3. General Random Variables
Part V: Conditioning

ECE 302 Fall 2009 TR 3-4:15pm

Purdue University, School of ECE

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Conditioning a continuous random variable on an event

- Conditional PDF of a continuous random variable *X*, conditioned on an event *A* with **P**(*A*)>0:

$$\mathbf{P}(X \in B \mid A) = \int_B f_{X|A}(x)\,dx,$$

$$\text{for "any" } B \subset R$$

Computing the conditional PDF of *X* from the marginal PDF of *X*

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If $A = \{X \in C\}$ where $C \subset R$, then, on the one hand,

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$$\text{Therefore, } f_{X|X \in C}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbf{P}(X \in C)}, & \text{if } x \in C \\ 0, & \text{otherwise} \end{cases}$$

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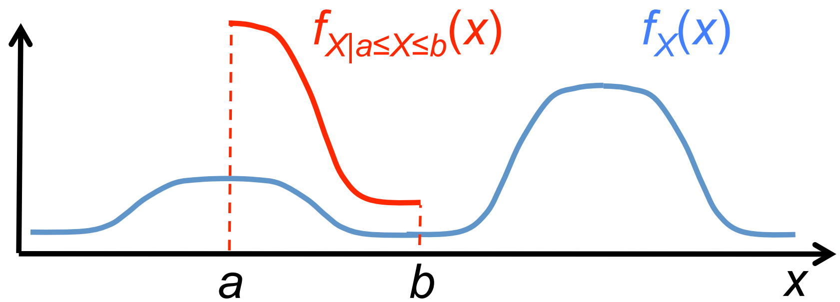

Example 3.13: memorylessness of the exponential random variable

- Time T between two successive buses is an exponential r.v. with mean $1/\lambda$
- You arrive t seconds after the last bus. Event $A=\{T>t\}$ = {there has been no buses in the t seconds since the last bus}
- $X = T - t$ the time you have to wait until the next bus
- Find $F_{X|A}(x)$.

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$$f_T(t) = \begin{cases} \lambda e^{-\lambda t}, \text{ for } t > 0 \\ 0, \text{ for } t \leq 0 \end{cases}$$

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A process like this where inter-arrival times are exponentially distributed, is called a *Poisson process* (to be studied later in the course).

Total Probability Theorem for PDFs

Recall that, according to the total probability theorem,

if $A_1, \ldots, A_n$ partition the sample space, then

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(B \mid A_i)\mathbf{P}(A_i)$$

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Differentiating both sides with respect to x, we obtain:

$$f_X(x)=\sum_{i=1}^{n}f_{X\mid A_i}(x)\mathbf{P}(A_i)$$

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This motivates the following definition:

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \text{ for all } y \text{ such that } f_Y(y) \neq 0.$$

Conditioning and independence

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}, \text{ for all } y \text{ such that } f_Y(y) \neq 0.$$

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$f_{X|Y}(x \mid y) = f_X(x)$ whenever $f_{X|Y}(x \mid y)$ is defined.

Similarly, if X and Y are independent, then

$f_{Y|X}(y \mid x) = f_Y(y)$ whenever $f_{Y|X}(y \mid x)$ is defined.

Conditional Expectation

If A is an event and X is a r.v., then

$$E[X \mid A] = \begin{cases} \int_{-\infty}^{\infty} x f_{X|A}(x)\,dx & \text{if } X \text{ is a continuous r.v.} \\ \sum_{x} x p_{X|A}(x) & \text{if } X \text{ is a discrete r.v.} \end{cases}$$

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If X and Y are r.v.'s, then

$$E[X \mid Y = y] = \begin{cases} \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\,dx & \text{if, given } Y{=}y,\, X \text{ is a continuous r.v.} \\ \sum_x x p_{X|Y}(x \mid y) & \text{if, given } Y{=}y,\, X \text{ is a discrete r.v.} \end{cases}$$

Example

$$f_{X,Y}(x,y) = \begin{cases} 3x, \ 0 \le \mathrm{y} \le \mathrm{x} \le 1 \\ 0, \text{otherwise} \end{cases}$$

Find $f_X(x)$ and $f_{Y|X}(y\,|\,x)$.

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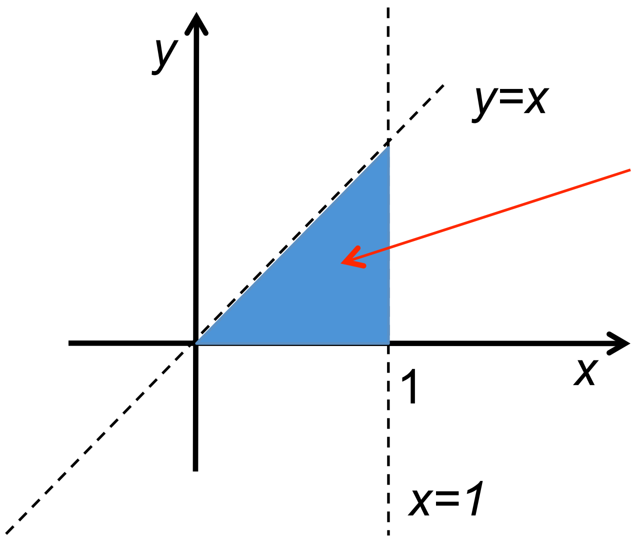

Here, $f_{X,Y}(x,y) = 3x$

Elsewhere, $f_{X,Y}(x,y) = 0$

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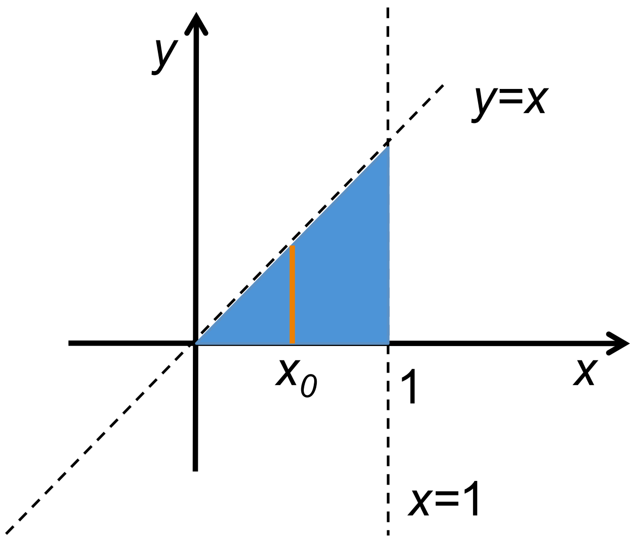

To find $f_X(x_0)$, integrate along the orange line.

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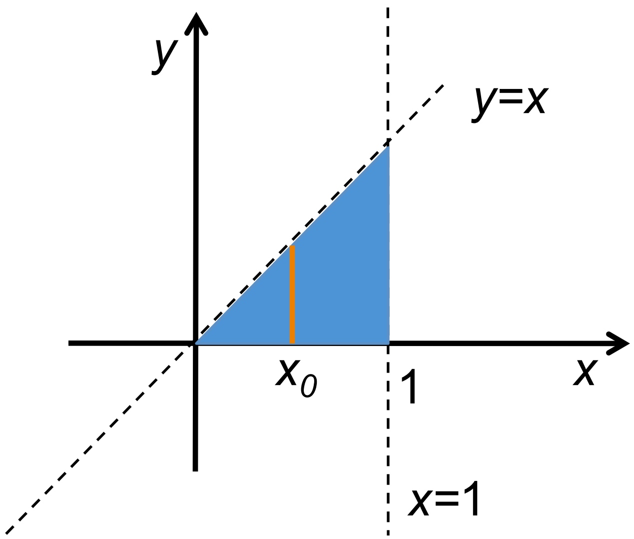

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$$f_X(x_0) = \int_{-\infty}^{\infty} f_{X,Y}(x_0, y)\,dy = \begin{cases} \int_0^{x_0} 3x_0\,dy = 3x_0^2, & 0 \le x_0 \le 1 \\ 0, & \text{otherwise} \end{cases}$$

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$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \begin{cases} \dfrac{3x}{3x^2} = \dfrac{1}{x}, & 0 \le y \le x \\ 0, & \text{otherwise} \end{cases}$$

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For all other values of x, $f_{Y|X}(y \mid x)$ is undefined.

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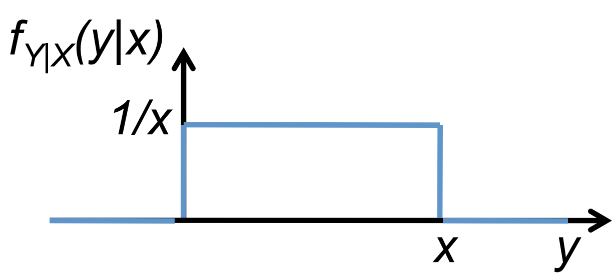

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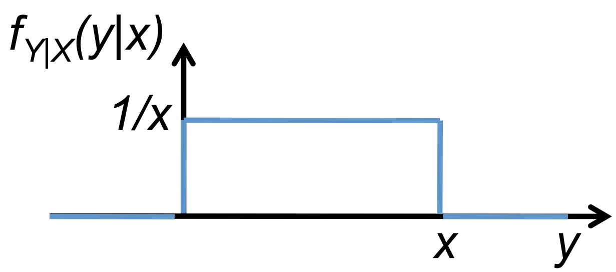
as a collection of functions of y, one for each x.
Viewed as a function of y, with x fixed, $f_{Y|X}(y\,|\,x)$
is a PDF (i.e., nonnegative and integrates to one).

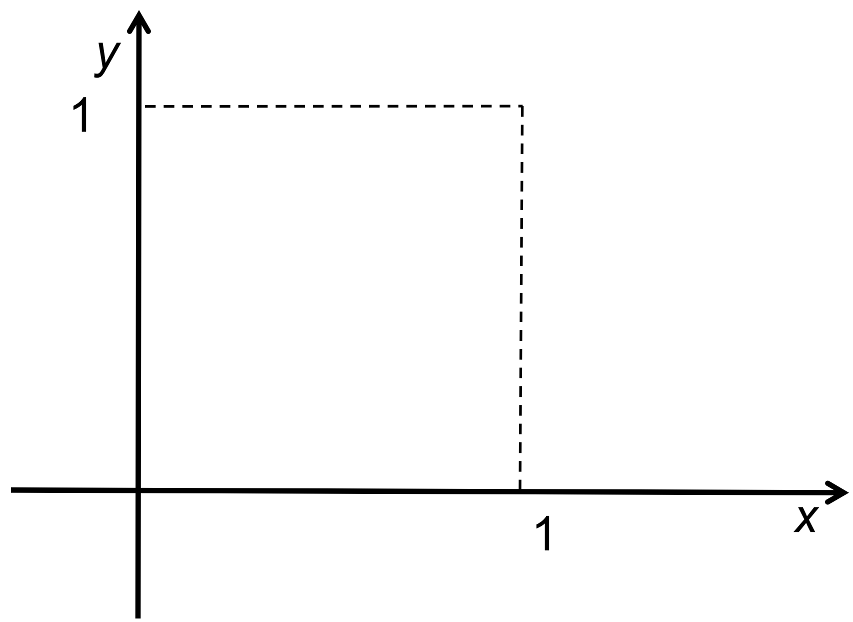


Example 3.9

- Romeo & Juliet have a date at a given time, and each will arrive with a delay between 0 and 1 hours.
- X = Romeo's delay.
- Y = Juliet's delay.
- X and Y are jointly uniform over [0,1]x[0,1].
- The first to arrive will wait for 15 minutes and will leave if the other has not yet arrived.
- **P**(they will meet) = ?

Example 3.9: solution

Their joint arrivals are uniform over the 1x1 square.



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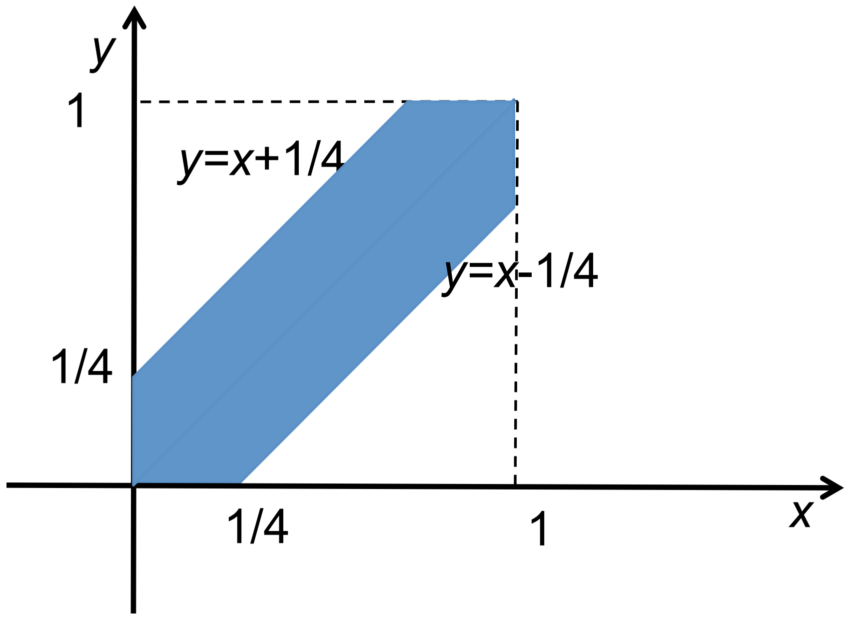
They meet in the blue region.



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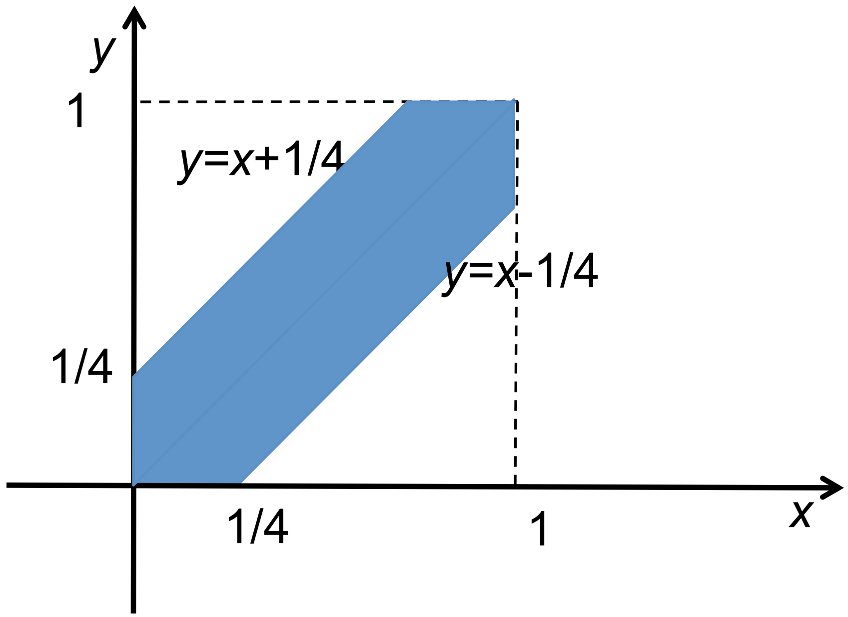

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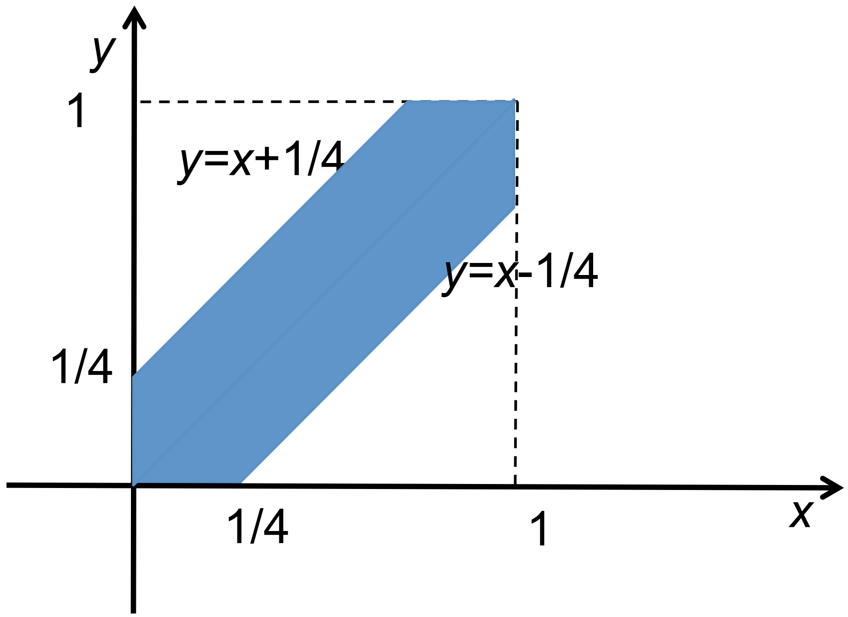

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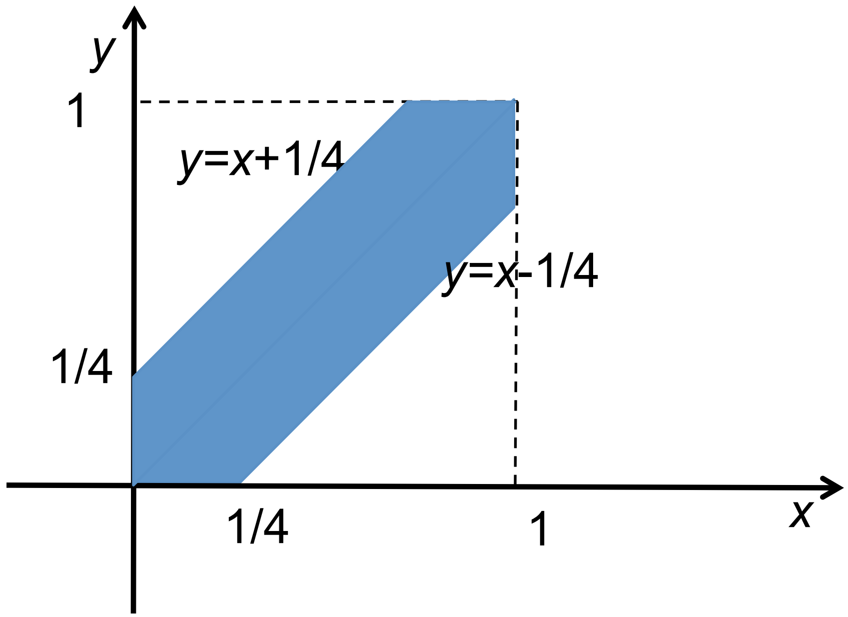

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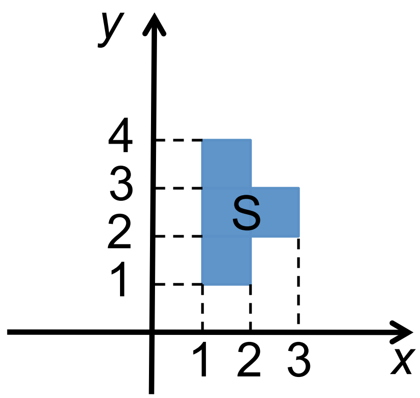
The joint density over the square is 1. This integral is therefore the area of the blue region, which is $1 - (3/4)^2/2 - (3/4)^2/2 = 7/16$.

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,\mathrm{y}) \in \mathrm{S} \\ 0, & \text{otherwise} \end{cases}$$

Find c, f_X, f_Y, $f_{X|Y}$, $f_{Y|X}$.

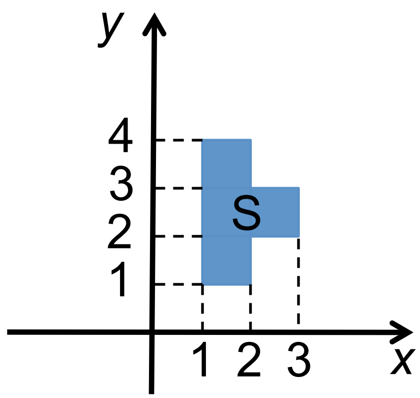


Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

The integral of the joint PDF over S must be equal to 1.

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.



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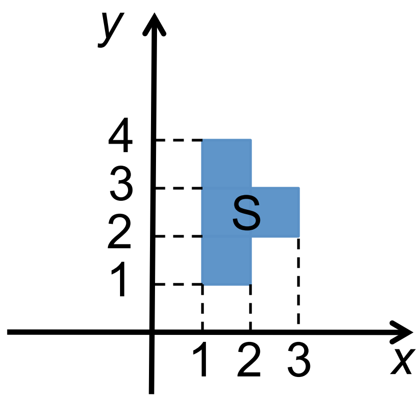

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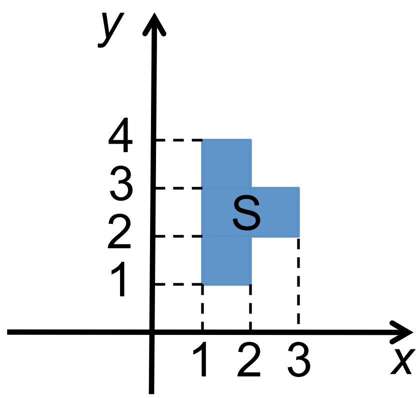

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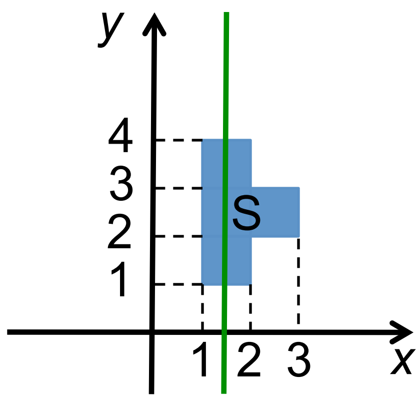
Hence, *c* = 1/area(S) = 1/4.

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,\mathrm{y}) \in \mathrm{S} \\ 0, & \text{otherwise} \end{cases}$$

Find c, f_X, f_Y, $f_{X|Y}$, $f_{Y|X}$.



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To find the marginal PDF of *X*, integrate the joint PDF over vertical lines

Example

$$f_{X,Y}(x,y)=\begin{cases} c, & (x,y)\in S \\ 0, & \text{otherwise} \end{cases}$$

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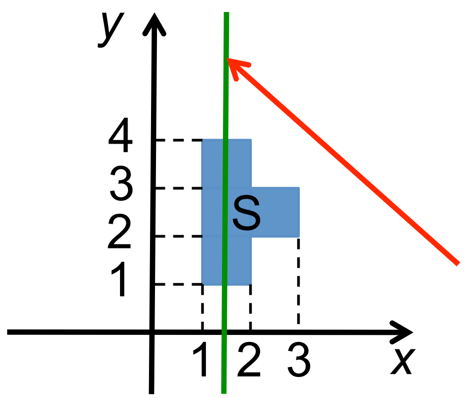

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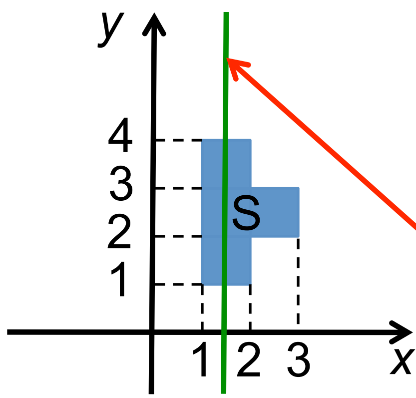
To find the marginal PDF of *X*, integrate the joint PDF over vertical lines

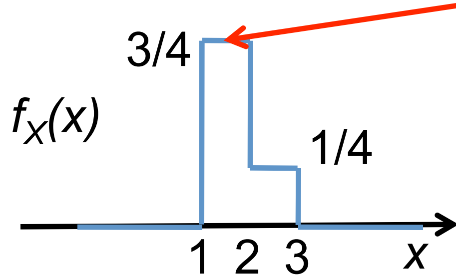
E.g., the integral of $f_{X,Y}(x,y)$ along the line x=1.5

Example

$$f_{X,Y}(x,y)=\begin{cases} c, & (x,y)\in \mathrm{S} \\ 0, & \text{otherwise} \end{cases}$$

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.





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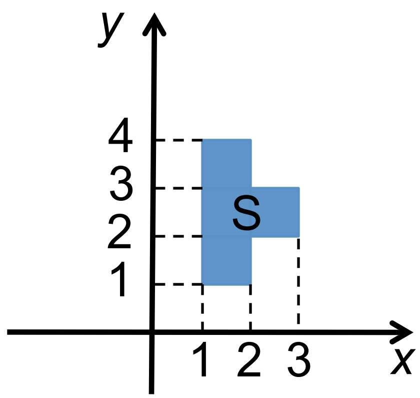
To find the marginal PDF of X, integrate the joint PDF over vertical lines

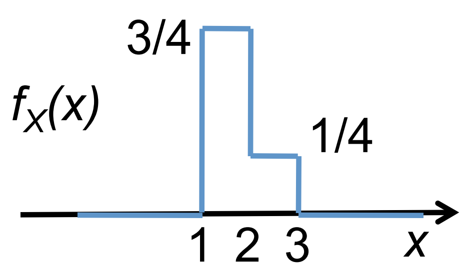
E.g., the integral of $f_{X,Y}(x,y)$ along the line x=1.5 is the marginal PDF $f_X(1.5)$.

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in \mathrm{S} \\ 0, & \text{otherwise} \end{cases}$$

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.





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To find the marginal PDF of X, integrate the joint PDF over vertical lines

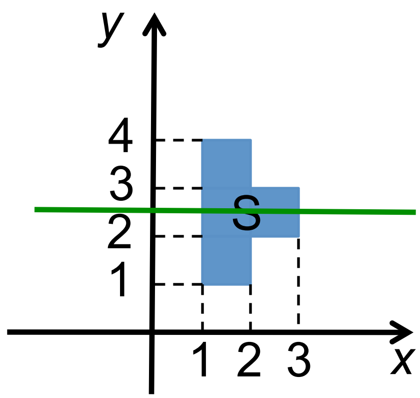
E.g., the integral of $f_{X,Y}(x,y)$ along the line x=1.5 is the marginal PDF $f_X(1.5)$.

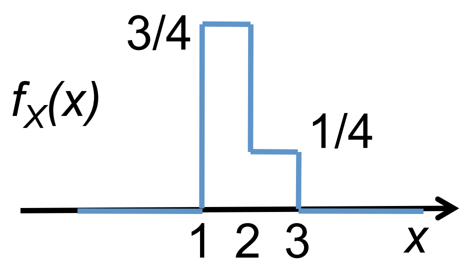
Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

To find the marginal PDF of *Y*, integrate the joint PDF over horizontal lines

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.





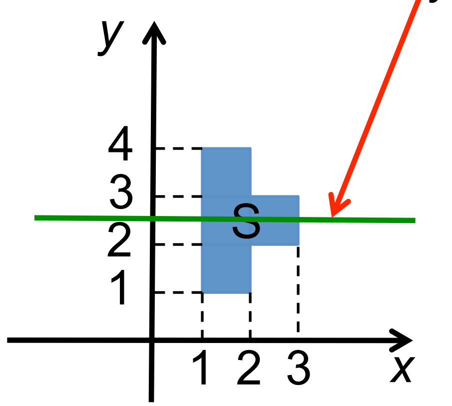
Example

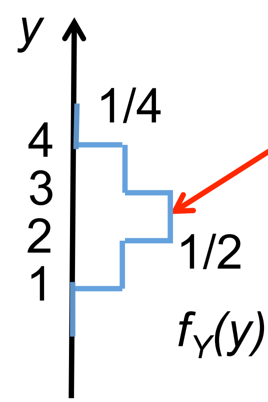
$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

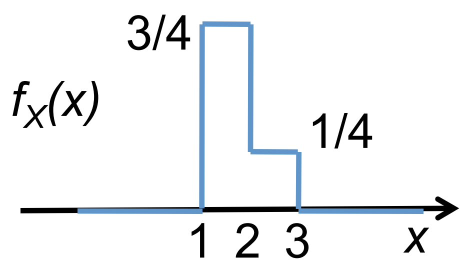
Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.

To find the marginal PDF of *Y*,
integrate the joint PDF over horizontal lines

E.g., the integral of $f_{X,Y}(x,y)$ along the line y=2.5 is the marginal PDF $f_Y(2.5)$.



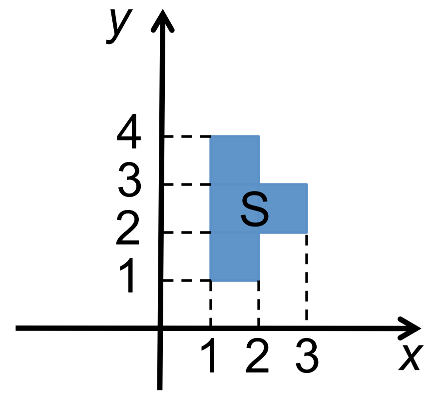


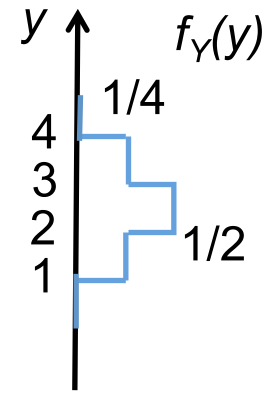


Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,\mathrm{y}) \in \mathrm{S} \\ 0, & \text{otherwise} \end{cases}$$

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.



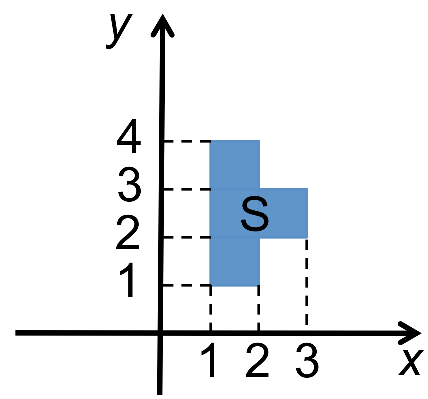


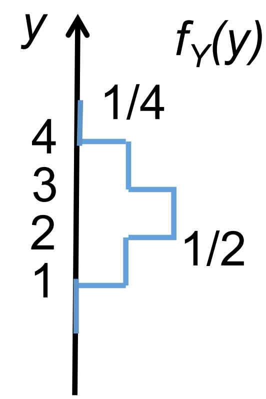
$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

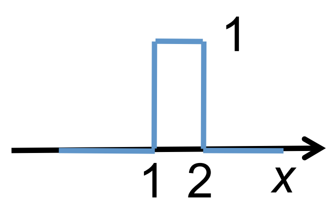
Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.





Recall: $c = 1/4$

$$f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} =$$



if $1 \le y \le 2$ or $3 \le y \le 4$

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,\mathrm{y}) \in \mathrm{S} \\ 0, & \text{otherwise} \end{cases}$$

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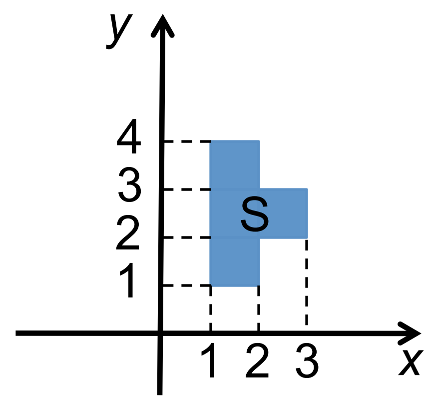

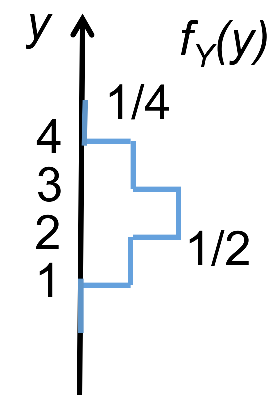

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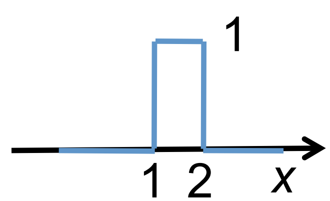

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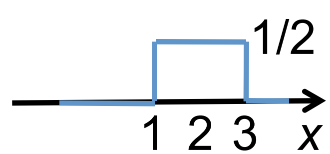
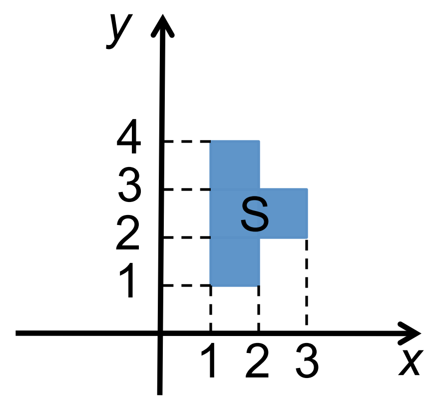


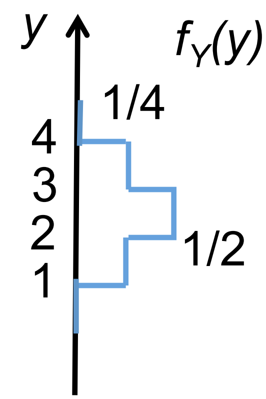
if $2 \le y \le 3$

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.





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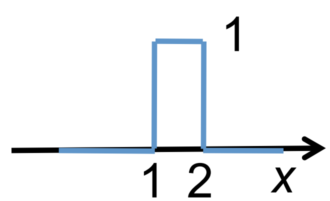

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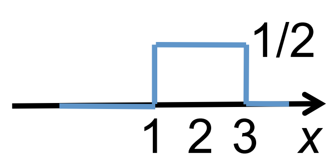


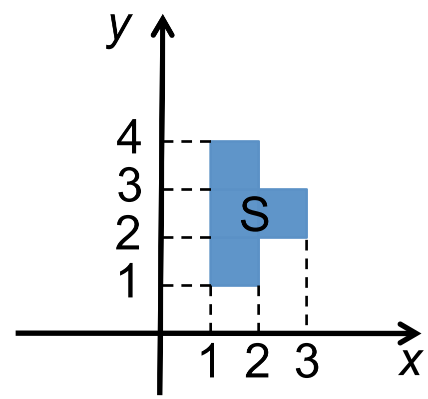
if $2 \le y \le 3$

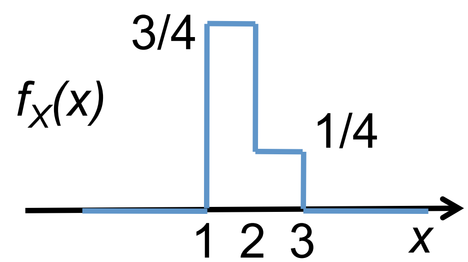
undefined if $y < 1$ or $y > 4$

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.





Recall: c = 1/4

$$f_{Y|X}(y \mid x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \left\{ \quad \right.$$

Example

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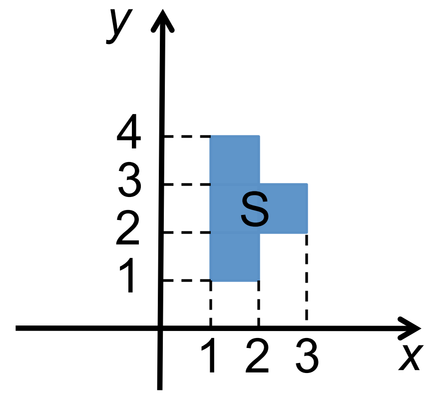

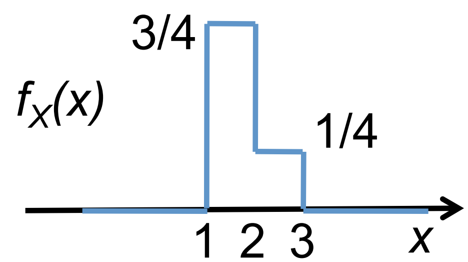

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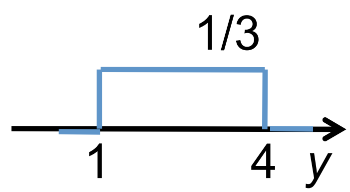

if $1 \le x \le 2$

Example

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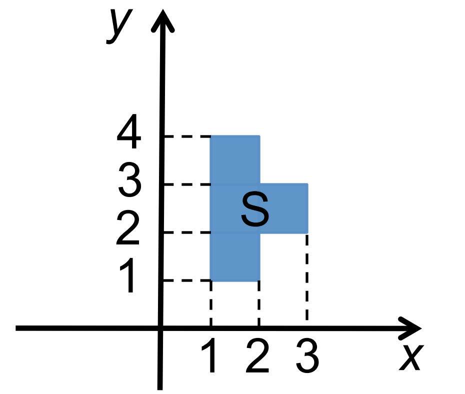

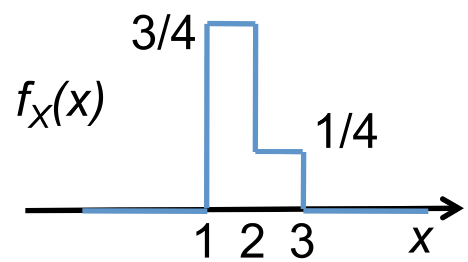

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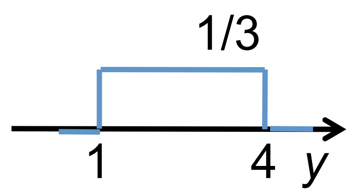

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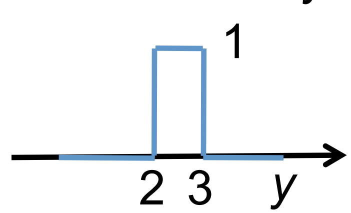
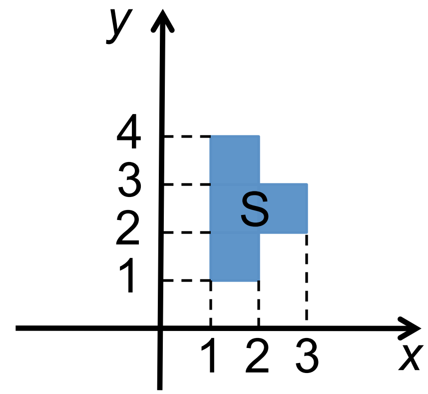


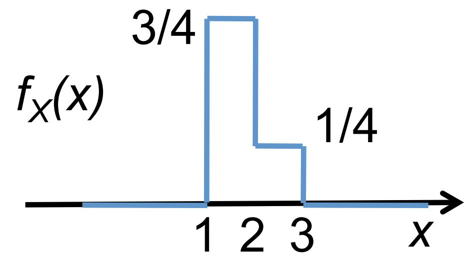
if $2 \leq x \leq 3$

Example

$$f_{X,Y}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{otherwise} \end{cases}$$

Find $c, f_X, f_Y, f_{X|Y}, f_{Y|X}$.





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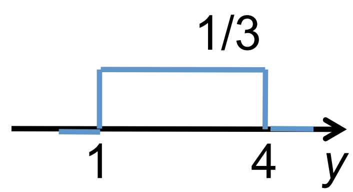

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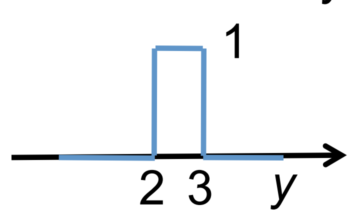


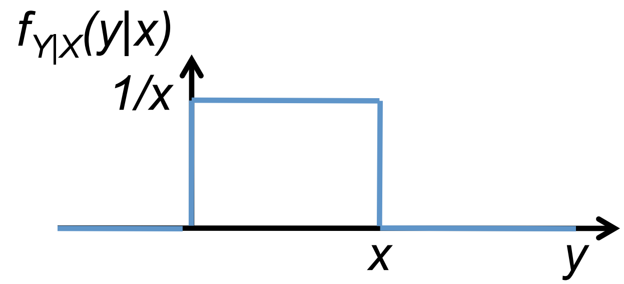
if $2 \le x \le 3$

undefined if $x < 1$ or $x > 3$

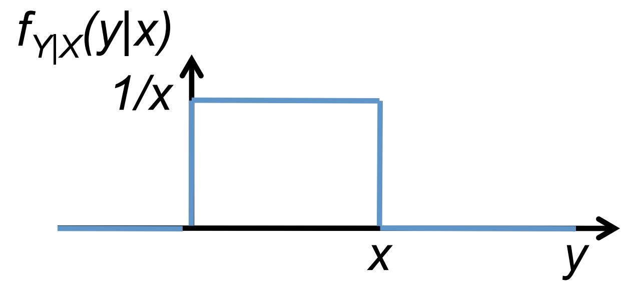
Problem 3.21

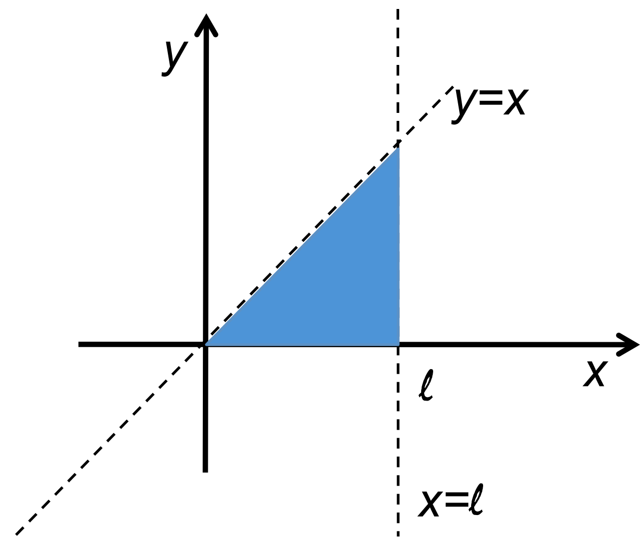
- Stick of length ℓ is broken at a random point, uniformly distributed over the length of the stick.
- X = length of the left piece.
- The left piece is broken at a random point, whose conditional distribution given the left piece is uniform over the left piece.
- Y = length of the resulting left piece.
- Find $f_{X,Y}$ and f_Y
- Find E[Y]

Problem 3.21: $f_{X,Y}$

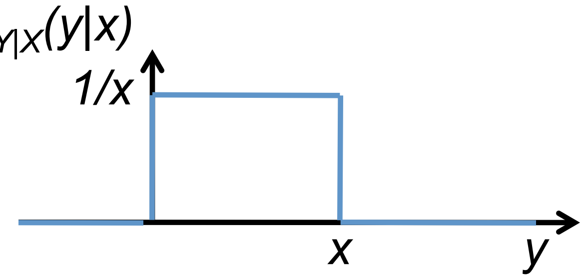


Problem 3.21: $f_{X,Y}$

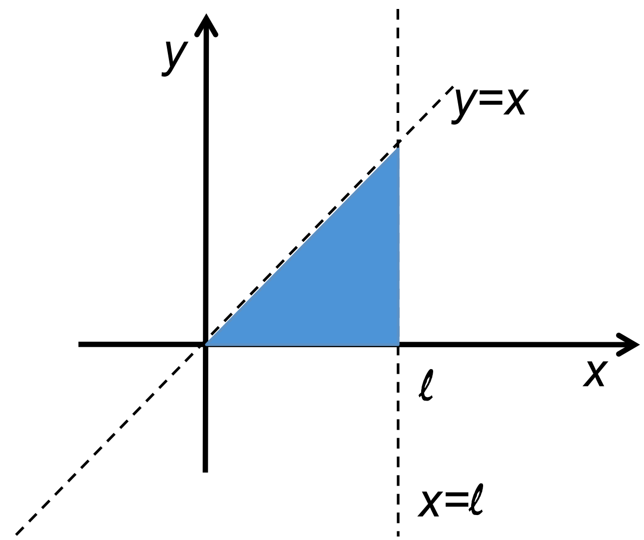




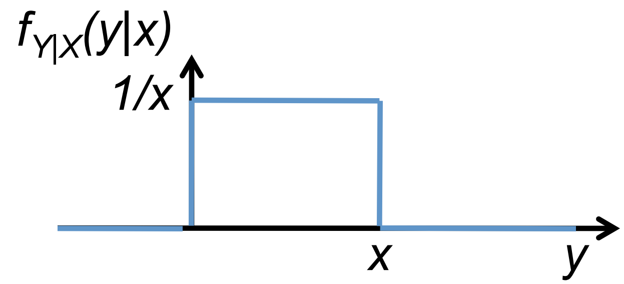
Problem 3.21: $f_{X,Y}$



$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) = \begin{cases} \dfrac{1}{\ell x}, & 0 \le y \le x \le \ell \\ 0, & \text{otherwise} \end{cases}$$



Problem 3.21: f_Y



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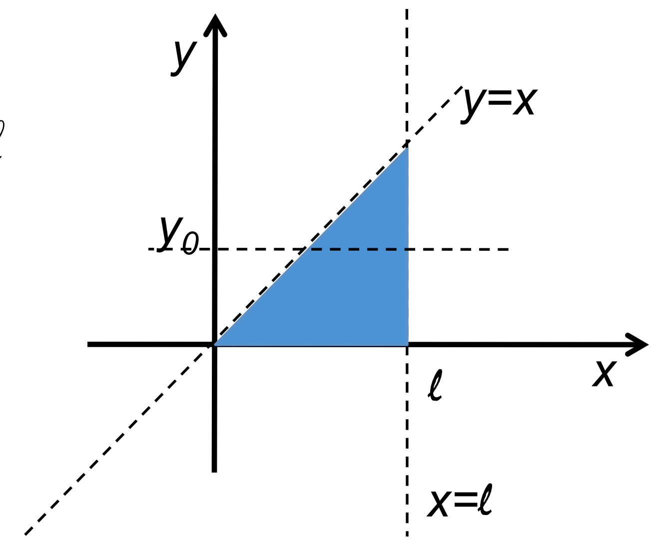
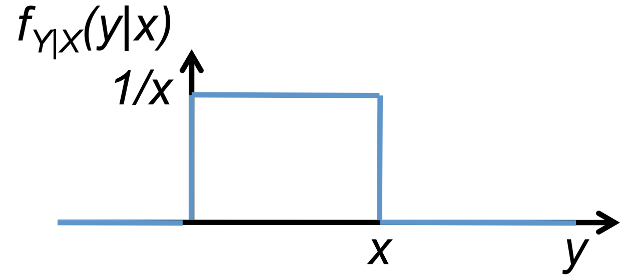
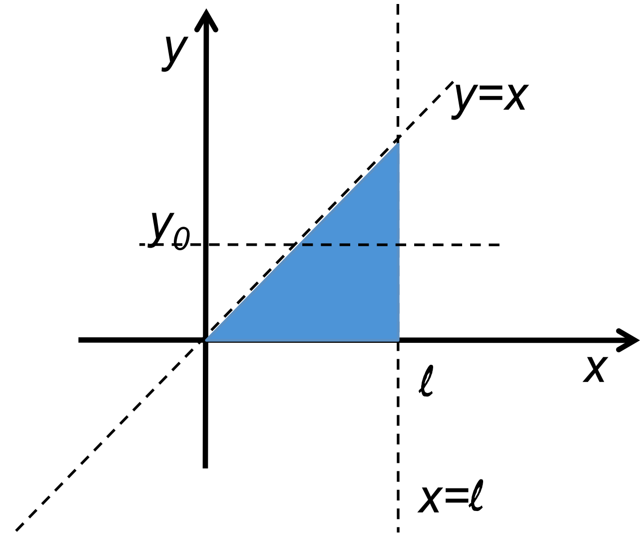


$$f_Y(y_0) = \int_{-\infty}^{\infty} f_{X,Y}(x, y_0)\,dx$$

Problem 3.21: f_Y



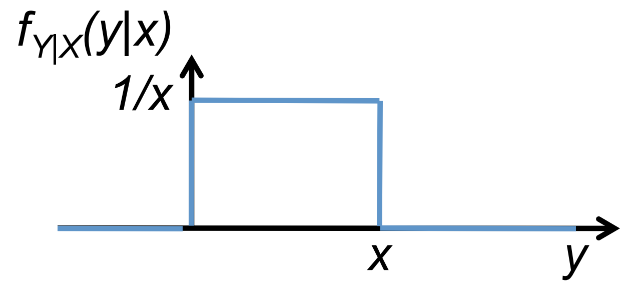
$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) = \begin{cases} \dfrac{1}{\ell x}, & 0 \le y \le x \le \ell \\ 0, & \text{otherwise} \end{cases}$$



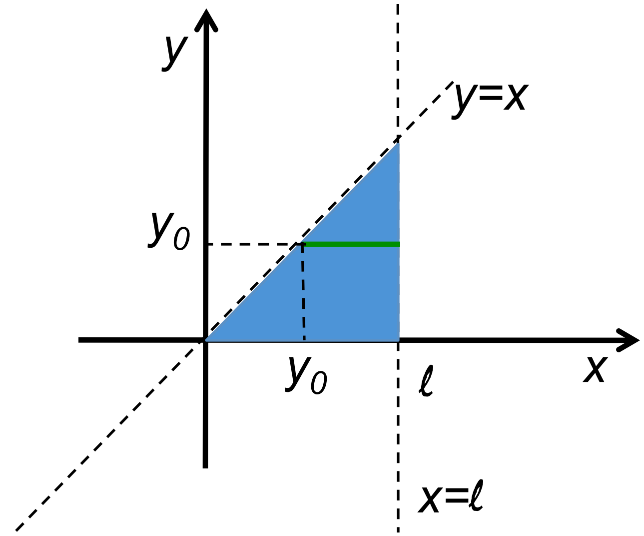
$$f_Y(y_0) = \int_{-\infty}^{\infty} f_{X,Y}(x,y_0)\,dx$$

$$= \begin{cases} ? & \text{if } 0 \le y_0 \le \ell \\ 0, & \text{if } y_0 < 0 \text{ or } y_0 > \ell \end{cases}$$

Problem 3.21: f_Y



$$f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) = \begin{cases} \dfrac{1}{\ell x}, & 0 \le y \le x \le \ell \\ 0, & \text{otherwise} \end{cases}$$



$$f_Y(y_0) = \int_{-\infty}^{\infty} f_{X,Y}(x, y_0)\, dx$$

$$= \begin{cases} \displaystyle\int_{y_0}^{\ell} \frac{1}{\ell x} dx & \text{if } 0 \le y_0 \le \ell \\ 0, & \text{if } y_0 < 0 \text{ or } y_0 > \ell \end{cases}$$

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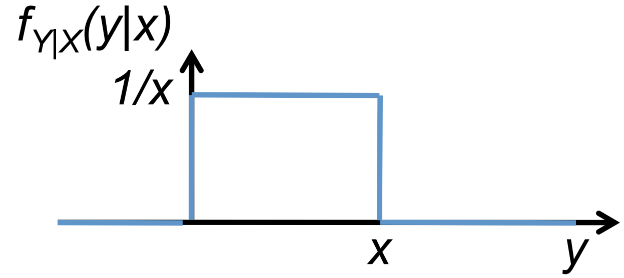

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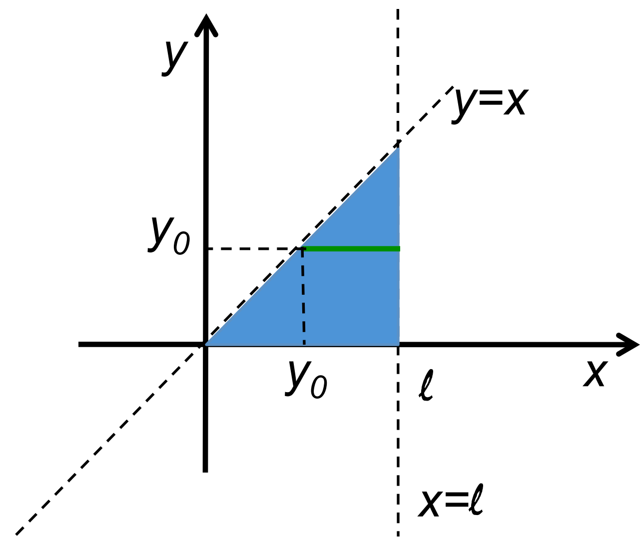


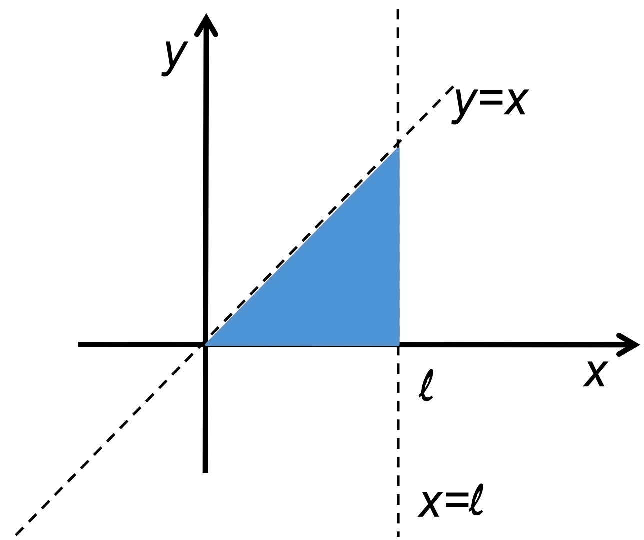
$$f_Y(y_0) = \int_{-\infty}^{\infty} f_{X,Y}(x,y_0)\,dx$$

$$= \begin{cases} \displaystyle\int_{y_0}^{\ell} \frac{1}{\ell x}\,dx = \frac{1}{\ell} \ln x \Big|_{y_0}^{\ell} = \frac{1}{\ell}\left(\ln \ell - \ln y_0\right) = \frac{1}{\ell} \ln \frac{\ell}{y_0} \quad \text{if } 0 \le y_0 \le \ell \\ 0, \quad \text{if } \ y_0 < 0 \text{ or } y_0 > \ell \end{cases}$$

Problem 3.21: E[*Y*], method 1

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{\ell x}, & 0\le y\le x\le \ell\\ 0, & \text{otherwise}\end{cases}$$

$$E[Y]=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{X,Y}(x,y)\,dx\,dy$$



Problem 3.21: E[*Y*], method 1

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{\ell x}, & 0 \le y \le x \le \ell \\ 0, & \text{otherwise} \end{cases}$$

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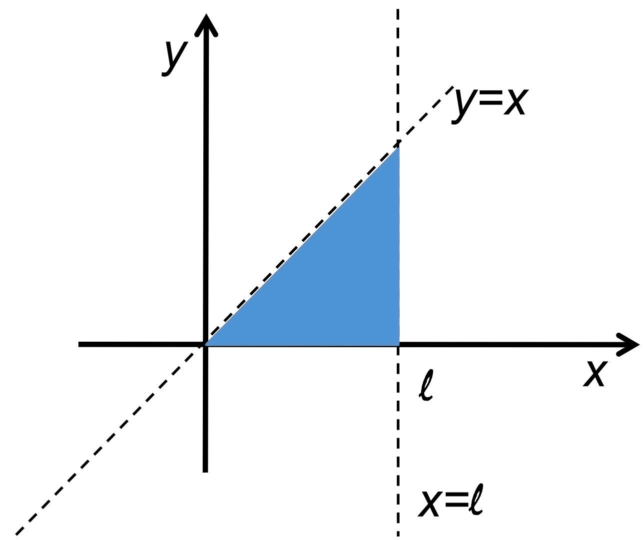

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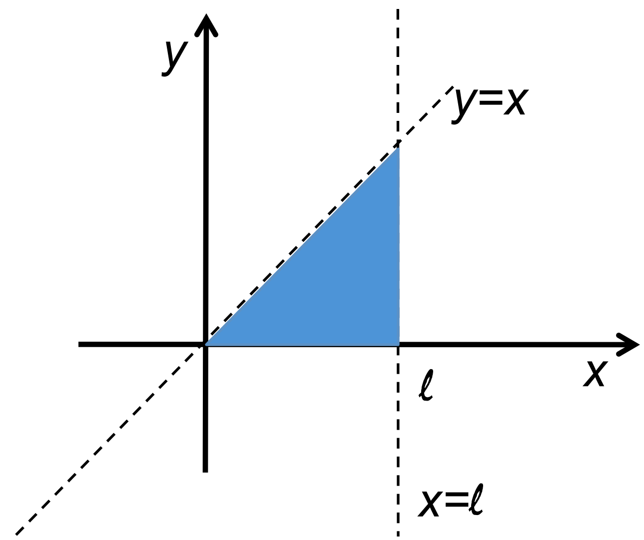
$$=\int_0^{\ell}\frac{1}{\ell x}\cdot\left.\frac{y^2}{2}\right|_0^x dx$$



Problem 3.21: E[Y], method 1

$$f_{X,Y}(x,y)=\begin{cases}\dfrac{1}{\ell x}, 0\le y\le x\le \ell\\ 0, \qquad \text{otherwise}\end{cases}$$

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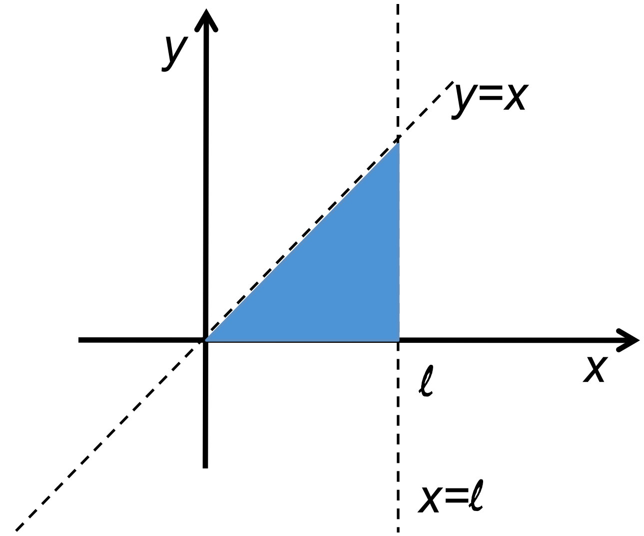
$$=\int_0^{\ell}\frac{1}{\ell x}\cdot\frac{y^2}{2}\bigg|_0^{x}dx=\int_0^{\ell}\frac{x}{2\ell}dx=\frac{x^2}{2\ell}\bigg|_0^{\ell}=\frac{\ell}{4}$$



Problem 3.21: E[*Y*], method 2

$$f_Y(y) = \begin{cases} \frac{1}{\ell}\ln\frac{\ell}{y} & \text{if } 0 \le y \le \ell \\ 0, & \text{otherwise} \end{cases}$$

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\,dy$$

Problem 3.21: E[Y], method 2

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Define $a = \ln\frac{\ell}{y}$, then $\frac{y}{\ell} = e^{-a}$, and $dy = -\ell e^{-a}da$

Problem 3.21: E[*Y*], method 2

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Define $a = \ln \frac{\ell}{y}$, then $\frac{y}{\ell} = e^{-a}$, and $dy = -\ell e^{-a}\,da$

$$E[Y] = -\ell \int_{\infty}^{0} a e^{-2a}\,da =$$

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$$E[Y] = -\ell\int_{\infty}^{0} a e^{-2a}\,da = \frac{\ell}{2}\int_0^{\infty} 2a e^{-2a}\,da$$

Problem 3.21: E[Y], method 2

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Problem 3.21: E[Y], method 3

- Proportion of 1st stick broken off is X/ℓ

Problem 3.21: E[Y], method 3

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- Proportion of 2nd stick broken off is Y/X

Problem 3.21: E[Y], method 3

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- The two are independent, and each is a continuous r.v., uniform between 0 and 1.

Problem 3.21: E[Y], method 3

- Proportion of 1st stick broken off is X/ℓ
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- $E[Y] = E[X(Y/X)]$

Problem 3.21: E[Y], method 3

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Problem 3.21: E[Y], method 3

- Proportion of 1st stick broken off is X/ℓ
- Proportion of 2nd stick broken off is Y/X
- The two are independent, and each is a continuous r.v., uniform between 0 and 1.
- $E[Y] = E[X(Y/X)] = E[X]\, E[Y/X] = (\ell/2)(1/2) = \ell/4$