3. The sample mean

Problem Set due May 1, 2020 05:29 IST Completed

Problem 3. The sample mean

5/5 points (graded)

Let X be a continuous random variable. We know that it takes values between 0 and 6, but we do not know its distribution or its mean and variance, although we know that its variance is at most 4. We are interested in estimating the mean of X, which we denote by h. To estimate h, we take n i.i.d. samples $X_1, \ldots, X_n$, which all have the same distribution as X, and compute the sample mean

$$H = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

1. Express your answers for this part in terms of h and n using standard notation.

   $\mathbf{E}[H] =$

   h

   ✔ **Answer:** h

   h

   Given the available information, the smallest upper bound for $\text{Var}(H)$ that we can assert/guarantee is:

   $\text{Var}(H) \leq$

   4/n

   ✔ **Answer:** 4/n

   $\frac{4}{n}$

2. Calculate the smallest possible value of n such that the standard deviation of H is guaranteed to be at most 0.01.

   This minimum value of n is: 40000 ✔ **Answer:** 40000

3. We would like to be at least 96% sure that our estimate is within 0.02 of the true mean h. Using the Chebyshev inequality, calculate the minimum value of n that will achieve this.

This minimum value of n is: 250000 ✔ **Answer:** 250000

4. Suppose now that X is uniformly distributed on $[h-3, h+3]$, for some unknown h. Using the Central Limit Theorem, identify the most appropriate expression for a 95% confidence interval for h. You may want to refer to the normal table.

Normal Table

The entries in this table provide the numerical values of $\Phi(z) = \mathbf{P}(Z \leq z)$, where Z is a standard normal random variable, for z between 0 and 3.49. For example, to find $\Phi(1.71)$, we look at the row corresponding to 1.7 and the column corresponding to 0.01, so that $\Phi(1.71) = .9564$. When z is negative, the value of $\Phi(z)$ can be found using the formula $\Phi(z) = 1 - \Phi(-z)$.

z	**0.00**	**0.01**	**0.02**	**0.03**	**0.04**	**0.05**	**0.06**	**0.07**	**0.08**	**0.09**
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767

2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

* For $z \geq 3.50$, the probability is greater than or equal to $.9998$.

Hide

- ○ $\left[H - \frac{\sqrt{1.96 \cdot 3}}{\sqrt{n}}, H + \frac{\sqrt{1.96 \cdot 3}}{\sqrt{n}}\right]$
- ○ $\left[H - \frac{1.96}{\sqrt{3n}}, H + \frac{1.96}{\sqrt{3n}}\right]$
- ◉ $\left[H - \frac{1.96 \cdot \sqrt{3}}{\sqrt{n}}, H + \frac{1.96 \cdot \sqrt{3}}{\sqrt{n}}\right]$
- ○ $\left[H - \frac{1.96 \cdot 3}{\sqrt{n}}, H + \frac{1.96 \cdot 3}{\sqrt{n}}\right]$

✔

STANDARD NOTATION

Solution:

1. We have

$$\begin{aligned} H &= \frac{X_1 + X_2 + \cdots + X_n}{n}, \\ \mathbf{E}[H] &= \frac{\mathbf{E}[X_1 + \cdots + X_n]}{n} = \frac{n \cdot \mathbf{E}[X]}{n} = h, \\ \sigma_H^2 &= \mathsf{Var}(H) = \frac{n \cdot \mathsf{Var}(X)}{n^2} \leq \frac{4}{n}. \end{aligned}$$

2. From the previous part, we know that $\sigma_H \leq 2/\sqrt{n}$. In order to guarantee that it is at most 0.01, we solve, $2/\sqrt{n} \leq 0.01$ for n to obtain $n \geq 40000$.

3. We apply the Chebyshev inequality to H, with $\mathbf{E}[H]$ and $\mathsf{Var}(H)$ from part (1):

$$\mathbf{P}(|H-h| \geq 0.02) \leq \frac{\sigma_H^2}{0.02^2} \quad \text{or} \quad \mathbf{P}(|H-h| \leq 0.02) \geq 1 - \frac{\sigma_H^2}{0.02^2}.$$

   Substituting in our upper bound on σ_H^2, we obtain

$$1 - \frac{\sigma_H^2}{0.02^2} \geq 1 - \frac{2^2}{n \cdot 0.02^2}.$$

   Hence, to guarantee that our estimate is within 0.02 of the true mean h with probability of at least 96%, it suffices to have,

$$1 - \frac{2^2}{n \cdot 0.02^2} \geq 0.96.$$

   Solving this for n, we have that n must satisfy,

$$n \geq 250000.$$

4. Since X is uniform in the interval $[h-3, h+3]$, we know that the expected value of X is h and its variance, denoted by σ_H^2, is 3. Using the standard normal table, and the Central Limit Theorem, we know that for sufficiently large n,

$$\mathbf{P}\left(\left|\frac{H-h}{\sigma_H/\sqrt{n}}\right| \leq 1.96\right) \approx 0.95.$$

Hence,

$$\mathbf{P}\left(H-\frac{1.96\cdot\sqrt{3}}{\sqrt{n}}\leq h\leq H+\frac{1.96\cdot\sqrt{3}}{\sqrt{n}}\right)\approx 0.95.$$

Therefore, the 95% confidence interval for h is $\left[H-\frac{1.96\cdot\sqrt{3}}{\sqrt{n}}, H+\frac{1.96\cdot\sqrt{3}}{\sqrt{n}}\right]$.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Discussion

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Topic: Unit 8: Limit theorems and classical statistics:Problem Set 8 / 3. The sample mean

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Confession
I am not gonna lie, I am at a 46% in the class after the midterm was a big debacle for me. It was big blow, but I am using the ti... 16

To find the confidence interval for h, do you find it for H and then transform the interval into one for h?
Is there a better way to solve #4? 6

The denominator in Question 4 1

Question 4 - Where is root of n? 2

[STAFF] The submit button is disabled.
I cannot submit these answers because the submit button is disabled but the deadline is until November 30th. Please help. 5

q3: precision question
What is your presicion for Q3? ***************** These are WRONG EXAMPLE VALUES HERE - Do not submit them ********... 3

Chebyshev, but how?
I understand what #3 is asking, but are we supposed to treat this like a confidence interval as well? And where does n come in? 10

Question 3 - minimum value of n
For Question 3, after plugging in Var (H) from Q1, and solving Chebyshev's inequality, I get an inequality with n <= (less than or ... 2

How to find Var(H) ?
I believe Var (H) should be = [Var(X)] / n But what is the Var (X) ? I'm wondering how to use the information that X is a continuo... 2

Hint
Look solved problem 7. It will be helpful. 2

? Hint for Var(H)

I'm stuck with this one. Any hint on how to proceed with "Given the available information, the smallest upper bound for Var(H)...

3