$D(a\xi + b\eta) = a^2 D\xi + b^2 D\eta + 2ab\,cov(\xi,\eta)$ $|r| \le 1$

$$D\tilde\theta_3 = D\left(\frac{\tilde\theta_1 + \tilde\theta_2}{2}\right) = \frac14 D\tilde\theta_1 + \frac14 D\tilde\theta_2 + \frac12 cov(\tilde\theta_1, \tilde\theta_2) =$$
$$= \frac12 \left(D\tilde\theta_1 + cov(\tilde\theta_1, \tilde\theta_2)\right)$$

$$r = \frac{cov(\tilde\theta_1, \tilde\theta_2)}{\sqrt{D\tilde\theta_1 \cdot D\tilde\theta_2}}$$

$$D\tilde\theta_3 = \frac12 \left| D\tilde\theta_1 + cov(\tilde\theta_1, \tilde\theta_2)\right| \le \frac12 D\tilde\theta_1 + \frac12 \left|cov(\tilde\theta_1, \tilde\theta_2)\right| \le \frac12 D\tilde\theta_1 + \frac12 D\tilde\theta_1 = D\tilde\theta_1$$

$$\frac{|cov(\tilde\theta_1, \tilde\theta_2)|}{\sqrt{D\tilde\theta_1 D\tilde\theta_2}} \le 1 \Rightarrow |cov(\tilde\theta_1, \tilde\theta_2)| \le D\tilde\theta_1$$

$\Rightarrow D\tilde\theta_3 = D\tilde\theta_1$ – тоже оптимальная

$$\Rightarrow D\tilde\theta_1 = \frac12 D\tilde\theta_1 + \frac12 cov(\tilde\theta_1, \tilde\theta_2) \Rightarrow cov(\tilde\theta_1, \tilde\theta_2) = D\tilde\theta_1$$

$$\Rightarrow r = \frac{D\tilde\theta_1}{D\tilde\theta_1} = 1 \Rightarrow \tilde\theta_2 = a\tilde\theta_1 + b, \quad a > 0$$

$$\underbrace{M\tilde\theta_2}_{\theta} = a\underbrace{M\tilde\theta_1}_{\theta} + b$$

$D\tilde\theta_2 = D\tilde\theta_1 \Rightarrow a = 1 \Rightarrow b = 0 \Rightarrow \tilde\theta_2 = \tilde\theta_1$?! ЧТД

№ 1

270 – угадали
276 – было

$\xi \sim R(0, \theta)$ равномерное распред.

вер. модель параметрическая, $\theta > 0$

$\theta \in \Theta = (0; +\infty)$

$p(x,\theta) = \frac{1}{\theta} I\{(0;\theta)\}$

выборка $\vec{x}_n$

$\tilde\theta_1 = 2\bar{x} = 2\frac1n \sum_{i=1}^n x_i$

$\tilde\theta_2 = x_{min}$

$\tilde\theta_3 = x_{max}$

$\tilde\theta_4 = x_1 + \frac{1}{(n-1)} \sum_{i=2}^{n} x_i$

$$M\xi = \int_{-\infty}^{+\infty} x\,dF(x,\theta) = \int_0^\theta x \frac1\theta dx = \frac\theta2$$
$$M\xi^2 = \int_0^\theta x^2 \frac1\theta dx = \frac{\theta^2}{3}$$
$$\Rightarrow D\xi = M\xi^2 - (M\xi)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$$

Несмещ. оценка: $\forall \theta \in \Theta \quad M\tilde{\theta} = \theta$

Сост. $\forall \theta \in \Theta \ \forall \varepsilon > 0 \quad P(|\tilde{\theta} - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

x_i – независ. случ. вел.: $x_i \sim R(0, \theta)$

① $\tilde{\theta}_1 \quad \forall \theta > 0 \quad M\tilde{\theta}_1 = M(2 \frac{1}{n} \sum x_i) =$

$= \frac{2}{n} \sum M x_i = \frac{2}{n} n M\xi = \theta$ ⇒ 1ая оценка несмещенная

$D\tilde{\theta}_1 = D(2 \frac{1}{n} \sum x_i) = \frac{4}{n^2} \sum D x_i = \frac{4}{n^2} n D\xi = \frac{\theta^2}{3n} \xrightarrow[n\to\infty]{} 0 \quad \forall \theta > 0$

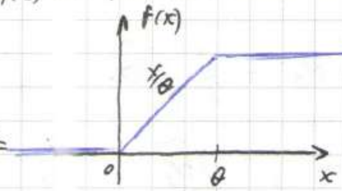
⇒ Оценка состоятельна по дост. условию

② $\tilde{\theta}_2 = x_{min}$

$\xi = F(x) \Rightarrow \xi_{min} \sim 1 - (1 - F(x))^n$

$\Phi(x) \Rightarrow \varphi(x) = \Phi'(x) = n(1 - F(x))^{n-1} F'(x) = n(1 - \frac{x}{\theta})^{n-1} \frac{1}{\theta} \mathbb{1}\{(0, \theta)\}$

$\forall \theta > 0 \quad M\tilde{\theta}_2 = M x_{min}$



$M x_{min} = \int_0^\theta x n (1 - \underbrace{\frac{x}{\theta}}_{t})^{n-1} \frac{1}{\theta} dx = -\int_1^0 n(1-t) t^{n-1} \theta dt =$

$= n \int_0^1 \theta (t^{n-1} - t^n) dt = n\theta \left(\frac{t^n}{n} - \frac{t^{n+1}}{n+1}\right)\Big|_0^1 = n\theta\left(\frac{1}{n} - \frac{1}{n+1}\right) = \frac{\theta}{n+1}$

не явл. несмещенной

$\tilde{\theta}_2' = (n+1) x_{min} \Rightarrow M\tilde{\theta}_2' = (n+1) M x_{min} = \theta$ несмещенная

$D\tilde{\theta}_2' = D((n+1) x_{min}) = (n+1)^2 D x_{min}$

$M x_{min}^2 = \int_0^\theta x^2 n \left(1 - \frac{x}{\theta}\right)^{n-1} \frac{1}{\theta} dx = \int_0^1 \theta^2 (1-t)^2 n t^{n-1} dt =$

$= n\theta^2 \int_0^1 (t^{n-1} - 2t^n + t^{n+1}) dt = n\theta^2 \left(\frac{1}{n} - \frac{2}{n+1} + \frac{1}{n+2}\right) = \frac{2\theta^2}{(n+1)(n+2)}$

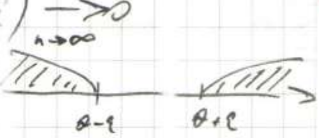
$\Rightarrow D x_{min} = \frac{2\theta^2}{(n+1)(n+2)} - \frac{\theta^2}{(n+1)^2} = \left(\frac{2(n+1) - (n+2)}{(n+1)^2 (n+2)}\right) \theta^2 = \frac{n\theta^2}{(n+1)^2 (n+2)} \Rightarrow$

$\Rightarrow (n+1)^2 D(x_{min}) = \frac{n\theta^2}{n+2} \not\xrightarrow[n\to\infty]{} 0$ Дост. признак не работает ⇒

⇒ По опред. проверяем

$\forall \theta > 0 \ \forall \varepsilon > 0 \quad P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

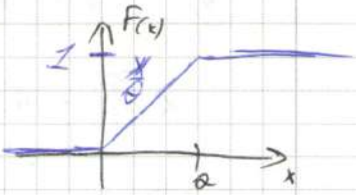
– Отрицание ...



$P(|\tilde{\theta}_2' - \theta| \geq \varepsilon) \geq P(\tilde{\theta}_2' \geq \theta + \varepsilon) = P((n+1) x_{min} \geq \theta + \varepsilon) =$

$$= P\left(x_{min} \geq \frac{\theta+\varepsilon}{n+1}\right) = 1 - \Phi\left(\frac{\theta+\varepsilon}{n+1}\right) =$$

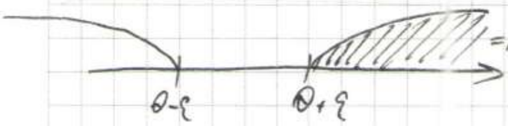
$$\Phi(x) = 1-(1-F(x))^n$$



$$= 1-\left(1-\left(1-\frac{\theta+\varepsilon}{\theta(n+1)}\right)^n\right) = \left(1-\frac{\theta+\varepsilon}{\theta(n+1)}\right) \xrightarrow[n\to\infty]{} e^{-\frac{\theta+\varepsilon}{\theta}} > 0$$

Не явл. состоятельной

$$P(|x_{min}-\theta| \geq \varepsilon) = P(x_{min} \leq \theta-\varepsilon) = \Phi(\theta-\varepsilon) =$$



=0, т.к. не может быть $> \theta$

$$= 1-(1-F(x-\varepsilon))^n = 1-\left(1-\frac{\theta-\varepsilon}{\theta}\right)^n = 1 \left(\frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 1$$

Не явл. состоятельной

(3) $\tilde{\theta}_3 = x_{max}$ $\quad x_{max} \sim (F(x))^n \Rightarrow \Psi'(x) = n\left(\frac{x}{\theta}\right)^{n-1}\frac{1}{\theta}\{(0,\theta)\}$

$$M\tilde{\theta}_3 = M x_{max} = \int_0^\theta x\Psi'(x)\,dx = \frac{n}{n+1}\theta \text{ — смещенная}$$

$$\Rightarrow \tilde{\theta}_3' = \frac{n+1}{n}(x)_{max} \Rightarrow M\tilde{\theta}_3' = \theta \text{ — не смещенная}$$

$$D\tilde{\theta}_3' = \left(\frac{n+1}{n}\right)^3 D x_{max}$$

$$M x_{max}^2 = \int x^2 \Psi'(x) = \dots = \frac{n}{\theta^n}\frac{\theta^{n+2}}{n+2} = \frac{\theta^2 n}{n+2}$$

$$\Rightarrow D x_{max} = M x_{max}^2 - (M x_{max})^2 = \dots = \frac{\theta^2}{n(n+2)} \xrightarrow[n\to\infty]{} 0$$ $\Rightarrow$ явл. состоятельной

В ДЗ док-ть, что $\tilde{\theta}_3'$ — состоятельная
+ посмотреть состоятельность $\tilde{\theta}_3$ — ? (x_{max})

④ $\tilde{\theta}_4 = x_1 + \frac{1}{(n-1)} \sum_{i=2}^{n} x_i$

$M \tilde{\theta}_4 = M x_1 + \frac{1}{(n-1)} \sum_{i=2}^{n} M x_i = M\xi + \frac{1}{(n-1)} (n-1) M\xi = \frac{\theta}{2} + \frac{\theta}{2} = \theta$

$D \tilde{\theta}_4 = D x_1 + \frac{1}{(n-1)^2} \sum_{i=2}^{n} D x_i = \frac{\theta^2}{12} + \frac{1}{(n-1)} \frac{\theta^2}{12} \not\xrightarrow[n\to\infty]{} 0$

не вып. достат

$\tilde{\theta}_4 \xrightarrow{P} \theta$

$x_1 + \frac{1}{(n-1)} \sum_{i=2}^{n} x_i \xrightarrow{P} \xi + \frac{\theta}{2}$ не явл. состоятельной

$\tilde{\theta}_3' = \frac{n+1}{n} x_{max}$

$\tilde{\theta}_1 = 2\bar{x}$

$D\tilde{\theta}_1 = \frac{\theta^2}{3n}$, $D\tilde{\theta}_3' = \frac{\theta^2}{(n+2)n}$

$\frac{1}{3n}$ $\frac{1}{(n+2)n}$

$n^2 + 2n > 3n$, для $n > 1$

$\Rightarrow$ $\tilde{\theta}_3'$ более эффективна, чем $\tilde{\theta}_1$

№1

θ: $\tilde{\theta}_1 = 2\bar{x} = \frac{2}{n}\sum_{i=1}^{n} x_i$

$\tilde{\theta}_2 = x_{min}$

$\tilde{\theta}_3 = x_{max}$

$\tilde{\theta}_4 = \left(\frac{x_1 + \sum_{k=2}^{n} x_k}{n-1}\right)$

Док-ть, что $\tilde{\theta}_3' = \frac{n+1}{n} x_{max}$ – состоятельна

+ проверить на состоятельность $\tilde{\theta}_3 = x_{max}$? (через определение)

Состоятельность: $\forall \varepsilon > 0 \quad P(|\tilde{\theta} - \theta| \geq \varepsilon) \xrightarrow[n\to\infty]{} 0$

1) $x_{max} \sim (F(x))^n \Rightarrow \varphi'(\,) = \left(\frac{x}{\theta}\right)^n \{(0;\theta]\}$

$\tilde{\theta}_3 \leq \theta$

$P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) \Leftrightarrow \tilde{\theta}_3 \leq \theta - \varepsilon \Rightarrow P(|\tilde{\theta}_3 - \theta| \geq \varepsilon) = P(\tilde{\theta}_3 \leq \theta - \varepsilon) =$

$= \left(\frac{\theta - \varepsilon}{\theta}\right)^n = \left(1 - \frac{\varepsilon}{\theta}\right)^n$ и т.к. $0 \leq 1 - \frac{\varepsilon}{\theta} < 1 \Rightarrow$

$\Rightarrow \left(1 - \frac{\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0 \Rightarrow$ $\tilde{\theta}_3$ - состоятельна

2) $\tilde{\theta}_3' = \frac{n+1}{n} x_{max}$

$P\left(\left|\frac{n+1}{n} x_{max} - \theta\right| \geq \varepsilon\right)$

$\frac{n+1}{n} x_{max} - \theta = \frac{n+1}{n}(x_{max} - \theta) - \frac{\theta}{n} \Rightarrow$

$\Rightarrow \left|\frac{n+1}{n} x_{max} - \theta\right| \leq \underbrace{\frac{n+1}{n}}_{\underset{n\to\infty}{<} 2} |x_{max} - \theta| + \underbrace{\frac{\theta}{n}}_{\underset{n\to\infty}{<} \frac{\varepsilon}{2}} \Rightarrow$

$\Rightarrow P\left(\left|\frac{n+1}{n} x_{max} - \theta\right| \geq \varepsilon\right) \leq P\left(|x_{max} - \theta| \geq \frac{\varepsilon}{4}\right) \xrightarrow[n\to\infty]{\text{Выше}} 0 \Rightarrow$

$\Rightarrow \tilde{\theta}_3'$ - тоже состоятельна

ЧТД