5. Estimating the parameter of a uniform r.v.

Problem Set due Apr 15, 2020 05:29 IST *Completed*

Problem 5. Estimating the parameter of a uniform r.v.

4/5 points (graded)

The random variable X is uniformly distributed over the interval $[\theta, 2\theta]$. The parameter θ is unknown and is modeled as the value of a continuous random variable Θ, uniformly distributed between zero and one.

1. Given an observation x of X, find the posterior distribution of Θ. Express your answers below in terms of θ and x. Use 'theta" to denote θ and 'ln" to denote the natural logarithm function. For example, $\ln(\theta)$ should be entered as 'ln(theta)'.

   For $0 \leq x \leq 1$ and $x/2 \leq \theta \leq x$:

   $f_{\Theta|X}(\theta \mid x) =$ 1/(theta*ln(2)) ✔ **Answer:** 1/(theta*ln(2))

   $\frac{1}{\theta \cdot \ln(2)}$

2. Find the MAP estimate of Θ based on the observation $X = x$ and assuming that $0 \leq x \leq 1$. Express your answer in terms of x.

   For $0 \leq x \leq 1$:

   $\hat{\theta}_{\text{MAP}}(x) =$ x/2 ✔ **Answer:** x/2

   $\frac{x}{2}$

3. Find the LMS estimate of Θ based on the observation $X = x$ and assuming that $0 \leq x \leq 1$. Express your answer in terms of x.

   For $0 \leq x \leq 1$:

$\hat{\theta}_{\text{LMS}}(x) =$ x/(2*ln(2)) ✔ **Answer:** x/(2*ln(2))

$\frac{x}{2 \cdot \ln(2)}$

4. Find the linear LMS estimate $\hat{\theta}_{\text{LLMS}}$ of Θ based on the observation $X = x$. Specifically, $\hat{\theta}_{\text{LLMS}}$ is of the form $c_1 + c_2 x$. Find c_1 and c_2.

$c_1 =$ 0.065 ✔ **Answer:** 0.06452

$c_2 =$ 0.058 ✖ **Answer:** 0.58065

Solution:

1. The prior PDF of Θ is

$$f_\Theta(\theta) = \begin{cases} 1, & \text{if } 0 \le \theta \le 1, \\ 0, & \text{otherwise,} \end{cases}$$

and the conditional PDF of the observation X is

$$f_{X|\Theta}(x \mid \theta) = \begin{cases} 1/\theta, & \text{if } \theta \le x \le 2\theta, \\ 0, & \text{otherwise.} \end{cases}$$

Using Bayes' rule, we find that for any $x \in [0, 1]$ and for $\theta \in [x/2, x]$, the posterior PDF is

$$\begin{aligned} f_{\Theta|X}(\theta \mid x) &= \frac{f_\Theta(\theta) f_{X|\Theta}(x \mid \theta)}{\int_{x/2}^{x} f_\Theta(\tilde{\theta}) f_{X|\Theta}(x \mid \tilde{\theta}) \, d\tilde{\theta}} \\ &= \frac{1/\theta}{\int_{x/2}^{x} \frac{1}{\tilde{\theta}} d\tilde{\theta}} \\ &= \frac{1}{\theta \cdot (\ln(x) - \ln(x/2))} \end{aligned}$$

$$= \frac{1}{\theta \cdot \ln(2)}.$$

2. In part (1), we saw that for $x \in [0,1]$ and $x/2 \leq \theta \leq x$, the posterior PDF is

$$f_{\Theta|X}(\theta \mid x) = \frac{1}{\theta \cdot \ln(2)},$$

which is decreasing in θ over the range $[x/2, x]$ of possible values of Θ. Thus, the MAP estimate for this case is equal to $x/2$.

3. The LMS estimate is the conditional expectation estimate. For $x \in [0,1]$,

$$\mathbf{E}[\Theta \mid X = x] = \int_{x/2}^{x} \theta \frac{1}{\theta \cdot \ln(2)} d\theta = \frac{x}{2 \cdot \ln(2)}.$$

4. The LLMS estimate is of the form

$$\hat{\theta}_{LLMS}(x) = \mathbf{E}[\Theta] + \frac{\operatorname{cov}(\Theta, X)}{\operatorname{Var}(X)}(x - \mathbf{E}[X]).$$

Here,

$$\begin{aligned}
\mathbf{E}[\Theta] &= 1/2, \\
\mathbf{E}[X] &= \mathbf{E}[\mathbf{E}[X \mid \Theta]] \\
&= \mathbf{E}\left[\frac{3}{2}\Theta\right] \\
&= \frac{3}{4}, \\
\mathbf{E}[X^2] &= \mathbf{E}[\mathbf{E}[X^2 \mid \Theta]] \\
&= \mathbf{E}\left[\frac{7}{3}\Theta^2\right]
\end{aligned}$$

$$= \frac{7}{9}.$$

Hence,

$$\begin{aligned}
\mathsf{Var}\,(X) &= \mathbf{E}\,[X^2] - (\mathbf{E}\,[X])^2 \\
&= \frac{31}{144}, \\
\mathbf{E}\,[\Theta X] &= \mathbf{E}\,[\mathbf{E}\,[X\Theta \mid \Theta]] \\
&= \mathbf{E}\left[\frac{3}{2}\Theta^2\right] \\
&= \frac{1}{2}, \\
\mathrm{cov}\,(\Theta, X) &= \mathbf{E}\,[\Theta X] - \mathbf{E}\,[\Theta]\,\mathbf{E}\,[X] \\
&= \frac{1}{2} - \frac{1}{2}\cdot\frac{3}{4} \\
&= \frac{1}{8}.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\hat{\Theta}_{LLMS} &= \mathbf{E}\,[\Theta] + \frac{\mathrm{cov}\,(\Theta, X)}{\mathsf{Var}\,(X)}(x - \mathbf{E}\,[X]) \\
&= \frac{1}{2} + \frac{1/8}{31/144}\left(x - \frac{3}{4}\right) \\
&= \frac{2}{31} + \frac{18}{31}x.
\end{aligned}$$

Submit You have used 4 of 4 attempts

Answers are displayed within the problem

Discussion

Hide Discussion

Topic: Unit 7: Bayesian inference:Problem Set 7b / 5. Estimating the parameter of a uniform r.v.

Show all posts by recent activity

- Problem Set 7b is Closed before the deadline - Why? — 8
  Problem set 7b is now showing as closed, none of the submit buttons are active. This is 1.5 hours bef...
- Am considering quitting the program - questions on other modules, capstone exams — 55
  As of right now, I am going back and forth on whether I should just quit the program. For me, educati...
- Some questions on the solutions — 3
  For (4) LLMS estimate the solution provides several steps. a. It calculates E[X] = E[E[X|Theta]] = E[3/2 T...
- STAFF - Regarding Extension. — 2
  If you are considering an extension, we lost power for 4 hours yesterday and internet for about 6. I co...
- [STAFF] Urgent Request: HW extension — 5
  Dear Staff, My friend can't access her account since yesterday, could you please extend the HW deadl...
- Why is the LLMS different from the LMS? — 10
  Hi, If I am not mistaken, the LMS for Theta is linear in X. Why is it not the LLMS then? Many thanks,
- Expectation of E[X^2|Theta] = 7/3Theta^2 — 3
  Can someone explain in detail how we get 7/3 here?
- [Staff] Getting problem set Markov chains (optional assignment) graded in lieu of problem set 7A — 1
  Dear Staff, I am a healthcare professional in Delhi,India.I could not do my assignment problem set 7A ...
- This was not an easy homework. — 2
  I was really looking forward to Bayesian Inference and the concepts seemed easy in general, but ever...
- Cant submit my answers — 5
  Theta (Θ) is included in my answer as mentioned in the Question , but when i submit it gave me ""Inv...
- Exam 2 - Is it till Unit 7a or 7b also included? — 1 new — 5
  Can you please let me know if Exam 2 is till Unit 7a or Unit 7b is also included. Thank you.
- [Staff] Why the the problem set was closed 1.5 hours before the announced end date and time?? — 1
  Now is 6:32 EDT and 10:32 UTC and submit button is already disabled.
- !Hint! — 1