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Dynamical study and robustness for a nonlinear wastewater treatment model

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This paper is dedicated to Serhani Khadija.

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ABSTRACT

In this work we deal with a wastewater treatment by using the activated sludge process. The problem is formulated as a nonlinear dynamical system. Firstly, we develop the dynamical study of the model when all parameters are well known. Hence, basic properties of invariance and dissipation are established and, under a suitable condition on parameters, a globally asymptotically stable equilibrium point occurs. Secondly, when the bacterium growth function and the substrate concentration in the feed stream are not well known, the robustness analysis provides the existence of an attractor domain to which all trajectories of the system converge. Finally, we prove that we can reduce the size (volume) of this attractor domain by increasing the recycle rate to a maximum fixed level.

1. Introduction

In this paper, we study a problem of wastewater treatment, namely that of the activated sludge process in a depuration station. The mathematical model is formulated as a nonlinear ordinary differential system. The working principle is described thoroughly in the literature (see for example [1–6]). Basically, the process can be summarized as follows: the wastewater is discharged into an aerator with a flow Q_{in} and a concentration s_{in} in the feed stream. The phase of biological oxidation of the polluted water (substrate), by a blend of a bacterial population in an aerobic reaction consuming the oxygen, begins in the aerator and completes in the settler tank. At this stage, due to gravity, the solid components will settle and concentrate at the bottom, while the sedimentation of soluble organic matter is assumed to be not significant. Part of the bacteria biomass is recycled into the aerator in order to stimulate the oxidation. A schematic of the process is shown in Fig. 1, where s, x and x_r are the state variables representing respectively the substrate biomass (pollutant), the bacteria biomass and the recycled bacteria biomass concentrations. Q_{in}, Q_{out}, Q_r, Q_w are the influent, effluent, recycle and waste flow rates, respectively. V_a and V_s represent the aerator and settler volumes and s_{in} corresponds to the substrate concentrations in the feed stream. Although many works showed interest in a similar problem, where biological aspects and aspects of observability [2–8], estimation, adaptive control, computer simulation [6,9–13], and numerical optimal control [5,14] were studied, there appears to be no work focused upon the dynamical behavior of a model involving a nonlinear bacterium growth function, Our goal in this work is to carry out a rigorous dynamical study for two cases: firstly, when all parameters of the model are known, and secondly, when some parameters are not known. Hence, in the first part of this paper, the basic properties of invariance and dissipation are established and we prove that under some conditions on parameters, there exists a globally asymptotically stable interior equilibrium point. In the second part, our attention is focused on the case where

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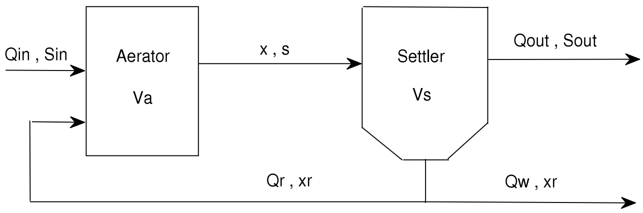

Fig. 1. Schematic diagram of the activated sludge process.

the bacterium growth function μ and the substrate concentration in the feed stream s_{in} are not known, only their upper and lower bounds. Indeed, due to metabolic variations and the influence of many physico-chemical factors (pH, temperature, oxygen, ...), it is very hard to obtain full information on the bioprocess kinetics and an accurate idea of μ (see e.g. [3,15]). In this framework, a robustness analysis of the model, invoking the monotone and cooperative system theory, leads to the fact that there exists an interior domain of stability U, instead a unique stable equilibrium point. Finally, we propose a simple way to reduce the size (volume) of the stability domain by increasing the value of the recycle rate r, and hence the stability domain reaches its minimal value when r achieves its maximum level.

2. Mathematical model

In the model, we assume that the aerator is sufficiently aerated to provide the dissolved oxygen necessary for the growth of the bacterial population; as a consequence the oxygen reaction is neglected. Three phenomena are considered: the reaction kinetics in the aerator linked to microbial growth, the substrate degradation and the recycling of the biomass from the settler. The mass balance of the various constituents gives the following set of equations:

$$(S) \quad \begin{cases} \dfrac{\mathrm{d}s}{\mathrm{d}t} = \dot{s} = -\dfrac{\mu(s)x}{Y} - (1+r)Ds + Ds_{\text{in}}; \quad s(0) = s_0 \\ \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dot{x} = \mu(s)x - (1+r)Dx + rDx_r; \quad x(0) = x_0 \\ \dfrac{\mathrm{d}x_r}{\mathrm{d}t} = \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r; \quad x_r(0) = x_{r0} \end{cases}$$

with

$$D = \frac{Q_{\text{in}}}{V_a}; \qquad r = \frac{Q_r}{Q_{\text{in}}}; \qquad w = \frac{Q_w}{Q_{\text{in}}}; \qquad \nu = \frac{V_a}{V_s},$$

and where Y refers to the yield coefficient of the growth of biomass on the substrate. μ is the specific growth rate function of the bacterium.

2.1. Hypotheses

We assume that

A_1 D, r, w and ν are positive constants.
A_2 $\mu : R^+ \to R^+$ is a C^2 monotone increasing function and satisfies

$$0 \le \mu(.) \le m$$
$$\mu(s) = 0 \quad \text{if } s = 0$$

where m is a given constant.

A classical example of the function μ is the so-called Monad function, given by the following formulation:

$$\mu(s) := \frac{ms}{k+s},$$

where k is a given constant.

3. Dynamical analysis

In this section we are interested in the asymptotic behavior of nonnegative solutions of the model (S) under the assumptions A_1 and A_2, i.e. when all parameters are fully known.

3.1. Invariance and dissipation

The aim is to establish the basic properties of invariance and dissipation.

Consider the open subset of $\mathbb{R}^3_+$, $\Omega := \{(s, x, x_r) \in \mathbb{R}^3_+ / s > 0, x > 0,\ x_r > 0,\}$. Firstly, we look for the invariance of Ω.

Proposition 1. *The set Ω is positively invariant for the system (S).*

Proof. On the boundary of Ω, $\partial\Omega$:

If $x = 0$ with $x_r > 0$ and $s \geq 0$ then $\dot{x} = rDx_r > 0$, so the vector field of x is pointed to inside Ω.

Likewise if $x_r = 0$ with $x > 0$ and $s \geq 0$ then $\dot{x}_r = \nu(1+r)D > 0$ and then the flow of x_r is pointed to inside Ω.

Now if $s = 0$, with ($x \geq 0$ and $x_r > 0$) or ($x > 0$ and $x_r \geq 0$) then $\dot{s} = Ds_{\text{in}} > 0$.

To conclude, it remains to show that the line $G := \{(s, x, x_r)\ s.t.\ s \geq 0,\ \ x = x_r = 0\}$ is invariant. But on G the system is reduced to

$$\begin{cases} \dot{s} = -(1+r)Ds + Ds_{\text{in}} \\ x = 0 \\ x_r = 0 \end{cases}$$

which shows by standard arguments for linear ODE that $s \geq 0$ and hence that G is invariant. □

We can now claim that the system (S) is dissipative, that is, all nonnegative trajectories are uniformly bounded in the positive octant $\mathbb{R}^+$.

Proposition 2. *All nonnegative solutions of (S) are uniformly bounded in $\overline{\Omega}$ (closure of Ω).*

Proof. Let $u(t) = (s(t), x(t), x_r(t))$ in Ω and consider

$$\alpha(t) := \frac{x_r(t)}{x_r(t) + \nu x(t)}.$$

We have $0 < \alpha < 1$.

Let us now prove the following result:

Lemma 3. *There exist a constant $\eta > 0$ sufficiently small and $T(\eta) > 0$ such that for all $t \geq T(\eta)$ we have $\eta \leq \alpha(t) \leq 1 - \eta$.*

Proof. By taking the derivative of α, we have

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = \dot{\alpha} = \nu \frac{\dot{x}_r x - x_r \dot{x}}{(x_r + \nu x)^2}.$$

But

$$\begin{aligned} \dot{x}_r x - x_r \dot{x} &= \nu(1+r)Dx^2 - \nu(w+r)Dx_r x - \mu(s)xx_r + (1+r)Dxx_r - rDx_r^2 \\ &= x(1+r)D[\nu x + x_r] - rDx_r[x_r + \nu x] - (\nu wD + \mu(s))xx_r. \end{aligned}$$

Hence

$$\dot{\alpha} = -rD\alpha + \nu(1+r)D\frac{x}{x_r + \nu x} - \nu(\nu wD + \mu(s))\frac{xx_r}{(x_r + \nu x)^2}.$$

Taking into account of the fact that $1 - \alpha = \frac{\nu x}{x_r + \nu x}$, it follows that

$$\dot{\alpha} = -rD\nu\alpha + (1+r)D(1-\alpha) - \alpha(1-\alpha)(\nu wD + \mu(s)).$$

By choosing $M = \nu wD + m$, where m is given by hypothesis A_1, we obtain that

$$\dot{\alpha} \leq -rD\nu\alpha + (1+r)D(1-\alpha) + \alpha(1-\alpha)M$$

and

$$\dot{\alpha} \geq -rD\nu\alpha + (1+r)D(1-\alpha) - \alpha(1-\alpha)M.$$

Consider two functions F^+ and F^- as

$$F^+(z) = -rD\nu z + (1+r)D(1-z) + z(1-z)M$$

and

$$F^-(z) = -rD\nu z + (1+r)D(1-z) - z(1-z)M.$$

Observe that $F^+(1) = -rD\nu < 0$ and $F^-(0) = (1+r)D > 0$, so, according to the continuity of F^- and F^+, there exist $\eta,\ \delta > 0$ such that

$$\forall z, \quad 1 - \eta \leq z \leq 1 \Longrightarrow F^+(z) \leq -\delta < 0$$

and

$$\forall z, \quad 0 \leq z \leq \eta \Longrightarrow F^-(z) \geq \delta > 0.$$

On the other hand, we have

$$\begin{cases}\dot{\alpha}(t) \le F^{+}(\alpha(t)) \\ \dot{\alpha}(t) \ge F^{-}(\alpha(t)).\end{cases}$$

So,

$$\forall \alpha(t), \quad 1-\eta \le \alpha(t) \le 1 \Longrightarrow \dot{\alpha}(t) \le F^{+}(\alpha(t)) \le -\delta < 0$$

and

$$\forall \alpha(t), \quad 0 \le \alpha(t) \le \eta \Longrightarrow \dot{\alpha}(t) \ge F^{-}(\alpha(t)) \ge \delta > 0$$

which implies that there exists a $T(\eta) > 0$ such that

$$\forall t \ge T(\eta) \quad \alpha(t) \in [\eta, 1-\eta]$$

as required. □

Return to the proof of the proposition and consider $\Sigma := \nu x + \nu Y s + x_r$.
We have

$$\begin{aligned}\dot{\Sigma} &= -(1+r)D[\nu x + \nu Y ds - \nu x] + rD\nu x_r - \nu(w+r)Dx_r + D\nu Y s_{\text{in}} \\ &= -(1+r)D\nu Y ds - \nu w D x_r + D\nu Y s_{\text{in}} \\ &\le -kD(x_r + \nu Y ds) + D\nu Y s_{\text{in}}\end{aligned}$$

where $k = \min(\nu w, 1+r)$.

Since $\eta s \le s$ and according to the previous lemma, the inequality $x_r \ge \eta(x_r + \nu x)$ holds and then

$$\dot{\Sigma} \le -kD(\eta(x_r + \nu x) + \nu Y \eta s) + D\nu Y s_{\text{in}}$$

which implies that

$$\dot{\Sigma} \le -kD\eta\Sigma + D\nu Y s_{\text{in}}$$

and this is true for all $t \ge T(\eta)$. We conclude that

$$\limsup_{t\to+\infty} \Sigma(t) \le \frac{\nu Y s_{\text{in}}}{k\eta}.$$

So, since according to Proposition 1, any trajectory is positively invariant, then all positive trajectories are uniformly bounded and hence the system is dissipative in $\bar{\Omega}$ which complete the proof of the theorem. □

3.2. Equilibria and local stability

3.2.1. Equilibria

In this subsection we focus upon the existence and the local stability of equilibria of (S). An equilibrium point must satisfy the following equations:

$$\begin{aligned}0 &= -\frac{\mu(s)x}{Y} - (1+r)Ds + Ds_{\text{in}} \\ 0 &= \mu(s)x - (1+r)Dx + rDx_r \\ 0 &= \nu(1+r)Dx - \nu(w+r)Dx_r.\end{aligned} \tag{1}$$

The system (1) always has a trivial solution $P_0 = \left(\frac{s_{\text{in}}}{1+r}, 0, 0\right)$ belonging to the boundary, $\partial\Omega$, of Ω. On the other hand, there exists an interior equilibrium point $P_1 \in \Omega$ under some conditions. Indeed:

Lemma 4. *Suppose that*

$$(1+r)D\frac{w}{w+r} < \mu\left(\frac{s_{\text{in}}}{1+r}\right). \tag{2}$$

Then, there exists an interior equilibrium point $P_1 = (s^*, x^*, x_r^*)$ such that

$$x_r^* = \frac{1+r}{w+r}x^*, \qquad x^* = \frac{1}{Y}\frac{w+r}{r}\left[\frac{s_{\text{in}}}{1+r} - s^*\right], \qquad s^* = \mu^{-1}\left((1+r)D\frac{w}{w+r}\right).$$

Proof. The third equation of system (1) implies that

$$x_r^* = \frac{1+r}{w+r}x^*. \tag{3}$$

From the second equation of system (1) we have, according to (3) (when $x \neq 0$), that

$$\mu(s^*) = (1+r)D\frac{w}{w+r}.$$

This is true only if

$$(1+r)D\frac{w}{w+r} < m. \tag{4}$$

Under the above condition, since $\mu(.)$ is C^2, increasing and bounded (and thus bijective), we can write

$$s^* = \mu^{-1}\left((1+r)D\frac{w}{w+r}\right).$$

By substituting $\mu(s^*)$ in the first equation of system (1), we obtain x^* as a function of s^* as follows:

$$x^* = \frac{1}{Y}\frac{w+r}{r}\left[\frac{s_{\text{in}}}{1+r} - s^*\right]. \tag{5}$$

But we are interest in positive equilibrium; in other words $x^* > 0$. Therefore we must have

$$\frac{s_{\text{in}}}{1+r} - s^* > 0$$

which is equivalent to condition (2):

$$(1+r)D\frac{w}{w+r} < \mu\left(\frac{s_{\text{in}}}{1+r}\right).$$

Remark that if condition (2) is satisfied then condition (4) is also satisfied. We conclude that under the required condition (2), the interior equilibrium point $P_1 = (s^*, x^*, x_r^*)$ holds, which completes the proof. □

3.2.2. Local stability

The local stability of $P_0 = \left(\frac{s_{\text{in}}}{1+r}, 0, 0\right)$ is determined by the eigenvalues of the Jacobian matrix of system (S) at P_0:

$$J(P_0) = \begin{pmatrix} -(1+r)D & -\dfrac{\mu\left(\frac{s_{\text{in}}}{1+r}\right)}{Y} & 0 \\ 0 & \mu\left(\dfrac{s_{\text{in}}}{1+r}\right) - (1+r)D & rD \\ 0 & \nu(1+r)D & -\nu(w+r)D \end{pmatrix}.$$

Proposition 5. *If $(1+r)D\frac{w}{w+r} > \mu\left(\frac{s_{\text{in}}}{1+r}\right)$, then P_0 is locally asymptotically stable.*

Otherwise, if $(1+r)D\frac{w}{w+r} < \mu\left(\frac{s_{\text{in}}}{1+r}\right)$ then P_0 is unstable.

Proof. Let us first examine the local stability.

The polynomial characteristic associated with $det(\lambda I - J(P_0)) = 0$ is given by

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$$

where $a_1 = 2(1+r)D - \mu\left(\frac{s_{\text{in}}}{1+r}\right) + \nu(w+r)D$, $a_2 = (1+r)D\left[-\mu\left(\frac{s_{\text{in}}}{1+r}\right) + (1+r)D + \nu(w+r)D\right] + \nu(w+r)D\left[-\mu\left(\frac{s_{\text{in}}}{1+r}\right) + (1+r)D\right] - r\nu(1+r)D^2$ and $a_3 = (1+r)D\left[\nu(w+r)D\left(-\mu\left(\frac{s_{\text{in}}}{1+r}\right) + (1+r)D\right) - r\nu(1+r)D^2\right]$. By the Routh–Hurwitz criterion it suffices to prove that $a_1 > 0$, $a_3 > 0$ and $a_1a_2 - a_3 > 0$ to conclude that all eigenvalues of $J(P_0)$ have negative real parts.

• **Claim** 1. $a_1 > 0$.

Since by hypothesis $\mu\left(\frac{s_{\text{in}}}{1+r}\right) < (1+r)D\frac{w}{w+r}$, then $\mu\left(\frac{s_{\text{in}}}{1+r}\right) < (1+r)D$, so $2(1+r)D - \mu\left(\frac{s_{\text{in}}}{1+r}\right) > 0$ and hence $a_1 > 0$.

• **Claim** 2. $a_3 > 0$.

$$\begin{aligned} a_3 &= (1+r)D\left[\nu(w+r)D\left(-\mu\left(\frac{s_{\text{in}}}{1+r}\right) + (1+r)D\right) - r\nu(1+r)D^2\right] \\ &= (1+r)D\left[-\nu D(w+r)\mu\left(\frac{s_{\text{in}}}{1+r}\right) + \nu w D^2(1+r)\right] \\ &= (1+r)D\nu D(w+r)\left[-\mu\left(\frac{s_{\text{in}}}{1+r}\right) + \frac{D(1+r)w}{w+r}\right]. \end{aligned}$$

It follows that $a_3 > 0$.

• **Claim** 3. $a_1a_2 - a_3 > 0$.

Set $A := \nu(w+r)D$, $B := -\mu\left(\frac{s_{\text{in}}}{1+r}\right) + (1+r)D$, $C := r\nu(1+r)D^2$ and $E := (1+r)D$, we can formulate a_1, a_2 and a_3 as follows: $a_1 = E + B + A$, $a_2 = E(B+A) + AB - C$ and $a_3 = E(AB - C)$;

$$\begin{aligned} a_1a_2 - a_3 &= E^2(B+A) + EAB - EC + E(A+B)^2 + (AB-C)(A+B) - E(AB-C) \\ &= E^2(B+A) + E(A+B)^2 + (AB-C)(A+B). \end{aligned}$$

To prove that $a_1a_2 - a_3 > 0$ it suffices to show that $AB - C > 0$, but

$$\begin{aligned} AB - C &= \nu(w+r)D\left[(1+r)D - \mu\left(\frac{s_{\text{in}}}{1+r}\right)\right] - r\nu(1+r)D^2 \\ &= \nu D\left[(w+r)(1+r)D - (w+r)\mu\left(\frac{s_{\text{in}}}{1+r}\right) - r(1+r)D\right] \\ &= \nu D\left[w(1+r)D - (w+r)\mu\left(\frac{s_{\text{in}}}{1+r}\right)\right] \\ &= \nu(w+r)D\left[\frac{w}{w+r}(1+r)D - \mu\left(\frac{s_{\text{in}}}{1+r}\right)\right] \end{aligned}$$

which is positive by hypothesis. We conclude that the equilibrium point (P_0) is locally stable.

Now, let us examine the case where (2) holds. In this case, as shown above,

$$a_3 = (1+r)D\nu D(w+r)\left[-\mu\left(\frac{s_{\text{in}}}{1+r}\right) + \frac{D(1+r)w}{w+r}\right] < 0$$

and hence P_0 is unstable as required. □

Recall that the equilibrium point P_1 exists under condition (2). Examine now its local stability.

Proposition 6. *If (2) holds then P_1 exists and it is locally stable.*

Proof. The Jacobian matrix associated with P_1 is given by

$$J(P_1) = \begin{pmatrix} -\frac{\mu'(s^*)x^*}{Y} - (1+r)D & -\frac{\mu(s^*)}{Y} & 0 \\ \mu'(s^*)x^* & \mu(s^*) - (1+r)D & rD \\ 0 & \nu(1+r)D & -\nu(w+r)D \end{pmatrix}.$$

The polynomial characteristic associated with $det(\lambda I - J(P_1)) = 0$ is

$$\lambda^3 + b_1\lambda^2 + b_2\lambda + b_3 = 0$$

where $b_1 = (1+r)D - \nu(w+r)D + (1+r)D\frac{r}{w+r} + \frac{x^*}{Y}\mu'(s^*)$, $b_2 = \nu(w+r)D\frac{x^*}{Y}\mu'(s^*) + (1+r)D[\nu(w+r)D + (1+r)D\frac{r}{w+r} + \frac{x^*}{Y}\mu'(s^*)]$ and $b_3 = (1+r)D[\nu(w+r)D\frac{x^*}{Y}\mu'(s^*)] - \nu r(1+r)D^2\frac{x^*}{Y}\mu'(s^*)$. Likewise, we use the Routh–Hurwitz criterion:

Claim 1. $b_1 > 0$.

Immediate since no negative terms appear in the formula.

Claim 2. $b_3 > 0$.

$$\begin{aligned} b_3 &= (1+r)D\left[\nu(w+r)D\frac{x^*}{Y}\mu'(s^*)\right] - \nu r(1+r)D^2\frac{x^*}{Y}\mu'(s^*) \\ &= (1+r)D^2\nu\frac{x^*}{Y}\mu'(s^*)w \\ &> 0. \end{aligned}$$

Claim 3. $b_1b_2 - b_3 > 0$.

Set $A := \nu(w+r)D$, $B := \frac{x^*}{Y}\mu'(s^*)$, $C := (1+r)D\frac{r}{w+r}$, $E := (1+r)D$ and $F := \nu rD$. It follows that

$$\begin{aligned} b_1 &= E + A + C + B \\ b_2 &= AB + E(A+B+C) \\ b_3 &= EAB - FEB. \end{aligned}$$

We obtain then $b_1b_2 - b_3 = E^2(A+C+B) + A^2B + E(A^2+CA+BA) + AB^2 + E(AB+CB+B^2)ABC + E(AC+C^2+BC) + FEB$ which is positive and therefore P_1 is locally asymptotically stable. □

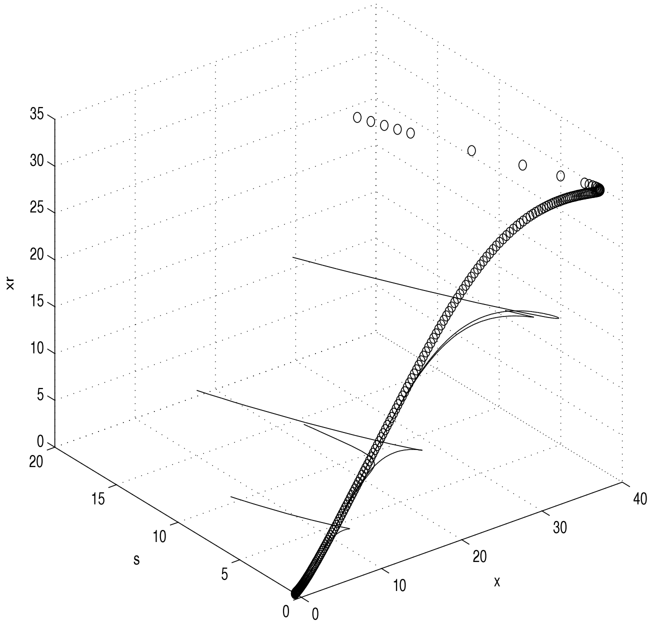

Fig. 2. Global stability of P_0 when condition (2) is not fulfilled.

3.3. Global stability

One expects the interior rest point P_1 to be globally asymptotically stable whenever it exists. This is established below. Before looking for this claim, let us first examine the global stability of P_0 whenever the condition (2) is not fulfilled.

Proposition 7. *If $\frac{w}{w+r}(1+r)D > \mu\left(\frac{s_{\text{in}}}{1+r}\right)$, then the equilibrium point P_0 is globally asymptotically stable.*

Proof. To prove global stability of P_0 whenever $\frac{w}{w+r}(1+r)D > \mu\left(\frac{s_{\text{in}}}{1+r}\right)$, we use the Lyapunov function.

Set $\bar{s} := \frac{s_{\text{in}}}{1+r}$ and consider

$$V(s, x, x_r) = Y\int_{\bar{s}}^{s}\left(1 - \frac{\mu(\bar{s})}{\mu(\xi)}\right)\mathrm{d}\xi + x + \frac{r}{\nu(w+r)}x_r.$$

It is easy to show that $V(\bar{s}, 0, 0) = 0$ and $V(s, x, x_r) > 0$ for $(s, x, x_r) \neq (\bar{s}, 0, 0)$. On the other hand for $u(t) = (s(t), x(t), x_r(t))$,

$$\begin{aligned}
\dot{V}(u(t)) &= Y\left(1 - \frac{\mu(\bar{s})}{\mu(s)}\right)\dot{s} + \dot{x} + \frac{r}{\nu(w+r)}\dot{x}_r \\
&= \left(1 - \frac{\mu(\bar{s})}{\mu(s)}\right)(-\mu(s)x - (1+r)DYs + DYs_{\text{in}}) + \mu(s)x - \frac{w}{w+r}(1+r)Dx \\
&= -\frac{\mu(\bar{s})}{\mu(s)}(-\mu(s)x - (1+r)DYs + DYs_{\text{in}}) - \frac{w}{w+r}(1+r)Dx - (1+r)DYs + DYs_{\text{in}} \\
&= x\left(\mu(\bar{s}) - \frac{w}{w+r}(1+r)D\right) + (1+r)DY\frac{\mu(s) - \mu(\bar{s})}{\mu(s)}(\bar{s} - s).
\end{aligned}$$

By hypothesis we have $\mu(\bar{s}) - \frac{w}{w+r}(1+r)D < 0$ and since $\mu(.)$ is increasing, we have $(\mu(s)) - \mu(\bar{s})(\bar{s} - s) < 0$. It follows that $\dot{V}(u(t)) < 0$.

It remains to show that $V(s, x, x_r) \longrightarrow +\infty$ when $\|(s, x, x_r)\| \longrightarrow +\infty$ which is immediate. We conclude that the equilibrium point P_0 is globally asymptotically stable as required. □

Fig. 2 shows the global stability of P_0 whenever the condition (2) is not fulfilled.

Suppose now that condition (2) is fulfilled and examine the global stability of P_1.

Theorem 8 (*Global Stability of P_1*)**.** *Under condition (2) the equilibrium point P_1 is globally asymptotically stable.*

Proof. We prove the global stability of P_1 by using the following Lyapunov function:

$$V(s,x,x_r) := Y\frac{w+r}{r}\int_{s^*}^{s}\left(1-\frac{\mu(s^*)}{\mu(\xi)}\right)\mathrm{d}\xi + \frac{w+r}{r}\int_{x^*}^{x}\left(1-\frac{x^*}{\xi}\right)\mathrm{d}\xi + \frac{1}{v}\int_{x_{r^*}}^{x_r}\left(1-\frac{x_{r^*}}{\xi}\right)\mathrm{d}\xi.$$

It is easy to show that $V(s^*,x^*,x_r^*) = 0$, $V(s,x,x_r) > 0$ for all $(s,x,x_r) \neq (s^*,x^*,x_r^*)$, and that $V(s,x,x_r) \longrightarrow +\infty$ as $(s,x,x_r) \longrightarrow +\infty$.

It remains to show that

$$\frac{\mathrm{d}V}{\mathrm{d}t}(s(t),x(t),x_r(t)) < 0 \quad \text{for all } (s(t),x(t),x_r(t)) \neq (s^*,x^*,x_r^*),$$

$$\begin{aligned}
\frac{\mathrm{d}V}{\mathrm{d}t}(s(t),x(t),x_r(t)) &= Y\frac{w+r}{r}\left(1-\frac{\mu(s^*)}{\mu(s)}\right)\dot{s} + \frac{w+r}{r}\left(1-\frac{x^*}{x}\right)\dot{x} + \frac{1}{v}\left(1-\frac{x_{r^*}}{x_r}\right)\dot{x}_r \\
&= \frac{w+r}{r}\left(1-\frac{\mu(s^*)}{\mu(s)}\right)[-\mu(s)x-(1+r)DYs+DYs_{\mathrm{in}}] \\
&\quad + \left(1-\frac{x^*}{x}\right)\left[\frac{w+r}{r}\mu(s)x-\frac{w+r}{r}(1+r)Dx+(w+r)Dx_r\right] \\
&\quad + \left(1-\frac{x_{r^*}}{x_r}\right)[(1+r)Dx-(w+r)Dx_r] \\
&= \frac{w+r}{r}\left(1-\frac{\mu(s^*)}{\mu(s)}\right)[-\mu(s)x-(1+r)DYs+DYs_{\mathrm{in}}] \\
&\quad + \left(1-\frac{x^*}{x}\right)\left[\frac{w+r}{r}\mu(s)x-\frac{w}{r}(1+r)Dx\right] \\
&\quad + \left(1-\frac{x^*}{x}\right)[-(1+r)Dx+(w+r)Dx_r] + \left(1-\frac{x_{r^*}}{x_r}\right)[(1+r)Dx-(w+r)Dx_r].
\end{aligned}$$

Set

$$A := \frac{w+r}{r}\left(1-\frac{\mu(s^*)}{\mu(s)}\right)[-\mu(s)x-(1+r)DYs+DYs_{\mathrm{in}}] + \left(1-\frac{x^*}{x}\right)\left[\frac{w+r}{r}\mu(s)x-\frac{w}{r}(1+r)Dx\right]$$

and

$$B := \left(1-\frac{x^*}{x}\right)[-(1+r)Dx+(w+r)Dx_r] + \left(1-\frac{x_{r^*}}{x_r}\right)[(1+r)Dx-(w+r)Dx_r].$$

Then,

$$\frac{\mathrm{d}V}{\mathrm{d}t}(s(t),x(t),x_r(t)) = A+B.$$

Let us examine separately B and A. For B, set $Q := (1+r)x-(w+r)x_r$; then we have

$$\begin{aligned}
B &= \left(1-\frac{x_{r^*}}{x_r}\right)QD - \left(1-\frac{x^*}{x}\right)QD \\
&= QD\frac{1}{x_r x}(x_r x^* - x x_r^*).
\end{aligned}$$

But, since $x_r^* = \frac{1+r}{w+r}x^*$, we have

$$\begin{aligned}
x_r x^* - x x_r^* &= x_r x^* - x\frac{1+r}{w+r}x^* \\
&= -Q\frac{x^*}{w+r}.
\end{aligned}$$

It follows, according to this fact, that

$$B = -Q^2 D\frac{x^*}{x_r x}\frac{x^*}{w+r} < 0.$$

Return now to A:

$$\begin{aligned}
A &= \frac{w+r}{r}\left(1-\frac{\mu(s^*)}{\mu(s)}\right)[-\mu(s)x-(1+r)DYs+DYs_{\mathrm{in}}] + \left(1-\frac{x^*}{x}\right)\left[\frac{w+r}{r}\mu(s)x-\frac{w}{r}(1+r)Dx\right] \\
&= \frac{w+r}{r}\frac{\mu(s)-\mu(s^*)}{\mu(s)}[DYs_{\mathrm{in}}-(1+r)DYs-\mu(s)x^*].
\end{aligned}$$

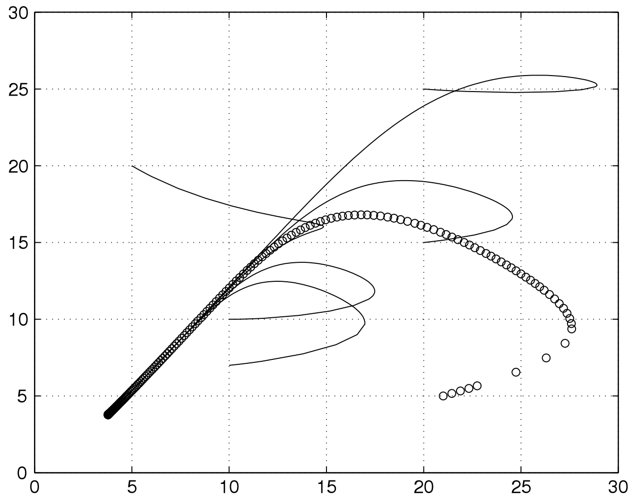

Fig. 3. Global stability of P_1 in phase space (x, x_r).

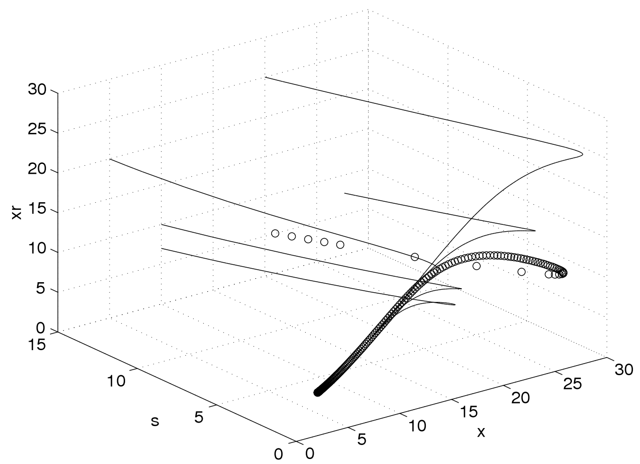

Fig. 4. Global stability of P_1 in phase space (s, x, x_r).

• **Case** 1. Suppose that $\mu(s) > \mu(s^*)$; this is equivalent to saying that $s > s^*$. So, according to both inequalities we have

$$DYs_{\text{in}} - (1+r)DYs - \mu(s)x^* < DYs_{\text{in}} - (1+r)DYs^* - \mu(s^*)x^*.$$

But $\mu(s^*) = (1+r)D\frac{w}{w+r}$ and $x^* = Y\frac{w+r}{w}\left[\frac{s_{\text{in}}}{1+r} - s^*\right]$; then by substituting $\mu(s^*)x^*$ by its value in the right hand side of the above inequality, we obtain that

$$DYs_{\text{in}} - (1+r)DYs^* - \mu(s^*)x^* = 0.$$

It follows that $DYs_{\text{in}} - (1+r)DYs - \mu(s)x^* < 0$, and hence that $A < 0$.

• **Case** 2. Suppose that $\mu(s) < \mu(s^*)$.
We proceed by the same way as for the first case.
So, we deduce that

$$\frac{\mathrm{d}V}{\mathrm{d}t}(s(t), x(t), x_r(t)) < 0$$

for all $(s(t), x(t), x_r(t)) \neq (s^*, x^*, x_r^*)$ as required. $\square$

Figs. 3 and 4 show (for in the phase spaces (x, x_r) and (s, x, x_r) respectively) the global stability of P_1 when the condition (2) is fulfilled.

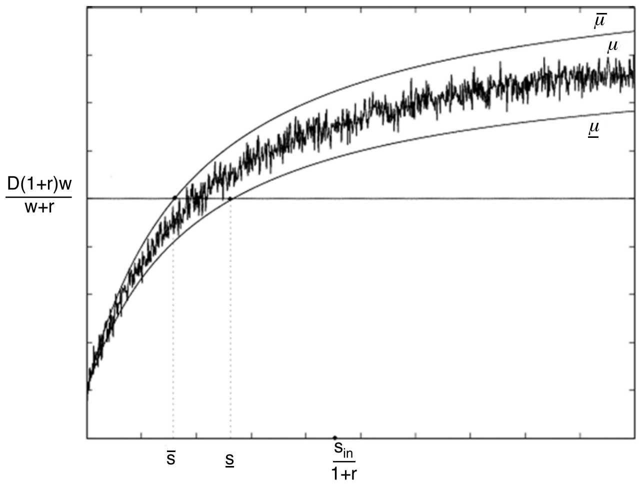

Fig. 5. Example with μ not well known.

4. Robustness

The object of this section is to extend the result obtained in Theorem 8 to establish the existence of a global attractor interior domain whenever full information about the growth function μ and the substrate concentrations in the feed stream s_{in} is not available. We will carry out our study in the framework of the following assumptions:

H_1 D, r, w and ν are positive constants.

H_2 There exist two functions $\underline{\mu}$ and $\overline{\mu}$ satisfying the hypothesis A_2 such that

$$\underline{\mu}(s) \leq \mu(s) \leq \overline{\mu}(s), \quad \forall\, s \geq 0.$$

This means that there exist $\overline{m}$ and $\underline{m}$ such that

$$\underline{\mu}(s) \leq \underline{m} \quad \text{and} \quad \overline{\mu}(s) \leq \overline{m} \quad \forall s \geq 0,$$

see Fig. 5.

H_3 $s_{\text{in}}(t)$ is an unknown time varying function but bounded by

$$s_{\text{in}}^{-} \leq s_{\text{in}}(t) \leq s_{\text{in}}^{+}, \quad \forall t \geq 0$$

where s_{in}^{-} and s_{in}^{+} are given positive constants.

Let us now introduce some tools necessary to establish our desired result. Consider the following two systems:

$$(S_+) \quad \begin{cases} \dot{s} = -\dfrac{\overline{\mu}(s)}{Y}x - (1+r)Ds + Ds_{\text{in}}^{+} \\ \dot{x} = \overline{\mu}(s)x - (1+r)Dx + rDx_r \\ \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r \end{cases}$$

and

$$(S_-) \quad \begin{cases} \dot{s} = -\dfrac{\underline{\mu}(s)}{Y}x - (1+r)Ds + Ds_{\text{in}}^{-} \\ \dot{x} = \underline{\mu}(s)x - (1+r)Dx + rDx_r \\ \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r. \end{cases}$$

The systems (S_+) and (S_-) fulfill Assumptions A_1 and A_2. So, from analysis developed in previous sections, we deduce that if

$$D(1+r)\frac{w}{w+r} < \underline{\mu}\left(\frac{s_{\text{in}}^{-}}{1+r}\right) \tag{6}$$

and

$$D(1+r)\frac{w}{w+r} < \overline{\mu}\left(\frac{s_{\text{in}}^{+}}{1+r}\right) \tag{7}$$

then there exist two equilibrium points $P_+ := (s^+, x^+, x_r^+)$ and $P_- := (s^-, x^-, x_r^-)$, globally asymptotically stable, for (S_+) and (S_-) respectively, with

$$\overline{\mu}(s^+) = D(1+r)\frac{w}{w+r}$$

and

$$\underline{\mu}(s^-) = D(1+r)\frac{w}{w+r},$$

and Eqs. (6) and (7) are equivalent to

$$s^+ < \frac{s_{\text{in}}^+}{1+r} \tag{8}$$

and

$$s^- < \frac{s_{\text{in}}^-}{1+r}. \tag{9}$$

Now, since $\overline{\mu}(s^+) = \underline{\mu}(s^-)$ and $\underline{\mu} \le \overline{\mu}$ we deduce that

$$s^+ \le s^-. \tag{10}$$

So, according to the fact that

$$s^+ \le s^- < \frac{s_{\text{in}}^-}{1+r} \le \frac{s_{\text{in}}^+}{1+r},$$

if condition (9) holds then condition (8) holds and hence conditions (6) and (7) hold also, We conclude that, in order to assure the existence of stable equilibrium points P_+ and P_-, it suffices to formulate the following additional assumption H_4:

H_4 Assume that $s^- < \frac{s_{\text{in}}^-}{1+r}$.

Consider now the following variable change: $z := x + Ys$, and reformulate the system (S) with the new variables

$$(S_R) \quad \begin{cases} \dot{z} = -(1+r)Dz + DYs_{\text{in}} + rDx_r; \quad z(0) = z_0 \\ \dot{x} = \mu\left(\dfrac{z-x}{Y}\right)x - (1+r)Dx + rDx_r; \quad x(0) = x_0 \\ \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r; \quad x_r(0) = x_{r0}. \end{cases}$$

Similarly the systems (S_+) and (S_-) are translated into

$$(S_{ov}) \quad \begin{cases} \dot{z} = -(1+r)Dz + DYs_{\text{in}}^+ + rDx_r \\ \dot{x} = \overline{\mu}\left(\dfrac{z-x}{Y}\right)x - (1+r)Dx + rDx_r \\ \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r \end{cases}$$

and

$$(S_{un}) \quad \begin{cases} \dot{z} = -(1+r)Dz + DYs_{\text{in}}^- + rDx_r; \quad z(0) = z_0 \\ \dot{x} = \underline{\mu}\left(\dfrac{z-x}{Y}\right)x - (1+r)Dx + rDx_r; \quad x(0) = x_0 \\ \dot{x}_r = \nu(1+r)Dx - \nu(w+r)Dx_r; \quad x_r(0) = x_{r0}. \end{cases}$$

Finally consider the set $\Omega_R := \{(z, x, x_r) \in R_+^3 : z > x, x > 0, x_r > 0\}$.

Remark 9. Properties of invariance and dissipation remain true on Ω_R.

We are now ready to give the main result of this section:

Theorem 10. *All trajectories of system S_R starting in Ω_R converge towards the domain $U_R := \{(z, x, x_r) \in \Omega_R \;:\; \underline{z} \le z \le \overline{z}\,,\ \underline{x} \le x \le \overline{x}\,,\ \underline{x}_r \le x_r \le \overline{x}_r\}$.*

Proof. It is clear that $S_{un} \leq S_R \leq S_{ov}$ in the sense that the vector field of S_R is bounded below by the vector field of S_{un} and bounded above by one of the systems S_{ov}. Furthermore, since the Jacobian matrix of the systems (S_{ov}) given by

$$J(S_{ov}) = \begin{pmatrix} -(1+r)D & 0 & rD \\ \dfrac{\overline{\mu}'\left(\frac{z-x}{Y}\right)x}{Y} & \overline{\mu}\left(\dfrac{z-x}{Y}\right) - (1+r)D - \dfrac{\overline{\mu}'\left(\frac{z-x}{Y}\right)x}{Y} & rD \\ 0 & \nu(1+r)D & -\nu(w+r)D \end{pmatrix}$$

and the Jacobian matrix of the systems (S_{un}) given by

$$J(S_{un}) = \begin{pmatrix} -(1+r)D & 0 & rD \\ \dfrac{\underline{\mu}'\left(\frac{z-x}{Y}\right)x}{Y} & \underline{\mu}\left(\dfrac{z-x}{Y}\right) - (1+r)D - \dfrac{\underline{\mu}'\left(\frac{z-x}{Y}\right)x}{Y} & rD \\ 0 & \nu(1+r)D & -\nu(w+r)D \end{pmatrix}$$

are nonnegative outside the diagonal elements ($\underline{\mu}'$ and $\overline{\mu}'$ are nonnegative according to assumption A_2), then these systems are cooperative (see for e.g. [16] and [17]). Therefore, by using arguments of cooperative inequalities (see Appendix B of [17]), we obtain that if

$$u(t,u_0) := \{(z(t),x(t),x_r(t)) \in {R_+}^3 \ / \ (z(0),x(0),x_r(0)) = u_0 := (z_0,x_0,x_{r0})\},$$

is the flow of (S_R) and similarly $\overline{u}(t,u_0)$ and $\underline{u}(t,u_0)$ are respectively the flows of (S_{ov}) and (S_{un}) starting at the same point u_0, then

$$\underline{u}(t,u_0) \leq u(t,u_0) \leq \overline{u}(t,u_0) \quad \forall t \geq 0. \tag{11}$$

On the other hand, with the change of variable z, the asymptotical global stable equilibrium points $P_- = (s^-,x^-,x_r^-)$ and $P_+ = (s^+,x^+,x_r^+)$ for the systems (S_-) and (S_+), respectively, are translated into asymptotical global stable equilibrium points $\underline{P} := (\underline{z},\underline{x},\underline{x}_r)$ and $\overline{P} := (\overline{z},\overline{x},\overline{x}_r)$ in the new systems (S_{ov}) and (S_{un}) respectively. In other words, all trajectories of system S_{un} (respectively S_{ov}) starting in Ω_R, converge towards $\underline{P}$ (respectively $\overline{P}$). Hence, according to (11) we conclude that all trajectories of system (S_R) starting in Ω_R converge towards a domain U_R bounded above by $\overline{P}$ and below by $\underline{P}$:

$$U_R := \{(z,x,x_r) \in \Omega_R : \underline{z} \leq z \leq \overline{z}, \underline{x} \leq x \leq \overline{x}, \underline{x}_r \leq x_r \leq \overline{x}_r\}$$

as required. □

It remains now to deduce the domain U towards which all trajectories of system (S) converge. To do this, we must define the variables of system (S) from those of (S_R). Let

$$s^- := \frac{\underline{z}-\underline{x}}{Y}; \qquad s := \frac{z-x}{Y}; \qquad s^+ := \frac{\overline{z}-\overline{x}}{Y} \tag{12}$$

and

$$x^- := \underline{x}; \qquad x^+ := \overline{x}; \qquad x_r^- := \underline{x}_r; \qquad x_r^+ := \overline{x}_r \tag{13}$$

where $\underline{P} = (\underline{z},\underline{x},\underline{x_r})$, (z,x,x_r) and $\overline{P} = (\overline{z},\overline{x},\overline{x_r})$ are respectively the equilibrium point of system (S_{un}), a trajectory of system (S_R) and the equilibrium point of system (S_{ov}). Hence, $\underline{P}$ corresponds to the equilibrium point of (S_-), $P_- = (s^-,x^-,x_r^-)$, and $\overline{P}$ corresponds to the equilibrium point of (S_+), $P_+ = (s^+,x^+,x_r^+)$.

We are now ready to establish the following result for a trajectory (s,x,x_r) of system (S).

Corollary 11. *Each trajectory (s,x,x_r) of system (S) starting in Ω converges towards the domain $U := \{(s,x,x_r) \in \Omega \ : s^- - \Delta \leq s \leq s^+ + \Delta\ ,\ x^- \leq x \leq x^+\ ,\ x_r^- \leq x_r \leq x_r^+\}$, where $\Delta := \frac{x^+-x^-}{Y}$.*

Proof. From the robustness Theorem 10, we know that there exists $T > 0$ such that

$$\underline{z} \leq z(t) \leq \overline{z}; \qquad \underline{x} \leq x(t) \leq \overline{x}; \qquad \underline{x}_r \leq x_r(t) \leq \overline{x}_r \quad \forall t \geq T$$

for each trajectory (z,x,x_r) of system (S_R). According to those inequalities, if (s,x,x_r) is a trajectory of system (S) obtained from (z,x,x_r) by the transformations (12) and (13), then

$$Ys + x \leq Ys^+ + x^+$$

which implies that

$$s \leq \frac{x^+ - x}{Y} + s^+$$

and hence

$$s \le \frac{x^+ - x^-}{Y} + s^+.$$

Conversely, following the same route, we obtain that

$$s \ge \frac{x^- - x^+}{Y} + s^-.$$

We conclude that

$$s^- - \varDelta \le s \le s^+ + \varDelta, \tag{14}$$

where $\varDelta := \frac{x^+ - x^-}{Y}$. As required. □

5. Reduction of the stability region *U*

The purpose of this section is to give a practical tool for reducing the size of stability domain U. The idea is to find an adequate value of the recycle rate r for which U reaches its minimal size (volume) value. Assume that:

H_5 $0 \le r \le R$, where R is the maximum level where r can be reached.

Theorem 12. *We can reduce the size of stability domain U by increasing the recycle rate r. Furthermore, under additional assumption H_5, the minimal size of U is achieved when $r = R$ and is given by*

$$L_R = \frac{w+R}{Y^2R^2(1+R)}(s_{\text{in}}^+ - s_{\text{in}}^-)^2(s^- - s^+) + 2\frac{w+R}{Y^2R^2}(s_{\text{in}}^+ - s_{\text{in}}^-)(s^- - s^+)^2 + \frac{(1+R)(w+R)}{Y^2R^2}(s^- - s^+)^3.$$

Proof. First, let us calculate the size L of stability domain U. We have that

$$L := (s^- - s^+)(x^+ - x^-)(x_r^+ - x_r^-)$$

and remark that, according to (10), $(s^- - s^+) \ge 0$. On the other hand, taking account of equilibrium equation (5), we can calculate $(x^+ - x^-)$ by using $(s^- - s^+)$. Indeed,

$$(x^+ - x^-) = \frac{w+r}{Yr}\left[\frac{s_{\text{in}}^+ - s_{\text{in}}^-}{1+r} + s^- - s^+\right]$$

and according to equilibrium equation (3), we can calculate $(x_r^+ - x_r^-)$ by using $(s^- - s^+)$. Indeed,

$$\begin{aligned}(x_r^+ - x_r^-) &= \frac{1+r}{w+r}(x^+ - x^-)\\ &= \frac{1+r}{Yr}\left[\frac{s_{\text{in}}^+ - s_{\text{in}}^-}{1+r} + s^- - s^+\right].\end{aligned}$$

Hence,

$$L = \frac{w+r}{Y^2r^2(1+r)}(s_{\text{in}}^+ - s_{\text{in}}^-)^2(s^- - s^+) + 2\frac{w+r}{Y^2r^2}(s_{\text{in}}^+ - s_{\text{in}}^-)(s^- - s^+)^2 + \frac{(1+r)(w+r)}{Y^2r^2}(s^- - s^+)^3.$$

It is easy to prove, by studying variations of $\frac{w+r}{Y^2r^2(1+r)}$, $2\frac{w+r}{Y^2r^2}$ and $\frac{(1+r)(w+r)}{Y^2r^2}$ with respect to r, that those functions are decreasing. Now, it remains to show that if r increases then $(s^- - s^+)$ decreases. Indeed,

$$\begin{aligned}(s^- - s^+) &= \underline{\mu}^{-1}\left(Dw\frac{1+r}{w+r}\right) - \overline{\mu}^{-1}\left(Dw\frac{1+r}{w+r}\right)\\ &= \underline{\mu}^{-1}\left(Dw\frac{1+r}{w+r}\right) - \underline{\mu}^{-1}(0) + \underline{\mu}^{-1}(0) - \overline{\mu}^{-1}(0) + \overline{\mu}^{-1}(0) - \overline{\mu}^{-1}\left(Dw\frac{1+r}{w+r}\right).\end{aligned}$$

Using Lipshitz properties of $\underline{\mu}^{-1}$ and $\overline{\mu}^{-1}$ and the fact that $\underline{\mu}^{-1}(0) = \overline{\mu}^{-1}(0) = 0$, we obtain

$$(s^- - s^+) \le (k_1 + k_2)Dw\frac{1+r}{w+r}$$

where k_1 and k_2 are, respectively, the Lipshitz constant of $\underline{\mu}^-1$ and $\overline{\mu}^-1$. So, since $Dw\frac{1+r}{w+r}$ is also decreasing with respect to r then $(s^- - s^+)$ is decreasing with respect to r and hence L is decreasing with respect to r. We conclude that we can reduce the size of stability domain U by increasing the recycle rate r. Furthermore, since $0 \leq r \leq R$, then the minimal size of U is obtained when $r = R$ and is given by

$$L_R = \frac{w+R}{Y^2R^2(1+R)}(s_{\text{in}}^+ - s_{\text{in}}^-)^2(s^- - s^+) + 2\frac{w+R}{Y^2R^2}(s_{\text{in}}^+ - s_{\text{in}}^-)(s^- - s^+)^2 + \frac{(1+R)(w+R)}{Y^2R^2}(s^- - s^+)^3. \quad \square$$

Corollary 13. *Using Lipshitz properties of* $\underline{\mu}^{-1}$ *and* $\overline{\mu}^{-1}$, *we can give an estimation of* L_R *as*

$$L \leq (k_1+k_2)\frac{Dw}{Y^2R^2}\left[(s_{\text{in}}^+ - s_{\text{in}}^-)^2 + 2\frac{Dw(1+R)^2}{w+R}(s_{\text{in}}^+ - s_{\text{in}}^-)(k_1+k_2) + \frac{(1+R)^4D^2w^2}{(w+R)^2}(k_1+k_2)^2\right].$$

Proof. It suffices to substitute, in the expression for L_R, $(s^- - s^+)$ by the Lipshitz inequality

$$(s^- - s^+) \leq (k_1+k_2)Dw\frac{1+r}{w+r},$$

proved in the proof of the previous theorem. $\square$

6. Conclusion

This work is concerned with an activated sludge problem. We have proved that in the case where all parameters are well known, the model admits a unique global stable equilibrium point. Otherwise, if information about the bacterium specific growth function μ and the concentration in the feed stream s_{in} is not available, the robustness analysis leads to the existence of a stability domain U. Finally, we propose a simple way to reduce the size (volume) of the stability domain by increasing the recycle rate r, and hence the stability domain reaches its minimal size when r achieves its maximum level. This work may be extended to take account of the cost of the recycle operation. This will be our next area of interest, using optimal control theory.

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