4. Joint PMF

Problem Set due Feb 28, 2020 05:29 IST *Completed*

Problem 4. Joint PMF

5/5 points (graded)

The joint PMF, $p_{X,Y}(x,y)$, of the random variables X and Y is given by the following table:

$y=1$	$4c$	0	$2c$	$8c$
$y=0$	$3c$	$2c$	0	$2c$
$y=-1$	$2c$	0	c	$4c$
	$x=-2$	$x=-1$	$x=0$	$x=1$

1. Find the value of the constant c.

   $c =$ 1/28 ✔ **Answer:** 0.03571

2. Find $p_X(1)$.

   $p_X(1) =$ 1/2 ✔ **Answer:** 0.5

3. Consider the random variable $Z = X^2 Y^3$. Find $\mathbf{E}[Z \mid Y=-1]$.

   $\mathbf{E}[Z \mid Y=-1] =$ -12/7 ✔ **Answer:** -1.71429

4. Conditioned on the event that $Y \neq 0$, are X and Y independent?

   Yes ✔ **Answer:** Yes

5. Find the conditional variance of Y given that $X=0$.

   $\text{Var}(Y \mid X=0) =$ 8/9 ✔ **Answer:** 0.88889

Solution:

1. We find c by using the fact that the probability of the entire sample space must equal 1.

$$\begin{aligned} 1 &= \sum_{x=-2}^{1} \sum_{y=-1}^{1} p_{X,Y}(x,y) \\ &= 2c+3c+4c+2c+c+2c+4c+2c+8c \\ &= 28c. \end{aligned}$$

   Therefore, $c = \frac{1}{28}$.

2. $$p_X(1) = \sum_{y=-1}^{1} p_{X,Y}(1,y) = 4c+2c+8c = 14c = \frac{1}{2}.$$

3. $$\begin{aligned} \mathbf{E}[Z \mid Y=-1] &= \mathbf{E}[X^2Y^3 \mid Y=-1] \\ &= \mathbf{E}[X^2(-1)^3 \mid Y=-1] \\ &= -\mathbf{E}[X^2 \mid Y=-1] \end{aligned}$$

   In order to calculate this conditional expectation, we need the conditional PMF of X given $Y=-1$:

$$p_{X|Y}(x \mid -1) = \frac{p_{X,Y}(x,-1)}{p_Y(-1)} = \begin{cases} \frac{2c}{7c} = \frac{2}{7}, & \text{if } x=-2, \\ \frac{c}{7c} = \frac{1}{7}, & \text{if } x=0, \\ \frac{4c}{7c} = \frac{4}{7}, & \text{if } x=1, \\ 0, & \text{otherwise.} \end{cases}$$

   Therefore,

$$\begin{aligned} \mathbf{E}[Z \mid Y=-1] &= -\sum_{x=-2}^{1} x^2 p_{X|Y}(x \mid -1) \\ &= -\left((-2)^2 \cdot \frac{2}{7} + 1^2 \cdot \frac{4}{7}\right) \\ &= -\frac{12}{7}. \end{aligned}$$

4. Yes. Given $Y \neq 0$, the conditional distribution of Y given $X=x$ is the same for all $x \in \{-2,-1,0,1\}$:
   $\mathbf{P}(Y=y \mid X=x, Y \neq 0) = \mathbf{P}(Y=y \mid Y \neq 0)$, for all $x \in \{-2,-1,0,1\}$.

For example,

$$\begin{aligned} \mathbf{P}(Y=1 \mid X=-2, Y\neq 0) &= \mathbf{P}(Y=1 \mid X=0, Y\neq 0) \\ &= \mathbf{P}(Y=1 \mid X=1, Y\neq 0) \\ &= \mathbf{P}(Y=1 \mid Y\neq 0) = \frac{2}{3}. \end{aligned}$$

5. We first find the conditional PMF of Y given $X=0$:

$$p_{Y|X}(y \mid 0) = \frac{p_{X,Y}(0,y)}{p_X(0)} = \begin{cases} \frac{c}{c+2c} = \frac{1}{3}, & \text{if } y=-1, \\ \frac{2c}{c+2c} = \frac{2}{3}, & \text{if } y=1, \\ 0, & \text{otherwise.} \end{cases}$$

We can then calculate the conditional expectation:

$$\mathbf{E}[Y \mid X=0] = \sum_{y=-1}^{1} y p_{Y|X}(y \mid 0) = (-1)\cdot\frac{1}{3} + (1)\cdot\frac{2}{3} = \frac{1}{3}.$$

Finally, the conditional variance can be calculated as

$$\begin{aligned} \mathsf{Var}(Y \mid X=0) &= \mathbf{E}\left[(Y-\mathbf{E}[Y \mid X=0])^2 \mid X=0\right] \\ &= \mathbf{E}\left[\left(Y-\frac{1}{3}\right)^2 \mid X=0\right] \\ &= \sum_{y=-1}^{1}\left(y-\frac{1}{3}\right)^2 p_{Y|X}(y \mid 0) \\ &= \left(-1-\frac{1}{3}\right)^2\cdot\left(\frac{1}{3}\right) + \left(1-\frac{1}{3}\right)^2\cdot\left(\frac{2}{3}\right) \\ &= \frac{8}{9}. \end{aligned}$$

Submit You have used 5 of 5 attempts

Answers are displayed within the problem

Discussion

Hide Discussion

Topic: Unit 4: Discrete random variables:Problem Set 4 / 4. Joint PMF

Show all posts by recent activity

- point4, contradictory intuition with independence of X and Y — 6
- Is the grader for point 4 incorrect? — 11
  conditioning on Y not equals 0, if X and Y are independent. Under the conditioning, what about x=-1? I can't say much. ...
- Q4 independence answer seems wrong. — 3
  If I understand correctly, test of independence for two variables X and Y should be as following PDF of X should be sam...
- #5: weights/probs of values — 4
  Hi, I am getting confused about calculating the conditional variance given that X=0. I find myself trying to make the p_yi...
- Q3: computation — 2 new 6
  My understanding is that E[Z] is -1 * E[X^2]. Where am I going wrong?
- Question 4 — 4
  My approach for q4 is to check whether p(x,y/y!=0) =p(x/y!=0)*p(y/y!=0), if the equality holds, then it is independent, oth...