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1. Determining the type of a lightbulb

Problem Set due Apr 15, 2020 05:29 IST Completed

Problem 1. Determining the type of a lightbulb

3/3 points (graded)

The lifetime of a type-A bulb is exponentially distributed with parameter λ . The lifetime of a type-B bulb is exponentially distributed with parameter μ , where $\mu>\lambda>0$. You have a box full of lightbulbs of the same type, and you would like to know whether they are of type A or B. Assume an **a priori** probability of 1/4 that the box contains type-B lightbulbs.

1. Assume that $\mu \geq 3\lambda$. You observe the value t_1 of the lifetime, T_1 , of a lightbulb. A MAP decision rule decides that the lightbulb is of type A if and only if $t_1 \geq \alpha$. Find α , and express your answer in terms of μ and λ . Use 'mu" and 'lambda" and 'ln" to denote μ, λ , and the natural logarithm function, respectively. For example, $\ln \frac{2\mu}{\lambda}$ should be entered as 'ln((2*mu)/lambda)".

 $\alpha =$ (1/(mu-lambda))*In(mu/(3*lambda))

Answer: (1/(mu-lambda))*ln(mu/(3*lambda))

$$\left(\frac{1}{\mu-\lambda}\right)\cdot \ln\left(\frac{\mu}{3\cdot\lambda}\right)$$

2. Assume again that $\mu \geq 3\lambda.$ What is the probability of error of the MAP decision rule?

 $\boxed{ \bigodot \frac{1}{4}e^{-\mu\alpha} + \frac{3}{4}(1-e^{-\lambda\alpha})}$

$$igcirc rac{3}{4}e^{-\mulpha}+rac{1}{4}(1-e^{-\lambdalpha})$$

$$igcirc rac{1}{4}(1-e^{-\mulpha})+rac{3}{4}e^{-\lambdalpha}$$

$$igcirc rac{3}{4}(1-e^{-\mulpha})+rac{1}{4}e^{-\lambdalpha}$$

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3. Assume that $\lambda=3$ and $\mu=4$. Find the LMS estimate of T_2 , the lifetime of another lightbulb from the same box, based on observing $T_1=2$. Assume that conditioned on the bulb type, bulb lifetimes are indeperture (For this part, you will need a calculator. Provide an answer with an accuracy of three decimal places.)

0.328 **✓ Answer**: 0.328

Solution:

1. With some abuse of notation, we let A and B be the events that the box contains lightbulbs of type A and type B, respectively. A MAP rule decides in favor of type A if and only if

$$egin{aligned} \mathbf{P}\left(A\mid T_{1}=t_{1}
ight) &\geq \mathbf{P}\left(B\mid T_{1}=t_{1}
ight) \ rac{f_{T_{1}\mid A}\left(t_{1}
ight)\mathbf{P}\left(A
ight)}{f_{T_{1}}\left(t_{1}
ight)} &\geq rac{f_{T_{1}\mid B}\left(t_{1}
ight)\mathbf{P}\left(B
ight)}{f_{T_{1}}\left(t_{1}
ight)}. \end{aligned}$$

Equivalently, we decide that the bulb is of type A if and only if

$$egin{align} f_{T_1|A}\left(t_1
ight)\mathbf{P}\left(A
ight) & \geq f_{T_1|B}\left(t_1
ight)\mathbf{P}\left(B
ight), \ \lambda e^{-\lambda t_1}rac{3}{4} & \geq \mu e^{-\mu t_1}rac{1}{4}, \ rac{\lambda}{\mu}e^{(\mu-\lambda)t_1} & \geq rac{1}{3}, \ \left(\mu-\lambda
ight)t_1 & \geq \ln\left(rac{\mu}{3\lambda}
ight). \end{split}$$

Thus, since $\mu-\lambda>0$, a MAP rule decides in favor of type A if and only if $t_1\geq \ln\left(\frac{\mu}{3\lambda}\right)\cdot\frac{1}{\mu-\lambda}$. Hence, we deduce that,

$$lpha = rac{1}{\mu - \lambda} \ln\left(rac{\mu}{3\lambda}
ight).$$

2. Let events A and B be defined as in part (1). Let \hat{A} be the event that the MAP rule decides in favor of type A, and let \hat{B} be the event that the MAP rule decides in favor of type B. An error occurs whenever the decision is different from the actual type of the bulb. Thus,

$$\begin{split} \mathbf{P} \left(\mathrm{error} \right) &= \mathbf{P} \left(\hat{A} \cap B \right) + \mathbf{P} \left(A \cap \hat{B} \right) \\ &= \mathbf{P} \left(\hat{A} \mid B \right) \cdot \mathbf{P} \left(B \right) + \mathbf{P} \left(\hat{B} \mid A \right) \cdot \mathbf{P} \left(A \right) \\ &= \mathbf{P} \left(T_1 \geq \alpha \mid B \right) \cdot \frac{1}{4} + \mathbf{P} \left(T_1 < \alpha \mid A \right) \cdot \frac{3}{4} \\ &= e^{-\mu \alpha} \cdot \frac{1}{4} + \left(1 - e^{-\lambda \alpha} \right) \cdot \frac{3}{4}. \end{split}$$

3. The LMS estimate of T_2 based on observing $T_1=t_1$ is

$$\mathbf{E} [T_2 \mid T_1 = t_1] = \mathbf{E} [T_2 \mid T_1 = t_1, A] \cdot \mathbf{P} (A \mid T_1 = t_1) + \mathbf{E} [T_2 \mid T_1 = t_1, B] \cdot \mathbf{P} (B \mid T_1 = t_1)$$

$$= \mathbf{E} [T_2 \mid A] \cdot \mathbf{P} (A \mid T_1 = t_1) + \mathbf{E} [T_2 \mid B] \cdot \mathbf{P} (B \mid T_1 = t_1)$$

$$egin{aligned} &=rac{1}{\lambda}\cdot\left(rac{f_{T_{1}|A}\left(t_{1}
ight)\cdot\mathbf{P}\left(A
ight)}{f_{T_{1}}\left(t_{1}
ight)}
ight)+rac{1}{\mu}\cdot\left(rac{f_{T_{1}|B}\left(t_{1}
ight)\cdot\mathbf{P}\left(B
ight)}{f_{T_{1}}\left(t_{1}
ight)}
ight) \ &=rac{rac{1}{\lambda}rac{3}{4}\lambda e^{-\lambda t_{1}}+rac{1}{\mu}rac{1}{4}\mu e^{-\mu t_{1}}}{rac{3}{4}\lambda e^{-\lambda t_{1}}+rac{1}{4}\mu e^{-\mu t_{1}}}. \end{aligned}$$

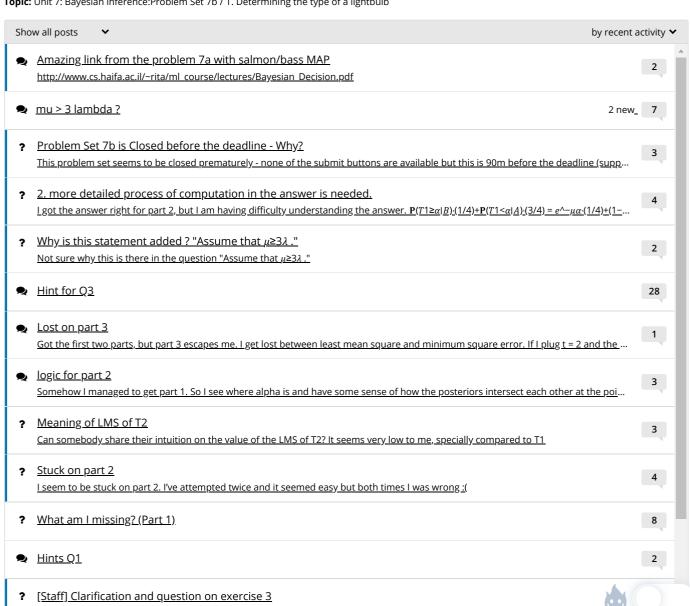
Inserting the values $\lambda=3$, $\mu=4$, and $t_1=2$, we obtain $\mathbf{E}\left[T_2\mid T_1=2\right]=0.328$.

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You have used 3 of 3 attempts

1 Answers are displayed within the problem

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