5. True or False

Problem Set due Mar 13, 2020 05:29 IST *Completed*

Problem 5. True or False

3/3 points (graded)

Determine whether each of the following statement is true (i.e., always true) or false (i.e., not always true).

1. Let X be a random variable that takes values between 0 and c only, for some $c \geq 0$, so that $\mathbf{P}(0 \leq X \leq c) = 1$. Then, $\mathsf{Var}(X) \leq c^2/4$.

   True ✔ **Answer:** True

2. Let X and Y be continuous random variables. If $X \sim N(\mu, \sigma^2)$ (i.e., normal with mean μ and variance σ^2), $Y = aX + b$, and $a > 0$, then $Y \sim N(a\mu + b, a\sigma^2)$.

   False ✔ **Answer:** False

3. The expected value of a non-negative continuous random variable X, which is defined by $\mathbf{E}[X] = \int_0^\infty x f_X(x)\, dx$, also satisfies $\mathbf{E}[X] = \int_0^\infty \mathbf{P}(X > t)\, \mathrm{d}t$.

   True ✔ **Answer:** True

Solution:

1. The statement is true. Since $0 \leq X \leq c$,

$$\begin{aligned}
\mathbf{E}[X^2] &= \mathbf{E}[XX] \\
&\leq \mathbf{E}[cX] \\
&= c\mathbf{E}[X].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\text{Var}(X) &= \mathbf{E}[X^2] - (\mathbf{E}[X])^2 \\
&\leq c\mathbf{E}[X] - (\mathbf{E}[X])^2 \\
&= c^2\left(\frac{\mathbf{E}[X]}{c}\right) - c^2\left(\frac{\mathbf{E}[X]}{c}\right)^2 \\
&= c^2\left(\frac{\mathbf{E}[X]}{c}\left(1 - \frac{\mathbf{E}[X]}{c}\right)\right) \\
&= c^2\left[\alpha(1-\alpha)\right] \\
&\leq c^2/4,
\end{aligned}$$

where $\alpha = \mathbf{E}[X]/c$. The last inequality is obtained by noticing that the function $\alpha(1-\alpha)$ is largest at $\alpha = 1/2$, where it takes a value of $1/4$.

2. The statement is false. The correct statement is: $Y \sim N(a\mu + b, a^2\sigma^2)$.

3. The statement is true. By changine the order of integration, we obtain

$$\begin{aligned}
\int_0^\infty \mathbf{P}(X > t)\,\mathrm{d}t &= \int_0^\infty \int_t^\infty f_X(x)\,dx\,dt \\
&= \int_0^\infty \int_0^x f_X(x)\,dt\,dx \\
&= \int_0^\infty x f_X(x)\,dx \\
&= \mathbf{E}[X].
\end{aligned}$$

This result is analogous to the result for discrete random variables that was the subject of a Unit 4 solved problem.

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You have used 1 of 1 attempt

Answers are displayed within the problem

Discussion

Topic: Unit 5: Continuous random variables:Problem Set 5 / 5. True or False

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- AAAARGH
  When you have only one attempt: solve the problem; save it; re-solve it with fresher mind; submit. Don't ... 4
- Q3 limits when switching order of integration?
  I don't understand why the limits on the inner integral after the switch are 0 to x, instead of x to infinity. I... 2
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  Hi. It is follows from Popoviciu's inequality on variances -- https://en.wikipedia.org/wiki/Popoviciu%27s_i... 1
  Community TA
- ? Hint: 5(3) on approach to E[X]? 3 new_ 7
- ? Notation
  Please forgive me, however what does this means: X~N(μ,σ2) ? I am not sure what the ~ means 3
- ? Q1 Clarification 9
- ? Q1 clarification (2)
  For Q1, should we assume that X is a uniform random variable, or could it be any arbitrary distribution b... 4