ORDINARY DIFFERENTIAL EQUATIONS

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PREAMBLE

```
from numpy import *
from numpy.linalg import *
from matplotlib.pyplot import *
from mpl_toolkits.mplot3d import *
```

INTRODUCTION

VECTOR FIELD

Let $n \in \mathbb{N}^*$ and $f : \mathbb{R}^n \to \mathbb{R}^n$.

Visualize $f(x)$ as an arrow with origin the point x.

In the plane ($n = 2$), use quiver (Matplotlib).

Q HELPER

```
def Q(f, xs, ys):
    X, Y = meshgrid(xs, ys)
    fx = vectorize(lambda x, y: f([x, y])[0])
    fy = vectorize(lambda x, y: f([x, y])[1])
    return X, Y, fx(X, Y), fy(X, Y)
```

👁 VECTOR FIELD / ROTATION

Consider $f(x, y) = (-y, x)$.

```
def f(xy):
    x, y = xy
    return array([-y, x])
```

```
figure()
x = y = linspace(-1.0, 1.0, 20)
gca().set_aspect(1.0); grid(True)
quiver(*Q(f, x, y))
```

1
0
−1
−1
0
1

ORDINARY DIFFERENTIAL EQUATIONS (ODES)

A solution of $\dot{x} = f(x)$ is

- a function $x : I \to \mathbb{R}^n$,
- defined on a (possibly unbounded) interval I of $\mathbb{R}$,
- such that for every $t \in I$,

$$\dot{x}(t) = dx(t)/dt = f(x(t)).$$

```
figure()
x = y = linspace(-1.0, 1.0, 20)
gca().set_aspect(1.0); grid(True)
streamplot(*Q(f, x, y), color="k")
```

1
0
−1
−1
0
1

INITIAL VALUE PROBLEM (IVP)

Solutions $x(t)$, for $t \geq t_0$, of

$$\dot{x} = f(x)$$

such that

$$x(t_0) = x_0 \in \mathbb{R}^n.$$

The **initial condition** (t_0, x_0) is made of

- the **initial time** $t_0 \in \mathbb{R}$ and
- the **initial value** or **initial state** $x_0 \in \mathbb{R}^n$.

The set $\mathbb{R}^n$ is the **state space**,
$x(t)$ the **state at time** t.

HIGHER-ORDER ODES

Scalar differential equations structured as

$$x^{(n)}(t) = f(x, \dot{x}, \ddot{x}, \dots, x^{(n-1)})$$

can be converted to the standard form with the state

$$y = (x, \dot{x}, \ddot{x}, \dots, x^{(n-1)}) \in \mathbb{R}^{\mathbb{n}}$$

$$
\begin{aligned}
\dot{y}_1 &= y_2 \\
\dot{y}_2 &= y_3 \\
\vdots \quad & \quad \vdots \quad \vdots \\
\dot{y}_n &= f(y_1, y_2, \ldots, y_{n-1})
\end{aligned}
$$

PENDULUM

ⓘ – MODEL / PENDULUM

- [⚙, x²] Establish the equations governing the pendulum dynamics when the mechanical energy of the system is constant.
- [⚙, x²] Generalize the dynamics when there is a friction torque $c = -b\dot{\theta}$ for some $b \geq 0$.

🔑 – RESULT

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = 0$$

Introduce the rotational frequency $\omega = \dot{\theta}$ in rad/sec.

The pendulum dynamics is equivalent to:

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -(b/m\ell^2)\omega - (g/\ell)\sin\theta \end{aligned}$$

```
m=1.0; b=0.0; l=1.0; g=9.81
def f(theta_d_theta):
    theta, d_theta = theta_d_theta
    J = m * l * l
    d2_theta  = - b / J * d_theta
    d2_theta += - g / l * sin(theta)
    return array([d_theta, d2_theta])
```

```
figure()
theta = linspace(-1.5 * pi, 1.5 * pi, 100)
d_theta = linspace(-5.0, 5.0, 100)
grid(True)
xticks([-pi, 0, pi], [r"$-\pi$", "$0$",
r"$\pi$"])
streamplot(*Q(f, theta, d_theta), color="k")
```

5
0
−5
−π
0
π

ⓘ – MODEL / PENDULUM

- [🧪, 👑] Determine an approximation of the least possible angular velocity $\omega_0 > 0$ such that when $\theta(0) = 0$ and $\dot{\theta}(0) = \omega_0$, the pendulum reaches (or overshoots) $\theta(t) = \pi$ for some $t > 0$.
- [💡, $\mathbf{x}^2$] Answer the same question analytically.

NUMERICAL SOLUTION (BASIC)

Given a finite **time span** $[t_0, t_f]$ and a small enough **time step** $\Delta t > 0$, we can use the approximation:

$$\begin{aligned} x(t + \Delta t) &\simeq x(t) + \Delta t \times \dot{x}(t) \\ &= x(t) + \Delta t \times f(x(t)) \end{aligned}$$

to compute a sequence of states $x_k \in \mathbb{R}^n$ such that:

$$x(t = t_0 + k\Delta t) \simeq x_k.$$

EULER SCHEME

(Fixed-step & explicit version)

```
def basic_solve_ivp(f, t0, x0, dt, t_f):
    ts, xs = [t0], [x0]
    while ts[-1] < t_f:
        t, x = ts[-1], xs[-1]
        t_next, x_next = t + dt, x + dt * f(x)
        ts.append(t_next); xs.append(x_next)
    return (array(ts), array(xs).T)
```

Why the final transposition (`.T`)?

So that when

```
t, x = basic_solve_ivp(...),
```

`x[i]` refers to the values of the `i`th component of `x`.

(without the `.T`, it would be `x[:][i]`)

👁 ROTATION / SOLUTION

$$\left| \begin{array}{l} \dot{x} = -y \\ \dot{y} = +x \end{array} \right. , \quad \text{with} \quad \left| \begin{array}{l} x(0) = 1 \\ y(0) = 0 \end{array} \right.$$

```
def f(xy):
    x, y = xy
    return array([-y, x])
t0, x0 = 0.0, array([1.0, 0.0])
dt, tf = 0.001, 5.0
t, x = basic_solve_ivp(f, t0, x0, dt, tf)
```

TRAJECTORIES

```
figure()
plot(t, x[0])
plot(t, x[1])
grid(True)
```

1
0
−1
0
2
4

TRAJECTORIES (STATE SPACE)

Represent solutions fragments in the background

```
figure()
xs = ys = linspace(-1.5, 1.5, 50)
streamplot(*Q(f, xs, ys), color="lightgrey")
```

...

```
plot(x[0], x[1], "k")
plot(x0[0], x0[1], "ko")
dx = x[0][-1] - x[0][-2]
dy = x[1][-1] - x[1][-2]
arrow(x[0][-1], x[1][-1], dx, dy, width=0.02,
color="k")
grid()
axis("equal")
```

1
0
−1
−2
0
2

BEYOND THE BASIC SOLVER

Many issues that our basic solver doesn’t address:

- time-dependent vector field,
- error control,
- dense outputs,
- and more …

Instead, use:

```
from scipy.integrate import solve_ivp
```

Documentation: `solve_ivp` (Scipy)

👁 IVP / ROTATION

Compute the solution $x(t)$ for $t \in [0, 2\pi]$ of the IVP:

$$\left| \begin{aligned} \dot{x}_1 &= -x_2 \\ \dot{x}_2 &= +x_1 \end{aligned} \right. \quad \text{with} \quad \left| \begin{aligned} x_1(0) &= 1 \\ x_2(0) &= 0 \end{aligned} \right.$$

👁 SOLVER INTERFACE / ROTATION

```
def fun(t, y):
    x1, x2 = y
    return array([-x2, x1])
t_span = [0.0, 2*pi]
y0 = [1.0, 0.0]
result = solve_ivp(fun=fun, t_span=t_span,
y0=y0)
```

RESULT STRUCTURE

The `result` is a dictionary-like object (a “bunch”).

Its fields:

- `t` : array, time points, shape `(n_points,)`,
- `y` : array, values of the solution at `t`, shape `(n, n_points)`,
- …

(See `solve_ivp` documentation)

NON-AUTONOMOUS SYSTEMS

⚠ The solver may apply to systems that are not time-invariant

$$\dot{y} = f(t, y)$$

The `t` argument in the definition of `fun` is mandatory, even if the returned value doesn't depend on it (time-invariant system).

```
r_t = result["t"]
x_1 = result["y"][0]
x_2 = result["y"][1]
```

```
figure()
t = linspace(0, 2*pi, 1000)
plot(t, cos(t), "k--")
plot(t, sin(t), "k--")
bold = {"lw": 2.0, "ms": 10.0}
plot(r_t, x_1, ".-", label="$x_1(t)$", **bold)
plot(r_t, x_2, ".-", label="$x_2(t)$", **bold)
xlabel("$t$"); grid(); legend()
```

1
0
−1
0.0
2.5
5.0
t
$x_1(t)$
$x_2(t)$

VARIABLE STEP SIZE

The step size is:

- variable: $t_{n+1} - t_n$ may not be constant,
- automatically selected by the algorithm,

The solver shall meet the user specification, but should select the largest step size to do so to minimize the number of computations.

Optionally, you can specify a `max_step` (default: $+\infty$).

ERROR CONTROL

We generally want to control the (local) error $e(t)$: the difference between the numerical solution and the exact one.

- `atol` is the **absolute tolerance** (default: 10^{-6}),
- `rtol` is the **relative tolerance** (default: 10^{-3}).

The solver ensures (approximately) that at each step:

$$|e(t)| \leq \text{atol} + \text{rtol} \times |x(t)|$$

```
options = {
    # at least 20 data points
    "max_step": 2*pi / 20,
    # standard absolute tolerance
    "atol"    : 1e-6,
    # very large relative tolerance
    "rtol"    : 1e9
}
```

```
result = solve_ivp(
    fun=fun, t_span=t_span, y0=y0,
    **options
)
r_t = result["t"]
x_1 = result["y"][0]
x_2 = result["y"][1]
```

```
figure()
t = linspace(0, 2*pi, 20)
plot(t, cos(t), "k--")
plot(t, sin(t), "k--")
bold = {"lw": 2.0, "ms": 10.0}
plot(r_t, x_1, ".-", label="$x_1(t)$", **bold)
plot(r_t, x_2, ".-", label="$x_2(t)$", **bold)
xlabel("$t$"); grid(); legend()
```

1
0
−1
0.0
2.5
5.0
t
$x_1(t)$
$x_2(t)$

DENSE OUTPUT

Using a small `max_step` is usually the wrong way to "get more data points" since this will trigger many (potentially expensive) evaluations of `fun`.

Instead, use dense outputs: the solver may return the discrete data `result["t"]` and `result["y"]` **and** an approximate solution `result["sol"]` **as a function of t** with little extra computations.

```
options = {
    "dense_output": True
}
```

```
result = solve_ivp(
    fun=fun, t_span=t_span, y0=y0,
    **options
)
r_t = result["t"]
x_1 = result["y"][0]
x_2 = result["y"][1]
sol = result["sol"]
```

```
figure()
t = linspace(0, 2*pi, 1000)
plot(t, cos(t), "k--")
plot(t, sin(t), "k--")
bold = {"lw": 2.0, "ms": 10.0}
plot(t, sol(t)[0], "-", label="$x_1(t)$",
**bold)
plot(t, sol(t)[1], "-", label="$x_2(t)$",
```

1
0
−1
0.0
2.5
5.0
t
$x_1(t)$
$x_2(t)$

WELL-POSEDNESS

WELL-POSEDNESS OF AN IVP

Well-Posedness = Existence + Uniqueness + Continuity

A set of properties that ensures that common problems with IVPs cannot happen.

We are going to detail each property

PREAMBLE: LOCAL VS GLOBAL

So far, we have only dealt with **global** solutions $x(t)$ of IVPs, defined for any $t \geq t_0$.

This concept is sometimes too stringent.

Consider for example:

$$\dot{x} = x^2 \text{ and } x(0) = 1.$$

```
def fun(t, y):
    return y * y
t0, tf, y0 = 0.0, 3.0, array([1.0])
result = solve_ivp(fun, t_span=[t0, tf], y0=y0)
figure()
plot(result["t"], result["y"][0], "k")
xlim(t0, tf); xlabel("$t$"); ylabel("$x(t)$")
```

$\times 10^{14}$
4
2
0
$x(t)$
0
1
2
3
t

Ouch.

There is actually no **global** solution.

However there is **local** solution $x(t)$, defined for $t \in [t_0, \tau[$ for some $\tau > t_0$.

Indeed, the function

$$x(t) = \frac{1}{1-t}$$

satisfies

$$\dot{x}(t) = \frac{d}{dt}x(t) = -\frac{-1}{(1-t)^2} = (x(t))^2.$$

But it's defined only for $t < 1$.

```
tf = 1.0
result = solve_ivp(fun, t_span=[t0, tf], y0=y0,
max_step=0.01)
figure()
plot(result["t"], result["y"][0], "k")
ylim(0.0, 10.0); grid(); xlabel("$t$");
ylabel("$x(t)$")
```

10
5
0
$x(t)$
0.0
0.5
1.0
t

This local solution is also **maximal**:

You cannot extend this solution beyond $\tau = 1.0$.

MAXIMAL SOLUTION

A solution $x : [0, \tau[$ to an IVP is **maximal** if there is no other solution

- defined on $[0, \tau'[$ with $\tau' > \tau$,
- whose restriction to $[0, \tau[$ is x.

ⓘ – LOCAL/MAXIMAL SOLUTION

- [**x²**] Find a local solution $x(t)$ of $\dot{x} = x^2$ such that $x(0) = x_0$ under the assumption that $x(t) \neq 0$ when $t \geq 0$.

  🔑 **Hint:** compute $d(1/x(t))/dt$.

- [💡] Find for every $x_0 \neq 0$ a maximal solution. When is it global?

BAD NEWS (1/3)

Sometimes things get worse than simply having no global solution.

👁 – NO LOCAL SOLUTION

Consider the scalar IVP with initial value $x(0) = (0, 0)$ and right-hand side

$$f(x_1, x_2) = \begin{cases} (+1, 0) & \text{if } x_1 < 0 \\ (-1, 0) & \text{if } x_1 \geq 0. \end{cases}$$

– NO LOCAL SOLUTION

```
def f(x1x2):
    x1, x2 = x1x2
    dx1 = 1.0 if x1 < 0.0 else -1.0
    return array([dx1, 0.0])
figure()
x1 = x2 = linspace(-1.0, 1.0, 20)
gca().set_aspect(1.0); grid(True)
quiver(*Q(f, x1, x2), color="k")
```

1
0
−1
−1
0
1

💡 – NO LOCAL SOLUTION

This system has no solution, not even a local one, when $x(0) = (0, 0)$.

PROOF

- Assume that $x : [0, \tau[\to \mathbb{R}$ is a local solution.
- Since $\dot{x}(0) = -1 < 0$, for some small enough $0 < \epsilon < \tau$ and any $t \in]0, \epsilon]$, we have $x(t) < 0$.
- Consequently, $\dot{x}(t) = +1$ and thus by integration

$$x(\epsilon) = x(0) + \int_0^\epsilon \dot{x}(t)\, dt = \epsilon > 0,$$

which is a contradiction.

GOOD NEWS (1/3)

However, a local solution exists under very mild assumptions.

EXISTENCE

If f is continuous,

- There is a (at least one) **local** solution to the IVP $\dot{x} = f(x)$ and $x(t_0) = x_0$.
- Any local solution on some $[t_0, \tau[$ can be extended to a (at least one) **maximal** one on some $[t_0, t_\infty[$.

Note: a maximal solution is **global** iff $t_\infty = +\infty$.

THEOREM – CHARACTERIZATION OF MAXIMAL SOLUTIONS

A solution on $[t_0, \tau[$ is maximal if and only if either

- $\tau = +\infty$: the solution is global, or
- $\tau < +\infty$ and $\lim_{t \to \tau} \|x(t)\| = +\infty$.

In plain words : a non-global solution cannot be extended further in time if and only if it “blows up”.

COROLLARY

Let's assume that a local maximal solution exists.

You wonder if this solution is defined in $[t_0, t_f[$ or blows up before t_f.

For example, you wonder if a solution is global (if $t_f = +\infty$ or $t_f < +\infty$.)

💡 PROVE EXISTENCE ON $[0, t_f[$

TODO. Show that any solution which defined on some sub-interval $[t_0, \tau]$ with $\tau < t_f$ would is bounded.

Then, no solution can be maximal on any such $[0, \tau[$ (since it doesn't blow up !). Since a maximal solution does exist, its domain is $[0, t_\infty[$ with $t_\infty \geq t_f$.

$\Rightarrow$ a solution is defined on $[t_0, t_f[$.

ⓘ – EXISTENCE / SIGMOID

Consider

$$\dot{x} = \frac{1}{1 + e^{-x}} \text{ with } x(0) = x_0 \in \mathbb{R}$$

- [**x²**] Show that there is a (at least one) maximal solution.
- [**x²**] Show that any such solution is global.

ⓘ – EXISTENCE / PENDULUM

Consider the pendulum, subject to a torque c

$$ml^2\ddot{\theta} + b\dot{\theta} + mg\ell \sin\theta = c$$

We assume that the torque provides a bounded power:

$$P(t) = c(t)\dot{\theta}(t) \leq M < +\infty.$$

- [**x²**] Show that for any initial condition $t_0, \theta(t_0) = \theta_0$ and $\dot{\theta}(t_0) = \omega_0$, there is a solution to the IVP which is global.

  Hint. Compute the derivative with respect to t of

$$E = \frac{1}{2} m\ell^2 \dot{\theta}^2 - mg\ell \cos\theta$$

ⓘ – EXISTENCE / LINEAR SYSTEMS

Let $A \in \mathbb{R}^{n \times n}$ and $x_0 \in \mathbb{R}^n$. Consider

$$\dot{x} = Ax \quad \text{and} \quad x(0) = x_0$$

◎ **Aim.** Show that any maximal solution is global.

- [**x²**] Show that $y(t) = \|x(t)\|^2$ is differentiable and satisfies $y(t) \geq 0$ and $\dot{y}(t) \leq \alpha y(t)$ for some $\alpha \geq 0$.
- [**x²**] Compute the derivative of $y(t)e^{-\alpha t}$ and conclude that $0 \leq y(t) \leq y(0)e^{\alpha t}$.
- [**x²**] Prove that any maximal solution $x(t)$ of the initial IVP is global.

BAD NEWS (2/3)

Uniqueness of solutions, even the maximal ones, is not granted either.

(Note: why does uniqueness of local solution does not make sense?)

👁 – NON-UNIQUENESS

The IVP

$$\dot{x} = \sqrt{x}, x(0) = 0$$

has several maximal (global) solutions.

PROOF

For any $\tau \geq 0, x_\tau$ is a solution:

$$x_\tau(t) = \begin{cases} 0 & \text{if } t \leq \tau, \\ 1/4 \times (t - \tau)^2 & \text{if } t > \tau. \end{cases}$$

GOOD NEWS (2/3)

However, uniqueness of maximal solution holds under mild assumptions.

NOTATION – JACOBIAN MATRIX

Let $x = (x_1, \ldots, x_n)$ and $f(x) = (f_1(x), \ldots, f_n(x))$.
The **Jacobian matrix** of f is defined as

$$\frac{\partial f}{\partial x} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$$

(when all the partial derivatives are defined).

UNIQUENESS

If $\partial f / \partial x$ exists and is continuous,
the maximal solution is unique.

BAD NEWS (3/3)

An infinitely small error in the initial value could result in a finite error in the solution, even in finite time.

That would undermine the utility of any approximation method.

DEFINITION – CONTINUITY

Instead of denoting $x(t)$ the solution, use $x(t, x_0)$ to emphasize the dependency w.r.t. the initial state.

Continuity w.r.t. the initial state means that if $x(t, x_0)$ is defined on $[t_0, \tau]$ and $t \in [t_0, \tau]$:

$$x(t, y) \to x(t, x_0) \text{ when } y \to x_0$$

and that this convergence is uniform w.r.t. t.

GOOD NEWS (3/3)

However, continuity wrt the initial value holds under mild assumptions.

CONTINUITY

Assume that $\partial f / \partial x$ exists and is continuous.

Then the dynamical system is continous w.r.t. the initial state.

👁 CONTINUITY / PREY-PREDATOR

```
alpha = 2 / 3; beta = 4 / 3; delta = gamma =
1.0
def f(t, y):
    x, y = y
    u = alpha * x - beta * x * y
    v = delta * x * y - gamma * y
    return array([u, v])
```

```
tf = 3.0
result = solve_ivp(f, t_span=[0.0, tf], y0=
[1.5, 1.5], max_step=0.01)
x = result["y"][0]
y = result["y"][1]
```

```
figure(); ax = gca()
xr = yr = linspace(0.0, 2.0, 1000)
streamplot(*Q(lambda y: f(0,y), xr, yr),
color="grey")
for xy in zip(x, y):
    x_, y_ = xy
    ax.add_artist(Circle((x_, y_), 0.2,
color="#d3d3d3"))
```

```
result = solve_ivp(f, t_span=[0.0, tf], y0=
[1.5, 1.575], max_step=0.01)
x = result["y"][0]
y = result["y"][1]
plot(x, y, "k--")
axis([0,2,0,2]); axis("square")
```

2
1
0
0
2

ⓘ – CONTINUITY / INITIAL VALUE

Let $h \geq 0$ and $x^h(t)$ be the solution of $\dot{x} = x$ and

$$x^h(0) = 1 + h.$$

- [**x²**]. Let $\epsilon > 0$ and $\tau \geq 0$; find the smallest $\delta > 0$ such that $|h| < \delta$ ensures that

$$\text{for any } t \in [t_0, \tau], |x^h(t) - x^0(t)| < \epsilon$$

- [$\mathbf{x}^2$]. What is the behavior of δ when τ goes to infinity?

ⓘ – CONTINUITY / INITIAL VALUE

Consider $\dot{x} = \sqrt{|x|}, x(0) = x_0$.

- [💻] Solve numerically this IVP for $t \in [0, 1]$ and $x_0 = 0$. Then, solve it again for $x_0 = 0.1, x_0 = 0.01$, etc.
- [🧪] Does the solution seem to be continuous with respect to the initial value?
- [💡] Explain this experimental result.

ⓘ – WELL-POSEDNESS / PREY-PREDATOR

Let

$$\begin{aligned} \dot{x} &= \alpha x - \beta x y \\ \dot{y} &= \delta x y - \gamma y \end{aligned}$$

with $\alpha = 2/3, \beta = 4/3, \delta = \gamma = 1.0.$

- [💡] Show that the system is well-posed.
- [💡, **x²**] Show that if $x(0) > 0$ and $y(0) > 0$, the maximal solution is global.

  🔑 **Hint.** Compute

$$d/dt(\delta x - \gamma \ln x + \beta y - \alpha \ln y)$$

ASYMPTOTIC BEHAVIOR

Asymptotic = Long-Term: when $t \to +\infty$

(note: from now on, all systems are well-posed.)

LORENZ SYSTEM

$$
\begin{aligned}
\dot{x} &= \sigma(y - x) \\
\dot{y} &= x(\rho - z) \\
\dot{z} &= xy - \beta z
\end{aligned}
$$

Visualized with Fibre

HADLEY SYSTEM

$$
\begin{aligned}
\dot{x} &= -y^2 - z^2 - ax + af \\
\dot{y} &= xy - bxz - y + g \\
\dot{z} &= bxy + xz - z
\end{aligned}
$$

Visualized with Fibre

EQUILIBRIUM

An **equilibrium** of system $\dot{x} = f(x)$ is a state x_e such that the maximal solution of this system such that $x(0) = x_e$ is $x(t) = x_e$ for any $t > 0$.

The state x_e is an equilibrium if and only if $f(x_e) = 0$.

ⓘ – EQUILIBRIUM / PENDULUM

Reminder: the pendulum is governed by the equation

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell\sin\theta = 0$$

- [**x²**] Find the equilibriums of this dynamics.

STABILITY

About the long-term behavior of solutions.

- “Stability” subtle concept,
- “Asymptotic Stability” simpler (and stronger),
- “Attractivity” simpler yet, (but often too weak).

ATTRACTIVITY

Consider a well-posed ODE $\dot{x} = f(x)$.

An equilibrium x_e is:

- **globally attractive** if for every x_0, the maximal solution $x(t)$ of the IVP with $x(0) = x_0$ exists for any $t \geq 0$ and
$$\lim_{t \to +\infty} x(t) = x_e.$$
- **locally attractive** if this property holds when x_0 is sufficiently close to the equilibrium x_e.

```
def f(xy):
    x, y = xy
    dx = -2*x + y
    dy = -2*y + x
    return array([dx, dy])
```

```
figure()
x = y = linspace(-5.0, 5.0, 1000)
streamplot(*Q(f, x, y), color="k")
plot([0], [0], "k.", ms=10.0)
```

GLOBALLY ATTRACTIVE

```
def f(xy):
    x, y = xy
    dx = -2*x + y**3
    dy = -2*y + x**3
    return array([dx, dy])
```

```
figure()
x = y = linspace(-5.0, 5.0, 1000)
streamplot(*Q(f, x, y), color="k")
plot([0], [0], "k.", ms=10.0)
```

LOCALLY ATTRACTIVE

ⓘ – EQUILIBRIUM / STABILITY

Consider a pendulum with a coefficient of friction b.

- [⚙] Is any equilibrium globally attractive?
- [🧪, 📉]
  - Assume that $b > 0$. Determine graphically the equilibriums which are locally attractive.
  - What happens when $b = 0$?

- [⚙⚙, **x²**] Prove that no equilibrium is locally attractive when $b = 0$.

  🔑 **Hint:** study the evolution in time

$$E = J\dot{\theta}^2/2 - mg\ell\cos\theta,$$

  with $J = m\ell^2$, the total mechanical energy of the pendulum.

ATTRACTIVITY

The equilibrium x_e is globally attractive iff:

- for any x_0 and for any $\epsilon > 0$, there is a $\tau \geq 0$ such that the maximal solution $x(t)$ such that $x(0) = x_0$ exists for all $t \geq 0$ and satisfies:

$$\|x(t) - x_e\| \leq \epsilon \quad \text{when} \quad t \geq \tau.$$

- Very close values of $x(0)$ could lead to very different "speed of convergence" towards the equilibrium.
- This is not contradictory with the well-posedness assumption: continuity w.r.t. the initial condition only works with finite time spans.

EXAMPLE

$$\left|\begin{array}{lcl} \dot{x} & = & x + xy - (x+y)\sqrt{x^2+y^2} \\ \dot{y} & = & y - x^2 + (x-y)\sqrt{x^2+y^2} \end{array}\right.$$

Equivalently, in polar coordinates:

$$\left| \begin{array}{lcl} \dot{r} & = & r(1-r) \\ \dot{\theta} & = & r(1-\cos\theta) \end{array} \right.$$

STREAM PLOT

```
def f(xy):
    x, y = xy
    r = sqrt(x*x + y*y)
    dx = x + x * y - (x + y) * r
    dy = y - x * x + (x - y) * r
    return [dx, dy]
```

```
figure()
x = y = linspace(-2.0, 2.0, 1000)
streamplot(*Q(f, x, y), color="k")
plot([1], [0], "k.", ms=10.0)
```

2
0
−2
−2
0
2

First, make sure that the right-hand side is time-dependent:

```
def fun(t, y):
    return f(y)
```

Then, pick a large time span and an initial state just above the equilibrium $(1.0, 0.0)$:

```
t_span = (0.0, 600.0)
theta_0 = 2 * pi / 1000
y0 = [cos(theta_0), sin(theta_0)]
```

```
y = solve_ivp(fun, t_span, y0=y0,
dense_output=True).sol
t = linspace(t_span[0], t_span[-1], 1000)
```

Plot the distance to the equilibrium as a function of time.

```
figure()
x1, x2 = y(t)[0], y(t)[1]
plot(t, (sqrt((x1-1.0)**2 + x2**2)))
grid()
xlabel("$t$")
```

2
1
0
0
250
500
t

ASYMPTOTIC STABILITY

Asymptotic stability is a stronger version of attractivity which is by definition robust with respect to the choice of the initial state.

GLOBAL ASYMPTOTIC STABILITY

The equilibrium x_e is **globally asymptotically stable** if:

- for any x_0 and for any $\epsilon > 0$, there is a $\tau \geq 0$ **and a $\boldsymbol{r > 0}$ such that if $\|\boldsymbol{x_1 - x_0}\| \leq \boldsymbol{r}$** , the maximal solution $x(t)$ **such that $\boldsymbol{x(0) = x_1}$** exists for all $t \geq 0$ and satisfies:

$$\|x(t) - x_e\| \leq \epsilon \quad \text{when} \quad t \geq \tau.$$

Equivalently:

- **for any $\boldsymbol{r} > \mathbf{0}$**, for any $\epsilon > 0$, there is a $\tau \geq 0$ such the maximal solution $x(t)$ **such that $\|\boldsymbol{x}(\mathbf{0}) - \boldsymbol{x_e}\| \leq \boldsymbol{r}$** exists for all $t \geq 0$ and satisfies:

$$\|x(t) - x_e\| \leq \epsilon \quad \text{when} \quad t \geq \tau.$$

G.A.S. IN PLAIN WORDS

- Pick any bounded set B in $\mathbb{R}^n$.
- Consider each point of the set as an initial value x_0, compute $x(t, x_0)$. Define B_t as the set of all such $x(t, x_0)$:

$$B_t = \{x(t, x_0) \mid x_0 \in B\}$$

- The set B_t will converge to the equilibrium x_e:

$$\sup \{d(x_e, x) \mid x \in B_t\} \to 0 \text{ when } t \to +\infty.$$

LOCAL ASYMPTOTIC STABILITY

A variant of the global asymptotic stability: the property may hold only for small enough balls.

- **for some $\boldsymbol{r} > \mathbf{0}$**, for any $\epsilon > 0$, there is a $\tau \geq 0$ such that any maximal solution $x(t)$ **such that** $\|\boldsymbol{x}(\mathbf{0}) - \boldsymbol{x_e}\| \leq \boldsymbol{r}$ exists for all $t \geq 0$ and satisfies:

$$\|x(t) - x_e\| \leq \epsilon \quad \text{when} \quad t \geq \tau.$$

ⓘ – ASYMPTOTIC STABILITY / VINOGRAD

Consider the ODE with right-hand side:

```
def f(xy):
    x, y = xy
    q = x**2 + y**2 * (1 + (x**2 + y**2)**2)
    dx = (x**2 * (y - x) + y**5) / q
    dy = y**2 * (y - 2*x) / q
    return array([dx, dy])
```

```
figure()
x = y = linspace(-1.0, 1.0, 1000)
streamplot(*Q(f, x, y), color="k")
xticks([-1, 0, 1])
plot([0], [0], "k.", ms=10.0)
```

1
0
−1
−1
0
1

- [🧪] Does the origin of the system look attractive ?
- [🧪, 💡] Does it seem to be asymptotically stable ?

🔑 **Hint.**

  - Show if the origin is asymptotically stable, then it is **stable**, that is: for any $r > 0$, there is a $\rho \leq r$ such that if $|x(0)| \leq \rho$, then $|x(t)| \leq r$ for any $t \geq 0$.
  - Is the system stable (graphically)?