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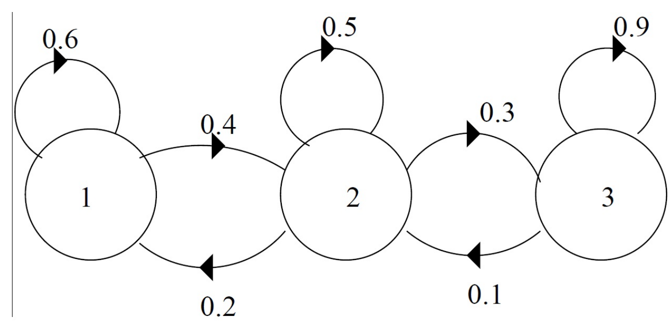
4. A simple Markov chain

Problem Set due May 29, 2020 05:29 IST

Problem 4. A simple Markov chain

10 points possible (ungraded)

Consider a Markov chain $\{X_0, X_1, \ldots\}$, specified by the following transition probability graph.



1. $\mathbf{P}(X_2 = 2 \mid X_0 = 1) =$ **Answer:** 0.44

2. Find the steady-state probabilities π_1, π_2, and π_3 associated with states 1, 2, and 3, respectively.

   - $\pi_1 =$ **Answer:** 0.11111
   - $\pi_2 =$ **Answer:** 0.22222
   - $\pi_3 =$ **Answer:** 0.66667

3. For $n = 1, 2, \ldots$, let $Y_n = X_n - X_{n-1}$. Thus, $Y_n = 1$ indicates that the nth transition was to the right, $Y_n = 0$ indicates that it was a self-transition, and $Y_n = -1$ indicates that it was a transition to the left.

   $\lim\limits_{n\to\infty} \mathbf{P}(Y_n = 1) =$ **Answer:** 0.11111

4. Is the sequence $Y_1, Y_2, \ldots$ a Markov chain?

   Select an option **Answer:** No

5. Given that the nth transition was a transition to the right ($Y_n = 1$), find (approximately) the probability that the state at time $n-1$ was state 1 (i.e., $X_{n-1} = 1$). Assume that n is large.

   Answer: 0.4

6. Suppose that $X_0 = 1$. Let T be the first **positive** time index n at which the state is equal to 1.

   $\mathbf{E}[T] =$ **Answer:** 9

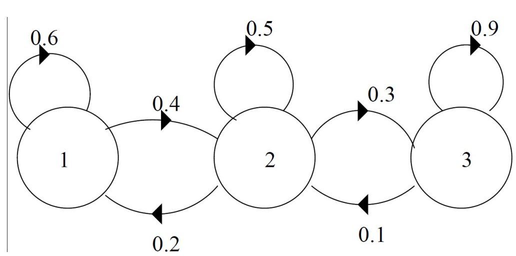
7. Does the sequence $X_1, X_2, X_3, \ldots$ converge in probability to a constant?

   Select an option **Answer:** No

8. Let $Z_n = \max\{X_1, \ldots, X_n\}$. Does the sequence $Z_1, Z_2, Z_3, \ldots$ converge in probability to a constant?

   Select an option **Answer:** Yes

Solution:



1. There are only two paths that go from state 1 to state 2 in two transitions: $1 \to 1 \to 2$ and $1 \to 2 \to 2$. The desired two-step transition probability is therefore

$$
\begin{aligned}
r_{12}(2) &= p_{11} \cdot p_{12} + p_{12} \cdot p_{22} \\
&= 0.6 \cdot 0.4 + 0.4 \cdot 0.5 \\
&= 0.44.
\end{aligned}
$$

2. We write down the local balance equations of a birth-death process and the normalization equation:

$$
\begin{aligned}
\pi_1 p_{12} &= \pi_2 p_{21} \\
\pi_2 p_{23} &= \pi_3 p_{32} \\
\pi_1 + \pi_2 + \pi_3 &= 1.
\end{aligned}
$$

   Solving this system of equations yields the following steady-state probabilities:

$$
\begin{aligned}
\pi_1 &= 1/9 \\
\pi_2 &= 2/9 \\
\pi_3 &= 6/9.
\end{aligned}
$$

3. Using the total probability theorem and the convergence to steady-state probabilities, we have

$$
\begin{aligned}
\lim_{n\to\infty} \mathbf{P}(Y_n = 1) &= \lim_{n\to\infty} \sum_{i=1}^{3} \mathbf{P}(X_{n-1} = i)\, \mathbf{P}(Y_n = 1 \mid X_{n-1} = i) \\
&= \sum_{i=1}^{3} \pi_i \cdot \mathbf{P}(Y_1 = 1 \mid X_0 = i) \\
&= \pi_1 p_{12} + \pi_2 p_{23} \\
&= 1/9.
\end{aligned}
$$

4. Note that $Y_1 = 1$, $Y_2 = 1$, and $Y_3 = 0$ implies that $X_3 = 3$.
   On the other hand, $Y_1 = -1$, $Y_2 = -1$, and $Y_3 = 0$ implies that $X_3 = 1$.
   Thus,

$$
\begin{aligned}
& \mathbf{P}(Y_4 = 1 \mid Y_1 = 1, Y_2 = 1, Y_3 = 0) = 0 \\
\neq\ & \mathbf{P}(Y_4 = 1 \mid Y_1 = -1, Y_2 = -1, Y_3 = 0) = p_{12} = 0.4,
\end{aligned}
$$

   even though $Y_3 = 0$ in both cases. Hence, the Markov property is violated.

5. Using Bayes' rule and the convergence to steady-state probabilities, we have

$$
\lim_{n\to\infty} \mathbf{P}(X_{n-1} = 1 \mid Y_n = 1) = \lim_{n\to\infty} \frac{\mathbf{P}(X_{n-1} = 1)\, \mathbf{P}(Y_n = 1 \mid X_{n-1} = 1)}{\sum_{i=1}^{3} \mathbf{P}(X_{n-1} = i)\, \mathbf{P}(Y_n = 1 \mid X_{n-1}}
$$

$$= \frac{\pi_1 p_{12}}{\pi_1 p_{12} + \pi_2 p_{23}}$$
$$= 2/5.$$

Hence, for large n, the desired probability is approximately $2/5$.

6. We are looking for the mean recurrence time of state 1. In order to calculate it, we first calculate the mean first passage times to state 1 by solving the following system of equations:

$$\begin{aligned} t_2 &= 1 + p_{22}t_2 + p_{23}t_3 \\ t_3 &= 1 + p_{32}t_2 + p_{33}t_3. \end{aligned}$$

Solving the system of equations yields $t_2 = 20$ and $t_3 = 30$. Hence, the mean recurrence time of state 1 is $t_1^* = \mathbf{E}[T] = 1 + p_{12}t_2 = 9$.

7. Even in steady state, X_n has positive probability of being equal to any of the three possible states. Hence the sequence $\{X_n\}$ does not converge in probability to a constant.

8. The sequence $\{Z_n\}$ converges to 3 in probability. Here is an intuitive explanation. For the original Markov chain, states $\{1,2,3\}$ form a single recurrent class. Therefore, the Markov chain will eventually visit state 3 at some time n^*, at which point $Z_{n^*} = 3$ and $Z_n = 3$ for all $n > n^*$.

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