Néhémy Lim

Distributions of Functions of Random Variables

1 Functions of One Random Variable

In some situations, you are given the pdf f_X of some rrv X. But you may actually be interested in some function of the initial rrv : $Y = u(X)$. In this chapter, we are going to study different techniques for finding the distribution of functions of random variables.

1.1 Distribution Function Technique

Assume that we are given a continuous rrv X with pdf f_X. We want to find the pdf of $Y = u(X)$. As seen previously when we studied the exponential distribution, we can apply the following strategy :

1. First, find the cdf (cumulative distribution function) $F_Y(y)$
2. Then, differentiate the cumulative distribution function $F_Y(y)$ to get the probability density function $f_Y(y)$. That is: $f_Y(y) = F'_Y(y)$

Example 1. Let X be a rrv with pdf :

$$f_X(x) = 3x^2 \mathbb{1}_{(0,1)}(x)$$

What is the pdf of $Y = X^2$?
Answer. The cdf of Y is : for $y \in (0, 1)$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(X^2 \leq y) \end{aligned}$$

Note that the transformation $u : x \mapsto x^2$ is strictly increasing on $(0, 1)$. Thus, u is invertible and its inverse $v : y \mapsto \sqrt{y}$ is also strictly increasing. Therefore, for $y \in (0, 1)$, we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(v(X^2) \leq v(y)) \\ &= \mathbb{P}(X \leq \sqrt{y}) \\ &= F_X(\sqrt{y}) \\ &= \int_0^{\sqrt{y}} 3x^2 \, dx \\ &= \left[x^3\right]_0^{\sqrt{y}} \\ &= y^{3/2} \end{aligned}$$

Hence, the pdf of Y is obtained as follows: for $y \in (0,1)$,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= \frac{3}{2}y^{3/2-1} \end{aligned}$$

In a nutshell,

$$f_Y(y) = \frac{3}{2}y^{1/2}\mathbb{1}_{(0,1)}(y)$$

Example 2. Let X be a rrv with pdf :

$$f_X(x) = 3(1-x)^2\mathbb{1}_{(0,1)}(x)$$

What is the pdf of $Y = (1-X)^3$?
Answer. The cdf of Y is : for $y \in (0,1)$

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}((1-X)^3 \leq y) \end{aligned}$$

Note that the transformation $u : x \mapsto (1-x)^3$ is strictly decreasing on $(0,1)$. Thus, u is invertible and its inverse $v : y \mapsto 1 - y^{1/3}$ is also strictly decreasing. Therefore, for $y \in (0,1)$, we have

$$\begin{aligned} F_Y(y) &= \mathbb{P}(v((1-X)^3) \geq v(y)) \\ &= \mathbb{P}(X \geq 1 - y^{1/3}) \\ &= 1 - F_X(1 - y^{1/3}) \\ &= 1 - \int_0^{1-y^{1/3}} 3(1-x)^2\,dx \\ &= 1 - \left[-(1-x)^3\right]_0^{1-y^{1/3}} \\ &= 1 + \left(\left(1-(1-y^{1/3})\right)^3 - (1-0)^3\right) \\ &= y \end{aligned}$$

Hence, the pdf of Y is obtained as follows: for $y \in (0,1)$,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}F_Y(y) \\ &= 1 \end{aligned}$$

In a nutshell,

$$f_Y(y) = \mathbb{1}_{(0,1)}(y)$$

That is Y follows a uniform distribution on $(0,1)$.

1.2 Change-of-Variable Technique

Theorem 1.1. *Let X be a continuous random variable on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with pdf $f_X = f \cdot \mathbb{1}_S$ where S is the support of f_X. If u is strictly monotonic with inverse function v, then the pdf of random variable $Y = u(X)$ is given by :*

$$f_Y(y) = f\left(v(y)\right) |v'(y)| \, \mathbb{1}_{u(S)}(y) \tag{1}$$

Proof. Assume u is strictly increasing. Then, u is invertible and its inverse v is also strictly increasing.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(u(X) \leq y) \\ &= \mathbb{P}(X \leq v(y)) \\ &= F_X(v(y)) \end{aligned}$$

Therefore,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} F_X(v(y)) \\ &= F_X'(v(y)) v'(y) \\ &= f_X(v(y)) v'(y) \end{aligned}$$

On the other hand, assume u is strictly decreasing. Then, u is invertible and its inverse v is also strictly decreasing.

$$\begin{aligned} F_Y(y) &= \mathbb{P}(Y \leq y) \\ &= \mathbb{P}(u(X) \leq y) \\ &= \mathbb{P}(X \geq v(y)) \\ &= 1 - F_X(v(y)) \end{aligned}$$

Therefore,

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} \left\{ 1 - F_X(v(y)) \right\} \\ &= -F_X'(v(y)) v'(y) \\ &= -f_X(v(y)) v'(y) \end{aligned}$$

We can merge the two cases since $v'(y) \geq 0$ if v is increasing and $v'(y) \leq 0$ if v is decreasing. □

For illustration, apply the Change-of-Variable Technique to Examples 1 and 2 and make sure you find the same results.

Case of two-to-one transformations.
Example 3. Let X be a rrv with pdf :

$$f_X(x) = \frac{x^2}{3}\mathbb{1}_{(-1,2)}(x)$$

What is the pdf of $Y = X^2$?
Answer. Note that the transformation $u : x \mapsto x^2$ is not strictly monotonic on $(-1, 2)$. Therefore we cannot apply Theorem 1.1 straight away. More precisely, u is two-to-one on $(-1, 1)$ and one-to-one on $(1, 2)$.

Let us focus on the interval $(-1, 1)$ and use the distribution technique. In that case, we have for $y \in (0, 1)$:

$$\begin{aligned}
F_Y(y) &= \mathbb{P}(Y \leq y) \\
&= \mathbb{P}(X^2 \leq y) \\
&= \mathbb{P}(-\sqrt{y} \leq X \leq \sqrt{y}) \\
&= F_X(\sqrt{y}) - F_X(-\sqrt{y})
\end{aligned}$$

Noting that

- u is strictly decreasing on $(-1, 0)$ with strictly decreasing inverse $v_1 : y \mapsto -\sqrt{y}$
- u is strictly increasing on $(0, 1)$ with strictly increasing inverse $v_2 : y \mapsto \sqrt{y}$

and by differentiating the cdf, we obtain : for $y \in (0, 1)$,

$$\begin{aligned}
f_Y(y) &= \frac{d}{dy}F_Y(y) \\
&= v_2'(y)f_X(v_2(y)) + (-v_1'(y))f_X(v_1(y)) \\
&= \frac{1}{2}y^{-1/2}f_X(\sqrt{y}) + \frac{1}{2}y^{-1/2}f_X(-\sqrt{y}) \\
&= \frac{1}{2}y^{-1/2}\frac{\sqrt{y}^2}{3} + \frac{1}{2}y^{-1/2}\frac{(-\sqrt{y})^2}{3} \\
&= \frac{\sqrt{y}}{3}
\end{aligned}$$

On the interval $(1, 2)$, u is strictly increasing, thus we can apply Theorem 1.1. After some calculations, you should find that for $y \in (1, 4)$,

$$f_Y(y) = \frac{\sqrt{y}}{6}$$

In a nutshell,

$$f_Y(y) = \begin{cases} \sqrt{y}/3 & \text{if } 0 < y < 1 \\ \sqrt{y}/6 & \text{if } 1 < y < 4 \\ 0 & \text{otherwise} \end{cases}$$

Let us generalize our finding. If the transformation u is two-to-one on some interval and can be *split* into two strictly monotonic functions with inverses v_1 and v_2. The the pdf of $Y = u(X)$ on that interval is :

$$f_Y(y) = |v_1'(y)| f_X(v_1(y)) + |v_2'(y)| f_X(v_2(y))$$

2 Transformations of Two Random Variables

Theorem 2.1. *Let X and Y be two continuous random variables on probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with joint pdf $f_{XY} = f \cdot \mathbb{1}_S$ where $S \subset \mathbb{R}^2$ is the support of f_{XY}. If $u = (u_1, u_2)$ is an invertible function on S with inverse function $v = (v_1, v_2)$, then the joint pdf of random variables $W = u_1(X, Y)$ and $Z = u_2(X, Y)$ is given by :*

$$f_{WZ}(w, z) = f\left(v_1(w, z), v_2(w, z)\right) |J| \, \mathbb{1}_{u(S)}(w, z) \tag{2}$$

where J is the Jacobian of v at point (s, t) defined by the following determinant :

$$J = \begin{vmatrix} \frac{\partial v_1(w,z)}{\partial w} & \frac{\partial v_1(w,z)}{\partial z} \\ \frac{\partial v_2(w,z)}{\partial w} & \frac{\partial v_2(w,z)}{\partial z} \end{vmatrix} = \frac{\partial v_1(w, z)}{\partial w} \frac{\partial v_2(w, z)}{\partial z} - \frac{\partial v_2(w, z)}{\partial w} \frac{\partial v_1(w, z)}{\partial z}$$

Example 4. Let X and Y be 2 rrv with joint pdf :

$$f_{XY}(x, y) = e^{-(x+y)} \mathbb{1}_{(0,\infty)^2}(x, y)$$

What is the joint pdf of $W = X + Y$ and $Z = \frac{X}{X+Y}$?
Answer. Let us solve the following system for X and Y :

$$\begin{aligned}
\begin{cases} W & = u_1(X, Y) = X + Y \\ Z & = u_2(X, Y) = \frac{X}{X+Y} \end{cases} & \Leftrightarrow \begin{cases} Y & = W - X \\ Z & = \frac{X}{W} \end{cases} \\
& \Leftrightarrow \begin{cases} Y & = W - X \\ X & = WZ \end{cases} \\
& \Leftrightarrow \begin{cases} Y & = W - WZ = v_2(W, Z) \\ X & = WZ = v_1(W, Z) \end{cases}
\end{aligned}$$

The determinant of the Jacobian of $v = (v_1, v_2)$ is thus given by :

$$\begin{aligned}
J &= \begin{vmatrix} \frac{\partial v_1(w,z)}{\partial w} & \frac{\partial v_1(w,z)}{\partial z} \\ \frac{\partial v_2(w,z)}{\partial w} & \frac{\partial v_2(w,z)}{\partial z} \end{vmatrix} \\
&= \begin{vmatrix} z & w \\ 1 - z & -w \end{vmatrix} \\
&= -wz - w(1 - z) \\
&= -w
\end{aligned}$$

The support of the joint pdf of X and Y is $S = (0, \infty)^2$. The transformation $u : (x, y) \mapsto (x + y, x/(x + y))$ maps S in the xy-plane into the domain $u(S)$ in the (w, z)-plane given by $w = x + y > 0$ and $z = x/(x + y) \in (0, 1)$. Thus, the joint pdf of W and Z is given by :

$$\begin{aligned} f_{WZ}(w, z) &= e^{-(v_1(w,z)+v_2(w,z))} \left|-w\right| \mathbb{1}_{(0,\infty)\times(0,1)}(w, z) \\ &= e^{-(wz+w-wz)} w \mathbb{1}_{(0,\infty)\times(0,1)}(w, z) \\ &= we^{-w} \mathbb{1}_{(0,\infty)\times(0,1)}(w, z) \end{aligned}$$