2. Three engines

Problem Set due May 13, 2020 05:29 IST Past Due

Problem 2. Three engines

7 points possible (graded)

Suppose that we have three engines, which we turn on at time 0. Each engine will eventually fail, and we model each engine's lifetime as exponentially distributed with parameter λ. The lifetimes of different engines are independent. One of the engines will fail first, followed by the second, and followed by the last. Let T_1 be the time of the first failure, T_2 be the time of the second failure, and T_3 be the time of the third failure. For answers involving algebraic expressions, enter "lambda" for λ and use "exp()" for exponentials. Follow standard notation.

1. Determine the PDF of T_1.
   For $t > 0$,

   $f_{T_1}(t) =$

   3*lambda*e^(-3*lambda*t)

   Answer: 3*lambda*e^(-3*lambda*t)

   $3 \cdot \lambda \cdot e^{-3 \cdot \lambda \cdot t}$

2. Let $X = T_2 - T_1$. Determine the conditional PDF $f_{X|T_1}(x|t)$.
   For $x, t > 0$,

   $f_{X|T_1}(x \mid t) =$

   2*lambda*e^(-2*lambda*t)

   Answer: 2*lambda*e^(-2*lambda*x)

   $2 \cdot \lambda \cdot e^{-2 \cdot \lambda \cdot t}$

3. Is X independent of T_1?

   Yes they are independent

   Answer: Yes they are independent

4. Let $Y = T_3 - T_2$. Find the PDF of $f_{Y|T_2}(y \mid T_2)$.
   For $y, t > 0$,

$f_{Y|T_2}(y \mid t) =$

Answer: lambda*e^(-lambda*y)

5. Is Y independent of T_2?

Select an option

Answer: Yes they are independent

6. Find the PDF $f_{T_3}(t)$ for $t \geq 0$.
   For $t \geq 0$,

   $f_{T_3}(t) =$

   Answer: 3*lambda*e^(-lambda*t)*(1-e^(-lambda*t))^2

   Hint: Think of an interpretation of T_3 as a maximum of some exponential random variables.

7. Find $\mathbf{E}[T_3]$.

   $\mathbf{E}[T_3] =$

   Answer: 11/(6*lambda)

STANDARD NOTATION

Solution:

1. Let M_i be the lifetime of i^{th} engine. Notice that T_1, the time until the first failure of an engine, is the smallest of M_1, M_2, and M_3, namely, $\min\{M_1, M_2, M_3\}$. Each M_i has the same exponential CDF, $F_M(m) = 1 - e^{-\lambda m}$, for $m \geq 0$.

   We will first find the CDF of T_1, and then differentiate with respect to m to find the PDF $f_{T_1}(t)$, for $t \geq 0$.

$$\begin{aligned}
F_{T_1}(t) &= \mathbf{P}\left(\min\{M_1, M_2, M_3\} \leq t\right) \\
&= 1 - \mathbf{P}\left(\min\{M_1, M_2, M_3\} > t\right) \\
&= 1 - \mathbf{P}(M_1 > t)\,\mathbf{P}(M_2 > t)\,\mathbf{P}(M_3 > t) \\
&= 1 - (1 - F_{M_1}(t))(1 - F_{M_2}(t))(1 - F_{M_3}(t))
\end{aligned}$$

$$= 1 - e^{-3\lambda t}.$$

Differentiating $F_{T_1}(t)$ with respect to t yields,

$$f_{T_1}(t) = 3\lambda e^{-3\lambda t} \quad \text{for } t \geq 0.$$

Note that this is the PDF of an exponential random variable with parameter 3λ.

For an alternative approach, we consider 3 independent Poisson processes, each with rate λ. We can then interpret M_i as the first arrival time in process i. If we merge the three processes, the first arrival time in the merged process corresponds precisely to T_1. Since the merged process has rate 3λ, the random variable T_1, an interarrival time, is exponentially distributed with parameter 3λ.

2. Conditioned on the time of the first failure, the time remaining until the second failure is an exponential random variable with parameter 2λ by the memorylessness property. (This can be interpreted in terms of the merged Poisson process as in solution to part 1. Indeed, after one engine fails at time T_1, the remaining merged process is a Poisson process with rate 2λ, and $X = T_2 - T_1$ is precisely the first arrival time in this merged Poisson process.) Consequently, for $t \geq 0$,

$$f_{X|T_1}(x \mid t) = 2\lambda e^{-2\lambda t}.$$

3. By the memorylessness property mentioned in part 2, X and T_1 are independent.

4. Conditioned on the second failure, and using the memorylessness property of the lifetime of the remaining engine, the time remaining until the next failure is an exponential random variable with parameter λ.

5. Yes, by the fresh start property of the Poisson process, Y and T_2 are independent.

6. With M_1, M_2, and M_3 defined as above, we notice that T_3 is the maximum of M_1, M_2, M_3. We will again compute the CDF of T_3 first, and then differentiate it with respect to t to obtain its PDF.

$$\begin{aligned} F_{T_3}(t) &= \mathbf{P}\left(\max\{M_1, M_2, M_3\} \leq t\right) \\ &= \mathbf{P}\left(M_1 \leq t\right)\mathbf{P}\left(M_2 \leq t\right)\mathbf{P}\left(M_3 \leq t\right) \\ &= \left(1 - e^{-3\lambda t}\right)^3. \end{aligned}$$

Differentiating $F_{T_3}(t)$ with respect to t, and using the chain rule, we obtain

$$f_{T_3}(t) = 3\lambda\left(1 - e^{-\lambda t}\right)^2 e^{-\lambda t}.$$

7. Notice that, $T_3 = T_1 + X + Y$, where T_1 is an exponential random variable with parameter 3λ, X is an exponential random variable with parameter 2λ, and Y is an exponential random variable with parameter λ. Hence,

$$\mathbf{E}[T_3] = \mathbf{E}[T_1 + X + Y] = \frac{1}{3\lambda} + \frac{1}{2\lambda} + \frac{1}{\lambda} = \frac{11}{6\lambda},$$

using the linearity of expectations and the fact that an exponential distribution with parameter μ has mean $1/\mu$.

Submit You have used 0 of 4 attempts

Answers are displayed within the problem

Discussion

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- Technical Hint 8
- Unconditioned PDFs 11
  Just curious, 1. What would be the unconditioned PDFs of 2nd and 3rd failures? 2. Can the first, second and 3rd engine f...
- [Staff] Typo in solution 2? 1
  Community TA
- Q2 and Q4 1 new_ 5
- (Staff) Error in the solution of (6) 4
- Possible typo in answer? 4
  In the solution of Q6, the final form of the CDF seems to have an additional 3 in the exponent of the exponential term. ...
- Caution, Wolfram-Alpha users! 15
  Those who use Wolfram-Alpha for arithmetic checking are advised to double check using another tool. One of the outp...
- part 6: reasoning from probabiity of max(X,Y,Z) < t? 2
- [STAFF] There is an error in the shown working for part 6 1
  There is an error in the exponent of "e" in the cumulative probability. It reads ...e^(-3*lamda*t)... instead of ...e^(-lamda...
- Hint on question 6
  I don't get how to use the hint on question 6. The max of T1, T2 and T3 is always going to be T3. We can express T2 = T1...

? Why exp()?
I just find the directive odd since plain old e should work per usual?
1 new_ 3

how to handle the conditional cases (2 and 4)
I'm having trouble understanding what the conditional PMF will be in these cases and how we should reach it. Did get s...
2

? Part 6
I’m using the CDF of Max{T1,T2,T3} so I’m using the Product of 3 CDFs and then taking the derivative to get the PDF of T...
2

For those who still get stuck with part 6
Revisit this solved problem [Mean and variance of the exponential][1] (Find the Mean and variance of the exponential p
3