20

Dynamics analysis of systems with distributed properties

The dynamic analysis of structures, modeled as lumped parameter systems with discrete coordinates was presented in Part I for single degree-of-freedom systems and in Parts II and III for multidegree-of-freedom systems. Modeling structures with discrete coordinates provides a practical approach for the analysis of structures subjected to dynamic loads. However, the results obtained from these discrete models can only give approximate solutions to the actual behavior of dynamic systems which have continuous distributed properties and, consequently, an infinite number of degrees of freedom.

The present chapter considers the dynamic theory of beams and rods having distributed mass and elasticity for which the governing equations of motion are partial differential equations. The integration of these equations is in general more complicated than the solution of ordinary differential equations governing discrete dynamic systems. Due to this mathematical complexity, the dynamic analysis of structures as continuous systems has limited use in practice. Nevertheless, the analysis, as continuous systems, of some simple structures provides, without much effort, results which are of great importance in assessing approximate methods based on discrete models.

20.1 FLEXURAL VIBRATION OF UNIFORM BEAMS

The treatment of beam flexure developed in this section is based on the simple bending theory as it is commonly used for engineering purposes. The method of analysis is known as the Bernoulli-Euler theory which assumes that a plane cross section of a beam remains plane during flexure.

Consider in Fig. 20.1 the free body diagram of a short segment of a beam. It is of length dx and is bounded by plane faces which are perpendicular to its axis. The forces and moments which act on the element are also shown in the figure: they are the shear forces V and $V + (\partial V/\partial x)$; the bending moments M and $M + (\partial M/\partial x)$; the lateral load pdx; and the inertia force $(mdx)\partial^2 y/dt^2$. In this notation m is the mass per unit length and $p = p(x, t)$ is the load per unit length. Partial derivatives are used to express acceleration and variations of shear and moment because these quantities are functions of two variables, position x along the beam and time t. If the deflection of the beam is small, as the theory presupposes, the inclination of the beam segment from the unloaded position is also small. Under these conditions, the equation of motion perpendicular to the x axis of the deflected beam obtained by equating to zero the sum of the forces in the free body diagram of Fig. 20.1(b) is

ppio D´alembert

$$V - \left(V + \frac{\partial V}{\partial x}\,dx\right) + p(x, t)\,dx - mdx\,\frac{\partial^2 y}{\partial t^2} = 0$$

which, upon simplification, becomes

$$\frac{\partial V}{\partial x} + m\,\frac{\partial^2 y}{\partial t^2} = p(x, t). \qquad (20.1)$$

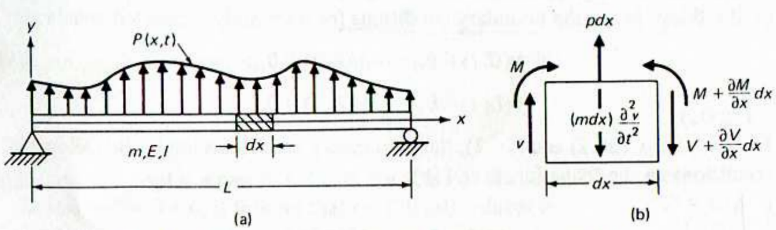


Fig. 20.1 Simple beam with distributed mass and load.

From simple bending theory, we have the relations

$$M = EI\,\frac{\partial^2 y}{\partial x^2} \qquad (20.2)$$

and

$$V = \frac{\partial M}{\partial x} \qquad (20.3)$$

where E is Young's modulus of elasticity and I is the moment of inertia of the cross-sectional area with respect to the neutral axis through the centroid. For a uniform beam, the combination of eqs. (20.1), (20.2), and (20.3) results in

$$V = EI\,\frac{\partial^3 y}{\partial x^3} \qquad (20.4)$$

and

$$\boxed{EI\,\frac{\partial^4 y}{\partial x^2} + m\,\frac{\partial^2 y}{\partial t^2} = p(x, t).} \qquad (20.5)$$

It is seen that eq. (20.5) is a partial differential equation of fourth order. It is an approximate equation. Only lateral flexural deflections were considered while the deflections due to shear forces and the inertial forces caused by the rotation of the cross section (rotary inertia) were neglected. The inclusion of shear deformations and rotary intertia in the differential equation of motion considerably increases its complexity. The equation taking into consideration shear deformation and rotary inertia is known as Timoshenko's equation. The differential equation, eq. (20.5), also does not include the flexural effects due to the presence of forces which may be applied axially to the beam. The axial effects will be discussed in Chapter 21.

20.2 SOLUTION OF THE EQUATION OF MOTION IN FREE VIBRATION

For free vibration ($p(x, t) = 0$), eq. (20.5) reduces to the homogeneous differential equation

$$EI\,\frac{\partial^4 y}{\partial x^4} + m\,\frac{\partial^2 y}{\partial t^2} = 0. \qquad (20.6)$$

The solution of eq. (20.6) can be found by the method of separation of variables. In this method, it is assumed that the solution may be expressed as the product of a function of position $\Phi(x)$ and a function of time $f(t)$, that is,

$$y(x, t) = \Phi(x) f(t). \qquad (20.7)$$

The substitution of eq. (20.7) in the differential equation, eq. (20.6), leads to

$$EIf(t)\frac{d^4\Phi(x)}{dx^4} + m\Phi(x)\frac{d^2f(t)}{dt^2} = 0. \tag{20.8}$$

This last equation may be written as

$$\frac{EI}{m}\frac{\Phi^{\mathrm{IV}}(x)}{\Phi(x)} = -\frac{\ddot{f}(t)}{f(t)} \;=\; \omega^2 = cte. \tag{20.9}$$

In this notation Roman indices indicate derivatives with respect to x and overdots indicate derivatives with respect to time. Since the left-hand side of eq. (20.9) is a function only of x while the right-hand side is a function only of t, each side of the equation must equal the same constant value; otherwise, the identity of eq. (20.9) cannot exist. We designate the constant by ω^2 which equated separately to each side of eq. (20.9) results in the two following differential equations:

$$\Phi^{\mathrm{IV}}(x) - a^4\Phi(x) = 0 \tag{20.10}$$

and

$$\ddot{f}(t) + \omega^2 f(t) = 0 \tag{20.11}$$

where

$$a^4 = \frac{m\omega^2}{EI}. \tag{20.12}$$

It is particularly convenient to solve eq. (20.12) for ω and to use the following notation, namely

$= a^4 = (aL)^2\sqrt{\frac{EI}{mL^4}}$

$$\omega = C\sqrt{\frac{EI}{mL^4}} \tag{20.13}$$

in which $C = (aL)^2$.

Equation (20.11) is the familiar free-vibration equation for the undamped single degree-of-freedom system and its solution from eq. (1.17) is

$$f(t) = A\cos\omega t + B\sin\omega t \tag{20.14}$$

where A and B are constants of integration. Equation (20.10) can be solved by letting (condic. iniciale)

$$\Phi(x) = Ce^{sx}. \tag{20.15}$$

The substitution of eq. (20.15) into eq. (20.10) results in

$$(s^4 - a^4)Ce^{sx} = 0$$

e^{sx}

which, for a nontrivial solution, requires that

$$s^4 - a^4 = 0. \tag{20.16}$$

The roots of eq. (20.16) are

$$s_1 = a, \qquad s_3 = ai,$$
$$s_2 = -a, \qquad s_4 = -ai. \tag{20.17}$$

The substitution of each of these roots into eq. (20.15) provides a solution of eq. (20.10). The general solution is then given by the superposition of these four possible solutions, namely

$$\Phi(x) = C_1e^{ax} + C_2e^{-ax} + C_3e^{iax} + C_4e^{-iax} \tag{20.18}$$

where C_1, C_2, C_3, and C_4 are constants of integration. The exponential functions in eq. (20.18) may be expressed in terms of trigonometric and hyperbolic functions by means of the relations

$$e^{\pm ax} = \cosh ax \pm \sinh ax$$
$$e^{\pm iax} = \cos ax \pm i\sin ax. \tag{20.19}$$

Substitution of these relations into eq. (20.18) yields

$$\Phi(x) = A\sin ax + B\cos ax + C\sinh ax + D\cosh ax \tag{20.20}$$

where A, B, C, D are new constants of integration. These four constants of integration define the shape and the amplitude of the beam in free vibration; they are evaluated by considering the boundary conditions at the ends of the beam as illustrated in the examples presented in the following section.

20.3 NATURAL FREQUENCIES AND MODE SHAPES FOR UNIFORM BEAMS

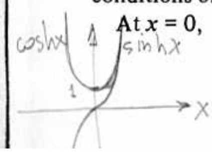
20.3.1 Both Ends Simply Supported

In this case the displacements and bending moments must be zero at both ends of the beam; hence the boundary conditions for the simply supported beams are

$$y(0, t) = 0, \qquad M(0, t) = 0,$$
$$y(L, t) = 0, \qquad M(L, t) = 0.$$

In view of eqs. (20.2) and (20.7), these boundary conditions imply the following conditions on the shape function $\Phi(x)$.

At $x = 0$,



$$\Phi(0) = 0, \qquad \Phi''(0) = 0. \tag{20.21}$$

$\Phi''(x) = -a^2A\sin ax - a^2B\cos ax + a^2C\sinh ax + a^2D\cosh ax$

At $x = L$,

$$\Phi(L) = 0, \qquad \Phi''(L) = 0. \tag{20.22}$$

The substitution of the first two of these boundary conditions into eq. (20.20) yields

$$\Phi(0) = A0 + B1 + C0 + D1 = 0,$$
$$\Phi''(0) = a^2(-A0 - B1 + C0 + D1) = 0$$

which reduce to

$$B + D = 0$$
$$-B + D = 0.$$

Hence

$$B = D = 0.$$

Similarly, substituting the last two boundary conditions into eq. (20.20) and setting $B = D = 0$ leads to

$$\Phi(L) = A \sin aL + C \sinh aL = 0$$
$$\Phi''(L) = a^2(-A \sin aL + C \sinh aL) = 0 \tag{20.23}$$

which, when added, give

$$2C \sinh aL = 0.$$

From this last relation, $C = 0$ since the hyperbolic sine function cannot vanish except for a zero argument. Thus eqs. (20.23) reduce to

$$A \sin aL = 0. \tag{20.24}$$

Excluding the trivial solution ($A = 0$), we obtain the frequency equation

$$\sin aL = 0 \qquad \text{Eq. frequency} \tag{20.25}$$

which will be satisfied for

$$a_n L = n\pi, \qquad n = 0, 1, 2 \ldots. \tag{20.26}$$

Substitution of the roots, eq. (20.26), into eq. (20.13) yields

$$\omega_n = n^2\pi^2 \sqrt{\frac{EI}{mL^4}} \tag{20.27}$$

where the subscript n serves to indicate the order of the natural frequencies.

Since $B = C = D = 0$, it follows that eq. (20.20) reduces to

$$\Phi_n(x) = A \sin \frac{n\pi x}{L}$$

or simply

$$\Phi_n(x) = \sin \frac{n\pi x}{L} \tag{20.28}$$

We note that in eq. (20.28) the constant A is absorbed by the other constants in the modal response given below by eq. (20.29).

From eq. (20.7) a modal shape or normal mode of vibration is given by

$$y_n(x, t) = \Phi_n(x) f_n(t)$$

or from eqs. (20.14) and (20.28) by

$$y_n(x, t) = \sin \frac{n\pi x}{L} \left[A_n \cos \omega_n t + B_n \sin \omega_n t\right]. \tag{20.29}$$

The general solution of the equation of motion in free vibration which satisfies the boundary conditions, eqs. (20.21) and (20.22), is the sum of all the normal modes of vibration, eq. (20.29), namely

$$y(x, t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{L} \left[A_n \cos \omega_n t + B_n \sin \omega_n t\right]. \tag{20.30}$$

The constants A_n and B_n are determined, as usual, from the initial conditions. If at $t = 0$, the shape of the beam is given by C.I.

$$y(x, 0) = \rho(x)$$

and the velocity by

$$\frac{\partial y(x, 0)}{\partial t} = \psi(x)$$

for $0 \leqslant x \leqslant L$, it follows from eq. (20.30) that

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = \rho(x)$$

and

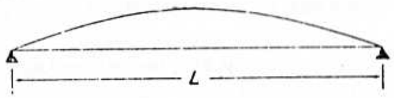
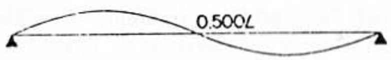
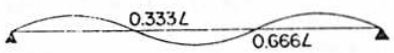
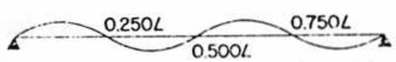
$$\sum_{n=1}^{\infty} B_n \omega_n \sin \frac{n\pi x}{L} = \psi(x).$$

Therefore, as shown in Chapter 5, Fourier coefficients are expressed as
pág 78

$$A_n = \frac{2}{L} \int_0^L \rho(x) \sin \frac{n\pi x}{L}\, dx$$

$$B_n = \frac{2}{L} \int_0^L \psi(x) \sin \frac{n\pi x}{L}\, dx. \tag{20.31}$$

TABLE 20.1 Natural Frequencies and Normal Modes for Simply Supported Beams.

Natural Frequencies			Normal Modes
$\omega_n = C_n \sqrt{\frac{EI}{mL^4}}$			$\Phi_n = \sin \frac{nx\pi}{L}$
n	$C_n = (aL)^2$	I_n^*	Shapes
1	π^2	$4/\pi$	
2	$4\pi^2$	0	
3	$9\pi^2$	$4/3\pi$	
4	$16\pi^2$	0	
5	$25\pi^2$	$4/5\pi$	

$^*I_n = \int_0^L \Phi_n(x)\,dx / \int_0^L \Phi^2(x)\,dx$.

The first five values for the natural frequencies and normal modes for the simply supported beam are presented in Table 20.1.

20.3.2 Both Ends Free (Free Beam)

The boundary conditions for a beam with both ends free are as follows.

At $x = 0$,

$$M(0, t) = 0 \quad \text{or} \quad \Phi''(0) = 0,$$
$$V(0, t) = 0 \quad \text{or} \quad \Phi'''(0) = 0.$$

At $x = L$,

$$M(L, t) = 0 \quad \text{or} \quad \Phi''(L) = 0,$$
$$V(L, t) = 0 \quad \text{or} \quad \Phi'''(L) = 0. \tag{20.33}$$

The substitutions of these conditions in eq. (20.20) yield

$$\Phi''(0) = a^2(-B + D) = 0,$$
$$\Phi'''(0) = a^3(-A + C) = 0,$$

and

$$\Phi''(L) = a^2(-A \sin aL - B \cos aL + C \sinh aL + D \cosh aL) = 0$$
$$\Phi'''(L) = a^3(-A \cos aL + B \sin aL + C \cosh aL + D \sinh aL) = 0.$$

From the first two equations we obtain

$$D = B, \qquad C = A \tag{20.34}$$

which, substituted into the last two equations, result in

$$(\sinh aL - \sin aL)A + (\cosh aL - \cos aL)B = 0,$$
$$(\cosh aL - \cos aL)A + (\sinh aL + \sin aL)B = 0. \tag{20.35}$$

For nontrivial solution of eqs. (20.35), it is required that the determinant of the unknown coefficients A and B be equal to zero; hence

$$\begin{vmatrix} \sinh al - \sin aL & \cosh al - \cos aL \\ \cosh aL - \cos aL & \sinh aL - \sin aL \end{vmatrix} = 0. \tag{20.36}$$

The expansion of this determinant provides the frequency equation for the free beam, namely

$$\cos aL \cdot \cosh aL - 1 = 0. \tag{20.37}$$

The first five natural frequencies which are obtained by substituting the roots of eq. (20.37) into eq. (20.13) are presented in Table 20.2. The corresponding normal modes are obtained by letting $A = 1$ (normal modes are determined only to a relative magnitude), substituting in eqs. (20.35) the roots of a_n of eq. (20.37), solving one of these equations for B, and finally introducing into eq. (20.20) the constants C, D from eq. (20.34) together with B. Performing these operations, we obtain

$$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n(\sinh a_n x + \sin a_n x) \tag{20.38}$$

where

$$\sigma_n = \frac{\cosh a_n L - \cos a_n L}{\sinh a_n L - \sin a_n L}. \tag{20.39}$$

TABLE 20.2 Natural Frequencies and Normal Modes for Free Beams.

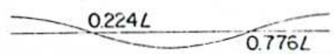
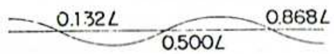
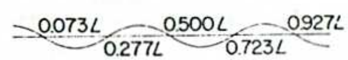
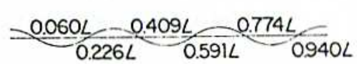
Natural Frequencies

$$\omega_n = C_n \sqrt{\frac{EI}{mL^4}}$$

Normal Modes

$$\Phi_n(x) = \cosh a_n x + \cos a_n x - \sigma_n(\sinh a_n x + \sin a_n L)$$

$$\sigma_n = \frac{\cosh a_n x - \cos a_n L}{\sinh a_n x - \sin a_n L}$$

n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shapes
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.000000	0.2323	

$^*I_n = \int_0^L \Phi_n(x)\,dx / \int_0^L \Phi_n^2(x)\,dx.$

20.3.3 Both Ends Fixed

The boundary conditions for a beam with both ends fixed are as follows.

At $x = 0$,

$$y(0, t) = 0 \quad \text{or} \quad \Phi(0) = 0,$$
$$y'(0, t) = 0 \quad \text{or} \quad \Phi'(0) = 0. \tag{20.40}$$

At $x = L$,

$$y(L, t) = 0 \quad \text{or} \quad \Phi(L) = 0,$$
$$y'(L, t) = 0 \quad \text{or} \quad \Phi'(L) = 0. \tag{20.41}$$

The use of the boundary conditions, eqs. (20.40), into eq. (20.20) gives

$$A = C = 0$$

while conditions, eqs. (20.41), yield the homogeneous system

$$(\cos aL - \cosh aL)B + (\sin aL - \sinh aL)D = 0,$$
$$-(\sin aL + \sinh al)B + (\cos aL - \cosh aL)D = 0. \tag{20.42}$$

Equating to zero the determinant of the coefficients of this system results in the frequency equation

$$\cos a_n L \cosh a_n L - 1 = 0. \tag{20.43}$$

TABLE 20.3 Natural Frequencies and Normal Modes for Fixed Beams.

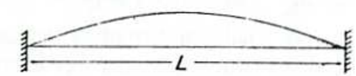
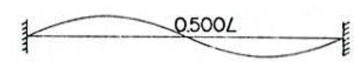
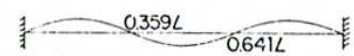
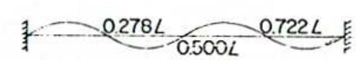
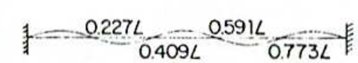
Natural Frequencies

$$\omega_n = C_n \sqrt{\frac{EI}{mL^4}}$$

Normal Modes

$$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n(\sinh a_n x - \sin a_n x)$$

$$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}$$

n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	22.3733	0.982502	0.8308	
2	61.6728	1.000777	0	
3	120.9034	0.999967	0.3640	
4	199.8594	1.000001	0	
5	298.5555	1.000000	0.2323	

$^*I_n = \int_0^L \Phi_n(x)\,dx / \int_0^L \Phi_n^2(x)\,dx.$

From the first of eqs. (20.42), it follows that

$$D = -\frac{\cos aL - \cosh aL}{\sin aL - \sinh aL} B, \tag{20.44}$$

where B is arbitrary. To each value of the natural frequency

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{mL^4}} \tag{20.45}$$

obtained by the substitution of the roots of eq. (20.43) into eq. (20.13), there corresponds a normal mode

$$\Phi_n(x) = \cosh a_n x - \cos a_n x - \sigma_n(\sinh a_n x - \sin a_n x), \tag{20.46}$$

$$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}. \tag{20.47}$$

The first five natural frequencies calculated from eqs. (20.43) and (20.45) and the corresponding normal modes obtained from eq. (20.46) are presented in Table 20.3

20.3.4 One End Fixed and the Other End Free (Cantilever Beam)

At the fixed end ($x = 0$) of the cantilever beam, the deflection and the slope must be zero and at the free end ($x = L$) the bending moment and the shear force must be zero. Hence the boundary conditions for this beam are as follows.

At $x = 0$,

$$\begin{aligned} y(0, t) &= 0 \quad \text{or} \quad \Phi(0) = 0, \\ y'(0, t) &= 0 \quad \text{or} \quad \Phi'(0) = 0. \end{aligned} \tag{20.48}$$

At $x = L$,

$$\begin{aligned} M(L, t) &= 0 \quad \text{or} \quad \Phi''(L) = 0, \\ V(L, t) &= 0 \quad \text{or} \quad \Phi'''(L) = 0. \end{aligned} \tag{20.49}$$

These boundary conditions when substituted into the shape equation, eq. (20.20), lead to the frequency equation

$$\cos a_n L \cosh a_n + 1 = 0. \tag{20.50}$$

To each root of eq. (20.50) corresponds a natural frequency

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{mL^4}} \tag{20.51}$$

and a normal shape

$$\Phi_n(x) = (\cosh a_n x - \cos a_n x) - \sigma_n(\sinh a_n x - \sin a_n x), \tag{20.52}$$

where

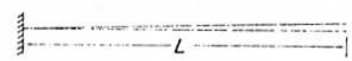
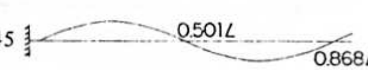
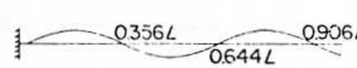
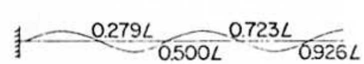
$$\sigma_n = \frac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L} \tag{20.53}$$

The first five natural frequencies and the corresponding mode shapes for cantilever beams are presented in Table 20.4.

TABLE 20.4 Natural Frequencies and Normal Modes for Cantilever Beams.

Natural Frequencies: $\omega_n = C_n \sqrt{\dfrac{EI}{mL^4}}$

Normal Modes: $\Phi_n = (\cosh a_n x - \cos a_n x) - \sigma_n(\sinh a_n x - \sin a_n x)$, $\sigma = \dfrac{\cos a_n L + \cosh a_n L}{\sin a_n L + \sinh a_n L}$

n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	15.4118	1.000777	0.8600	
2	49.9648	1.000001	0.0826	
3	104.2477	1.000000	0.03345	
4	178.2697	1.000000	0.0434	
5	272.0309	1.000000	0.2076	

$^*I_n = \int_0^L \Phi_n(x)\,dx / \int_0^L \Phi_n^2(x)\,dx.$

20.3.5 One End Fixed and the Other Simply Supported

The boundary conditions for a beam with one end fixed and the other simply supported are as follows.

At $x = 0$,

$$y(0, t) = 0 \quad \text{or} \quad \Phi(0) = 0,$$
$$y'(0, t) = 0 \quad \text{or} \quad \Phi'(0) = 0. \tag{20.54}$$

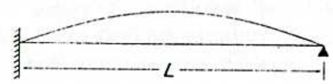
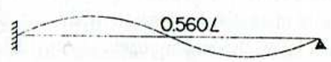
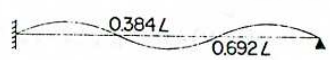
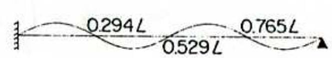
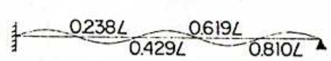
At $x = L$,

$$y(L, t) = 0 \quad \text{or} \quad \Phi(L) = 0,$$
$$M(L, t) = 0 \quad \text{or} \quad \Phi''(L) = 0. \tag{20.55}$$

TABLE 20.5 Natural Frequencies and Normal Modes for Fixed-Simply Supported Beams.

Natural Frequencies: $\omega_n = C_n \sqrt{\dfrac{EI}{mL^4}}$

Normal Modes: $\Phi(x) = \cosh a_n x - \cos a_n x + \sigma_n(\sinh a_n x - \sin a_n x)$, $\quad \sigma_n = \dfrac{\cos a_n L - \cosh a_n L}{\sinh a_n L - \sin a_n L}$

n	$C_n = (a_n L)^2$	σ_n	I_n^*	Shape
1	15.4118	1.000777	0.8600	
2	49.9648	1.000001	0.0826	
3	104.2477	1.000000	0.3345	
4	178.2697	1.000000	0.0434	
5	272.0309	1.000000	0.2076	

$^*I_n = \int_0^L \Phi_n(x)\,dx \big/ \int_0^L \Phi_n^2(x)\,dx.$

The substitution of these boundary conditions into the shape equation, eq. (20.20), results in the frequency equation

$$\tan a_n L - \tanh a_n L = 0. \tag{20.56}$$

To each root of this last equation corresponds a natural frequency

$$\omega_n = (a_n L)^2 \sqrt{\frac{EI}{mL^4}} \tag{20.57}$$

and a normal mode

$$\Phi_n(x) = (\cosh a_n x - \cos a_n x) + \sigma_n(\sinh a_n x - \sin a_n x) \tag{20.58}$$

where

$$\sigma_n = \frac{\cos a_n L - \cosh a_n L}{\sin a_n L - \sinh a_n L}. \tag{20.59}$$

The first five natural frequencies for the fixed simply supported beam and corresponding mode shapes are presented in Table 20.5.

20.4 ORTHOGONALITY CONDITION BETWEEN NORMAL MODES

The most important property of the normal modes is that of orthogonality. It is this property which makes possible the uncoupling of the equations of motion as it has previously been shown for discrete systems. The orthogonality property for continuous systems can be demonstrated in essentially the same way as for discrete parameter systems.

Consider in Fig. 20.2 a beam subjected to the inertial forces resulting from the

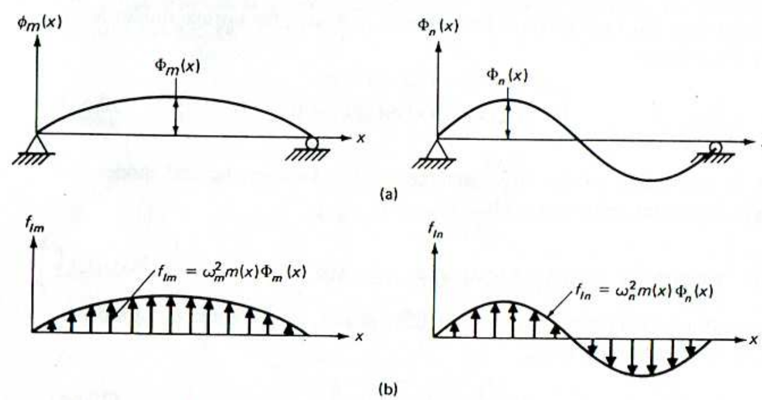


Fig. 20.2 Beam showing two modes of vibration and inertial forces. (a) Displacements. (b) Inertial forces.

$\omega_n^2 m(x) \phi_n(x) = EI\phi_n^{IV}(x) = a^4 EI \Phi_n(x)$

$\omega_n^2 \int \phi_m(x)\, m(x)\, \phi_n(x)\, dx = \int \Phi_m(x)\, a^4 EI \Phi_n(x)\, dx$ = 2do cond. ortog.

vibrations of two different modes, $\Phi_m(x)$ and $\Phi_n(x)$. The deflection curves for these two modes and the corresponding inertial forces are depicted in the same figure. Betti's law is applied to these two deflection patterns. Accordingly, the work done by the inertial force, f_{In}, acting on the displacements of mode m is equal to the work of the inertial forces, f_{Im}, acting on the displacements of mode n, that is,

$$\int_0^L \Phi_m(x)\, f_{In}(x)\, dx = \int_0^L \Phi_n(x)\, f_{Im}(x)\, dx. \tag{20.60}$$

The inertial force f_{In} per unit length along the beam is equal to the mass per unit length times the acceleration. Inasmuch as the vibratory motion in a normal mode is harmonic, the amplitude of the acceleration is given by $\omega_n^2 \Phi_n(x)$. Hence the inertial force per unit length along the beam for the nth mode is

$$f_{In} = \omega_n^2 m(x)\, \phi_n(x)$$

and for the mth mode

$$f_{Im} = \omega_m^2 m(x)\, \Phi_m(x). \tag{20.61}$$

Substituting these expressions in eq. (20.60), we obtain

$$\omega_n^2 \int_0^L \Phi_m(x)\, m(x)\, \Phi_n(x)\, dx = \omega_m^2 \int_0^L \Phi_n(x)\, m(x)\, \Phi_m(x)\, dx$$

which may be written as

$$(\omega_n^2 - \omega_m^2) \int_0^L \Phi_m(x)\, \Phi_n(x)\, m(x)\, dx = 0. \tag{20.62}$$

It follows that, for two different frequencies $\omega_n \neq \omega_m$, the normal modes must satisfy the relation

1ro cond ortog.

$$\int_0^L \Phi_m(x)\, \Phi_n(x)\, m(x)\, dx = 0 \tag{20.63}$$

which is equivalent to the orthogonal condition between normal modes for discrete parameter systems, eq. (10.27).

20.5 FORCED VIBRATION OF BEAMS (Superposición modal)

For a uniform beam acted on by lateral forces $p(x, t)$, the equation of motion, eq. (20.5), may be written as

$$EI \frac{\partial^4 y}{\partial x^4} = p(x, t) - m \frac{\partial^2 y}{\partial t^2} \tag{20.64}$$

(20.10) $\Phi^{IV}_{(x)} - a^4 \Phi_{(x)} = 0$ (20.12) $a^4 = \frac{m\omega^2}{EI}$

in which $p(x, t)$ is the external load per unit length along the beam. We assume that the general solution of this equation may be expressed by the summation of the products of the normal modes $\Phi_n(x)$ multiplied by factors $z_n(t)$ which are to be determined. Hence

$$y(x, t) = \sum_{n=1}^{\infty} \Phi_n(x)\, z_n(t). \tag{20.65}$$

The normal modes $\Phi_n(x)$ satisfy the differential equation, eq. (20.10), which by eq. (20.12) may be written as

ó ec(20.9)

$$EI\, \Phi_n^{IV}(x) = m\omega_n^2\, \Phi_n(x), \qquad n = 1, 2, 3, \ldots . \tag{20.66}$$

The normal modes should also satisfy the specific force boundary conditions at the ends of the beam. Substitution of eq. (20.65) in eq. (20.64) gives

$$EI \sum_n \Phi_n^{IV}(x)\, z_n(t) = p(x, t) - m \sum_n \phi_n(x)\, \ddot{z}_n(t). \tag{20.67}$$

In view of eq. (20.66), we can write eq. (20.67) as

$$\sum_n m\omega_n^2\, \Phi_n(x)\, z_n(t) = p(x, t) - m \sum_n \Phi_n(x)\, \ddot{z}_n(t). \tag{20.68}$$

Multiplying both sides of eq. (20.68) by $\Phi_m(x)$ and integrating between 0 and L result in

$$\omega_m^2 z_m(t) \underbrace{\int_0^L m\Phi_m^2(x)\, dx}_{M_n} = \underbrace{\int_0^L \phi_m(x)\, p(x, t)\, dx}_{F_n} - \ddot{z}_m(t) \underbrace{\int_0^L m\Phi_m^2(x)\, dx.}_{M_n} \tag{20.69}$$

We note that all the terms which contain products of different indices ($n \neq m$) vanish from the summations in eq. (20.68) in view of the orthogonality conditions, eq. (20.63), between normal modes. Equation (20.69) may conveniently be written as

$$M_n \ddot{z}_n(t) + \omega_n^2 M_n z_n(t) = F_n(t), \qquad n = 1, 2, 3, \ldots, m, \ldots \tag{20.70}$$

where

$$M_n = \int_0^L m\Phi_n^2(x)\, dx \tag{20.71}$$

is the modal mass, and

$$F_n(t) = \int_0^L \Phi_n(x)\, p(x, t)\, dx \tag{20.72}$$

is the modal force.

The equation of motion for the nth normal mode, eq. (20.70), is completely analogous to the modal equation, eq. (12.9), for discrete systems. Modal damping could certainly be introduced by simply adding the damping term in eq. (20.70); hence we would obtain

$$M_n\ddot{z}_n(t) + C_n\dot{z}_n(t) + K_n z_n(t) = F_n(t) \tag{20.73}$$

which, upon dividing by M_n, gives

$$\ddot{z}_n(t) + 2\xi_n\omega_n\dot{z}_n(t) + \omega_n^2 z_n(t) = \frac{F_n(t)}{M_n} \tag{20.74}$$

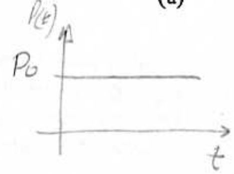
where $\xi_n = C_n/C_{n,cr}$ is the modal damping ratio and $K_n = M_n\omega_n^2$ is the modal stiffness. The total response is then obtained from eq. (20.65) as the superposition of the solution of the modal equation, eq. (20.74), for as many modes as desired. Though the summation in eq. (20.65) is over an infinite number of terms, in most structural problems only the first few modes have any significant contribution to the total response and in some cases the response is given essentially by the contribution of the first mode alone.

The modal equation, eq. (20.74), is completely general and applies to beams with any type of load distribution. If the loads are concentrated rather than distributed, the integral in eq. (20.72) merely becomes a summation having one term for each load. The computation of the integral in eqs. (20.71) and (20.72) becomes tedious except for the simply supported beam because the normal shapes are rather complicated functions. Values of the ratios of these integrals needed for problems with uniform distributed load are presented in the last columns of Tables 20.1 through 20.5 for some common types of beams.

Illustrative Example 20.1. Consider in Fig. 20.3 a simply supported uniform beam subjected to a concentrated constant force suddenly applied at a section x_1 units from the left support. Determine the response using modal analysis.

The modal shapes of a simply supported beam by eq. (20.28) are

$$\Phi_n = \sin\frac{n\pi x}{L}, \qquad n = 1, 2, 3, \ldots \tag{a}$$



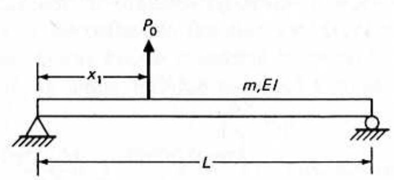


Fig. 20.3 Simply supported beam subjected to a suddenly applied force.

and the modal force by eq. (20.72) is

$$F_n(t) = \int_0^L \Phi_n(x)\, p(x, t)\, dx.$$

$\int_0^L \phi_n(x) P_0\, \delta(x_1 - x)$ $= P_0\, \phi_n(x_1)$

In this problem $p(x, t) = P_0$ at $x = x_1$; otherwise, $p(x, t) = 0$. Hence

$$F_n(t) = P_0\Phi_n(x_1)$$

or using eq. (a), we obtain

$$F_n(t) = P_0 \sin\frac{n\pi x_1}{L}. \tag{b}$$

The modal mass by eq. (20.71) is

$$M_n = \int_0^L m\Phi^2(x)\, dx$$

$\sin^2\frac{n\pi x}{L} = \frac{1}{2}\left(1 - \cos\frac{2n\pi x}{L}\right)$

$$= \int_0^L m \sin^2\frac{n\pi x}{L}\, dx = \frac{mL}{2}. \tag{c}$$

$= m\left[\frac{L}{2} - \frac{L\sin(2n\pi x)}{4n\pi}\right]_0^L = m\frac{L}{2}$

Substituting the modal force, eq. (b), and the modal mass, eq. (c), into the modal equation, eq. (20.70), results in

$=0$

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{P_0 \sin\frac{n\pi x_1}{L}}{mL/2}. \tag{d}$$

F_n / M_n

For initial conditions of zero displacement and zero velocity, the solution of eq. (d) from eqs. (4.5) is

$$z_n = (z_{st})_n(1 - \cos\omega_n t) \tag{e}$$

in which

$\frac{F_m}{K_m}$

$$(z_{st})_n = \frac{2P_0 \sin\frac{n\pi x_1}{L}}{\omega_n^2 mL} \tag{f}$$

$P_0 \sin\frac{n\pi x_1}{L} = F_0$

$\omega_n^2\, m\frac{L}{2} = k$

so that

$$z_n = \frac{2P_0 \sin\frac{n\pi x_1}{L}}{\omega_n^2 mL}(1 - \cos\omega_n t). \tag{g}$$

Eq 4.5 $y = \frac{F_0}{k}(1 - \cos\omega t) = y_{st}(1 - \cos\omega t)$

The modal deflection at any section of the beam is

$$y_n(x, t) = \Phi_n(x) z_n(t) \tag{h}$$

which, upon substitution of eqs. (a) and (g), becomes

$$y_n(x, t) = \frac{2P_0 \sin \dfrac{n\pi x_1}{L}}{\omega_n^2 mL} (1 - \cos \omega_n t) \sin \frac{n\pi x}{L}. \tag{i}$$

By eq. (20.65), the total deflection is then

$$y(x, t) = \frac{2P_0}{mL} \sum_n \left[\frac{1}{\omega_n^2} \sin \frac{n\pi x_1}{L} (1 - \cos \omega_n t) \sin \frac{n\pi x}{L}\right]. \tag{j}$$

As a special case, let us consider the force applied at midspan, i.e., $x_1 = L/2$. Hence eq. (j) becomes in this case

$$y(x, t) = \frac{2P_0}{mL} \sum_n \left[\frac{1}{\omega_n^2} \sin \frac{n\pi}{2} (1 - \cos \omega_n t) \sin \frac{n\pi x}{L}\right]. \tag{20.75}$$

From the latter (due to the presence of the factor $\sin n\pi/2$) it is apparent that all the even modes do not contribute to the deflection at any point. This is true because such modes are antisymmetrical (shapes in Table 20.1) and are not excited by a symmetrical load.

It is also of interest to compare the contribution of the various modes to the deflection at midspan. This comparison will be done on the basis of maximum modal displacement disregarding the manner in which these displacements combine. The amplitudes will indicate the relative importance of the modes. The dynamic load factor $(1 - \cos \omega_n t)$ in eq. (20.75) has a maximum value of 2 for all the modes. Furthermore, since all sines are unity for odd modes and zero for even modes, the modal contributions are simply in proportion to $1/\omega_n^2$.

$= n^2\pi^2\sqrt{\frac{EI}{mL^4}}$ Hence the maximum modal deflections are in proportion to 1, 1/81, and 1/625 for the first, third, and fifth modes, respectively. It is apparent, in this example, that the higher modes contribute very little to the midspan deflection.

$\propto \frac{1}{n^4}$

Illustrative Example 20.2. Determine the maximum deflection at the mid-point of the fixed beam shown in Fig. 20.4 subjected to a harmonic load $p(x, t) = p_0 \sin 300t$ lb/in uniformly distributed along the span. Consider in the analysis the first three modes contributing to the response.

The natural frequencies for uniform beams are given by eq. (20.13) as

$$\omega_n = C_n \sqrt{\frac{EI}{mL^4}}$$

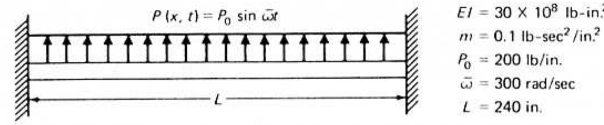


Fig. 20.4 Fixed beam with uniform harmonic load.

or, substituting numerical values for this example, we get

$$\omega_n = C_n \sqrt{\frac{30 \times 10^8}{0.1(240)^4}} \tag{a}$$

where the values of C_n are given for the first five modes in Table 20.3. The deflection of the beam is given by eq. (20.65) as

$$y(x, t) = \sum_{n=1}^{\infty} \Phi_n(x) z_n(t) \tag{b}$$

in which $\Phi_n(x)$ is the modal shape defined for a fixed beam by eq. (20.46) and $z_n(t)$ is the modal response.

The modal equation by eq. (20.70) (neglecting damping) may be written as

$$\ddot{z}_n(t) + \omega_n^2 z(t) = \frac{\int_0^L p(x, t)\phi_n(x)\, dx}{\int_0^L m\phi_n^2(x)\, dx}$$

Then, substituting numerical values to this example, we obtain

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = \frac{200 \int_0^L \phi_n(x)\, dx}{0.1 \int_0^L \phi_n^2(x)\, dx} \sin 300t$$

or

$$\ddot{z}_n(t) + \omega_n^2 z_n(t) = 2000 I_n \sin 300t \tag{c}$$

in which

$$I_n = \frac{\int_0^L \phi_n(x)\, dx}{\int_0^L \phi^2(x)\, dx}$$

TABLE 20.6 Modal Response at Midspan for the Beam in Fig. 20.4.

Mode	$\omega_n \left(\frac{\text{rad}}{\text{sec}}\right)$	$a_n L$	I_n	$z_n = \frac{2000\, I_n}{\omega_n^2 - \bar{\omega}^2}$ (in)	$\Phi_n\left(x = \frac{L}{2}\right)$
1	67.28	4.730	0.8380	−0.0194	1.588
2	185.45	7.853	0	0	0
3	363.56	10.996	0.3640	0.0173	−1.410
4	500.98	14.137	0	0	0
5	897.76	17.279	0.2323	0.00065	1.414

is given for the first five modes in Table 20.3. The modal steady-state response is

$$z_n(t) = \frac{2000 I_n}{\omega_n^2 - (300)^2} \sin 300t. \tag{d}$$

The numerical calculations are conveniently presented in Table 20.6.

The deflections at midspan of the beam are then calculated from eq. (b) and values in Table 20.6 as

$$y\left(\frac{L}{2}, t\right) = [(1.588)(-0.0194) + (-1.410)(0.0173) + (1.414)(0.00065)] \sin 300t$$

$$y\left(\frac{L}{2}, t\right) = -0.0266 \sin 300t \text{ (in)}.$$

20.6 DYNAMIC STRESSES IN BEAMS

To determine stresses in beams, we apply the following well-known relationships for bending moment M and shear force V, namely

$$M = EI \frac{\partial^2 y}{\partial x^2},$$

$$V = \frac{\partial M}{\partial x} = EI \frac{\partial^3 y}{\partial x^2}.$$

Therefore, the calculation of the bending moment or the shear force requires only differentiation of the deflection function $y = y(x, t)$ with respect to x. For example, in the case of the simple supported beam with a concentrated load suddenly applied at its center, differentiation of the deflection function, eq. (20.75), gives

$$M = -\frac{2\pi^2 P_0 EI}{mL^3} \sum_n \left[\frac{n^2}{\omega_n^2} \sin\frac{n\pi}{2}(1 - \cos\omega_n t) \sin\frac{n\pi x}{L}\right] \tag{20.76}$$

$$V = -\frac{2\pi^3 P_0 EI}{mL^4} \sum_n \left[\frac{n^3}{\omega_n^2} \sin\frac{n\pi}{2}(1 - \cos\omega_n t) \cos\frac{n\pi x}{L}\right]. \tag{20.77}$$

We note that the higher modes are increasingly more important for moments than for deflections and even more so for shear force, as indicated by the factors $1, n^2$, and n^3, respectively in eqs. (20.75), (20.76), and (20.77).

To illustrate, we compare the amplitudes for the first and third modes at their maximum values. Noting that ω_n^2 is proportional to n^4, we obtain from eqs. (20.75), (20.76), and (20.77) the following ratios: eq 20.

$$\frac{y_1}{y_3} = 3^4 = 81 \qquad \frac{y_1}{y_5} = 5^4 = 625$$

$$\frac{M_1}{M_3} = 3^2 = 9 \qquad \frac{M_1}{M_5} = 5^2 = 25$$

$$\frac{V_1}{V_3} = 3. \qquad \frac{V_1}{V_3} = 5$$

$\omega_n = n^2\pi^2\sqrt{\frac{E}{m}}$

$\omega_n^2 = n^4\pi^4\frac{E}{m}$

This tendency in which higher modes have increasing importance in moment and shear calculation is generally true of beam response.

In those cases in which the first mode dominates the response, it is possible to obtain approximate deflections and stresses from static values of these quantities amplified by the dynamic load factor. For example, the maximum deflection of a simple supported beam with a concentrated force at midspan may be closely approximated by

$$y\left(x = \frac{L}{2}\right) = \frac{P_0 L^3}{48EI}(1 - \cos\omega_1 t).$$

If we consider only the first mode, the corresponding value given by eq. (20.75) is

$$y\left(x = \frac{L}{2}\right) = \frac{2P_0}{mL\omega_1^2}(1 - \cos\omega_1 t).$$

Since $\omega_1^2 = \pi^4 EI/mL^4$, it follows that

$$y\left(x = \frac{L}{2}\right) = \frac{2P_0 L^3}{\pi^4 EI}(1 - \cos\omega_1 t)$$

$$= \frac{P_0 L^3}{48.7EI}(1 - \cos\omega_1 t).$$

The close agreement between these two computations is due to the fact that static deflections can also be expressed in terms of modal components, and for a beam supporting a concentrated load at midspan the first mode dominates both static and dynamic response.

20.7 SUMMARY

The dynamic analysis of single span beams with distributed properties (mass and elasticity) and subjected to flexural loading was presented in this chapter. The extension of this analysis to multispan or continuous beams and other structures, though possible, becomes increasingly complex and impractical. The results obtained from these single span beams are particularly important in evaluating approximate methods based on discrete models as those presented in preceding chapters. From such evaluation, it has been found that the stiffness method of dynamic analysis in conjunction with the consistent mass formulation provides in general satisfactory results even with a rather coarse discretization of the structure.

The natural frequencies and corresponding normal modes of single-span beams with different supports are determined by solving the differential equation of motion and imposing the corresponding boundary conditions. The normal modes satisfy the orthogonality condition between any two modes m and n, namely,

$$\int_0^L \phi_m(x)\,\phi_n(x)\,m\,dx = 0. \qquad (m \neq n).$$

The response of a continuous system may be determined as the superposition of nodal contributions, that is

$$y(x,t) = \sum_n \phi_n(x)\,z_n(t)$$

where $z_n(t)$ is the solution of n modal equation

$$\ddot{z}(t) + 2\xi_n\omega_n\dot{z}(t) + \omega_n^2 z(t) = F_n(t)/M_n$$

in which

$$F_n(t) = \int_0^L \phi_n(x)\,p(x,t)\,dx$$

and

$$M_n = \int_0^L m\phi_n^2(x)\,dx.$$

The bending moment M and the shear for V at any section of a beam are calculated from the well-known relations

$$M = EI\frac{\partial^2 y}{\partial x^2},$$

$$V = EI\frac{\partial^3 y}{\partial x^3}.$$

PROBLEMS

20.1 Determine the first three natural frequencies and corresponding modal shapes of a simply supported reinforced concrete beam having a cross section 10 in wide by 24 in deep with a span of 36 ft. Assume the flexural stiffness of the beam, $EI = 3.5 \times 10^9$ lb · in^2 and weight per unit volume $W = 150$ lb/ft^3. (Neglect shear distortion and rotary inertia.)

20.2 Solve Problem 20.1 for the beam with its two ends fixed.

20.3 Solve Problem 20.1 for the beam with one end fixed and the other simply supported.

20.4 Determine the maximum deflection at the center of the simply supported beam of Problem 20.1 when a constant force of 2000 lb is suddenly applied at 9 ft from the left support.

20.5 A simply supported beam is prismatic and has the following properties: $m = 0.3$ lb · sec^2/in per inch of span, $EI = 10^6$ lb · in^2, and $L = 150$ in. The beam is subjected to a uniform distributed static load p_0 which is suddenly removed. Write the series expression for the resulting free vibration and determine the amplitude of the first mode in terms of p_0.

20.6 The beam of Problem 20.5 is acted upon by a concentrated force given by $P(t) = 1000 \sin 500t$ lb applied at its midspan. Determine the amplitude of the steady-state motion at a quarter point from the left support in each of the first two modes. Neglect damping.

20.7 Solve Problem 20.6 assuming 10% of critical damping in each mode. Also determine the steady-state motion at the quarter point considering the first two modes.

20.8 The cantilever beam shown in Fig. P20.8 is prismatic and has the following properties: $m = 0.5$ lb · sec^2/in per inch of span, $E = 30 \times 10^6$ psi,

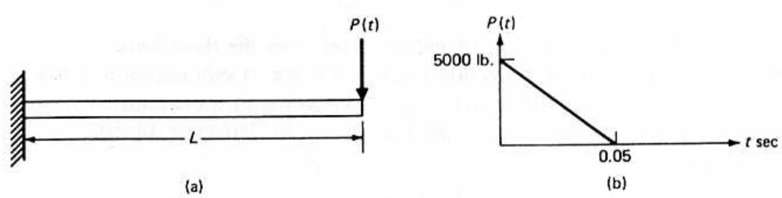


Fig. P20.8