile on the path) in any A-, B-, C-, or D-square which is recursively divided into quadrants. Exactly one rule in the table can be applied to each tile. This is true for all other tiles on the path except the last one, for which none of the rules is applicable.

Let us now fix the directions to the basic tiles introduced in the previous section. As mentioned earlier the final tiles will be composed of 2×2 blocks of basic tiles. Then every tile will have exactly one single cross situated in its upper right corner. So, it is enough if we attach directions only to the single crosses, and the directions refer to one of the four single crosses situated to the north, west, south, or east. Then every 2×2 block will have exactly one direction pointing to one of the neighboring blocks.

The idea of the directions is that the paths they define through the $(2^{n+1}-1)$-squares are the paths $A_{2^n}, ..., D_{2^n}$ that visit all the single crosses in the square. Note that a $(2^{n+1}-1)$-square contains exactly $2^n \times 2^n$ single crosses. In order to control which one of the four possible paths (A-, B-, C-, or D-path) the directions define in a specific square, we give the arrows of the basic tiles new labels. Each arrow (single and double) can be labeled with A, B, C, or D. In a valid tiling the meeting arrow heads and tails must have the same label. The new labeling is independent of the old labeling with the directions NW, NE, SW, and SE.

The only restrictions we impose for the new labels concern crosses and mixed arms. In each cross all four arrow heads must have the same label. The purpose of the labeling is that the label on the central cross of a $(2^{n+1}-1)$-square (called the label of the square from now on) tells which one of the four paths the directions define on the square.

The mixed arms connect the label of the $(2^{n+1}-1)$-square and the labels of the four (2^n-1)-squares inside it to each other. The idea is that the mixed arms force the labels of the four quadrants to be the correct ones, that is, the ones given in Fig. 10. If the $(2^{n+1}-1)$-square is labeled, for example, with A, then the (2^n-1)-square in its upper left quadrant must have label A, the square in the lower left quadrant must be labeled with D, the square in the lower right quadrant with C and the square in the upper right quadrant with A (see Fig. 10). This is obtained by restricting the mixed arms whose principal arrow is labeled with A to those in Fig. 12. For example, the mixed arm pointing to the right, whose principal arrow is labeled with A, must have the labels A and C on the side arrows in its upper and lower edge, respectively. This forces the (2^n-1)-squares situated in the upper right and lower right quadrants of the $(2^{n+1}-1)$-square to have labels A and C. Similar restrictions are imposed on the mixed arms whose principal arrow is labeled with B, C, or D. The restrictions can be read in the same way from Fig. 10.

What direction a specific single cross on the plane should have can be seen in Table I. The conditions in the table should, of course, be checked locally at each single cross, so that the neighbors of the cross determine its direction. This is indeed possible since the rules in Table I are equivalent to the ones presented in Table II. For example, a single cross is situated in the upper right corner of the lower right

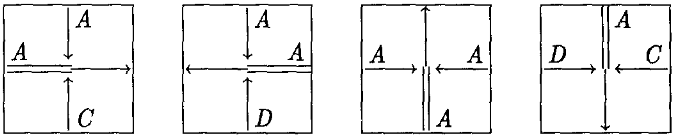

FIG. 12. The labeling of mixed arms whose principal arrow has the label A.

quadrant of a (2^n-1)-square labeled with A if and only if its NE-neighbor is a vertical arm whose left edge has a side arrow with the label A. Naturally we have to assume that the single cross (and its Moore neighborhood) is inside some legally tiled (2^m-1)-square, $m>n$. In a similar way all 12 rules of Table I can be checked locally at each single cross as summarized in Table II.

We can make the convention that there is a tiling error in a single cross, if none of the rules in Table II can be applied, or if more than one of them are applicable. There still exist legal tilings of the plane after this convention, because inside every (2^n-1)-square exactly one of the rules can be applied to each single cross. This fact is stated in the following lemma.

LEMMA 4. *Let S be a (2^n-1)-square on the plane. For each single cross whose Moore neighborhood is contained in S there is exactly one rule in Table II that can be applied.*

Proof. We skip the exact proof—we just point out that the rules in Table II are equivalent to those in Table I inside properly tiled (2^n-1)-squares. It was mentioned earlier that for each tile in an A-, B-, C-, or D-square (except one tile in a corner, which is the last tile on the path through the square), which is recursively divided into quadrants with proper labels, there is exactly one rule in Table I that can be applied. ∎

The construction of the directions was made in such a way that the directions define an A_{2^n}-, B_{2^n}-, C_{2^n}-, or D_{2^n}-path (depending on what is the label of the central cross) through the single crosses of each $(2^{n+1}-1)$-square.

TABLE II

The Rules Defining the Directions of the Single Crosses

The direction of the single cross is	
N	if its NE-neighbor is a double cross with the label C or a vertical arm whose left edge has a side arrow with label A or B.
E	if its NE-neighbor is a double cross with the label B or a horizontal arm whose lower edge has a side arrow with label C or D.
S	if its SW-neighbor is a double cross with the label D or a vertical arm whose right edge has a side arrow with label A or B.
W	if its SW-neighbor is a double cross with the label A or a horizontal arm whose upper edge has a side arrow with label C or D.

The final requirement we make for correct tilings is the following: A tiling is incorrect in a tile unless exactly one of the surrounding four single crosses points towards the tile. So, each single cross has exactly one possible predecessor on a path. Note that this condition can be checked locally at each tile, since the four single crosses are contained in the neighborhood of the original tile, as the final tiles are blocks of 2×2 basic tiles. Note also that this requirement is satisfied on the A-, B-, C-, and D-paths.

3.4. *The Plane-Filling Property*

In this section we show that the directed tiles defined in the previous sections possess the plane-filling property. This will conclude the proof of Theorem 1. First, one should note that the entire plane can be tiled legally with the tiles. This follows from the fact that the tiling property is valid in the $(2^n - 1)$-squares for every n. This is true also after the diagonal arrows and directions are added, as shown in Lemmas 2 and 4.

To prove the plane-filling property we need to consider an infinite path following the directions of the tiles on the plane (we do not assume that the path does not repeat itself cyclically—it will follow from the plane-filling property that this cannot be the case). It is supposed that there is no tiling error on any of the tiles on this path. This means that inside the Moore neighborhoods of the tiles meeting arrow heads and tails match each other. It will be shown that for each n there is a $(2^n - 1)$-square on the plane, whose single crosses are all visited by the path.

LEMMA 5. *Let n be a natural number and let t be a single cross on the plane. Consider the path that goes via t and is defined by the directions. Suppose that there are no tiling errors in any of the 4^n tiles that precede and the 4^n tiles that succeed t on this path. Then t belongs to a $(2^n - 1)$-square, whose single crosses are all visited by the path. The path through the $(2^n - 1)$-square is a A-, B-, C-, or D-path if the central cross of the square has the label A, B, C, or D, respectively.*

Lemma 5 proves the plane-filling property. For any n take the single cross that is the $(4^n + 1)$th on the infinite path considered. According to the lemma this cross is on a $(2^n - 1)$-square whose single crosses are all on the path.

Proof. The lemma is proved using induction on n. If n is one then the single cross t is the required one-square.

Suppose that the lemma has been proved for n and assume that the 4^{n+1} single crosses preceding and succeeding t on the path do not contain tiling errors. We already know, by the inductive hypothesis, that t belongs to a path through a $(2^n - 1)$-square S_1 (see Fig. 13). Suppose without loss of generality that the square S_1 has the label SW (the other cases are symmetrical). According to Lemma 3 the tile in the upper right corner of the Moore neighborhood of the square S_1 (denoted with X in Fig. 13) is a double cross. The double cross can have any of the labels A, B, C, and D. The different cases are not completely symmetric, but the proof proceeds in a similar way in all cases. Let us show here the case when the double cross has the label C.

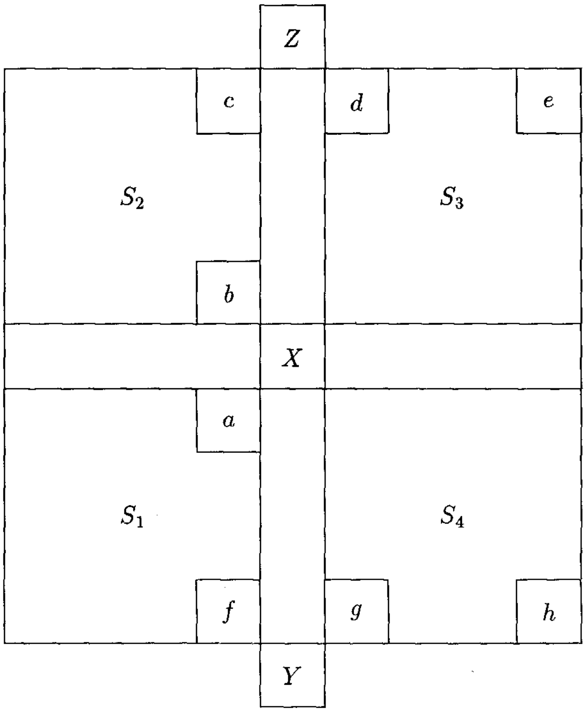

FIG. 13. The $(2^{n+1}-1)$-square constructed during the proof of Lemma 5.

Because the double cross X is labeled with C the central cross of the square S_1 has to have the label C as well (the mixed arm above S_1 takes care of that). This means that the path through the S_1 square is a C-path, starting from the tile f and ending up in the tile a (see Fig. 13). The single cross a has as its NE-neighbor the double cross X whose label is C, so that the direction of a must be N (see Table II). So the next tile on the path is the single cross b of Fig. 13.

The inductive hypothesis can be applied to b, because the number of tiles succeeding and preceding b on the correctly tiled path is at least $4^{n+1}-4^{n-1}>4^n$ (the number of single crosses in the square S_1 is 4^{n-1}). According to the inductive hypothesis b belongs to a (2^n-1)-square S_2. If the tile a would also belong to S_2 then the squares S_1 and S_2 would overlap which is impossible according to Lemma 1. We conclude that b must be the first tile on the path through the square S_2, so b must be situated in one of its corners. More precisely, b has to be in the SE-corner of S_2. (If $n>1$ then the label of a must be NE, which means that the label of b has to be SE. This follows from the fact that the tile between them is a horizontal arm, where the labels of the two side arrows must satisfy the conditions presented in Fig. 3. Because b has the label SE, it cannot be in any other corner but in the SE-corner of S_2. If $n=1$ then S_2 consists of b alone, in which case b is trivially in the SE-corner of the square.) We conclude that S_2 is situated above S_1, as given in Fig. 13.

The central cross of S_2 has to be labeled with C. This means that the path continues from b through all the single crosses of S_2 to the tile c in the upper right corner of S_2. According to Lemma 3 (or its counterpart for NW-squares) the tile Z is a horizontal arm. This means that the direction of tile c has to point to the right, to the tile d of Fig. 13 (see Table II). Again, in the same way as for the tile b and square S_2, we can conclude that d must be the first tile on a B-path through the single crosses of the (2^n-1)-square S_3 that is situated on the right side of S_2. The path ends in the upper right corner of S_3 (tile e in Fig. 13).

Consider then the tile f, where the C-path through S_1 starts. The single cross directly on the right side of f (the tile g in Fig. 13) has to have the direction W; that is, it has to point towards f. This follows because the tile Y, the SW-neighbor of g, is a horizontal arm whose upper edge has a side arrow labeled with C (see Lemma 3 and Table II). It is important to note that, because the tiles are 2×2 blocks of the basic tiles, tiles g and Y are contained in the Moore neighborhood of f, and the tiling is known to be correct in the Moore neighborhood of f—otherwise the tiling in g could be incorrect, and g could point to some other direction. In a correct tiling each single cross has only one tile pointing towards it, so that g must be the unique predecessor of f on the path.

Again, according to the inductive hypothesis, g is known to belong to a (2^n-1)-square S_4. In the same way as above we conclude that the square S_4 is on the right side of S_1, and its central cross has the label A. The A-path through S_4 starts from h, the tile situated in the lower right corner of S_4. Altogether the A-, C-, C-, and B-paths through the squares S_4, S_1, S_2, and S_3, respectively, form a C_{2^n}-path through the $(2^{n+1}-1)$-square whose central cross is X. The tile t is on this path, as required.

The other cases (the label of X is A, B, or D) are similar. ∎

Lemma 5 shows that the tile set constructed in Sections 3.2 and 3.3 has the plane-filling property. This proves Proposition 2 and completes the proof of Theorem 1 in Section 3.1. ∎

4. Surjective CA

This section investigates the problem of testing the surjectivity of CA. Like the injectivity also the surjectivity of one-dimensional CA was proved decidable by Amoroso and Patt [1]. The higher dimensional cases turn out to be undecidable.

Recall from Section 2.1 the definitions for a quiescent state and finite configurations. Remember that G_f^F denotes the restriction of the global function G_f to the set of finite configurations. The well-known Garden of Eden theorem states that the global function G_f is surjective if and only if its restriction G_f^F is injective. This fact was proved by Moore [9] and Myhill [11]. Explicitly the form above was used by Richardson [12]. The Garden of Eden theorem reduces the testing of the surjectivity of G_f to the testing of the injectivity of G_f^F. The definitions in Section 2.1

allow us to talk about G_f^F only in case there is a quiescent state. It might well be so that the CA has no state q that satisfies the special property

$$f(q, q, \ldots, q) = q.$$

In this context, however, the special property above is superfluous. We can name any state s as the quiescent state and term as finite those configurations that have only finitely many cells in states other than s. Still the Garden of Eden theorem holds true. This can be seen as follows. Let φ be the permutation of the state set that exchanges s and $f(s, s, \ldots, s)$ and keeps the other states unchaged. Then s satisfies the quiescent state property with respect to $f \circ \varphi$. According to the Garden of Eden theorem, $G_{f \circ \varphi}$ is surjective if and only if its restriction to the set of finite configurations is injective. But $G_{f \circ \varphi}$ is surjective iff G_f is, and its restriction is injective iff G_f^F is injective.

4.1. *The Finite Tiling Problem*

In the following the problem of testing whether or not G_f^F is injective is proved undecidable. The method used resembles the method used in the previous section to prove the similar result for G_f. However, there is one basic difference between the two injectivity problems. It was mentioned in Section 2.1 that a CA is injective iff it has an inverse automaton. It follows from this fact that we can *effectively enumerate* all injective CA: just enumerate all cellular automata and check for each pair of automata if they are inverses of each other (this can be checked easily). If so, add them to the list of injective CA. On the other hand, the non-injective CA cannot be effectively enumerated because the injectivity problem is undecidable.

With G_f^F the situation is just the opposite. The cellular automata with a non-injective G_f^F can be effectively enumerated while this is not true for the set of CA with injective G_f^F.

In Section 3 the reduction from the tiling problem was done in such a way that the tiling problem had a solution iff the corresponding G_f was not injective. This reduction can be done since the tile sets that possess a solution can not be effectively enumerated. Doing the same with G_f^F is not possible—a similar reduction would map the tile sets with valid tilings into the effectively enumerable set of non-injective G_f^F.

This problem is solved by introducing a new tiling problem, called *the finite tiling problem*. The tile sets that possess the finite tiling property can be effectively enumerated. So this problem seems better suited for our purpose.

In the finite tiling problem we are given a finite collection of tiles $\mathcal{T} = (T, N, R)$ as before. The tile set must contain a special *blank tile* $b \in T$ that satisfies the property

$$(b, b, \ldots, b) \in R.$$

If the relation R is defined using colors on the edges of the tiles, then b has the same color on all four sides. Valid tilings are defined as before—the neighbors of each tile must satisfy the relation R. There is, of course, at least one valid tiling, namely the one where all the tiles are blank. This tiling is called *trivial*. A tiling is called *finite* if only finitely many of the tiles used are not blank. The problem is to decide whether there exists a valid, finite, and non-trivial tiling of the plane. This problem is undecidable.

PROPOSITION 3. *It is undecidable whether a given collection $\mathcal{T} = (T, N_M, R)$ of tiles with the Moore neighborhood, that contains a blank tile b can be used to form a finite tiling that is valid but not trivial.*

Proof. The proof is a simple reduction from the halting problem for Turing machines started on a blank tape. This is a well-known undecidable problem.

A Turing machine consists of a finite state set S and a finite alphabet A of tape symbols. The tape is infinite in both directions. A special tape symbol a_0 in A is called blank. In S there are two special states: s_0 is the initial state and s_h is the halting state of the machine. The machine works under some rules of the form

$$(a, s) \rightarrow (a', s', d), \qquad \text{where} \quad a, a' \in A,\ s, s' \in S,\ d \in \{L, R, S\}.$$

The rule says that if the machine is in state s and scans the tape symbol a, it will overprint a by a', change its state to s', and move its read/write head as d indicates. If d is S then the head remains in the current position, if d is L the head moves one symbol to the left, and if d is R the head moves one symbol to the right. The halting problem asks whether a given Turing machine, when started on a blank tape in the initial state s_0, can eventually change into the halting state s_h.

We now describe how the operation of a Turing machine can be represented with tiles on the plane. The tiles we use are similar to the tiles presented in [13]. The tape of the Turing machine runs horizontally on the plane while the vertical direction represents time. Two configurations of the Turing machine at two consecutive time steps are represented on the lower and upper edges of a row of tiles.

The tiles have, once again, labeled arrows on their edges. As before, an arrow head must meet an arrow tail with the same label in the neighboring tile.

The initial configuration of the Turing machine is represented using the initial tiles of Fig. 14. The horizontal arrows of the initial tiles have always the same label I. The possible labels on the vertical arrows are L, R, a_0 (representing the blank symbol) and s_0a_0 (representing the read/write head in the initial state reading a blank symbol). The labels L and R denote the ends of the tape. Although the tape is infinite the machine uses only a finite portion of it, provided the machine halts after a finite number of time steps.

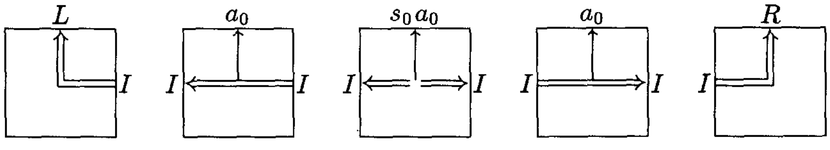

FIG. 14. The initial tiles.

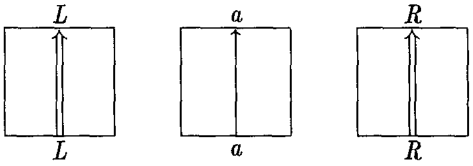

FIG. 15. The static tiles—a denotes any tape symbol.

The subsequent configurations of the Turing machine are represented by the static tiles of Fig. 15 together with the action tiles of Fig. 16. The static tiles correspond to those symbols on the tape that are not scanned by the read/write head just before or after the time step considered. They remain unaltered on that instant. There is a static tile for every tape symbol a as well as for L and R.

The action tiles of Fig. 16 represent computation steps of the Turing machine. For every rule

$$(a, s) \to (a', s', d)$$

of the machine there is one tile of the type 16a, b, or c (see Fig. 16). If the direction d is L then the tile is the one in 16a, if d equals S then the tile is 16b and in case of R it is 16c. For every state s and tape symbol a there are the two merging tiles of Fig. 16d.

A configuration where the machine is in the halting state s_h is represented by the halting tiles of Fig. 17. The horizontal arrows have the label H, and a denotes any tape symbol.

The last tile we need is the blank tile. It does not have any arrows in any of its four sides. Next we show that the tiles above can be used to make a finite, valid, and non-trivial tiling of the plane if and only if the corresponding Turing machine halts when started on a blank tape.

First, suppose that the Turing machine halts. Let n be the length of the halting computation. Construct a row of initial tiles, where there are at least n tiles between the read/write head and the right and left ends of the tape. The rows of tiles above the initial row simulate the computation of the Turing machine. The nth row is constructed using the halting tiles. All the tiles used are inside a rectangle bordered by double arrows. The tiles outside the rectangle are blank. This is the required tiling.

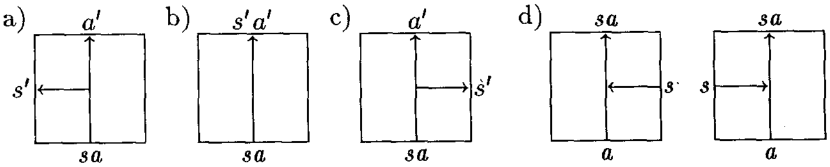

FIG. 16. The action tiles. The head (a) moves left; (b) stays at the current position; or (c) moves right. The merging tiles are depicted in (d).

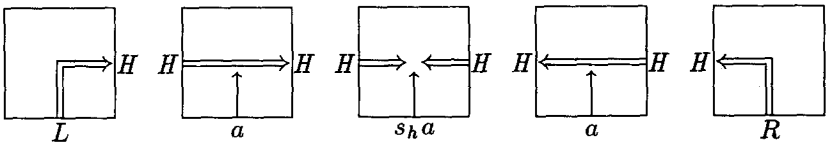

FIG. 17. The halting tiles.

On the other hand, suppose there is a valid, finite, and non-trivial tiling of the plane using the tiles above. The initial tiles are the only tiles where there is no arrow on the lower edge of the tile. So there must be an initial tile on the plane. Obviously this tile must belong to a row of initial tiles, starting from the left end labeled L, containing one read/write head, and ending to the right end labeled R. The tiles above the left end must be static tiles with the label L, until a halting tile is encountered. The same is true for the right edge as well. The halting tiles of the two edges must be at the same level, and all tiles between them must be halting tiles.

The two edges together with the initial and halting tiles isolate a rectangle on the plane. Inside the rectangle computation steps of the Turing machine must be simulated. The computation halts because the halting state is eventually encountered. ∎

4.2. *A Modified Set of Tiles*

Next we turn our attention to the set $\mathscr{D} = (D, N_M, R_0, d)$ of directed tiles with the plane-filling property that was presented in Section 3. The tiles must be changed slightly to allow finite tilings of the plane. In valid finite tilings the paths that the directions define on the plane should be closed loops.

More precisely, a set $\mathscr{D}' = (D', N_M, R_0', d)$ of directed tiles is said to satisfy the *finite version* of the plane-filling property, if there are two disjoint subsets $\mathscr{X}$ and $\mathscr{Y}$ of the tile set $\mathscr{D}'$ such that the following conditions are fulfilled:

1. For every $n > 0$ there exists a valid tiling of the plane containing a hollow square whose sides are longer than n tiles and are composed of tiles belonging to $\mathscr{Y}$. The tile in the center of the square belongs to $\mathscr{X}$. There is a closed path of finite length following the directions of the tiles that passes through all tiles inside and on the borders of the square. (It may also visit tiles outside the square.) The only tiles in the square that belong to $\mathscr{X}$ or $\mathscr{Y}$ are the tiles in the center and on the sides of the square, respectively.

2. For an arbitrary tiling of the plane and an arbitrary path following the directions one of the following three alternatives is true:

   (a) there is a tiling error on the path,

   (b) the length of the path is infinite; that is, it does not form a closed loop, or

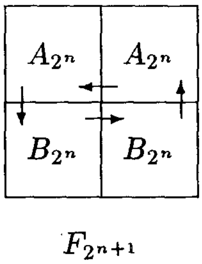

FIG. 18. Constructing the closed $F_{2^{n+1}}$-path from the shorter A_{2^n}- and B_{2^n}-paths.

(c) the path forms a closed loop that visits all tiles of a square of $n \times n$ tiles, for some $n > 0$, whose center tile belongs to $\mathscr{X}$ and the tiles on the border belong to $\mathscr{Y}$.

This complex behavior is obtained if the set $\mathscr{D}$ presented in Section 3 is modified by introducing new labels F and G that can be used instead of the labels A, B, C, and D on the arrows. The directions of the single crosses inside a $(2^n - 1)$-square whose central cross has the label F define a closed path through the single crosses of the square. This path is called an F-path and it is composed of the A- and B-paths as presented in Fig. 18.

The F-path is forced in the same way as the A-, B-, C-, and D-paths through the $(2^n - 1)$-squares with the corresponding label in the central cross. First, the labels of the two side arrows on the mixed arms whose principal arrow is labeled F are restricted as depicted in Fig. 19. This forces the four quadrants inside the $(2^n - 1)$-square to have correct labels. Next, we add to the rules of Table II that govern the directions of the single crosses the following ones (readable from Fig. 18): The direction of a single cross is

1. W, if its SW-neighbor is a double cross with the label F,
2. E, if its NE-neighbor is a double cross with the label F,
3. S, if its SW-neighbor is a vertical arm whose right edge has a side arrow with label F,
4. N, if its NE-neighbor is a vertical arm whose left edge has a side arrow with label F.

The label G is needed only to make sure that a $(2^n - 1)$-square labeled with F can exist in a valid tiling of the whole plane. The mixed arms whose principal arrow is labeled with G are restricted in the following way: The two side arrows may be labeled either both with F or both with G.

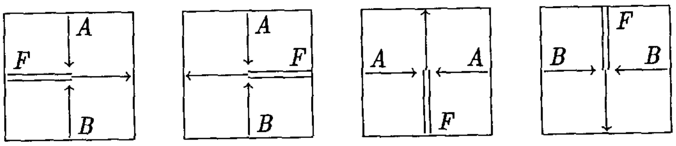

FIG. 19. The labeling of mixed arms whose principal arrow is labeled with F.

To avoid the trivial case of a 1-square labeled with F, we add the restriction that the label of a single cross can never be F or G.

LEMMA 6. *For each $n > 1$ there exists a tiling that is valid everywhere and contains a $(2^n - 1)$-square labeled with F. The directions define a closed F-path through the $(2^n - 1)$-square.*

Proof. Consider a valid tiling of the plane with the basic tiles of Section 3.2. The plane can be tiled with the basic tiles in such a way that each single cross on the plane belongs to one 1-square, one 3-square, one 7-square, etc. Let us label all single and double crosses in such a way that each cross which is the central cross of a $(2^n - 1)$-square will be labeled with F and each cross which is the central cross of a $(2^m - 1)$-square for some $m > n$ will be labeled with G. The labels of the other crosses (which are central crosses of $(2^m - 1)$-squares for $m < n$) are chosen so that the tiling remains valid. This means that the upper left quadrant of an F-square will be labeled with A (see Fig. 18), the lower left quadrant with B, etc. The labels of the arms are uniquely defined by the labels of the crosses.

The tiling property is obviously satisfied everywhere after this labeling. Also the direction of each single cross is uniquely determined, and each single cross has exactly one predecessor as required. Note especially that the directions define closed loops inside the $(2^n - 1)$-squares. The loops visit all single crosses of the squares. ∎

The following lemma is the counterpart of Lemma 5 for the extended set of tiles. It states that every infinite path defined by the directions through tiles with valid tilings either goes through arbitrarily large squares or it forms a closed path through a $(2^n - 1)$-square whose central cross is labeled with F.

LEMMA 7. *Let n be a natural number and let t be a single cross on the plane. Suppose that the directions define a path on the plane that goes via t and where there are no tiling errors in any one of the 4^n tiles that precede and the 4^n tiles that succeed t on this path. (The tiles do not need to be separate—the path can form a loop.) Then there are two possibilities:*

(1) *t belongs to a $(2^n - 1)$-square, whose single crosses are all visited by the path. The path through the (2^{n-1})-square is a A-, B-, C-, or D-path if the central cross of the square has the label A, B, C, or D, respectively.*

(2) *t belongs to a $(2^m - 1)$-square, for some $m \leqslant n$, whose central cross is labeled with F and whose single crosses are all visited by the path. The path is the closed F-path.*

Proof. The proof is carried out in the same way as the proof of Lemma 5. We use induction on n. If $n = 1$ then t alone is the $(2^n - 1)$-square and the case (1) of the lemma holds true.

Next, assume that the lemma has been proved for n and suppose that the 4^{n+1} single crosses preceding and succeeding t on the path do not contain tiling errors.

According to the inductive hypothesis either t belongs to a path through a (2^n-1)-square S_1 that is labeled with A, B, C, or D, or t belongs to an F-path through a (2^m-1)-square, for some $m \leqslant n$. In the latter case the condition (2) of the lemma is satisfied (of course $m \leqslant n+1$). Let us then assume the first case. We can suppose without loss of generality that S_1 has the label SW.

Lemma 3 states that the tile in the upper right corner of the Moore neighborhood of the square S_1 is a double cross. The double cross can have any of the labels A, B, C, D, and F. If the label is A, B, C, or D then the proof continues exactly in the same way as in Lemma 5. (Actually only the case of the label C was presented in the proof of Lemma 5—the others are similar.) In these cases condition (1) of the lemma is true.

The proof proceeds in the same way also if the label of the double cross in the upper rigt corner is F. In this case the path is forced to follow the F-path through the four quadrants of the $(2^{n+1}-1)$-square whose center is the double cross. In this case condition (2) of the lemma is satisfied. ∎

We need to add one more marker to the arrows. Any arrow in any tile can be marked to be a *border* arrow. In arms, if the tail (or, head) of the principal arrow is a border arrow then also the head (tail, respectively) is marked to be a border arrow. In valid tilings, if one of the meeting arrow tails and heads is a border arrow, then both of them are. These requirements guarantee that in valid tilings in each row of arrows that follow each other, either all arrows are border arrows or none of them is. The essential restriction we make concerning border arrows is the following: In a mixed arm whose principal arrow is labeled with F, the two side arrows must be border arrows. This restriction forces the tiles between the central crosses of the four quadrants of a (2^n-1)-square labeled with F to contain border arrows. These border arrows form a hollow square around the center of the (2^n-1)-square.

Now the modifications of the set $\mathscr{D}=(D, N_M, R_0, d)$ of directed tiles are complete. We obtain a new set $\mathscr{D}'=(D', N_M, R_0', d')$ that consists of 2×2 blocks of the tiles described above.

Define subsets $\mathscr{X}$ and $\mathscr{Y}$ of the tile set D' as follows: $\mathscr{X}$ contains tiles with double crosses that are labeled with F, and $\mathscr{Y}$ contains all tiles with a border arrow. The finite version of the plane filling property is satisfied by $\mathscr{D}'$. Condition 1 follows from Lemma 6 and from the facts that the center of the F-square is the only tile belonging to $\mathscr{X}$ in the square, and that the border arrows form a hollow square around the center. Condition 2, on the other hand, follows from Lemma 7. Lemma 7 states namely that a path following the directions either contains a tiling error, does not form a finite loop, or forms a closed loop through the tiles of an F-square.

Now we are ready to prove the following theorem.

THEOREM 2. *It is undecidable whether a given two-dimensional cellular automaton with the Moore neighborhood is surjective.*

Proof. The proof is carried out by reducing the finite tiling problem to the injectivity problem of G_f^F. Therefore, suppose that $\mathscr{T} = (T, N_M, R)$ is a given tile set with the blank tile. Let $\mathscr{D}' = (D', N_M, R_0', d')$ be a set of directed tiles that satisfies the finite version of the plane-filling property. It can be, for example, the set described above. We construct a two-dimensional CA

$$\mathscr{A}_{\mathscr{T}'} = (2, S, N_M, f_{\mathscr{T}'}),$$

such that $G_{f_{\mathscr{T}'}}^F$ is not injective (or, equivalently, $\mathscr{A}_{\mathscr{T}'}$ is not surjective) if and only if the tile set $\mathscr{T}$ can be used to form a valid, finite, and non-trivial tiling of the plane.

The state set S of the CA $\mathscr{A}_{\mathscr{T}'}$ is a subset of

$$(D' \cup \{B\}) \times T \times \{0, 1\},$$

where B is a new symbol different from all elements of D'. S contains all triples $(d, t, b) \in (D' \cup \{B\}) \times T \times \{0, 1\}$ with the following restrictions:

1. If d belongs to $\mathscr{Y}$ then t must be blank, and
2. if d belongs to $\mathscr{X}$ then t is not blank.

The local rule of the CA $\mathscr{A}_{\mathscr{T}'}$ is the same as the local rule of $\mathscr{A}_{\mathscr{T}}$ in Section 3.1, in the proof of Theorem 1: The tile components of the states, both d and t, do not change. Only the bits may be altered. If there is a tiling error in either of the tile components, or if the neighborhood of the cell contains a state with B in the first component, then the bit of the state does not change. If, on the other hand, the tilings with both components are valid at the cell, then the cell adds the bit of the succeeding tile on the path to its current bit modulo 2. The state where the first component is B, the second component is the blank tile and the bit is 0 is the quiescent state. In the following it will be shown that the restriction $G_{f_{\mathscr{T}'}}^F$ of the global function of this CA to the finite configurations is not injective if and only if the tile set $\mathscr{T}$ can be used to make a valid, finite, and non-trivial tiling of the plane.

Suppose first that the tiles of $\mathscr{T}$ can be used to form a valid, finite, and non-trivial tiling of the plane. Let $\psi_1 : \mathbb{Z}^2 \to T$ be such a tiling. We can assume without loss of generality that $\psi_1(0, 0)$ is not blank. Let n be so large that the non-blank tiles of the tiling fit inside the hollow square whose sides are n tiles long and whose center is in position $(0, 0)$. According to condition 1 of the finite version of the plane-filling property there is a tiling $\psi_2 : \mathbb{Z}^2 \to D'$ with the directed tiles $\mathscr{D}'$ such that $\psi_2(0, 0) \in \mathscr{X}$, there is a hollow square around the origin consisting of elements of $\mathscr{Y}$ whose sides are at least n tiles long, and the directions define a closed loop L that visits all tiles inside and on the border of the hollow square. Inside the square none of the tiles belongs to $\mathscr{Y}$, and the only tile in the square belonging to $\mathscr{X}$ is the tile at the origin. Let $\psi_2' : \mathbb{Z}^2 \to D' \cup \{B\}$ denote the mapping obtained from ψ_2 by replacing by B every tile that is not in the neighborhood of the closed path L.

We form two finite configurations c_0 and c_1 as follows: The first components of both c_0 and c_1 are given by ψ'_2 and the second components by ψ_1. The bit-components of c_0 are all 0, while the bit-components in c_1 are 1 on the tiles that belong to the loop L, and are 0 outside the loop.

In one time step all the bits in configurations c_0 and c_1 are changed to 0: On the loop L the tiling is correct and every bit 1 of c_1 is changed to 0; outside the loop the bits do not change—they remain 0. We conclude that c_0 and c_1 turn into the same configuration in one time step, so $G^F_{f_{\mathscr{T}'}}$ is not injective.

Conversely, suppose that there are two different finite configurations c_0 and c_1 such that $G^F_{f_{\mathscr{T}'}}(c_0) = G^F_{f_{\mathscr{T}'}}(c_1)$. The tile components of c_0 and c_1 must be the same. The difference between the two configurations is in the bit-components. Consider a cell whose bit-component is different in c_0 and c_1. The tilings must be valid at this cell, and the bits in the cell pointed by the direction of the $\mathscr{D}'$-component are different in c_0 and c_1. We repeat the reasoning for this new cell and proceed to the following cells on the path defined by the directions. We conclude that the tiling property is not violated at any of the cells on the path.

There are two possibilities: Either the path is infinite or it forms a closed loop. The first alternative is impossible because c_0 and c_1 are finite configurations. The second alternative implies, according to condition 2 of the finite version of the plane-filling property, that the path visits all cells of a square whose center cell has an element of $\mathscr{X}$ and border cells have elements of $\mathscr{Y}$ as their first components. In this square the tiling with the $\mathscr{T}$-components is valid. In the center the $\mathscr{T}$-component is not blank (restriction 2 above), but the tiles on the border of the square are all blank (restriction 1). This means that the tile set $\mathscr{T}$ can be used to form a valid, finite, and non-trivial tiling of the plane. ∎

5. Conclusions

In this article we have shown that the injectivity and surjectivity problems of two-dimensional CA are undecidable. Theorem 1 that states the undecidability of the problem of deciding whether a given cellular automaton rule is injective has interesting consequences. Because injectivity and reversibility are equivalent notions, we know that every injective CA has an inverse automaton. But even though the original automaton uses the Moore neighborhood, its inverse may need a much wider neighborhood. In fact, no computable function of the size of the state set can be an upper bound for the inverse neighborhood. If there were a computable bound, then one could generate all candidates for the inverse CA and check one after the other whether some of them really is the inverse automaton. Note that it is an easy matter to check whether two given CA are the inverses of each other. This would yield an algorithm for the injectivity problem, which is not possible.

Theorem 1 implies that given a reversible two-dimensional CA rule finding its inverse is very difficult in general. There is no algorithm for finding the inverse rule with a time complexity bounded by a computable function. This fact may have direct applications in public-key cryptography. Reversible CA rules can be used to encrypt messages in a natural way: The plaintext is expressed as a configuration of the CA, and the encryption is done by applying the CA rule for a fixed number of time steps. The configuration obtained is the cryptotext. One may use periodic boundary conditions to make the configurations finite in size. The decryption is done simply by applying the inverse rule to the cryptotext for the same number of steps. The operations can be done very efficiently in parallel if proper hardware implementations are available.

If one is using an arbitrary two-dimensional reversible cellular automaton $\mathscr{A}$ to encrypt and its inverse automaton $\mathscr{A}^{-1}$ to decrypt messages, then one can make $\mathscr{A}$ public. Making $\mathscr{A}$ public does not reveal its inverse $\mathscr{A}^{-1}$.

Note that the CA constructed in the proof of Theorem 1 were all surjective. This means that the reversibility problem remains undecidable even in the restricted class of surjective CA.

In Theorems 1 and 2 the cellular automata were two-dimensional, and they used the Moore neighborhood. The results are naturally valid for higher dimensional automata as well, since any two-dimensional automaton $\mathscr{A}$ can be simulated by a higher dimensional one that uses only two dimensions and ignores the others.

The results are valid for more restricted neighborhoods as well. Take the other widely used neighborhood, the von Neumann neighborhood (see Fig. 1b). It is not surprising that the injectivity and surjectivity problems are undecidable also when restricted to CA with the von Neumann neighborhood.

Let $\mathscr{A} = (2, S, N_M, f)$ be any CA using the Moore neighborhood. We construct a CA $\mathscr{A}' = (2, S_{2\times 2}, N_{vN}, g)$ that simulates the computation of $\mathscr{A}$ (actually two computations at the same time). The state set $S_{2\times 2}$ consists of 2×2 blocks of the states of $\mathscr{A}$. In the following the local rule g is described.

Let the states of the four von Neumann neighbors of a cell c be arbitrary, like in the left side of Fig. 20. The cell's own current state does not affect its next state. Reshape the neighbors to form a 4×4 block as in the right side of the figure. Apply then the local rule of $\mathscr{A}$ and compute the new states for the four cells in the interior

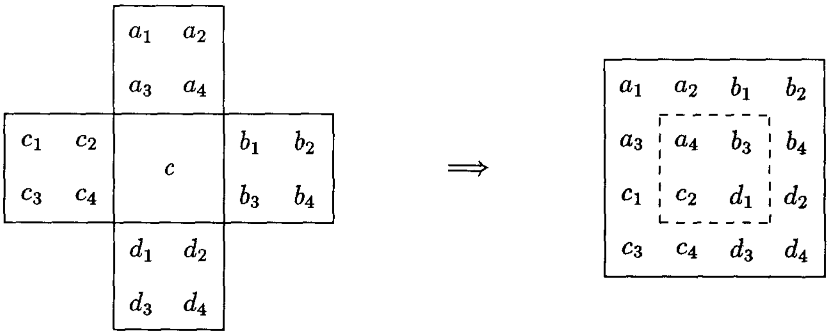

FIG. 20. Reshaping the von Neumann neighborhood of c.

of the block. The states obtained for this 2×2 block form the new state of c produced by the local rule g.

The automaton $\mathscr{A}'$ simulates two computations of $\mathscr{A}$ at the same time. Color the plane like a checkerboard, half of the cells black, the other half white. Let c' be any configuration of $\mathscr{A}'$, and let c'_b denote its restriction to the black cells and c'_w to the white cells. Reshape c'_b to form a configuration c_b of $\mathscr{A}$ by rotating it 45° in the same way as above when defining the local rule g. Because of the way g is defined, the white cells of $G_g(c')$ represent the same configuration as $G_f(c_b)$. Similarly, the black cells of $G_g(c')$ form the configuration $G_f(c_w)$. So, $\mathscr{A}'$ executes two computations of $\mathscr{A}$ at the same time. It is obvious that $\mathscr{A}'$ is injective (surjective) if and only if $\mathscr{A}$ is. We conclude that the injectivity and surjectivity problems for CA using the von Neumann neighborhood are undecidable.

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