LECTURE 7: Conditioning on a random variable; Independence of r.v.'s

This is the most important topic pay attention!

- Conditional PMFs
  - Conditional expectations
  - Total expectation theorem
- Independence of r.v.'s
  - Expectation properties
  - Variance properties
- The variance of the binomial
- The hat problem: mean and variance

Conditional PMFs

$A = \{Y = y\}$

$$p_{X|A}(x \mid A) = \mathbf{P}(X = x \mid A)$$

$$\underline{p_{X|Y}(x \mid y)} = \mathbf{P}(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

$$\boxed{p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} \qquad \text{defined for } y \text{ such that } p_Y(y) > 0}$$

$$\sum_x p_{X|Y}(x \mid y) = 1$$

y \ x	1	2	3	4
4	1/20	2/20	2/20	
3	2/20	4/20	1/20	2/20
→ 2		1/20	3/20	1/20
1		1/20		

$Y = 2$

$P_Y(2) = 5/20$

$P_{X|Y}(1 \mid 2) = 0$

$P_{X|Y}(2 \mid 2) = 1/5$

$P_{X|Y}(\cdot \mid 2)$

This is the important fact following:

$$p_{X,Y}(x,y) = p_Y(y)\, p_{X|Y}(x \mid y)$$

$$p_{X,Y}(x,y) = p_X(x)\, p_{Y|X}(y \mid x)$$

Conditional PMFs involving more than two r.v.'s

- Self-explanatory notation

$$p_{X|Y,Z}(x \mid y, z) = \mathbf{P}(X=x \mid Y=y, Z=z) = \frac{\mathbf{P}(X=x, Y=y, Z=z)}{\mathbf{P}(Y=y, Z=z)} = \frac{p_{X,Y,Z}(x,y,z)}{p_{Y,Z}(y,z)}$$

$$p_{X,Y|Z}(x, y \mid z) = \mathbf{P}(X=x, Y=y \mid Z=z)$$

- Multiplication rule

$$\mathbf{P}(A \cap B \cap C) = \mathbf{P}(A)\,\mathbf{P}(B \mid A)\,\mathbf{P}(C \mid A \cap B)$$

$A = \{X=x\}$ $\quad B = \{Y=y\}$ $\quad C = \{Z=z\}$

$$p_{X,Y,Z}(x, y, z) = p_X(x)\, p_{Y|X}(y \mid x)\, p_{Z|X,Y}(z \mid x, y)$$

Conditional expectation

$A = \{Y = y\}$

$$\mathbf{E}[X] = \sum_x x p_X(x) \qquad \mathbf{E}[X \mid A] = \sum_x x\, p_{X|A}(x) \qquad \mathbf{E}[X \mid Y = y] = \sum_x x p_{X|Y}(x \mid y)$$

- Expected value rule

$$\mathbf{E}[g(X)] = \sum_x g(x)\, p_X(x) \qquad \mathbf{E}[g(X) \mid A] = \sum_x g(x)\, p_{X|A}(x)$$

$$\mathbf{E}[g(X) \mid Y = y] = \sum_x g(x)\, p_{X|Y}(x \mid y)$$

Total probability and expectation theorems

- $A_1, \ldots, A_n$: partition of Ω

$Y = \{y_1, \ldots, y_n\}$ $A_i = \{Y = y_i\}$

- $p_X(x) = \mathbf{P}(A_1)\, p_{X|A_1}(x) + \cdots + \mathbf{P}(A_n)\, p_{X|A_n}(x)$

$$p_X(x) = \sum_y p_Y(y)\, p_{X|Y}(x \mid y)$$

- $\mathbf{E}[X] = \mathbf{P}(A_1)\, \mathbf{E}[X \mid A_1] + \cdots + \mathbf{P}(A_n)\, \mathbf{E}[X \mid A_n]$

$$\boxed{\mathbf{E}[X] = \sum_y p_Y(y)\, \mathbf{E}[X \mid Y = y]}$$

same

E[X|Y] = $\sum_y$ PY(y)* E[X|Y=y]

- Fine print:
  Also valid when Y is a discrete r.v. that ranges over an infinite set, as long as $\mathbf{E}[|X|] < \infty$

Independence

- of two events: $\mathbf{P}(A \cap B) = \mathbf{P}(A) \cdot \mathbf{P}(B)$ $\qquad$ $\mathbf{P}(A \mid B) = \mathbf{P}(A)$

- of a r.v. and an event: $\mathbf{P}(\underline{X = x} \text{ and } \underline{A}) = \mathbf{P}(X = x) \cdot \mathbf{P}(A)$, $\underline{\text{for all } x}$

$p_{X|A}(x) = p_X(x)$, for all x $\qquad$ $\mathbf{P}(A \mid X = x) = \mathbf{P}(A)$, for all x

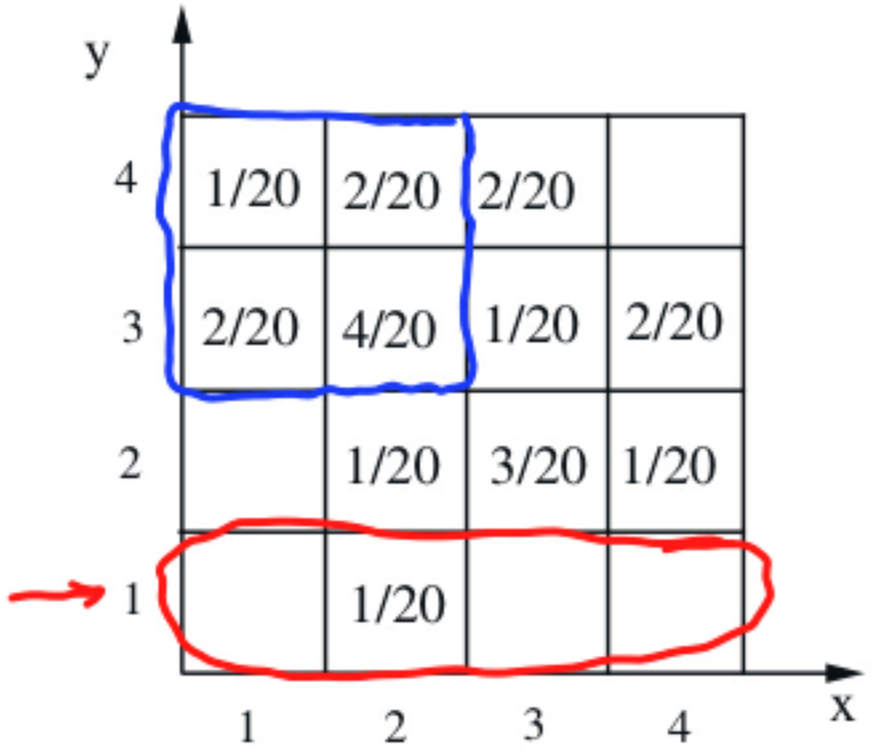
- of two r.v.'s: $\mathbf{P}(\underline{X = x} \text{ and } \underline{Y = y}) = \mathbf{P}(X = x) \cdot \mathbf{P}(Y = y)$, for all $\underline{x, y}$

$p_{X|Y}(x|y) = p_X(x)$ $\qquad$ $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$, for all x, y

$p_{Y|X}(y|x) = p_Y(y)$

> X, Y, Z are **independent** if:
> $p_{X,Y,Z}(x, y, z) = p_X(x)\, p_Y(y)\, p_Z(z)$, for all x, y, z

Example: independence and conditional independence



1/3	1/9	2/9
2/3	2/9	4/9
	1/3	2/3

- Independent? No

$p_X(1) = 3/20$

$p_{X|Y}(1|1) = 0$

- What if we condition on $X \leq 2$ and $Y \geq 3$?

Yes

Independence and expectations

- In general: $\mathbf{E}[g(X,Y)] \neq g\big(\mathbf{E}[X], \mathbf{E}[Y]\big)$

always true

- Exceptions: $\mathbf{E}[aX+b] = a\mathbf{E}[X]+b$ $\qquad \mathbf{E}[X+Y+Z] = \mathbf{E}[X]+\mathbf{E}[Y]+\mathbf{E}[Z]$

If X, Y are **independent:** $\mathbf{E}[XY] = \mathbf{E}[X]\mathbf{E}[Y]$

$g(X)$ and $h(Y)$ are also independent: $\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)] \cdot \mathbf{E}[h(Y)]$

$E[g(X,Y)] \qquad g(x,y) = xy$

$$= \sum_x \sum_y xy\, p_{X,Y}(x,y) = \sum_x \sum_y x y\, p_X(x) p_Y(y)$$

$$= \sum_x x\, p_X(x) \underbrace{\sum_y y\, p_Y(y)} = E[X]\, E[Y]$$

Independence and variances

- Always true: $\text{var}(aX) = a^2 \text{var}(X)$ $\qquad$ $\text{var}(X + a) = \text{var}(X)$

- In general: $\text{var}(X + Y) \neq \text{var}(X) + \text{var}(Y)$

$$\boxed{\text{If } X, Y \text{ are independent: } \text{var}(X + Y) = \text{var}(X) + \text{var}(Y)}$$

assume
$E[X] = E[Y] = 0$
$E[XY] = E[X]E[Y] = 0$

$$\text{var}(X+Y) = E[(X+Y)^2] = E[X^2 + 2XY + Y^2]$$
$$= E[X^2] + 2\underbrace{E[XY]}_{} + E[Y^2] = \text{var}(X) + \text{var}(Y)$$

- Examples:
  - If $X = Y$: $\text{var}(X + Y) =$ $\text{var}(2X) = 4\text{var}(X)$
  - If $X = -Y$: $\text{var}(X + Y) =$ $\text{var}(0) = 0$
  - If X, Y independent: $\text{var}(X - 3Y) =$ $\text{var}(X) + \text{var}(-3Y) = \text{var}(X) + 9\text{var}(Y)$

Variance of the binomial

- X: binomial with parameters n, p
  - number of successes in n independent trials

$X_i = 1$ if ith trial is a success; (indicator variable) independent
$X_i = 0$ otherwise

$$X = X_1 + \cdots + X_n$$

$$\boxed{\mathrm{var}(X)} = \mathrm{var}(X_1) + \cdots + \mathrm{var}(X_n)$$

$$= n \cdot \mathrm{var}(X_1) = \boxed{np(1-p)}$$

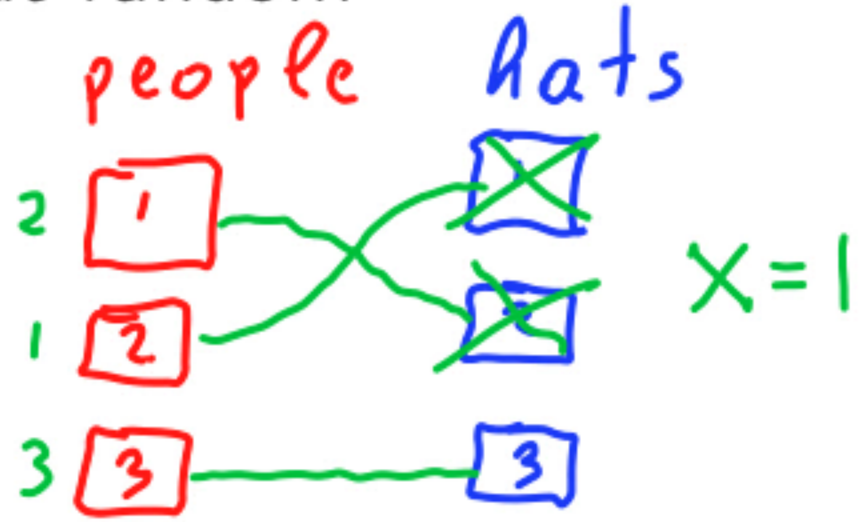
The hat problem

- n people throw their hats in a box and then pick one at random
  - All permutations equally likely $1/n!$
  - Equivalent to picking one hat at a time
- X: number of people who get their own hat
  - Find $\boxed{\mathbf{E}[X]} = E[X_1] + \cdots + E[X_n] = n \cdot \frac{1}{n} = \boxed{1}$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

- $\mathbf{E}[X_i] = E[X_1] = P(X_1 = 1) = \frac{1}{n}$



$\frac{1}{3} \cdot \frac{1}{2} \cdot 1 = \frac{1}{3!}$

$p_X(k)$: hard!

$\sum_k k\, p_X(k)$

The variance in the hat problem

$n = 2$

$X_1 = 1 \Rightarrow X_2 = 1$

$X_1 = 0 \Rightarrow X_2 = 0$

- X: number of people who get their own hat
  - Find $\text{var}(X)$

$$X_i = \begin{cases} 1, & \text{if } i \text{ selects own hat} \\ 0, & \text{otherwise.} \end{cases}$$

$$X = X_1 + X_2 + \cdots + X_n$$

$$X^2 = \overbrace{\sum_i X_i^2}^{n} + \overbrace{\sum_{i,j:i \neq j} X_i X_j}^{n^2 - n}$$

$n(n-1)$

- $\boxed{\text{var}(X)} = \mathbf{E}[X^2] - (\mathbf{E}[X])^2 = 2 - 1 = \boxed{1}$
- $\mathbf{E}[X_i^2] = E[X_1^2] = E[X_1] = 1/n$

$E[X^2] = n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n} \cdot \frac{1}{n-1}$

- For $i \neq j$: $\mathbf{E}[X_i X_j] = E[X_1 X_2] = P(X_1 X_2 = 1) = P(X_1 = 1, X_2 = 1)$

$= P(X_1 = 1)\, P(X_2 = 1 \mid X_1 = 1) = \frac{1}{n} \cdot \frac{1}{n-1}$