1. Determining the type of a lightbulb

Problem Set due Apr 15, 2020 05:29 IST Completed

Problem 1. Determining the type of a lightbulb

3/3 points (graded)

The lifetime of a type-A bulb is exponentially distributed with parameter λ. The lifetime of a type-B bulb is exponentially distributed with parameter μ, where $\mu > \lambda > 0$. You have a box full of lightbulbs of the same type, and you would like to know whether they are of type A or B. Assume an **a priori** probability of $1/4$ that the box contains type-B lightbulbs.

1. Assume that $\mu \geq 3\lambda$. You observe the value t_1 of the lifetime, T_1, of a lightbulb. A MAP decision rule decides that the lightbulb is of type A if and only if $t_1 \geq \alpha$. Find α, and express your answer in terms of μ and λ. Use 'mu" and 'lambda" and 'ln" to denote μ, λ, and the natural logarithm function, respectively. For example, $\ln \frac{2\mu}{\lambda}$ should be entered as 'ln((2*mu)/lambda)".

   $\alpha =$

   (1/(mu-lambda))*ln(mu/(3*lambda)) ✔

   Answer: (1/(mu-lambda))*ln(mu/(3*lambda))

   $\left(\frac{1}{\mu-\lambda}\right) \cdot \ln\left(\frac{\mu}{3\cdot\lambda}\right)$

2. Assume again that $\mu \geq 3\lambda$. What is the probability of error of the MAP decision rule?

   - (●) $\frac{1}{4}e^{-\mu\alpha} + \frac{3}{4}(1 - e^{-\lambda\alpha})$
   - ( ) $\frac{3}{4}e^{-\mu\alpha} + \frac{1}{4}(1 - e^{-\lambda\alpha})$
   - ( ) $\frac{1}{4}(1 - e^{-\mu\alpha}) + \frac{3}{4}e^{-\lambda\alpha}$
   - ( ) $\frac{3}{4}(1 - e^{-\mu\alpha}) + \frac{1}{4}e^{-\lambda\alpha}$

   ✔

3. Assume that $\lambda = 3$ and $\mu = 4$. Find the LMS estimate of T_2, the lifetime of another lightbulb from the same box, based on observing $T_1 = 2$. Assume that conditioned on the bulb type, bulb lifetimes are indep (For this part, you will need a calculator. Provide an answer with an accuracy of three decimal places.)

LMS estimate of T_2:

0.328 ✔ **Answer:** 0.328

Solution:

1. With some abuse of notation, we let A and B be the events that the box contains lightbulbs of type A and type B, respectively. A MAP rule decides in favor of type A if and only if

$$\begin{aligned} \mathbf{P}(A \mid T_1 = t_1) &\geq \mathbf{P}(B \mid T_1 = t_1) \\ \frac{f_{T_1|A}(t_1)\,\mathbf{P}(A)}{f_{T_1}(t_1)} &\geq \frac{f_{T_1|B}(t_1)\,\mathbf{P}(B)}{f_{T_1}(t_1)}. \end{aligned}$$

Equivalently, we decide that the bulb is of type A if and only if

$$\begin{aligned} f_{T_1|A}(t_1)\,\mathbf{P}(A) &\geq f_{T_1|B}(t_1)\,\mathbf{P}(B), \\ \lambda e^{-\lambda t_1}\frac{3}{4} &\geq \mu e^{-\mu t_1}\frac{1}{4}, \\ \frac{\lambda}{\mu}e^{(\mu-\lambda)t_1} &\geq \frac{1}{3}, \\ (\mu-\lambda)\,t_1 \geq \ln\left(\frac{\mu}{3\lambda}\right). \end{aligned}$$

Thus, since $\mu - \lambda > 0$, a MAP rule decides in favor of type A if and only if $t_1 \geq \ln\left(\frac{\mu}{3\lambda}\right) \cdot \frac{1}{\mu-\lambda}$. Hence, we deduce that,

$$\alpha = \frac{1}{\mu-\lambda}\ln\left(\frac{\mu}{3\lambda}\right).$$

2. Let events A and B be defined as in part (1). Let $\hat{A}$ be the event that the MAP rule decides in favor of type A, and let $\hat{B}$ be the event that the MAP rule decides in favor of type B. An error occurs whenever the decision is different from the actual type of the bulb. Thus,

$$\begin{aligned} \mathbf{P}(\text{error}) &= \mathbf{P}(\hat{A} \cap B) + \mathbf{P}(A \cap \hat{B}) \\ &= \mathbf{P}(\hat{A} \mid B) \cdot \mathbf{P}(B) + \mathbf{P}(\hat{B} \mid A) \cdot \mathbf{P}(A) \\ &= \mathbf{P}(T_1 \geq \alpha \mid B) \cdot \frac{1}{4} + \mathbf{P}(T_1 < \alpha \mid A) \cdot \frac{3}{4} \\ &= e^{-\mu\alpha} \cdot \frac{1}{4} + (1 - e^{-\lambda\alpha}) \cdot \frac{3}{4}. \end{aligned}$$

3. The LMS estimate of T_2 based on observing $T_1 = t_1$ is

$$\begin{aligned} \mathbf{E}[T_2 \mid T_1 = t_1] &= \mathbf{E}[T_2 \mid T_1 = t_1, A] \cdot \mathbf{P}(A \mid T_1 = t_1) + \mathbf{E}[T_2 \mid T_1 = t_1, B] \cdot \mathbf{P}(B \mid T_1 = t_1) \\ &= \mathbf{E}[T_2 \mid A] \cdot \mathbf{P}(A \mid T_1 = t_1) + \mathbf{E}[T_2 \mid B] \cdot \mathbf{P}(B \mid T_1 = t_1) \end{aligned}$$

$$= \frac{1}{\lambda} \cdot \left(\frac{f_{T_1|A}(t_1) \cdot \mathbf{P}(A)}{f_{T_1}(t_1)} \right) + \frac{1}{\mu} \cdot \left(\frac{f_{T_1|B}(t_1) \cdot \mathbf{P}(B)}{f_{T_1}(t_1)} \right)$$

$$= \frac{\frac{1}{\lambda}\frac{3}{4}\lambda e^{-\lambda t_1} + \frac{1}{\mu}\frac{1}{4}\mu e^{-\mu t_1}}{\frac{3}{4}\lambda e^{-\lambda t_1} + \frac{1}{4}\mu e^{-\mu t_1}}.$$

Inserting the values $\lambda = 3$,$\mu = 4$, and $t_1 = 2$, we obtain $\mathbf{E}[T_2 \mid T_1 = 2] = 0.328$.

Submit You have used 3 of 3 attempts

Answers are displayed within the problem

Discussion

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- Amazing link from the problem 7a with salmon/bass MAP — 2
  http://www.cs.haifa.ac.il/~rita/ml_course/lectures/Bayesian_Decision.pdf
- mu > 3 lambda ? — 2 new_ 7
- Problem Set 7b is Closed before the deadline - Why? — 3
  This problem set seems to be closed prematurely - none of the submit buttons are available but this is 90m before the deadline (supp...
- 2. more detailed process of computation in the answer is needed. — 4
  I got the answer right for part 2, but I am having difficulty understanding the answer. $\mathbf{P}(T1 \geq \alpha|B)\cdot(1/4)+\mathbf{P}(T1<\alpha|A)\cdot(3/4)$ = e^−$\mu\alpha\cdot$(1/4)+(1−...
- Why is this statement added ? "Assume that $\mu \geq 3\lambda$." — 2
  Not sure why this is there in the question "Assume that $\mu \geq 3\lambda$."
- Hint for Q3 — 28
- Lost on part 3 — 1
  Got the first two parts, but part 3 escapes me. I get lost between least mean square and minimum square error. If I plug t = 2 and the ...
- logic for part 2 — 3
  Somehow I managed to get part 1. So I see where alpha is and have some sense of how the posteriors intersect each other at the poi...
- Meaning of LMS of T2 — 3
  Can somebody share their intuition on the value of the LMS of T2? It seems very low to me, specially compared to T1
- Stuck on part 2 — 4
  I seem to be stuck on part 2. I've attempted twice and it seemed easy but both times I was wrong :(
- What am I missing? (Part 1) — 8
- Hints Q1 — 2
- [Staff] Clarification and question on exercise 3

Struggling to understand intuition behind Q2 P(error)

Apologies for the simple question, but it is where I'm at. How do you use CDF's in P(error) calculations (generally speaking)?

2