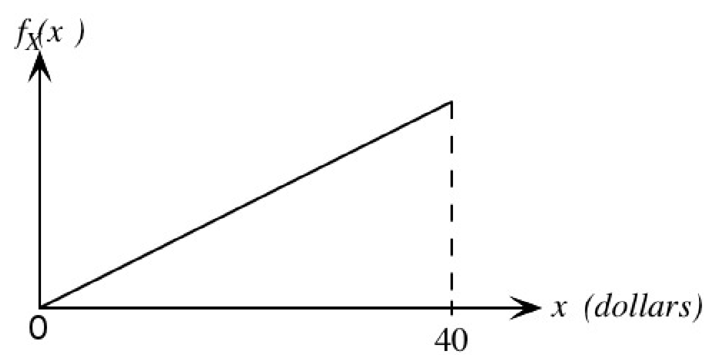
4. Sophia's vacation

Problem Set due Mar 13, 2020 05:29 IST *Completed*

Problem 4. Sophia's vacation

6/7 points (graded)

Sophia is vacationing in Monte Carlo. On any given night, she takes X dollars to the casino and returns with Y dollars. The random variable X has the PDF shown in the figure. Conditional on $X = x$, the continuous random variable Y is uniformly distributed between zero and $3x$.



1. Determine the joint PDF $f_{X,Y}(x, y)$.

   If $0 < x < 40$ and $0 < y < 3x$:

   $f_{X,Y}(x, y) =$ 1/2400 ✔ **Answer:** 1/2400

   If $y < 0$ or $y > 3x$:

   $f_{X,Y}(x, y) =$ 0 ✔ **Answer:** 0

2. On any particular night, Sophia makes a profit $Z = Y - X$ dollars. Find the probability that Sophia makes a positive profit, that is, find $\mathbf{P}(Z > 0)$.

$\mathbf{P}(Z > 0) =$ 2/3 Answer: 2/3

3. Find the PDF of Z. Express your answers in terms of z using standard notation.

   Hint: Start by finding $f_{Z|X}(z \mid x)$.

   If $-40 < z < 0$:

   $f_Z(z) =$ (z+40)/2400 **Answer:** (40+z)/2400

   $\frac{z+40}{2400}$

   If $0 < z < 80$:

   $f_Z(z) =$ (80-2*z)/2400 **Answer:** (80-z)/4800

   $\frac{80-2\cdot z}{2400}$

   If $z < -40$ or $z > 80$:

   $f_Z(z) =$ 0 **Answer:** 0

   0

4. What is $\mathbf{E}[Z]$?

   $\mathbf{E}[Z] =$ 40/3 **Answer:** 40/3

STANDARD NOTATION

Solution:

1. For this part, we will use the fact that $f_{X,Y}(x,y) = f_X(x)\, f_{Y|X}(y \mid x)$. Let us start by revealing $f_X(x)$. Clearly, $f_X(x) = ax$ for some ax, as shown in figure. Hence,

$$1 = \int_{-\infty}^{\infty} f_X(x)\ dx = \int_0^{40} ax\ dx = 800a.$$

   Hence, $f_X(x) = \frac{x}{800}$. Using $f_{Y|X}(y \mid x) = \frac{1}{3x}$, for $0 < y < 3x$, we obtain the following expression for the joint density:

$$f_{X,Y}(x,y) = \begin{cases} \dfrac{1}{2400}, & \text{if } 0 < x < 40 \text{ and } 0 < y < 3x \\ 0, & \text{otherwise.} \end{cases}$$

2. The first approach is to consider the region where Sophia makes positive profit. Notice that, this region consists of pairs (x, y), where $y > x$. Intersecting this region with the region where the joint density is non-negative, we need to consider

   $$\{(x,y) : 0 < x < 40, x < y < 3x\}.$$

   Thus,

   $$\mathbf{P}(Y > X) = \int_0^{40} \int_x^{3x} f_{X,Y}(x,y)\ dy\ dx = \int_0^{40} \int_x^{3x} \frac{1}{2400}\ dy\ dx = \int_0^{40} \frac{x}{1200} = \frac{2}{3}.$$

   We could have also arrived at this answer by realizing that for each possible value of X, there is a $2/3$ probability that Y>X, and therefore by the total probability theorem,

   $$\begin{aligned} \mathbf{P}(Y > X) &= \int_0^{40} \mathbf{P}(Y > X \mid X = x) f_X(x)\ dx \\ &= \int_0^{40} \frac{2}{3} f_X(x)\ dx \\ &= \frac{2}{3}, \end{aligned}$$

   where the last equality follows because a PDF always integrates to 1, over the region where it is nonzero.

3. Given $X = x$, Y is uniformly distributed on $[0, 3x]$, hence $Z = Y - x$ is uniform over $[-x, 2x]$. Thus,

   $$f_{Z|X}(z \mid x) = \frac{1}{3x}, \qquad \text{for } -x \le z \le 2x.$$

   Therefore,

   $$f_{X,Z}(x,z) = f_X(x) f_{Z|X}(z \mid x) = \frac{x}{800} \frac{1}{3x} = \frac{1}{2400}, \text{ for } 0 < x < 40 \text{ and } -x \le z \le 2x$$

Now, we will integrate over x to compute the marginal density $f_Z(z)$. Note that, $x \geq -z$ and $x \geq \frac{z}{2}$ must be satisfied at the same time (in order for $f_{X,Z}$ to be non-zero).

If $-40 < z < 0$, the range of integration is $-z < x < 40$. Hence,

$$f_Z(z) = \int_{-z}^{40} \frac{1}{2400}\, dx = \frac{40+z}{2400}.$$

If $0 < z < 80$, the range of integration is $z/2 \leq x \leq 40$. Hence,

$$f_Z(z) = \int_{z/2}^{40} \frac{1}{2400}\, dx = \frac{80-z}{4800}.$$

Therefore, the pdf of Z is

$$f_Z(z) = \begin{cases} \dfrac{40+z}{2400}, & -40 < z < 0 \\ \dfrac{80-z}{4800}, & 0 < z < 80 \\ 0, & \text{otherwise.} \end{cases}$$

4. First, note that $\mathbf{E}[Y|X=x] = \frac{3x}{2}$, for any $x \in [0,40]$. Thus, using the total expectation theorem,

$$\begin{aligned} \mathbf{E}[Y] &= \int_0^{40} \mathbf{E}[Y|X=x]\, f_X(x)\ dx \\ &= \frac{3}{2}\int_0^{40} x f_X(x)\ dx \\ &= \frac{3}{2}\mathbf{E}[X]. \end{aligned}$$

Since, $Z = Y - X$, we have, using linearity of expectation, $\mathbf{E}[Z] = \mathbf{E}[Y] - \mathbf{E}[X] = \frac{1}{2}\mathbf{E}[X]$.

Now,

$$\mathbf{E}[X] = \int_0^{40} x f_X(x)\ dx = \int_0^{40} \frac{x^2}{800}\, dx = \frac{80}{3}.$$

Hence, $\mathbf{E}[Z] = 40/3$.

Submit You have used 6 of 6 attempts

Answers are displayed within the problem

Discussion

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why not use fz in question 4?
why the right answer is get from the indirect aproach using fx and fy, if we have fz? 1

Additional resources to supplement the material
Hi all, I'm hoping that we can all share resources that we are using in addition to this Edx course. I personally, am using... 2

Q1 Solution
Hi, My answer to Q1 was in decimals (.0004) and equivalent to 1/2400. Is there a reason the answer in decimal was not... 3

How many hours did you spend on Sophia's vacation, Seraph's umbrella, and Oscar's dog?
How many hours did you spend on Sophia's vacation, Seraph's umbrella, and Oscar's dog? These people need to be jai... 14

Can Sophia just skip her vacation?
Can Sophia just skip her vacation and we get perfect score for this question? There's currently a Corona outbreak. 3 new 9

Part 3: How to find the range of integration for x
I have a feeling this is much simpler than I'm realizing, but I can't see how to translate the range of z values, -40 < z < 0,... 4

Q3 - help!
f_z(z) is a constant, right? Since it is uniformly distributed? Then why is it dependent on z? Is there anyway to confirm th... 2

[Staff] 3.2 integration interval question.
Community TA 2

Can the submission deadline be extended for just like few hours?
I work full time and this week has been very busy for me, so could not get this exercise done. I am hoping to finish it af... 6

Q3 Study Material
Hi, I have tried a lot of things but can't get the Q3 right. I guess it is the same principle as the last question in the probl... 4

Part 3 2 new 12

Slope of f_X(x)
Hi. I think I understand this, but I'm getting a very slow rising f_X(x). Is it true that the area of the triangle = 1? That is, th... 2

how does one do the first question

very tired but wont stop until i get this question how does one understand the question?