5. Covariance of the multinomial

Problem Set due Apr 1, 2020 05:29 IST *Completed*

Problem 5. Covariance of the multinomial

3/3 points (graded)

Consider n independent rolls of a k-sided fair die with $k \geq 2$: the sides of the die are labelled $1, 2, \ldots, k$, and each side has probability $1/k$. Let the random variable X_i denote the number of rolls that result in side i. Thus, the random vector $(X_1, \ldots, X_k)$ has a multinomial distribution.

1. Which of the following statements is correct? Try to answer without doing any calculations.

- X_1 and X_2 are uncorrelated.
- X_1 and X_2 are positively correlated.
- (selected) X_1 and X_2 are negatively correlated.

✔

2. Find the covariance, $\textbf{cov}\,(X_1, X_2)$, of X_1 and X_2. Express your answer as a function of n and k using standard notation. *Hint:* Use indicator variables to encode the result of each roll.

$\textbf{cov}\,(X_1, X_2) =$ -n/(k^2) ✔ **Answer:** -n/(k^2)

$-\frac{n}{k^2}$

3. Suppose now that the die is biased, with a probability $p_i \neq 0$ that the result of any given die roll is i, for $i = 1, 2, \ldots, k$. We still consider n independent rolls of this biased die and define X_i to be the number of rolls that result in side i.

Generalize your answer to part 2: Find $\mathrm{cov}\left(X_1, X_2\right)$ for this case of a biased die. Express your answer as a function of n, k, p_1, p_2 using standard notation. Write p_1 and p_2 as p_1 and p_2, respectively, for example, $2p_1p_2$ must be entered as 2*p_1*p_2.

$\mathrm{cov}\left(X_1, X_2\right) =$ -n*p_1*p_2 ✔ **Answer:** -n*(p_1)*(p_2)

$-n \cdot p_1 \cdot p_2$

STANDARD NOTATION

Solution:

1. The random variables X_1 and X_2 are negatively correlated. There is a fixed number, n, of rolls of the die. Intuitively, a large number of rolls that result in a 1 uses up many of the n total rolls, which leaves fewer remaining rolls that could result in a 2.

2. Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the tth roll resulted in a 1 (respectively, 2). Note that $X_1 = \sum_{t=1}^{n} A_t$ and $X_2 = \sum_{t=1}^{n} B_t$, and so

$$\mathbf{E}\left[X_1\right] = \mathbf{E}\left[X_2\right] = \mathbf{E}\left[\sum_{t=1}^{n} A_t\right] = n\mathbf{E}\left[A_1\right] = \frac{n}{k}.$$

Since a single roll of the die cannot result in both a 1 and a 2, at least one of A_t and B_t must equal 0. Thus, $\mathbf{E}\left[A_t B_t\right] = 0$. Furthermore, since different rolls are independent, A_t and B_s are independent when $t \neq s$. Therefore,

$$\mathbf{E}\left[A_t B_s\right] = \mathbf{E}\left[A_t\right]\mathbf{E}\left[B_s\right] = \frac{1}{k} \cdot \frac{1}{k} = \frac{1}{k^2} \qquad \text{for} \quad t \neq s,$$

and so

$$\mathbf{E}\left[X_1 X_2\right] \;=\; \mathbf{E}\left[\left(A_1 + \cdots + A_n\right)\left(B_1 + \cdots + B_n\right)\right]$$

$$
\begin{aligned}
&= \mathbf{E}\left[\sum_{t=s} A_t B_t + \sum_{t \neq s} A_t B_s\right] \\
&= n \cdot 0 + n(n-1) \cdot \mathbf{E}[A_1 B_2] \\
&= n(n-1) \cdot \frac{1}{k^2}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\operatorname{cov}(X_1, X_2) &= \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1]\,\mathbf{E}[X_2] \\
&= n(n-1) \cdot \frac{1}{k^2} - \frac{n}{k} \cdot \frac{n}{k} \\
&= -\frac{n}{k^2}.
\end{aligned}
$$

The covariance of X_1 and X_2 is negative as expected.

3. We follow the same reasoning as in part 2. Let A_t (respectively, B_t) be a Bernoulli random variable that is equal to 1 if and only if the tth roll resulted in a 1 (respectively, 2). As in part 2, a single roll of the die cannot result in both a 1 and a 2, so $\mathbf{E}[A_t B_t] = 0$. Different rolls of the die are independent, and so $\mathbf{E}[A_t B_s] = \mathbf{E}[A_t]\,\mathbf{E}[B_s] = p_1 \cdot p_2$, for $t \neq s$. Thus,

$$
\begin{aligned}
\mathbf{E}[X_1 X_2] &= \mathbf{E}[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)] \\
&= \mathbf{E}\left[\sum_{t=s} A_t B_t + \sum_{t \neq s} A_t B_s\right] \\
&= n \cdot 0 + n(n-1) \cdot \mathbf{E}[A_1 B_2] \\
&= n(n-1)\, p_1 p_2.
\end{aligned}
$$

Note that $X_1 = \sum_{t=1}^{n} A_t$ and $X_2 = \sum_{t=1}^{n} B_t$, and so $\mathbf{E}[X_1] = \mathbf{E}\left[\sum_{t=1}^{n} A_t\right] = n\mathbf{E}[A_1] = np_1$. Similarly, $\mathbf{E}[X_2] = np_2$.

Therefore,

$$
\operatorname{cov}(X_1, X_2) = \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1]\, E[X_2]
$$

$$
\begin{aligned}
&= n(n-1)\, p_1 p_2 - (np_1)(np_2) \\
&= -np_1 p_2.
\end{aligned}
$$

The covariance of X_1 and X_2 is again negative, even when the die is not fair.

Submit You have used 1 of 3 attempts

Answers are displayed within the problem

Discussion

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Topic: Unit 6: Further topics on random variables:Problem Set 6 / 5. Covariance of the multinomial

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Hint for 2:
I found the hat problem - lecture 7, No. 14 very useful in understanding how to solve this problem. I am ...
2 new_

[STAFF]About Deadlines
Dear Sir/Madam, Greetings.Due to immense reshuffle in last 10 days here in India due to COVID-19,, for l...
3

? Confused about cov(X1,X2) = E[X1X2]−E[X1]E[X2] property
May I know where is this property cov(X1,X2)= E[X1X2]−E[X1]E[X2] shown in the lecture videos please?
2

? Can we please have more problems and/or examples using indicator variables?
After completing this problem, indicator variables seem incredibly useful. Would very much like to practi...
2

Hint: read carefully on the model & consider Var(X+Y)
As above.
3

Perfect Strategy with Indicator random variable.
I found is very useful strategy here with indicator random variable https://math.stackexchange.com/que...
3
Following

Hint: wikipedia on multinomial distribution
check if hint is needed
1

Defying and difficult but stil,...
a BEAUTIFUL problem!!
1

? How to use indicator variables?

I'm stuck on part 2 ... I'm using Cov (A,B) = E[X1*X2] - E[X1]*E[X2], and I think it's the E[X1*X2] term we sh...	6
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