

# CONTROLLERS

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# PREAMBLE

```
from numpy import *  
from numpy.linalg import *  
from numpy.testing import *  
from matplotlib.pyplot import *  
from scipy.integrate import *
```

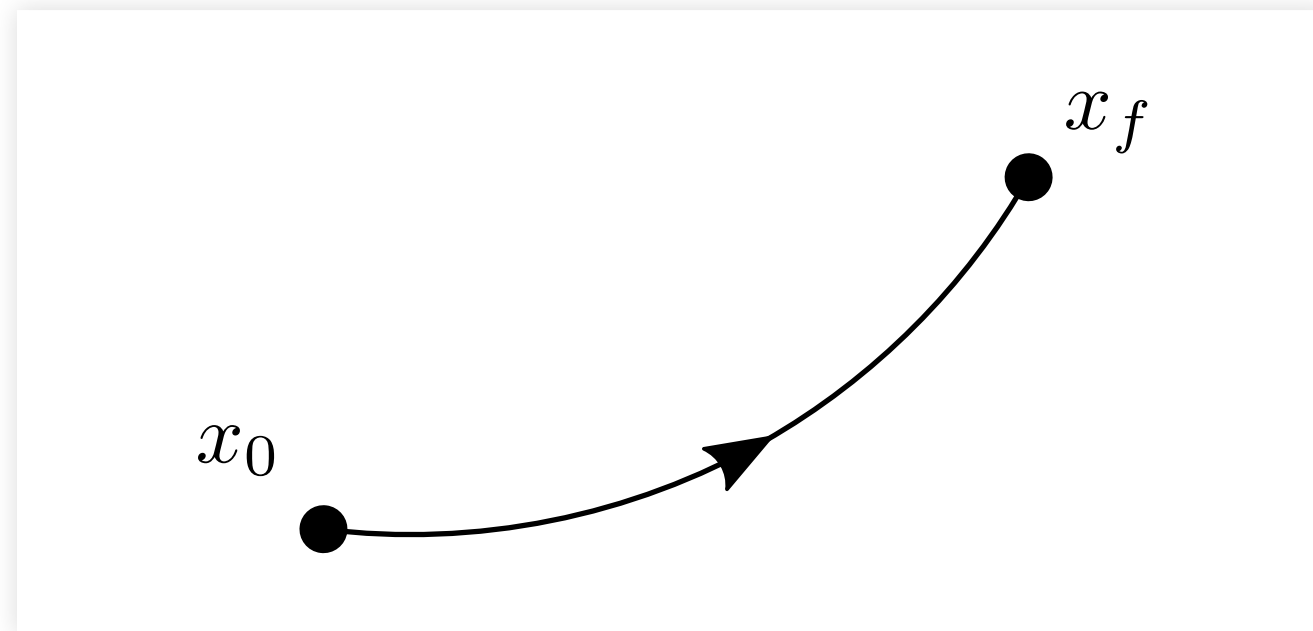
# CONTROLLABILITY

# DEFINITION

The system  $\dot{x} = f(x, u)$  is **controllable** if

- for any  $t_0 \in \mathbb{R}, x_0 \in \mathbb{R}^n$  and  $x_f \in \mathbb{R}^n$ ,
- there are  $t_f > 0$  and  $u : [t_0, t_f] \rightarrow \mathbb{R}^m$  such that
- the solution  $x(t)$  such that  $x(t_0) = x_0$  satisfies

$$x(t_f) = x_f.$$



# CONTROLLABILITY / CAR

The position  $x$  (in meters) of a car of mass  $m$  (in kg) on a straight road is governed by

$$m\ddot{x} = u$$

where  $u$  the force (in Newtons) generated by its motor.

The car is initially at the origin of a road and motionless. We would like to drive it to across the location  $x_f > 0$  at speed  $v_f$  and at time  $t_f > 0$ .

### Numerical values:

- $m = 1500$  kg,
- $t_f = 10$  s,  $x_f = 100$  m and  $v_f = 100$  km/h.



# STRATEGY

## STEP 1 – TRAJECTORY PLANNING

We search for a **reference trajectory** for the state

$$X_r(t) = (x_r(t), \dot{x}_r(t))$$

such that:

- $x_r(0) = 0, \dot{x}_r(0) = 0,$
- $x_r(t_f) = x_f, \dot{x}_r(t_f) = v_f.$

## STEP 2 – ADMISSIBILITY

We check that this reference trajectory is **admissible**,  
i.e. that we can find a control  $u_r(t)$  such that the  
solution of the IVP is  $X(t) = X_r(t)$  when  $X(0) = X_r(0)$ .

# ADMISSIBLE TRAJECTORY

Here, if  $x_r$  is smooth and if we apply the control

$$u(t) = m\ddot{x}_r(t),$$

$$m\frac{d^2}{dt^2}(x - x_r) = 0,$$

$$(x - x_r)(0) = 0, \quad \frac{d}{dt}(x - x_r)(0) = 0.$$

Thus,  $x(t) = x_r(t)$  – and thus  $\dot{x}(t) = \dot{x}_r(t)$  – for every  $t \geq 0$ .

# REFERENCE TRAJECTORY

We can find  $x_r$  as a third-order polynomial in  $t$

$$x_r(t) = at^3 + bt^2 + ct + d$$

with

$$a = \frac{v_f}{t_f^2} - 2\frac{x_f}{t_f^3}, \quad b = 3\frac{x_f}{t_f^2} - \frac{v_f}{t_f}, \quad c = 0, \quad d = 0.$$

(equivalently, with  $u(t)$  as an affine function of  $t$ ).

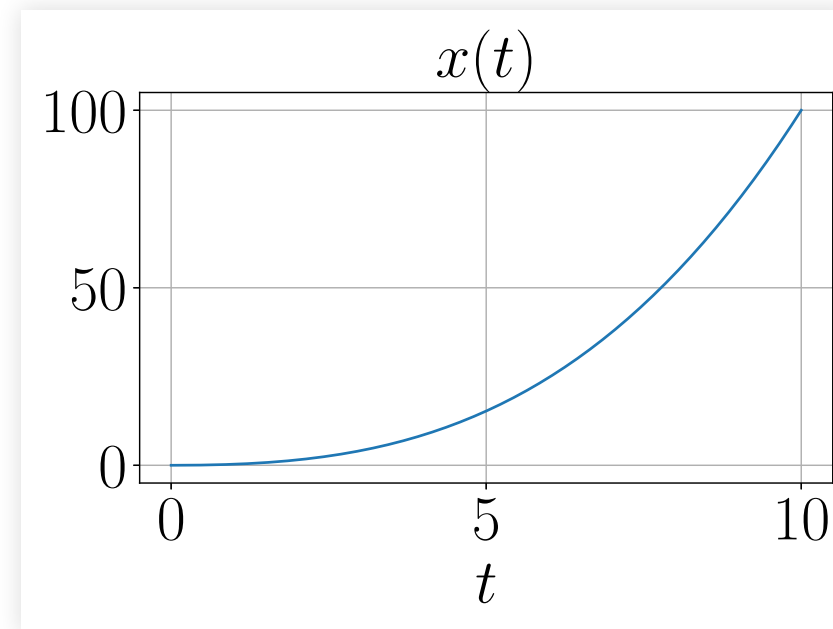
```
m = 1500.0
xf = 100.0
vf = 100.0 * 1000 / 3600 # m/s
tf = 10.0
a = vf/tf**2 - 2*xf/tf**3
b = 3*xf/tf**2 - vf/tf
```

```
def x(t):  
    return a * t**3 + b * t**2  
def d2_x(t):  
    return 6 * a * t + 2 * b  
def u(t):  
    return m * d2_x(t)
```

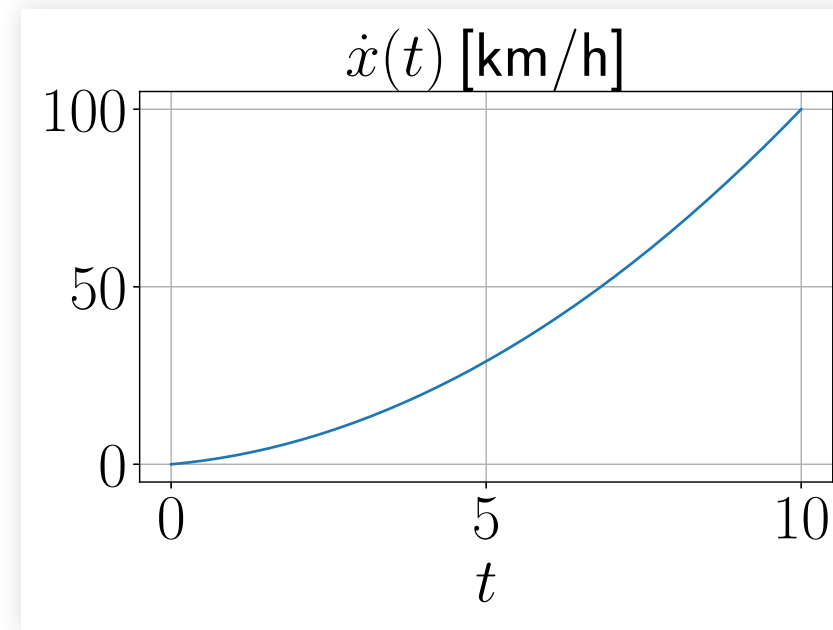
```
y0 = [0.0, 0.0]
def fun(t, y):
    x, d_x = y
    d2_x = u(t) / m
    return [d_x, d2_x]
result = solve_ivp(fun, [0.0, tf], y0,
dense_output=True)
```

```
figure()  
t = linspace(0, tf, 1000)  
xt = result["sol"](t)[0]  
plot(t, xt)  
grid(True); xlabel("$t$"); title("$x(t)$")
```





```
figure()  
vt = result["sol"](t)[1]  
plot(t, 3.6 * vt)  
grid(True); xlabel("$t$")  
title(r"$\dot{x}(t) \ , \ \mbox{[km/h]}$")
```



# ② NON-ADMISSIBLE TRAJECTORY

Let

$$\dot{x} = Ax \text{ with } x \in \mathbb{R}^2, A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Find a smooth reference trajectory

$$x_r(t), t \in [0, 1]$$



which is not admissible.

# ② PENDULUM

Consider the pendulum with dynamics:

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell \sin \theta = u$$

- [💡,  $\mathbf{x}^2$ ] Find a smooth reference trajectory that leads the pendulum from  $\theta(0) = 0$  and  $\dot{\theta}(0) = 0$  to  $\theta(t_f) = \pi$  and  $\dot{\theta}(t_f) = 0$ .
- [💡,  $\mathbf{x}^2$ ] Show that the reference trajectory is admissible and compute the corresponding input  $u(t)$  as a function of  $t$  and  $\theta(t)$ .

- [, ] Simulate the result with standard and high-precision (small steps). What should happen theoretically after  $t = t_f$  if  $u(t) = 0$  is applied ? What does happen in practice ?

Numerical Values:

$$m = 1.0, l = 1.0, b = 0.1, g = 9.81, t_f = 10.$$

# CONTROLLABILITY / LTI SYSTEM

For a LTI system, it is sufficient to check that

- from the origin  $x_0 = 0$  at  $t_0 = 0$ ,
- we can reach any state  $x_f \in \mathbb{R}^n$ .

# KALMAN CRITERION

The system  $\dot{x} = Ax + Bu$  is controllable iff:

$$\text{rank} \begin{bmatrix} B, AB, \dots, A^{n-1}B \end{bmatrix} = n$$

$[B, \dots, A^{n-1}B]$  is the **Kalman controllability matrix**.



# KALMAN CONTROLLABILITY MATRIX

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

# COMPUTATION

```
def KCM(A, B):  
    n = shape(A)[0]  
    mp = matrix_power  
    cs = column_stack  
    return cs([mp(A, k) @ B for k in range(n)])
```

```
n = 3
A = zeros((n, n))
for i in range(0, n-1):
    A[i, i+1] = 1.0

B = zeros((n, 1))
B[n-1, 0] = 1.0
```

```
C = KCM(A, B)
```

```
C_expected = [[0, 0, 1], [0, 1, 0], [1, 0, 0]]
```

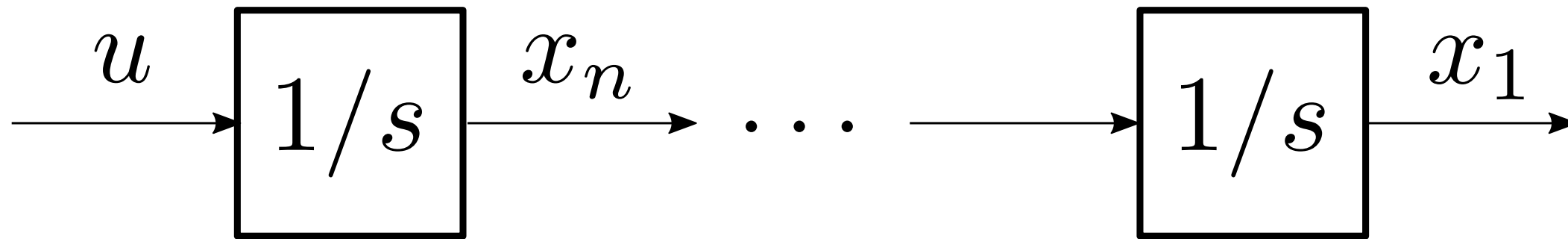
```
assert_almost_equal(C, C_expected)
```

# ② FULLY ACTUATED SYSTEM

Consider  $\dot{x} = Ax + Bu$  with  $x \in \mathbb{R}^n$ ,  $u \in \mathbb{R}^n$  and  $\text{rank } B = n$ .

- [💡,  $\mathbf{x}^2$ ] Is the systems controllable ?
- [💡,  $\mathbf{x}^2$ ] Given  $x_0, x_f$  and  $t_f > 0$ , show that any smooth trajectory that leads from  $x_0$  to  $x_f$  in  $t_f$  seconds is admissible.

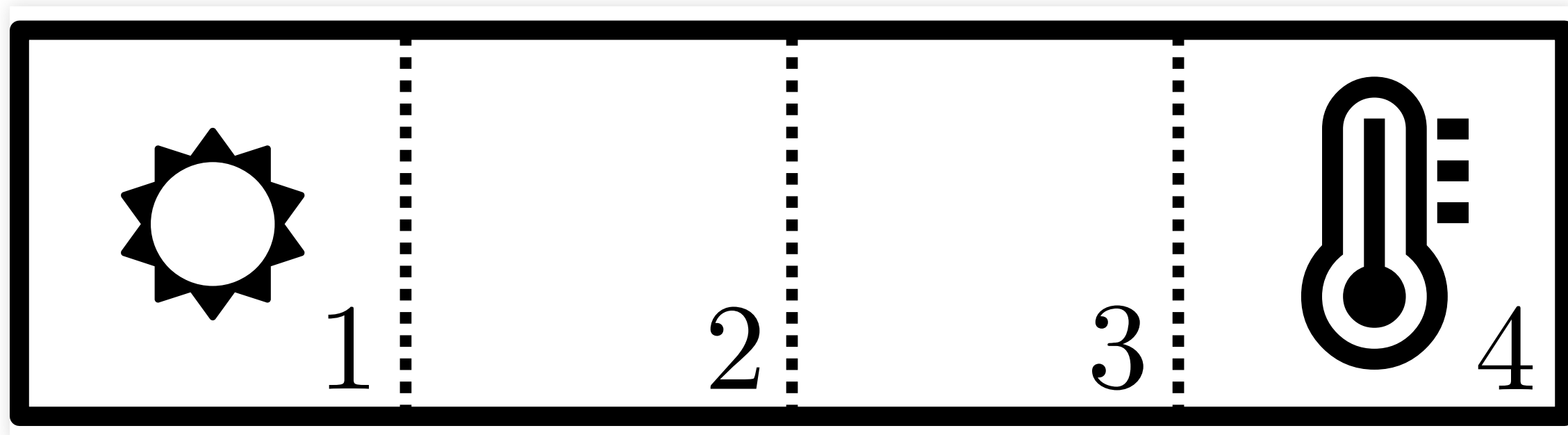
# ② INTEGRATOR CHAIN



$$\dot{x}_n = u, \dot{x}_{n-1} = x_n, \dots, \dot{x}_1 = x_2.$$

- [💡,  $\mathbf{x}^2$ ] Show that the system is controllable

# ② HEAT EQUATION





- $dT_1/dt = u + (T_2 - T_1)$
- $dT_2/dt = (T_1 - T_2) + (T_3 - T_2)$
- $dT_3/dt = (T_2 - T_3) + (T_4 - T_3)$
- $dT_4/dt = (T_3 - T_4)$

- [💡,  $\mathbf{x}^2$ ] Show that the system is controllable.
- [💡,  $\mathbf{x}^2$ ] Is it still true if the four cells are organized as a square and the heat sink/source is in any of the corners ? How many independent sources do you need to make the system controllable and where can you place them?

# EXTRA EXERCICES

- Unreachable states
- Brunovsky form
- Controllability in prey-predator systems (via the invariant)
- etc.

# ASYMPTOTIC STABILIZATION

# STABILIZATION

When the system

$$\dot{x} = Ax, \quad x \in \mathbb{R}^n$$

is not asymptotically stable at the origin,

maybe there are some inputs  $u \in \mathbb{R}^m$  such that

$$\dot{x} = Ax + Bu$$

that we can use to stabilize asymptotically the system?

# LINEAR FEEDBACK

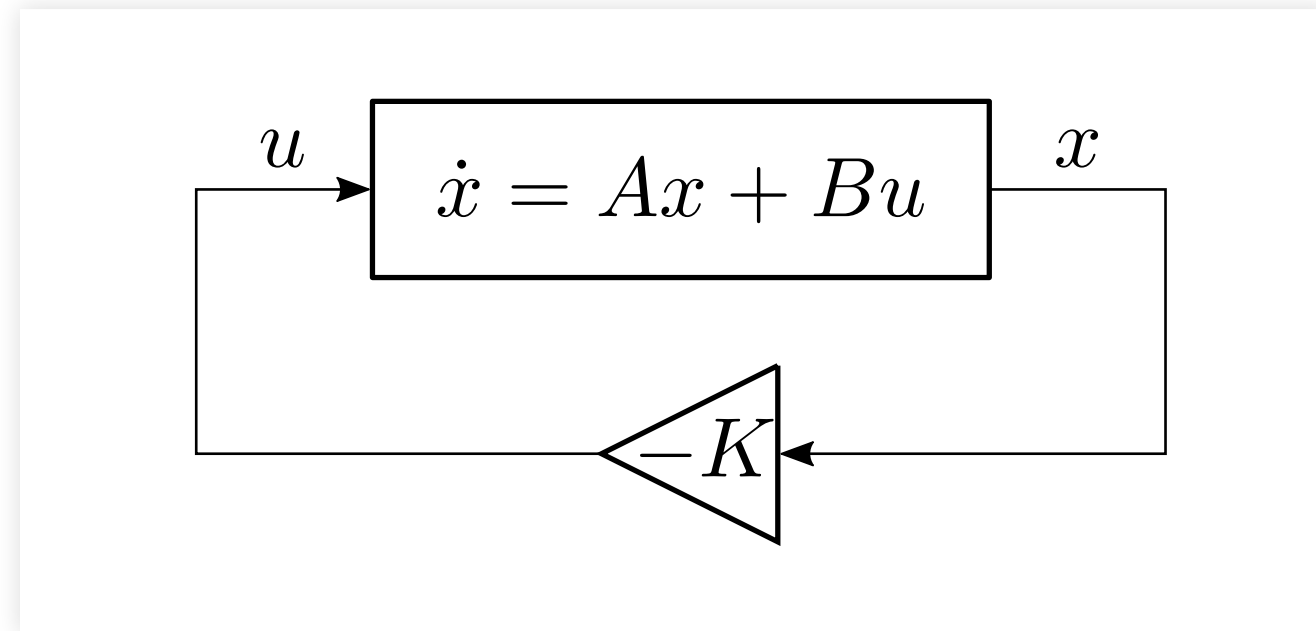
We can try to compute  $u$  as

$$u(t) = -Kx(t)$$

for some  $K \in \mathbb{R}^{m \times n}$

■ **Note.** This strategy requires the system state  $x(t)$  to be known (measured); this information is then **fed back** into the system.

# CLOSED-LOOP DIAGRAM



# CLOSED-LOOP DYNAMICS

When

$$\dot{x} = Ax + Bu$$

$$u = -Kx$$

the state  $x \in \mathbb{R}^n$  evolves according to:

$$\dot{x} = (A - BK)x$$



The closed-loop system is asymptotically stable iff  
every eigenvalue of the matrix

$$A - BK$$

is in the open left-hand plane.

# POLE ASSIGNMENT

Assume that:

- The systems  $\dot{x} = Ax + Bu$  is controllable.
- Let  $\Lambda = \{\lambda_1, \dots, \lambda_n\} \in \mathbb{C}^n$  be a (multi-)set of complex numbers (a value may appear several times) which is symmetric:  
if  $\lambda \in \Lambda$ , then  $\bar{\lambda} \in \Lambda$  (with the same multiplicity)

- Let  $\sigma(A - BK)$  denote the (multi-)set of eigenvalues of  $A - BK$  (eigenvalues are counted with their multiplicity).
- Then there is a matrix  $K$  such that

$$\sigma(A - BK) = \Lambda.$$

# STABILIZATION/POLE ASSIGNMENT

Consider the double integrator  $\ddot{x} = u$

$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

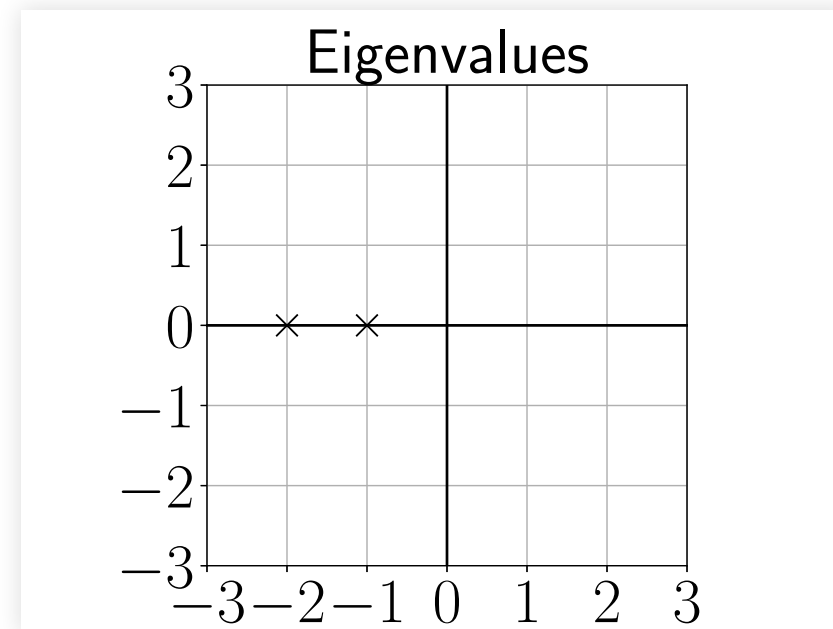
(in standard form)

```
from scipy.signal import place_poles  
A = array([[0, 1], [0, 0]])  
B = array([[0], [1]])  
poles = [-1, -2]  
K = place_poles(A, B, poles).gain_matrix
```

```
assert_almost_equal(K, [[2.0, 3.0]])  
eigenvalues, _ = eig(A - B @ K)  
assert_almost_equal(eigenvalues, [-1, -2])
```



```
figure()
x = [real(s) for s in eigenvalues]
y = [imag(s) for s in eigenvalues]
plot(x, y, "kx", ms=12.0)
xticks([-3, -2, -1, 0, 1, 2, 3])
yticks([-3, -2, -1, 0, 1, 2, 3])
plot([0, 0], [-3, 3], "k")
plot([-3, 3], [0, 0], "k")
```





# IMPLEMENTATION DETAIL

- The `place_poles` function will not accept eigenvalues whose multiplicity is higher than the rank of  $B$ .
- So here `poles = [-1, -1]` won't work.
- But `poles = [-1, -1.001]` may work.

# ② POLE ASSIGNMENT / DEFAULT

Consider system with dynamics

$$\dot{x}_1 = x_1 - x_2 + u$$

$$\dot{x}_2 = -x_1 + x_2 + u$$

- [💡,  $\mathbf{x}^2$ ]. We apply the control law

$$u = -k_1 x_1 - k_2 x_2;$$

can we move the poles of the system where we want by a suitable choice of  $k_1$  and  $k_2$ ?

- [💡] Explain this result.

# ② PENDULUM

Consider the pendulum with dynamics:

$$m\ell^2\ddot{\theta} + b\dot{\theta} + mg\ell \sin \theta = u$$

- [💡,  $\mathbf{x}^2$ ] Compute the linearized dynamics of the system around the equilibrium  $\theta = \pi$  and  $\dot{\theta} = 0$ .



- [💡,  $\mathbf{x}^2$ ] Design a control law

$$u = -k_1(\theta - \pi) - k_2\dot{\theta}$$

such that the closed-loop linear system is asymptotically stable, with a time constant smaller than 10 sec.

Numerical Values:

$$m = 1.0, l = 1.0, b = 0.1, g = 9.81$$

- [, \theta(0) = 0 and  $\dot{\theta}(0) = 0$ ; compare with the open-loop strategy that we have already considered.

# ② DOUBLE SPRING SYSTEM





Consider the dynamics:

$$m_1 \ddot{x}_1 = -k_1 x_1 - k_2(x_1 - x_2) - b_1 \dot{x}_1$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - b_2 \dot{x}_2 + u$$

Numerical values:

$$m_1 = m_2 = 1, \quad k_1 = 1, k_2 = 100, \quad b_1 = 0, \quad b_2 = 20$$

- [, - [, 

# OPTIMAL CONTROL



# WHY ?

## Limitations of Pole Assignment

- it is not always obvious what set of poles we should target (especially for large systems),
- we do not control explicitly the trade-off between “speed of convergence” and “intensity of the control” (large input values maybe costly or impossible).

Let

$$\dot{x} = Ax + Bu$$

where

- $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{m \times n}$  and
- $x(0) = x_0 \in \mathbb{R}^n$  is given.

Find  $u(t)$  that minimizes

$$J = \int_0^{+\infty} x(t)^t Q x(t) + u(t)^t R u(t) dt$$

where:

- $Q \in \mathbb{R}^{n \times n}$  and  $R \in \mathbb{R}^{m \times m}$ ,
- (to be continued ...)

- $Q$  and  $R$  are **symmetric** ( $R^t = R$  and  $Q^t = Q$ ),
- $Q$  and  $R$  are **positive definite** (denoted “ $> 0$ ”)

$$x^t Q x \geq 0 \text{ and } x^t Q x = 0 \text{ iff } x = 0$$

and

$$u^t R u \geq 0 \text{ and } u^t R u = 0 \text{ iff } u = 0.$$

# HEURISTICS / SCALAR CASE

If  $x \in \mathbb{R}$  and  $u \in \mathbb{R}$ ,

$$J = \int_0^{+\infty} qx(t)^2 + ru(t)^2 dt$$

with  $q > 0$  and  $r > 0$ .

When we minimize  $J$ :

- Only the relative values of  $q$  and  $r$  matters.
- Large values of  $q$  penalize strongly non-zero states:  
 $\Rightarrow$  fast convergence.
- Large values of  $r$  penalize strongly non-zero inputs:  
 $\Rightarrow$  small input values.

# HEURISTICS / VECTOR CASE

If  $x \in \mathbb{R}^n$  and  $u \in \mathbb{R}^m$  and  $Q$  and  $R$  are diagonal,

$$Q = \text{diag}(q_1, \dots, q_n), \quad R = \text{diag}(r_1, \dots, r_m),$$

$$J = \int_0^{+\infty} \sum_i q_i x_i(t)^2 + \sum_j r_j u_j(t)^2 dt$$

with  $q_i > 0$  and  $r_j > 0$ .

Thus we can control the cost of each component of  $x$  and  $u$  independently.

# OPTIMAL SOLUTION

Assume that  $\dot{x} = Ax + Bu$  is controllable.

There is an optimal solution; it is a linear feedback

$$u = -Kx$$

The corresponding closed-loop dynamics is asymptotically stable.



# ALGEBRAIC RICCATI EQUATION

The gain matrix  $K$  is given by

$$K = R^{-1} B^t \Pi,$$

where  $\Pi \in \mathbb{R}^{n \times n}$  is the unique matrix such that

$$\Pi^t = \Pi, \Pi > 0 \text{ and}$$

$$\Pi B R^{-1} B^t \Pi - \Pi A - A^t \Pi - Q = 0.$$

## ② VALUE OF $J$

Consider the dynamics  $\dot{x} = Ax + Bu$  where  $u = -Kx$  is the optimal control associated to

$$J = \int_0^{+\infty} j(x(t), u(t)) dt$$

where

$$j(x, u) = x^t Q x + u^t R u.$$

- [💡,  $\mathbf{x}^2$ ] Show that

$$\dot{j}(x(t), u(t)) = -\frac{d}{dt}x(t)^t \Pi x(t)$$

- [💡,  $\mathbf{x}^2$ ] What is the value of  $J$  in the optimal case?

# STABILIZATION/OPTIMAL CONTROL

Consider the double integrator  $\ddot{x} = u$

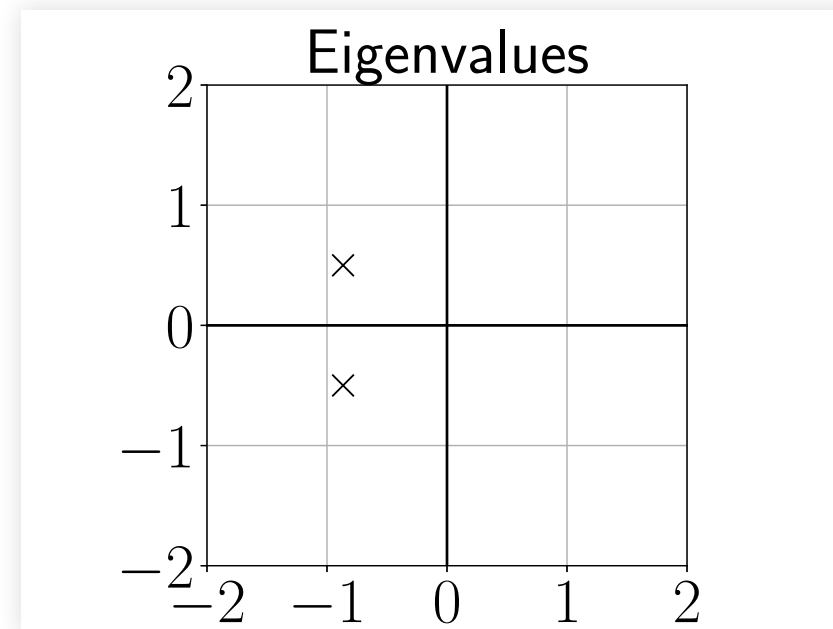
$$\frac{d}{dt} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

(in standard form)

```
from scipy.linalg import solve_continuous_are
A = array([[0, 1], [0, 0]])
B = array([[0], [1]])
Q = array([[1, 0], [0, 1]]); R = array([[1]])
Pi = solve_continuous_are(A, B, Q, R)
K = inv(R) @ B.T @ Pi
eigenvalues, _ = eig(A - B @ K)
assert all([real(s) < 0 for s in eigenvalues])
```



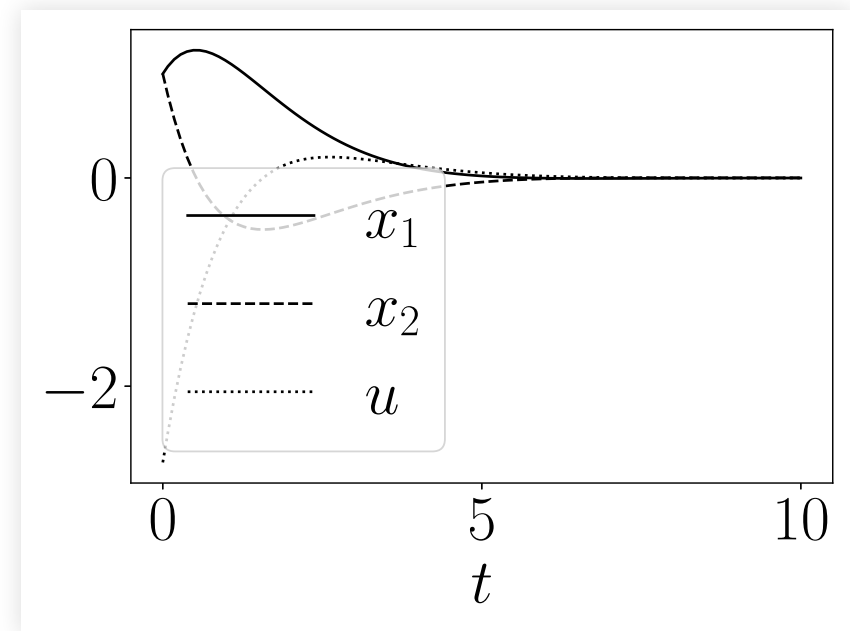
```
figure()
x = [real(s) for s in eigenvalues]
y = [imag(s) for s in eigenvalues]
plot(x, y, "kx", ms=12.0)
xticks([-2, -1, 0, 1, 2])
yticks([-2, -1, 0, 1, 2])
plot([0, 0], [-2, 2], "k")
plot([-2, 2], [0, 0], "k")
```



```
y0 = [1.0, 1.0]
def f(t, x):
    return (A - B.dot(K)).dot(x)
result = solve_ivp(f, t_span=[0, 10], y0=y0,
max_step=0.1)
t = result["t"]
x1 = result["y"][0]
x2 = result["y"][1]
```



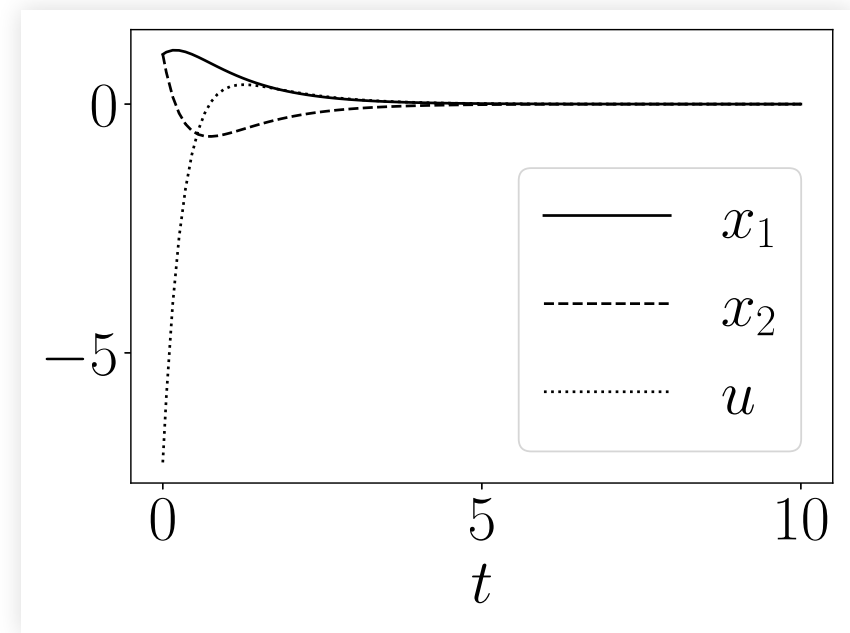
```
figure()  
plot(t, x1, "k-", label="$x_1$")  
plot(t, x2, "k--", label="$x_2$")  
plot(t, u, "k:", label="$u$")  
xlabel("$t$")  
legend()
```



```
Q = array([[10, 0], [0, 10]]); R = array([[1]])  
Pi = solve_continuous_are(A, B, Q, R)  
K = inv(R) @ B.T @ Pi
```

```
result = solve_ivp(f, t_span=[0, 10], y0=y0,  
max_step=0.1)  
t = result["t"]  
x1 = result["y"][0]  
x2 = result["y"][1]  
u = -K.dot(result["y"]).flatten()
```

```
figure()  
plot(t, x1, "k-", label="$x_1$")  
plot(t, x2, "k--", label="$x_2$")  
plot(t, u, "k:", label="$u$")  
xlabel("$t$")  
legend()
```

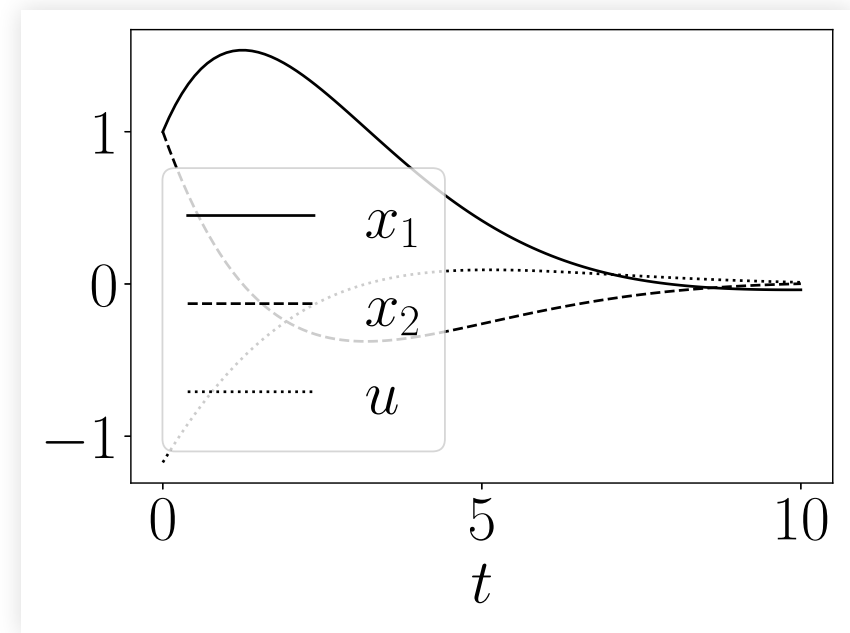


```
Q = array([[1, 0], [0, 1]]); R = array([[10]])  
Pi = solve_continuous_are(A, B, Q, R)  
K = inv(R) @ B.T @ Pi
```

```
result = solve_ivp(f, t_span=[0, 10], y0=y0,  
max_step=0.1)  
t = result["t"]  
x1 = result["y"][0]  
x2 = result["y"][1]  
u = -K.dot(result["y"]).flatten()
```



```
figure()  
plot(t, x1, "k-", label="$x_1$")  
plot(t, x2, "k--", label="$x_2$")  
plot(t, u, "k:", label="$u$")  
xlabel("$t$")  
legend()
```



# EXTRA EXERCISE

- “Lunar lander” for “rendez-vous” with limited fuel?