① 1- ¿$n^2 \in O(n^3)$? Verdadero, pues:

$$n^2 \in O(n^2) \subseteq O(n^3) \quad y \quad n^2 \leq c \cdot n^3 \quad \forall c \in \mathbb{R} \wedge c > 0$$

2- ¿$n^3 \in O(n^2)$? Falso, ya que:

No existe c y n_0 tal que $n^3 \leq c \cdot n^2$ siendo $n \geq n_0$, además $n^3 \in O(n^3) \nsubseteq O(n^2)$

3- ¿$2^{n+1} \in O(2^n)$? Verdadero, por definición:

$$2^{n+1} = 2 \cdot 2^n \in O(2^n)$$

4- ¿$(n+1)! \in O(n!)$ Verdadero, por definición:

$$(n+1)! = (n+1) \cdot n! \in O(n!)$$

5- ¿$f(n) \in O(n) \rightarrow 2^{f(n)} \in O(2^n)$? Verdadero, por la primera parte

$f(n) \leq c \cdot n$, que implica: $2^{f(n)} \leq 2^{c \cdot n} \in O(2^n)$

6- ¿$3^n \in O(2^n)$? Falso, pues:

$$\cancel{3^n \leq c \cdot 2^n \rightarrow 3^n = c \cdot 2^n \rightarrow \log_3 3^n = \log_3 c + \log_3 2^n}$$

$$\cancel{n = \log_3 c + n \cdot \log_3 2 \rightarrow n(1 - \log_3 2) = \log_3 c}$$ TACHAR

$$\cancel{3^n \leq c \cdot 2^n \rightarrow 3^n = c \cdot 2^n \rightarrow c = 3^n / 2^n}$$

$\lim_{n \to \infty} \frac{2^n}{3^n} = 0$, entonces $2^n \in O(3^n)$ y $3^n \notin O(2^n)$

7- ¿$\log(n) \in O(n^{1/2})$? Verdadero, pues

$$\lim_{n \to \infty} \frac{\log(n)}{n^{1/2}} = \frac{\infty}{\infty} \xrightarrow{L'Hôpital} \lim_{n \to \infty} \frac{1/n}{1/2 \cdot (n^{-1/2})} = \frac{1}{2} \cdot \lim_{n \to \infty} \frac{1}{n} \div \frac{1}{n^{1/2}} = 0$$

Que implica: $\log(n) \in O(\sqrt{n}) \wedge \sqrt{n} \notin O(\log(n))$

8- ¿$\sqrt{n} \in O(\log(n))$? Falso, por el razonamiento anterior

9- ¿ $n^2 \in \Omega(n^3)$? Falso, ya que

$\cancel{n^2 \geq c \cdot n^3 \Rightarrow n^2(nc-1) \geq 0 \Rightarrow nc-1=0 \Rightarrow n = 1/c} \begin{cases} n_0 = 1 \\ c = 1 \end{cases}$

$\lim_{n \to \infty} \frac{n^2}{n^3} = 0$ y entonces $n^3 \in \Omega(n^2)$ y $n^2 \notin \Omega(n^3)$

10- ¿ $n^3 \in \Omega(n^2)$? Cierto, por la demonstración anterior

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$O(1) \subseteq O(\log(n)) \Rightarrow \lim_{n \to \infty} \frac{1}{\log(n)} = 0$

$O(\log n) \subseteq O(n) \Rightarrow \lim_{n \to \infty} \frac{\log(n)}{n} = \lim_{n \to \infty} \frac{1/n}{1} = 0$

$O(n) \subseteq O(\log(n) \cdot n) \Rightarrow \lim_{n \to \infty} \frac{n}{n \cdot \log(n)} = \lim_{n \to \infty} \frac{1}{\log(n)} = 0$

$O(n^2) \subseteq O(n^3) \Rightarrow \lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n} = 0$

$O(n^3) \subseteq O(n^k) \Rightarrow \lim_{n \to \infty} \frac{n^3}{n^k} = \lim_{n \to \infty} \frac{1}{n^{k-3}} = 0 \quad \forall k > 3$

$O(n^k) \subseteq O(2^n) \Rightarrow \lim_{n \to \infty} \frac{n^k}{2^n} = 0$; * (orden superior)

$O(2^n) \subseteq O(n!) \Rightarrow$ n! es de orden superior a 2^n

* Sea $f(n) = \frac{n!}{2^n}$, $\frac{f(n+1)}{f(n)} = \frac{(n \cdot n!)/(2 \cdot 2^n)}{n!/2^n} = \frac{2}{n}$, para $n \to \infty$ $f(n)$ tiende a 0

$\cancel{\text{* Sea } f(n) = \frac{n^k}{2^n} , \frac{f(n+1)}{f(n)} = \frac{(n+1)^k}{2 \cdot 2^n} \div \frac{n^k}{2^n} = \frac{(n+1)^k}{2 \cdot n^k}}$

(4) Por definición $f_1(n) = n^2 \in O(n^2)$ y $\Omega(n^2)$

Por la regla de la suma $f_2(n) = n^2 + 1000n \in O(n^2)$ y $\Omega(n^2)$

$f_3(n) = \begin{cases} n, & \text{si } n \text{ impar} \\ n^3, & \text{si } n \text{ par} \end{cases} \in O(n^3)$ y $\Omega(n^3)$ (equivale a un if/else)

$f_4(n) = \begin{cases} n & \text{si } n \leq 100 \\ n^3 & \text{si } n > 100 \end{cases} \in O(n^3)$ y $\Omega(n^3)$ (consideramos que n tiende a ∞)

(5) a) $T(n) = 3T(n-1) + 4T(n-2)$ $\quad n > 1 \quad T(0) = 0 \quad T(1) = 1$

$T(n) - 3T(n-1) - 4T(n-2) = 0$

$\downarrow x^2 = T(n)$

$x^2 - 3x - 4 = 0$

$(x+1)(x-4) = 0$

$T(n) = c_1 \cdot 4^n - c_2 \cdot 1^n = \frac{1}{3} \cdot 4^n - \frac{1}{3} \cdot 1^n \in \Theta(4^n)$

$\left.\begin{array}{l} T(0) = 0 = c_1 - c_2 \\ T(1) = 1 = 4c_1 - c_2 \end{array}\right\} \begin{array}{l} c_1 = 1/3 \\ c_2 = 1/3 \end{array}$

b) $T(n) = 2T(n-1) - (n+5) \cdot 3^n$ $\quad n > 0 \quad T(0) = 0$

$T(n) - 2T(n-1) = -(n+5) \cdot 3^n$

$\downarrow x = T(n)$ $\qquad \downarrow -n \cdot 3^n = b^n \cdot p(n) \begin{cases} b = 3 \\ p(n) = -n \\ d = 1 \end{cases}$ $\qquad \downarrow 5 \cdot 3^n \begin{cases} b = 3 \\ p(n) = 5 \\ d = 0 \end{cases}$

$(x-2)(x-3)^2(x-3) = 0$

$T(n) = c_1 \cdot 2^n + c_2 \cdot 3^n + c_3 \cdot n \cdot 3^n + c_4 \cdot n^2 \cdot 3^n$

$\left.\begin{array}{l} T(0) = 0 = c_1 + c_2 \\ T(1) = -5 = c_1 \cdot 2^1 + c_2 \cdot 3^1 + c_3 \cdot 3^1 + c_4 \cdot 3^1 \\ T(2) = -73 = c_1 \cdot 2^2 + c_2 \cdot 3^2 + 2c_3 \cdot 3^2 + 4c_4 \cdot 3^2 \\ T(3) = -362 = c_1 \cdot 2^3 + c_2 \cdot 3^3 + 3c_3 \cdot 3^3 + 9c_4 \cdot 3^3 \end{array}\right\}$

c) $T(n) = 4 \cdot T(n/2) + n^2 \qquad n>4 \qquad$ n potencia de 2 $\qquad T(1)=1 \qquad T(2)=8$

$T(n) - 4 \cdot T(n/2) = n^2$

$\downarrow n = 2^m \quad \rightsquigarrow n^2 = (2^m)^2 = 4^m$

$T(2^m) - 4\,T(2^{m-1}) = 4^m$

$\downarrow x = T(2^m) \qquad \downarrow 4^m = b^m \cdot p(m) \quad \begin{cases} b = 4 \\ p(m) = 1 \\ d = 0 \end{cases}$

$(x-4)(x-4) = 0$

$T(2^m) = c_1 \cdot 4^m + c_2 \cdot m \cdot 4^m$

$T(n) = c_1 \cdot n^2 + c_2 \cdot \log(n) \cdot n^2 = n^2 + n^2 \cdot \log_2 n \in \Theta(n^2 \cdot \log_2(n))$

$\left.\begin{array}{l} T(1) = 1 = c_1 \\ T(2) = 8 = 4c_1 + 4c_2 \end{array}\right\} \begin{array}{l} c_1 = 1 \\ c_2 = 1 \end{array}$

d) $T(n) = 2\,T(n/2) + n \cdot \log n \qquad n>1 \qquad$ n potencia de 2

$T(n) - 2T(n/2) = n \cdot \log n$

$\downarrow n = 2^m$

$T(2^m) - 2T(2^{m-1}) = 2^m \cdot \log 2^m = m \cdot 2^m$

$\downarrow x = T(2^m) \qquad \downarrow m \cdot 2^m = b^m \cdot p(m) \quad \begin{cases} b = 2 \\ p(m) = m \\ d = 1 \end{cases}$

$(x-2)(x-2)^2 = 0$

$T(2^m) = c_1 \cdot 2^m + c_2 \cdot m \cdot 2^m + c_3 \cdot m^2 \cdot 2^m$

$T(n) = c_1 \cdot n + c_2 \cdot \log_2(n) \cdot n + c_3 \cdot n \cdot (\log_2 n)^2$

Si c_1 y c_2 y $c_3 > 0$, $T(n) \in \Theta(n \cdot (\log_2(n))^2)$

e) $T(n) = 3T(n/2) + 5n + 3 \qquad n > 1 \qquad n$ potencia 2

$T(n) - 3T(n/2) = 5n + 3$

$\downarrow n = 2^m \rightarrow m = \log_2 n$

$T(2^m) - 3T(2^{m-1}) = 5 \cdot 2^m + 3$

$\downarrow x = T(2^m) \qquad \downarrow 5 \cdot 2^m = \begin{cases} b = 2 \\ p(m) = 5 \\ d = 0 \end{cases} \qquad \downarrow 3 = \begin{cases} b = 1 \\ p(m) = 3 \\ d = 0 \end{cases}$

$(x-3)(x-2)(x-1) = 0$

$T(2^m) = c_1 \cdot 3^m + c_2 \cdot 2^m + c_3 \cdot 1^m$

$T(n) = c_1 \cdot 3^{\log_2 n} + c_2 \cdot n + c_3 \cdot 1^{\log_2 n}$

$= c_1 \cdot n^{\log_2 3} + c_2 \cdot n + c_3$

Si $c_i, \forall i > 0 \quad \in \Theta(n \cdot \log_2 n)$

f) $T(n) = 2T(n/2) + \log n \qquad n > 1 \qquad n$ potencia de 2

$T(n) - 2T(n/2) = \log n$

$\downarrow n = 2^m$

$T(2^m) - 2T(2^{m-1}) = m$

$\downarrow x = T(2^m) \qquad \downarrow m = b^m \cdot p(m) \begin{cases} b = 1 \\ p(m) = m \\ d = 1 \end{cases}$

$(x-2)(x-1)^2$

$T(2^m) = c_1 \cdot 2^m + c_2 \cdot 1^m + c_3 \cdot m \cdot 1^m$

$T(n) = c_1 \cdot n + c_2 + c_3 \cdot \log_2 n$

Si $c_i > 0 \ \forall i \rightarrow T(n) \in \Theta(n)$

g) $T(n) = 2T(\sqrt{n}) + \log n \qquad n = 2^{2^k} \qquad T(2) = 1$

$T(n) - 2T(n^{1/2}) = \log n$

$\downarrow n = 2^{2^m} \rightarrow \sqrt{n} = n^{1/2} = 2^{2^{m-1}}$

$T(2^{2^m}) - 2T(2^{2^{m-1}}) = \log 2^{2^m} = 2^m$

$\downarrow x = T(2^{2^m}) \qquad \downarrow 2^m = b^m \cdot p(m) \begin{cases} b = 2 \\ p(m) = 1 \\ d = 0 \end{cases}$

$(x-2)(x-2) = 0$

$T(2^{2^m}) = c_1 \cdot 2^m + c_2 \cdot m \cdot 2^m$

$T(n) = c_1 \cdot \log_2 n + c_2 \cdot \log_2(\log_2 n) \cdot \log_2(n)$

$T(2) = 1 = c_1 \cdot \log_2(2)$

$T(4) = 4 = c_1 \cdot 2 + c_2 \cdot 2$

$T(16) = 12 = c_1 \cdot 4 + c_2 \cdot 4 \cdot 2$

$\Big\} \begin{matrix} c_1 = 1 \\ c_2 = 1 \end{matrix} \Big\} \; T(n) = \log_2(n) + \log_2(n) \cdot \log_2(\log_2(n))$

$\in \Theta(\log_2(n) \cdot \log_2(\log_2(n)))$

h) $T(n) = 5T(n/2) + (n \cdot \log_2 n)^2 \qquad n > 1$, n potencia de 2, $T(1) = 1$

$T(n) - 5T(n/2) = (n \cdot \log_2 n)^2$

$\downarrow n = 2^m \rightarrow m = \log_2 n$

$T(2^m) - 5T(2^{m-1}) = (m \cdot 2^m)^2 = (m^2 \cdot (2^m)^2) = (m^2 \cdot 4^m)$

$\downarrow x = T(2^m) \qquad \downarrow m^2 \cdot 4^m = b^m \cdot p(m) \begin{cases} b = 4 \\ p(m) = m^2 \\ d = 2 \end{cases}$

$(x-5)(x-4)^3 = 0$

$T(2^m) = c_1 \cdot 5^m + c_2 \cdot 4^m + c_3 \cdot m \cdot 4^m + c_4 \cdot m^2 \cdot 4^m$

$T(n) = c_1 \cdot n^{\log_2 5} + c_2 \cdot n^2 + c_3 \cdot \log_2(n) \cdot n^2 + c_4 \cdot n^2 \cdot (\log_2(n))^2 \qquad \in \Theta(n^2 \cdot (\log_2(n))^2)$

$T(1) = 1 = c_1 + c_2 + 0 + 0$

$T(2) = 9 = c_1 \cdot 2^{\log_2 5} + c_2 \cdot 4 + c_3 \cdot 4 + c_4 \cdot 4$

$T(4) = 73 = c_1 \cdot 4^{\log_2 5} + c_2 \cdot 16 + c_3 \cdot 32 + c_4 \cdot 64$

$T(8) = 644 = c_1 \cdot 8^{\log_2 5} + c_2 \cdot 64 + c_3 \cdot 64 \cdot 2 + c_4 \cdot 64 \cdot 16$

i) $T(n) = T(n-1) + 2T(n-2) - 2T(n-3) \quad n > 2, \quad T(n) = 9n^2 + 15n + 106 \quad n = 0, 1, 2$

$T(n) - T(n-1) - 2T(n-2) + 2T(n-3) = 0$

$\downarrow x^3 = T(n)$

$(x^3 - x^2 - 2x + 2) = 0 \rightsquigarrow (x-1)(x+\sqrt{2})(x-\sqrt{2}) = 0$

$$\begin{array}{r|rrrr} & 1 & -1 & -2 & 2 \\ 1 & & 1 & 0 & -2 \\ \hline & 1 & 0 & -2 & \boxed{0} \end{array} \rightsquigarrow (x^2-2) \in (x+\sqrt{2})(x-\sqrt{2})$$

$T(n) = c_1 \cdot 1^n + c_2 \cdot \sqrt{2}^n + c_3 \cdot (-\sqrt{2})^n$

$$\left.\begin{array}{l} T(0) = 106 = c_1 + c_2 - c_3 \\ T(1) = 100 = c_1 + \sqrt{2}c_2 - \sqrt{2}c_3 \\ T(2) = 112 = c_1 + 2c_2 + 2c_3 \end{array}\right\} \left.\begin{array}{l} c_1 = 120{,}51 \\ c_2 = -9{,}4 \\ c_3 = 5{,}1 \end{array}\right\} T(n) = \in \Theta(\sqrt{2}^n)$$

j) $T(n) = \frac{3}{2} T(n/2) - \frac{1}{2} T(n/4) - \frac{1}{n} \quad n > 2 \quad T(1) = 1 \quad T(2) = \frac{3}{2}$

$\downarrow n = 2^m$

$T(2^m) - \frac{3}{2} T(2^{m-1}) + \frac{1}{2} T(2^{m-2}) = -\frac{1}{2^m} = -(\frac{1}{2})^m$

$\downarrow x^2 = T(2^m) \qquad \downarrow -(\frac{1}{2})^m = b^m \cdot p(m) \quad \begin{cases} b = \frac{1}{2} \\ p(m) = -1 \\ d = 0 \end{cases}$

$(x^2 - \frac{3}{2}x + \frac{1}{2})(x - \frac{1}{2}) = 0$

$(x-1)(x-\frac{1}{2})^2 = 0$

$T(2^m) = c_1 \cdot 1^m + c_2 \cdot m \cdot (\frac{1}{2})^m + c_3 \cdot (\frac{1}{2})^m$

$= c_1 + c_2 \cdot \log_2(n) \cdot n^{-1} + c_3 \cdot (n^{-1})$

$$\left.\begin{array}{l} T(1) = 1 = c_1 + c_3 \\ T(2) = \frac{3}{2} = c_1 + \frac{1}{2}c_2 + \frac{1}{2}c_3 \\ T(4) = \frac{3}{2} = c_1 + \frac{1}{2}c_2 + \frac{1}{4}c_3 \end{array}\right\} \left.\begin{array}{l} c_1 = 1 \\ c_2 = 1 \\ c_3 = 0 \end{array}\right\} T(n) = 1 + \frac{\log_2(n)}{n} \in \Theta\left(\frac{\log_2(n)}{n}\right)$$

B) $T(n) = 2T(n/4) + n^{1/2}$ $n > 4$ n potencia de 4

$T(n) - 2T(n/4) = n^{1/2}$

$\downarrow n = 4^m \rightarrow m = \log_4 m$

$T(4^m) - 2T(4^{m-1}) = 2^m$

$\downarrow x = T(4^m)$ $\downarrow 2^m \begin{cases} b = 2 \\ p(m) = 1 \\ d = 0 \end{cases}$

$(x-1)(x-2) = 0$

$T(4^m) = c_1 \cdot 1^m + c_2 \cdot 2^m$

$T(n) = c_1 \cdot 1^{\log_2 n} + c_2 \cdot 2^{\log_4 n} = c_1 + c_2 \cdot n^{1/2}$

Si c_1 y $c_2 > 0 \rightarrow T(n) \in \Theta(\sqrt{n})$

B $T(n) = 4T(n/3) + n^2$ $n > 3$ n potencia de 3

$\downarrow n = 3^m$

$T(3^m) - 4T(3^{m-1}) = 9^m$

$\downarrow x = T(3^m)$ $\downarrow 9^m = \begin{cases} b = 9 \\ p(m) = 1 \\ d = 0 \end{cases}$

$(x-4)(x-9) = 0$

$T(3^m) = c_1 \cdot 4^m + c_2 \cdot 9^m$

$T(n) = c_1 \cdot n^{\log_3 4} + c_2 \cdot n^2$

Si c_1 y $c_2 > 0 \rightarrow T(n) \in \Theta(n^2)$

(6) $T_1 \in O(f)$ y $T_2 \in O(f)$

a) ¿ $T_1 + T_2 \in O(f)$? Cierto por la regla de la suma

b) ¿ $T_1 - T_2 \in O(f)$? Verdadero, $T_1 - T_2$ puede llegar a pertenecer a un orden inferior a $O(f)$, pero nunca superior

c) ¿ $T_1/T_2 \in O(1)$? Falso, sea:

$$\left.\begin{array}{l} T_1 = n^2 \in O(n^2) \\ T_2 = n \in O(n) \in O(n^2) \end{array}\right\} T_1/T_2 = n^2/n = n \in O(n) \notin O(1)$$

d) ¿ $T_1 \in O(T_2)$? Cierto, por la regla:

$$O(T_1) = O(T_2) \rightarrow T_1 \in O(T_2) \wedge T_2 \in O(T_1)$$

(7) Demostrar $\log^k n \in O(n)$ $\forall k$

$$\log^k n \leq c \cdot n \xrightarrow[n=2^m]{} (\log_2 2^m)^k \leq c \cdot 2^m$$

$$m^k \leq c \cdot 2^m$$

(8) - Árbol equilibrado 

$T(n) = 2\,T(n/2) + 1$

$T(n) - 2\,T(n/2) = 1$

$\downarrow\ n = 2^m$

$T(2^m) - 2\,T(2^{m-1}) = 1$

$\downarrow\ x = T(2^m) \qquad \downarrow\ 1 = b^m \cdot p(m) \left\{ \begin{array}{l} b = 1 \\ p(m) = 1 \\ d = 0 \end{array} \right.$

$(x-2)(x-1)$

$T(2^m) = c_1 \cdot 2^m + c_2 \cdot 1^m$

$T(n) = c_1 \cdot n + c_2$

- Árbol no equilibrado 

$T(n) = T(n-1) + 1$

$T(n) - T(n-1) = 1$

$\downarrow\ x = T(n) \qquad \downarrow\ 1 = \left\{ \begin{array}{l} b = 1 \\ p(n) = 1 \\ d = 0 \end{array} \right.$

$(x-1)(x-1)$

$T(n) = c_1 \cdot 1^n + c_2 \cdot n \cdot 1^n = c_1 + c_2 \cdot n$

Suponiendo que c_1 y $c_2 > 0$, vemos que la eficiencia del algoritmo es de n tanto para el mejor, como el peor y el caso promedio.

(9) El desarrollo y los resultados son los mismos que los del ejercicio anterior

(12) $T(n) = 1 + T(n/3) + T(n/3)$

$T(n) = 2\,T(n/3) + 1$

$\downarrow n = 3^m$

$T(3^m) - 2\,T(3^{m-1}) = 1$

$\downarrow x = T(3^m) \qquad \downarrow 1 = b^m \cdot p(m) \quad \begin{cases} b = 1 \\ p(m) = 1 \\ d = 0 \end{cases}$

$(x-2)(x-1)$

$T(3^m) = c_1 \cdot 2^m + c_2 \cdot 1^m$

$T(n) = c_1 \cdot n^{\log_3 2} + c_2$

Si c_1 y $c_2 > 0 \rightarrow T(n) \in \Theta(n^{\log_3 2})$

~~(13) $n \log(n) \in$~~

(10) $17,\ \log(\log n),\ \log n,\ \log^2 n,\ n/\log n,\ \sqrt{n},\ n,\ \sqrt{n}\cdot\log^2 n,\ n^2,\ (3/2)^n$

(2)

$O(n \cdot \log n) \subseteq O\left(\frac{n^2}{\log n}\right) \subseteq O(n^{(1+a)}) \subseteq O(n^2 \cdot \log(n)) \subseteq O(n^2 + 8n + \log^3 n) = O(n^2) \subseteq$

$\subseteq O(n^8) \subseteq O((1+a)^n) \subseteq O(2^n)$