



## Lecture 2

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# Sample from the Distribution

Slide  
2.1



# Selected Topics

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1. Independent events and random variables
2. Sample from the distribution
3. Confidence intervals
4. Hypothesis testing

Slide  
2.2



# Independent Events

Two events are **independent** if the occurrence of one does not in any way affect the probability of the occurrence of the other.

**Example:** You toss a quarter and a dime. Let  $A$  be the event “*Head* on the quarter” and  $B$  the event “*Head* on the dime.”

Then the probability of occurrence of both  $A$  and  $B$  is  $1/4$ .

Event “both  $A$  and  $B$ ” is denoted as  $A \cap B$ . Hence  $P(A \cap B) = 1/4$ .

Notice that  $P(A) = P(B) = 1/2$ .

**Definition:** Events  $A$  and  $B$  are **independent** if  $P(A \cap B) = P(A) \cdot P(B)$



# Conditional Probability

**Definition:** Conditional probability of event  $A$  occurring in an experiment given that event  $B$  occurred, denoted by  $P(A|B)$  (“probability of  $A$  given  $B$ ”), is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Notice that  $P(B) > 0$  (think what “given that event  $B$  occurred” implies).

**Example:** A fair die is rolled and you are guessing the outcome. You want to claim the outcome is six. Your weird buddy is on the side and he claims that although he did not see the outcome, he is certain that no *dot* was in the middle. Does this information help you? Quantify it.

Solution: Let  $S$  be the event “the outcome is six” and  $E$  “the outcome is even.”

Your buddy knows that event  $E$  occurred in this experiment.

Absent your buddy, your probability of guessing the outcome is  $P(S) = 1/6$ .

If your buddy whispers to you, your probability of guessing the outcome is

$$P(S|E) = \frac{P(S \cap E)}{P(E)} = \frac{P(S)}{P(E)} = \frac{1/6}{1/2} = \frac{1}{3}.$$



# Independent Events and Cond. Probability

**Note:** If events  $A$  and  $B$  are independent and  $P(B) > 0$  then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A).$$

(and similarly  $P(B|A) = \dots = P(B)$  if  $P(A) > 0$ )

Hence the knowledge of occurrence of event  $B$  does not change the probability of occurrence of  $A$ .

The converse also holds: if the knowledge of occurrence of event  $B$  does not change the probability of occurrence of  $A$ , i.e., if

$$P(A|B) = P(A) \quad (\text{note that } P(B) > 0)$$

then the events  $A$  and  $B$  are independent.



# Independence of random variables

Two random variables are **independent** if the outcome of one variable does not in any way influence the outcome of another.

**Example:** Roll a die and toss a coin. If  $X$  is the outcome of the die (1, 2, 3, 4, 5, or 6) and  $Y$  the outcome of the coin (0 or 1), then  $X$  and  $Y$  are independent random variables.

**Definition:** Random variables  $X$  and  $Y$  are **independent** if

$$P(a < X \leq b, c < Y \leq d) = P(a < X \leq b) \cdot P(c < Y \leq d),$$

for all real numbers  $a, b, c, d$  such that  $a < b$  and  $c < d$ .

**Note:** Discrete random variables  $X$  and  $Y$  are independent if and only if

$$P(X = k, Y = m) = P(X = k) \cdot P(Y = m),$$

for all discrete values  $k$  that  $X$  assumes and all discrete values  $m$  that  $Y$  assumes.



# Sample from the distribution

Random variables  $X_1, X_2, \dots, X_n$  are a **sample from distribution  $F$**  (CDF) if they are independent and all have distribution  $F$ .

Any function  $g(X_1, X_2, \dots, X_n)$  is a **statistic** defined on the sample. Such function is, on its own, a random variable.

Two most common statistics are:

**Sample Mean:**  $\bar{X} = \bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$

**Sample Variance:**  $S^2 = S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2$

If  $F$  is a distribution with **population mean  $\mu$**  and **population variance  $\sigma^2$** , it can be shown that

$$E(\bar{X}_n) = \mu, \quad Var(\bar{X}_n) = \frac{\sigma^2}{n}, \quad \text{and} \quad E(S_n^2) = \sigma^2.$$

*Sample Mean is an **unbiased estimator** of the **population mean***

*Sample Variance is an **unbiased estimator** of the **population variance***



# Sum of independent random variables

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For any random variables  $X$  and  $Y$ ,  $E(X+Y) = E(X) + E(Y)$ .

Slide 1.40:  
properties  
(5) and (6)

Moreover, for *independent*  $X$  and  $Y$ ,  $Var(X+Y) = Var(X) + Var(Y)$ .

The identities extend to any finite sequence  $X_1, X_2, \dots, X_n$ :

$$(E) \quad E(X_1 + X_2 + \dots + X_n) = E(X_1) + E(X_2) + \dots + E(X_n).$$

And for independent random variables:

$$(V) \quad Var(X_1 + X_2 + \dots + X_n) = Var(X_1) + Var(X_2) + \dots + Var(X_n).$$

Consequently, if  $X_1, X_2, \dots, X_n$  is a sample from the distribution  $F$  with mean  $\mu$  and variance  $\sigma^2$ :

$$E(\bar{X}_n) = E\left(\frac{1}{n} \sum_{k=1}^n X_k\right) \stackrel{(*)}{=} \frac{1}{n} E\left(\sum_{k=1}^n X_k\right) \stackrel{\text{by (E)}}{=} \frac{1}{n} \sum_{k=1}^n E(X_k) = \frac{1}{n} \cdot n\mu = \mu$$

$$Var(\bar{X}_n) = Var\left(\frac{1}{n} \sum_{k=1}^n X_k\right) \stackrel{(**)}{=} \frac{1}{n^2} Var\left(\sum_{k=1}^n X_k\right) \stackrel{\text{by (V)}}{=} \frac{1}{n^2} \sum_{k=1}^n Var(X_k) = \frac{1}{n^2} \cdot n\sigma^2 = \frac{\sigma^2}{n}$$

(\*) property (2) on slide 1.40.

(\*\*) property (4) on slide 1.40.

Slide  
2.8





## Examples

**Example:** Suppose that  $X_1, X_2, \dots, X_{100}$  is a sample from  $N(0,25)$ . What is the expected value and the variance of  $\bar{X}_{100}$ ?

Solution: Population mean  $\mu = 0$  and variance  $\sigma^2 = 25$ .

$$\text{Then } E(\bar{X}_{100}) = \mu = 0 \text{ and } \text{Var}(\bar{X}_{100}) = \frac{\sigma^2}{n} = \frac{25}{100} = 0.25$$

**Example:** Suppose that  $X_1, X_2, \dots, X_{144}$  is a sample from  $\text{Exp}(0.25)$ . What is the expected value and the variance of  $3(\bar{X}_{144} - 4)$ ?

Solution: Population mean  $\mu = \frac{1}{0.25} = 4$  and variance  $\sigma^2 = \frac{1}{0.25^2} = 16$ .

$$\text{Thus } E(3(\bar{X}_{144} - 4)) = 3E(\bar{X}_{144} - 4) = 3(E(\bar{X}_{144}) - 4) = 3 \cdot 0 = 0$$

$$\text{and } \text{Var}(3(\bar{X}_{144} - 4)) = 3^2 \text{Var}(\bar{X}_{144} - 4) = 9 \text{Var}(\bar{X}_{144}) = 9 \cdot \frac{16}{144} = 1$$

Note:  $3(\bar{X}_{144} - 4)$  is the **Transformed Sample Mean** of this sample:

$$\bar{Z}_n = \frac{\bar{X}_n - \mu}{\sqrt{\frac{\sigma^2}{n}}} = \frac{\bar{X}_{144} - 4}{\sqrt{\frac{16}{144}}} = \frac{\bar{X}_{144} - 4}{\frac{1}{3}} = 3(\bar{X}_{144} - 4)$$



# Transformed Sample Mean

If  $X_1, X_2, \dots, X_n$  is a sample from the distribution  $F$  with population mean  $\mu$  and population variance  $\sigma^2$ , then we define the *Transformed Sample Mean*

$$\bar{Z}_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \frac{\sqrt{n}}{\sigma} (\bar{X}_n - \mu)$$

Then  $E(\bar{Z}_n) = E\left(\frac{\sqrt{n}}{\sigma} (\bar{X}_n - \mu)\right) \stackrel{(2)}{=} \frac{\sqrt{n}}{\sigma} E(\bar{X}_n - \mu) \stackrel{(1)}{=} \frac{\sqrt{n}}{\sigma} (\overset{=\mu}{E\bar{X}_n}) - \mu = 0$ , and

$$Var(\bar{Z}_n) = Var\left(\frac{\sqrt{n}}{\sigma} (\bar{X}_n - \mu)\right) \stackrel{(4)}{=} \frac{n}{\sigma^2} Var(\bar{X}_n - \mu) \stackrel{(3)}{=} \frac{n}{\sigma^2} \overset{=\frac{\sigma^2}{n}}{Var\bar{X}_n} = 1$$

where (1) – (4) are properties on slide 1.40.

Recap: *Transformed Sample Mean* of a sample  $X_1, X_2, \dots, X_n$  from  $F$  with population mean  $\mu$  and population variance  $\sigma^2$  is a random variable with expected value zero and variance 1.



# Normal Samples

For independent random variables  $X \sim N(\mu, \sigma^2)$  and  $Y \sim N(\nu, \rho^2)$ ,  $X + Y$  is a normal random variable (*'independent normal's are closed under summation'*).

From (E) and (V) it follows that

$$X + Y \sim N(\mu + \nu, \sigma^2 + \rho^2).$$

Similarly, for any finite sequence of independent normal's,

$$X_1 + X_2 + \dots + X_n \sim N(\mu_1 + \mu_2 + \dots + \mu_n, \sigma_1^2 + \sigma_2^2 + \dots + \sigma_n^2).$$

With help of (2) and (4) on slide 1.40 we conclude that the *Sample Mean* of the sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  is

$$\bar{X}_n \sim N\left(\mu, \frac{\sigma^2}{n}\right).$$

By *transformation to standard normal* (see (4), slide 1.41):  $\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1).$

Note: the latter is the *Transformed Sample Mean* and from the previous slide we know its expected value is zero and variance 1. However, for the **normal** sample more can be said: the Transformed Sample Mean is a standard **normal** variable.



# Examples

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2.12

Recall: Suppose  $X \sim N(3, 2)$ . What is the distribution of  $2X - 5$ ?

Solution:  $2X - 5 \sim N(1, 8)$ .

Slide 1.42

**Example:** Assume that  $X \sim N(4, 12)$  and  $Y \sim N(2, 8)$  are independent. What is the distribution of: (a)  $X + Y$ , (b)  $2X - 3Y$ .

Solution (a): Sum of independent normals is normal, so  $X + Y$  is a normal random variable. Then  $E(X + Y) = E(X) + E(Y) = 4 + 2 = 6$ , and

$$\text{Var}(X + Y) = \text{Var}(X) + \text{Var}(Y) = 12 + 8 = 20.$$

Therefore  $X + Y \sim N(6, 20)$ .

Solution (b):  $2X$  and  $-3Y$  are normal random variables (prop. (3) on 1.41) and they are also independent, so their sum is a normal random variable.

Then  $E(2X - 3Y) = E(2X) + E(-3Y) = 2E(X) - 3E(Y) = 2 \cdot 4 - 3 \cdot 2 = 2$ , and

$$\text{Var}(2X - 3Y) = \text{Var}(2X) + \text{Var}(-3Y) = 2^2 \text{Var}(X) + (-3)^2 \text{Var}(Y) = 4 \cdot 12 + 9 \cdot 8 = 120.$$

Therefore  $2X - 3Y \sim N(2, 120)$ .



## Examples

**Example:** Let  $X_1, X_2, \dots, X_{100}$  be a sample from  $N(-10, 4)$ .

(a) What is the distribution of its Sample Mean?

(b) What is the distribution of  $5(\bar{X}_{100} + 10)$ ?

Solution (a): *Sample Mean*  $\bar{X}_n$  of the sample from  $N(\mu, \sigma^2)$  has  $N\left(\mu, \frac{\sigma^2}{n}\right)$  distribution (see 2.11). In this case we have population mean  $\mu = -10$ , population variance  $\sigma^2 = 4$ , and sample size  $n = 100$ , therefore

$$\bar{X}_{100} \sim N\left(-10, \frac{4}{100}\right) = N\left(-10, \frac{1}{25}\right)$$

Solution (b): Notice that  $5(\bar{X}_{100} + 10)$  is normal since  $\bar{X}_{100}$  is normal.

$$E(5(\bar{X}_{100} + 10)) = 5E(\bar{X}_{100}) + 50 = 5 \cdot (-10) + 50 = 0, \text{ and}$$

$$Var(5(\bar{X}_{100} + 10)) = 5^2 Var(\bar{X}_{100} + 10) = 25 Var(\bar{X}_{100}) = 25 \cdot \frac{1}{25} = 1.$$

Note: This is the Transformed Sample Mean of a sample  $X_1, X_2, \dots, X_{100}$ :

$$5(\bar{X}_{100} + 10) = \frac{\bar{X}_{100} + 10}{\frac{1}{5}} = \frac{\bar{X}_{100} - (-10)}{\sqrt{\frac{1}{25}}} = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0, 1)$$

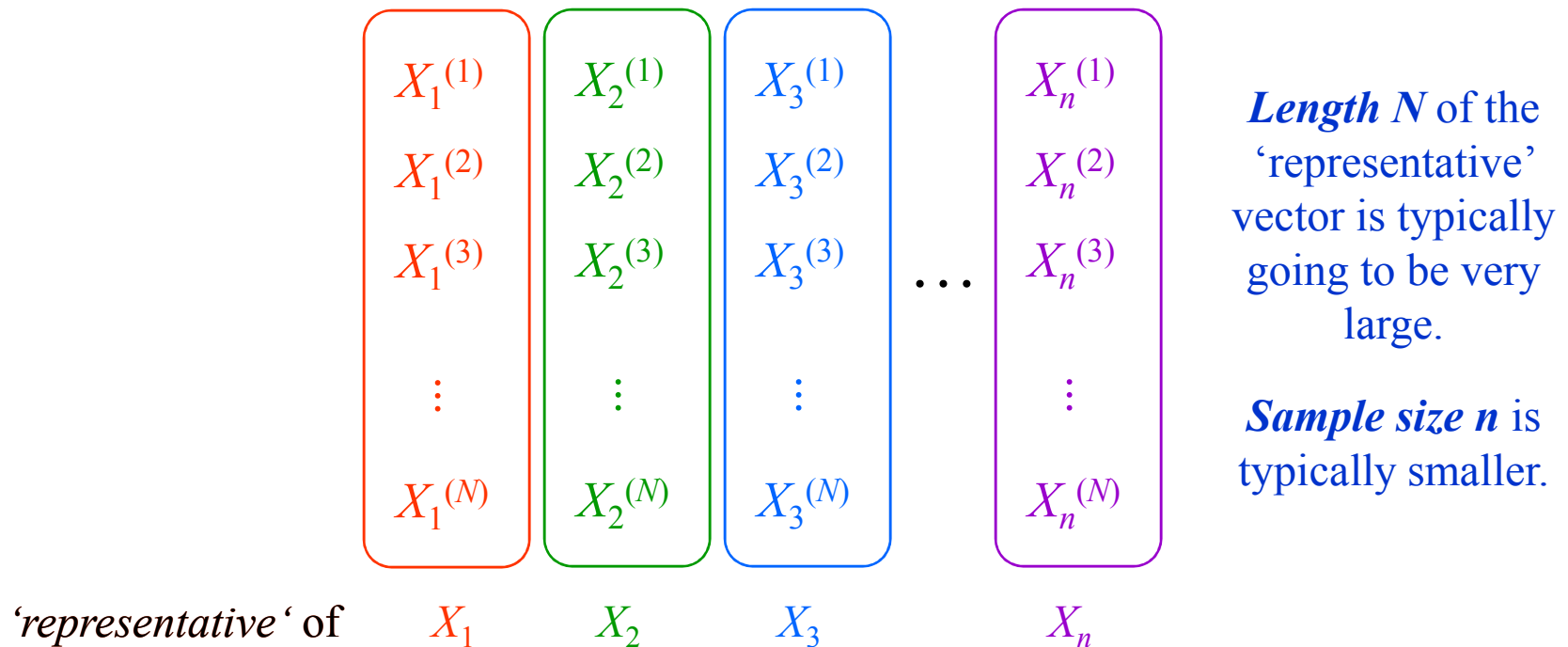


# How to “create” a sample in $R$ ?

We can use random number generators in  $R$ . But they produce sequences (vectors) of *numbers*, not random variables.

Idea: Generate a vector of  $N$  random numbers from a given distribution (use `rnorm()`, `runif()`, ...) and think of it as *a representative* of the first random variable,  $X_1$ , from the sample.

Repeat this for random variables  $X_2, \dots, X_n$ .

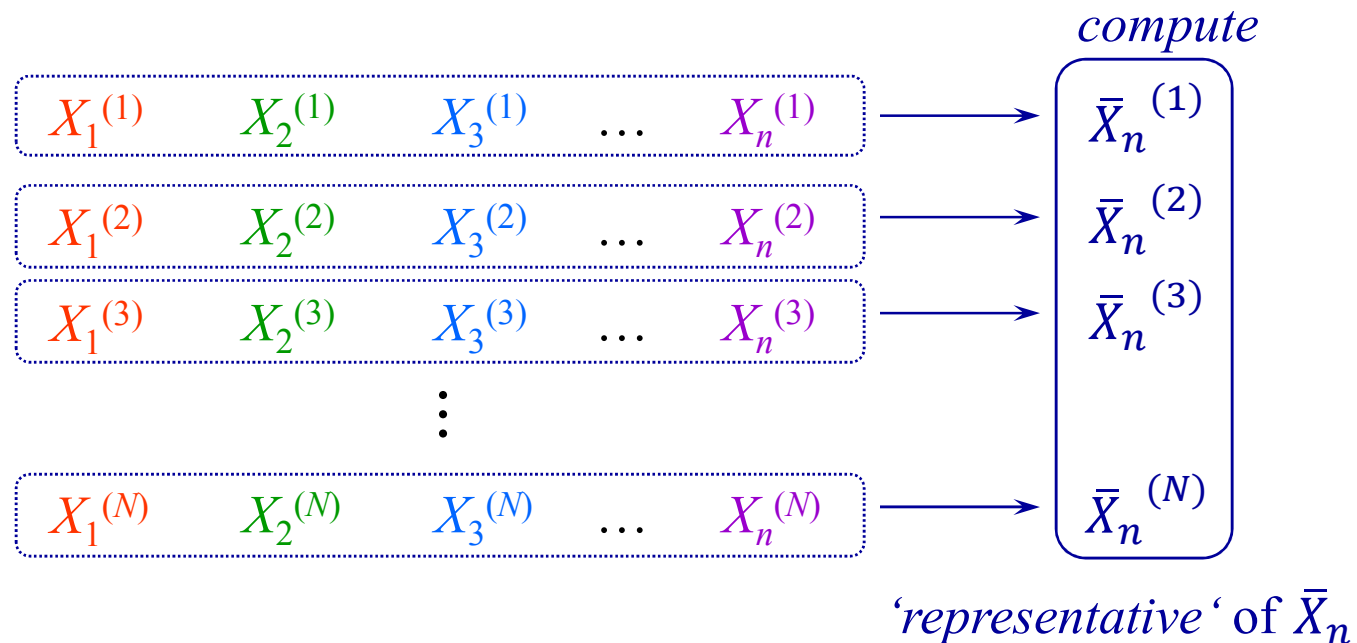




## Slight modification of the idea

With the sample representation as described, in order to compute the statistics of interest (sample mean, variance, median, ...) we need to store  $N \cdot n$  numbers in the memory. This is fine if  $n$  is a 'small' number.

If we put some faith in our random number generator, we can 'transpose' this procedure, i.e., generate the matrix row-by-row, and at each step calculate the desired statistic(s).



The illustration is for the *Sample Mean*, but can be mimicked for any desired statistic.



## A brief detour into CLT

Clearly by taking larger  $N$  we obtain better representation of our sample, and consequently a better representation of the statistic(s) of interest.

But what happens as we increase the sample size  $n$ ?

For the sample mean this is answered by *Central Limit Theorem*.

CLT: Let  $X_1, X_2, \dots$  be a sequence of i.i.d. random variables from the distribution  $F$  with population mean  $\mu$  and variance  $\sigma^2$ . Then the sequence of random variables  $Z_1, Z_2, \dots$  defined by

*Transformed  
Sample Means*

$$Z_n = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} = \sqrt{n} \frac{\bar{X}_n - \mu}{\sigma} = \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{X_k - \mu}{\sigma}$$

converges *in distribution* to  $N(0,1)$  as  $n \rightarrow \infty$ .

*Convergence in distribution:* if we denote by  $F_n$  the CDF of  $Z_n$  and the CDF of the standard normal by  $F$  then

$$\lim_{n \rightarrow \infty} F_n(x) = F(x), \text{ for all real numbers } x.$$





# Expected value of Sample Variance: $R$

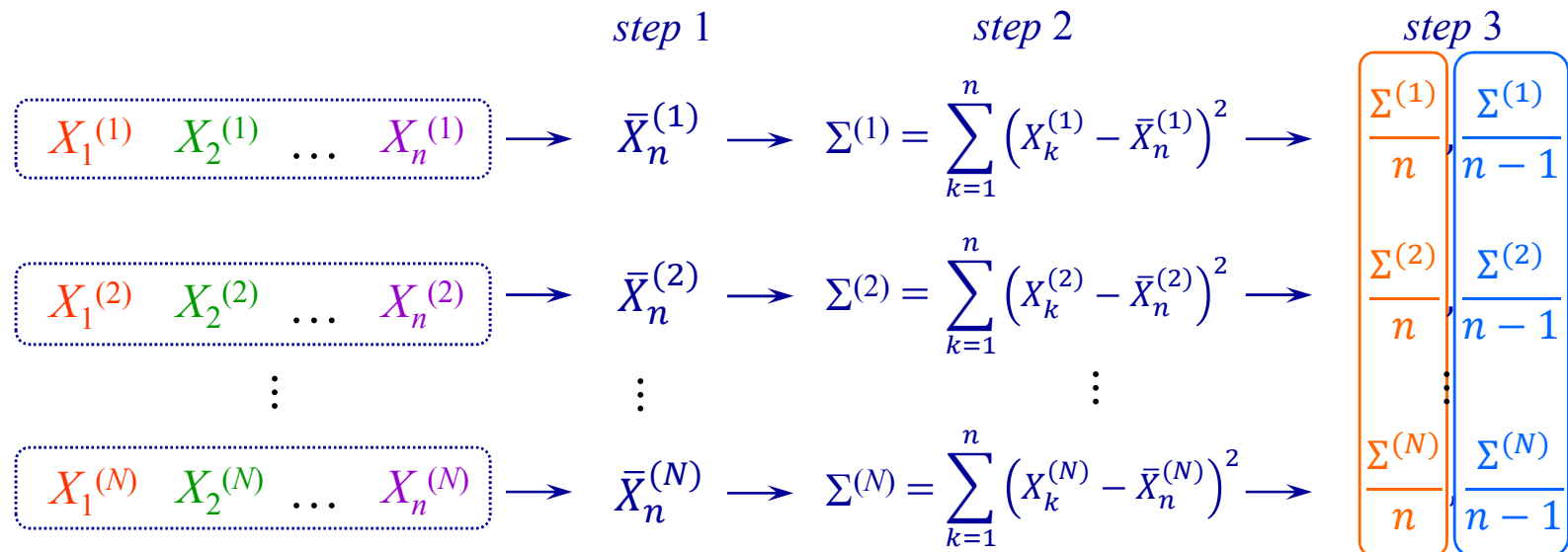
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Given a sample  $X_1, X_2, \dots, X_n$  from distribution  $F$  we define

Sample Variance  $S^2 = S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2 = \frac{\Sigma}{n-1}$

Question: Why is  $\Sigma$  divided by  $n-1$  instead of  $n$ ?

We can use  $R$  to estimate expected values of random variables  $\frac{\Sigma}{n}$  and  $\frac{\Sigma}{n-1}$ :



Our estimates for expected values of  $\frac{\Sigma}{n}$  and  $\frac{\Sigma}{n-1}$  are averages of these columns.



# Sample Variance

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Given a sample  $X_1, X_2, \dots, X_n$  from distribution  $F$  we define

Sample Variance  $S^2 = S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2$

WHY?  $n-1$

Recall: This is a random variable (like any statistic on sample).

Were the population mean  $\mu$  known, then

$$\begin{aligned} E \left( \frac{1}{n} \sum_{k=1}^n (X_k - \mu)^2 \right) &= \frac{1}{n} E \left( \sum_{k=1}^n (X_k - \mu)^2 \right) = \frac{1}{n} \sum_{k=1}^n E(X_k - \mu)^2 \\ &= \frac{1}{n} \sum_{k=1}^n \text{Var}(X_k) = \frac{1}{n} n \sigma^2 = \sigma^2. \end{aligned}$$

But the population mean  $\mu$  is unknown and our best estimate (*unbiased estimator*) for it is the *Sample Mean*.

$$E(\bar{X}_n) = \mu$$



## Sample Variance cont.

Idea: we could try to estimate the population variance with

$$\frac{1}{n} \sum_{k=1}^n (X_k - \bar{X}_n)^2$$

Unfortunately  $E \left( \frac{1}{n} \sum_{k=1}^n (X_k - \bar{X}_n)^2 \right) = \frac{n-1}{n} \sigma^2$ . (simulations confirm this)

By multiplying both sides by  $\frac{n}{n-1}$  we get

$$E(S_n^2) = E \left( \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2 \right) = \sigma^2.$$

Answer to **WHY?**: an unbiased estimator of population variance  $\sigma^2$  is obtained when the sum of squares is divided by  $n - 1$ , not  $n$ .



## Intuitive answer to WHY?

Given a sample  $X_1, X_2, \dots, X_n$  from distribution  $F$  with population mean  $\mu$ , it is reasonable to expect that

$$\sum_{k=1}^n (X_k - \bar{X}_n)^2 \leq \sum_{k=1}^n (X_k - \mu)^2.$$

In fact, given any  $n$  numbers  $x_1, x_2, \dots, x_n$  one can show that the function

$$\begin{aligned} f(u) &= (x_1 - u)^2 + (x_2 - u)^2 + \dots + (x_n - u)^2 = \\ &= (u - x_1)^2 + (u - x_2)^2 + \dots + (u - x_n)^2 \end{aligned}$$

attains minimum at point  $u$  that is the average of  $x_k$ 's.

(basic calculus: take the derivative of  $f(u)$ , set it equal zero and solve for  $u$ )

Divide both sides by  $n$ , and notice that the inequality extends to expected values of those random variables above

$$E\left(\frac{1}{n} \sum_{k=1}^n (X_k - \bar{X}_n)^2\right) \leq E\left(\frac{1}{n} \sum_{k=1}^n (X_k - \mu)^2\right) = \sigma^2.$$

The correction factor turns out to be  $\frac{n}{n-1}$ .



# Degrees of Freedom

Given a sample  $X_1, X_2, \dots, X_n$  from distribution  $F$  the *residual* random variables

$$r_1 = X_1 - \bar{X}_n$$

$$r_2 = X_2 - \bar{X}_n$$

$$r_3 = X_3 - \bar{X}_n$$

...

$$r_n = X_n - \bar{X}_n$$

are NOT independent. **Why?** Because they sum up to zero!

However, they are independent for any choice of  $n - 1$  of them.

This yields a notion of  $n - 1$  *degrees of freedom*!



# Probabilistic foundation

## Confidence Interval for Normal sample Population Mean

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If  $X_1, X_2, \dots, X_n$  is a sample from  $N(\mu, \sigma^2)$  then:

$$\frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{"known variance"}$$

$$\frac{\bar{X}_n - \mu}{\boxed{\frac{S_n}{\sqrt{n}}}} \sim t(n-1) \quad \text{"unknown variance"}$$

*Standard Error of a sample*

The first claim we proved on slide 2.11.

The second claim is much harder to prove.



# Confidence intervals for population mean

**Population mean confidence interval** is an open interval which, with probability **C** (**confidence level**), includes an **unknown** population mean  $\mu$ .

It is calculated from the sample and quite likely will be different for different samples.

If  $X_1, X_2, \dots, X_n$  is a sample from  $N(\mu, \sigma^2)$  and  $\sigma$  is known then

$$Z = \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1)$$

Given a confidence level  $C$  (for example  $C = 95\% = 0.95$ ), find the point  $z_C = z_{0.95}$  on the  $x$ -axis such that  $P(-z_C < Z < z_C) = 0.95$ .

If we can find such point we are done! **Why?**



# Confidence intervals for population mean

$$-z_C < Z < z_C$$

$$\Leftrightarrow -z_C < \frac{\bar{X}_n - \mu}{\frac{\sigma}{\sqrt{n}}} < z_C$$

$$\Leftrightarrow -\frac{\sigma}{\sqrt{n}} z_C < \bar{X}_n - \mu < \frac{\sigma}{\sqrt{n}} z_C$$

$$\Leftrightarrow -\frac{\sigma}{\sqrt{n}} z_C - \bar{X}_n < -\mu < \frac{\sigma}{\sqrt{n}} z_C - \bar{X}_n$$

$$\Leftrightarrow \bar{X}_n + \frac{\sigma}{\sqrt{n}} z_C > \mu > \bar{X}_n - \frac{\sigma}{\sqrt{n}} z_C$$

How to interpret “with probability 0.95?”

Imagine you sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  independently thousand times.

You get a thousand open intervals of same length centered at different points.

Population mean  $\mu$  would be inside approximately 950 of those intervals, and outside of approximately 50 of them.

$$\text{Hence } 0.95 = P(-z_C < Z < z_C) = P\left(\bar{X}_n - \frac{\sigma}{\sqrt{n}} z_C < \mu < \bar{X}_n + \frac{\sigma}{\sqrt{n}} z_C\right)$$

i.e., with probability 0.95 the population mean  $\mu$  lies inside

$$\left(\bar{X}_n - \frac{\sigma}{\sqrt{n}} z_C, \bar{X}_n + \frac{\sigma}{\sqrt{n}} z_C\right)$$

*Confidence Interval* for confidence  $C$   
and given normal sample  
(known variance case)

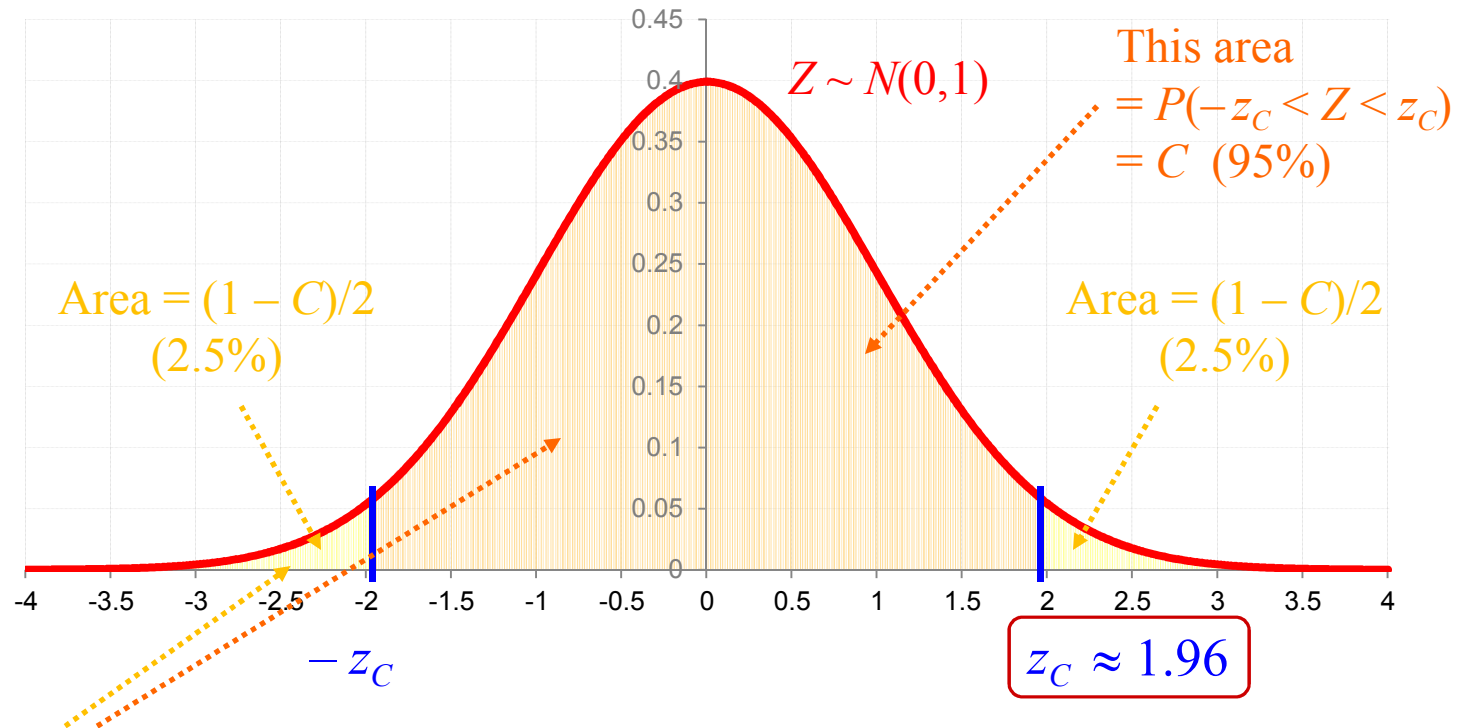




## Question: how do we compute $z_C$ ?

Given a confidence level  $C$  (e.g.,  $C = 95\% = 0.95$ ),  
find the point  $z_C$  on the  $x$ -axis such that  $P(-z_C < Z < z_C) = C$ .

Idea:



These two areas =  $P(Z < z_C) = F(z_C) = (1 - C)/2 + C = 2.5\% + 95\% = 97.5\%$

$$F(z_C) = 0.975 \Leftrightarrow z_C = F^{-1}(0.975) = \text{qnorm}(0.975) \approx 1.95996398454$$

Note: In order to compute  $z_C$  all we used was that  $N(0,1)$  is a symmetric distribution.



## Example

Numbers 7.85, 6.7, 9.83, 4, 8.61, 6.05, 5.39, 7.82, 5.63, 7.07, 11.2, and 6.8 are drawn from  $N(\mu, 4)$ . Calculate confidence interval for confidence level of 90%.

Solution:  $n = 12$  and  $\sigma^2 = 4$ . Compute the sample mean:

$$\bar{X}_n = \frac{7.85 + 6.7 + \dots + 6.8}{12} \approx 7.245833$$

For a conf. level  $C = 90\% = 0.9$ ,  $z_C = z_{0.9} = \text{qnorm}(0.05 + 0.9) \approx 1.644854$

$$\Rightarrow \frac{\sigma}{\sqrt{n}} \cdot z_C = \frac{2}{\sqrt{12}} \cdot z_{0.9} \approx 0.9496567$$

Confidence interval:  $\left( \bar{X}_n - \frac{\sigma}{\sqrt{n}} z_C, \bar{X}_n + \frac{\sigma}{\sqrt{n}} z_C \right)$

$$\approx (7.24583 - 0.9496567, 7.24583 + 0.9496567) = (6.296177, 8.195490)$$

Conclusion: with probability 0.9 the population mean  $\mu$  is inside the interval (6.296177, 8.19549).



# Confidence intervals: unknown $\sigma$

If  $X_1, X_2, \dots, X_n$  is a sample from  $N(\mu, \sigma^2)$  and  $\sigma$  is NOT known then

$$T = \frac{\bar{X}_n - \mu}{\frac{S_n}{\sqrt{n}}} \sim t(n-1) \quad \text{Student } t \text{ distributions are symmetric!}$$

Given a confidence level  $C$  (for example  $C = 95\% = 0.95$ ), find the point  $t_C = t_{0.95}$  on the  $x$ -axis such that  $P(-t_C < T < t_C) = 0.95$ .

$$\text{As before: } 0.95 = P(-t_C < T < t_C) = P\left(\bar{X}_n - \frac{S_n}{\sqrt{n}} t_C < \mu < \bar{X}_n + \frac{S_n}{\sqrt{n}} t_C\right)$$

i.e., with probability 0.95 the population mean  $\mu$  lies inside

$$\left( \bar{X}_n - \frac{S_n}{\sqrt{n}} t_C, \bar{X}_n + \frac{S_n}{\sqrt{n}} t_C \right)$$

*Confidence Interval for confidence  $C$  and given normal sample (unknown variance case)*

Note: Unlike the known variance case, the lengths of the confidence intervals vary, since the *Sample Standard Deviation*  $S_n$  (unlike population standard deviation  $\sigma$ ) depends on the sample.



## Confidence intervals: unknown $\sigma$

Note: If you sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  independently thousand times, you get thousand open intervals centered at different points but this time *their lengths vary*, since the *Sample Standard Deviation*  $S_n$  depends on the sample.

Population mean  $\mu$  will be inside approximately 950 of those intervals, and outside of approximately 50 of them.

To compute  $t_c$  we use the CDF of the *Student*  $t(n-1)$  distribution.

For example, if the sample is of the size  $n = 30$ , then

$$F(t_c) = 0.975 \Leftrightarrow t_c = F^{-1}(0.975) = \text{qt}(0.975, 29) \approx 2.04522964213$$



## Example

Numbers 7.85, 6.7, 9.83, 4, 8.61, 6.05, 5.39, 7.82, 5.63, 7.07, 11.2, and 6.8 are drawn from  $N(\mu, \sigma^2)$ . Calculate confidence interval for confidence level of 90%.

Solution:  $n = 12$ . Compute the sample mean and sample variance

$$\bar{X}_n = \frac{7.85 + 6.7 + \dots + 6.8}{12} \approx 7.245833$$

$$S_n^2 = \frac{(7.85 - \bar{X}_n)^2 + (6.8 - \bar{X}_n)^2 + \dots + (6.7 - \bar{X}_n)^2}{12 - 1} \approx 3.947008$$

For a conf. level  $C = 90\% = 0.9$ ,  $t_C = t_{0.9} = \text{qt}(0.05 + 0.9, 11) \approx 1.795885$

$$\Rightarrow \frac{S_n}{\sqrt{n}} \cdot t_C \approx \frac{\sqrt{3.947008}}{\sqrt{12}} \cdot t_{0.9} \approx 1.029964$$

Confidence interval:  $\left( \bar{X}_n - \frac{S_n}{\sqrt{n}} t_C, \bar{X}_n + \frac{S_n}{\sqrt{n}} t_C \right)$

$$\approx (7.24583 - 1.029964, 7.24583 + 1.029964) = (6.21587, 8.275797)$$

Conclusion: with probability 0.9 the population mean  $\mu$  is inside the interval  $(6.21587, 8.275797)$ .



# Hypothesis testing (intro)

A botanist is trying to determine whether an experimental growth substance applied to seeds affected the average height of plants. The standard average height is 306.9 mm. The botanist treated a random sample of 420 seeds with the extract and subsequently obtained the height data.

She decides to make a conclusion in the following way:

- If the height population mean is inside 95% confidence interval generated by her sample she will conclude that the substance did not affect the growth.
- Otherwise, she will conclude the substance affected the growth.

The 95% confidence interval is (306.94, 307.53). Consequently the botanist concludes that the substance affected the plant growth.

Another take: A botanist makes a

*hypothesis:  $\mu = 306.9$*

and either accepts it, or rejects it – in favor of the opposite (*alternative*):

*alternative hypothesis:  $\mu \neq 306.9$*

In this context  $\mu$  is the (height) population mean of the treated plant population, while  $\mu_0 = 306.9$  is its *hypothesized value*. Notice that this value is likely a ‘commonly known’ fact among the botanists.



# (Classical) Two-tailed t-test

Suppose we have a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  and the population parameters are not known. We make a hypothesis

$$H_0: \mu = \mu_0, \text{ where } \mu_0 \text{ is our guess}^1.$$

This is called a **null hypothesis**. It could be an accepted fact based on past experience or it may represent a theory that has been put forward because it is believed to be true.

Its counterpart is the **alternative hypothesis**  $H_a: \mu \neq \mu_0$

For example, in the botanist example the null hypothesis was that the substance did not affect the plant growth:

$H_0$ : *there is no difference* between the mean height of treated and untreated plant populations,

$H_a$ : *there is a difference* between the two.

A null hypothesis is either *accepted* or *rejected*. How?

---

<sup>1</sup> as seen in botanist example, a guess could be a commonly known or accepted fact.



# Two-tailed t-test (cont.)

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, compute the  $T$  statistic

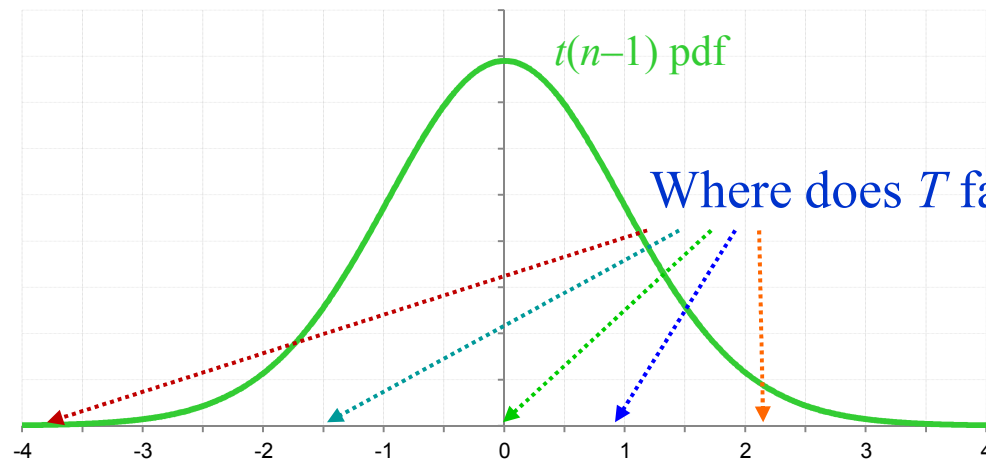
$$T = \frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}}.$$

has  $t(n-1)$  distribution

Notice: 
$$T = \frac{\bar{X}_n - \mu + \mu - \mu_0}{\frac{S_n}{\sqrt{n}}} = \frac{\bar{X}_n - \mu}{\frac{S_n}{\sqrt{n}}} + \frac{\mu - \mu_0}{\frac{S_n}{\sqrt{n}}} \sim t(n-1) + \frac{\sqrt{n}}{S_n}(\mu - \mu_0)$$

If  $H_0: \mu = \mu_0$  is true, then  $T \sim t(n-1)$ . Thus it is reasonable to establish where this numeric value resides relative to the pdf of  $t(n-1)$  distribution:

Idea: If  $T$  came from  $t(n-1)$  distribution, then it is more likely that it will be closer to zero, i.e., the mean of  $t(n-1)$ .





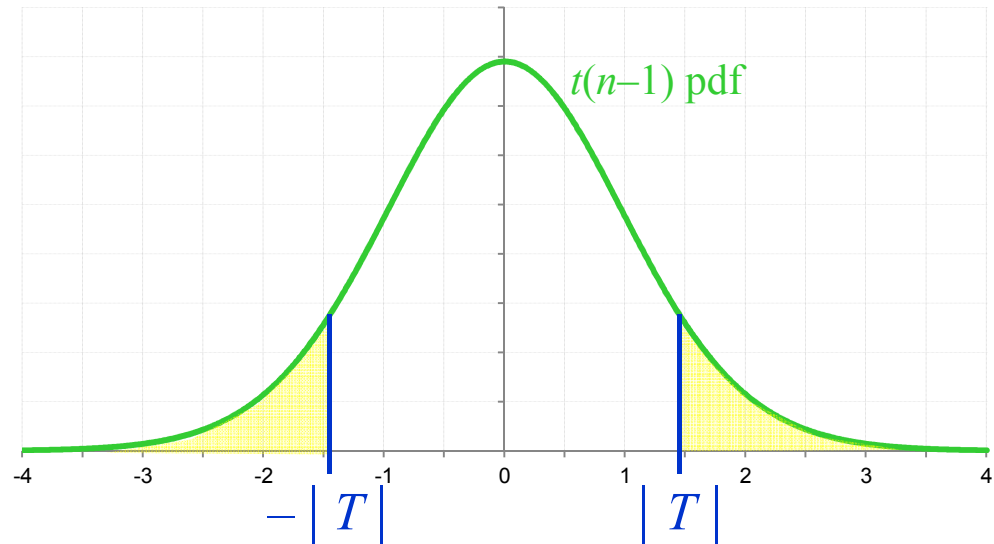


## Two-tailed t-test (cont.)

Idea: ‘measure’ how likely is that the  $T$  value we computed is that much away from zero.

Note: Assume  $\alpha = 0.05$ ; if we have 1000 samples from the normal distribution with population mean  $\mu_0$ , for  $\sim 950$  of those samples null hypothesis will be accepted.

However for  $\sim 50$  *it will be rejected*, despite the fact that the null hypothesis is correct!



That measure is exactly the sum of two yellow areas:

$$P(t(n-1) \leq -|T|) + P(t(n-1) \geq |T|) = 2 P(t(n-1) \leq -|T|)$$

This is called the *p value*. If the  $p$  value is ‘large’, it is more likely that the  $T$  statistic value came from the  $t(n-1)$  distribution, i.e., more likely that  $H_0$  is true.

Thus if  $p$  value is larger than some agreed *significance level*  $\alpha$  we *accept the null hypothesis*  $H_0$ . Typically  $\alpha = 0.05$  (5%) or  $0.01$  (1%).



## Summary: Two-tailed t-test

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, formulate the hypotheses

$$H_0: \mu = \mu_0$$

$$H_a: \mu \neq \mu_0$$

and compute:

$$T \text{ statistic} \quad T = \frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}}$$

$$p \text{ value} \quad P(t(n-1) \leq -|T|) + P(t(n-1) \geq |T|)$$

$$= \text{pt}(-\text{abs}(T), n-1) + 1 - \text{pt}(\text{abs}(T), n-1)$$

Given the agreed significance level  $\alpha$ ,

if  $p \text{ value} > \alpha$ , *accept the null hypothesis!*

if  $p \text{ value} \leq \alpha$ , *reject the null hypothesis in favor of the alternative!*



## Example

Numbers 7.85, 6.7, 9.83, 4, 8.61, 6.05, 5.39, 7.82, 5.63, 7.07, 11.2, and 6.8 are drawn from  $N(\mu, \sigma^2)$ . Test the null hypothesis  $\mu = 7$  for 10% significance level.

$$\text{Solution: } \bar{X}_n = \frac{7.85 + 6.7 + \dots + 6.8}{12} \approx 7.245833$$

$$S_n^2 = \frac{(7.85 - \bar{X}_n)^2 + (6.8 - \bar{X}_n)^2 + \dots + (6.7 - \bar{X}_n)^2}{12 - 1} \approx 3.947008$$

$$T = \frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}} \approx \frac{7.245833 - 7}{\sqrt{\frac{3.947008}{12}}} \approx 0.4286446$$

$$p \text{ value} = \text{pt}(-T, n-1) + 1 - \text{pt}(T, n-1)$$

$$\approx \text{pt}(-0.4286446, 11) + 1 - \text{pt}(0.4286446, 11)$$

$$\approx 0.6764553 > 0.1$$

Conclusion: At 10% significance accept the null hypothesis  $\mu = 7$ .



# Two-tailed t-test and Confidence Interval

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, then

$$\frac{\bar{X}_n - \mu}{\frac{S_n}{\sqrt{n}}} \sim t(n-1).$$

For confidence  $C = 0.95$  find the point  $t_{0.95}$  on the  $x$ -axis such that

$$P(-t_{0.95} < t(n-1) < t_{0.95}) = P(|t(n-1)| < t_{0.95}) = 0.95.$$

Then the Confidence Interval for the population mean  $\mu$  is

$$\left( \bar{X}_n - \frac{S_n}{\sqrt{n}} t_{0.95}, \bar{X}_n + \frac{S_n}{\sqrt{n}} t_{0.95} \right), \text{ i.e.,}$$

$$P\left( \bar{X}_n - \frac{S_n}{\sqrt{n}} t_{0.95} < \mu < \bar{X}_n + \frac{S_n}{\sqrt{n}} t_{0.95} \right) = 0.95.$$

Take  $\mu_0$  from this interval and formulate the null hypothesis  $H_0: \mu = \mu_0$ , then

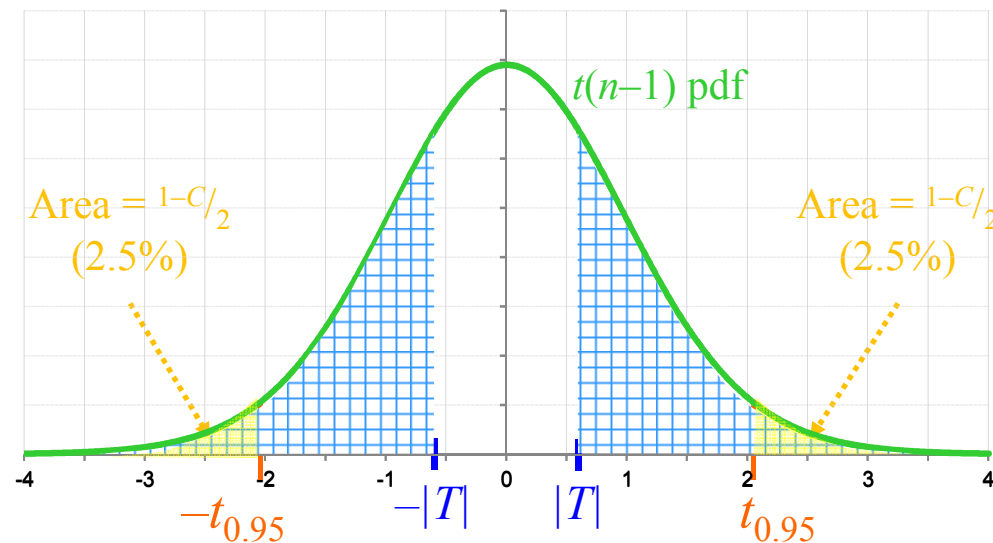
$$\bar{X}_n - \frac{S_n}{\sqrt{n}} t_{0.95} < \mu_0 < \bar{X}_n + \frac{S_n}{\sqrt{n}} t_{0.95} \iff -t_{0.95} < \boxed{\frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}}} < t_{0.95}$$

‘our’  $T$  statistic



# Two-tailed t-test and Confidence Interval

Recap: If  $\mu_0$  is from the confidence interval then  $-t_{0.95} < T < t_{0.95}$ .



But then the  $p$  value is certainly larger than  $0.05 = 1 - 0.95$ .

Thus if we take any  $\mu_0$  from the confidence interval generated by confidence level  $C$ , the null hypothesis will be accepted at significance level  $\alpha = 1 - C$ .

One can easily show the converse: if the null hypothesis  $H_0: \mu = \mu_0$  is accepted at significance level  $\alpha$  then  $\mu_0$  will be in the confidence interval generated by the confidence level  $C = 1 - \alpha$ .



# Alternative formulation for two-tailed t-test

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, formulate the hypotheses

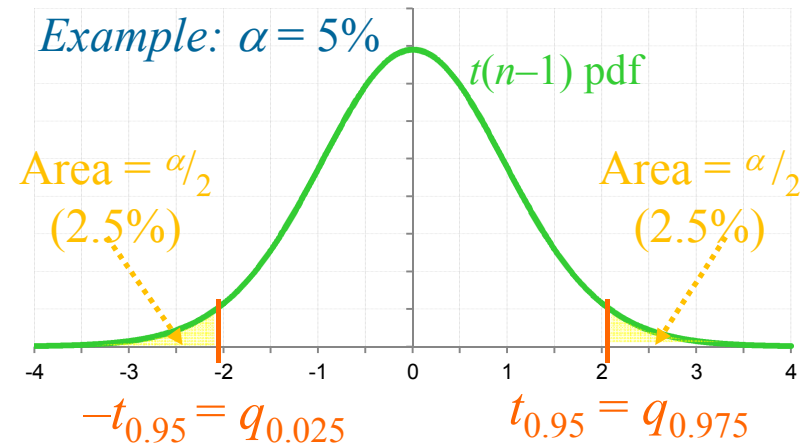
$$H_0: \mu = \mu_0 \quad H_a: \mu \neq \mu_0$$

and compute

$$T \text{ statistic } T = \frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}}$$

For significance level  $\alpha$  compute

$$t_{1-\alpha} = \text{qt}(1 - \alpha/2, n - 1)$$



Then:

If  $|T| < t_{1-\alpha}$ , **accept**  $H_0$   $(-t_{1-\alpha} < T < t_{1-\alpha})$

If  $|T| \geq t_{1-\alpha}$ , **reject**  $H_0$  **in favor of**  $H_a$   $(T \leq -t_{1-\alpha} \text{ or } T \geq t_{1-\alpha})$

Note: This formulation ignores the  $p$  value altogether.



## Upper-tailed $t$ -test (motivation)

*Salinity* is a measure of the concentration of dissolved salts in water, commonly defined in “grams of salt in 1 kilogram of water.” An environmentalist is collecting samples on the lake near the ocean. A recent hurricane caused the high tide and it is necessary to determine whether the salt content in the lake has increased significantly (with significance of 5% in the usual statistical testing sense). In the process 58 samples are taken and their salinity is measured. Use the  $t$ -test to determine whether the water salinity has increased significantly if the normal salinity of the lake is 0.5.

Note: the salinity of the ocean water is typically around 35.

Notice that the statement of the problem indicates that the salinity, had it changed, **should have increased**. The goal of the environmentalist is the same:  
to show whether the salinity **truly increased** or **not**!

The two-tailed test with  $H_0: \mu = 0.5$  and  $H_a: \mu \neq 0.5$  can only try to answer whether the salinity changed or not.

If we try to set the test with  $H_0: \mu \geq 0.5$  and  $H_a: \mu < 0.5$ , if the null-hypothesis is accepted it still includes the possibility of salinity staying the same (‘=’ is a part of ‘ $\geq$ ’)

The ‘test’ we need is:  $H_0: \mu \leq 0.5$  and  $H_a: \mu > 0.5$ .

The environmentalist will be convinced if the null-hypothesis is rejected.



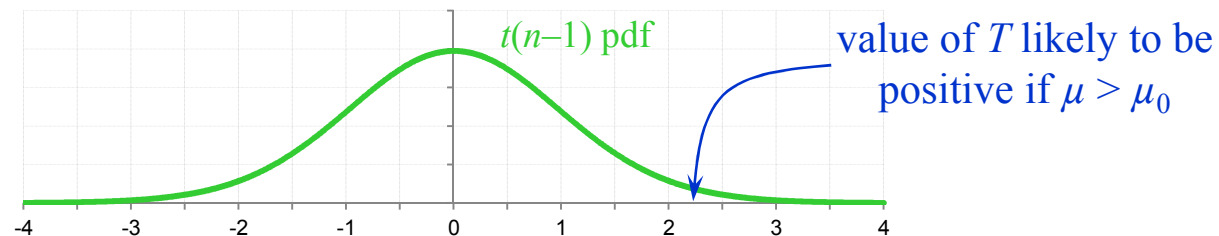
## Upper-tailed t-test (math)

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, recall that  $T$  has  $t(n-1)$  distribution

$$T = \frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}} = \frac{\bar{X}_n - \mu + \mu - \mu_0}{\frac{S_n}{\sqrt{n}}} = \underbrace{\frac{\bar{X}_n - \mu}{\frac{S_n}{\sqrt{n}}}}_{\sim t(n-1)} + \frac{\mu - \mu_0}{\frac{S_n}{\sqrt{n}}} \sim t(n-1) + \underbrace{\frac{\sqrt{n}}{S_n}}_{> 0} (\mu - \mu_0)$$

for any ‘estimate’  $\mu_0$  of the population mean  $\mu$ .

Clearly if  $\mu = \mu_0$  the  $T$  statistic value will have  $t(n-1)$  distribution. But if  $\mu > \mu_0$  its value will (tend to) be positive, since the second term is positive.



Accordingly, we construct the test with the following hypotheses:

$$H_0: \mu \leq \mu_0 \qquad H_a: \mu > \mu_0$$

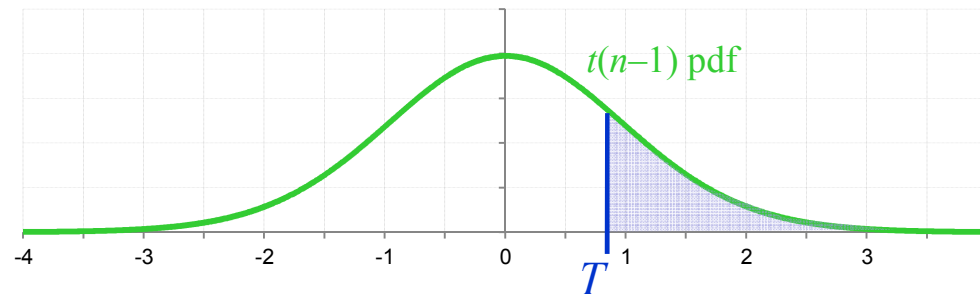
Note: If you intend to show  $\mu > \mu_0$  you construct a test whose *alternative hypothesis* is exactly that!





## Upper-tailed t-test

Back to the salinity example: the environmentalist is only interested in finding that the (new) population mean is truly larger than  $\mu_0 = 0.5$  g/l. From the data the  $T$  statistic value is computed and this value is placed ‘under’ the  $t(n-1)$  pdf.



If the  $T$  value happens to be negative or positive but ‘not too large’, there is no justification to the belief that  $\mu$  is truly larger than  $\mu_0 = 0.5$ .

In other words, we should be accepting the null hypothesis  $H_0: \mu \leq \mu_0$ .

But how do we quantify ‘too large’? We can measure the area to the right of  $T$  (‘upper tail’ of the distribution). That area is the upper-tailed test  $p$  value.

$$p \text{ value: } P(t(n-1) \geq T) = 1 - pt(T, n-1)$$

Given the agreed significance level  $\alpha$ ,

if  $p \text{ value} > \alpha$ , accept  $H_0: \mu \leq \mu_0$  (effectively  $\mu = \mu_0$ )

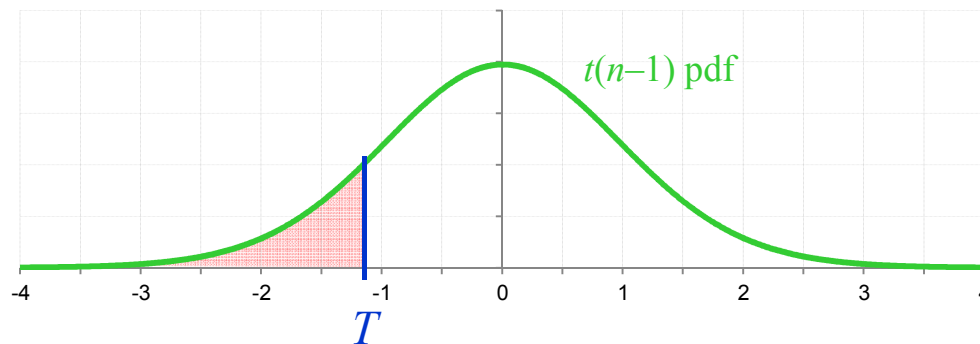
if  $p \text{ value} \leq \alpha$ , reject  $H_0$  in favor of  $H_a: \mu > \mu_0$



## Lower-tailed t-test

Following the same idea: we are trying to establish that the population mean is smaller than some value. E.g., “the average height ... is less than 71 inches.”

Accordingly we construct the test:  $H_0: \mu \geq \mu_0$      $H_a: \mu < \mu_0$



If the  $T$  value happens to be positive or negative but ‘not too large’, we have no reason to doubt that the population mean  $\mu$  is truly not smaller than  $\mu_0$ .

We quantify ‘too large negative’ as the area to the left of  $T$  (‘lower tail’ of the distribution), which is the lower-tailed test  $p$  value:

$$p \text{ value: } P(t(n-1) \leq T) = pt(T, n-1)$$

Given the agreed significance level  $\alpha$ ,

if  $p \text{ value} > \alpha$ , accept  $H_0: \mu \geq \mu_0$  (effectively  $\mu = \mu_0$ )

if  $p \text{ value} \leq \alpha$ , reject  $H_0$  in favor of  $H_a: \mu < \mu_0$



# Lower-tailed t-test alternative formulation

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, formulate the hypotheses

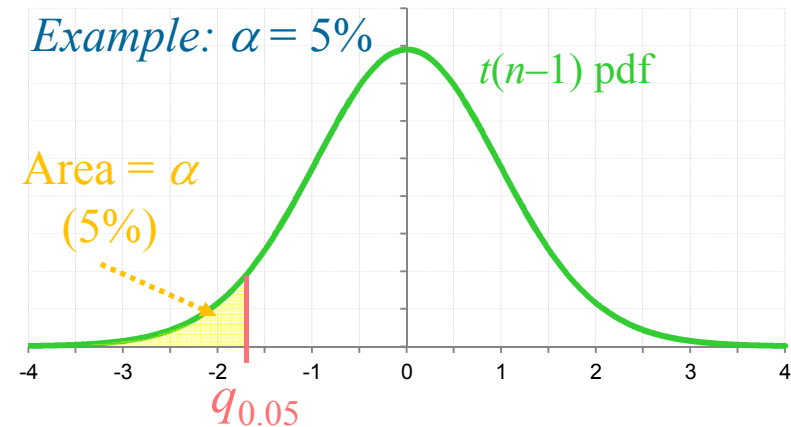
$$H_0: \mu \geq \mu_0 \quad H_a: \mu < \mu_0$$

and compute  $T$  statistic

$$T = \frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}}$$

For significance level  $\alpha$  compute

$$q_\alpha = \text{qt}(\alpha, n - 1)$$



If  $T > q_\alpha$ , accept  $H_0: \mu \geq \mu_0$  (effectively  $\mu = \mu_0$ )

Then:

If  $T \leq q_\alpha$ , reject  $H_0$  in favor of  $H_a: \mu < \mu_0$

Note: This formulation ignores the  $p$  value.



# Upper-tailed t-test alternative formulation

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, formulate the hypotheses

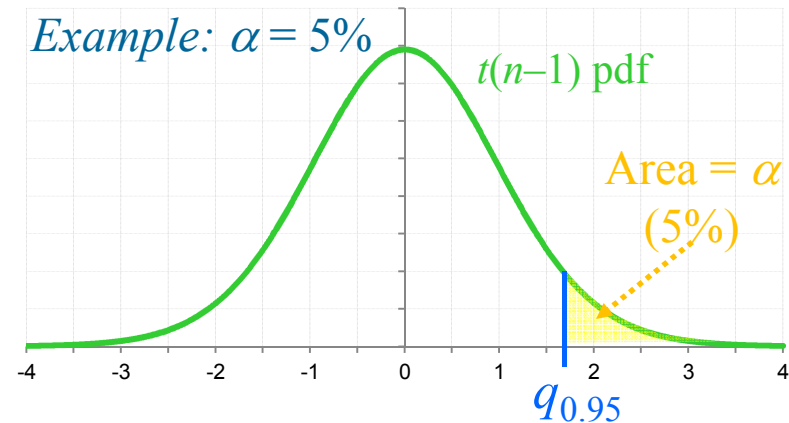
$$H_0: \mu \leq \mu_0 \quad H_a: \mu > \mu_0$$

and compute  $T$  statistic

$$T = \frac{\bar{X}_n - \mu_0}{\frac{S_n}{\sqrt{n}}}$$

For significance level  $\alpha$  compute

$$q_{1-\alpha} = \text{qt}(1 - \alpha, n - 1)$$



If  $T < q_{1-\alpha}$ , accept  $H_0: \mu \leq \mu_0$  (effectively  $\mu = \mu_0$ )

Then:

If  $T \geq q_{1-\alpha}$ , reject  $H_0$  in favor of  $H_a: \mu > \mu_0$

Note: This formulation ignores the  $p$  value.



## Two-tailed z-test

Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population mean and a known population variance  $\sigma^2$ , formulate the hypotheses

$$H_0: \mu = \mu_0 \quad H_a: \mu \neq \mu_0$$

and compute:

$$\text{Z statistic} \quad Z = \frac{\bar{X}_n - \mu_0}{\frac{\sigma}{\sqrt{n}}} \quad \text{Recall: If } H_0 \text{ is true, then } Z \sim N(0,1)$$

$$\begin{aligned} p \text{ value} \quad & P(N(0,1) \leq -|Z|) + P(N(0,1) \geq |Z|) \\ & = \text{pnorm}(-\text{abs}(Z)) + 1 - \text{pnorm}(\text{abs}(Z)) \end{aligned}$$

Given the agreed significance level  $\alpha$ ,

if  $p \text{ value} > \alpha$ , accept  $H_0$

if  $p \text{ value} \leq \alpha$ , reject  $H_0$  in favor of  $H_a$

Derive on your own the one-tailed versions of the z-test, as well as the alternative formulations analogous to those given for the  $t$ -tests.

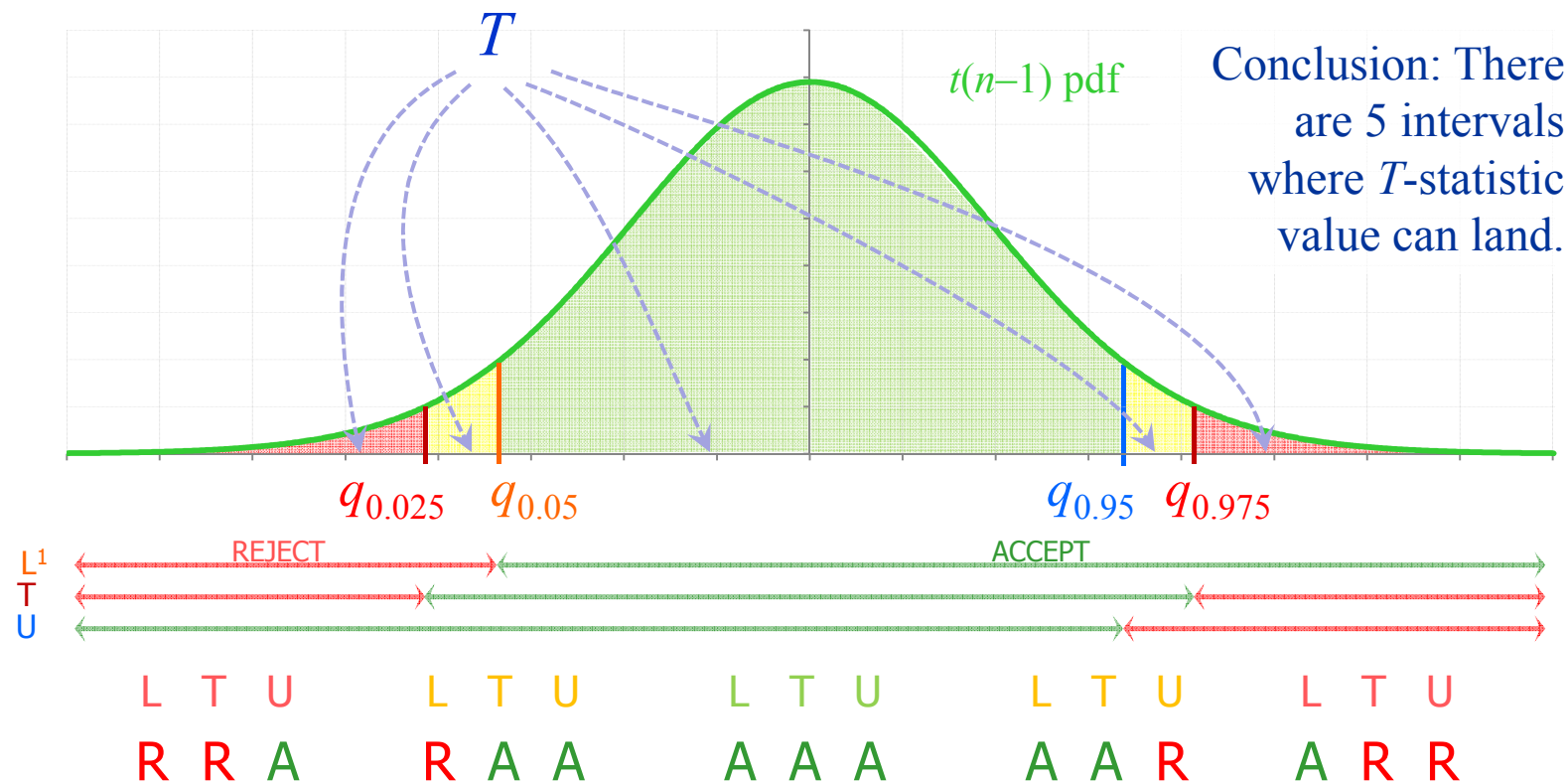


# Three tests combined

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Given a sample  $X_1, X_2, \dots, X_n$  from  $N(\mu, \sigma^2)$  with unknown population parameters, what are the possible outcomes of all three  $t$ -tests if we run them simultaneously? Namely, we compute  $T$  statistic based on assumed  $\mu_0$  and the sample and use the same significance level  $\alpha$  (for simplicity assume  $\alpha = 0.05$ ).



<sup>1</sup> L: Lower-tailed test      T: Two-tailed test      U: Upper-tailed test



## Example

*Salinity* is a measure of the concentration of dissolved salts in water, commonly defined in “grams of salt in 1 kilogram of water.” An environmentalist is collecting samples on the lake near the ocean. A recent hurricane caused the high tide and it is necessary to determine whether the salt content in the lake has increased significantly (with significance of 5% in the usual statistical testing sense). In the process 58 samples are taken and their salinity is measured. Use the *t-test* to determine whether the water salinity has increased significantly if the normal salinity of the lake is 0.5.

Notice that the statement of the problem indicates that the salinity, had it changed, should have increased. The goal of the environmentalist is the same: to show whether the salinity increased or not!

Thus we apply the upper-tailed test with  $H_0: \mu \leq 0.5$  and  $H_a: \mu > 0.5$ .

```
R output:  t.test(u$Salinity, mu=0.5, alternative="greater")

           One Sample t-test

data:  u$Salinity
t = 1.7207, df = 57, p-value = 0.04537
alternative hypothesis: true mean is greater than 0.5
95 percent confidence interval:  0.5007201  Inf
sample estimates:
mean of x:  0.5254655
```

Conclusion: Reject the null-hypothesis in favor of the alternative:  $\mu > 0.5$

Answer: the salinity **has increased!**



# Question: what would two-tailed test yield?

The two-tailed test, however, accepts its null-hypothesis:

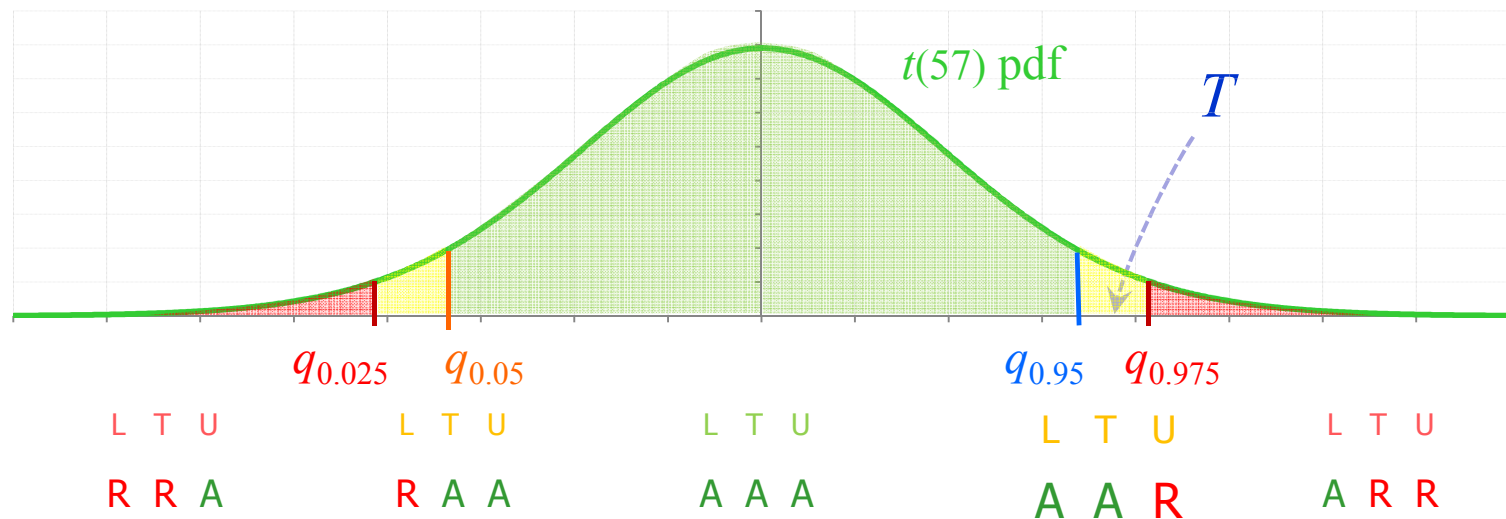
$$H_0: \mu = 0.5 \quad H_a: \mu \neq 0.5$$

R output: `t.test(u$Salinity, mu=0.5)`  
`t = 1.7207, df = 57, p-value = 0.09073`  
`alternative hypothesis: true mean is not equal to 0.5`  
`95 percent confidence interval: 0.4958298 0.5551013`

Conclusion: Accept the null-hypothesis at 5% significance:  $\mu = 0.5$

**Answer: the salinity has not changed!**

Recall:







# The botanist example revisited

A botanist is trying to determine whether an experimental growth substance applied to seeds affected the average height of plants. The standard average height is 306.9 mm. The botanist treated a random sample of 420 seeds with the extract and subsequently obtained the height data.

Lets apply three t-tests at 5% significance level:

The botanist is secretly hoping to show that the substance increases plants' heights, thus we first apply the upper-tailed test with

$$H_0: \mu \leq 306.9 \text{ and } H_a: \mu > 306.9.$$

R output: `t.test(ph, mu=306.9, alternative="greater")`

One Sample t-test

```
data: ph
t = 2.2557, df = 419, p-value = 0.0123
alternative hypothesis: true mean is greater than 306.9
95 percent confidence interval: 306.9911 Inf
sample estimates:
mean of x: 307.2383
```

Conclusion: At 5% significance the botanist rejects the null-hypothesis in favor of the alternative:  $\mu > 306.9$

The substance helps the plants grow!



# Check

Dig in a bit more: The two-tailed test rejects its null-hypothesis as well:

$$H_0: \mu = 306.9 \quad H_a: \mu \neq 306.9$$

R output: `t.test(ph, mu=306.9)`  
...  
`t = 2.2557, df = 419, p-value = 0.0246`  
`alternative hypothesis: true mean is not equal to 306.9`  
`95 percent confidence interval: 306.9435 307.5332`

Conclusion: Reject the null-hypothesis at 5% significance.

The substance has effect on the plant growth!

Finally, the lower-tailed test accepts its null-hypothesis:

R output: `t.test(ph, mu=306.9, alternative = "less")`  
...  
`t = 2.2557, df = 419, p-value = 0.9877`  
`alternative hypothesis: true mean is less than 306.9`  
`95 percent confidence interval: -Inf 307.4856`



## Rule of thumb

Two-tailed test should be used whenever our intention is to establish (within given significance level) that the population mean is equal to some hypothesized value. The null-hypothesis is either accepted or rejected in favor the alternative: population mean is not equal to the hypothesized value.

Using two-tailed test is the cautious way to conduct the hypothesis testing.

If an additional information (physics, past experience, sound intuition, etc.) indicates that the population mean is either strictly greater or strictly less than the hypothesized value, one of the one-tailed tests can be used:

(U) If the additional information indicates that the population mean is greater than the hypothesized value, use the upper-tailed test. The null-hypothesis is either accepted (in which case we effectively conclude that population mean is equal to the hypothesized value) or rejected in favor the alternative:

population mean is **strictly greater than** the hypothesized value.

(L) If the additional information indicates that the population mean is smaller than the hypothesized value, use the lower-tailed test. The null-hypothesis is either accepted (in which case we effectively conclude that population mean is equal to the hypothesized value) or rejected in favor the alternative:

population mean is **strictly less than** the hypothesized value.



# Two-sample t-tests

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Purpose: comparison of population means of two samples.

Two distinct cases: *paired samples* and *independent samples*.

Examples: (1) Randomly select 50 women and 50 men and record the average time (per week) they spend watching TV.

(2) Randomly select 50 couples and record the average time (per week) the wives spend watching TV, and then the same data for the husbands.

In a paired sample each entity is measured twice, resulting in pairs of observations. Common applications include *case-control studies*: a famous one was the demonstration of the link between tobacco smoking and lung cancer.

The tests are fundamentally different for these two cases. For example, for paired samples clearly the sample sizes must be equal. For independent samples this is not the requirement.



# Paired samples tests

**Paired samples:** The sample selected from the first population is *related* to the corresponding sample from the second population.

Idea: Given two samples,  $X_1, X_2, \dots, X_n$  and  $Y_1, Y_2, \dots, Y_n$ , consider their differences

$$D_1 = X_1 - Y_1, \quad D_2 = X_2 - Y_2, \quad \dots, \quad D_n = X_n - Y_n.$$

Then  $D_1, D_2, \dots, D_n$  is a sample (they are independent r.v.'s) from some distribution with population mean  $\mu$  and variance  $\sigma^2$ . If that happens to be  $N(\mu, \sigma^2)$ , or if the sample size is large enough, we can use any classical (one-sample) test; if  $H_0: \mu_1 = \mu_2$  is true, then

$$Z = \frac{\overline{D}_n}{\frac{\sigma}{\sqrt{n}}} \sim N(0,1) \quad \text{and} \quad T = \frac{\overline{D}_n}{\frac{S_n}{\sqrt{n}}} \sim t(n-1)$$

( $S_n$  is the sample standard deviation of the 'difference sample')

Note: The only assumption we need is that the differences are normally distributed, or that the samples are large enough (CLT).



# Independent samples tests

**Independent samples:** The sample from the first population is *independent* of the sample from the second population.

Idea: if we have two samples,  $X_1, X_2, \dots, X_n$  from  $N(\mu_1, \sigma_1^2)$  and  $Y_1, Y_2, \dots, Y_m$  from  $N(\mu_2, \sigma_2^2)$ , then

$$\bar{X}_n \sim N\left(\mu_1, \frac{\sigma_1^2}{n}\right) \quad \text{and} \quad \bar{Y}_m \sim N\left(\mu_2, \frac{\sigma_2^2}{m}\right)$$

Note: By the CLT, the normality of the samples is not required if both samples are large enough.

and furthermore they are independent, thus their difference

$$\bar{X}_n - \bar{Y}_m \sim N\left(\mu_1 - \mu_2, \frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}\right).$$

Consequently if the population variances are known, the z-test with the null hypothesis  $H_0: \mu_1 = \mu_2$  and the Z statistic

$$Z = \frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{\sigma_1^2}{n} + \frac{\sigma_2^2}{m}}} \quad \text{is straightforward.} \quad (\text{and both one-tailed z-tests as well})$$



# Independent samples t-test (cont)

**Problem:** Unknown population variances.

Recall: Classical  $t$ -test was based on the theoretical result: for a normal sample (or, by CLT, for sample large enough),

$$\frac{\bar{X}_n - \mu}{\frac{S_n}{\sqrt{n}}} \sim t(n-1).$$

Similar result for differences of two independent sample means?

Somewhat: there are two subcases to consider:

**Pooled Variances:** Sample sizes are “nearly equal” and neither sample standard deviation is more than twice the other, i.e.,

$$\frac{1}{2} \leq \frac{S_1}{S_2} \leq 2.$$

**Separate Variances:** All other cases (*Welch test*).



# Independent samples t-test (cont)

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**Pooled Variances:** If the null hypothesis  $H_0: \mu_1 = \mu_2$  is true, then

$$\frac{\bar{X}_n - \bar{Y}_m}{S_p \sqrt{\frac{1}{n} + \frac{1}{m}}} \sim t(n+m-2) \quad \text{with} \quad S_p = \sqrt{\frac{(n-1)S_n^2 + (m-1)S_m^2}{n+m-2}}.$$

**Separate Variances:** If the null hypothesis  $H_0: \mu_1 = \mu_2$  is true, then

$$\frac{\bar{X}_n - \bar{Y}_m}{\sqrt{\frac{S_n^2}{n} + \frac{S_m^2}{m}}} \sim t(df) \quad \text{with} \quad df = \frac{(n-1) \cdot (m-1)}{(m-1)C^2 + (1-C)^2(n-1)} \quad \& \quad C = \frac{\frac{S_n^2}{n}}{\frac{S_n^2}{n} + \frac{S_m^2}{m}}.$$

An alternate (conservative) option is to use  $df = \min(n, m) - 1$ .

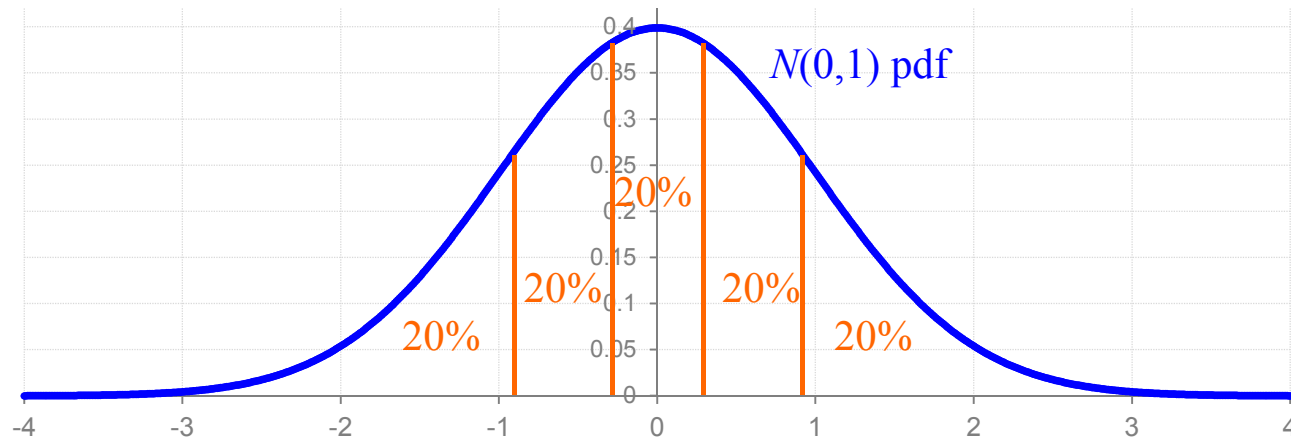
We will rely on  $R$  built in *t-test* function for applications of these tests.



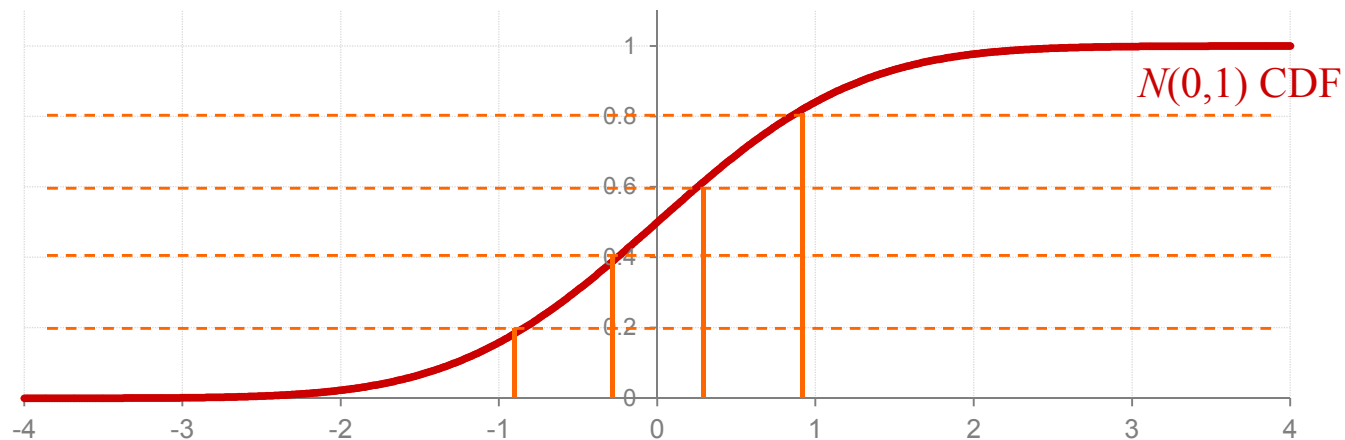


# Q: Is sample from Normal distribution?

Idea: Generate five random numbers from *Standard Normal* distribution.



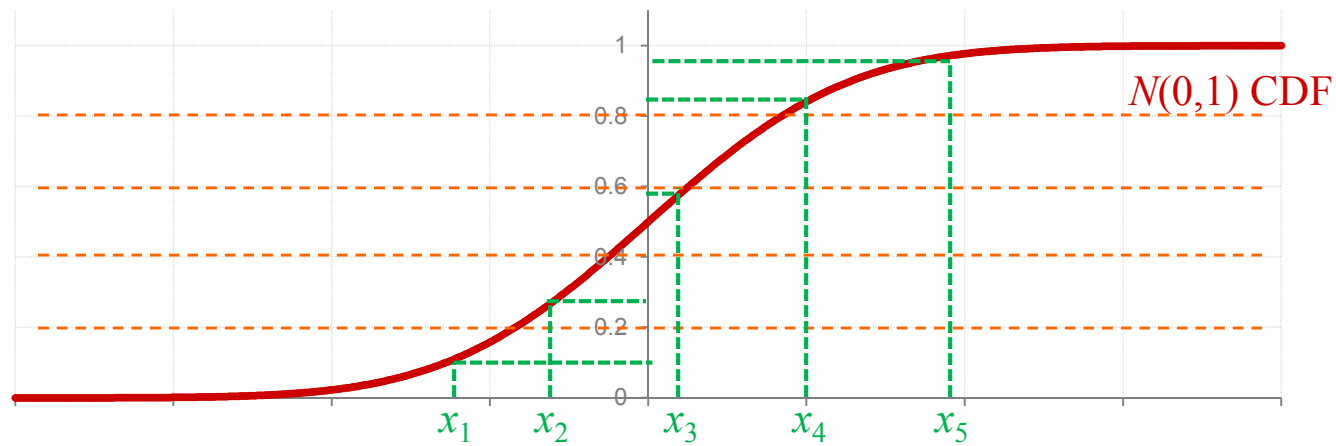
Where should they ‘fall in’? If we were to repeat generation many times, they should “uniformly” fill the five intervals above!





# Is sample from Normal distribution? (cont)

We can sort the five random numbers and compute their CDF values.



Those values  $F(x_1), F(x_2), \dots, F(x_5)$  we can compare to the mid-points of their respective interval on the  $y$ -axis: 0.1, 0.3, 0.5, 0.7, and 0.9.

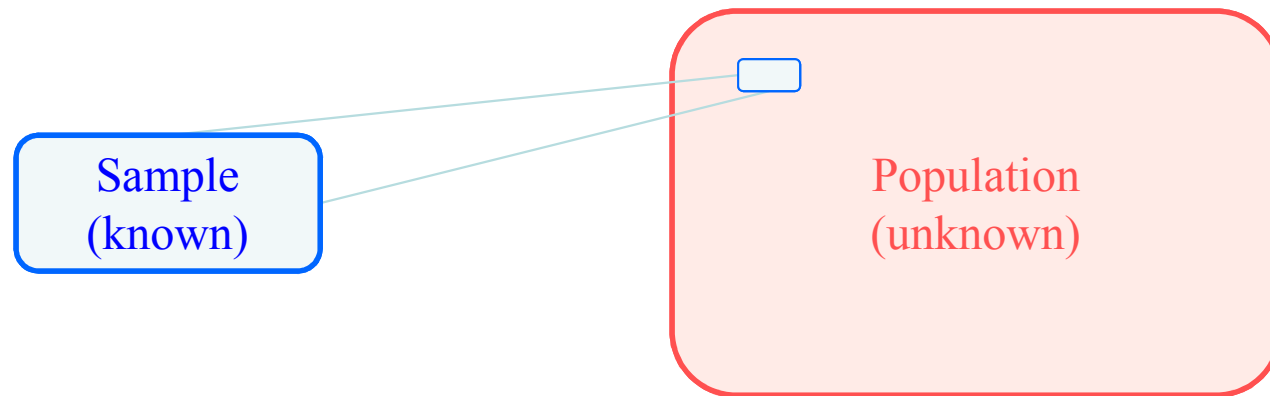
Alternatively, we can compare the original sorted values  $x_1, x_2, \dots, x_5$  to values of  $F^{-1}(0.1), F^{-1}(0.3), \dots, F^{-1}(0.9)$ , i.e., the 0.1, 0.3, ..., 0.9 quantiles.

The scatter plot of ‘theoretical’ quantiles (quantiles of mid-points) against the sorted random numbers is called the **quantile plot** (or *q-q plot*). Quantile plot is generate by the distribution; for the standard normal this is called *standard normal quantile plot*.



# Hypothesis Testing (a view from 30 thousand feet)

**Summary:** use a sample of data to draw *inferences* about the wider population from which the sample is drawn. In such settings, information about the entire population is, or may be, unknown (and possibly unknowable).



Choose a *test statistic* of interest to describe the population.

For example, if your population is the height of people you may be interested in *average height*; if it is the income you may be interested in *median income*.

Then create a *null hypothesis* (typically a conjecture, i.e., guess<sup>1</sup>) about the population: for example, “the population mean equals...” or “the population variance equals ...”

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<sup>1</sup> guess could be a commonly accepted fact: for example, the body temperature of a healthy human has a population mean of 98.6° F (37° C).



# Classical Statistical Testing

The distribution of the test statistic is known!

For example, the *Sample Mean* of the normal sample has a normal distribution with parameters that are tied to the usually unknown population parameters.

- (1) Choose the test statistic for which the distribution is known,
- (2) Create a null hypothesis and the alternative hypothesis. These are made about some characteristic of the population related to test statistic in (1),
- (3) Compute the test-statistic for the given sample and locate the value on the theoretical distribution from (1). A value in the tail would rarely occur by chance, and the null hypothesis can be *rejected* in favor of the alternative. Otherwise, there is no evidence to reject the null hypothesis – it is *accepted*.

Problem with this approach: many restrictions on *test statistic* and on *population distribution*! It was represented by the classical *t-test*: the hypothesis is about the unknown *population mean*, and the test is devised for the “transformed” *Sample Mean* of the *Normal sample* with unknown population variance.



# Statistical Testing via Sampling Distributions

Relies on computing power (“simulations”).

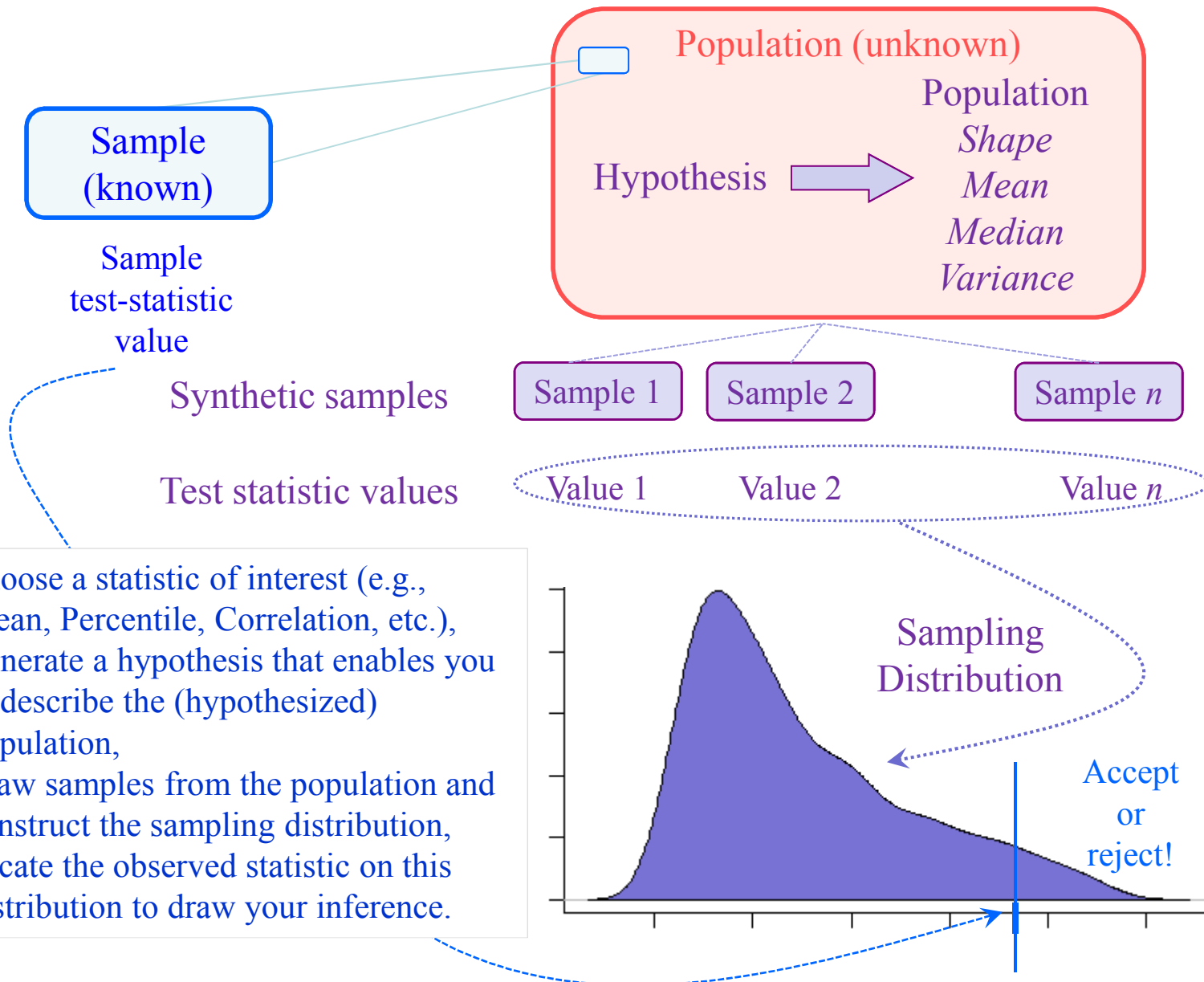
- (1) Choose a *population property* of interest (e.g., mean, variance, percentile, etc.),
- (2) Formulate a null hypothesis about the population property,
- (3) Assume population has some parametric distribution and draw synthetic samples from that distribution OR draw synthetic samples by *resampling* from the obtained data (i.e., sample).
- (4) Compute the *test statistic* value (that corresponds to population property of interest) for each of the synthetic samples drawn and create a distribution of these values. This is usually called the *sampling distribution*.
- (5) Compute the test statistic for the obtained data and locate the value on the sampling distribution. A value in the tail would rarely occur by chance and the null hypothesis should be rejected. Otherwise, it should be accepted.

With the computing power of  $R$  you can construct sampling distribution for the test that you are interested in conducting. This approach of creating your own sampling distributions is termed *permutation* (or *randomization*) *testing*.

Instead of comparing the actual value of a test statistic to a known distribution (which works under many restrictions), the sampling distribution is generated from the data.



# Statistical Testing via Sampling Distributions





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Optional



## Proof of $E(S^2) = \sigma^2$

*Sample Variance* of a sample  $X_1, X_2, \dots, X_n$  from a distribution  $F$  is an unbiased estimator of the population variance  $\sigma^2$ .

$$E(S_n^2) = E\left(\frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2\right) = \frac{1}{n-1} \sum_{k=1}^n \boxed{E(X_k - \bar{X}_n)^2} \quad \text{take an arbitrary } k$$

$$\begin{aligned} E(X_k - \bar{X}_n)^2 &= E(X_k - \mu + \mu - \bar{X}_n)^2 = E((X_k - \mu) - (\bar{X}_n - \mu))^2 \\ &= E((X_k - \mu)^2 - 2(X_k - \mu)(\bar{X}_n - \mu) + (\bar{X}_n - \mu)^2) \\ &= \boxed{E(X_k - \mu)^2} - 2\boxed{E(X_k - \mu)(\bar{X}_n - \mu)} + \boxed{E(\bar{X}_n - \mu)^2} \end{aligned}$$

$$\begin{aligned} &= \text{Var}(X_k) = \sigma^2, \text{ since } E(X_k) = \mu &= \text{Var}(\bar{X}_n) = \frac{\sigma^2}{n}, \text{ since } E(\bar{X}_n) = \mu \end{aligned}$$

$$\begin{aligned} E(X_k - \mu)(\bar{X}_n - \mu) &= E\left((X_k - \mu)\left(\frac{1}{n} \sum_{i=1}^n X_i - \mu\right)\right) = \frac{1}{n} \sum_{i=1}^n E((X_k - \mu)(X_i - \mu)) \\ &= \frac{1}{n} \sum_{i=1}^n \boxed{\text{Cov}(X_k, X_i)} = \frac{1}{n} \text{Var}(X_k) = \frac{\sigma^2}{n} \end{aligned}$$

$\text{Cov}(X_k, X_i) = 0$  if  $i \neq k$ , since  $X_k$  and  $X_i$  are independent





## Proof cont.

Hence:  $E(X_k - \bar{X}_n)^2 = \sigma^2 - 2\frac{\sigma^2}{n} + \frac{\sigma^2}{n} = \sigma^2 - \frac{\sigma^2}{n} = \frac{n-1}{n}\sigma^2$ , for any  $k$ .

$$E(S_n^2) = \frac{1}{n-1} \sum_{k=1}^n E(X_k - \bar{X}_n)^2 = \frac{1}{\cancel{n-1}} \sum_{k=1}^n \frac{\cancel{n-1}}{n} \sigma^2 = \frac{1}{n} \sum_{k=1}^n \sigma^2 = \sigma^2.$$